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for road vehicle aerodynamics

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BOUNDARY ELEMENT METHODS FOR
ROAD VEHICLE AERODYNAMICS

by

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A Doctoral Thesis
Submitted in partial fulfilment of the requirements
for the award of
DOCTOR OF PHILOSOPHY
of the Loughborough University of Technology
October, 1985

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CERTIFICATE OF ORIGINALITY

This is to certify that I am responsible for the work submitted in this thesis, that the original work is my own except as specified in acknowledgements or in footnotes, and that neither the thesis nor the original work contained therein has been submitted to this or any other institution for a higher degree.
DEDICATION

To my Parents, wife and children
ACKNOWLEDGEMENTS

I begin in the name of Allah the most Beneficient, the most Merciful. All praise be to Allah, who is the Lord of the whole universe and the source of all knowledge. I thank Allah for giving me the strength and energy to complete this work and making everything possible.

I am deeply indebted to Dr. K.S. Peat for his supervision, guidance, help and encouragement at every stage during the progress of my research work and in preparation of the manuscript of this thesis. His invaluable suggestions and comments helped considerably in the development of the work presented here. He has always been friendly and cooperative and his readiness to assist me at all times has been remarkable.

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I wish to take this opportunity to express my gratitude to my parents and mother-in-law for their patience, encouragement and prayful blessings for my welfare and success. I am deeply grieved for the death of my father-in-law in my absence from Pakistan, who had always wished for my higher education but can no longer witness this thesis; my ultimate gratitude.
My deepest gratitude goes to my wife for her moral support, constant encouragement and help throughout the course of my study. Special thanks for my children who have made my stay in the U.K. joyful.

I would like to thank BL Technology Ltd. for providing me with the results of their wind tunnel tests over the models of car bodies. Thanks are also due to the British Council for providing me the financial support for the period of my studies in the U.K.

Finally, my thanks go to Mrs. Barbara Bell for her beautiful typing of this thesis.
The technique of the boundary element method consists of subdividing the boundary of the field of a function into a series of discrete elements, over which the function can vary. This technique offers important advantages over domain type solutions such as finite elements and finite differences. One of the most important features of the method is the much smaller system of equations and the considerable reduction in data required to run a program. Furthermore, the method is well-suited to problems with an infinite domain. Boundary element methods can be formulated using two different approaches called the "direct" and the "indirect" methods.

In this thesis, the author has considered various formulations of the boundary element method in the calculation of the potential flow field around a road vehicle. In practice the potential flow solution must be followed by a boundary layer analysis, and repeated iteration of both is required for a complete solution. Thus computational efficiency of the boundary element method is of paramount importance for this solution technique. The flowfield around a road vehicle is semi-infinite, due to the presence of the ground plane, and is characterized by a semi-infinite wake extending downstream of the vehicle.

The results obtained using various boundary element methods have been compared with analytical solutions for unbounded bodies, bodies in the presence of a ground plane, and semi-infinite bodies, in
order to investigate their relative advantages in all relevant situations. This has resulted in the development of a direct boundary element method that can model the flow about several bodies, each of which may have a three-dimensional wake.

Finally, the 'direct' boundary element method has been applied to calculate the flow around car body shapes and comparison between computed results and experimental results which have been obtained from models of car bodies is presented.
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LIST OF SYMBOLS

\( \hat{\mathbf{A}} \) = vector point function
\( C_p \) = pressure coefficient
\( g_1, g_2, g_3 \) = components of the normal vector
\( |\mathbf{G}| \) = magnitude of the normal vector
\( i \) = the fixed point
\( J \) = Jacobian
\( \ell \) = length of a two-dimensional jth boundary element
\( L \) = maximum number of nodes on an element
\( m \) = number of boundary elements
\( M \) = total number of nodes on the body
\( \mathbf{n} \) = normal vector
\( \mathbf{n} \) = unit normal vector to an element
\( N_i \) = shape functions
\( P \) = number of Gaussian points
\( q \) = magnitude of the velocity
\( r, \hat{r} \) = distance and vector position of any variable point from the fixed point 'i'
\( \mathbf{R} \) = position vector from the origin
\( S \) = surface enclosing the volume \( V \)
\( S_B \) = surface of the actual body
\( S_W \) = surface of the wake
\( S^* \) = surface of the sphere of radius \( \rho \)
\( S_j, -i \) = surface area of the element 'j' excluding the fixed point 'i'
\[ \hat{t} \quad = \quad \text{tangent vector} \]
\[ \hat{\tau} \quad = \quad \text{unit tangent vector} \]
\[ U \quad = \quad \text{speed of the uniform stream} \]
\[ V \quad = \quad \text{volume of the region R. Also used as the velocity} \]
\[ w_p \quad = \quad \text{Gaussian weighting coefficients} \]
\[ x_0 \quad = \quad \text{starting point of the wake} \]
\[ x, y \quad = \quad \text{cartesian plane coordinates} \]
\[ (x_q', y_q') \quad = \quad \text{coordinates of the variable point 'Q'} \]
\[ (x_i', y_i') \quad = \quad \text{coordinates of the fixed point 'i'. Also used as the nodal values of the variables x and y} \]
\[ (x_k', y_k') \quad = \quad \text{coordinates of the extreme points of the boundary elements} \]
\[ (x_m', y_m') \quad = \quad \text{coordinates of the mid-point of the two-dimensional boundary elements} \]
\[ x, y, z \quad = \quad \text{cartesian space coordinates} \]
\[ \phi \quad = \quad \text{velocity potential in the region R. Also used as a scalar point function} \]
\[ \phi' \quad = \quad \text{velocity potential in the region R'} \]
\[ \phi_i \quad = \quad \text{value of the velocity potential } \phi \text{ at the fixed point 'i'. Also used to denote the nodal values of the variable } \phi \]
\[ \phi_{c.c} \quad = \quad \text{perturbation velocity potential of the circular cylinder} \]
\[ \phi_{sphere} \quad = \quad \text{perturbation velocity potential of the sphere} \]
\[ \phi_{o.e} \quad = \quad \text{perturbation velocity potential of an ovary ellipsoid} \]
\[ \phi_{s.inf} \quad = \quad \text{perturbation velocity potential of the semi-infinite body} \]
\[ \phi_{u.s} \quad = \quad \text{velocity potential of the uniform stream} \]
\[ \phi_{\infty} \quad = \quad \text{value of the velocity potential at infinity} \]
\[ \Phi \quad = \quad \text{total velocity potential of the flow} \]
\( \Gamma \) = boundary of a two-dimensional region
\( \Gamma_j \) = surface of the jth element
\( \rho \) = radius of the sphere
\( \sigma \) = source strength
\( \mu \) = doublet strength
\( \psi \) = stream function. Also used as a scaler point function.
\( \theta \) = angle between the radius vector and the x-axis
\( r, \theta \) = polar coordinates in two-dimensions
\( \xi, \eta, \zeta \) = local coordinate system on an element
\( \xi_k, \eta_k \) = coordinates of the kth node on the element
\( \xi_p, \eta_p \) = coordinates of the pth Gaussian point
\( \rho, \theta, \eta \) = spherical polar coordinates in section (2.3)
\( x, \omega, \omega \) = cylindrical polar coordinates in section (4.4.3)
1.1 STATEMENT OF THE PROBLEM

The problem considered in this thesis is to calculate the flow field around a road vehicle using the boundary element method. Basically the aim is to calculate the velocity and the pressure distribution on the surface of the vehicle which result from its motion through the atmosphere. It will be assumed that the speed at which the vehicle is moving is constant. In fluid mechanics, the problem of the uniform rectilinear motion of a body through a fluid at rest at infinity and that of the onset flow uniform at infinity disturbed by the introduction of the same body held at rest are dynamically equivalent. The force experienced by the body in the first case is equal in magnitude and opposite in direction to the force required to hold the body at rest in the second case. For convenience, the vehicle body is regarded as being stationary and immersed in an infinite volume of the uniform onset flow. The uniform onset flow is defined as the one which would exist in the absence of the body and is taken as a uniform stream of unit magnitude. The component of the resultant force exerted by the fluid on the body in the direction of the uniform onset flow of the fluid is called the drag force, whilst the component of the resultant force in the direction normal to the flow direction is
called the lift force.

The flow around road vehicles is of low Mach number, therefore it can be considered as incompressible. Furthermore, this flow is of high Reynolds number which implies that the viscosity of the fluid has negligible effect except very near to the body surface, where a thin boundary layer will exist. Thus, provided that there is no input vorticity in the free stream, the whole of the flow field exterior to the thin boundary layer can be considered to be irrotational. The assumptions of incompressible and inviscid flow cause great simplifications to the general equations of aerodynamics. The resultant potential flow analysis will give a first approximation to the real flow field, and this approximation can later be refined through the use of boundary layer solutions. Since the flow can be considered incompressible, the equation of continuity becomes:

\[ \text{div } \vec{V} = 0 \]  \hspace{1cm} (1.1)

where \( \vec{V} \) is the velocity of the fluid at any point of the flowfield. Further, if the flow is taken to be irrotational, then

\[ \vec{V} = -\text{grad } \phi \]  \hspace{1cm} (1.2)

where \( \phi \) is the total velocity potential of the flowfield. From equations (1.1) and (1.2), it follows that

\[ \nabla^2 \phi = 0 \]  \hspace{1cm} (1.3)

which is Laplace's equation.

The boundary condition on the surface \( S \) of the body is derived from the requirement that on a stationary impervious surface \( S \), the normal component of the fluid velocity is zero. Thus,
\[ \hat{n} \cdot \nabla \bigg|_S = \hat{n} \cdot \text{grad} \phi \bigg|_S = \frac{\partial \phi}{\partial n} \bigg|_S = 0 \]  

\[ (1.4) \]

where \( \hat{n} \) is the outward drawn unit normal vector to the surface \( S \).

For the exterior flow problems, the condition that at infinity the function \( \phi \) approaches the uniform stream potential must also be satisfied, that is

\[ \phi \rightarrow \phi_{u.s} \text{ at infinity} \]

\[ (1.5) \]

where \( \phi_{u.s} \) is the velocity potential of the uniform stream.

The problem is to find the function \( \phi \) which satisfies equation (1.3) together with the boundary conditions in equations (1.4) and (1.5). It is convenient to write \( \phi \) as

\[ \phi = \phi_{u.s} + \phi' \]

\[ (1.6) \]

where \( \phi' \) is the disturbance velocity potential due to the body. Since \( \phi_{u.s} \) is known, the problem reduces to finding the disturbance or perturbation velocity potential of the body, \( \phi' \), which satisfies the following equations:

\[ \nabla^2 \phi' = 0 \]

\[ (1.7) \]

\[ \frac{\partial \phi'}{\partial n} \bigg|_S = - \frac{\partial \phi_{u.s}}{\partial n} \bigg|_S \]

\[ (1.8) \]

and \( \phi' \rightarrow 0 \) at infinity.

\[ (1.9) \]

Equations (1.7), (1.8) and (1.9) comprise a well-posed boundary value problem for the perturbation velocity potential \( \phi' \). Once \( \phi' \) is known,
the total velocity potential $\phi$ of the flow can be found from equation (1.6) and the velocity $\vec{V}$ of the flow field is given by equation (1.2). The pressure coefficient $C_p$ at any point on the surface of the body can then be found from Bernoulli's equation which reduces to

$$C_p = 1 - \left( \frac{V}{U} \right)^2 = 1 - V^2$$

(1.10)
since the speed $U$ of the uniform stream has been taken as unity.

For a body such as a road vehicle, which lies near the ground, simulation of the ground flow is an important feature to be incorporated into the theory. The inviscid flow over the ground, which is considered to be a plane, remains everywhere tangential to its surface. This flow is simulated by the so called "mirror image" principle. A mirror image of the vehicle is imagined to be present below the ground. The flow around the vehicle and its image is then symmetrical with respect to the ground plane. The plane of symmetry is a streamline or stream surface and represents the inviscid flow over the ground.

The real flow field around a road vehicle is further characterized by the presence of a wake which starts from the rear end of the body and extends to infinity in the direction of the onset flow. This wake is due to the separation of the boundary layer from the surface of the body and the boundary layer is caused by the viscosity of the fluid. If the fluid were truly ideal then there would be no boundary layer and no wake and hence there would be no drag on an object immersed into the flowing fluid. Whilst the wake is caused by separation of the boundary layer, it is of much larger scale and hence larger influence on the total flow field than is the boundary
layer. Thus even a first attempt at a potential flow solution must include some approximation to account for the existence of a wake. The technique used in this thesis is to model the separation stream surface containing the wake as a continuation of the true body surface, which will then extend to infinity.

1.2 LITERATURE SURVEY

The role of vehicle aerodynamics is of growing importance for the design of vehicles, see for example Hucho [1] and Landahl [2]. At present, the study of vehicle aerodynamics is based almost solely on empirical methods, the wind tunnel being the most essential design tool. Almost all major automanufacturers operate their own wind tunnels, most of which are suitable for full-sized cars. The current capital cost of a wind tunnel for full-sized cars is around ten million pounds and the running costs of a tunnel are of the order of thousands of pounds per hour. Thus there is great potential cost benefit in the development of any system which can eliminate or reduce the need for wind tunnel tests. In recent years, a few attempts have been made to calculate the flow field around road vehicles with the aid of theoretical methods. The work presented in this thesis is a step in this direction. The author is well aware of the fact that a more sophisticated flow model than the one outlined here is needed, if it should be applicable as a design tool. It seems doubtful that the theoretical methods will completely replace the wind tunnel tests on road vehicles in the foreseeable future. However, theoretical methods can feasibly be developed to a stage where they
can be used as a guide to the basic aerodynamic design of road
vehicles, thus reducing the amount of wind tunnel testing required
and the cost of major design changes at a late stage of development.
It therefore appears that the theoretical methods will be of
increasing importance in the calculation of the flow field around
road vehicles. At present the available theoretical methods for the
flow field calculations are the methods of finite difference (Hirt and
Ramshaw [3], Markatos [4] and Demuren and Rodi [5]), finite elements
Ecer [6] and boundary elements.

Boundary element methods offer important advantages over the
'domain' type methods such as finite elements or finite differences.
One of the advantages is that with boundary elements one only has to
define the surface of the body, whereas with field methods it is
necessary to mesh the entire flow field. The amount of input data
for a boundary element method is therefore significantly less than for
a field method, which is a very important advantage in practice, as
many hours can be spent in preparing and checking the data for finite
element or finite difference programs. Furthermore, for a given
problem the boundary element method will have a much smaller system
size than a field method, and will therefore be computationally more
efficient. Boundary element methods are also well-suited to solve
problems where some boundary conditions are applied at infinity, as is
the case for exterior vehicle aerodynamics. In this thesis
attention will be given only to boundary element methods.

Over the past few years the importance of boundary elements have
been widely recognised and numerous papers and other works have been
published in this field. These methods are presented under different
names such as 'panel methods', 'surface singularity methods',

'boundary integral equation methods' or 'boundary integral solutions'.
The name 'boundary element methods', which has originated from
Southampton University, describes the basic technique of the method
which consists in subdividing the boundary of the region under
consideration into a series of elements. These methods have been
successfully applied in a number of fields, for example elasticity,
potential theory, elastostatics and elastodynamics, see Brebbia [26].
Boundary element methods can be formulated using two different
approaches called the 'direct' method and the 'indirect' method.
For potential flow problems, the direct method can be expressed in the
form of an integral equation which relates the value of the potential
at any point within the flow field with the values of the potential
and potential derivatives over the surface of the body and thus the
unknowns are calculated in the form of potential and potential
derivatives over the body surface. On the other hand, the indirect
method is based on the distribution of singularities, such as sources
or doublets, over the surface of the body and computes the unknowns
in the form of singularity strengths.

The indirect method has been used for many years in the past for
flow field calculations due to its simplicity. The first work on
flow field calculations around three-dimensional bodies is probably
that by Hess and Smith [7] and [8]. Their method utilized a constant
source distribution over the surface of the body and is therefore
classified as a 'lower-order indirect' method. Most of the work on
flowfield calculations using boundary element methods has been done in
the field of aircraft aerodynamics. Over the past decade, boundary
element methods have seen a trend towards higher-order formulations,
for which the singularity strength and/or geometry can vary over an
element according to some higher-order polynomial, see for example Hess and Martin [9] and Stafford [16]. It was argued that the higher-order boundary element methods would give comparable accuracy to low-order methods at lower computing costs. Hess and Martin [9] developed a higher-order indirect method and applied it to calculate the flow past axisymmetric bodies. They showed that for the flows which they considered, the higher-order methods were computationally more efficient than low-order methods. In contrast Stafford [16] and others have not directly compared their high-order methods with low-order methods. Hess [10] has presented a higher-order indirect boundary element method for the numerical solutions of two-dimensional potential flow problems. The method utilizes a source distribution over the surface of the body. He has established the superiority of the higher order methods by considering the accuracy with which methods, of various order, model specific cases. He has, unfortunately, used the same number of elements for each order method, and thus his conclusion is biased, in that for a given computing effort one could use more low-order elements than high-order elements. Furthermore, Hess concludes that elements should have one higher degree of variation in geometry than source distribution. He reaches this conclusion on the assumption that the actual variation of body geometry and the required source distribution over it are polynomials of similar form. In practice the velocity potential is likely to vary more significantly than does the body geometry. For instance, the flow about a cube would need only six zero order elements to model the geometry exactly, but such a source distribution on the elements would not possibly approximate the variation of potential.
A direct boundary element method for potential flow problems has been applied only once in the past, by Morino et al [11]. The method does not require the concept of sources or doublets, but instead utilizes a distribution of constant potential over the quadrilateral elements of the body surface. From the results obtained with this method, they conclude that this method is not only more general and flexible than other existing methods, but is also efficient and accurate. In practice, the direct method is less general and flexible than the indirect method as there is no basic freedom of choice in the solution of a given problem. With the indirect method one can choose a singularity type to 'best' model a given system. One of the disadvantages in the formulation given by Morino et al [11] is that in his analytical evaluation of the integrals he has made an approximation which effectively implies that the element is flat, thus creating possible 'holes' in the surface of the body. Furthermore, it becomes impossible to use analytical integration when the variation of the potential is taken to be of higher order. Finally, with the use of a constant potential distribution on each element, the velocity at the centroid of each element ultimately has to be found by a finite-difference method. In contrast, if a higher order variation of the potential is used, the velocity at the centroid of each element follows simply through use of the shape functions.

Maskew [12] has presented a low-order indirect boundary element method for the calculation of the aerodynamic characteristics of general configurations. The method uses constant source and doublet singularities. He demonstrates that by choosing a particular type of the interior flow, his indirect method coincides with the direct method given by Morino et al [11]. He says that the higher order
methods are not advantageous in regions of close interaction between singularities, where the main factor affecting the solution accuracy is the density of control points (points where the zero normal velocity boundary condition is satisfied), while the order of the singularity distribution has only a small influence. Maskew has formed the judgement on the comparison of high order versus low-order formulations for the numerical solution of a two-dimensional problem of a vortex positioned close to a right-angled corner. From the comparisons of his computed results with an analytical solution, he concludes that the low-order boundary element method gives comparable accuracy at lower computing costs than higher-order methods. It should be noted that the choice of a non-curved boundary for this test case favours the low-order methods, in the sense that one of the main theoretical advantages of a high-order method is its ability to closely model curved boundaries.

In addition to the main application of boundary element methods in the calculation of the flow field around aircraft, they have also been applied to calculate the flow field around trains and road vehicles. The first work on calculating the flow field around road vehicles was by Stafford [13]. He used the indirect boundary element method with a distribution of vortex circuits over the surface of the body and over a postulated wake. The method predicts adequately the pressure distribution only in the attached flow region. Ahmed and Hucho [14] and Berta et al [15] have also used the low-order indirect boundary element methods for the calculation of flow field around road vehicles. In the former case the length of the wake was taken to be equal to the length of the vehicle and both of these were discretized into flat quadrilateral elements. The body surface was covered with a
distribution of sources and the wake surface was covered with doublets. It was found necessary to extend the wake inside the body surface in order to remove numerical problems at the body/wake interaction region. The method then gave fairly good predictions of the pressure distribution over that part of the body where the flow remained attached. The drawback of this method is that the wake model contributes as much to the overall system size as does the body surface, which is very poor from the point of view of computer storage and time. Berta et al [15] completely ignored the wake, with the result that the potential flow solution only modelled the attached flow over the front portion of the body.

Stafford [16] has introduced a higher-order indirect boundary element method for the calculations of the flow field around road vehicles. In this method the separation over the body surface was taken to start from some assumed position. The wake, which is of infinite length, has been taken to be of an assumed shape. The surface of the vehicle was covered with a source distribution on the forebody and a doublet distribution on the afterbody, while the wake surface was represented by a distribution of doublets. Again, the results obtained were not satisfactory in the separated flow region as the method is not able to take account of the real flow phenomena. The comparisons of these computed results with those using a low-order method were not presented, thus the efficiency of this higher order method cannot be established.

Djojodihardjo [17] has developed an indirect boundary element method for the calculation of the flow field around road vehicles. The method uses a distribution of constant doublet singularities over the surfaces of the body and the wake. The wake has been assumed to
be a sheet of zero thickness emerging from the upper trailing edge of the vehicle and extending to infinity downstream, parallel to the free-stream. The surfaces of the body and that of the wake have been discretized into quadrilateral elements. He concludes that the method can predict the velocity potential with reasonable accuracy, but numerical instabilities exist in the evaluation of the tangential velocity distribution and therefore further refinement is required to predict surface pressure distributions accurately. Furthermore, the comparisons of the results obtained using this method with those of the analytical results, even in the case of the flow past a sphere, indicate that the agreement in pressure distribution is only fair. The poor agreement in the case of the sphere might be due to taking an insufficient number of elements over the surface of the sphere. For the case of a road vehicle, the wake has not been modelled correctly. It has been taken as a sheet of zero thickness, but in reality a road vehicle is a bluff body and will have a three-dimensional wake. Further, it is not clear how much length of the wake has been discretized. The method has again the drawback of taking large computing effort.

In recent years attention has been given to the calculation of incompressible separated flows and wakes behind bluff bodies. Losito et al [18] have discussed a method for the numerical solutions of potential and viscous flows around road vehicles. The inviscid flow has been calculated with the indirect boundary element method using a constant source distribution. Viscous flow calculations have been restricted to the prediction of two-dimensional boundary layers along three-dimensional streamlines. The basic potential method can reasonably predict the pressure distribution, which agrees with the
experiments in the attached flow region, while realistic prediction
of separation lines can be achieved by using two-dimensional finite
difference viscous computations along three-dimensional inviscid
stream lines. The accuracy and validity of the viscous model adopted
cannot be checked due to the lack of experimental data. Losito and
Nicola [19] have used the same method mentioned above for the
simulation of axisymmetric wakes behind cylindrical blunt-based
bodies. They conclude that the proposed method can be easily applied
for the simulation of wakes behind road vehicles, but, as this would
require a fully three-dimensional boundary layer analysis, it is not
a simple extension of their previous work. Chomenton [20] has
presented an indirect boundary element method for calculating three-
dimensional separated flow around road vehicles. His method can
predict accurately the structure of the wake only when the separation
line is determined empirically. Hirschel et al [21] have used
boundary layer theory to study the three-dimensional turbulent
boundary layer development on car bodies. The boundary layer
computation follows from a potential flow solution which has been
obtained using the indirect boundary element method with a source
distribution. This potential flow method cannot be used to model a
lifting wake but the complete method does predict location of the
separation line to good accuracy.

Summa and Maskew [22] have presented an indirect boundary element
method for the prediction of automobile aerodynamic characteristics.
The method couples the potential flow solution with the integral
boundary layer solution. The wake of the vehicle is initially
assumed to have some prescribed geometry and is taken to be equal to
the length of the vehicle, which has been discretized into flat
quadrilateral elements. An iterative potential flow/boundary layer solution scheme is adopted, within which the wake pattern is determined at each iteration. The surface of the vehicle has a combined distribution of constant sources and doublets while the wake surface has been represented by a linear doublet distribution. The source and doublet distribution on the body was chosen such that the flow internal to the body has zero potential. The unseparated flow, pressure distribution on general body geometries has been calculated accurately, while the separated flow model has to be improved for general body shapes. The method has the usual drawback of large computing effort.

As mentioned above the real flow field around road vehicles is found by first calculating the inviscid flow field. The calculated velocity distribution is then used in a boundary layer analysis to determine the displacement thickness and separation line of the boundary layer. This analysis can in turn be used to alter the boundary condition of the potential flow solution, which will then give a new velocity distribution for use in the boundary layer analysis program. These two solutions are repeated iteratively until convergence occurs. Thus the potential flow solution can be regarded as the basic solution of the flow field problem. The accuracy and efficiency of the method to calculate the real flow field therefore depends upon the accuracy and efficiency of the basic inviscid flow solution. The aim of this thesis is to establish which is the most accurate and efficient potential flow solution for use with vehicle aerodynamics. In the past the inviscid flow field around road vehicles has always been obtained using an indirect boundary element method. As mentioned previously the direct boundary element method
has been used only by Morino et al [11] to calculate the flow field around aircraft. Thus the need arises to apply the direct boundary element method to calculate the inviscid flow around road vehicles and compare it with indirect solutions. Furthermore, the relative advantage of low-order to high-order elements for both direct and indirect methods has not been clearly established in previous work.

1.3 BRIEF DESCRIPTION OF THE METHOD OF SOLUTION

Despite the fact that Laplace's equation is one of the simplest and best known of all partial differential equations, the number of exact analytical solutions is quite small. The difficulty of course lies in satisfying the boundary conditions. Since exact solutions are scarce, the boundary value problem defined by equations (1.7), (1.8) and (1.9) will be solved using numerical methods. The method for the numerical solution is based on the reduction of the problem to an integral equation over the surface of the body under consideration. It has been shown in section (2) how the Laplace's equation (1.7) using Green's theorem can be reduced to the boundary integral equations for the direct and indirect boundary element methods. Once this is achieved, the next step is to solve the resulting boundary integral equation. The method adopted for the numerical solution of the integral equation consists in approximating the integral equation by a set of linear algebraic equations. This is achieved by subdividing the boundary surface of the body about which the flow is to be computed, into a large number of elements. If it were desired to represent the body surface exactly by means of analytic expressions, the type of the
bodies that could be handled would have to be restricted. To allow arbitrary bodies to be considered, it is, therefore natural to require the body surface to be specified by a set of points distributed over the surface. These points should be distributed in such a way that the best representation of the body is obtained with the fewest possible points. In particular, points should be concentrated in regions where the curvature of the body is large and in regions where the flow velocity is expected to change rapidly, while points may be distributed sparsely in regions where neither the body geometry nor the flow properties are varying significantly. Once the body surface has been approximated by the elements, a known variation of the function is assumed over each of these elements. The integrals over each of the boundary elements are evaluated using numerical quadrature and thus a set of simultaneous linear equations is obtained. In this work, a Gauss-quadrature rule has been used to evaluate the integrals over the elements. The set of simultaneous linear equations can always be solved by the Gauss Elimination method. In some formulations of the boundary element method, the equations have diagonal dominance and have been solved by an iterative method, Gauss-Seidel, with large savings on computer time. Once the system of linear equations have been solved, the velocity of the flow field at any point can be obtained by summing the contributions of all the surface elements and adding the contribution of the uniform onset flow. The pressure coefficient at any point of the flow field can then be obtained from equation (1.10).

In section (2) the constitutive equations for the boundary element methods for the interior and exterior domains are derived, and in section (3) the various types of boundary elements, the matrix formulation for the equation of the direct method, the numerical
evaluation of the integrals over an element and the solution of the resultant system of equations are discussed. In section (4) the direct boundary element method is applied to the two and three-dimensional potential problems and the comparisons of the analytical and computed results are presented. Also, the ground plane problem and the modelling of wake are discussed in this section. In section (5) comparison of the direct and indirect methods is presented, while in section (6) different quadrature schemes are compared over the various types of boundary elements. In section (7), the direct method is applied to calculate the flow field around models of car bodies and the comparisons of the experimental and the computed pressure coefficients are presented. Finally, conclusions and suggestions for further work are given in section (8).
2.1 INTRODUCTION

Boundary element problems can be formulated using two different approaches called the 'direct' and indirect' methods. The direct method takes the form of a statement which provides the values of the unknown variables at any field point in terms of the complete set of all the boundary data. The indirect method utilizes a distribution of singularities over the surface of the body and computes this distribution as the solution of an integral equation. For example, a source density distribution is obtained as the solution of a Fredholm integral equation of the second kind. The equation of the direct method can be formulated using either an approach based on Green's theorem (see for example Lamb [23], Milne-Thomson [24], Kellogg [25]) or as a particular case of the weighted residual methods (Brebbia [26]). The equation for the indirect method can be deduced from the equation for the direct method and can also be interpreted as a weighted residual formulation.

In the following section the derivation of the constitutive equations for the direct method will be outlined. The formulation follows the Green's theorem approach and is extended to consider problems of infinite domain. Finally, the derivation of the equations for the indirect method is discussed and it is shown that the direct method is equivalent to one particular case of the many possible formulations which follow from the indirect method.
2.2 DERIVATION OF THE EQUATION FOR THE DIRECT BOUNDARY ELEMENT METHOD

Let \( V \) be the volume of a region \( R \) bounded by a closed surface \( S \) and let \( \hat{n} \) be the outward drawn unit normal to \( S \). If \( \vec{A} \) is any vector function of position with continuous partial derivatives in \( R \), then Gauss' divergence theorem (Green's theorem in space) states that

\[
\iiint_V \nabla \cdot \vec{A} \, dV = \iint_S \vec{A} \cdot \hat{n} \, dS \tag{2.2.1}
\]

Let \( \vec{A} = \phi \vec{V} \psi \) in equation (2.2.1), where \( \phi \) and \( \psi \) are scalar functions of position with continuous derivatives of the second order at least then

\[
\iiint_V \nabla \cdot (\phi \vec{V} \psi) \, dV = \iint_S (\phi \vec{V} \psi) \cdot \hat{n} \, dS
\]

\[
= \iint_S (\phi \vec{V} \psi) \cdot d\vec{S} \tag{2.2.2}
\]

But \( \nabla \cdot (\phi \vec{V} \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \vec{V} \psi \)

Thus equation (2.2.2) becomes

\[
\iiint_V [\phi \nabla^2 \psi + (\nabla \psi) \cdot (\vec{V} \psi)] \, dV = \iint_S (\phi \vec{V} \psi) \cdot d\vec{S} \tag{2.2.3}
\]

Equation (2.2.3) is usually called Green's first identity.
Interchanging $\phi$ and $\psi$ in equation (2.2.3),

$$\iiint_{V} [\phi \nabla^{2} \phi + (\nabla \phi) \cdot (\nabla \phi)] dV = \iiint_{S} (\phi \nabla \phi) \cdot d\hat{S}$$  \hspace{1cm} (2.2.4)$$

and subtracting (2.2.4) from (2.2.3), then

$$\iiint_{V} (\psi \nabla^{2} \psi - \psi \nabla^{2} \phi) dV = \iiint_{S} (\psi \nabla \psi - \psi \nabla \phi) \cdot d\hat{S}$$  \hspace{1cm} (2.2.5)$$

Equation (2.2.5) is known as Green's second identity or symmetrical theorem.

Since $\hat{n} \cdot \nabla = \frac{\partial}{\partial n}$, equation (2.2.5) can be written as

$$\iiint_{V} (\psi \nabla^{2} \psi - \psi \nabla^{2} \phi) dV = \iiint_{S} \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] dS$$  \hspace{1cm} (2.2.6)$$

Let $i(x,y,z)$ be any fixed point exterior to $S$, and take $\psi = \frac{1}{r}$, where $r$ is the distance from the point 'i' to any variable point $Q$ within $V$.

![Figure 2.1](image_url)

The function $\left( \frac{1}{r} \right)$ is termed the fundamental solution of Laplace's equation in three dimensions in that $\nabla^{2} \left( \frac{1}{r} \right) = 0$. 
Then equation (2.2.6) reduces to

\[ \iint_S \left( \frac{\partial}{\partial n} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial \phi}{\partial n} \right) dS + \iiint_V \frac{1}{r} \nabla^2 \phi \, dV = 0 \]  

(2.2.7)

Suppose now that the point 'i' is taken within S, so that

\[ \frac{1}{r} \]  

is infinite at 'i'. Now identity (2.2.7) cannot be applied to the whole region within S.

![Figure 2.2](image)

To overcome this difficulty, consider a small sphere E with centre at 'i' and radius \( \varepsilon \) so small that \( \Xi \) lies entirely within S as shown in figure (2.2). Let \( V' \) be the volume of the region \( R' \) between S and \( \Xi \) and let \( S' \) be the surface bounding \( V' \). Let \( \hat{n} \) denote the unit normal to \( S' \), drawn outwards from \( V' \), then since 'i' is outside \( S' \), it follows from equation (2.2.7) that

\[ \iint_{S'} \left( \frac{\partial}{\partial n} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial \phi}{\partial n} \right) dS + \iiint_{V'} \frac{1}{r} \nabla^2 \phi \, dV = 0 \]

or

\[ \iint_{S + \Xi} \left( \frac{\partial}{\partial n} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial \phi}{\partial n} \right) dS + \iiint_{V'} \frac{1}{r} \nabla^2 \phi \, dV = 0 \]  

(2.2.8)
On $\Sigma$ the unit normal outwards from $V'$ goes into the interior of $\Sigma$. Thus on $\Sigma$, $r = \varepsilon$ and

$$\frac{\partial}{\partial n} \left( \frac{1}{r} \right) \bigg|_{r=\varepsilon} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \right) \bigg|_{r=\varepsilon} = 1/\varepsilon^2$$

since $r$ and $\hat{n}$ are in opposite direction in $I$. Hence

$$\iint_{\Sigma} \left( \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial \phi}{\partial n} \right) dS \sim \left[ \phi \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \frac{2\phi}{\partial n} \right] 4\pi\varepsilon^2$$

$$\sim 4\pi\phi_i \text{ as } \varepsilon \to 0$$

where $\phi_i$ is the value of $\phi$ at 'i'.

Thus in the limit as $\varepsilon \to 0$, equation (2.2.8) becomes

$$4\pi\phi_i = \iint_{S} \left( \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \right) dS - \iiint_{V} \frac{1}{r} \nabla^2 \phi \ dV \quad (2.2.9)$$

Equation (2.2.9) is often called Green's third identity.

Consider lastly the case when the point 'i' lies on a surface $S$ bounding a region $R$ of volume $V$.

Figure 2.3
Again, in this case, the function $\frac{1}{r}$ becomes infinite at 'i'. The singular point 'i' can be excluded from the region $R$ by the construction of a hemisphere $\Sigma$ of radius $\varepsilon$, centred on 'i' on the assumption that the surface $S$ is smooth. Let $V'$ be the volume of the region bounded by $S$ and $\Sigma$ and $S'$ be the surface bounding $V'$. Then from equation (2.2.7),

$$\iiint_{S'} \left( \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial \phi}{\partial n} \right) dS + \iiint_{V'} \frac{1}{r} \nabla^2 \phi \, dV = 0$$

or

$$\iiint_{(S-\Sigma)+\Sigma} \left( \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial \phi}{\partial n} \right) dS + \iiint_{V'} \frac{1}{r} \nabla^2 \phi \, dV = 0 \quad (2.2.10)$$

Once again on $\Sigma$, $r = \varepsilon$ and $\mathbf{r}$ and $\mathbf{n}$ are in opposite directions. Thus

$$\iiint_{\Sigma} \left( \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) - \frac{1}{r} \frac{\partial \phi}{\partial n} \right) dS \approx \left( \phi \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \frac{\partial \phi}{\partial n} \right) \cdot 2\pi \varepsilon^2$$

$$\approx 2\pi \phi_{i} \quad \text{as} \quad \varepsilon \to 0$$

Thus in the limit when $\varepsilon \to 0$, equation (2.2.10) becomes

$$2\pi \phi_{i} = \iiint_{S-i} \left( \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \right) dS - \iiint_{V} \frac{1}{r} \nabla^2 \phi \, dV \quad (2.2.11)$$

where $S-i$ signifies that the point 'i' is excluded from the surface integral.
Equations (2.2.7), (2.2.9) and (2.2.11) can be combined into a single form

\[ c_i \phi_i = \frac{1}{4\pi} \int_{S-i} \left[ \frac{1}{r} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \right] dS - \frac{1}{4\pi} \int_V \frac{1}{r} \nabla^2 \phi \, dV \quad (2.2.12) \]

where \( c_i = 0 \) when \( i \) is exterior to \( R \)

\( = 1 \) " " " interior " "

\( = \frac{1}{2} \) when \( i \) lies on \( S \) and \( S \) is smooth.

In particular, if \( \phi \) is harmonic in \( R \), then \( \nabla^2 \phi = 0 \) and equation (2.2.12) reduces to

\[ c_i \phi_i = \frac{1}{4\pi} \int_{S} \frac{1}{r} \frac{\partial \phi}{\partial n} \, dS - \frac{1}{4\pi} \int_{S-i} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS \quad (2.2.13) \]

This is the equation for the direct boundary element method and it relates the value of \( \phi \) at any point 'i' with the values of \( \phi \) and \( \frac{\partial \phi}{\partial n} \) over the surface \( S \). The first term on the R.H.S of equation (2.2.13) is the velocity potential due to a surface distribution of sources of strength \( \frac{\partial \phi}{\partial n} \) per unit area. The second term is the velocity potential for a surface distribution of doublets with axes normal to \( S \) and of strength \(-\phi\) per unit area. Equation (2.2.13) holds for three-dimensional solutions of Laplace's equation.

For two-dimensional problems, the fundamental solution is \( \log \left( \frac{1}{r} \right) \). Let \( \Gamma \) be the closed boundary of a two-dimensional region \( R \), then the equation for the direct method takes the form
\[ c_i \phi_i = \frac{1}{2\pi} \int_{\Gamma} \log \left( \frac{1}{r} \right) \frac{\partial \phi}{\partial n} \, d\Gamma - \frac{1}{2\pi} \int_{\Gamma-i} \phi \frac{\partial}{\partial n} \left( \log \left( \frac{1}{r} \right) \right) \, d\Gamma \tag{2.2.14} \]

where
- \( c_i = 0 \) when \( i \) is exterior to \( \Gamma \)
- \( = 1 \) " " " interior " "
- \( = \frac{1}{4} \) when \( i \) lies on \( \Gamma \) and \( \Gamma \) is smooth.

If the unit normal vector \( \hat{n} \) is drawn inwards to the domain, then the equations for the direct method for three- and two-dimensional problems can be written from equation (2.2.13) and (2.2.14) by replacing \( n \) by \( -n \).

### 2.3 FORMULATION OF THE DIRECT BOUNDARY ELEMENT METHOD FOR AN INFINITE DOMAIN

The equations for the direct boundary element method given in section (2.2) were derived for a finite domain bounded by a surface \( S \). In problems of external vehicle aerodynamics, the flow field is exterior to \( S \), the surface of the vehicle, and is considered to be of infinite extent.

![Figure 2.4](image-url)
Let $S$, the surface of interest, be enclosed by a sphere $S'$ of radius $\rho$, whose centre lies within $S$. The construction of a cross-cut to connect $S$ and $S'$ makes the region $R$, bounded by $S$ and $S'$, simply connected. Let the unit normal $\hat{n}$ be directed into the region $R$. Green's theorem is now applicable to the region $R$, thus equation (2.2.13) for the direct method can be used in $R$ with the sign of $n$ changed due to the change in sense of the unit normal. Then

$$c_i \phi_i = -\frac{1}{4\pi} \iint_S \frac{1}{r} \frac{\partial \phi}{\partial n} \, dS + \frac{1}{4\pi} \iint_{S-i} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, dS$$

$$+ \frac{1}{4\pi} \iint_S \left( -\frac{1}{r} \frac{\partial \phi}{\partial n} + \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \right) \, dS \tag{2.3.1}$$

As $\rho$ tends to infinity, the region $R$ becomes the unbounded field surrounding the surface $S$. Over the surface $S'$, $r$ is effectively $\rho$ and is measured in the opposite sense to the inward unit normal $\hat{n}$.

Consider the last integral on the R.H.S. of equation (2.3.1) in the limit when $\rho \to \infty$. Denoting this integral by $I$, then

$$I = \lim_{\rho \to \infty} \frac{1}{4\pi} \left( \frac{2\pi}{\rho} \right) \left( -\frac{1}{\rho} \frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \right) \right) \rho^2 \sin \theta \, d\theta \, d\eta$$

where $d\theta$ and $d\eta$ are the differentials of the angles in spherical polar coordinates.
This integral must remain bounded in the limit \( p \rightarrow \infty \), which imposes a restraint upon the possible form \( \phi \) can take as \( p \rightarrow \infty \). In general there will be either no flow at infinity, hence \( \phi \) is constant, say \( \phi_\infty \), or else a steaming flow in, say, the negative \( x \) direction, which would imply that

\[
\phi = U_x + \phi_\infty \tag{2.3.2}
\]

where \( \phi \) is the velocity potential of the uniform stream. Clearly the former case is a particular instance of the latter \((U = 0)\).

Now in spherical polar coordinates

\[
x = \rho \sin \theta \cos \eta
\]

Therefore equation (2.3.2) becomes

\[
\phi = U_\rho \sin \theta \cos \eta + \phi_\infty
\]

\[
\frac{3\phi}{3\rho} = - \frac{3\phi}{3\eta} = -U \sin \theta \cos \eta
\]

\[
I = \lim_{\rho \rightarrow \infty} \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left[ U_\rho \sin^2 \theta \cos \eta + \sin \theta (U_\rho \sin \theta \cos \eta + \phi_\infty) \right] d\theta d\eta
\]

\[
= \lim_{\rho \rightarrow \infty} \frac{1}{4\pi} \int_0^{2\pi} \left[ U_\rho \cos \eta + 2\phi_\infty \right] d\eta
\]
\[ \lim_{\rho \to \infty} \frac{1}{4\pi} \left( \left| \mathcal{U}_0 \sin \eta \right|_0^{2\pi} + 2\phi_\infty \right) = \phi_\infty \]

Hence equation (2.3.1) takes the form

\[ \frac{c_i \phi_i}{\phi_\infty} = \frac{1}{4\pi} \int \frac{1}{r} \frac{\partial \phi}{\partial n} \, dS + \frac{1}{4\pi} \int \phi \frac{3}{\partial n} \left( \frac{1}{r} \right) dS \quad (2.3.3) \]

This equation is the statement of the direct boundary element method for regions of infinite domain.

For two-dimensional problems, the equation of the direct boundary element method for an infinite domain takes the form

\[ \frac{c_i \phi_i}{\phi_\infty} = \frac{1}{2\pi} \int \log \left( \frac{1}{r} \right) \frac{\partial \phi}{\partial n} \, d\Gamma + \frac{1}{2\pi} \int \phi \frac{3}{\partial n} \left( \log \left( \frac{1}{r} \right) \right) d\Gamma \quad (2.3.4) \]

2.4 DERIVATION OF THE EQUATION FOR THE INDIRECT BOUNDARY ELEMENT METHOD

Frequently the boundary element method is implemented via the application of a distribution of sources and/or doublets on the boundary and the problem is then solved in terms of the unknown strengths of the sources or doublets. This technique is called the indirect method and can be derived from the equation of the direct boundary element method.
Let $S$ be a connected closed surface which divides the space into two regions $R$ and $R'$, where $R$ is of infinite extent. Let $\hat{n}$ denote the outward drawn unit normal to $S$. Let $\phi$ and $\phi'$ denote the velocity potentials of acyclic irrotational motions in the regions $R$ and $R'$ respectively. Then if the point 'i' be internal to $R'$ and therefore external to $R$, equation (2.2.13) gives

$$\phi'_i = \frac{1}{4\pi} \iint_{S} \frac{1}{r} \frac{\partial \phi'}{\partial n} dS - \frac{1}{4\pi} \iint_{S-i} \phi' \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS$$

whilst equation (2.3.3) gives

$$0 = \phi_\infty - \frac{1}{4\pi} \iint_{S} \frac{1}{r} \frac{\partial \phi}{\partial n} dS + \frac{1}{4\pi} \iint_{S-i} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS$$

Addition of these two equations gives

$$\phi'_i = \phi_\infty + \frac{1}{4\pi} \iint_{S} \frac{1}{r} \left( \frac{\partial \phi'}{\partial n} - \frac{\partial \phi}{\partial n} \right) dS - \frac{1}{4\pi} \iint_{S-i} (\phi' - \phi) \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS \quad (2.4.1)$$
Similarly in the case when \( i \) is internal to \( R \) and hence external to \( R' \), the same equation results with \( \phi_i \) replacing \( \phi_i' \) on the L.H.S.

Most importantly, when \( i \) lies on the surface \( S \), then \( \phi_i' \) is replaced by \( \frac{1}{2}(\phi_i + \phi_i') \). The above mentioned three cases can be combined by writing

\[
\left( c_1\phi_i + (1 - c_1)\phi_i' \right) = \phi_\infty + \frac{1}{4\pi} \iint_S \frac{1}{r} \left( \frac{\partial \phi'}{\partial n} - \frac{\partial \phi}{\partial n} \right) dS
\]

\[
- \frac{1}{4\pi} \iint_{S-i} (\phi' - \phi) \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS \quad (2.4.2)
\]

where \( c_i = 0 \) when \( i \) is within \( R' \)

\[= 1 \quad \text{" " " " } R \]

\[= \frac{1}{2} \text{ when } i \text{ is on } S \text{ and } S \text{ is smooth.} \]

This is the equation for the indirect boundary element method in terms of distributions of sources and doublets. Two particular forms of this expression result in a distribution of either sources only, or of doublets only, over the surface \( S \).

Consider firstly the situation when \( \phi' = \phi \) over \( S \), so that the second integral in equation (2.4.2) vanishes. This gives a continuous tangential velocity but discontinuous normal velocity over \( S \), and equation (2.4.2) becomes

\[
\left( c_i\phi_i + (1 - c_i)\phi_i' \right) = \phi_\infty + \frac{1}{4\pi} \iint_S \frac{1}{r} \left( \frac{\partial \phi}{\partial n} - \frac{\partial \phi}{\partial n} \right) dS \quad (2.4.3)
\]
for all positions of 'i'. The motion is that due to a surface distribution of sources of strength \( \frac{\partial \phi'}{\partial n} - \frac{\partial \phi}{\partial n} \) per unit area.

This is the form of the indirect boundary element method in terms of sources alone.

Secondly suppose that \( \frac{\partial \phi'}{\partial n} = \frac{\partial \phi}{\partial n} \) over the surface \( S \). The first integral in equation (2.4.2) now vanishes and it reduces to

\[
\left[ c_i \phi_i + (1 - c_i) \phi'_i \right] = \phi_\infty - \frac{1}{4\pi} \iint_{S-i} (\phi' - \phi) \frac{3}{\partial n} \left( \frac{1}{r} \right) dS \quad (2.4.4)
\]

There is now continuous normal velocity but discontinuous tangential velocity over the surface \( S \), which implies that the motion is due to a surface distribution of doublets of strength \( -(\phi' - \phi) \) per unit area, with axes normal to \( S \). Equation (2.4.4) is the indirect formulation of the boundary element method in terms of doublets alone. It may be shown (Lamb [23]) that the representations (2.4.3) and (2.4.4) are unique, whereas the representation (2.4.2) is not unique.

Of the infinitely many possible formulations of combined source and doublet distributions, one worthy of special note is the case when the interior flow potential \( \phi' = 0 \). Equation (2.4.2) then reduces to equation (2.3.3), establishing the formal equivalence of the direct and indirect methods.
SECTION 3

DISCRETE ELEMENT FORMULATION OF THE

BOUNDARY INTEGRAL EQUATION

3.1 INTRODUCTION

The equation (2.2.13) for the direct boundary element method cannot usually be solved analytically and recourse is made to numerical methods. Since this equation involves integrals to be evaluated at the surface of the body under consideration, the surface of the body is divided into a large number of boundary elements and the integral over the surface is then replaced by the sum of the integrals taken over all the elements. The boundary integral equation thus reduces to a set of simultaneous linear algebraic equations, which can be solved by simple numerical methods.

The following sections outline the various procedures which can be followed for element discretization, matrix formulation, integration over an element and solution of the resultant set of equations.
3.2 TYPES OF BOUNDARY ELEMENTS

In this section the types of the boundary elements that are used to approximate the boundary of the body for two and three-dimensional problems will be discussed. The boundary of the body is specified as a set of discrete data points. The points where the unknown functional values are sought will be termed 'nodes'. Figure 3.1 shows various simple boundary elements which can be used in the analysis of two-dimensional problems.

one-noded line element

Two-noded line element

Three-noded line element

Four-noded line element

Figure 3.1

To distinguish between the variations of the actual unknown function and that of element geometry, the following symbols are used.

(i) the points marked with a square are used to define the element geometry.

(ii) the points (nodes) marked with a filled circle are used to define the variation of the unknown function.

The number and the position of the points defining the functional variation (nodes) may or may not be the same as those
of the points defining the element geometry. In figure 3.1
the geometry of the element in each case is defined by the two
end points while the functional variation is defined by one, two,
three and four nodes in (a), (b), (c) and (d) respectively.

The interpolation functions or 'shape functions', for the
functional variation over the elements in (b), (c) and (d) are
discussed in Huebner [27] and are given as follows:

For the two-noded line element shown in figure 3.1(b), the
shape functions in terms of the local coordinate $\xi$ are

\[
N_1(\xi) = \frac{1}{2}(1-\xi), \quad N_2(\xi) = \frac{1}{2}(1+\xi), \quad -1 \leq \xi \leq 1 \quad (3.2.1)
\]

the origin being at the centroid of the element.

For the three-noded line element shown in figure 3.1(c), the
shape functions are

\[
N_1(\xi) = \frac{1}{6}\xi(\xi-1), \quad N_2(\xi) = 1-\xi^2, \quad N_3(\xi) = \frac{1}{6}\xi(\xi+1) \quad (3.2.2)
\]

and for the four-noded line element shown in figure 3.1(d), the
shape functions are

\[
N_1(\xi) = \frac{1}{16}(1-\xi)(9\xi^2-1), \quad N_2(\xi) = \frac{9}{16}(1-\xi^2)(1-3\xi)
\]

\[
N_3(\xi) = \frac{9}{16}(1-\xi^2)(1+3\xi), \quad N_4(\xi) = \frac{1}{16}(1+\xi)(9\xi^2-1) \quad (3.2.3)
\]

Any variable $\phi$ defined on the two-noded element say, can
then be approximated by
\[ \phi = [N_1 \hspace{1em} N_2] \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \]

where \( \phi_i \) are the nodal values of the variable \( \phi \). Similarly the variable \( \phi \) can be represented on the three and four-noded elements.

The element geometry can also be represented by the use of shape functions i.e. the variations of the \( x \) and \( y \) coordinates on the boundary can be expressed in terms of \( \xi \), by

\[
x = [M_1 \hspace{1em} M_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

\[
y = [M_1 \hspace{1em} M_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]

where \( M_1 \) and \( M_2 \) are the shape functions in terms of \( \xi \) for the two-noded line element and \( x_i \) and \( y_i \) are the values of \( x \) and \( y \) respectively at the end points of the element.

If the special choice is made that these shape functions \( M_i \) are exactly the same as \( N_i \) then the resulting element is called an isoparametric element. Thus for an isoparametric element

\[
x = [N_1 \hspace{1em} N_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

\[
y = [N_1 \hspace{1em} N_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]

(3.2.4)
The element shown in figure 3.1(b) is isoparametric since the same points define the geometry as well as the functional variation.

If the number of nodes defining the functional variation is less than that used to define the geometry, then the element is called super-parametric. Thus the element shown in figure 3.1(a) is super-parametric, noting that the variation of geometry is more general than that of the actual unknown. Similarly if the number of nodes used to define the variation of the unknown is greater than that used to define the geometry, then the resulting element is called sub-parametric. The elements shown in (c) and (d) of figure 3.1 are sub-parametric.

The functional variation shown in (a), (b), (c) and (d) of figure 3.1 is termed constant, linear, quadratic and cubic respectively.

For three dimensional problems, the boundary elements are part of the external surface of the body. They are usually of two types, either quadrilateral or triangular elements. In boundary element or finite element work there is a tendency to prefer quadrilateral elements, which will be used exclusively in this thesis. Figure 3.2 shows different types of quadrilateral elements used for three dimensional problems.
In figure 3.2 the geometry of quadrilateral element in each case is defined by four corner points while the functional variation is defined by one, four, eight and twelve nodes in (a), (b), (c) and (d) respectively. The functional variation in (a), (b), (c) and (d) of figure 3.2 is termed constant, linear, quadratic and cubic respectively. The elements shown in (a) and (b) of figure 3.2 are superparametric and isoparametric respectively while the elements in (c) and (d) are sub-parametric.

The shape functions for the functional variation over the elements shown in (b), (c) and (d) of figure 3.2 are discussed in Zienkiewicz [28] and are given as follows:

For the element shown in figure 3.2(b), the shape functions in terms of the local coordinates $\xi$ and $\eta$ are
where \((\xi_k', \eta_k')\) are the coordinates of the kth node.

For the element shown in figure 3.2(c), the shape functions are:

Corner nodes:

\[
N_k(\xi, \eta) = \frac{1}{4} (1 + \xi \xi_k)(1 + \eta \eta_k) (\xi \xi_k + \eta \eta_k - 1)
\]

Typical midside node: \(\xi = 0, \eta_k = \pm 1\) \hspace{1cm} (3.2.6)

\[
N_k(\xi, \eta) = \frac{1}{2} (1 - \xi^2)(1 + \eta \eta_k)
\]

where \((\xi_k', \eta_k')\) are the coordinates of the kth node.

For the element shown in figure 3.2(d), the shape functions are:

Corner nodes:

\[
N_k(\xi, \eta) = \frac{1}{32} (1 + \xi \xi_k)(1 + \eta \eta_k)[ -10 + 9(\xi^2 + \eta^2)]
\]

Typical midside node: \(\xi_k = \pm 1, \eta_k = \pm \frac{1}{3}\) \hspace{1cm} (3.2.7)

\[
N_k(\xi, \eta) = \frac{9}{32} (1 + \xi \xi_k)(1 - \eta^2)(1 + 9 \eta \eta_k)
\]

where \((\xi_k', \eta_k')\) are the coordinates of the kth node.

The boundary elements considered so far in figures 3.1 and 3.2 have been of linear geometrical variation only; in problems where the boundary is curved, the true boundary shape can be modelled more precisely for a given number of elements by using
higher order shape functions for the geometrical variation.

Curved boundary elements for two and three-dimensional problems can be obtained by transforming the standard regions shown in (a) and (b), respectively, of figure 3.3.

Figure 3.3

Figure 3.4 shows some two and three-dimensional curved boundary elements.
Each of the elements shown in figure 3.4 is isoparametric since the same points define the element geometry as well as the functional variation. The elements shown in (a) and (b) of figure 3.1 will be termed the constant and linear boundary elements and those shown in (a) and (b) of figure 3.4 as the quadratic and cubic boundary elements for the two-dimensional problems. Similarly the elements shown in (a) and (b) of figure 3.2 and those shown in (c) and (d) of figure 3.4 will be termed the constant, linear, quadratic and cubic boundary elements respectively, for the three-dimensional problems.

Figure 3.5 shows a two-dimensional body discretized into constant, linear and quadratic boundary elements, while figure 3.6 shows a three-dimensional body discretized into constant linear, quadratic and cubic boundary elements.

Figure 3.5: Discretization of a two-dimensional body into
(a) constant;
(b) linear and
(c) quadratic boundary elements.
Figure 3.6: Discretization of a three-dimensional body into
(a) constant; (b) linear; (c) quadratic; and
(d) cubic boundary elements.
3.3 MATRIX FORMULATION

In this section it will be shown how equation (2.2.13) can be transformed into matrix form. The matrix formulation of equation (2.2.14) for two-dimensional problems is discussed in Brebbia [26] and [29]. Let the surface of the body be discretized into \( m \) quadrilateral elements, then equation (2.2.13) can be written in the discretized form as

\[
s_{j-i} \sum_{j=1}^{m} \int_{S_{j-i}} \phi \frac{\partial}{\partial n} \left( \frac{\Phi}{4\pi r} \right) dS = \sum_{j=1}^{m} \int_{S_{j}} \frac{\Phi}{4\pi r} \frac{\partial}{\partial n} dS \quad (3.3.1)
\]

where \( S_{j-i} \) is the surface area of the element \( j \) excluding the point 'i'.

For the constant element case, the values of \( \phi \) and \( \frac{\partial \phi}{\partial n} \) are assumed to be constant on each element and equal to the values at the mid-node of the element. The number of nodes in this case will be the same as the number of elements \( m \). On each element one of the two variables \( \phi \) or \( \frac{\partial \phi}{\partial n} \) is specified as a boundary condition. As \( \phi \) and \( \frac{\partial \phi}{\partial n} \) are constant over each element they can be taken out of the integrals. This gives

\[
c_{i} \phi_{i} + \sum_{j=1}^{m} \left( \int_{S_{j-i}} \frac{\partial}{\partial n} \left( \frac{1}{4\pi r} \right) dS \right) \phi_{j} = \sum_{j=1}^{m} \int_{S_{j}} \frac{1}{4\pi r} \frac{\partial \phi}{\partial n} dS \quad (3.3.2)
\]
Equation (3.3.2) applies for a particular node 'i' and the integrals
\[
\iint_{S_{-i}} \frac{\partial}{\partial n} \left( \frac{1}{4\pi r} \right) dS \text{ relate the node 'i' with the element 'j' over which integrals are evaluated. These integrals will be denoted by } \hat{H}_{ij}. \text{ The integrals on the R.H.S. are of the type }
\[
\iint_{S_j} \left( \frac{1}{4\pi r} \right) dS \text{ and will be denoted by } G_{ij}. \text{ Hence equation (3.3.2) can be written as}
\[
\sum_{i=1}^{m} c_{i} \phi_{i} + \sum_{j=1}^{m} \hat{H}_{ij} \phi_{j} = \sum_{j=1}^{m} G_{ij} \frac{\partial \phi_{j}}{\partial n} \quad (3.3.3)
\]

The integrals \( \hat{H}_{ij} \) and \( G_{ij} \) in equation (3.3.3) are difficult to evaluate analytically and are usually evaluated numerically. Morino, Chen, and Suciu [11] have obtained approximate analytical solutions of these integrals for quadrilateral elements with constant functional value and linear geometrical variation, but the integrals become more difficult for higher order elements. In this thesis, the integrals over the quadrilateral elements will be calculated numerically.

Let \( H_{ij} = \begin{cases} \hat{H}_{ij} & \text{when } i \neq j \\ \hat{H}_{ij} + c_{i} & \text{when } i = j \end{cases} \quad (3.3.4) \)

Equation (3.3.3) can be rewritten as
\[
\sum_{j=1}^{m} H_{ij} \phi_{j} = \sum_{j=1}^{m} G_{ij} \frac{\partial \phi_{j}}{\partial n} \quad (3.3.5)
\]
The whole set of equations can be expressed in matrix form as

\[ [H][\Psi] = [G][Q] \quad (3.3.6) \]

Assuming that either the value of \( \phi \) or \( \frac{\partial \phi}{\partial n} \) is given as a boundary condition on each element of \( S \), then equation (3.3.6) has a set of \( m \) unknowns. Equation (3.3.6) can be reordered in such a way that all the unknowns are on the left hand side and can then be written as

\[ [A][X] = [B] \quad (3.3.7) \]

where \( \{X\} \) is a vector of the unknown values of \( \phi \) and \( \frac{\partial \phi}{\partial n} \) and \( [A] \) is the coefficient matrix. Equation (3.3.7) represents a set of \( m \) simultaneous linear equations in \( m \) unknowns and can be solved by standard methods to give the values of \( \phi \) and \( \frac{\partial \phi}{\partial n} \) on the surface \( S \).

Once this is done, the values of \( \phi \) at any point can be calculated from equation (2.2.13), where the boundary \( S \) is discretized into elements as before.

The values of the fluxes \( \frac{\partial \phi}{\partial x} \), \( \frac{\partial \phi}{\partial y} \), \( \frac{\partial \phi}{\partial z} \) can be calculated by differentiating equation (2.2.13), for example at any point 'i'

\[ c_i \left( \frac{\partial \phi}{\partial x} \right)_i = \iint_S \frac{\partial}{\partial x} \left( \frac{1}{4\pi r} \right) \frac{\partial \phi}{\partial n} \, dS - \iiint_{S-i} \frac{\partial^2}{\partial x \partial n} \left( \frac{1}{4\pi r} \right) \, dS \quad (3.3.8) \]
Thus far the matrix formulation has been restricted to the simple case for which $\phi$ and $\frac{\partial \phi}{\partial n}$ are constant over each element.

Consider next a surface which is divided into $m$ linear, quadratic or cubic quadrilateral elements i.e. quadrilaterals with four, eight or twelve nodes. In this case the number of nodes will be more than the number of elements. Suppose that $M$ is the number of nodes in this case. Using the shape functions as defined in equation (3.2.5) or (3.2.6) or (3.2.7), $\phi$ and $\frac{\partial \phi}{\partial n}$ can be written as

$$\phi(\xi, \eta) = \sum_{k=1}^{L} N_k \phi_k$$

$$\frac{\partial \phi}{\partial n}(\xi, \eta) = \sum_{k=1}^{L} N_k \frac{\partial \phi_k}{\partial n}$$

where $L$ denotes the number of nodes on each element.

The integrals along the element 'j' on the L.H.S of equation (3.3.1) can now be written as

$$\int \int_{S_j-i} \phi \frac{\partial}{\partial n} \left[ \frac{1}{4\pi r} \right] dS = \int \int_{S_j-i} \left[ \sum_{k=1}^{L} N_k \phi_k \right] \frac{\partial}{\partial n} \left[ \frac{1}{4\pi r} \right] dS$$

$$= \begin{bmatrix} N_1 & N_2 & \cdots & N_L \end{bmatrix} \frac{\partial}{\partial n} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} dS \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_L \end{bmatrix}$$
\[ h_{ij}^k = \int \int_{S_{j-i}} N_k \frac{\partial}{\partial n} \left( \frac{1}{4\pi r} \right) \, dS, \quad k = 1, 2, \ldots, L \quad (3.3.10) \]

The \( h_{ij}^k \) are influence coefficients defining the interaction between the node 'i' and a particular \( k \) node on an element 'j'.

Similarly, the integrals on the R.H.S. of equation (3.3.1) can be written as

\[
\int \int_{S_j} \frac{\partial \phi}{\partial n} \left( \frac{1}{4\pi r} \right) \, dS = \int \int_{S_j} \left\{ \sum_{k=1}^{L} N_k \frac{\partial \phi_k}{\partial n} \left( \frac{1}{4\pi r} \right) \right\} \, dS
\]

\[
= \int \int_{S_j} \begin{bmatrix} N_1 & N_2 & \cdots & N_L \end{bmatrix} \frac{1}{4\pi r} \, dS \begin{bmatrix} \frac{\partial \phi_1}{\partial n} \\ \frac{\partial \phi_2}{\partial n} \\ \vdots \\ \frac{\partial \phi_L}{\partial n} \end{bmatrix}
\]

\[
= \begin{bmatrix} g_{1j}^1 & g_{1j}^2 & \cdots & g_{1j}^L \\ g_{2j}^1 & g_{2j}^2 & \cdots & g_{2j}^L \\ \vdots & \vdots & \ddots & \vdots \\ g_{Lj}^1 & g_{Lj}^2 & \cdots & g_{Lj}^L \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_1}{\partial n} \\ \frac{\partial \phi_2}{\partial n} \\ \vdots \\ \frac{\partial \phi_L}{\partial n} \end{bmatrix} \quad (3.3.11)
\]
where \( g_{ij} = \int \int_{S_j} N_k \left( \frac{1}{4\pi r} \right) \, dS \), \( k = 1, 2, \ldots, L \) \hspace{1cm} (3.3.12)

To write equation (3.3.1) corresponding to the node 'i', the contributions from all of the elements associated with the node 'i' are to be added into one term, defining the nodal coefficients. This will give the following equation:

\[
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_M
\end{bmatrix}
= 
\begin{bmatrix}
\hat{H}_{i1} & \hat{H}_{i2} & \cdots & \hat{H}_{iM}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_M
\end{bmatrix}
= 
\begin{bmatrix}
\frac{\partial \phi_1}{\partial n} \\
\frac{\partial \phi_2}{\partial n} \\
\vdots \\
\frac{\partial \phi_M}{\partial n}
\end{bmatrix}
\]

\hspace{1cm} (3.3.13)

where each \( \hat{H}_{ij} \) and \( G_{ij} \) term is the sum of the contributions from all the adjoining elements of the node 'i'. Hence equation (3.3.13) represents the assembled equation for node 'i' and can be written as

\[
c_i \phi_i + \sum_{j=1}^{M} \hat{H}_{ij} \phi_j = \sum_{j=1}^{M} G_{ij} \frac{\partial \phi_j}{\partial n}
\]

or

\[
\sum_{j=1}^{M} \hat{H}_{ij} \phi_j = \sum_{j=1}^{M} G_{ij} \frac{\partial \phi_j}{\partial n}
\]

\hspace{1cm} (3.3.14)
where

\[
H_{ij} = \begin{cases} 
\hat{H}_{ij} & \text{for } i \neq j \\
\hat{H}_{ij} + c_i & \text{for } i = j 
\end{cases}
\]  

(3.3.15)

When all the M nodes are taken into consideration, equation (3.3.14) produces an \(M \times M\) system of equations which can be written in matrix form as

\[
[H] \{U\} = [G] \{Q\}
\]  

(3.3.16)

which is of the same form as equation (3.3.6). This set of equations can then be rearranged and solved in the same manner as outlined previously.

In general the surface will not be smooth at the point \('i'\), such that \(c_i \neq 1\) in equation (3.3.15). One can, however, calculate the diagonal terms of \([H]\) by using the fact that when a uniform potential is applied over the whole boundary, including that at infinity, there will be no flux of \(\psi\) through the surface at any point. Thus equation (3.3.16) becomes

\[
[H] \{U\} = \{0\}
\]  

(3.3.17)

Equation (3.3.17) indicates that the sum of all the terms of \([H]\) in a row must be zero, hence the values of the coefficients in a diagonal can easily be calculated once the off-diagonal coefficients are all known, as
Thus for the matrix formulation of the equation (2.2.13), which holds for an interior domain has been considered. For exterior flow problems, equation (2.3.3) for the direct method can be written as

\[
\sum_{j=1}^{M} H_{ij} \phi_j + \phi_\infty = \sum_{j=1}^{M} G_{ij} \frac{\partial \phi_j}{\partial n} \tag{3.3.19}
\]

where

\[
H_{ij} = \begin{cases} 
H_{ij} & \text{when } i \neq j \\
H_{ij} - c_i & \text{when } i = j 
\end{cases} \tag{3.3.20}
\]

following the same method outlined previously.

When all the nodes are taken into consideration, equation (3.3.19) produces an \( M \times (M + 1) \) system of equations which can again be put in the matrix form as

\[
[H]{\phi} = [G]{\phi_\infty} \tag{3.3.21}
\]

Note that now \( {\phi} \) in equation (3.3.21) has \( (M + 1) \) unknowns \( \phi_1, \phi_2, \ldots, \phi_m, \phi_\infty \). To solve precisely this system of equations, the value of \( \phi \) at some position must be specified. For convenience, \( \phi_\infty \) is chosen as zero. Thus the \( M \times (M + 1) \) system reduces to an \( M \times M \) system of equations which can be solved as before, but now the diagonal coefficients of \( [H] \) will be found by

\[
\Pi_{ii} = - \sum_{j=1}^{M} (H_{ij} - 1) \tag{3.3.22}
\]
3.4 EVALUATION OF INTEGRALS

The integrals in equations (3.3.10) and (3.3.12) are difficult to evaluate analytically, therefore their numerical evaluation will be considered in this section. Since these integrals involve shape functions which depend upon the local coordinates $\xi$ and $\eta$ defined on the standard element shown in figure 3.3, it is required to find the transformation from the cartesian $x, y, z$ system to the $\xi, \eta, \zeta$ system defined over the body.

Following Brebbia [26] or Zienkiewicz [28], consider the systems defined in the figure 3.7. For a function $u$, the general transformation is given by

![Figure 3.7](image-url)
\[
\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \xi}
\]

\[
\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \eta}
\]

\[
\frac{\partial u}{\partial \zeta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \zeta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \zeta} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial \zeta}
\]

which can be put in the matrix form as

\[
\begin{bmatrix}
\frac{\partial u}{\partial \zeta} \\
\frac{\partial u}{\partial \eta} \\
\frac{\partial u}{\partial \zeta}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial z}
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
\frac{\partial u}{\partial \zeta} \\
\frac{\partial u}{\partial \eta} \\
\frac{\partial u}{\partial \zeta}
\end{bmatrix} = J
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial z}
\end{bmatrix}
\]

where \( J \) denotes the 3 x 3 matrix in the above equation. Thus the inverse relationship can be found as

\[
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial z}
\end{bmatrix} = J^{-1}
\begin{bmatrix}
\frac{\partial u}{\partial \zeta} \\
\frac{\partial u}{\partial \eta} \\
\frac{\partial u}{\partial \zeta}
\end{bmatrix}
\]
Replacing \( u \) by \( \vec{R} \) for instance, the differential of area (such as \( dS \)) can be defined as

\[
dS = \left| \frac{\partial \vec{R}}{\partial 
\right| d\xi d\eta
\]

\[
= |G| d\xi d\eta
\]

(3.4.1)

Since \( \vec{R} = x\hat{i} + y\hat{j} + z\hat{k} \)

and \( x = x(\xi, \eta, \zeta) \)
\( y = y(\xi, \eta, \zeta) \)
\( z = z(\xi, \eta, \zeta) \)

\[
\frac{\partial \vec{R}}{\partial \xi} = \frac{\partial \vec{R}}{\partial x} x_{\xi} + \frac{\partial \vec{R}}{\partial y} y_{\xi} + \frac{\partial \vec{R}}{\partial z} z_{\xi}
\]

\[
= \frac{\partial x}{\partial \xi} \hat{i} + \frac{\partial y}{\partial \xi} \hat{j} + \frac{\partial z}{\partial \xi} \hat{k}
\]

\[
= \left( \frac{\partial x}{\partial \xi}, \frac{\partial y}{\partial \xi}, \frac{\partial z}{\partial \xi} \right)
\]

Similarly \( \frac{\partial \vec{R}}{\partial \eta} = \left( \frac{\partial x}{\partial \eta}, \frac{\partial y}{\partial \eta}, \frac{\partial z}{\partial \eta} \right) \)

Note that \( |G| \) is the magnitude of the normal vector,

\[
\vec{n} = \vec{\frac{\partial \vec{R}}{\partial \xi}} = \frac{\vec{\partial \vec{R}}}{\partial \xi} \times \frac{\vec{\partial \vec{R}}}{\partial \eta}
\]

\[
= (g_1, g_2, g_3)
\]

(3.4.2)
where \( g_1 = \left( \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \eta} \right) \)

\[ g_2 = \left( \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta} \right) \]

\[ g_3 = \left( \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} \right) \]

and \( |G| = \sqrt{g_1^2 + g_2^2 + g_3^2} \) \hspace{1cm} (3.4.3)

Using these relationships, the integrals in equations (3.3.10) and (3.3.12) over the linear quadrilateral elements become

\[ \int_{-1}^{1} \int_{-1}^{1} N_k \frac{\partial}{\partial \eta} \left( \frac{1}{4\pi r} \right) |G| d\xi d\eta, \quad k = 1, 2, \ldots, L \hspace{1cm} (3.4.4) \]

and

\[ \int_{-1}^{1} \int_{-1}^{1} N_k \left( \frac{1}{4\pi r} \right) |G| d\xi d\eta, \quad k = 1, 2, \ldots, L \hspace{1cm} (3.4.5) \]

The integral in equation (3.4.4) and (3.4.5) are now on the standard element \(-1 \leq \xi \leq 1, -1 \leq \eta \leq 1\), as shown in figure 3.3(b), which can then be evaluated using two-dimensional Gaussian quadrature, which approximates a definite integral by summation over a finite number of points. Thus one can write

\[ \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) d\xi d\eta \approx \sum_{p=1}^{P} \Sigma W_p f(\xi_p, \eta_p) \hspace{1cm} (3.4.6) \]
where \( w_p \) are the weighting coefficients and \((\xi_p, \eta_p)\) are the coordinates of the associated \( P \) Gaussian points. Thus the region of integration for the Gaussian quadrature is precisely the same as the standard element, namely \(-1 \leq \xi \leq 1, -1 \leq \eta \leq 1\).

Equation (3.4.4) and (3.4.5) can be written as

\[
\begin{align*}
\hat{h}_{ij}^k &= -\frac{1}{4\pi} \sum_{p=1}^{P} w_p \left[ N_k \frac{1}{r^2} (\hat{\mathbf{n}} \cdot \hat{\mathbf{r}}) |G| \right] \left( \xi_p, \eta_p \right) \\
\hat{\varepsilon}_{ij}^k &= \frac{1}{4\pi} \sum_{p=1}^{P} w_p \left[ N_k \frac{1}{r} |G| \right] \left( \xi_p, \eta_p \right)
\end{align*}
\]

(3.4.7) (3.4.8)

Now \( r \) is the distance from the fixed point 'i' to any general point \( Q(x_q, y_q, z_q) \) on the surface of the body under consideration.

\[
\hat{r} = (x_q - x_i)\hat{i} + (y_q - y_i)\hat{j} + (z_q - z_i)\hat{k}
\]

(3.4.9)

and for an isoparametric quadrilateral element,

\[
\begin{align*}
x_q &= \sum_{k=1}^{L} N_k x_k \\
y_q &= \sum_{k=1}^{L} N_k y_k \\
z_q &= \sum_{k=1}^{L} N_k z_k
\end{align*}
\]

(3.4.10)

Thus \( \hat{\mathbf{n}} \cdot \hat{\mathbf{r}} \) can be found from equation (3.4.2), (3.4.9) and (3.4.10).
In general Gauss-quadrature formulae give good results, but not when the singularity is present on the element over which the integration is being performed. In such cases alternative numerical quadrature schemes have been developed. For three-dimensional problems, a modified quadrature rule for this case is discussed in section (6). In two-dimensional formulations for this case, either a higher order integration rule or a special logarithmically weighted integration formula, Stroud and Secrest [30], should be used.

3.5 SOLUTION OF THE SYSTEM OF EQUATIONS

When the boundary integral equation (2.2.13) is transformed into a system of linear algebraic equations (3.3.7), then the solution of this system is an important part of the total calculation. The system of equations (3.3.7) will be solved using the Gauss elimination method as well as the Gauss-Seidel method. The detailed discussion of these methods can be seen from many texts e.g. Nobel and Daniel [31]. Only a brief comparison of these two methods will be presented here.

Gauss elimination works for any non-singular system of M equation in M unknowns, while the Gauss-Seidel method works for only specific cases where convergence occurs. In situations where convergence does occur for the Gauss-Seidel method, the choice between using this or Gauss elimination depends upon several factors, in particular the system size of the equations M. This is because the number of arithmetic operations required for
the Gauss elimination method to solve a system of $M$ linear equations is proportional to $M^3$, and the number of operations for one iteration of the Gauss-Seidel method is proportional to $M^2$. Thus if the Gauss-Seidel method converges in fewer than $M$ iterations, it is computationally more efficient than Gauss elimination. The speed of convergence depends upon the precise format of the coefficients of the equation, as well as the guessed starting values for iteration. In general, the larger the system size $M$ is, the more likely it is that Gauss-Seidel will require less computing time than Gauss elimination. Furthermore the round-off error for the Gauss-Seidel is generally smaller than that for Gauss elimination. Finally, when $M$ is very large, then the storage of $M^2$ entries of the coefficient matrix $[A]$ on the computer becomes a problem. It then becomes necessary to store the coefficients on disc and read them into the program line by line for solution of the equations. The Gauss-Seidel method is much more convenient for line by line solution than Gauss elimination. Comparison of the Gauss-Seidel and Gauss elimination methods of solution for problems of three-dimensional exterior unbounded flows is given in section (4).
SECTION 4

A DIRECT BOUNDARY ELEMENT METHOD APPLIED TO POTENTIAL FLOW PROBLEMS

4.1 INTRODUCTION

In this section the direct boundary element method is applied to obtain the potential flow solution around two and three-dimensional bodies, for which analytical solutions are available. To check the accuracy of the method, the calculated flow velocities or the pressure distributions are compared with the analytical solutions for flows over the surfaces of a sphere and ellipsoids of revolution in the three-dimensional case, and over the boundary of a circular cylinder in the two-dimensional case.

Since the flow field around a body lying in the proximity of the ground is different from the flowfield around the same body when it is lying in an unbounded space, the ground plane problem is an important feature which is also discussed in this section.

A further feature of the real flow field about a bluff road vehicle is the presence of a wake downstream of the body. A method of accounting for this feature in the potential flow model is presented at the end of this section.
4.2 UNBOUNDED FLOW FIELD CALCULATIONS

Consideration is given firstly to the uniform streaming flow about a circular cylinder, a sphere and ellipsoids of revolution. In the case of a sphere and ovary ellipsoids, the surface of the body under consideration is discretized into linear isoparametric quadrilateral elements and the integrals over each element have been evaluated numerically using Gauss-quadrature with four integration points. In the case of a circular cylinder, the boundary of the cylinder is discretized into constant and linear boundary elements and the integrals over these elements are calculated numerically using Gauss-quadrature with four integration points except for the element corresponding to the node under consideration. For this element the integrals are calculated analytically.

4.2.1 FLOW PAST A CIRCULAR CYLINDER

Boundary element methods can be used to solve two-dimensional interior or exterior flow problems. As an example of an exterior flow problem, consider the flow past a circular cylinder. Let a circular cylinder be of radius 'a' with centre at the origin and let the onset flow be the uniform stream with velocity \( U \) in the positive direction of the x-axis as shown in figure 4.1.
The magnitude of the exact velocity distribution over the boundary of the circular cylinder is given in Milne-Thomson [24] as

\[ |\mathbf{V}| = 2aU \sin \theta \] (4.2.1.1)

where \( \theta \) is the angle between the radius vector and the positive direction of the \( x \)-axis.

Now the condition to be satisfied on the boundary of the circular cylinder is

\[ \hat{n} \cdot \mathbf{V} = 0 \] (4.2.1.2)

where \( \hat{n} \) is the unit normal vector to the boundary of the cylinder.

Since the motion is irrotational,

\[ \mathbf{V} = -\nabla \phi \]

where \( \phi \) is the total velocity potential. Thus equation (4.2.1.2) becomes
\[ \nabla \cdot \mathbf{V} = 0 \]

or \[ \frac{\partial \Phi}{\partial n} = 0 \quad (4.2.1.3) \]

Now the total velocity potential \( \Phi \) is the sum of the perturbation velocity potential \( \Phi_{c.c} \) and the velocity potential of the uniform stream \( \Phi_{u.s} \)

i.e. \[ \Phi = \Phi_{u.s} + \Phi_{c.c} \quad (4.2.1.4) \]

or \[ \frac{\partial \Phi}{\partial n} = \frac{\partial \Phi_{u.s}}{\partial n} + \frac{\partial \Phi_{c.c}}{\partial n} \]

which on using equation (4.2.1.3) becomes

\[ \frac{\partial \Phi_{c.c}}{\partial n} = - \frac{\partial \Phi_{u.s}}{\partial n} \quad (4.2.1.5) \]

But the velocity potential of the uniform stream, given in Milne-Thomson [24], is

\[ \Phi_{u.s} = -Ux \quad (4.2.1.6) \]

\[ \frac{\partial \Phi_{u.s}}{\partial n} = -U \frac{\partial x}{\partial n} \]

\[ = -U (\hat{n} \cdot \hat{i}) \quad (4.2.1.7) \]

Thus from equations (4.2.1.5) and (4.2.1.7),

\[ \frac{\partial \Phi_{c.c}}{\partial n} = U (\hat{n} \cdot \hat{i}) \quad (4.2.1.8) \]
or \[ \frac{\partial \phi}{\partial n} = U \frac{x}{\sqrt{x^2 + y^2}} \] (4.2.1.9)

Equation (4.2.1.9) is the boundary condition which must be satisfied over the boundary of the circular cylinder.

Now for the approximation of the boundary of the circular cylinder, the coordinates of the extreme points of the boundary elements can be generated within the computer program as follows:

Divide the boundary of the circular cylinder into \( m \) elements in the clockwise direction by using the formula

\[ \theta_k = \frac{(m + 3) - 2k \pi}{m}, \quad k = 1, 2, \ldots, m \] (4.2.1.10)

Then the coordinates of the extreme points of these \( m \) elements are calculated from

\[
\begin{align*}
    x_k &= a \cos \theta_k \\
    y_k &= a \sin \theta_k
\end{align*}
\] (4.2.1.11)

Take \( m = 8 \) and \( a = 1 \).

First consider the case of constant boundary elements where there is only one node at the middle of the element and the potential \( \phi \) and the potential derivative \( \frac{\partial \phi}{\partial n} \) are constant over each element and equal to the value at the middle node of the element.
Figure 4.2 shows the discretization of the circular cylinder of unit radius into 8 constant boundary element. The coordinates of the middle node of each boundary element are given by

\[
\begin{align*}
    x_m &= \frac{x_k + x_{k+1}}{2} \\
    y_m &= \frac{y_k + y_{k+1}}{2}
\end{align*}
\]

and therefore the boundary condition (4.2.1.9) in this case takes the form

\[
\frac{\partial \phi_{c.c}}{\partial n} = U \frac{x_m}{\sqrt{x_m^2 + y_m^2}}.
\]

The velocity \( U \) of the uniform steam is also taken as unity.

The equation (2.3.4) for the direct boundary element method can be written in the discretized form as
\[-c_i \phi_i + \sum_{j=1}^{m} \left( \int_{r_j-i}^{\infty} \frac{\partial}{\partial n} \left( \frac{1}{2\pi} \log \frac{1}{r} \right) d\Gamma \right) \phi_j + \phi_\infty \]

\[= \sum_{j=1}^{m} \left( \int_{r_j-i}^{\infty} \frac{1}{2\pi} \log \frac{1}{r} d\Gamma \right) \frac{\partial \phi_j}{\partial n} \tag{4.2.1.13} \]

The integrals in equation (4.2.1.13) on the elements can be calculated numerically except the element on which the fixed point 'i' is located. For this element the integrals are calculated analytically. Denoting the integrals on the L.H.S. of equation (4.2.1.13) by \(H_{ij}\) and that on the R.H.S. by \(G_{ij}\), then

\[\hat{H}_{ij} = \int_{r_j-i}^{\infty} \frac{\partial}{\partial n} \left( \frac{1}{2\pi} \log \frac{1}{r} \right) d\Gamma \]

\[= -\frac{1}{2\pi} \int_{r_j-i}^{\infty} \frac{1}{r} \frac{\partial r}{\partial n} d\Gamma \]

\[= -\frac{1}{2\pi} \int_{-1}^{1} \frac{1}{r} \left( \hat{n} \cdot \hat{r} \right) \frac{\xi}{2} d\xi \tag{4.2.1.14} \]

where \(\xi\) is the length of the element \(r_j\).

To find \(\hat{n} \cdot \hat{r}\) let the coordinates of the fixed point 'i' be \((x_i, y_i)\) and that of the variable point 'Q' be \((x_q, y_q)\), then

\[r = \sqrt{(x_q - x_i)^2 + (y_q - y_i)^2}. \]

But \(x_q\) and \(y_q\) in terms of \(\xi\) and the global coordinates \(x_k\) and \(y_k\) can be written as
\[ x_q = \sum_{k=1}^{2} N_k x_k, \quad y_q = \sum_{k=1}^{2} N_k y_k \]

where \( N_k \) are the shape functions at the end points of the elements and are defined in equation (3.2.1).

\[
x_q = \frac{1}{4}(1 - \xi)x_1 + \frac{1}{4}(1 + \xi)x_2
\]

\[
= \frac{1}{4}(x_2 - x_1)\xi + \frac{1}{4}(x_2 + x_1)
\]

Similarly, \( y_q = \frac{1}{4}(y_2 - y_1)\xi + \frac{1}{4}(y_2 + y_1) \)

\[
\mathbf{r} = \sqrt{\left[\frac{1}{4}(x_2 - x_1)\xi + \frac{1}{4}(x_2 + x_1) - x_1\right]^2 + \left[\frac{1}{4}(y_2 - y_1)\xi + \frac{1}{4}(y_2 + y_1) - y_1\right]^2}
\]

and

\[
\hat{r} = \frac{\left[\frac{1}{4}(x_2 - x_1)\xi + \frac{1}{4}(x_2 + x_1) - x_1\right]\hat{i} + \left[\frac{1}{4}(y_2 - y_1)\xi + \frac{1}{4}(y_2 + y_1) - y_1\right]\hat{j}}{r}
\]

(4.2.1.15)

The unit normal vector \( \hat{n} \) can be found as follows:

![Figure 4.3](image-url)
From the figure 4.3, the tangent vector to the point \( Q \) is
\[
\mathbf{t} = (x_2 - x_1)i + (y_2 - y_1)j
\]
then the unit tangent vector \( \hat{\mathbf{t}} \) is given by
\[
\hat{\mathbf{t}} = \frac{(x_2 - x_1)i + (y_2 - y_1)j}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}
\]
The numbering system in figure 4.2 is taken as clockwise, therefore to get the outward unit normal to \( \Gamma \), one must rotate the unit tangent vector \( \hat{\mathbf{t}} \) through 90° anticlockwise, thus
\[
\hat{n} = \frac{-(y_2 - y_1)i + (x_2 - x_1)j}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{4.2.1.16}
\]
Hence \( \hat{n} \cdot \hat{r} \) can be found by taking the dot product of equations (4.2.1.15) and (4.2.1.16).

Using \( \hat{n} \cdot \hat{r} \) in Equation (4.2.1.14), \( \hat{H}_{ij} \) can be calculated numerically using a Gauss-quadrature scheme with say four integration points. Similarly \( G_{ij} \) can be found. These are simple as they involve no derivative term i.e.
\[
G_{ij} = \int_{\Gamma_i} \frac{1}{2\pi} \log \left( \frac{1}{r} \right) \, d\Gamma'
\]
\[
= -\frac{1}{2\pi} \int_{-1}^{1} \log x \, \frac{\xi}{2} \, d\xi
\]
For the case of that element on which the fixed point 'i' is lying, the integrals are calculated analytically and are given in Brebbia and Walker [26].

Having found these integrals, equation (4.2.1.13) can be written as
Defining

\[ H_{ij} = \begin{cases} \hat{H}_{ij} & \text{when } i \neq j \\ \hat{H}_{ij} - c_i & \text{when } i = j \end{cases} \]

Equation (4.2.1.17) takes the form

\[ \sum_{j=1}^{m} H_{ij} \phi_j + \phi_\infty = \sum_{j=1}^{m} G_{ij} \frac{\partial \phi_j}{\partial n} \]

which can be expressed in matrix form as

\[ [H][\Phi] = [G][Q] \]

(4.2.1.18)

Since \( \frac{\partial \phi}{\partial n} \) is specified at each node of the element, the values of the perturbation velocity potential \( \phi \) are found at each node on the boundary via equation (4.2.1.18). The total potential \( \phi \) is then found from equation (4.2.1.4) which will then be used to calculate the velocity on the circular cylinder.

Figure 4.4
The velocity midway between two nodes on the boundary, as shown in the figure 4.4 can then be approximated by using the formula

\[ \text{velocity } \phi = \frac{\phi_{k+1} - \phi_k}{\text{Length from node } k \text{ to } k+1} \]  

(4.2.1.19)

The table (1) below shows the comparison of the analytical and calculated velocity distributions over the boundary of the circular cylinder for 8 boundary elements.

<table>
<thead>
<tr>
<th>Element</th>
<th>x- Coordinate</th>
<th>y- Coordinate</th>
<th>Calculated Velocity</th>
<th>Analytical Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.79</td>
<td>0.33</td>
<td>0.80718E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>2</td>
<td>-0.33</td>
<td>0.79</td>
<td>0.19487E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>3</td>
<td>0.33</td>
<td>0.79</td>
<td>0.19487E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>4</td>
<td>0.79</td>
<td>0.33</td>
<td>0.80718E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>5</td>
<td>0.79</td>
<td>-0.33</td>
<td>0.80718E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>6</td>
<td>0.33</td>
<td>-0.79</td>
<td>0.19487E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>7</td>
<td>-0.33</td>
<td>-0.79</td>
<td>0.19487E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>8</td>
<td>-0.79</td>
<td>-0.33</td>
<td>0.80718E+00</td>
<td>0.76537E+00</td>
</tr>
</tbody>
</table>

In representing the boundary of the circular cylinder as shown in figure 4.2, it can be seen that the nodes do not actually lie on the boundary itself and hence the calculated results are a rough approximation to the analytical ones. To increase the accuracy of the method, the number of elements can be increased on the boundary. An increase in the number of elements allows the actual continuous variation of potential to be modelled more closely. Furthermore, the nodes are brought nearer the true body surface, which again enables a
closer approximation to the real variation of $\phi$ on the boundary to be obtained. The improvement gained by using 24 elements can be seen from Table (2), where only the results corresponding to the same values of $\phi$ as shown in Table (1) are given.

Table (2)

<table>
<thead>
<tr>
<th>Element</th>
<th>x-</th>
<th>y-</th>
<th>Calculated Velocity</th>
<th>Analytical Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-0.91</td>
<td>0.38</td>
<td>0.76975E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>5</td>
<td>-0.38</td>
<td>0.91</td>
<td>0.18583E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>8</td>
<td>0.38</td>
<td>0.91</td>
<td>0.18583E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>11</td>
<td>0.91</td>
<td>0.38</td>
<td>0.76975E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>14</td>
<td>0.91</td>
<td>-0.38</td>
<td>0.76975E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>17</td>
<td>0.38</td>
<td>-0.91</td>
<td>0.18583E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>20</td>
<td>-0.38</td>
<td>-0.91</td>
<td>0.18583E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>23</td>
<td>-0.91</td>
<td>-0.38</td>
<td>0.76975E+00</td>
<td>0.76537E+00</td>
</tr>
</tbody>
</table>

Consider next the case where the boundary of the circular cylinder is divided into linear elements. In this case the nodes where the boundary conditions are specified are at the intersection of the elements. The boundary of the cylinder can be divided into elements by the formula

$$\theta_k = [(m + 2) - 2k] \pi / m, \quad k = 1, 2, \ldots, m \quad (4.2.1.20)$$

The figure 4.5 shows the discretization of the circular cylinder into 8 linear boundary elements.
The coordinates of the extreme points of the elements and the coordinates of the mid-point of each element where the velocity will be calculated can be found from the equations (4.2.1.11) and (4.2.1.12) respectively. The reason for distributing the elements in this way is such that the velocities are calculated at the same values of $\theta$ in both figures 4.4 and 4.5, so that the calculated results could be easily compared.

The boundary condition (4.2.1.9) in this case takes the form

$$\frac{\partial \phi}{\partial n} = U_x$$
$$= x \text{ taking } U = 1$$

(4.2.1.21)

Equation (2.3.4) can be written for this case as

$$-c_i \phi_i + \sum_{j=1}^{m} \left( \phi \frac{\partial}{\partial n} \left( \frac{1}{2\pi} \log \frac{1}{r} \right) d\Gamma + \phi_\infty \right) = \sum_{j=1}^{m} \left( \frac{\partial \phi}{\partial n} \left( \frac{1}{2\pi} \log \frac{1}{r} \right) d\Gamma \right)$$

(4.2.1.22)
Since \( \phi \) and \( \frac{\partial \phi}{\partial n} \) vary linearly over the element, their values at any point on the element can be defined in terms of their nodal values and the shape functions \( N_1 \) and \( N_2 \) as

\[
\phi = [N_1 \ N_2] \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
\]

\[\frac{\partial \phi}{\partial n} = [N_1 \ N_2] \begin{pmatrix} \frac{\partial \phi_1}{\partial n} \\ \frac{\partial \phi_2}{\partial n} \end{pmatrix}\]

(4.2.1.23)

The integrals along an element 'j' on the L.H.S. of equation (4.2.1.22) can now be written as

\[
\int_{\Gamma_{j-i}} \phi \frac{\partial}{\partial n} \left( \frac{1}{2\pi} \log \frac{1}{r} \right) d\Gamma = \int_{\Gamma_{j-i}} [N_1 \ N_2] \frac{\partial}{\partial n} \left( \frac{1}{2\pi} \log \frac{1}{r} \right) d\Gamma \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
\]

\[= [h^1_{ij} \ h^2_{ij}] \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\]

where \( h^k_{ij} = \int_{\Gamma_{j-i}} N_k \frac{\partial}{\partial n} \left( \frac{1}{2\pi} \log \frac{1}{r} \right) d\Gamma, \ k = 1, 2. \) (4.2.1.24)

The integrals on the R.H.S. of equation (4.2.1.22) can be written as
\[
\int_{\Gamma_j} \frac{\partial \left( \frac{1}{2\pi} \log \frac{1}{r} \right)}{\partial n} d\Gamma = \int_{\Gamma_j} \left[ N_1 N_2 \left( \frac{1}{2\pi} \log \frac{1}{r} \right) \right] d\Gamma \left\{ \begin{array}{c}
\frac{\partial \phi_1}{\partial n} \\
\frac{\partial \phi_2}{\partial n}
\end{array} \right\} = \left[ g^1_{ij} \ g^2_{ij} \right] \left\{ \begin{array}{c}
\frac{\partial \phi_1}{\partial n} \\
\frac{\partial \phi_2}{\partial n}
\end{array} \right\}
\]

where \( g^k_{ij} = \int_{\Gamma_j} N_k \left( \frac{1}{2\pi} \log \frac{1}{r} \right) d\Gamma, \ k = 1, 2. \) (4.2.1.25)

Again the integrals in equations (4.2.1.24) and (4.2.1.25) are calculated numerically as before except for the element on which the fixed point 'i' is lying. For this element the integrals are calculated analytically. The integrals \( h^1_{ii} \) and \( h^2_{ii} \) are zero because \( r \) and \( \hat{n} \) are orthogonal to each other over the element. The integral \( g^1_{ii} \) and \( g^2_{ii} \) can be evaluated as follows:

\[
g^1_{ii} = \int_{\Gamma_i} N_1 \left( \frac{1}{2\pi} \log \frac{1}{r} \right) d\Gamma \] (4.2.1.26)

Since \( N_1 \) and \( r \) are functions of \( \xi \), and the integral is w.r.t \( \Gamma \), therefore the evaluation of this integral requires the use of a Jacobian \( |J| \) of the transformation. As with Brebbia and Walker [26], this Jacobian is given by

\[
|J| = \sqrt{\left( \frac{dx}{d\xi} \right)^2 + \left( \frac{dy}{d\xi} \right)^2} \] (4.2.1.27)
so that \( d\Gamma = |J| d\xi \).

Now to calculate \(|J|\),

\[
x = N_1 x_1 + N_2 x_2
\]

\[
= \frac{1}{2}(1 - \xi)x_1 + \frac{1}{2}(1 + \xi)x_2
\]

Similarly \( y = \frac{1}{2}(1 - \xi)y_1 + \frac{1}{2}(1 + \xi)y_2 \),

\[
\text{fixed Point}
\]

\[
\text{node (1)} \quad \text{node (2)}
\]

\[
\xi = -1 \quad \xi = 0 \quad \xi = 1
\]

\[
\ell / 2
\]

Figure 4.6

where \( x_i, y_i \) are the coordinates of the nodes referred to the global system.

\[
\therefore \frac{dx}{d\xi} = \frac{1}{2}(x_2 - x_1), \quad \frac{dy}{d\xi} = \frac{1}{2}(y_2 - y_1)
\]

and thus from equation (4.2.1.27),

\[
|J| = \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2} = \frac{\ell}{2} \quad (4.2.1.28)
\]

where \( \ell \) is the length of the element shown in the figure 4.6. For the linear element shown this is obvious, but the same procedure yields the jacobian for more complicated elements.

Thus the integral in equation (4.2.1.26) becomes

\[
g^1_{11} = \frac{1}{2\pi} \int_{-1}^{1} \frac{1}{2}(1 - \xi) \log \left( \frac{1}{x} \right) \frac{\ell}{2} d\xi \quad (4.2.1.29)
\]
If node (1) is taken as the fixed point 'i', then r is measured from node (1), therefore

\[ r = \frac{\varphi}{2} (1 + \xi) \quad (4.2.1.30) \]

From equations (4.2.1.29) and (4.2.1.30),

\[ \frac{g^1_{ii}}{\beta} = \frac{\varphi}{2 \pi} \int_{-1}^{1} (1 - \xi) \log \frac{1}{\varphi/2} (1 + \xi) \, d\xi \]

\[ = \frac{\varphi}{2 \pi} \lim_{\varepsilon \to 0^+} \int_{-1+\varepsilon}^{1} (1 - \xi) \log \frac{1}{\varphi/2} (1 + \xi) \, d\xi \]

Integrating by parts and simplifying, then

\[ g^1_{ii} = \frac{\varphi}{8 \pi} [3 - 2 \log \varepsilon] \quad (4.2.1.31) \]

Similarly,

\[ g^2_{ii} = \int_{\Gamma_i} N_2 \left( \frac{1}{2 \pi} \log \frac{1}{\varphi} \right) \, d\Gamma \]

\[ = \frac{1}{2 \pi} \int_{-1}^{1} \frac{1}{4}(1 + \xi) \log \frac{1}{\varphi} |J| \, d\xi \quad (4.2.1.32) \]

Using equations (4.2.1.28) and (4.2.1.30), equation (4.2.1.32) takes the form

\[ g^2_{ii} = \frac{\varphi}{8 \pi} \int_{-1}^{1} (1 + \xi) \log \frac{1}{\varphi/2} (1 + \xi) \, d\xi \]

\[ = \frac{\varphi}{8 \pi} \lim_{\varepsilon \to 0^+} \int_{-1+\varepsilon}^{1} (1 + \xi) \log \frac{1}{\varphi/2} (1 + \xi) \, d\xi \]
Again integrating by parts and simplifying we get

\[ a_{ii}^2 = \frac{8}{\pi^2} [1 - 2 \log \ell] \]  

(4.2.1.33)

The table (3) below shows the comparison of the analytical and calculated velocities at the mid points of 8 boundary elements over the circular cylinder.

<table>
<thead>
<tr>
<th>Element</th>
<th>Coordinate x</th>
<th>Coordinate y</th>
<th>Calculated Velocity</th>
<th>Analytical Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.85</td>
<td>0.35</td>
<td>0.71640E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>2</td>
<td>-0.35</td>
<td>0.85</td>
<td>0.17295E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>3</td>
<td>0.35</td>
<td>0.85</td>
<td>0.17295E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>4</td>
<td>0.85</td>
<td>0.35</td>
<td>0.71640E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>5</td>
<td>0.85</td>
<td>-0.35</td>
<td>0.71640E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>6</td>
<td>0.35</td>
<td>-0.85</td>
<td>0.17295E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>7</td>
<td>-0.35</td>
<td>-0.85</td>
<td>0.17295E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>8</td>
<td>-0.85</td>
<td>-0.35</td>
<td>0.71640E+00</td>
<td>0.76537E+00</td>
</tr>
</tbody>
</table>

In this case also, the accuracy of the results can be improved by increasing the number of elements on the boundary. On comparison of these results with those in table (1) it can be seen that the results in the case of linear elements are not better than those with the constant elements. However, this is not always the case and in general the accuracy of the results improves in passing from constant to linear or higher order elements using the boundary element methods.
4.2.2 FLOW PAST A SPHERE

The body shape that received the greatest attention is naturally the sphere because it is one of the simplest three-dimensional symmetrical bodies and has an analytical solution with which the calculated solutions can be compared. Let the sphere be of radius 'a' with centre at the origin and let the onset flow be the uniform stream of velocity $U$ in the positive direction of the x-axis as shown in the figure 4.7.

The pressure distributions over the surface of the sphere can be calculated analytically, Vallentine [32], and is given by the formula

$$C_p = 1 - \frac{4}{9}\sin^2\theta$$

(4.2.2.1)

The boundary condition to be satisfied over the surface of the sphere can be obtained in a similar manner as for the case of the
circular cylinder and takes the form

\[ \frac{\partial \phi_{\text{sphere}}}{\partial n} = U \frac{x}{\sqrt{x^2 + y^2 + z^2}} \]  

(4.2.2.2)

The equation of the surface of the sphere is

\[ x^2 + y^2 + z^2 = 1 \]  

(4.2.2.3)

where the radius \( a \) is taken as 1.

Therefore equation (4.2.2.2) becomes

\[ \frac{\partial \phi_{\text{sphere}}}{\partial n} = Ux \]

\[ = x, \text{ taking } U = 1 \]  

(4.2.2.4)

The flowfield around the sphere is calculated using different numbers of boundary elements to approximate the body surface. The method of element distribution which has been applied is as follows:

Consider the surface of the sphere in one octant to be divided into three quadrilateral elements by joining the centroid of the surface with the mid points of the curves in the coordinate planes as shown in

Figure 4.8
the figure 4.8. Then each element is divided further into four elements by joining the centroid of the surface with the mid point of each side of the element. Thus one octant of the surface of the sphere is divided into 12 elements and the whole surface of the body is divided into 96 boundary elements. One can then continue to divide each element into four separate elements by the method mentioned above such that the total surface of the sphere would be divided in 384 boundary elements, and so on. The above mentioned method is adopted in order to produce a uniform distribution of elements over the surface of the body. Figures 4.16 and 4.17 show the discretization of the sphere into 96 and 384 boundary elements respectively.

As the body possesses planes of symmetry, this fact may be used in the input to the program and only the non-redundant portion need be specified by input points. The other portions are automatically taken into account. The planes of symmetry are taken to be the coordinate planes of the reference coordinate system. The advantage of the use of symmetry is that it reduces the order of the resulting system of equations and consequently reduces the computing time in running a program. As a sphere is symmetric with respect to all the three coordinate planes of the reference coordinate system, only one eighth of the body surface need be specified by the input points, while the other seven-eighths can be accounted for by symmetry. However, since the boundary element methods are to be applied to road vehicles which are symmetric with respect to only one plane, it was decided to make use of only this one plane of symmetry for all the bodies considered, namely the xy-plane. Where the body intersects its plane of symmetry, the points in this plane, (i.e. the points having z = 0) must be included among the input points.

After the integrals of the equation (2.3.3) for the direct
boundary element method have been calculated numerically over each of
the boundary elements, the equation (2.3.3) can be put in the matrix
form (3.3.21).

In order to evaluate the potential at each node on the surface it
is necessary to specify \[
\frac{\partial \phi}{\partial n}_{\text{sphere}} \]
at each node, given by (4.2.2.4).

The total velocity potential of the flow, \( \phi \), can then be obtained in a
similar way to that for the circular cylinder [equation (4.2.1.4)],
and can then be used to calculate the velocity. The velocity at any
point on an element can then be found as follows:
The x, y and z components of the velocity are given by

\[
\left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)
\]
respectively. Now

\[
\frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial \xi}
\]

and

\[
\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} \frac{\partial x}{\partial n} + \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial n} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial n}
\]

Furthermore as there must be zero normal velocity on the surface of the
body, then

\[
\hat{n} \cdot \nabla \phi = 0
\]
or

\[
\hat{n}_1 \frac{\partial \phi}{\partial x} + \hat{n}_2 \frac{\partial \phi}{\partial y} + \hat{n}_3 \frac{\partial \phi}{\partial z} = 0 \tag{4.2.2.5}
\]

The complete system of the equations can be written in matrix form as
\[
\begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\hat{n}_1 & \hat{n}_2 & \hat{n}_3
\end{bmatrix} \begin{bmatrix}
\frac{\partial \Phi}{\partial x} \\
\frac{\partial \Phi}{\partial y} \\
\frac{\partial \Phi}{\partial z}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \Phi}{\partial \xi} \\
\frac{\partial \Phi}{\partial \eta} \\
0
\end{bmatrix}
\] (4.2.2.6)

and can then be solved for \(\frac{\partial \Phi}{\partial x}\), \(\frac{\partial \Phi}{\partial y}\) and \(\frac{\partial \Phi}{\partial z}\). Once these are known, the magnitude of the velocity is given by

\[V = |\vec{V}| = \sqrt{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2} \] (4.2.2.7)

and the pressure coefficient is defined as

\[C_p = 1 - \left(\frac{V}{U}\right)^2 \] (4.2.2.8)

The calculated pressure coefficients at the mid-points of all the elements over the surface of the sphere are compared with the analytic solutions. Figure 4.18 shows the comparison of the analytical and calculated pressure coefficients for (a) 96 boundary elements and (b) 384 boundary elements. In both the cases the accuracy is seen to be good. The 384 element case agrees with the analytical solution even where the velocity changes rapidly with position.

4.2.3 \textbf{FLOW PAST AN ELLIPSOID OF REVOLUTION}

The body shape that is considered next is an ovary ellipsoid, also called a prolate spheroid, which is generated by the rotation of an ellipse about its major axis. The flow past an ellipsoid of
revolutions is an axisymmetric flow and has some analogies with the two-dimensional flow; in particular, a stream function $\psi$ can be defined, and, of course, when the motion is irrotational, a velocity potential always exists.

The accuracy of the computed results is best checked by comparing them with analytical results. The stream function for flow past an ovary ellipsoid is given in Milne-Thomson [24] and the velocity distribution about the ellipsoid can be determined from this, as shown below.

Let an ovary ellipsoid be generated by revolving an ellipse of semi-major axis $a$ and semi minor axis $b$ about the $x$-axis and let the uniform stream of velocity $U$ be in the positive direction of $x$-axis as shown in the figure 4.9.

An axisymmetric flow is most conveniently formulated in cylindrical polar coordinates. Following Milne-Thomson [24], the cylindrical polar coordinates are taken as $(x, \omega, \bar{w})$. 
In Milne-Thomson [24], the ovary ellipsoid is defined by the transformation

\[ x + iw = c \cosh \zeta \]
\[ = c \cosh (\xi + i\eta) \]
\[ = c \cosh \xi \cos \eta + ic \sinh \xi \sin \eta \]

Comparison of real and imaginary parts gives

\[ x = c \cosh \xi \cos \eta \]
\[ \dot{w} = c \sinh \xi \sin \eta \]  \hspace{1cm} (4.2.3.1)

and therefore the curve \( \xi = \xi_0 \) is an ellipse in the \( xw \)-plane whose semi-axes are

\[ a = c \cosh \xi_0 \]
\[ b = c \sinh \xi_0 \]  \hspace{1cm} (4.2.3.2)

and so \( \xi = \xi_0 \) is an ovary ellipsoid.

The stream function \( \psi \) for an ovary ellipsoid moving in the negative direction of the \( x \)-axis with velocity \( U \) is given in Milne-Thomson [24], art 16.57 as

\[ \psi = \frac{1}{2} mb^2 (\cosh \xi + \sinh^2 \xi \log \tanh \frac{\xi}{2}) \]
\[ \frac{a + \frac{b^2}{c^2} \log \frac{a + b - c}{a + b + c}} \]  \hspace{1cm} (4.2.3.3)

Also, the stream function \( \psi \) for the uniform stream with velocity \( U \), in the positive direction of \( x \)-axis is given in Milne-Thomson [24], art 16.22 as

\[ \psi = -\frac{1}{4} Uw^2 \]  \hspace{1cm} (4.2.3.4)
Therefore the stream function $\psi$ for the streaming motion past a fixed ovary ellipsoid in the positive direction of the x-axis becomes

$$\psi = -\frac{1}{2} Uc^2 \sinh^2 \xi \sin^2 \eta$$

$$+ \frac{1}{2} Ub^2 (\cosh \xi + \sinh^2 \xi \log \tanh \frac{\xi}{2}) \sin^2 \eta$$

which on using equation (4.2.3.1) becomes

$$\psi = -\frac{1}{2} Uc^2 \sinh^2 \xi \sin^2 \eta$$

$$+ \frac{1}{2} Ub^2 (\cosh \xi + \sinh^2 \xi \log \tanh \frac{\xi}{2}) \sin^2 \eta$$

To find the formula for the velocity, the following relation from Milne-Thomson [24], art 16.51 is used

$$q^2 \omega^2 (\zeta) \bar{f}'(\zeta) = \left(\frac{\partial \psi}{\partial \xi}\right)^2 + \left(\frac{\partial \psi}{\partial \eta}\right)^2$$

Since $f(\zeta) = c \cosh (\zeta)$

$$f'(\zeta) = c \sinh (\zeta) = c \sinh (\xi + in)$$

$$\bar{f}'(\zeta) = c \sinh \bar{\zeta} = c \sinh (\xi - in)$$

$$\therefore f'(\zeta) \bar{f}'(\bar{\zeta}) = c^2 (\sinh^2 \xi \cos^2 \eta + \cosh^2 \xi \sin^2 \eta)$$

Thus when $\xi = \xi_0$, we have from equations (4.2.3.1), (4.2.3.6) and (4.2.3.7)

$$q^2 c^4 \sinh^2 \xi_0 \sin^2 \eta (\sinh^2 \xi_0 \cos^2 \eta + \cosh^2 \xi_0 \sin^2 \eta)$$

$$= \left(\frac{\partial \psi}{\partial \xi}\right)^2_{\xi=\xi_0} + \left(\frac{\partial \psi}{\partial \eta}\right)^2_{\xi=\xi_0}$$

(4.2.3.8)
Now from equation (4.2.3.5)

\[
\frac{\partial \psi}{\partial \xi} \bigg|_{\xi = \xi_0} = -uc^2 \sinh \xi_0 \cosh \xi_0 \sin^2 \eta
\]

\[
+ \left( a \frac{b^2}{c^2} \log \frac{a + b - c}{a + b + c} \right) \sin^2 \eta
\]

which on using equation (4.2.3.2) takes the form

\[
\frac{\partial \psi}{\partial \xi} \bigg|_{\xi = \xi_0} = U \sin^2 \eta \left[ -ab + \left( \frac{b^3}{c} + b \frac{a b}{c} \frac{\log \tanh \frac{\xi_0}{2}}{c} \right) \right]
\]

(4.2.3.9)

Let us find \( \tanh \frac{\xi_0}{2} \) in terms of \( a, b \) and \( c \).

From equation (4.2.3.2),

\[
\frac{a}{c} = \cosh \xi_0 , \quad \frac{b}{c} = \sinh \xi_0
\]

or \( \frac{a}{c} = 2 \cosh^2 \frac{\xi_0}{2} - 1 \), \( \frac{b}{c} = 2 \sinh \frac{\xi_0}{2} \cosh \frac{\xi_0}{2} \)

\[
\tanh \frac{\xi_0}{2} = \frac{\sinh \frac{\xi_0}{2}}{\cosh \frac{\xi_0}{2}} = \frac{2 \sinh \frac{\xi_0}{2} \cosh \frac{\xi_0}{2}}{2 \cosh^2 \frac{\xi_0}{2}}
\]

\[
= \frac{b/c}{a/c + 1} = \frac{b}{a + c}
\]

(4.2.3.10)

Now \( \frac{a + b - c}{a + b + c} - \frac{b}{a + c} = \frac{a^2 - b^2 - c^2}{(a + b + c)(a + c)} = 0 \)

since \( a^2 - b^2 = c^2 \)
\[ \therefore \quad \frac{a + b - c}{a + b + c} = \frac{b}{a + c} \quad (4.2.3.11) \]

From equation (4.2.3.10) and (4.2.3.11),

\[ \tanh \frac{\xi_0}{2} = \frac{a + b - c}{a + b + c} \quad (4.2.3.12) \]

\[ \therefore \quad \text{Equation (4.2.3.9) takes the form} \]

\[ \left( \frac{\partial \psi}{\partial \xi} \right)_{\xi = \xi_0} = U \sin^2 \eta \left[ -ab + \frac{b^3 + ab^3}{c} \log \frac{a + b - c}{a + b + c} \right. \]

\[ \left. + \frac{a}{c} + \frac{b^2}{c^2} \log \frac{a + b - c}{a + b + c} \right] \quad (4.2.3.13) \]

Also from equation (4.2.3.5),

\[ \left( \frac{\partial \psi}{\partial n} \right)_{\xi = \xi_0} = -U b^2 \sinh^2 \xi_0 \sin \eta \cos \eta \]

\[ + \frac{U b^2 [\cosh \xi_0 + \sinh \xi_0 \log \tanh \frac{\xi_0}{2} ]}{c} \sin \eta \cos \eta \]

and from equations (4.2.3.2) and (4.2.3.12), this becomes

\[ \left( \frac{\partial \psi}{\partial n} \right)_{\xi = \xi_0} = U \sin \eta \cos \eta \left[ -b^2 + \frac{b^2}{c} \left[ \frac{a + b^2}{c} \log \frac{a + b - c}{a + b + c} \right. \right. \]

\[ \left. \left. + \frac{a}{c} + \frac{b}{c} \log \frac{a + b - c}{a + b + c} \right] \right] \]

\[ = 0 \quad (4.2.3.14) \]

as clearly should be the case.

Using equations (4.2.3.13) and (4.2.3.14), equation (4.2.3.8) becomes
\[ q^2c^2 \sinh^2 \xi_0 \sin^2 \eta [c^2 \sinh^2 \xi_0 \cos^2 \eta + c^2 \cosh^2 \xi_0 \sin^2 \eta] \]

\[ \frac{\frac{U^2b^2c^2 \sinh \eta}{a + b - c}}{ \log \frac{a + b - c}{a + b + c}} \]  

(4.2.3.15)

But on the ellipsoid \( \xi = \xi_0 \), equations (4.2.3.1) and (4.2.3.2) become

\[ \frac{x}{a} = \cos \eta, \quad \frac{\phi}{b} = \sin \eta \]

Using these relationships, equation (4.2.3.15) becomes

\[ q^2 = \frac{c^2U^2b^2a^2}{\left\{ \frac{a + b - c}{b + c} \log \frac{a + b - c}{a + b + c} \right\}^2 (b^4x^2 + a^4\phi^2)} \]  

(4.2.3.16)

and \( q \) will be found by taking the square root of equation (4.2.3.16).

Now the boundary condition to be satisfied over the surface of an ovary ellipsoid will be similar to that in equation (4.2.1.8) for the case of the circular cylinder i.e.

\[ \frac{\partial \phi_{o,e}}{\partial n} = U(n \cdot i) \]  

(4.2.3.17)

where \( \phi_{o,e} \) denotes the perturbation velocity potential of an ovary ellipsoid and \( \hat{n} \) is the outward drawn unit normal to the surface of the ellipsoid. The equation of an ovary ellipsoid of figure 4.9 is

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1 \]

Let \( f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} - 1 \)

\[ \hat{n} = \frac{\nabla f}{|\nabla f|} = \left( \frac{x}{a^2} \right)^i + \left( \frac{y}{b^2} \right)^j + \left( \frac{z}{b^2} \right)^k \]  

\[ \frac{\sqrt{(x/a^2)^2 + (y/b^2)^2 + (z/b^2)^2}}{\sqrt{(x/a^2)^2 + (y/b^2)^2 + (z/b^2)^2}} \]
Therefore the boundary condition in equation (4.2.3.17) takes the form

\[
\hat{n} \cdot \hat{\mathbf{i}} = \frac{xb^2}{\sqrt{b^4 x^2 + a^4 (y^2 + z^2)}}
\]

The calculated velocity distributions are compared with analytical solutions for the ellipsoids of fineness ratios 2 and 10. In both cases of ellipsoids of revolution, the input points are distributed on a sphere in the manner described in the section (4.2.2), and the y and z coordinates of these points are then divided by the fineness ratios to generate the points for the ovary ellipsoids. This procedure is easy and tends to concentrate the points in the regions of high curvature of the body surface, but the uniformity of the element distribution becomes poor. The number of elements used to obtain the calculated velocity distribution are the same as were used for the sphere. Figures 4.19 and 4.20 show the discretization of the ovary ellipsoid of fineness ratio 2 into 96 and 384 boundary elements respectively. Figure 4.21 shows the comparison of the analytical and calculated velocity distributions over the surface of an ovary ellipsoid of fineness ratio 2 for (a) 96 boundary elements and (b) 384 boundary elements.

Similarly figures 4.22 and 4.23 show the discretization of the ovary ellipsoid of fineness ratio 10 into 96 and 384 boundary elements respectively. Figure 4.24 shows the comparison of the analytical and calculated velocity distributions over the surface of an ovary ellipsoid of fineness ratio 10 for (a) 96 boundary elements and (b) 384 boundary elements. The accuracy of the calculated results in both cases of
ellipsoids of fineness ratio 2 and 10 is seen to be quite good. The 384 element case in both ovary ellipsoids agree with the analytical solutions even in the region near the stagnation point where the velocity varies rapidly with position.

4.3 EFFECTS OF A GROUND PLANE

Thus far the body under consideration has always been assumed to be positioned in an infinite space. The flow field around any body lying in a free atmosphere is different from the flow field around the same body at the same attitude when it is lying very close to the ground. Therefore simulation of the ground flow is an important feature to be incorporated in the theory. The inviscid flow over the ground, which is considered to be a plane, remains everywhere tangential to its surface. This flow is simulated by the so-called 'mirror image' principle. A mirror image of the body is imagined to be present below the ground plane. The flow field produced by a uniform stream, parallel to the ground plane, which is perturbed by the body and its image is then symmetrical with respect to the ground plane. The plane of symmetry is a stream surface and represents the ground, see figure 4.10.
Figure 4.11 illustrates how the influences of the image body on the real body are calculated.
Figure 4.11

Suppose side (1) is the side at which the values of the potential are to be evaluated at the fixed points 'i'. At a given point, the components of the $\hat{H}_{ij}$ and $G_{ij}$ integrals due to an element on side (1) are evaluated first. The z-coordinates of all the nodes of this element are then changed (side (2)) and the components from this reflected element are calculated. The y-coordinates of all the nodes of this element are then changed (side (3)) and the components from this element are evaluated. Position (4) is then reached by changing the z coordinates again and evaluating the components due to the element. The integral components from the corresponding elements
on all the four sides are then summed to calculate the total integral values at the point on side (1) due to one particular element on side (1). The process is repeated for all the elements on side (1) and then for all the fixed points. The pressure distribution over the surface of a sphere of radius 1 unit with ground clearance of 0.1 units and 0.3 units has been calculated using the above method. The pressure coefficient over the surface of the sphere is calculated for 96, 384 and 1536 boundary elements. To check the accuracy of the calculated results, these have been compared with the analytical results given in Peat [33]. Figures 4.25 and 4.26 show the comparisons of the calculated and analytical pressure coefficients over the surface of the sphere for (a) 96 and (b) 384 boundary elements, with ground clearances of 0.1 units and 0.3 units respectively. The analytical solutions are based upon a truncated series of images of doublets and line doublets. The truncation causes no serious error for a ground clearance of 0.3 units, but in the case of 0.1 units the results near the ground may be inaccurate in the first decimal place. For this reason, the pressure distribution over the surface of the sphere was also obtained for 1536 boundary elements, to hopefully give a more accurate standard for comparison of the low element cases than is possible from the analytical solution near the ground. Figure (4.27) shows the discretization of the sphere into 1536 boundary elements, whilst figure (4.28) shows the comparison of the analytical and computed pressure coefficients with the ground clearances of 0.1 and 0.3 units. From these figures one can deduce that the basic modelling of the ground plane for the boundary element method is correct.
4.4 MODELLING OF A WAKE

Thus for in the analysis of the flow field, the viscosity of the fluid has been neglected. However, all fluids are viscous to some extent, and thus in problems of aerodynamics, there is a region of the flow field near the body where the viscosity plays an important role. If the body under consideration is streamlined, then the boundary layer will cover the body from front to the rear end, producing a wake of negligible thickness. On the other hand, in vehicle aerodynamics, the road vehicle is a bluff body and thus the boundary layer separates from the body producing a thick wake.

Let the body under consideration be as shown in the figure 4.12.

In this case, the surface of the body is $S = S_B + S_W$, where $S_B$ denotes the surface of the actual body and $S_W$ the surface of the tubular wake which starts from the back of the body and extends to infinity in the
direction of the onset flow. Let \( \phi \) be the velocity potential of the acyclic irrotational motion in the region \( R \), with \( \phi \) regular at infinity. Equation (2.3.3) of the direct boundary element method for an infinite domain can be written as

\[
c_i \phi_i = \phi_\infty - \frac{1}{4\pi} \int \frac{1}{r} \frac{\partial \phi}{\partial n} \, ds + \frac{1}{4\pi} \int \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, ds
\]

\[
(4.4.1)
\]

or

\[
c_i \phi_i = \phi_\infty - \frac{1}{4\pi} \int \frac{1}{r} \frac{\partial \phi}{\partial n} \, ds - \frac{1}{4\pi} \int \frac{1}{r} \frac{\partial \phi}{\partial n} \, ds
\]

\[
+ \frac{1}{4\pi} \int \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, ds + \frac{1}{4\pi} \int \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, ds
\]

\[
(4.4.2)
\]

Since \( \frac{\partial \phi}{\partial n} = 0 \) over the surface of the wake \( S_W \), the second integral on the R.H.S. of equation (4.4.2) is zero, hence

\[
c_i \phi_i = \phi_\infty - \frac{1}{4\pi} \int \frac{1}{r} \frac{\partial \phi}{\partial n} \, ds + \frac{1}{4\pi} \int \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, ds
\]

\[
+ \frac{1}{4\pi} \int \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, ds
\]

\[
(4.4.3)
\]

Equation (4.4.3) differs from equation (2.3.3) in that now there is one additional integral on the R.H.S. of equation (4.4.3) which is to be evaluated over the surface of the wake. Equation (4.4.3) can again be expressed in the matrix formulation (3.3.21), but now the contribution of the wake integral will be added in the \( H_{ij} \) integrals of equation...
The matrix system can again be solved for unknown velocity potentials and thus the pressure coefficients over the surface of the body can be calculated by the same method outlined in the preceding sections.

The integral over the surface of the wake can be evaluated as follows:

Denoting this integral by I, then

\[
I = \frac{1}{4\pi} \iint_{S_W} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS
\]

\[
= \frac{1}{4\pi} \int_{-\infty}^{x_0} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dx d\Gamma
\]

(4.4.4)

where \( x_0 \) is the starting point of the wake and \( \Gamma \) is the closed curve bounding the tubular wake at its point of origin.

Since the wake is assumed to be tubular (i.e. parallel to the \( x \)-axis), then on the surface of the wake the velocity is the free-stream velocity, that is

\[
\frac{\partial \phi}{\partial x} = \frac{\partial \phi_{u,s}}{\partial x}
\]

But \( \phi = \phi + \phi_{u,s} \)

\[
\frac{\partial \phi}{\partial x} = 0, \text{ over the surface of the wake, where } \phi \text{ is the perturbation velocity potential.}
\]

Thus \( \phi \) is constant over the wake, or, as the wake is semi-infinite it is possible for \( \phi \) to have a finite linear variation with respect to 'x' over the complete wake strip. In the first instance, consider that \( \phi \) is constant, then equation (4.4.4) can be written as
\[ I = \frac{1}{4\pi} \int_{\Gamma} \phi \int_{-\infty}^{x_0} \frac{\partial}{\partial \mathbf{n}} \left( \frac{1}{r} \right) dx d\Gamma \] (4.4.5)

The inner integral in equation (4.4.5) can be calculated analytically. Denoting this integral by \( I_1 \), then

\[ I_1 = \int_{-\infty}^{x_0} \frac{\partial}{\partial \mathbf{n}} \left( \frac{1}{r} \right) dx \]

\[ = -\int_{-\infty}^{x_0} \frac{1}{r^2} (\mathbf{n} \cdot \mathbf{r}) dx \quad (\therefore \frac{\partial r}{\partial \mathbf{n}} = \mathbf{n} \cdot \mathbf{r}) \] (4.4.6)

Since \( r \) is the distance from the fixed point \( i(x_i, y_i, z_i) \) to any general point \( (x, y, z) \), then

\[ r = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} \] (4.4.7)

\[ \therefore \mathbf{n} \cdot \mathbf{r} = \frac{(x - x_i)\mathbf{n} \cdot \mathbf{i} + (y - y_i)\mathbf{n} \cdot \mathbf{j} + (z - z_i)\mathbf{n} \cdot \mathbf{k}}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} \]

Since \( \mathbf{n} \cdot \mathbf{i} = 0 \), over the wake surface, then

\[ \mathbf{n} \cdot \mathbf{r} = \frac{(y - y_i)\mathbf{n} \cdot \mathbf{j} + (z - z_i)\mathbf{n} \cdot \mathbf{k}}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}} \] (4.4.8)

Using equations (4.4.7) and (4.4.8), equation (4.4.6) takes the form

\[ I = -[(y - y_i)\mathbf{n} \cdot \mathbf{j} + (z - z_i)\mathbf{n} \cdot \mathbf{k}] \int_{-\infty}^{x_0} \frac{dx}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}^{3/2}} \] (4.4.9)

Now making use of the general formula
\[
\int \frac{dx}{(ax^2 + bx + c)^{3/2}} = \frac{-2(2ax + b)}{(b^2 - 4ac)^{1/2}ax^2 + bx + c}
\]

the integral in equation (4.4.9) can be evaluated as follows:

\[
\begin{align*}
\int_{-\infty}^{x_0} \frac{dx}{[x^2 - 2x_1 x + x_1^2 + (y - y_1)^2 + (z - z_1)^2]^{3/2}} &= \left[ \frac{(x - x_1)}{[(y - y_1)^2 + (z - z_1)^2][x^2 - 2x_1 x + x_1^2 + (y - y_1)^2 + (z - z_1)^2]^{1/2}} \right]_{-\infty}^{x_0} \\
&= \frac{1}{[(y - y_1)^2 + (z - z_1)^2]} \left[ \frac{(x_0 - x_1)}{[(x_0 - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{1/2}} + 1 \right]
\end{align*}
\]

(4.4.10)

From equation (4.4.9) and (4.4.10)

\[
I_1 = -\frac{[(y - y_1)\hat{n}_j + (z - z_1)\hat{n}_k]}{[(y - y_1)^2 + (z - z_1)^2]} \left[ \frac{(x_0 + x_1)}{[(x_0 - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{1/2}} + 1 \right]
\]

(4.4.11)

Substituting this value of \( I_1 \) in equation (4.4.5), then

\[
I = -\frac{1}{4\pi} \int_{\Gamma} \frac{(y - y_1)\hat{n}_j + (z - z_1)\hat{n}_k}{[(y - y_1)^2 + (z - z_1)^2]} \left[ \frac{(x_0 - x_1)}{[(x_0 - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{1/2}} + 1 \right] d\Gamma
\]

(4.4.12)

Suppose that the closed curve \( \Gamma \) is divided into \( m_1 \) two dimensional boundary elements. \( m_1 \) is also the number of boundary elements on the
surface of the body associated with the wake. The two-dimensional isoparametric boundary elements can be linear, quadratic or cubic, corresponding to the variation chosen for the quadrilateral elements over the surface of the body. Thus $\phi$, over the two-dimensional boundary elements, can be written as

$$\phi = \sum_{k=1}^{L} N_k \phi_k$$  \hspace{1cm} (4.4.13)

where $L$ is the number of nodes and is equal to two, three or four in the case of linear quadratic or cubic elements respectively, $\phi_k$ are the nodal values of $\phi$ and $N_k$ are the shape functions as defined in equations (3.2.1) or (3.2.2) or (3.2.3).

Using equation (4.4.13), equation (4.4.12) becomes

$$I = - \frac{1}{4\pi} \sum_{j=1}^{m_1} \int \frac{(y - y_i) \hat{n}_j + (z - z_i) \hat{n}_k}{[(y - y_i)^2 + (z - z_i)^2]^2} \left[ \frac{(x_0 - x_i)}{[(x_0 - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^2} + 1 \right] \, \mathrm{d}r \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_L \end{bmatrix}$$

$$= \sum_{j=1}^{m_1} \begin{bmatrix} w_{ij}^1 & w_{ij}^2 & \cdots & w_{ij}^L \end{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_L \end{bmatrix}$$  \hspace{1cm} (4.4.14)
\[ w_{ij}^k = - \frac{1}{4\pi} \int_{\Gamma_j} N_k \left[ \frac{(y-y_i)^\hat{n}.j + (z-z_i)^\hat{n}.k}{(y-y_i)^2 + (z-z_i)^2} \right] \left[ \frac{(x_0 - x_i)}{(x_0 - x_i)^2 + (y-y_i)^2 + (z-z_i)^2} + 1 \right] d\Gamma \]

\[ k = 1, 2, \ldots, L \]

\( \hat{n}.j \) and \( \hat{n}.k \) in equation (4.4.15) can be found as follows:

For the sake of simplicity assume that the surface of the body is divided into linear quadrilateral elements so that the two-dimensional boundary elements are also linear. The same method can be applied to the two-dimensional quadratic or cubic boundary elements. Figure 4.13 shows a two-dimensional linear boundary element in the plane \( x = x_0 \), where \( x_0 \) is a point from where the wake of body starts. Let \( \hat{r}_0 \) be the position vector of any point on the element, then \( \hat{r}_0 \) can be written as

\[ \hat{r}_0 = yj + zk \]

But for an isoparametric element \( y \) and \( z \) can be written as
\[ y = \sum_{k=1}^{2} N_k y_k, \quad z = \sum_{k=1}^{2} N_k z_k \]

or

\[ y = \frac{1}{2}(y_2 - y_1) \xi + \frac{1}{2}(y_2 + y_1) \]
\[ z = \frac{1}{2}(z_2 - z_1) \xi + \frac{1}{2}(z_2 + z_1) \] \hspace{1cm} (4.4.17)

From equations (4.4.16) and (4.4.17),

\[ \hat{r}_0 = \left[ \frac{1}{2}(y_2 - y_1) \xi + \frac{1}{2}(y_2 + y_1) \right] \hat{j} + \left[ \frac{1}{2}(z_2 - z_1) \xi + \frac{1}{2}(z_2 + z_1) \right] \hat{k} \]

The tangent vector \( \hat{t} \) at any point to the element can be written as

\[ \hat{t} = \frac{d\hat{r}_0}{d\xi} = \frac{1}{2}(y_2 - y_1) \hat{j} + \frac{1}{2}(z_2 - z_1) \hat{k} \]

or the unit tangent vector \( \hat{t} \) can be given as

\[ \hat{t} = \frac{(y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k}}{\sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}} \]

Since the numbering system for the elements on the wake is anti-clockwise as viewed from the front of the body, then rotating this unit tangent clockwise through 90°, the outwardly drawn unit normal \( \hat{n} \) to the element is given by

\[ \hat{n} = \frac{(z_2 - z_1) \hat{j} - (y_2 - y_1) \hat{k}}{\sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}} \] \hspace{1cm} (4.4.18)

and thus \( \hat{n} \cdot \hat{j} \) and \( \hat{n} \cdot \hat{k} \) can be written as
\[ n.j = \frac{(z_2 - z_1)}{\sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}} \]

and

\[ n.k = \frac{-(y_2 - y_1)}{\sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}} \]  \hspace{1cm} (4.4.19)

With these values of \( n.j \) and \( n.k \), the integrals of equation (4.4.15) can be evaluated numerically using one-dimensional Gauss-quadrature rule.

Thus far, the value of \( \phi \) in equation (4.4.4) has been taken to be constant with respect to 'x' over the surface of the wake. If one considers \( \phi \) to have a linear variation with respect to 'x', it can be shown that the value of the integral I remains the same, thus the results obtained above hold in this case as well.

4.4.1 FLOW PAST A SEMI INFINITE BODY

Before the previous analysis was used to calculate the flow around a ground vehicle, it was felt advisable to compare it with an analytical solution so that the accuracy of the method could be checked. Whilst there is no analytical solution to flow around a bluff body with a wake, the technique used in the boundary element method was to model the wake as a physical extension of the body, giving an effectively semi-infinite solid body, and an analytical solution can be found for uniform flow past some semi-infinite bodies.

Let the semi-infinite body be as shown in the figure 4.14.
This semi-infinite body is obtained by considering a source of unit strength lying at the origin in a uniform stream of velocity 1 in the negative direction of x-axis. It has been shown in Vallentine [32], that the surface of this semi-infinite body is

\[ r = \sec \frac{\theta}{2} \]  

(4.4.1.1)

and the pressure distribution over the surface of this semi-infinite body is calculated analytically from the formula

\[ C_p = \frac{3}{r^4} - \frac{2}{r^2} \]  

(4.4.1.2)

The surface of the semi-infinite body is approximated using different number of boundary elements. The method of distribution of elements is as follows:
Divide the surface of a hemisphere of unit radius into boundary elements in the manner as described in section (4.2.2). Multiply the coordinates of the input points by 2 to generate the data for the hemisphere of radius 2. Stretch the hemisphere of radius 2 in the negative direction of x-axis in such a way that the surface of the hemisphere is transformed into the shape of a semi-infinite body and cut the body by a vertical plane at the point $x = -a$, where $a$ is any positive constant. In this way the point $(2, 0)$ and the plane $x = 0$ of the hemisphere of radius 2 coincide with the point $(1, 0)$ and the plane $x = -a$ respectively of the deformed body. The y and z coordinates of the deformed body are the same as that of the hemisphere of radius 2, while the x-coordinate of the deformed body is calculated as follows:

From the figure 4.15,

$$x_1 = 2 \cos \phi, \quad R_1 = 2 \sin \phi$$

and $$x_2 = \sec \frac{\theta}{2} \cos \theta, \quad R_2 = \sec \frac{\theta}{2} \sin \theta = 2 \sin \frac{\theta}{2}$$

Therefore the stretch $= x_1 - x_2$

$$= 2 \cos \phi - \sec \frac{\theta}{2} \cos \theta$$

or

$$x_2 = x_1 + \sec \frac{\theta}{2} \cos \theta - 2 \cos \phi$$

where

$$\phi = \tan^{-1} \left( \frac{R_1}{x_1} \right).$$

To calculate $\theta$, it can be seen from the figure 4.15 that $R_1 = R_2$

$$\therefore 2 \sin \phi = 2 \sin \frac{\theta}{2} \quad \text{and} \quad \theta = 2\phi$$
Next, to approximate the deformed body with the semi-infinite body, a wake will be added to the surface of the deformed body starting from the plane \(x = -a\) and extending to \(x = -\infty\) in the direction of the uniform stream.

The boundary condition to be satisfied on the surface of the semi-infinite body is

\[
\frac{\partial \phi_{s, \text{inf}}}{\partial n} = -U(n \cdot \hat{n})
\]

where \(\phi_{s, \text{inf}}\) denotes the perturbation velocity potential of the semi-infinite body and \(\hat{n}\) is the outwardly drawn unit normal at each node on the surface of the semi-infinite body. The problem of finding this unit normal is slightly more difficult than in previous examples. The actual method used, given below, was developed for the general case of road vehicles where it is impossible to define analytically a unit normal.

Each node on the surface of the semi-infinite body, except in the symmetry plane, has a maximum of four boundary elements associated with it and corresponding to each of these boundary elements, the normal at this node will, in general, have a different direction. Thus the normal at the node under consideration is calculated with respect to each of the associated elements and then the mean of all these normals is taken as the true normal to the surface at that node. For a node in the symmetry plane, there are a maximum of two boundary elements associated with this node and to find the true normal, the elements corresponding to this node on the opposite side of the symmetry plane are also taken into account. Thus the mean of all the normals at
this node corresponding to these elements will be taken as the true normal to the surface at this node.

Figures 4.29, 4.30 and 4.31 show the discretization of the semi-infinite body into 48, 192 and 768 boundary elements respectively. In order to have the uniform distribution of elements at the back of the body, it is cut by the plane \( x = -5, x = -10 \) and \( x = -20 \) for the case of 48, 192 and 768 boundary elements respectively. The pressure distribution over the surface of these semi-infinite bodies are calculated using the direct boundary element method. Figure 4.32 shows the comparison of the analytical and calculated pressure distributions over the surface of the semi-infinite body for (a) 48 boundary elements (b) 192 boundary elements, whilst figure 4.33 shows the comparison for 768 boundary elements. It is seen that as the number of boundary elements is increased, the accuracy in the results is also increased. In the case of 768 boundary elements, there is an excellent agreement between the analytical and the calculated results.

4.5 COMPARISON OF GAUSS-ELIMINATION AND GAUSS-SEIDEL METHODS

In section (3.5), the general comparison of Gauss-elimination and Gauss-Seidel iterative methods for solving the system of linear algebraic equations was presented. It was concluded that in solving a system of \( M \) linear equations, the Gauss-Seidel method is computationally more efficient if it converges in fewer than \( M \) iterations. In this section the comparison of these two methods for solving the system of equations which result from the discretization of the body in the case of three-dimensional exterior flow problems is presented. The following table
shows the comparison of computing time for the solution stage of equation systems of different orders. The actual results presented are for flows about a sphere and a semi-infinite body, but for flow about any other bodies, the computing time is not significantly different when the system size is kept the same.

**TABLE**

<table>
<thead>
<tr>
<th>System Size</th>
<th>Gauss-Elimination</th>
<th>Gauss-Seidel</th>
<th>Iterations Required</th>
</tr>
</thead>
<tbody>
<tr>
<td>57 x 57</td>
<td>0.602</td>
<td>0.448</td>
<td>7</td>
</tr>
<tr>
<td>113 x 113</td>
<td>4.636</td>
<td>1.559</td>
<td>8</td>
</tr>
<tr>
<td>209 x 209</td>
<td>28.823</td>
<td>4.383</td>
<td>7</td>
</tr>
<tr>
<td>417 x 417</td>
<td>-</td>
<td>20.416</td>
<td>9</td>
</tr>
<tr>
<td>801 x 801</td>
<td>-</td>
<td>58.048</td>
<td>7</td>
</tr>
</tbody>
</table>

It can be seen from this table that in all cases the Gauss-Seidel method is more computationally efficient than the Gauss-elimination method. The gain in efficiency of the Gauss-Seidel method over the Gauss-elimination method is seen to increase with increasing system size. The number of iterations required for the convergence of the Gauss-Seidel method is very small even for large system size. Convergence of Gauss-Seidel method has been taken to occur when the greatest iterative change of any single value of potential is less than 0.00001.
Figure 4.16: Discretization of the sphere into 96 boundary elements. The point of observation is (a) on the z-axis; (b) at 45° to all axes.
Figure 4.17: Discretization of the sphere into 384 boundary elements. The point of observation is
(a) on the z-axis; (b) at 45° to all axes.
Figure 4.18: Comparison of analytical and computed pressure coefficients on a sphere for (a) 96 boundary elements; (b) 384 boundary elements.
Figure 4.19: Discretization of an ovary ellipsoid of fineness ratio 2 into 96 boundary elements. The point of observation is 
(a) on the minor axis; (b) at 45° to all axes.
Figure 4.20: Discretization of an ovary ellipsoid of fineness ratio 2 into 384 boundary elements. The point of observation is (a) on the minor axis; (b) at 45° to all axes.
Figure 4.21: Comparison of analytical and computed velocity distributions on an ovary ellipsoid of fineness ratio 2 for (a) 96 boundary elements; (b) 384 boundary elements.
Figure 4.22: Discretization of an ovary ellipsoid of fineness ratio 10 into 96 boundary elements. The point of observation is (a) on the minor axis; (b) at $45^\circ$ to all axes.
Figure 4.23: Discretization of an ovary ellipsoid of fineness ratio 10 in 384 boundary elements. The point of observation is (a) on the minor axis; (b) at 45° to all axes.
Figure 4.24: Comparison of analytical and computed velocity distributions on an ovary ellipsoid of fineness 10 for (a) 96 boundary elements; (b) 384 boundary elements.
Figure 4.25: Comparison of analytical and computed pressure coefficients over the surface of a sphere with ground clearance 0.1 units for (a) 96 boundary elements; (b) 384 boundary elements.
Figure 4.26: Comparison of analytical and computed pressure coefficients over the surface of a sphere with ground clearance 0.3 units for (a) 96 boundary elements; (b) 384 boundary elements.
Figure 4.27: Discretization of the sphere into 1536 boundary elements. The point of observation is (a) on the z-axis; (b) at 45° to all axes.
Figure 4.28: Comparison of analytical and computed pressure coefficients over the surface of a sphere for 1536 boundary elements with ground clearance (a) 0.1 units; (b) 0.3 units.
Figure 4.29: Discretization of a semi-infinite body into 48 boundary elements. The point of observation is (a) on the z-axis; (b) at $45^\circ$ to all axes.
Figure 4.30: Discretization of a semi-infinite body into 192 boundary elements. The point of observation is (a) on the minor axis; (b) at 45° to all axes.
Figure 4.31: Discretization of a semi-infinite body into 768 boundary elements. The point of observation is (a) on the z-axis; (b) at 45° to all axes.
Figure 4.32: Comparison of analytical and computed pressure coefficients on a semi-infinite body using (a) 48 boundary elements; (b) 192 boundary elements.
Figure 4.33: Comparison of analytical and computed pressure coefficients on a semi-infinite body using 768 boundary elements.
SECTION 5

COMPARISON OF THE DIRECT AND INDIRECT BOUNDARY ELEMENT METHODS

5.1 INTRODUCTION

In this section the indirect boundary element method is discussed and is applied to obtain the flowfield calculations around the bodies which were considered in section 4. The results obtained by this method are compared with those which were obtained using the direct boundary element method in section 4. In the usual indirect boundary element method, the surface of the body under consideration is covered with a distribution of singularities such as sources or doublets and the basic unknowns are then calculated in the form of source or doublet strengths over the body. The calculated singularity distribution can then be used to calculate the tangential velocity of the flow over the body. For three-dimensional problems consideration will be given only to doublet distributions and source distributions will not be considered, due to the following reasons.

The boundary element methods are to be applied to calculate the flowfield around road vehicles which generally have some net circulation of flow around them which gives rise to a lift force on the vehicle. A source distribution alone cannot produce any circulation and must therefore be used in conjunction with a doublet distribution. If the
surface of the body is covered with a source distribution and the wake surface with a doublet distribution, then, as discussed by Ahmed and Hucho [14], this representation leads to numerical difficulties at the end of the body where the wake starts. The problem can be overcome to some extent by an artificial extension of the wake interior to the body but this still represents extra complications as compared to the case for a distribution of doublets alone. Furthermore, if the surface of the body is covered with a source distribution, then the resulting matrix of influence coefficients is not always diagonally dominant and then an iterative method of solution is not possible.

A combined distribution of sources and doublets could be used to cover the surface of the body and the wake, which would introduce twice as many unknowns as compared with the distribution of sources alone or doublets alone. This technique is often used in two dimensional aerodynamics, where the extra set of unknowns are eliminated by equating the strength of the singularities on pairs of elements on the top and bottom surface of the body respectively. This clearly has no counterpart in three-dimensions, and in any case is essentially aimed at dealing with sharp trailing edges of wings, a problem which does not occur with bluff road vehicles. Thus, keeping in view the above mentioned difficulties, the surface of the body as well as the surface of the wake will be covered with a distribution of doublets alone.

In section (5.2), alternative formulations of the indirect boundary element method in the case of doublet distributions alone are discussed while the matrix formulation is given in section (5.3). In section (5.4), the method is applied to calculate the flow past a circular cylinder, a sphere and ellipsoids of revolution, while in section (5.5) the flow around the sphere is calculated taking the
ground effects into consideration. Finally, in section (5.6), the modelling of the wake is discussed and the method is applied to a semi-infinite body. The results obtained in section (5.4) to (5.6) are compared with the corresponding results in section 4 with regard to accuracy. The comparison of the computing times for both the direct and indirect methods is given in section (5.7) and the conclusions of this chapter are presented in section (5.8).

5.2 ALTERNATIVE FORMS OF THE INDIRECT BOUNDARY ELEMENT METHOD USING DOUBLET DISTRIBUTIONS

The equation (2.4.4) for the indirect boundary element method in the case of a doublet distribution alone can be written as

\[ (1 - c_i)[\phi'_i - \phi_i] + \phi_i = \phi_\infty - \frac{1}{4\pi} \oint (\phi' - \phi) \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS \]

Denoting \(- (\phi' - \phi) = \mu\), the strength of the doublet, the above equation takes the form

\[ \phi_i = \phi_\infty + (1 - c_i)\mu_i + \frac{1}{4\pi} \oint \mu \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS \]  \hspace{1cm} (5.2.1)

In order to apply the boundary condition of zero normal velocity, equation (5.2.1) is differentiated with respect to 'n' to give a second derivative on the R.H.S. In this formulation, the unknowns are calculated in the form of doublet strengths which are then used to calculate the tangential velocity via the derivative of equation (5.2.1) with respect to 't'. This approach is the 'usual' indirect
method formulation for the doublet distribution alone.

An alternative formulation of the indirect method for this case follows from equation (2.4.2) by specifying the interior velocity potential $\phi'$ equal to the negative of the velocity potential $\phi_{u.s}$ of the uniform stream. This approach has been discussed by Maskew [12]. Let $\phi' = -\phi_{u.s}$ in equation (2.4.2), then

\[
\begin{align*}
\frac{c_i}{n} \left[ \phi_i + (\phi_{u.s})_i \right] - (\phi_{u.s})_i &= \phi_\infty + \frac{1}{4\pi} \int \int \frac{1}{r} \frac{\partial}{\partial n} \left[ -\phi_{u.s} \right] - \frac{\partial \phi}{\partial n} dS \\
&- \frac{1}{4\pi} \int \int_{S-i} \left[ -\phi_{u.s} - \phi \right] \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS
\end{align*}
\]

Now with $\phi = \phi + \phi_{u.s}$, this equation becomes

\[
\begin{align*}
-c_i \phi_i - \frac{1}{4\pi} \int \int \frac{1}{r} \frac{\partial \phi}{\partial n} dS + \frac{1}{4\pi} \int \int_{S-i} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS + \phi_\infty &= -(\phi_{u.s})_i \\
(5.2.2)
\end{align*}
\]

The boundary condition of zero normal velocity now implies that $\frac{\partial \phi}{\partial n} = 0$, and the first integral on the L.H.S. of equation (5.2.2) vanishes. Thus

\[
\begin{align*}
-c_i \phi_i + \frac{1}{4\pi} \int \int_{S-i} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS + \phi_\infty &= -(\phi_{u.s})_i \\
(5.2.3)
\end{align*}
\]

In this formulation, the unknowns are calculated in the form of total potentials at the surface of the body which are then used to calculate the velocity in a similar manner to that discussed in section 4 for the direct boundary element method.
The formulation of the indirect method given by equation (5.2.3) is much easier to implement than the 'formal' doublet approach and will be used throughout this section to calculate the flow field around three-dimensional bodies.

For two-dimensional problems, the equation of the indirect boundary element method in the case of a doublet distribution alone with this formulation takes the form

\[-c_i \phi_i + \frac{1}{2\pi} \int_{\Gamma-i} \phi \frac{3}{2n} \left( \log \frac{1}{r} \right) d\Gamma + \phi_\infty = -(\phi_{u,s})_i\]  (5.2.4)

5.3 MATRIX FORMULATION

In this section the matrix formulation of the equation (5.2.3) will be discussed. The same procedure can be adopted for the matrix formulation of the equation (5.2.4). Divide the surface of the body into \( m \) quadrilateral boundary elements, then equation (5.2.3) can be written as

\[-c_i \phi_i + \frac{1}{4\pi} \sum_{j=1}^{m} \int_{S_{j-i}} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS + \phi_\infty = -(\phi_{u,s})_i\]  (5.3.1)

The integrals on the L.H.S. of equation (5.3.1) are of the same form as the integrals on the L.H.S. of equation (3.3.1) except that \( \phi \) in the integrals of equation (3.3.1) has now been replaced by the total potential \( \phi \). Proceeding as in section (3.3) for the case of exterior flow problems, equation (5.3.1) in the case of constant, linear or higher order elements can be expressed in the matrix form as
\[ [H][\mathbf{u}] = [R] \quad (5.3.2) \]

where as usual \([H]\) is a matrix of influence coefficients, \([\mathbf{u}]\) is a vector of unknown total potentials \(\phi_i\) and \([R]\) on the R.H.S. is a known vector whose elements are the negative of the values of the velocity potential of the uniform stream at the nodes on the surface of the body. Note the difference in the matrix formulations (3.3.21) and (5.3.2). In the latter case, the matrix \([G]\) of influence coefficients on the R.H.S. is absent as opposed to the matrix formulation (3.3.21) for the direct boundary element method. Again in the formulation (5.3.2), for the case of linear or higher order elements, the values of the coefficients in the diagonal of the matrix \([H]\) can be calculated from the equation (3.3.22), once the off-diagonal coefficients are all known.

### 5.4 UNBOUNDED FLOW FIELD CALCULATIONS

In this section, the indirect boundary element method with formulation (5.2.3) has been applied to calculate the flow past a circular cylinder, a sphere and ellipsoids of revolution. In the case of a circular cylinder, the flow is also calculated using the source distribution alone and a comparison of this formulation with that of formulation (5.2.3) has been presented. Again the boundary of the circular cylinder is discretized into the constant and the linear boundary elements and the integrals over these elements are calculated numerically using Gauss-quadrature rule with four integration points except for the element corresponding to the node under consideration. For this element the integrals are calculated
analytically. In the case of a sphere and ellipsoids of revolution, the surface of the body under consideration is discretized into four noded linear isoparametric quadrilateral elements and the integrals over each element have been evaluated using Gaussian-quadrature rule with four integration points.

5.4.1 FLOW PAST A CIRCULAR CYLINDER

The flow past a circular cylinder has been calculated using the doublet distribution alone as well as the source distribution alone. Consider a circular cylinder of radius 'a' with centre at the origin lying in a uniform stream, which is flowing with velocity $U$ in the positive direction of the x-axis as shown in the figure 4.1.

Consider firstly the case of the doublet distribution over the constant boundary elements. The discretization of the boundary of the circular cylinder for this case is shown in figure 4.2. The equation of the indirect boundary element method in the case of two dimensional problems takes the form

$$-\frac{1}{2} \phi_i + \frac{1}{2\pi} \int_{\Gamma_i} \phi \frac{3}{2n} (\log \frac{1}{r}) d\Gamma + \phi_\infty = -(\phi_{u.s})_i$$

Since $(\phi_{u.s})_i = -x_i$, the above equation becomes

$$-\frac{1}{2} \phi_i + \frac{1}{2\pi} \int_{\Gamma_i} \phi \frac{3}{2n} (\log \frac{1}{r}) d\Gamma + \phi_\infty = x_i$$  \hspace{1cm} (5.4.1.1)

where $\hat{n}$ is the unit normal vector drawn into the domain, as was the case for the direct boundary element method. The evaluation of the
integrals in equation (5.4.1.1) has already been discussed in section (4.2.1). Once these integrals are calculated, equation (5.4.1.1) can now be written in the matrix form (5.3.2) which can be solved for the total potentials $\Phi_i$. Again the velocity at the 8 points shown in figure 4.4 is calculated using the formula (4.2.1.19). The table (1) below shows the comparison of the analytical and the calculated velocities over the boundary of the circular cylinder for 8 points.

<table>
<thead>
<tr>
<th>Element</th>
<th>$x$-Coordinate</th>
<th>$y$-Coordinate</th>
<th>Calculated Velocity</th>
<th>Analytical Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.79</td>
<td>0.33</td>
<td>0.82884E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>2</td>
<td>-0.33</td>
<td>0.79</td>
<td>0.20010E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>3</td>
<td>0.33</td>
<td>0.79</td>
<td>0.20010E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>4</td>
<td>0.79</td>
<td>0.33</td>
<td>0.82884E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>5</td>
<td>0.79</td>
<td>-0.33</td>
<td>0.82884E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>6</td>
<td>0.33</td>
<td>-0.79</td>
<td>0.20010E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>7</td>
<td>-0.33</td>
<td>-0.79</td>
<td>0.20010E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>8</td>
<td>-0.79</td>
<td>-0.33</td>
<td>0.82884E+00</td>
<td>0.76537E+00</td>
</tr>
</tbody>
</table>

On comparison of the results of this table with those in table (1) on page (81), which were obtained using the direct boundary element method, it is seen that the results obtained with the direct method are more close to the analytical ones.

Next consider the case of a doublet distribution over linear boundary elements. The discretization of the boundary of the circular cylinder into elements is shown in figure 4.5. The evaluation of the integrals in equation (5.4.1.1) for this case is
discussed in section (4.2.1) and again equation (5.4.1.1.) can be put in the matrix form as in equation (5.3.2), which will be solved for the total potential $\phi_1$. The velocity at the mid-point of each element, as shown in figure 4.5 is calculated using formula (4.2.1.19). The table (2) below shows a comparison of the analytical and calculated velocities for the 8 boundary elements.

<table>
<thead>
<tr>
<th>Element</th>
<th>$x$- Coordinate</th>
<th>$y$- Coordinate</th>
<th>Calculated Velocity</th>
<th>Analytical Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.85</td>
<td>0.35</td>
<td>0.71971E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>2</td>
<td>-0.35</td>
<td>0.85</td>
<td>0.17375E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>3</td>
<td>0.35</td>
<td>0.85</td>
<td>0.17375E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>4</td>
<td>0.85</td>
<td>0.35</td>
<td>0.71971E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>5</td>
<td>0.85</td>
<td>-0.35</td>
<td>0.71971E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>6</td>
<td>0.35</td>
<td>-0.85</td>
<td>0.17375E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>7</td>
<td>-0.35</td>
<td>-0.85</td>
<td>0.17375E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>8</td>
<td>-0.85</td>
<td>-0.35</td>
<td>0.71971E+00</td>
<td>0.76537E+00</td>
</tr>
</tbody>
</table>

On comparison of the results of this table with those in table (3) on page (88), which were obtained using the direct boundary element method, it is seen that in this case the results obtained with the indirect method are more close to the analytical results, in contrast to the situation for constant elements. Also it can be seen that with this formulation in passing from constant boundary elements to linear boundary elements, the accuracy is increased, as one expects. Once again, an increase in the number of elements leads to an increase in accuracy.
Let us now consider the case of a source distribution over the constant boundary elements. Discretize the boundary of the circular cylinder as shown in the figure 5.1.

![Figure 5.1](image)

The equation of the indirect boundary element method for the source distribution alone in the case of two-dimensional exterior flow problems can be written as

\[
\phi_i = \phi_w + \frac{1}{2\pi} \int_{\Gamma} \left( \log \frac{1}{r} \right) \sigma \mathrm{d}\Gamma
\]  

(5.4.1.2)

In order to apply the boundary condition of zero normal velocity at the surface, equation (5.4.1.2) is differentiated with respect to 'n' to give

\[
\left( \frac{\partial \phi}{\partial n} \right)_i = \frac{1}{2\pi} \int_{\Gamma} \sigma \frac{1}{r} \left( \log \frac{1}{r} \right) d\Gamma
\]

If the point 'i' is on the boundary, the integral on the R.H.S. becomes singular and thus
where $\Gamma - i$ indicates that the point 'i' has been excluded from the boundary $\Gamma$. In this case, contrary to the direct boundary element method or the indirect boundary element method with formulation (5.3.2), $\hat{n}$ is the unit normal vector to the fixed point 'i' on the boundary.

The boundary of the circular cylinder is discretized into $m$ boundary elements, such that equation (5.4.1.3) can be written as

$$\left[ \frac{\partial \phi}{\partial n} \right]_i = \frac{1}{2} \sigma_i + \frac{1}{2\pi} \int_{\Gamma - i} \sigma \frac{\partial}{\partial n} \left( \log \frac{1}{r} \right) dr \quad (5.4.1.3)$$

and

$$\left[ \frac{\partial \phi}{\partial n} \right]_i = \frac{1}{2} \sigma_i + \sum_{j=1}^{m} \left[ \frac{1}{2\pi} \int_{\Gamma - i} \frac{\partial}{\partial n} \left( \log \frac{1}{r} \right) dr \right] \sigma_j \quad (5.4.1.4)$$

since $\sigma_j$ are constant over the elements.

The integrals on the R.H.S. of equation (5.4.1.4) on the elements are evaluated numerically except for the element on which the fixed point 'i' is lying. For this element the integrals are calculated analytically. The reason for adopting this choice is that although in this particular case of constant elements the integrals over all the elements could be evaluated analytically, in the case of higher order boundary elements the integrals become too difficult to evaluate analytically. On the other hand, whilst all these integrals could be evaluated numerically for any elements, including that over which the fixed point 'i' is lying, this latter procedure gives poor results.

It will be shown in section (6) that the normal procedure for numerical evaluation of integrals on the boundary elements for three-dimensional problems over which the fixed point 'i' is lying gives poor results. As the integrals cannot be evaluated analytically in that case, an
alternative quadrature scheme has been developed which gives much better results. Equation (5.4.1.4) then becomes

\[
\left\{ \frac{\partial \phi}{\partial n} \right\}_i = \frac{1}{2} \sigma_i + \sum_{j=1}^{m} H_{ij} \sigma_j
\]  

(5.4.1.5)

where

\[
\hat{H}_{ij} = \frac{1}{2\pi} \int_{-1}^{1} \frac{1}{r^2} \left[ (x_i - x_q) \cos \alpha + (y_i - y_q) \sin \alpha \right] \frac{d\xi}{2}
\]

in which \((x_i, y_i)\) are the coordinates of the fixed point 'i' and \((x_q, y_q)\) are the coordinates of any general point 'Q' on the element over which the integration is being performed.

Let

\[
\hat{H}_{ij} = \begin{cases} 
\hat{H}_{ij} & \text{when } i \neq j \\
\hat{H}_{ij} + \frac{1}{2} & \text{when } i = j 
\end{cases}
\]

then equation (5.4.1.5) takes the form

\[
\sum_{j=1}^{m} H_{ij} \sigma_j = \left\{ \frac{\partial \phi}{\partial n} \right\}_i
\]

which can be expressed in the matrix form

\[
[H] \{\sigma\} = \{Q\}
\]  

(5.4.1.6)

From this equation the unknowns are calculated in the form of the source strengths \(\sigma_i\) over the boundary of the circular cylinder which are then used to find the tangential velocities.

Now the tangential velocity at the mid-point of each element can be found by differentiating equation (5.4.1.2) with respect to 't', thus
\[
\left\{ \frac{\partial \phi}{\partial t} \right\}_i = \frac{1}{2\pi} \int_\Gamma \sigma \frac{\partial}{\partial t} \left( \log \frac{1}{r} \right) d\Gamma
\]

\[
= - \frac{1}{2\pi} \sum_{j=1}^{m} \int_{\Gamma_j} \sigma_j \left( \frac{1}{r} \frac{\partial r}{\partial t} \right) d\Gamma
\]

\[
= - \frac{1}{2\pi} \sum_{j=1}^{m} \int_{-1}^{1} \sigma_j \frac{1}{r} \left( \hat{t} \cdot \hat{r} \right) \frac{\xi}{2} d\xi
\]

Since \( \hat{r} = -\hat{r} \), the above equation becomes

\[
\left\{ \frac{\partial \phi}{\partial t} \right\}_i = \frac{1}{2\pi} \sum_{j=1}^{m} \int_{-1}^{1} \sigma_j \frac{1}{r} \left( \hat{t} \right) \frac{\xi}{2} d\xi \tag{5.4.1.7}
\]

where \( \hat{t} \) is the unit tangent vector at the fixed point \( 'i' \). Equation (5.4.1.7) can be written as

\[
\left\{ \frac{\partial \phi}{\partial t} \right\}_i = \frac{1}{2\pi} \sum_{j=1}^{m} \int_{-1}^{1} \sigma_j \frac{1}{r^2} \left[ (x_i - x_j) \sin \alpha - (y_i - y_j) \cos \alpha \right] \frac{\xi}{2} d\xi \tag{5.4.1.8}
\]

Once again the integrals are evaluated numerically using a Gauss-quadrature rule, except over the element containing the fixed point \( 'i' \). For that element, it can be shown analytically that the integral is zero. Thus equation (5.4.1.8) gives the tangential velocity at the mid-point of each element due to the source distribution. The total tangential velocity of the flow past a circular cylinder can then be calculated by adding the tangential component of the free stream velocity at each point.
The table (3) below shows the comparison of the analytical and calculated velocities at the mid point of 8 boundary elements over the circular cylinder.

<table>
<thead>
<tr>
<th>Element</th>
<th>x- Coordinate</th>
<th>y- Coordinate</th>
<th>Calculated Velocity</th>
<th>Analytical Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.85</td>
<td>0.35</td>
<td>0.76538E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>2</td>
<td>-0.35</td>
<td>0.85</td>
<td>0.18478E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>3</td>
<td>0.35</td>
<td>0.85</td>
<td>0.18478E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>4</td>
<td>0.85</td>
<td>0.35</td>
<td>0.76538E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>5</td>
<td>0.85</td>
<td>-0.35</td>
<td>0.76538E+00</td>
<td>0.76537E+00</td>
</tr>
<tr>
<td>6</td>
<td>0.35</td>
<td>-0.85</td>
<td>0.18478E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>7</td>
<td>-0.35</td>
<td>-0.85</td>
<td>0.18478E+01</td>
<td>0.18478E+01</td>
</tr>
<tr>
<td>8</td>
<td>-0.85</td>
<td>-0.35</td>
<td>0.76538E+00</td>
<td>0.76537E+00</td>
</tr>
</tbody>
</table>

On comparison of the results of this table with those which were obtained using the direct method or the indirect method with a doublet distribution, using either the constant or the linear boundary elements, it can be seen that the results obtained for the indirect method using a source distribution are in excellent agreement with the analytical results.

Now consider the case of a source distribution over the linear boundary elements. The discretization of the boundary of the circular cylinder into elements is shown in the figure 4.5. Since $\sigma$ varies linearly over the element, its value at any point can be defined in terms of the nodal values and the two shape functions $N_1, N_2$, that is
The detailed formulation follows closely that given for the constant source distribution. In this particular case, the influence coefficients matrix with an even number of boundary elements becomes singular, therefore the tangential velocities at the mid points of 9 boundary elements are calculated. The table (4) below shows the comparison of the analytical and computed velocities at the mid-points of 9 boundary elements.

<table>
<thead>
<tr>
<th>Element</th>
<th>x- Coordinate</th>
<th>y- Coordinate</th>
<th>Calculated Velocity</th>
<th>Analytical Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.88</td>
<td>0.32</td>
<td>0.58077E+00</td>
<td>0.68404E+00</td>
</tr>
<tr>
<td>2</td>
<td>-0.47</td>
<td>0.81</td>
<td>0.14706E+01</td>
<td>0.17321E+01</td>
</tr>
<tr>
<td>3</td>
<td>0.16</td>
<td>0.93</td>
<td>0.16723E+01</td>
<td>0.19696E+01</td>
</tr>
<tr>
<td>4</td>
<td>0.72</td>
<td>0.60</td>
<td>0.10915E+01</td>
<td>0.12856E+01</td>
</tr>
<tr>
<td>5</td>
<td>0.94</td>
<td>0.00</td>
<td>0.40494E-06</td>
<td>0.00000E+00</td>
</tr>
<tr>
<td>6</td>
<td>0.72</td>
<td>-0.60</td>
<td>0.10915E+01</td>
<td>0.12856E+01</td>
</tr>
<tr>
<td>7</td>
<td>0.16</td>
<td>-0.93</td>
<td>0.16723E+01</td>
<td>0.19696E+01</td>
</tr>
<tr>
<td>8</td>
<td>-0.47</td>
<td>-0.81</td>
<td>0.14706E+01</td>
<td>0.17321E+01</td>
</tr>
<tr>
<td>9</td>
<td>-0.88</td>
<td>-0.32</td>
<td>0.58077E+00</td>
<td>0.68404E+00</td>
</tr>
</tbody>
</table>

On comparison of the results of this table with those in table (3) on page (150), which were obtained using constant boundary elements, it is seen that the results with linear elements in the source formulation are poor.
More importantly, it can be seen that in using a 'usual' indirect method, after calculating the unknowns in the form of source/doublet strengths over the body, one has to formulate and solve further integrals to calculate the tangential velocity. This is in contrast with the formulation (5.2.3) of the indirect method with a doublet distribution where, after finding the unknowns in the form of total potentials, one can easily calculate the tangential velocity by taking the derivative. Certainly, it is one of the reasons for which the 'usual' indirect method is rejected.

5.4.2 FLOW PAST A SPHERE

Consider the sphere placed in a uniform stream as shown in the figure 4.7. Once again the sphere is discretized into 96 and 384 linear boundary elements by the method discussed in section (4.2.2). Since the uniform stream is in the positive direction of the x-axis, the velocity potential \( \phi_{u.s} \) of the uniform stream is given by

\[
\phi_{u.s} = -Ux = -x \quad \text{with} \quad U = 1
\]

\[
\therefore (\phi_{u.s})_i = -x_i
\]

Thus equation (5.2.3) takes the form

\[
- \frac{1}{2} \phi_i + \frac{1}{4\pi} \oint_{S-i} \phi \frac{3}{3\pi} \left( \frac{1}{r} \right) dS + \phi_{\infty} = x_i \quad (5.4.2.1)
\]

The matrix formulation of this equation is discussed in section (5.3). Since with this formulation the unknowns are calculated in the form of
total potentials $\phi_i$, the pressure coefficients at the surface of the sphere are found by the same method as discussed in section (4.2.2). The pressure coefficients obtained by this method for 96 and 384 boundary elements are again compared with the analytical ones. The results obtained by this method appear like those shown in figure 4.18 for the direct boundary element method, but are in fact more accurate, the difference being manifest in the third decimal place of the solution.

5.4.3 FLOW PAST AN ELLIPSOID OF REVOLUTION

Consider next flow past the ovary ellipsoids (prolate spheroids) which were discussed in section 4; namely ovary ellipsoids of fineness ratios 2 and 10, with a distribution of elements as discussed in section (4.2.3). The discretizations of the ovary ellipsoid of fineness ratio 2 into 96 and 384 boundary elements are shown in figures 4.19 and 4.20 respectively, while figures 4.22 and 4.23 show the discretizations of the ovary ellipsoid of fineness ratio 10 into 96 and 384 boundary elements respectively. The velocity distributions over the surface of the ovary ellipsoids of fineness ratios 2 and 10 are calculated using formulation (5.2.3) for 96 and 384 boundary elements and are compared with their analytical solutions. The results obtained by this method for an ellipsoid of fineness ratio 2 are better than those obtained by using the direct boundary element method in the fourth decimal place, whereas the results obtained by this method for the case of an ellipsoid of fineness ratio 10 become worse than those obtained by using the direct boundary element method.
in the fourth decimal place. The reason for this could be that the
distribution of elements on the surface of an ellipsoid of fineness
eratio 2 remains almost uniform, whereas in the case of an ellipsoid
of fineness ratio 10, the distribution becomes worse. This situation
agrees with the results of Maskew [12], who has concluded that the
direct method appears to be the most forgiving when faced with poor
element distributions. Since the results obtained by this method for
both ellipsoids differ from results obtained by the direct method only
in the fourth decimal place, their graphical comparison with analytical
results appears the same as that shown for the direct method in
figures 4.21 and 4.24.

5.5 EFFECTS OF A GROUND PLANE

The indirect boundary element method with formulation (5.2.3) has
also been applied to calculate the flow past a sphere in the proximity
of the ground. Consider a sphere of unit radius with centre at the
origin lying near the ground in a uniform stream of velocity \( U \) in the
positive direction of the \( x \)-axis as shown in figure 4.10. The sphere
is discretized in turn into 96, 384 and 1536 boundary elements as
shown in figures 4.16, 4.17 and 4.27. The pressure coefficients over
the surface of the sphere for 96, 384 and 1536 boundary elements are
calculated once again with ground clearances of 0.1 and 0.3 units.

Since the results obtained by using this method differ in the
second or third decimal place from those which were obtained using the
direct method, their graphical comparisons with the analytical ones
look the same as those shown in figures 4.25, 4.26 and 4.28. It should
be remembered that the analytical results in the above mentioned figures, with which the computed results are compared, do themselves have some error. In general the error is in the fourth decimal place, but at points very near the ground in the case of ground clearance of 0.1 units, the error could be in the second or even first decimal place. The detailed comparison of the computed results obtained using the direct and the indirect methods with the analytical results show that the results with the indirect method are close to the analytical ones at half or slightly more points shown in the above mentioned figures. Furthermore, the results with the indirect method are better near the line joining the centres of the spheres, whilst the results with the direct method are better on the remaining points.

5.6 MODELLING OF A WAKE

The equation (5.2.3) for the indirect boundary element method holds for an infinite domain surrounding the closed body. In this section, this equation will again be modified as in the case of the direct boundary element method to take into account the wake of the body. Let the body under consideration be as shown in the figure 5.2.
Figure 5.2

The surface of body is $S = S_B + S_w + S^*_1$, where $S_B$ is the surface of the actual body, $S_w$ the surface of the tubular wake which starts from the rear end of the body and extends to infinity in the direction of the onset flow and $S^*_1$ the surface of that portion of the sphere $S^*$ cut by the wake of the body.

Let the surface $S$ divide the space into two regions $R$ and $R'$ and let $\hat{n}$ be the outward drawn unit normal to the surface $S$. Let $\phi$ and $\phi'$ denote the velocity potentials of the acyclic irrotational motions in the regions $R$ and $R'$ respectively, with $\phi$ regular at infinity. Then if the point $'i'$ be internal to $R'$ and therefore external to $R$, then from section (4.4) with $c_i = 0$,

$$0 = \phi_{\infty} - \frac{1}{4\pi} \iint_{S_B} \frac{1}{r} \frac{\partial \phi}{\partial n} \, dS + \frac{1}{4\pi} \iint_{S_B^1} \phi \frac{3}{\partial n} \left( \frac{1}{r} \right) dS + \frac{1}{4\pi} \iint_{S_w} \phi \frac{3}{\partial n} \left( \frac{1}{r} \right) dS.$$

(5.6.1)
for the region R.

Further, from section (2.2) with \( c_i = 1 \),

\[
\phi_i' = \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \phi'}{\partial n} \, dS - \frac{1}{4\pi} \int_{S-i} \phi' \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, dS
\]  \hspace{1cm} (5.6.2)

for the region \( R' \).

In this region, \( S = S_B + S_w + S_i^* \), therefore

\[
\phi_i' = \frac{1}{4\pi} \int_{S_B} \frac{1}{r} \frac{\partial \phi'}{\partial n} \, dS + \frac{1}{4\pi} \int_{S_w} \frac{1}{r} \frac{\partial \phi'}{\partial n} \, dS + \frac{1}{4\pi} \int_{S_i^*} \frac{1}{r} \frac{\partial \phi'}{\partial n} \, dS
\]

\[- \frac{1}{4\pi} \int_{S_B-i} \phi' \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, dS - \frac{1}{4\pi} \int_{S_w} \phi' \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, dS
\]

\[- \frac{1}{4\pi} \int_{S_i^*} \phi' \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, dS
\]  \hspace{1cm} (5.6.3)

Since \( \frac{\partial \phi'}{\partial n} = 0 \) on \( S_w \), the second integral on the R.H.S. of equation (5.6.3) is zero.

Adding equations (5.6.1) and (5.6.3), then

\[
\phi_i' = \phi_\infty + \frac{1}{4\pi} \int_{S_B} \frac{1}{r} \left( \frac{\partial \phi'}{\partial n} - \frac{\partial \phi}{\partial n} \right) \, dS - \frac{1}{4\pi} \int_{S_B-i} (\phi' - \phi) \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, dS
\]

\[- \frac{1}{4\pi} \int_{S_w} (\phi' - \phi) \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \, dS + \frac{1}{4\pi} \int_{S_i^*} \left[ \frac{1}{r} \frac{\partial \phi'}{\partial n} - \phi' \frac{\partial}{\partial n} \left( \frac{1}{r} \right) \right] \, dS
\]

Similarly in the case when \( 'i' \) is internal to \( R \) and hence external to
\( R' \), the same equation results with \( \phi_i \) replacing \( \phi_i' \) on the L.H.S. In particular, when \( 'i' \) lies on the surface \( S \), then \( \phi_i' \) is replaced by \( \frac{1}{2}(\phi_i + \phi_i') \). The above mentioned three cases can be combined by writing

\[
[c_i \phi_i + (1 - c_i) \phi_i'] = \phi_{\infty} + \frac{1}{4\pi} \iint_{S_B} \left( \frac{\partial \phi'}{\partial n} - \frac{\partial \phi}{\partial n} \right) dS
\]

\[
- \frac{1}{4\pi} \iint_{S_B - i} (\phi' - \phi) \frac{3}{3n} \left( \frac{1}{r} \right) dS - \frac{1}{4\pi} \iint_{S_W} (\phi' - \phi) \frac{3}{3n} \left( \frac{1}{r} \right) dS
\]

\[
+ \frac{1}{4\pi} \iint_{S_i^*} \left( \frac{1}{r} \frac{\partial \phi'}{\partial n} - \phi' \frac{3}{3n} \left( \frac{1}{r} \right) \right) dS
\]

(5.6.4)

where \( c_i = 0 \) when \( 'i' \) is within \( R' \)

\[
= 1 \quad \text{" } \quad \text{" } \quad \text{" } \quad \text{" }
\]

\[
= \frac{1}{2} \quad \text{when \( 'i' \) is on \( S \) and \( S \) is smooth.}
\]

Let the interior velocity potential \( \phi' \) be equal to the negative of the velocity potential of the uniform stream, \( \phi_{u.s.} \), then

\[
c_i \phi_i - (\phi_{u.s.})_i = \phi_{\infty} - \frac{1}{4\pi} \iint_{S_B} \frac{1}{r} \frac{\partial \phi}{\partial n} dS + \frac{1}{4\pi} \iint_{S_B - i} \phi \frac{3}{3n} \left( \frac{1}{r} \right) dS
\]

\[
+ \frac{1}{4\pi} \iint_{S_W} \phi \frac{3}{3n} \left( \frac{1}{r} \right) dS + \frac{1}{4\pi} \iint_{S_i^*} \left( \frac{-1}{r} \frac{\partial \phi_{u.s.}}{\partial n} + \phi_{u.s.} \frac{3}{3n} \left( \frac{1}{r} \right) \right) dS
\]

(5.6.5)

where \( \phi = \phi + \phi_{u.s.} \). The boundary condition of zero normal velocity on the body now implies
\( \frac{\partial \phi}{\partial n} = 0 \) on \( S_B \), thus the first integral on the R.H.S. becomes zero.

Also the uniform stream in the negative direction of the \( x \)-axis has the velocity potential

\[ \phi_{u.s} = Ux \]

\[ = x \quad \text{when} \quad U = 1 \]

Further, since the outwardly drawn unit normal \( \hat{n} \) to \( S_i \) is in the negative direction of the \( x \)-axis, then \( \frac{\partial}{\partial n} = -\frac{\partial}{\partial x} \) and the last integral on the R.H.S. of equation (5.6.5) becomes

\[
\frac{1}{4\pi} \iint_{S} \left( \frac{1}{r} \frac{\partial x}{\partial x} - x \frac{\partial}{\partial x} \left( \frac{1}{r} \right) \right) dS
\]

\[ = \frac{1}{4\pi} \iint_{S} \left( \frac{1}{r} + x \frac{1}{r^2} \frac{\partial r}{\partial x} \right) dS \]

\( \Rightarrow 0 \) as \( -x \) and \( r \rightarrow \infty \).

Hence equation (5.6.5) reduces to

\[
- c_i \phi_i + \frac{1}{4\pi} \iint_{S} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS + \frac{1}{4\pi} \iint_{S_w} \phi \frac{\partial}{\partial n} \left( \frac{1}{r} \right) dS + \phi_{\infty}
\]

\[ = - (\phi_{u.s})_i \quad \text{(5.6.6)} \]

On comparison of this equation with that of equation (5.2.3), it can be seen that now there is an additional integral on the L.H.S of equation (5.6.6) which is to be evaluated over the surface of the wake. Equation (5.6.6) can be put in the matrix formulation (5.3.2) by following the procedure discussed in section (4.4) which can then be
solved for the unknown total velocity potentials and thus the pressure coefficients over the surface of the body can be calculated by the same method outlined in the preceding sections. Again in this case the wake integral can be evaluated by the method discussed in section (4.4) and its value turns out to be the same as the value of the wake integral for the direct method found in section (4.4).

5.6.1 FLOW PAST A SEMI-INFINITE BODY

The equation (5.6.6) for the indirect boundary element method has been applied to calculate the flow past the semi-infinite body of figure 4.14. The method of distribution of elements on the semi-infinite body has been discussed in section (4.4.1). Once again the semi-infinite body is discretized in turn into 48, 192 and 768 boundary elements as shown in figures 4.29, 4.30 and 4.31 respectively.

The pressure distribution over the surface of the semi-infinite body has been calculated for each element distribution and the comparisons with the analytical solutions are shown in figures 5.3 and 5.4. As compared with the direct method, results obtained with this indirect method are much worse on the elements near the rear end of the body, but slightly better on the remainder of the body.

Thus it appears that the formulation of the indirect boundary element method given by Maskew [12] is not suitable for calculating the flow around bodies having tubular wakes. Maskew only considered the case of two-dimensional wakes of negligible thickness, shed from the trailing edge of a wing.

It therefore appears that the direct method is better than the
indirect method. It has also been concluded by Maskew [12] that the
direct formulation (4.4.3) is superior over the indirect formulation
(5.6.6). Writing equation (4.4.3) as

$$-c_i \phi_i + \frac{1}{4\pi} \int_{S_B-i} \phi \frac{2}{3n} \left( \frac{1}{r} \right) dS + \frac{1}{4\pi} \int_{S_W} \phi \frac{2}{3n} \left( \frac{1}{r} \right) dS + \phi_\infty$$

$$= \frac{1}{4\pi} \int_{S_B} 1 \frac{\partial \phi}{\partial n} dS \quad (5.6.1.1)$$

it can be seen that the left hand sides of equations (5.6.1.1) and
(5.6.6) have the same integral operators, which implies that the linear
algebraic systems, resulting from the discretization of the body, have
the same influence matrix. Only the right hand sides of these
equations are different, leading to different solution vectors.
Groh [34] states that if the solution based on the indirect formulation
(5.6.6) is inaccurate, while the solution based on the direct
formulation (5.6.1.1) is acceptable, then the inaccuracy can be
caused only by the term \((-\phi_{u,s})_i\). He further says that by writing
the equation (5.6.6) in the mathematically equivalent form as

$$-c_i \phi_i + \frac{1}{4\pi} \int_{S_B-i} \phi \frac{2}{3n} \left( \frac{1}{r} \right) dS + \frac{1}{4\pi} \int_{S_W} \phi \frac{2}{3n} \left( \frac{1}{r} \right) dS + \phi_\infty$$

$$= c_i (\phi_{u,s})_i - \frac{1}{4\pi} \int_{S_B-i} \phi_{u,s} \frac{2}{3n} \left( \frac{1}{r} \right) dS - \frac{1}{4\pi} \int_{S_W} \phi_{u,s} \frac{2}{3n} \left( \frac{1}{r} \right) dS \quad (5.6.1.2)$$

the superiority of the indirect method can be restored. Groh [34] claims
that equation (5.6.1.2) is numerically superior to equation (5.6.6) and
even equation (5.6.1.1). The flow past a semi-infinite body was calculated using equation (5.6.1.2), but the results obtained were exactly the same as those which were obtained already using the indirect formulation (5.6.6). This implies that the analysis used already to calculate the flow past bodies with tubular wakes was correct, and the poor results near the rear of the body are due to the formulation. Thus it can be concluded that in this application the indirect formulation (5.6.6) given by Maskew [12] is not numerically inferior as compared to the formulation (5.6.1.2). Maskew [12] makes the valid assertion that the 'best' formulation is very much problem dependent, as an 'internal flow' should be chosen which requires the least perturbation from the applied doublet distribution in a particular case.

5.7 COMPARISON OF COMPUTING TIME

The total computing time taken by the direct or indirect boundary element method in producing the results is almost the same. About 90 to 95 percent of the total computing time has been utilized in setting up the influence coefficient matrices, while the remaining 5 to 10 percent is used in reading the data, solving the system of equations, calculation and writing of the results. It may, however, be mentioned that the percentage of the total computing time in setting up the two matrices in the direct method is very nearly the same as the percentage of the total computing time taken to set up a single matrix in the indirect method. Furthermore, in order to model the effect of a boundary layer in future stages of analysis it is necessary to use the non-zero normal velocities over the boundary and then the indirect
method will also require the same two influence matrices to be formed and hence will take the same computing time as that of the direct method.

5.8 CONCLUSIONS

On comparison of the direct boundary element method with the indirect boundary element method used in this section, it can be concluded that the indirect method is not suitable for the road vehicle aerodynamics due to the following reasons:

(i) it gives poor results for the body with a tubular wake near the rear end of the body

(ii) it is not as forgiving to poor panelling representations as the direct method

(iii) it is not significantly different in respect of the computing time required.

Due to the above reasons the direct boundary element method will be used and analysed in the remaining sections for road vehicle aerodynamics.
Figure 5.3: Comparison of analytical and computed pressure coefficients on a semi-infinite body using
(a) 48 boundary elements; (b) 192 boundary elements.
Figure 5.4: Comparison of analytical and computed pressure coefficients on a semi-infinite body using 768 boundary elements.
SECTION 6

COMPARISON OF INTEGRATION SCHEMES FOR

VARIOUS TYPES OF ELEMENTS

6.1 INTRODUCTION

This section is devoted to the analysis of the direct boundary element method for various element types and integration schemes. The analysis is carried out by comparing the results of the flow field calculations around the same three-dimensional bodies which were considered in sections (4) and (5). The aim of these comparisons is to establish which integration scheme over which type of element is 'best' with respect to accuracy for given computing time. The outcome of these comparisons will then be used to calculate the flow field around road vehicles.

In section (6.2) modified quadrature formulae for elements with singularities are discussed and modified integration points and weights are derived for these elements. The accuracy of the modified quadrature scheme is checked over a flat square element by evaluating the integrals of equations (3.3.10) and (3.3.12) and the results are also compared with those obtained using Gauss-quadrature rule over this element. In section (6.3) the comparison of the Guass-quadrature and the modified quadrature is presented over general linear and quadratic boundary elements. Finally, section (6.4) is given to the comparison of various types of boundary elements.
6.2 MODIFIED QUADRATURE FORMULAE FOR ELEMENTS WITH SINGULARITIES

So far the integrals in the equations of the boundary element methods over the quadrilateral elements on the surface of the body for three-dimensional problem have been evaluated numerically using a two-dimensional Gauss-quadrature rule. In this rule the integral over the quadrilateral element is first transformed to an integral over the standard element \(-1 \leq \xi \leq 1, -1 \leq \eta \leq 1\), as shown in figure 3.3 (b). This is simply achieved by the use of the shape functions for isoparametric elements, as considered in this thesis. It has been reported by Cristescu and Loubignac [35] that the use of a Gaussian-quadrature rule over elements containing singularities give poor results. To overcome this difficulty they have derived modified quadrature formulae for these elements and have calculated some modified integration points and weights in the xy coordinate system over the standard elements as shown in figure 6.1.

![Diagram](image)

Figure 6.1
In figure 6.1(a), the singularity is at the corner node of the element, while in figure 6.1(b) the singularity is at the mid point of the lower side of the element. As previously mentioned, the shape functions transform an isoparametric element into the standard element shown in figure 3.3(b). Further transformation of this standard element to a 'modified quadrature element' is simplest when the latter is of the form shown in figure 6.1(b) rather than that of figure 6.1(a). Thus it is desirable to obtain the modified integration points and weights in terms of x and y coordinates for the element shown in figure 6.2 where the singularity is at the bottom left corner of the element.

![Figure 6.2](image.png)

In a Gauss-quadrature rule which integrates exactly a polynomial of degree $2n - 1$, the Gaussian points are the roots of the Legendre polynomial of degree $n$. Similarly to find the modified integration points for a formula which integrates exactly a polynomial of degree $2n - 1$ divided by $\sqrt{x^2 + y^2}$, the orthogonal polynomial of degree $n$ will be calculated. In [35] the polynomial $R_{nm}(x, y)$ is defined as a polynomial of degree $n + m$ with $x^n y^m$ as highest degree term.
\[ R_{nm}(x, y) = x^n y^m + \text{lower order terms} \quad (6.2.1) \]

For the four modified integration points the coefficients of \( R_{20}(x, y) \) and \( R_{02}(x, y) \) need to be calculated, while for the nine integration points the coefficients of \( R_{30}(x, y) \) and \( R_{03}(x, y) \) will be calculated and so on. From equation (6.2.1), \( R_{20}(x, y) \) and \( R_{02}(x, y) \) can be written as

\[
R_{20}(x, y) = x^2 + R_{200} + R_{201}x + R_{202}y \\
R_{02}(x, y) = y^2 + R_{020} + R_{021}x + R_{022}y \quad (6.2.2)
\]

The polynomial \( R_{nm}(x, y) \) must be orthogonal to all polynomials of degree less than \( n + m \). These conditions give exactly \((n + m)(n + m + 1)/2\) equations from which the coefficients of \( R_{nm}(x, y) \) can be calculated.

Defining \( I_{nm} = \int_0^2 \int_0^2 \frac{x^n y^m}{\sqrt{x^2 + y^2}} \, dx \, dy \quad (6.2.3) \)

the equations determining the coefficients of \( R_{20}(x, y) \) and \( R_{02}(x, y) \) are given in [33] and thus the four modified integration points are the common distinct roots of the polynomials \( R_{20}(x, y) \) and \( R_{02}(x, y) \) given in equation (6.2.2). The four modified integration points for the element shown in figure 6.2 are given in [35] and the four weights for this element can be found as follows:

Proceeding as in [35], the four modified weights are calculated from the equation
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
x_1 & x_2 & x_3 & x_4 \\
x_1^2 & x_2^2 & x_3^2 & x_4^2 \\
x_1^3 & x_2^3 & x_3^3 & x_4^3
\end{bmatrix}
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\omega_4
\end{bmatrix}
= 
\begin{bmatrix}
I_{00} \\
I_{10} \\
I_{20} \\
I_{30}
\end{bmatrix}
\tag{6.2.4}
\]

where \( x_i \), \( i = 1, \ldots, 4 \) are the modified integration points, \( \omega_i \), \( i = 1, \ldots, 4 \) are the unknown modified weights and \( I_{00}, I_{10}, I_{20}, I_{30} \) are defined in equation (6.2.3) and can be found as follows:

\[
I_{00} = \int_{0}^{2} \int_{0}^{2} \frac{1}{\sqrt{x^2 + y^2}} \, dx \, dy
\]

Transforming to polar coordinates

\[
I_{00} = 2 \int_{0}^{\pi/4} \int_{0}^{2 \sec \theta} r \, dr \, d\theta
\]

\[= 4 \log (\sqrt{2} + 1).\]

\[
I_{10} = \int_{0}^{2} \int_{0}^{2} \frac{x}{\sqrt{x^2 + y^2}} \, dx \, dy
\]

\[= \int_{0}^{2} (\sqrt{4 + y^2} - y) \, dy
\]

\[= 2(\sqrt{2} - 1) + 2 \log (\sqrt{2} + 1).\]

\[
I_{20} = \int_{0}^{2} \int_{0}^{2} \frac{x^2}{\sqrt{x^2 + y^2}} \, dx \, dy
\]
In polar coordinates

\[ I_{20} = \int_0^{\pi/4} \int_0^r r^2 \cos^2 \theta \, dr \, d\theta + \int_{\pi/4}^{\pi/2} \int_0^r r^2 \cos^2 \theta \, dr \, d\theta \]

\[ = \frac{8}{3} \int_0^{\pi/4} \sec \theta d\theta + \frac{8}{3} \int_{\pi/4}^{\pi/2} \cos^2 \theta \csc^2 \theta \, d\theta \]

\[ = \frac{4}{3} \left[ \sqrt{2} + \log (\sqrt{2} + 1) \right]. \]

\[ I_{30} = \int_0^2 \int_0^2 \frac{x^3}{\sqrt{x^2 + y^2}} \, dx \, dy \]

\[ = \int_0^2 \left[ 4\sqrt{4 + y^2} - \frac{2}{3} (4 + y^2)^{3/2} + \frac{2}{3} y^2 \right] dy \]

\[ = \frac{8}{3} - \frac{4\sqrt{2}}{3} + 4 \log (\sqrt{2} + 1) \]

With these values of \( I_{00}, I_{10}, I_{20}, I_{30} \) the system of equations (6.2.4) can be solved and the required modified weights are obtained and are given in table (1).

Similarly, for nine integration points, the coefficients of \( R_{30}(x, y) \) and \( R_{03}(x, y) \) need to be calculated, where

\[ R_{30}(x, y) = x^3 + R_{300} + R_{301}x + R_{302}y + R_{303}x^2 + R_{304}xy + R_{305}y^2 \]

(6.2.5)

\[ R_{03}(x, y) = y^3 + R_{030} + R_{031}x + R_{032}y + R_{033}x^2 + R_{034}xy + R_{035}y^2 \]

(6.2.6)
As in [35] the coefficients of $R_{30}$ and $R_{03}$ are calculated from the following systems of equations respectively.

\[
\begin{pmatrix}
I_{00} & I_{10} & I_{01} & I_{20} & I_{11} & I_{02} \\
I_{10} & I_{20} & I_{11} & I_{30} & I_{21} & I_{12} \\
I_{01} & I_{11} & I_{02} & I_{21} & I_{12} & I_{03} \\
I_{20} & I_{30} & I_{21} & I_{40} & I_{31} & I_{22} \\
I_{11} & I_{21} & I_{12} & I_{31} & I_{22} & I_{13} \\
I_{02} & I_{12} & I_{03} & I_{22} & I_{13} & I_{04}
\end{pmatrix}
\begin{pmatrix}
R_{300} \\
R_{301} \\
R_{302} \\
R_{303} \\
R_{304} \\
R_{305}
\end{pmatrix} =
\begin{pmatrix}
-I_{30} \\
-I_{40} \\
-I_{31} \\
-I_{50} \\
-I_{41} \\
-I_{32}
\end{pmatrix}
\]

(6.2.7)

\[
\begin{pmatrix}
I_{00} & I_{10} & I_{01} & I_{20} & I_{11} & I_{02} \\
I_{10} & I_{20} & I_{11} & I_{30} & I_{21} & I_{12} \\
I_{01} & I_{11} & I_{02} & I_{21} & I_{12} & I_{03} \\
I_{20} & I_{30} & I_{21} & I_{40} & I_{31} & I_{22} \\
I_{11} & I_{21} & I_{12} & I_{31} & I_{22} & I_{13} \\
I_{02} & I_{12} & I_{03} & I_{22} & I_{13} & I_{04}
\end{pmatrix}
\begin{pmatrix}
R_{030} \\
R_{031} \\
R_{032} \\
R_{033} \\
R_{034} \\
R_{035}
\end{pmatrix} =
\begin{pmatrix}
-I_{03} \\
-I_{13} \\
-I_{04} \\
-I_{23} \\
-I_{14} \\
-I_{05}
\end{pmatrix}
\]

(6.2.8)

The elements of the coefficient matrices and that of the R.H.S. vectors in equations (6.2.7) and (6.2.8) can be found from equation (6.2.3) giving different values to $n$ and $m$. $I_{00}, I_{10}, I_{20}$ and $I_{30}$ have already been found. The remaining elements are found as follows:
\[ I_{01} = I_{10} \]
\[ I_{02} = I_{20} \]
\[ I_{03} = I_{30} \]

\[ I_{11} = \int_0^2 \int_0^2 \frac{xy}{\sqrt{x^2 + y^2}} \, dxdy \]

\[ = \int_0^2 (y\sqrt{4 + y^2} - y^2) \, dy \]

\[ = \frac{16}{3} (\sqrt{2} - 1). \]

\[ I_{21} = \int_0^2 \int_0^2 \frac{x^2y}{\sqrt{x^2 + y^2}} \, dxdy \]

\[ = \int_0^2 (x^2 \sqrt{4 + x^2} - x^3) \, dx \]

\[ = 6\sqrt{2} - 2 \log (\sqrt{2} + 1) - 4. \]

\[ I_{12} = I_{21} \]

\[ I_{22} = \int_0^2 \int_0^2 \frac{x^2y^2}{\sqrt{x^2 + y^2}} \, dxdy \]

In polar coordinates
\[ I_{22} = 2 \int_{0}^{\pi/4} \int_{0}^{2} 2\sec \theta \, r^4 \cos^2 \theta \, \sin^2 \theta \, dr \, d\theta \]
\[ = \frac{64}{5} \int_{0}^{\pi/4} \sin^2 \theta \, \sec^3 \theta \, d\theta \]
\[ = \frac{32}{5} [\sqrt{2} - \log (\sqrt{2} + 1)] \]
\[ I_{31} = \int_{0}^{2} \int_{0}^{2} \frac{x^3 \, y}{\sqrt{x^2 + y^2}} \, dx \, dy \]
\[ = \int_{0}^{2} (x^3 \sqrt{4 + x^2 - x^4}) \, dx \]
\[ = \frac{32}{15} (2\sqrt{2} - 1) \]
\[ I_{13} = I_{31} \]
\[ I_{40} = \int_{0}^{2} \int_{0}^{2} \frac{x^4}{\sqrt{x^2 + y^2}} \, dx \, dy \]

Transforming to polar coordinates

\[ I_{40} = \int_{0}^{\pi/4} \int_{0}^{\pi/4} 2\sec \theta \, r^4 \cos^4 \theta \, d\theta \, dr \, d\theta + \int_{\pi/4}^{\pi/2} \int_{0}^{\pi/4} 2\sec \theta \, r^4 \cos^4 \theta \, d\theta \, dr \, d\theta \]
\[ = \frac{32}{5} \int_{0}^{\pi/4} \sec \theta \, d\theta + \frac{32}{5} \int_{\pi/4}^{\pi/2} \cos^4 \theta \, \csc^5 \theta \, d\theta \]
\[ = \frac{4}{5} [11 \log (\sqrt{2} + 1) - 4\sqrt{2}] \]
Transforming to polar coordinates

\[
I_{04} = I_{40}
\]

\[
I_{32} = \iint_{0}^{2} \frac{x^3 y^2}{\sqrt{x^2 + y^2}} \, dx \, dy
\]

\[
I_{23} = I_{32}
\]

\[
I_{41} = \iint_{0}^{2} \frac{x^4 y}{\sqrt{x^2 + y^2}} \, dx \, dy
\]

Transforming to polar coordinates

\[
I_{41} = \int_{0}^{\pi/4} \int_{0}^{\pi/2} r^5 \cos^4 \theta \sin \theta \, dr \, d\theta + \int_{\pi/4}^{\pi/2} \int_{0}^{\pi/4} r^5 \cos^4 \theta \sin \theta \, dr \, d\theta
\]

\[
= \frac{32}{3} \int_{0}^{\pi/4} \sec \theta \tan \theta \, d\theta + \frac{32}{3} \int_{\pi/4}^{\pi/2} \cos^4 \theta \csc^5 \theta \, d\theta
\]

\[
= \frac{28 \sqrt{2}}{3} - \frac{32}{3} + 4 \log (\sqrt{2} + 1).
\]
\[ I_{14} = I_{41} \]

\[ I_{50} = \int_{0}^{2} \int_{0}^{2} \frac{x^5}{\sqrt{x^2 + y^2}} \, dx \, dy \]

In polar coordinates

\[ I_{50} = \int_{0}^{\pi/4} \int_{0}^{\pi/4} r^5 \cos^5 \theta \, dr \, d\theta + \int_{\pi/4}^{\pi/2} \int_{0}^{\pi/4} r^5 \cos^5 \theta \, dr \, d\theta \]

\[ = \frac{32}{3} \int_{0}^{\pi/4} \sec \theta \, d\theta + \frac{32}{3} \int_{\pi/4}^{\pi/2} \cos^5 \theta \csc^6 \theta \, d\theta \]

\[ = \frac{32}{3} \log (\sqrt{2} + 1) - \frac{256}{45} + \frac{224\sqrt{2}}{45} \]

\[ I_{05} = I_{50} \]

With these values of \( I_{00}, \ldots, I_{50} \), the unknowns in the systems of equations (6.2.7) and (6.2.8) which are respectively the coefficients in the polynomials \( R_{30} \) and \( R_{03} \) can be calculated and are given in table (2). From this table it can be seen that

\[ R_{300} = R_{030}, \quad R_{301} = R_{032}, \quad R_{302} = R_{031} \]

\[ R_{303} = R_{035}, \quad R_{304} = R_{034}, \quad R_{305} = R_{033} \]

(6.2.9)

Having found the unknowns of the systems of equations (6.2.7) and (6.2.8), the nine integration points are the distinct \( 3^2 \) common roots of the equations (6.2.5) and (6.2.6). i.e.

\[ x^3 + R_{300} + R_{301}x + R_{302}y + R_{303}x^2 + R_{304}xy + R_{305}y^2 = 0 \]

(6.2.10)
and
\[ y^3 + R_{030} + R_{031}x + R_{032}y + R_{033}x^2 + R_{034}xy + R_{035}y^2 = 0 \]  \hspace{1cm} (6.2.11)

Subtraction of (6.2.11) from (6.2.10) using (6.2.9) gives
\[ (x^3 - y^3) + (R_{301} - R_{302})(x - y) + (R_{303} - R_{305})(x^2 - y^2) = 0 \]  \hspace{1cm} (6.2.12)

Addition of (6.2.10) and (6.2.11) using (6.2.9) gives
\[
(x^3 + y^3) + 2R_{300} + (R_{301} + R_{302})(x + y) + (R_{303} + R_{305})(x^2 + y^2) \\
+ 2R_{304}xy = 0
\]  \hspace{1cm} (6.2.13)

It can be seen that \( x = y \) is a root of the equation (6.2.12).

Substituting \( y = x \) in equation (6.2.13), then
\[ x^3 + (R_{303} + R_{304} + R_{305})x^2 + (R_{301} + R_{302})x + R_{300} = 0 \]  \hspace{1cm} (6.2.14)

which is a third order polynomial in \( x \).

Let \( x_1, x_2, x_3 \) be the three roots of equation (6.2.14), then
\[ (x_1, x_1), \ (x_2, x_2), \ (x_3, x_3) \]  \hspace{1cm} (6.2.15)

are the three modified integration points on the line \( y = x \).

The remaining roots of the equation (6.2.12) are given by
\[ (x^2 + y^2 + xy) + (R_{301} - R_{302}) + (R_{303} - R_{305})(x + y) = 0 \]  \hspace{1cm} (6.2.16)

From this equation it is not possible to express \( y \) explicitly as a function of \( x \). Therefore, to make this equation easy and
understandable, transform the xy-coordinate system to the uv-coordinate system by rotating the axes through 45° anticlockwise as shown in the figure 6.3.

![Figure 6.3](image)

Then

\[
\begin{align*}
  u &= \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \\
  v &= \frac{-x}{\sqrt{2}} + \frac{y}{\sqrt{2}}
\end{align*}
\] (6.2.17)

and

\[
\begin{align*}
  x &= \frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}} \\
  y &= \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}
\end{align*}
\] (6.2.18)

Substituting for x and y from equation (6.2.18), equations (6.2.16) and (6.2.13) become

\[
\frac{1}{4} (3u^2 + v^2) + (R_{301} - R_{302}) + \sqrt{2}(R_{303} - R_{305})u = 0
\] (6.2.19)
\[(u^3 + 3uv^2) + 2\sqrt{2} R_{300} + 2(R_{301} + R_{302})u + \sqrt{2} R_{304}(u^2 - v^2) + \sqrt{2} (R_{303} + R_{305})(u^2 + v^2) = 0\] (6.2.20)

Solving for \(v^2\) from equation (6.2.19), then

\[v^2 = -3u^2 - 2(R_{301} - R_{302}) - 2\sqrt{2} (R_{303} - R_{305})u\] (6.2.21)

Substituting this value of \(v^2\) into equation (6.2.20), then

\[-4\sqrt{2} u^3 + 4(R_{305} + R_{304} - 2R_{303})u^2 + [4\sqrt{2} R_{302} - 2\sqrt{2} R_{301} - 2\sqrt{2} R_{304}(R_{305} - R_{303}) + 2\sqrt{2}(R_{305}^2 - R_{303}^2)]u + 2R_{300} - 2R_{304}(R_{302} - R_{301}) + 2(R_{302} - R_{301})(R_{303} + R_{305}) = 0\] (6.2.22)

which is a third order polynomial in \(u\). Let the three roots of \(u\) be \(u_1, u_2, u_3\).

For each of these roots, there are two roots for \(v\) from equation (6.2.21) say \(\pm v_1, \pm v_2, \pm v_3\).

The remaining six modified integration points are given by \((u_1, v_1), (u_1, -v_1), (u_2, v_2), (u_2, -v_2), (u_3, v_3)\) and \((u_3, -v_3)\).

These points in the xy-system can be found from equation (6.2.18) which are given as
Thus the nine modified integration points given in equations (6.2.15) and (6.2.23) have the values as given in table (3).

Having found these integration points, the nine weights can be found by solving the following system of equations:
\[
\begin{bmatrix}
1 & 1 \\
x_1 & x_2 \\
y_1 & y_2 \\
x_1y_1 & x_2y_2 \\
x_1^2 & x_2^2 \\
y_1^2 & y_2^2 \\
x_1^2y_1 & x_2^2y_2 \\
x_1^3 & x_2^3 \\
y_1^3 & y_2^3 \\
x_1^3y_1 & x_2^3y_2 \\
x_1^4 & x_2^4 \\
y_1^4 & y_2^4 \\
x_1^4y_1 & x_2^4y_2 \\
x_1^5 & x_2^5 \\
y_1^5 & y_2^5 \\
\end{bmatrix}
= 
\begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\omega_4 \\
\omega_5 \\
\omega_6 \\
\omega_7 \\
\omega_8 \\
\omega_9 \\
\end{bmatrix}
\begin{bmatrix}
I_00 \\
I_{10} \\
I_{01} \\
I_{11} \\
I_{20} \\
I_{02} \\
I_{21} \\
I_{12} \\
I_{22} \\
I_{30} \\
I_{03} \\
I_{31} \\
I_{13} \\
I_{40} \\
I_{04} \\
I_{32} \\
I_{23} \\
I_{41} \\
I_{14} \\
I_{50} \\
I_{05} \\
\end{bmatrix}
\]

(6.2.24)
The nine weights found are also given in table (3).

To check the accuracy of the modified quadrature scheme, the integrals of equations (3.3.10) and (3.3.12) are evaluated over a flat square element. The reason for choosing the flat square element is that the integrals can be calculated analytically over this element for which modified quadrature scheme should be exact. Consider first a flat square linear isoparametric element and suppose that the singularity is at the left bottom corner of the element as shown in figure 6.4.

The integrals of the equation (3.3.12) can be calculated analytically as follows:

\[ g_{i1}^{l} = \iint_{S_i} N_1 \left( \frac{1}{4\pi r} \right) dS \]
\[
= \frac{1}{16\pi} \int_{-1}^{1} \int_{-1}^{1} (1 - \xi)(1 - \eta) \frac{1}{r} |G| \, d\xi \, d\eta
\]

From figure 6.4, the jacobian \(|G| = 1\) and

\[
r = \sqrt{(1 + \xi)^2 + (1 + \eta)^2}
\]

\[
(6.2.25)
\]

\[
\therefore \quad g_{ii}^1 = \frac{1}{16\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 - \xi)(1 - \eta)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} \, d\xi \, d\eta
\]

\[
= \frac{1}{16\pi} \int_{-1}^{1} (1 - \eta) \left\{ 2 \log \left[ \frac{2 + \sqrt{4 + (1 + \eta)^2}}{1 + \eta} \right] - \sqrt{4 + (1 + \eta)^2} + (1 + \eta) \right\} \, d\eta
\]

\[
= \frac{1}{2\pi} \left[ \log\left( \sqrt{2} + 1 \right) + \frac{1}{3} (1 - \sqrt{2}) \right]
\]

\[
g_{ii}^2 = \int \int_{S_1} N_2 \left( \frac{1}{4\pi r} \right) \, dS
\]

Using the value of \(r\) from equation (6.2.25) it becomes

\[
\therefore \quad g_{ii}^2 = \frac{1}{16\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 + \xi)(1 - \eta)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} \, d\xi \, d\eta
\]

\[
= \frac{1}{16\pi} \int_{-1}^{1} \left( 1 - \eta \right) \left[ \sqrt{4 + (1 + \eta)^2} - (1 + \eta) \right] \, d\eta
\]

\[
= \frac{1}{4\pi} \left[ \log(\sqrt{2} + 1) + \frac{1}{3} (1 - \sqrt{2}) \right]
\]
\( g_{i}^{3} = \iint_{S_{i}} N_{3} \left( \frac{1}{4\pi r} \right) dS \)

\[
= \frac{1}{16\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 + \xi)(1 + \eta)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} d\xi d\eta
\]

\[
= \frac{1}{16\pi} \int_{-1}^{1} (1 + \eta) \left[ \frac{\sqrt{4 + (1 + \eta)^2} - (1 + \eta)}{\xi} \right] d\eta
\]

\[
= -\frac{1}{3\pi} (1 - \sqrt{2})
\]

\( g_{i}^{4} = \iint_{S_{i}} N_{4} \left( \frac{1}{4\pi r} \right) dS \)

\[
= \frac{1}{16\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 - \xi)(1 + \eta)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} d\xi d\eta
\]

\[
= \frac{1}{16\pi} \int_{-1}^{1} (1 - \xi) \left[ \frac{\sqrt{4 + (1 + \xi)^2} - (1 + \xi)}{\eta} \right] d\xi
\]

\[
= \frac{1}{4\pi} \left[ \log(\sqrt{2} + 1) + \frac{1}{3} (1 - \sqrt{2}) \right]
\]

\( = g_{i}^{2} \)

The integrals are zero since \( r \) and \( \hat{n} \) are orthogonal to each other over this element. The above integrals are calculated numerically using Gauss–quadrature rule with four, nine and sixteen integration points as well as using the modified quadrature rule with four and nine integration points. The comparison of the computed and analytical
results is shown in table (4). It can be seen from this table that the results using Gauss-quadrature rule even with sixteen integration points are not as good as those obtained with the modified quadrature rule using four integration points. However, the accuracy in the results with the Gauss-quadrature rule increases with the increase in the number of integration points, as one would expect. The results obtained with the modified quadrature rule using four and nine integration points are exactly the same as those of the analytical results for this particular element. This is as expected and indicates that there is no error in formulation of the scheme and that there is negligible computational error.

Consider next the case of a flat square element with quadratic functional variation on each side, i.e. an 8 noded square element and suppose that the singularity is at the bottom left corner of the element as shown in the figure 6.5.

![Figure 6.5](image)

Figure 6.5

Now the integrals of equation (3.3.12) can be calculated analytically
as follows:

\[ g_{ii} = \iint_{S_1} N_1 \left( \frac{1}{4\pi r} \right) dS \]

\[ = \frac{1}{16\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 - \xi)(1 - \eta)(-\xi - \eta - 1)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} d\xi d\eta \]

Let \( 1 + \xi = x \) and \( 1 + \eta = y \), then

\[ g_{ii} = \frac{1}{16\pi} \int_{0}^{2} \int_{0}^{2} \frac{(2 - x)(2 - y)(1 - x - y)}{\sqrt{x^2 + y^2}} dx dy \]

Transforming to polar coordinates, it takes the form

\[ g_{ii} = \frac{1}{16\pi} \int_{0}^{\pi/4} \int_{0}^{\pi/4} (2 - r\cos\theta)(2 - r\sin\theta)(1 - r\cos\theta - r\sin\theta) dr d\theta \]

\[ + \int_{\pi/4}^{\pi/2} \int_{0}^{\pi/4} (2 - r\cos\theta)(2 - r\sin\theta)(1 - r\cos\theta - r\sin\theta) dr d\theta \]

\[ = \frac{1}{16\pi} \left[ \int_{0}^{\pi/4} \left( \frac{4}{3} \sec\theta - \frac{8}{3} \sec\theta \tan\theta + \frac{4}{3} \sin^2\theta \right) d\theta \right. \]

\[ + \int_{\pi/4}^{\pi/2} \left( \frac{4}{3} \cosec\theta - \frac{8}{3} \cosec\theta \cot\theta + \frac{4}{3} \cos^2\theta \right) d\theta \]

\[ = \frac{1}{12\pi} \left[ 4 - 3\sqrt{2} + \log(\sqrt{2} + 1) \right] \]
\[ g_{ii}^2 = \int \int_{S_i} N_2 \left( \frac{1}{4\pi} \right) dS. \]

\[ = \frac{1}{8\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 - \xi^2)(1 - \eta)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} d\xi d\eta. \]

Let \( 1 + \xi = x \) and \( 1 + \eta = y \), then

\[ g_{ii}^2 = \frac{1}{8\pi} \int_{0}^{2} \int_{0}^{2} \frac{(2x - x^2)(2 - y)}{\sqrt{x^2 + y^2}} dxdy. \]

Transforming to polar coordinates,

\[ g_{ii}^2 = \frac{1}{8\pi} \int \int_{0}^{\pi/4} 2\sec^2 \theta (2r \cos \theta - r^2 \cos^2 \theta)(2 - r \sin \theta)drd\theta \]

\[ + \int_{\pi/4}^{\pi/2} 2\cosec \theta (2r \cos \theta - r^2 \cos^2 \theta)(2 - r \sin \theta)drd\theta \]

\[ = \frac{1}{8\pi} \left[ \int_{0}^{\pi/4} \left( \frac{8}{3} \sec \theta - \frac{4}{3} \sec \theta \tan \theta \right) d\theta \right. \]

\[ + \int_{\pi/4}^{\pi/2} \left( \frac{8}{3} \cosec \theta \cot \theta - \frac{4}{3} \cos^2 \theta \right) d\theta \]

\[ = \frac{1}{24\pi} \left[ 10 \log (\sqrt{2} + 1) + 2\sqrt{2} - 4 \right] \]

\[ g_{ii}^3 = \int \int_{S_i} N_3 \left( \frac{1}{4\pi} \right) dS. \]
\[
\frac{1}{16\pi} \int_{1}^{1} \int_{1}^{1} \frac{(1 + \xi)(1 - \eta)(\xi - \eta - 1)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} \, d\xi d\eta
\]

With \(1 + \xi = x\) and \(1 + \eta = y\), this becomes

\[
g_{ii}^3 = \frac{1}{16\pi} \int_{0}^{2} \int_{0}^{2} \frac{x(2 - y)(x - y - 1)}{\sqrt{x^2 + y^2}} \, dx dy
\]

and in polar coordinates, it takes the form

\[
g_{ii}^3 = \frac{1}{16\pi} \left[ \int_{0}^{\pi/4} \int_{0}^{\pi/2} r \cos\theta (2 - r \sin\theta)(r \cos\theta - r \sin\theta - 1) \, dr d\theta 
+ \int_{\pi/4}^{\pi/2} \int_{0}^{\pi/2} r \cos\theta (2 - r \sin\theta)(r \cos\theta - r \sin\theta - 1) \, dr d\theta \right]
\]

\[
= \frac{1}{16\pi} \left[ \int_{0}^{\pi/4} \left( \frac{4}{3} \sec\theta - \frac{20}{3} \sec\theta \tan\theta + 4\frac{\sin^2\theta}{\cos^3\theta} \right) d\theta
+ \int_{\pi/4}^{\pi/2} \left( -\frac{8}{3} \csc\theta \cot\theta + \frac{4}{3} \frac{\cos^2\theta}{\sin^3\theta} \right) d\theta \right]
\]

\[
= \frac{1}{12\pi} \left[ 7 - 5\sqrt{2} - \log(\sqrt{2} + 1) \right]
\]

\[
g_{ii}^4 = \int_{S_i} \int_{S_i} N_i^3 \left( \frac{1}{4\pi} \right) \, dS
\]

\[
= \frac{1}{8\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 + \xi)(1 - \eta^2)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} \, d\xi d\eta
\]

With \(1 + \xi = x\) and \(1 + \eta = y\), it becomes
In polar coordinates, it takes the form

\[
g_{ii}^{4} = \frac{1}{8\pi} \int_{0}^{\pi/4} \int_{0}^{\pi/2} r \cos\theta (2 r \sin \theta - r^2 \sin^2 \theta) \, dr \, d\theta
\]

\[+ \int_{\pi/4}^{\pi/2} \int_{0}^{\pi/2} r \cos\theta (2 r \sin \theta - r^2 \sin^2 \theta) \, dr \, d\theta\]

\[= \frac{1}{8\pi} \left[ \int_{0}^{\pi/4} \left( \frac{16}{3} \sec \theta \tan \theta - 4 \frac{\sin^2 \theta}{\cos^3 \theta} \right) \, d\theta \right.
\]

\[+ \int_{\pi/4}^{\pi/2} \frac{4}{3} \csc \theta \cot \theta \, d\theta \]

\[= \frac{1}{12\pi} \left[ -10 + 7\sqrt{2} + 3 \log\left(\sqrt{2} + 1\right) \right]\]

\[g_{ii}^{5} = \int \int_{S_{ij}} N_{5} \left( \frac{1}{4\pi r} \right) \, dS
\]

\[= \frac{1}{16\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 + \xi)(1 + \eta)(\xi + \eta - 1)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} \, d\xi \, d\eta
\]

Let \(1 + \xi = x\) and \(1 + \eta = y\), it becomes

\[g_{ii}^{5} = \frac{1}{16\pi} \int_{0}^{2} \int_{0}^{2} \frac{xy(x + y - 3)}{\sqrt{x^2 + y^2}} \, dx \, dy\]
Transforming to polar coordinates, then

\[ g_{ii}^5 = \frac{1}{16\pi} \left[ \int_{\pi/4}^{\pi/2} \int_0^0 r^2 \cos^2 \theta \sin \theta (r \cos \theta + r \sin \theta - 3) \, dr \, d\theta 
+ \int_{\pi/4}^{\pi/2} \int_0^0 r^2 \cos^2 \theta \sin \theta (r \cos \theta + r \sin \theta - 3) \, dr \, d\theta \right] 
= \frac{1}{16\pi} \left[ \int_0^{\pi/4} \int_0^0 (-4 \sec \theta \tan \theta + 4 \sin^2 \theta \cos^3 \theta) \, d\theta 
+ \int_{\pi/4}^{\pi/2} \int_0^0 (-4 \csc \theta \cot \theta + 4 \cos^2 \theta \sin^3 \theta) \, d\theta \right] 
= \frac{1}{4\pi} \left[ 2 - \sqrt{2} - \log (\sqrt{2} + 1) \right] 
\]

\[ g_{ii}^6 = \int \int_{S_i} N_6 \left( \frac{1}{4\pi r} \right) \, dS \n= \frac{1}{8\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 - \xi^2)(1 + \eta)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} \, d\xi \, d\eta \n= \frac{1}{8\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(2x - x^2)y}{\sqrt{x^2 + y^2}} \, dx \, dy \n\]

Let \( 1 + \xi = x \) and \( 1 + \eta = y \), then

\[ g_{ii}^6 = \frac{1}{8\pi} \int_{0}^{2} \int_{0}^{2} \frac{(2x - x^2)y}{\sqrt{x^2 + y^2}} \, dx \, dy \n\]

In polar coordinates, it becomes
\[ g_{ii}^6 = \frac{1}{8\pi} \left[ \int_0^{\pi/4} \int_0^{2\sec\theta} (2r \cos\theta - r^2 \cos^2\theta)r \sin\theta \, dr \, d\theta \right. \\
\left. + \int_{\pi/4}^{\pi/2} \int_0^{2\cosec\theta} (2r \cos\theta - r^2 \cos^2\theta)r \sin\theta \, dr \, d\theta \right] \\
= \frac{1}{8\pi} \left[ \int_0^{\pi/4} \frac{4}{3} \sec\theta \tan\theta \, d\theta + \int_{\pi/4}^{\pi/2} \left[ \frac{16}{3} \cosec\theta \cot\theta - 4 \frac{\cos^2\theta}{\sin^3\theta} \right] d\theta \right] \\
= \frac{1}{12\pi} \left[ -10 + 7\sqrt{2} + 3 \log (\sqrt{2} + 1) \right] \\
= g_{ii}^4 \\

\[ g_{ii}^7 = \int_S N_\gamma \left( \frac{1}{4\pi r} \right) dS \\
= \frac{1}{16\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{(1 - \xi)(1 + \eta)(-\xi + \eta - 1)}{\sqrt{(1 + \xi)^2 + (1 + \eta)^2}} \, d\xi \, d\eta \\
\]

With \( 1 + \xi = x \) and \( 1 + \eta = y \), it becomes

\[ g_{ii}^7 = \frac{1}{16\pi} \int_{0}^{2} \int_{0}^{2} \frac{-(2 - x)y(-x + y - 1)}{\sqrt{x^2 + y^2}} \, dx \, dy \\
\]

Transforming to polar coordinates,

\[ g_{ii}^7 = \frac{1}{16\pi} \left[ \int_0^{\pi/4} \int_0^{2\sec\theta} (2 - r \cos\theta)r \sin\theta (-r \cos\theta + r \sin\theta - 1) \, dr \, d\theta \right. \\
\left. + \int_{\pi/4}^{\pi/2} \int_0^{2\cosec\theta} (2 - r \cos\theta)r \sin\theta (-r \cos\theta + r \sin\theta - 1) \, dr \, d\theta \right] \\
= \frac{1}{16\pi} \left[ \int_0^{\pi/4} \frac{4}{3} \sec\theta \tan\theta \, d\theta + \int_{\pi/4}^{\pi/2} \left[ \frac{16}{3} \cosec\theta \cot\theta - 4 \frac{\cos^2\theta}{\sin^3\theta} \right] d\theta \right] \\
= \frac{1}{12\pi} \left[ -10 + 7\sqrt{2} + 3 \log (\sqrt{2} + 1) \right] \\
= g_{ii}^4 \\

\]
\[
\int_0^{\pi/2} \int_0^{\pi/4} \left( 2 \csc \theta \right) (2 - r \cos \theta) r \sin \theta (-r \cos \theta + r \sin \theta - 1) \, dr \, d\theta \\
= \frac{1}{16\pi} \left[ \int_0^{\pi/4} \left( -\frac{8}{3} \sec \theta \tan \theta + \frac{4}{3} \sin^2 \theta \right) d\theta \\
+ \int_0^{\pi/2} \left( \frac{4}{3} \csc \theta - \frac{20}{3} \csc \theta \cot \theta + \frac{4 \cos^2 \theta}{\sin^2 \theta} \right) d\theta \right] \\
= \frac{1}{12\pi} \left[ 7 - 5\sqrt{2} - \log (\sqrt{2} + 1) \right] \\
= g_{ii}^3
\]

\[
g_{ii}^3 = \int_0^1 \int_0^1 N_8 \left( \frac{1}{4\pi r} \right) dS \\
= \frac{1}{8\pi} \int_{-1}^1 \int_{-1}^1 \frac{(1 - \xi)(1 - 1)}{(1 + \xi)^2 + (1 + 1)^2} \, d\xi \, dn
\]

Let \( 1 + \xi = x \), and \( 1 + \eta = y \), then

\[
g_{ii}^3 = \frac{1}{8\pi} \int_0^2 \int_0^2 \frac{(2 - x)(2y - y^2)}{\sqrt{2} + y^2} \, dx \, dy
\]

Now transforming to polar coordinates, it becomes

\[
g_{ii}^3 = \frac{1}{8\pi} \int_0^{\pi/4} \int_0^{2\sec \theta} \left( 2 - r \cos \theta \right) (2r \sin \theta - r^2 \sin^2 \theta) \, dr \, d\theta
\]
\[ \int_{0}^{\pi/4} \int_{\pi/4}^{\pi/2} 2 \csc \theta \cos \theta (2 - r \cos \theta) (2r \sin \theta - r^2 \sin^2 \theta) \, dr \, d\theta \]

\[ = \frac{1}{8\pi} \left( \int_{0}^{\pi/4} \left( \frac{8}{3} \sec \theta \tan \theta - \frac{4}{3} \frac{\sin^2 \theta}{\cos^3 \theta} \right) \, d\theta \right) \]

\[ + \int_{\pi/4}^{\pi/2} \left( \frac{8}{3} \csc \theta - \frac{4}{3} \csc \theta \cot \theta \right) \, d\theta \]

\[ = \frac{1}{24\pi} [10 \log (\sqrt{2} + 1) + 2\sqrt{2} - 4] \]

\[ = 8_{i_{ii}}^2 \]

Again the \( h_{i_{ii}} \) integrals of equation (3.3.10) are zero, since \( r \) and \( n \) are orthogonal over this element. The above eight integrals were calculated numerically using Gauss-quadrature rule with four, nine and sixteen integration points as well as using modified quadrature rule with four and nine integration points. The comparison of the computed and analytical values of these integrals is shown in table (5). It can be seen from this table that the same type of behaviour is observed in the results for this element as was observed for the four noded flat square element. Once again it can be concluded that the weights and points for the modified quadrature rule have been evaluated correctly. The same technique could be used to evaluate the integrals having a singularity of \( \frac{1}{r^2} \ln r \) in them, but in this work the above formulae have been used to evaluate the \( h_{i_{ii}}^k \) integrals as well as \( g_{i_{ii}}^k \) integrals.

This has been done because the integrand in the case of \( h_{i_{ii}}^k \) contains an
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...term in the numerator which tends to zero as r approaches zero, thus the overall singularity is effectively that of \( \frac{1}{r} \).

6.3 COMPARISON OF GAUSS-QUADRATURE AND MODIFIED QUADRATURE SCHEMES

In section 4, the flow field calculations for the sphere, ellipsoids of fineness ratios 2 and 10, sphere in the proximity of the ground and the semi-infinite body were obtained using Gauss-quadrature rule with four integration points. The surface of each of these bodies was discretized into linear quadrilateral boundary elements. In this section, the flow field calculations around the above-mentioned bodies are obtained using Gauss-quadrature rule with nine and sixteen integration points as well as using the modified quadrature rule with four and nine integration points. These bodies will be discretized into linear and quadratic isoparametric quadrilateral elements. The number of elements and the method of distribution of these elements for each of these bodies is the same as discussed in section 4. Tables (6) and (7) show the comparison of the Gauss-quadrature and the modified quadrature rule for 96 and 384 linear boundary elements respectively, while tables (8) and (9) show the comparison of these two quadrature rules for 96 and 384 quadratic boundary elements respectively. It should be noted that the semi-infinite body only has half as many elements as all the other bodies in each table. In the cases of a sphere, a sphere in the proximity of the ground and the semi-infinite body, the results are compared for the pressure distributions, while in the case of ellipsoids of revolution, the results are compared for the velocity distributions. Furthermore,
in the case of sphere near the ground, the results were obtained with
the ground clearance of 0.3 units.

In tables (6), (7), (8) and (9), for each of the bodies, the maximum
error, average error and the computing time for setting up the matrices
\([G]\) and \([H]\) of equations (3.3.21) are given.

Before comparisons are made, it is pointed out that since the
boundary element methods are to be applied to calculate the flow field
around road vehicles, which lie near the ground and have tubular wakes
behind them, and as isolated spheres and ellipsoids of revolution are
very special types of bodies, therefore particular consideration will
be given to the results for the sphere near the ground and the semi­
infinite body.

It can be seen from the above mentioned tables that in general,
for each of the bodies with either linear or quadratic boundary
elements, the results with the modified quadrature rule are better than
those using the Gauss-quadrature rule. The results with the modified
quadrature rule over linear boundary elements show a slight gain in
accuracy in going from a four point to a nine point scheme, but the
computing time is more than doubled. In the case of quadratic
boundary elements with the modified quadrature rule, there is little
to choose between the four point and the nine point schemes in terms of
accuracy, but the four point scheme is computationally much quicker.
In some instances the sixteen point Gauss-quadrature rule gives
slightly more accurate results than the four point modified
quadrature rule, but at the expense of much greater computing time.

Thus, keeping in view the accuracy obtained for given computational
effort, it can be concluded that the modified quadrature rule with four
integration points is the best scheme to use for linear and quadratic
6.4 **COMPARISON OF TYPES OF ELEMENTS**

In this section comparison of different types of boundary elements will be presented. For this purpose, in this section the flow field calculations around the bodies considered in section (6.3) are also obtained using cubic boundary elements. The integrals over the cubic elements are evaluated numerically using Gauss-quadrature rule with four, nine and sixteen integration points. The surface of the sphere and the ellipsoids of revolution will be discretized into 96 cubic boundary elements, whereas the surface of the semi-infinite body will be discretized into 48 cubic boundary elements. The method of distribution of elements is the same as discussed in section 4.

On comparison of the results over 96 quadratic boundary elements in table (8) with those over 96 cubic boundary elements in table (10), it can easily be seen that the results over the cubic elements are in general worse than those for the quadratic elements. In particular, the results for the semi-infinite body in the case of cubic elements are very poor. The average error and the maximum error increases with the increase in the number of integration points on the element. These worse results are on the elements near the rear end of the body. Furthermore the computing time in the case of cubic elements is more than double the computing time for the quadratic elements. Hence quadratic elements are to be preferred to cubic elements on the grounds both of accuracy and computing time.
Now comparing the results over 384 linear boundary elements in table (7) with those over 384 quadratic boundary elements in table (9), it can be seen that the average error for the sphere and ellipsoids of revolution, is in general, less in the case of linear elements. On the other hand, for the sphere near the ground and a semi-infinite body, the average error is less in the case of 384 quadratic elements, but the computing time has increased more than four times as compared to the computing time in the case of linear elements.

Next comparing the results over 384 linear boundary elements in table (7) with those over 96 quadratic boundary elements in table (8), it can be seen that the average error is least for the case of linear elements, but the computing time for 384 linear elements is more than three times than that for 96 quadratic elements.

Finally, comparing the results over the 96 quadratic boundary elements in table (8) with those over 96 linear boundary elements in table (6), it can be seen that in general, the results for quadratic elements are slightly better than those of the linear elements, but the computing time is more than four times that required for the linear elements.

It therefore appears from the limited data available, that on the basis of accuracy for a given computing time, the linear boundary elements are better than quadratic elements for the flow field calculations in the case of three dimensional problems. This situation agrees with Maskew [12] who states that the lower order methods give comparable accuracy at lower computing costs as compared to the higher order formulations. Hess [10] has used the higher order boundary elements to calculate the flow past two-dimensional bodies. He concludes that the higher order solutions give very little increase in accuracy for the case of exterior flow about a convex body.
### TABLE (1)

**MODIFIED WEIGHTS**

<table>
<thead>
<tr>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>1.599081</td>
</tr>
<tr>
<td>$\omega_2 = \omega_4$</td>
<td>0.624407</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>0.677597</td>
</tr>
</tbody>
</table>

### TABLE (2)

<table>
<thead>
<tr>
<th>$R_{30}$</th>
<th>Coefficients</th>
<th>$R_{03}$</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{300}$</td>
<td>-0.211575005921497</td>
<td>$R_{030}$</td>
<td>-0.211575005921497</td>
</tr>
<tr>
<td>$R_{301}$</td>
<td>1.9694544555252099</td>
<td>$R_{031}$</td>
<td>-0.104865856848484</td>
</tr>
<tr>
<td>$R_{302}$</td>
<td>-0.104865856848484</td>
<td>$R_{032}$</td>
<td>1.9694544555252099</td>
</tr>
<tr>
<td>$R_{303}$</td>
<td>-2.805208321585304</td>
<td>$R_{033}$</td>
<td>0.029785120454849</td>
</tr>
<tr>
<td>$R_{304}$</td>
<td>0.046559886100398</td>
<td>$R_{034}$</td>
<td>0.046559886100398</td>
</tr>
<tr>
<td>$R_{305}$</td>
<td>0.029785120454849</td>
<td>$R_{035}$</td>
<td>-2.805208321585304</td>
</tr>
</tbody>
</table>
**Table (3)**

<table>
<thead>
<tr>
<th>Modified Integration Points</th>
<th>Modified Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 = 1.711823931756158$</td>
<td>$\omega_1 = 0.187442978417879$</td>
</tr>
<tr>
<td>$y_1 = x_1$</td>
<td></td>
</tr>
<tr>
<td>$x_2 = 0.875937756252321$</td>
<td>$\omega_2 = 0.666750710927685$</td>
</tr>
<tr>
<td>$y_2 = x_2$</td>
<td></td>
</tr>
<tr>
<td>$x_3 = 0.141101627021577$</td>
<td>$\omega_3 = 0.845686715223277$</td>
</tr>
<tr>
<td>$y_3 = x_3$</td>
<td></td>
</tr>
<tr>
<td>$x_4 = 0.897460382010956$</td>
<td>$\omega_4 = 0.273178975447147$</td>
</tr>
<tr>
<td>$y_4 = 1.745327820461991$</td>
<td></td>
</tr>
<tr>
<td>$x_5 = y_4$</td>
<td>$\omega_5 = \omega_4$</td>
</tr>
<tr>
<td>$y_5 = x_5$</td>
<td></td>
</tr>
<tr>
<td>$x_6 = 0.198000853722974$</td>
<td>$\omega_6 = 0.168998274274658$</td>
</tr>
<tr>
<td>$y_6 = 1.750050403616516$</td>
<td></td>
</tr>
<tr>
<td>$x_7 = y_6$</td>
<td>$\omega_7 = \omega_6$</td>
</tr>
<tr>
<td>$y_7 = x_7$</td>
<td></td>
</tr>
<tr>
<td>$x_8 = 0.184880078528939$</td>
<td>$\omega_8 = 0.470629721991967$</td>
</tr>
<tr>
<td>$y_8 = 0.911042111294785$</td>
<td></td>
</tr>
<tr>
<td>$x_9 = y_8$</td>
<td>$\omega_9 = \omega_8$</td>
</tr>
<tr>
<td>$y_9 = x_9$</td>
<td></td>
</tr>
</tbody>
</table>
**TABLE (4)**

**COMPARISON OF THE GAUSS-QUADRATURE AND THE MODIFIED QUADRATURE SCHEMES OVER A FOUR NODED FLAT SQUARE ELEMENT**

<table>
<thead>
<tr>
<th>INTEGRAL</th>
<th>GAUSS-QUADRATURE</th>
<th>MODIFIED QUADRATURE</th>
<th>ANALYTICAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 Points</td>
<td>9 Points</td>
<td>16 Points</td>
</tr>
<tr>
<td>$g_{ii}$</td>
<td>0.1006484</td>
<td>0.1108358</td>
<td>0.1142003</td>
</tr>
<tr>
<td>$g_{ii}$, $g_{ii}$</td>
<td>0.0606223</td>
<td>0.0595966</td>
<td>0.0593198</td>
</tr>
<tr>
<td>$g_{ii}$</td>
<td>0.0443786</td>
<td>0.0439831</td>
<td>0.0439595</td>
</tr>
</tbody>
</table>
TABLE (5)

COMPARISON OF THE GAUSS–QUADRATURE AND THE MODIFIED QUADRATURE SCHEMES OVER AN EIGHT NODED FLAT SQUARE ELEMENT

<table>
<thead>
<tr>
<th>INTEGRAL</th>
<th>GAUSS–QUADRATURE</th>
<th>MODIFIED QUADRATURE</th>
<th>ANALYTICAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 Points</td>
<td>9 Points</td>
<td>16 Points</td>
</tr>
<tr>
<td>$g_{ii}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.0068654</td>
<td>0.0076111</td>
<td>0.0121403</td>
</tr>
<tr>
<td>$g_{ii}', g_{ii}$</td>
<td>0.1075138</td>
<td>0.1032247</td>
<td>0.1020600</td>
</tr>
<tr>
<td>$g_{ii}'$, $g_{ii}$</td>
<td>-0.0281349</td>
<td>-0.0258322</td>
<td>-0.0254658</td>
</tr>
<tr>
<td>$g_{ii}'$, $g_{ii}$</td>
<td>0.0700006</td>
<td>0.0676330</td>
<td>0.0675111</td>
</tr>
<tr>
<td>$g_{ii}$</td>
<td>-0.0256220</td>
<td>-0.0236499</td>
<td>-0.0235516</td>
</tr>
</tbody>
</table>
### Comparison of Gauss-Quadrature and Modified Quadrature Schemes Over 96 Linear Boundary Elements

<table>
<thead>
<tr>
<th>Type of Body</th>
<th>Integration Points</th>
<th>Gauss-Quadrature</th>
<th>Modified Quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 Points</td>
<td>9 Points</td>
<td>16 Points</td>
</tr>
<tr>
<td>Sphere</td>
<td>Ave. Error = 0.0778</td>
<td>0.0620</td>
<td>0.0563</td>
</tr>
<tr>
<td></td>
<td>Max. Error = 0.1418</td>
<td>0.1177</td>
<td>0.1087</td>
</tr>
<tr>
<td></td>
<td>CPU Time = 11.6</td>
<td>24.8</td>
<td>43.5</td>
</tr>
<tr>
<td>Ellipsoid of Finesness</td>
<td>Ave. Error = 0.0324</td>
<td>0.0284</td>
<td>0.0269</td>
</tr>
<tr>
<td></td>
<td>Max. Error = 0.0940</td>
<td>0.0884</td>
<td>0.0866</td>
</tr>
<tr>
<td></td>
<td>CPU Time = 11.6</td>
<td>24.9</td>
<td>43.5</td>
</tr>
<tr>
<td>Ellipsoid of Finesness</td>
<td>Ave. Error = 0.0099</td>
<td>0.0091</td>
<td>0.0086</td>
</tr>
<tr>
<td></td>
<td>Max. Error = 0.0603</td>
<td>0.0541</td>
<td>0.0507</td>
</tr>
<tr>
<td></td>
<td>CPU Time = 11.8</td>
<td>24.9</td>
<td>43.5</td>
</tr>
<tr>
<td>Sphere</td>
<td>Ave. Error = 0.0846</td>
<td>0.0737</td>
<td>0.0696</td>
</tr>
<tr>
<td>Near the</td>
<td>Max. Error = 0.3091</td>
<td>0.2806</td>
<td>0.2702</td>
</tr>
<tr>
<td>Ground</td>
<td>CPU Time = 23.4</td>
<td>49.6</td>
<td>86.9</td>
</tr>
<tr>
<td>Semi-Infinite Body</td>
<td>Ave. Error = 0.0985</td>
<td>0.0933</td>
<td>0.0919</td>
</tr>
<tr>
<td></td>
<td>Max. Error = 0.2407</td>
<td>0.2250</td>
<td>0.2201</td>
</tr>
<tr>
<td></td>
<td>CPU Time = 3.7</td>
<td>7.5</td>
<td>13.2</td>
</tr>
</tbody>
</table>
TABLE (7)

COMPARISON OF GAUSS-QUADRATURE AND MODIFIED QUADRATURE SCHEMES OVER 384 LINEAR BOUNDARY ELEMENTS

<table>
<thead>
<tr>
<th>TYPE OF BODY</th>
<th>INTEGRATION POINTS</th>
<th>GAUSS-QUADRATURE</th>
<th>MODIFIED QUADRATURE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4 POINTS</td>
<td>9 POINTS</td>
<td>16 POINTS</td>
</tr>
<tr>
<td>SPHERE</td>
<td>AVE. ERROR = 0.0347</td>
<td>0.0257</td>
<td>0.0224</td>
</tr>
<tr>
<td></td>
<td>MAX. ERROR = 0.1215</td>
<td>0.0919</td>
<td>0.0801</td>
</tr>
<tr>
<td></td>
<td>CPU TIME = 169.9</td>
<td>362.1</td>
<td>634.1</td>
</tr>
<tr>
<td>ELLIPSOID OF</td>
<td>AVE. ERROR = 0.0097</td>
<td>0.0079</td>
<td>0.0072</td>
</tr>
<tr>
<td>FINENESS</td>
<td>MAX. ERROR = 0.0387</td>
<td>0.0360</td>
<td>0.0351</td>
</tr>
<tr>
<td></td>
<td>CPU TIME = 169.2</td>
<td>360.6</td>
<td>632.8</td>
</tr>
<tr>
<td>ELLIPSOID OF</td>
<td>AVE. ERROR = 0.0051</td>
<td>0.0042</td>
<td>0.0041</td>
</tr>
<tr>
<td>FINENESS</td>
<td>MAX. ERROR = 0.0986</td>
<td>0.0904</td>
<td>0.0876</td>
</tr>
<tr>
<td></td>
<td>CPU TIME = 171.7</td>
<td>366.6</td>
<td>641.2</td>
</tr>
<tr>
<td>SPHERE</td>
<td>AVE. ERROR = 0.0320</td>
<td>0.0270</td>
<td>0.0253</td>
</tr>
<tr>
<td>NEAR THE GROUND</td>
<td>MAX. ERROR = 0.1472</td>
<td>0.1319</td>
<td>0.1264</td>
</tr>
<tr>
<td></td>
<td>CPU TIME = 347.9</td>
<td>728.0</td>
<td>1273.4</td>
</tr>
<tr>
<td>SEMI-INFINITE BODY</td>
<td>AVE. ERROR = 0.0304</td>
<td>0.0284</td>
<td>0.0278</td>
</tr>
<tr>
<td></td>
<td>MAX. ERROR = 0.0789</td>
<td>0.0663</td>
<td>0.0627</td>
</tr>
<tr>
<td></td>
<td>CPU TIME = 47.5</td>
<td>100.7</td>
<td>176.6</td>
</tr>
<tr>
<td>TYPE OF BODY</td>
<td>INTEGRATION POINTS</td>
<td>GAUSS-QUADRATURE</td>
<td>MODIFIED QUADRATURE</td>
</tr>
<tr>
<td>--------------</td>
<td>--------------------</td>
<td>-------------------</td>
<td>--------------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4 POINTS</td>
<td>9 POINTS</td>
</tr>
<tr>
<td>SPHERE</td>
<td>AVE. ERROR = 0.0519</td>
<td>0.0453</td>
<td>0.0463</td>
</tr>
<tr>
<td></td>
<td>MAX. ERROR = 0.1111</td>
<td>0.1340</td>
<td>0.0652</td>
</tr>
<tr>
<td></td>
<td>CPU TIME = 50.9</td>
<td>108.7</td>
<td>190.5</td>
</tr>
<tr>
<td>ELLIPSOID OF FINESSNESS</td>
<td>AVE. ERROR = 0.0204</td>
<td>0.0265</td>
<td>0.0127</td>
</tr>
<tr>
<td></td>
<td>MAX. ERROR = 0.0521</td>
<td>0.0698</td>
<td>0.0368</td>
</tr>
<tr>
<td></td>
<td>CPU TIME = 50.8</td>
<td>108.5</td>
<td>189.5</td>
</tr>
<tr>
<td>ELLIPSOID OF FINESSNESS</td>
<td>AVE. ERROR = 0.0122</td>
<td>0.0149</td>
<td>0.0074</td>
</tr>
<tr>
<td></td>
<td>MAX. ERROR = 0.0501</td>
<td>0.0499</td>
<td>0.0399</td>
</tr>
<tr>
<td></td>
<td>CPU TIME = 51.2</td>
<td>109.3</td>
<td>189.6</td>
</tr>
<tr>
<td>SPHERE</td>
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<td>0.0603</td>
<td>0.0498</td>
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<td>MAX. ERROR = 0.1887</td>
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<td>0.1439</td>
</tr>
<tr>
<td></td>
<td>CPU TIME = 102.1</td>
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<td>376.6</td>
</tr>
<tr>
<td>SEMI-INFINITE BODY</td>
<td>AVE. ERROR = 0.0551</td>
<td>0.0313</td>
<td>0.0438</td>
</tr>
<tr>
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<td>MAX. ERROR = 0.1924</td>
<td>0.0394</td>
<td>0.1410</td>
</tr>
<tr>
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<td>CPU TIME = 14.9</td>
<td>31.6</td>
<td>53.4</td>
</tr>
<tr>
<td>TYPE OF BODY</td>
<td>INTEGRATION POINTS</td>
<td>GAUSS-QUADRATURE</td>
<td>MODIFIED QUADRATURE</td>
</tr>
<tr>
<td>----------------------</td>
<td>--------------------</td>
<td>------------------</td>
<td>---------------------</td>
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**TABLE (9)**

Comparison of Gauss-Quadrature and Modified Quadrature Schemes over 384 Quadratic Boundary Elements.
**TABLE (10)**
COMPARISON OF GAUSS-QUADRATURE SCHEMES OVER 96 CUBIC BOUNDARY ELEMENTS

<table>
<thead>
<tr>
<th>TYPE OF BODY</th>
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<th>GAUSS-QUADRATURE</th>
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7.1 INTRODUCTION

In this section the direct boundary element method with linear quadrilateral elements and using the modified quadrature rule with four integration points will be used to calculate the pressure distributions over the surfaces of models of road vehicles. The calculated pressure coefficients will be compared with the experimental pressure coefficients which were obtained in wind tunnel tests on models of three different shapes of car bodies. The experimental work was commissioned some years ago by BL Technology Ltd. and was carried out at Imperial College, London.

7.2 COMPARISON OF EXPERIMENTAL AND COMPUTED PRESSURE COEFFICIENTS OVER THE SURFACE OF A MODEL CAR BODY

The shapes of the car bodies for which the experimental results are available are of the following types:

(i) car body with vertical back
(ii) " " " short taper
(iii) " " " long taper
See figure 7.1.

Each of the models of the car bodies is simplified in that there were no wheels, wheel cavities, bumpers and mirrors etc. The experimental results over the model of each of these types of car bodies were obtained in a wind tunnel measuring 1.2 meters by 1.37 meters at a test velocity of 20 meters per second. The wind tunnel had a moving ground plane which also ran at 20 meters per second. The ground clearance for each of the models of these three shapes was 18mm, with the underside set parallel to the ground. Each of the models of the car bodies has been discretized into linear quadrilateral boundary elements. The boundary condition of zero normal velocity at the surface of the car body has the form

\[
\frac{\partial \Phi}{\partial n} = \hat{n} \cdot \hat{i} \tag{7.2.1}
\]

where the unit normal \( \hat{n} \) can be found by the same method as discussed in section (4.4.1).

Figure 7.1
It should be noted that the plane of symmetry for the car bodies shown in figure 7.1 has been taken as the xz-plane rather than the xy-plane as used in previous sections. The edges of the car bodies are rounded and are of 20mm radius.

In order to make the comparisons easy, the top, side and bottom views of each of the types of car body are shown in figure 7.6. In this figure different lines have been drawn along which the experimental and computed results will be compared. The lines, A, B, C and D in the top view are the lines in the planes $y = 0$, $y = 40$, $y = 50$ and $y = 76$ respectively. The corresponding lines in the above mentioned planes in the bottom view have been indicated by $A'$, $B'$, $C'$ and $D'$ respectively.

Consider now in turn each of the above mentioned types of car body.

(i) **CAR BODY WITH VERTICAL BACK**

Consider first the model of the car body which is cut at $x = -10$mm to give a bluff tailed shape. Figures 7.2 and 7.3 show the discretization of this car model into 464 linear quadrilateral boundary elements. The pressure coefficient at the mid-point of each quadrilateral element is calculated by the method discussed in section (4). Interpolation is used to obtain the calculated pressure coefficients at any intermediate points. Figures 7.7 to 7.10 show the comparisons of the experimental and the computed pressure coefficients over the surface of the car body with a vertical back. Figure 7.7 shows the comparisons along the lines A and $A'$, while figure 7.8 shows the comparisons along the lines $B$, $C$ and $B'$, $C'$. In figure 7.8 the graphs for the experimental and computed results are shown to be discontinuous, indicating that the right-most portions of the graphs are the
comparisons along the lines C and C' and the left-most portions of the graphs are the comparisons along the lines B and B'. Similarly, figure 7.9 shows the comparisons along the lines D and D' and figure 7.10 shows the comparison along the line E. In all these figures, there is quite good agreement between the experimental and the computed results. In figure 7.10 there is some doubt about the experimental values in the middle of the body, but the computed results seem to be consistent.

(ii) CAR BODY WITH SHORT TAPER

Consider next the model of the car body with a 25° slant back which is cut off to give a bluff base of half model height. Figure 7.4 shows the discretization of this model into 618 linear quadrilateral boundary elements. Again the pressure coefficient is calculated at the mid-point of each quadrilateral element and interpolated as required. Figures 7.11 to 7.14 show the comparisons of the calculated and experimental pressure coefficients over the surface of the car body with short taper. In all the figures showing the comparisons on the top surface of the car body, it can be seen that there is excellent agreement between the experimental and computed results except over the slant-back part of the body, where the computed results differ from the experimental results. This situation might be due to the separation of the boundary layer from the body, because in obtaining the results for the car bodies it was assumed that the boundary layer remains attached right to the rear end of the body. Even if the boundary layer does remain attached, it will be considerably thicker here than elsewhere, and the computed results at present ignore the displacement effects of boundary layer growth. On
the other hand, for the results on the bottom surface of the body in these figures, where the boundary layer was always attached and of small displacement thickness, there is an excellent agreement between the experimental and the computed results right from the front to the rear end of the body. In figure 7.12 (b), the second experimental value from the left seems to be incorrect as noted by the experimentors. Finally, figure 7.14 shows the comparison along the line E. Again, as in the case of the car body with a vertical back, the experimental value near the middle of the body seems to be incorrect, possibly due to a faulty pressure tapping.

(iii) CAR BODY WITH LONG TAPER

Lastly consider the model of the car body with a 25\degree slant back which continues to a point. The boundary layer on the upper surface is not likely to remain attached right up to the trailing edge of the body, and the experimental results show the pressure coefficient tending towards a constant value near the rear of the body, as happens in an area of separated flow. Therefore, in this case the car body shape was truncated at \( x = -215\text{mm} \) and then it was assumed that a tubular wake emanated from this position. Figure 7.5 shows a discretization of this model car body into 750 linear quadrilateral boundary elements. The pressure distribution is calculated at the mid-point of each quadrilateral element. Figure 7.15 to 7.18 show the comparisons of the experimental and computed pressure coefficients over the surface of the car body with long taper. It can be seen from these figures that the same type of behaviour in the results is observed as was observed in the case of a car body with short taper.
Thus far, in producing the results shown in figures 7.7 to 7.18 over the models of the car bodies shown in figure 7.1, it was assumed that the tubular wake behind the body started immediately from the rear end of the body. It can be seen from the results for a car body with a vertical back from figures 7.7 to 7.10, that near the rear of the body, the computed pressure coefficient is greater than the experimental pressure coefficient. This shows that the computed velocity is less than that of the experimental at the rear of the body. A complete analysis including the effects of the boundary layer and the wake should remove this discrepancy between the experimental and computed results over the rear portion of the car body. The reason for this discrepancy could be that the wake for the computed results has been assumed to be parallel to the \(x\)-axis immediately from the rear end of the body, while in reality there would be a slight contraction of the wake at the rear of the body which would give rise to higher velocities over the rear portion of the body than would be predicted from parallel flow. To check the validity of this argument, it was decided to model a slight contraction of the wake, starting from the rear end of the body, as a part of the actual body, followed by a normal tubular wake parallel to the \(x\)-axis. This guessed portion of the flowfield was also discretized into elements in the same manner in which the actual body was discretized. The pressure distribution over the surface of the car body with a vertical back was again obtained. The results obtained for this case are shown in figure 7.19 to 7.22. On comparison of these results with those in figures 7.7 to
7.10, it can be seen that the results in this case show a behaviour in the right direction as expected.

Next, consider the case of a car body with a short or long taper. In this case it can be seen from figures 7.11 to 7.14 or 7.15 to 7.18 that at the rear of the body the difference in the experimental and the computed pressure coefficients on the top surface of the body is of greater magnitude than on the bottom surface of the body, in comparison to the case of a car body with a vertical back where this difference was of the same order of magnitude on all sides of the body. In the present case, the experimental and the computed pressure coefficients are in good agreement on the bottom surface of the body, while they are much different on the slant back part on the top surface of the body. The reason for this is that in the case of the computed results, the velocity distribution is that for potential flow about two sharp corners which gives rise to large velocities at the expansion corner and low velocities at the contraction corner, where the tubular wake is assumed to begin. In reality, the boundary layer and wake separation effects would produce a much smoother expansion and contraction of the flow, which would give rise to less extreme values of pressure coefficient as is observed in the experimental results.
Figure 7.2: Discretization of the car body shape with vertical back into 464 boundary elements. Top View.
Figure 7.3: Bottom view of the car body shape with vertical back.
Figure 7.4: Discretization of the car body shape with short taper into 618 boundary elements. Top View.
Figure 7.5: Discretization of the car body shape with long taper into 750 boundary elements. Top View.
Figure 7.6: The lines on the car body surfaces along which the comparisons of pressure coefficients are made.
Figure 7.7: Comparison of experimental and computed pressure coefficients over the surface of the car body with a vertical back along the lines (a) A; (b) A' of figure 7.6.
Figure 7.8: Comparison of experimental and computed pressure coefficients over the surface of the car body with a vertical back along the lines (a) B and C; (b) B' and C' of figure 7.6.
Figure 7.9: Comparison of experimental and computed pressure coefficients over the surface of the car body with a vertical back along the lines (a) D; (b) D' of figure 7.6.
Figure 7.10: Comparison of experimental and computed pressure coefficients over the surface of the car body with a vertical back along the line E of figure 7.6.
Figure 7.11: Comparison of experimental and computed pressure coefficients over the surface of the car body with short taper along the lines (a) A; (b) A' of figure 7.6.
Figure 7.12: Comparison of experimental and computed pressure coefficients over the surface of the car body with short taper along the lines (a) B and C; (b) B' and C' of figure 7.6.
Figure 7.13: Comparison of experimental and computed pressure coefficients over the surface of the car body with short taper along the lines (a) D; (b) D' of figure 7.6.
Figure 7.14: Comparison of experimental and computed pressure coefficients over the surface of the car body with short taper along the line E of figure 7.6.
Figure 7.15: Comparison of experimental and computed pressure coefficients over the surface of the car body with long taper along the lines (a) A; (b) A' of figure 7.6.
Figure 7.16: Comparison of experimental and computed pressure coefficients over the surface of the car body with long taper along the lines (a) B and C; (b) B' and C' of figure 7.6.
Experimental results
x Computed solution

Figure 7.17: Comparison of experimental and computed pressure coefficients over the surface of the car body with long taper along the lines (a) D; (b) D' of figure 7.6.
Figure 7.18: Comparison of experimental and computed pressure coefficients over the surface of the car body with long taper along the line E of figure 7.6.
Figure 7.19: Comparison of experimental and computed pressure coefficients over the surface of the car body with a vertical back along the lines (a) A; (b) A' of figure 7.6.
Figure 7.20: Comparison of experimental and computed pressure coefficients over the surface of the car body with a vertical back along the lines (a) B and C; (b) B' and C' of figure 7.6.
Figure 7.21: Comparison of experimental and computed pressure coefficients over the surface of the car body with a vertical back along the lines (a) D; (b) D' of figure 7.6.
Figure 7.22: Comparison of experimental and computed pressure coefficients over the surface of the car body with a vertical back along the line E of figure 7.6.
SECTION 8

CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

A direct boundary element method for predicting the flow field around road vehicles has been presented. The method has been compared with the indirect boundary element method and it is concluded that the direct method gives better results for a body with a tubular wake and is also more forgiving when faced with poor element distribution over the body. Furthermore, the direct method does not differ significantly in comparison with the indirect method in respect of the computing time required.

The boundary integrals have been evaluated numerically by use of a Gauss-quadrature rule over all the elements except those on which the singularities lie. For integrals of the latter type, a modified quadrature rule has been developed which gives better results as compared to the Gauss-quadrature rule. Furthermore, the incorporation of the modified quadrature rule does not add significantly to the computing time required to set up the system matrices.

The direct method has also been compared for different types of boundary elements and it is concluded that on the basis of accuracy for a given computing time, the direct method with linear isoparametric quadrilateral elements is better for the flowfield calculations in the case of three-dimensional flow problems.

The use of the direct boundary element method gives rise to a system of algebraic equations which is diagonally dominant and which
can therefore be solved by an iterative method of solution. Convergence of the solution of this system of equations has been found to be very rapid for the Gauss-Seidel iterative method which is, therefore, more computationally efficient than, say the Gauss-elimination method.

The boundary element method presented here has been shown to be accurate when applied to simple body shapes for which the analytical solutions are available for comparison. When applied to more complicated body shapes, such as road vehicles, the results obtained with the present method are in good agreement with the experimental results over most of the body surface, where the real flow remains attached. Over the rear portion of the vehicle, where the boundary layer rapidly thickens and separates to form the wake, the theoretical and experimental results differ more markedly. It has been shown that the theoretical results in this area change significantly in response to small changes in the initial shape of the wake and they will also vary with the precise positioning of the separation of the wake. Thus the complete analysis of the boundary layer growth and the position of boundary layer separation is required to remove this discrepancy between the computed and the experimental results. The most urgent need for further work is thus the implementation of a boundary layer analysis scheme. Such a scheme would require repeated use of the potential flow analysis scheme developed in this thesis, which therefore needs to be as efficient as possible.

As has been mentioned previously in section 5, about 90 to 95 percent of the total computing time for the current method is used on setting up the matrices of influence coefficients. These matrices need only be created once for a full solution scheme including boundary
layer iteration, but even so the most significant future gain in computational efficiency must come from effort in this area. All the results produced in this thesis have been obtained using an 'optimize' option at compile time, which significantly reduces the run time of the program. Other than this, no great attention has been paid to the efficiency of the algorithm which produces the influence coefficients. Thus significant gain may be possible in this area, particularly by using simpler approximations to integrals over elements at large distances from the fixed point.

For all the work presented in this thesis the onset flow has been taken to be a uniform stream in the direction of the x-axis, i.e. parallel to the body. However, this restriction is not necessary and the present method can be applied with an onset flow in any desired direction. In particular, vehicle motion in a cross-wind could be analysed, but some difficulty would be experienced in deciding upon the line of wake separation in the absence of a complete boundary layer analysis.

Furthermore all the results presented in this thesis have been for situations where there is no net normal velocity at the body surface. The direct method can be applied with equal ease to situations with any chosen normal velocity, thus the modelling of boundary layer thickness via surface transpiration or the modelling of inflow and outflow at the radiator grille and ventilation grilles could easily be considered in future developments.
REFERENCES


