The forced vibration of a partially delaminated beam

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The Forced Vibration of a Partially Delaminated Beam

by

Roger Menday

A DOCTORAL THESIS.

Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University

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Abstract

The forced vibration of a partially delaminated structure such as an aircraft wing can result in catastrophic crack growth. In order to look at the underlying mechanism of the dynamics and failure of the material, a simplified model of a cantilever beam with a single delamination at its free end is considered. We investigate a number of aspects of this system, using mathematical models to gain insight into its behaviour.

This first model makes a number of broad, general assumptions. The model produced is known mathematically as an impact oscillator, a non-linear system where the nonlinearity is caused by the discontinuity of an impact between the main beam and its delamination.

In the earlier chapters of the thesis, we examine both numerically and analytically, the resulting motions from the model subject to a harmonic forcing of the main beam. The model displays several types of motion; periodic, chattering-periodic and chaotic motion. Experimental evidence, a sample of which is provided, also shows some of these features.

In the later chapters of the thesis, we consider the impact between the main beam and its delamination in more detail. In order to study the detailed nature of an impact between the two bodies, we construct a series of models, of increasing complexity. This serves to establish the validity of the assumptions made when deriving the initial model of the system.

Keywords

impact, delamination, impact oscillator, wave propagation, coefficient of restitution, vibrating beams, discrete models, coupled systems
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Nomenclature

Roman letters

$A_c$ cross-sectional area of beam/area of 'sides' of slug

$A_T$ impacting surface area between delamination and main beam

$b$ breadth of beam

$C$ modal amplitude function, Chapters 2-4

$c$ speed of wave propagation

$E$ Young's modulus

$h_s$ depth of beam/slug

$I$ second moment of area of the beam

$L$ mode shape function of beam (this is a continuous function in Chapters 2-4, and a discrete function in Chapters 6-9)

$l$ length of beam

$t$ time

$w$ displacement of beam

$x$ displacement of beam relative to forcing (Chapters 2-4), displacement of slug/rod (Chapters 6-9)

$X_0$ cross-section of the slug or rod

$Z$ measures the distance along the beam from the clamped end

$z$ defined as $\frac{\theta}{\delta}$

Greek letters

$\epsilon$ strain

$\gamma$ non-dimensional damping parameter, Chapters 2-4

$\kappa$ dimensional damping parameter, Chapters 2-4

$\omega$ angular frequency

$\omega_1$ frequency of first mode

$\omega_d$ damped natural frequency

$\phi$ impacting phase
\( \rho \) density

\( \sigma \) stress

Notes

Throughout the text, the addition of the symbol \( \hat{\cdot} \) placed over a variable indicates it is in dimensional form. The symbol \( \hat{\cdot} \) represents a variable after taking Laplace transforms. Finally, in chapter 6 onwards, the symbol \( \hat{\cdot} \) placed above a variable refers to a rod, rather than a slug.

A vector is identified by placing an arrow at the top of a variable, for example \( \vec{U} \).

Throughout the text matrices are always displayed in the sans serif font, for example, the matrix \( M \). However, the sans serif font is sometimes used for non matrix variables.

Subscripts are placed on variables where appropriate. For example, when there is a collection of \( i \) separate bodies, \( \sigma_i \) is the stress in the \( i \)'th body. Conversely, \( \sigma_z \) is the stress in the \( z \)-direction, and sometimes a variable will not have subscripts. In all cases, the context of a variable is established in the text.
Chapter 1

Introduction

The use of fibre composite laminated structures is a rapidly growing area of engineering design. The issue of delaminations between the layers of the material and the implications regarding the performance and strength of the material must be addressed. When used in high stress situations, such as an aircraft wing, the forced vibration of a partially delaminated structure can result in catastrophic crack growth. We suggest that the configuration, materials and external influences will all effect the dynamics and the likelihood of crack growth, or 'tearing' between the layers of the material. In this thesis we address the problem of modelling the forced vibration of a beam with a delamination, and we discuss the motion of the system. We could then continue the work and calculate forces acting at crucial points (particularly at the join between the delamination and the main beam), and this would be important in order to extend this work and consider the growth of the delamination.

1.1 The single DOF model of the laminated beam

We consider a simple configuration where a cantilever beam has a single delamination at the free end. We examine the possible motions of the delamination when the main beam is subjected to a harmonic forcing. Firstly, we develop a single degree-of-freedom model of the system, and then examine the resulting motions. We then present some experimental results, and we find that the mathematical model is at least partially accurate in predicting physical behaviour.

We model a system where a small beam is clamped to a point near the end of a larger beam, where these two components represent the delamination and the main beam. In its unforced, equilibrium position, the main beam and its delamination are in contact. If the main beam is forced harmonically it is clear that this is an example of a system which includes impacts.

There are two major assumptions made when deriving the model. Firstly we assume that
since the length of the delamination is small compared with the main beam, the main beam has zero curvature along the length of the delamination. Secondly we assume that the delamination moves in its first mode of vibration only. Coupling the two assumptions we can assume that when an impact occurs, it occurs along the entire length of the delamination. We decide to use a very simple model for the impact itself, namely a simple restitution law relating the impacting and rebounding velocities.

It turns out that once we include these assumptions in the model the motion between impacts is described by an amplitude function associated with the fundamental mode of vibration, the evolution of which is defined by a second order ordinary differential equation. Also, due to the coefficient of restitution model, the impact is represented by a instantaneous change of speed. Thus, we obtain a mathematical system, commonly known as an impact oscillator, and it is a nonlinear system.

The widespread interest in nonlinear dynamics in recent years, has been due in part to the advances in computer technology. Many dynamical systems are so complex that analytical solutions cannot be found and numerical simulation is the only solution. Advances in computer power and ability have made such progress possible and have opened up many interesting and relevant research possibilities.

In a non-linear system the nonlinearities arise either smoothly or non-smoothly and many treatments of dynamical systems assume that the nonlinearity arises smoothly. However, an interesting sub-category are non-smooth or discontinuous dynamical systems. Examples of such systems are common in engineering, and hence their study has application in a wide range of engineering problems. A simple prototype of such a system is a single degree-of-freedom forced mechanical oscillator with a piecewise linear restoring force. An example of such a system with discontinuous stiffness can be found in marine structures such as a ship moored to a mooring tower, see [25], [26]. The towers are essentially inverted pendulums fixed to the sea bed, with buoyancy causing them to remain upright. The ship is regarded as a fixed object, while the tower oscillates due to forcing from the waves. For a positive displacement of the tower towards the ship, the restoring force is simply the buoyancy of the tower, whilst for a negative displacement the restoring force is the buoyancy of the tower plus the stiffness of the mooring line. This is an example of a bilinear oscillator.

Another example from the marine environment is a ship, subjected to harmonic forcing from the waves, moored in harbour to a fixed harbour wall. We assume that the ship is moored with no slackness in the mooring line. Now the stiffness for positive displacement of the ship away from wall is provided by the mooring line. We assume that the stiffness for a negative displacement of the ship is effectively infinite, i.e. the wall is immovable, and a negative displacement is not possible! Here the ship experiences an impact with the harbour wall with its velocity changing instantaneously. This system is another example of an impact oscillator.

Many other practical examples of non-smooth dynamical systems exist, such as gearing
systems with free-play or backlash, the motion of rigid blocks in earthquakes [13], a ball bouncing on an oscillating table [12], the motion of a pendulum with amplitude limiting stops [18] and the motion of a beam with an amplitude constraint in the form of a ‘stop’, [22].

Single degree of freedom impact or bilinear oscillators display surprisingly complex behaviour. Indeed it is testimony to their complex nature that only low dimensional systems have been thoroughly studied. It is hoped that a thorough understanding of low dimensional systems will be applicable to higher dimensional systems. Luckily many physical systems can be reduced, through the modelling process, to a low degree of freedom system. Indeed the subject of this thesis is a continuous system with infinitely many degrees of freedom, and through the modelling process with major simplifying assumptions, it is reduced to a single degree of freedom impact oscillator. However, we must determine whether we have developed a realistic model.

Low dimensional bilinear and impact oscillators are studied as mathematical systems in their own right without establishing any links to physical counterparts. There is a great deal of research published on these mathematical systems, see [2], [3], [4] and [19]. In reference [20], a periodically forced single DOF impact oscillator is investigated through experimental and theoretical simulation. The experimental set-up of the single DOF impact oscillator is represented by a stiff rod free to rotate in one plane about a bearing, with a restoring force provided by linear springs. The impact occurs with a stop placed near the end of the rod. In contrast to a delaminated beam, the rigidity of the rod ensures that complications arising from the contributions to the solution from the higher modes of vibration is largely eliminated. Essentially, the experiment is a close physical representation of the theoretical model, and the results do indicate this. The investigation of an impacting pendulum system [18], is also an example of a system where the higher modes have a small influence.

In [17] and [22] the impacting body is a flexible cantilever beam, where the impact occurs with a stop placed at the free end of the beam. In the model, movement only in the fundamental mode is considered. When the end of the beam is not in contact with the stop the motion is in the fundamental mode for a clamped-free beam and when the beam and the stop are in contact, the motion of the beam is in the fundamental mode for a clamped-pinned beam. Switching between the clamped-pinned motion and clamped-free motion occurs when the contact force at the stop becomes tensile. As concluded in [17], the effect of the higher modes of vibration remain a subject for future study, and that ‘the simulation presented based on the single DOF model is only an attempt to capture some of the qualitative features of the experimental results’.

As already mentioned we propose an experimental investigation, the purpose of which is to compare theoretical and experimental results. Hence, we devise a physical representation of a main beam with a delamination at its free end. By choosing a small coefficient of restitution in the model, and we find that the model is good for predicting physical behaviour for a range of forcing frequencies.
If we set the coefficient of restitution to zero, we obtain an inelastic impact oscillator. We find that the results from the inelastic model, can be applied to the physical system where the coefficient is small but not necessarily zero. In some aspects the mathematical analysis of inelastic impact oscillators is less complicated than for more general impact oscillators, and such analysis is the subject of Chapter 5.

1.2 The impact in further detail

In the initial stages of modelling we used a coefficient of restitution to model the impact. Later on in the thesis we look in greater detail at the impact itself. We cautiously use the coefficient of restitution model; as commented in [27] 'multi DOF models are important for an accurate representation of the actual system behaviour, although a single DOF model captures important first-order information about a lot of the non-linear phenomena in the low-frequency range'. Hence, we develop models of some impacting systems, and by considering wave propagation from the impact we examine impacting behaviour and hence we can document the inaccuracies of the coefficient of restitution model.

When two solids collide, wave propagation occurs in both bodies. The bodies transmit tensile and compressive stress, where the particles move in the direction of the wave propagation. The bodies may also transmit shear stress, where the motion of the particles is transverse to the direction of propagation. It is this continual propagation and subsequent reflections of the waves in a bounded solid that brings about a state of static equilibrium. When the loading rate is slow compared with the transit time of the waves, the wave effects are of no consequence. However, in an impacting situation the wave propagation from the impact is instrumental in determining the behaviour after the initial collision.

The consequence of the assumption that the delamination moves only in the fundamental mode and the main beam has zero curvature, is that impacts occur at the same time along the length of the beam. Without making either of these assumptions, it will almost always be the case that impacts occur at specific single points somewhere along the length of the delamination. The Hertz theory of impact can be applied in situations of this type, [7], [27]. However, this procedure is quite complex, and is valid only if the duration of the impact is long in comparison with the transit time for elastic wave propagation through the thickness of the beam, since the Hertz theory is based on a quasi-static approach.

In the impacting beams the stress waves set up would travel through the depth of each beam away from the contact surfaces. It is then necessary to derive the exact equations of elasticity and thus obtain comprehensive models of the beams. However, in an attempt to simplify the problem we propose a discrete model of a beam where a ‘train’ of masses are interconnected with shear springs. We can use so-called ‘strength of materials’ theories to model the wave propagation. These theories approximate the detailed behaviour of the solid which is otherwise obtainable using the elasticity equations. The motion for simple
shapes can be described by plane wavefronts without consideration of the individual waves spreading from point sources.

Hence, in chapter 6 we consider the impact between a slug and a rod. By considering a small element of each body, and making the assumption that plane sections remain plane through the deformations, we find that the motion of the solid is described by the wave equation. References [8] and [15] discuss wave propagation issues in impacting systems.

The slug and the rod are the building blocks for the model of the main beam and its delamination. The delamination is represented as series of interconnected slugs, and the main beam as a series of unconnected rods. The wave equation is used to model the wave propagation in each element of the model. In principle it is then possible to have impacts occurring where the delamination is not moving in the fundamental mode, i.e., impacts do not occur at the same time along the length of the discretised delamination. However, in chapters 7 and 8 we model the impact between the delamination moving in the fundamental mode and a stationary row of rods. For given parameters we establish the time after the impact that each of the slugs and rods part company (and if they part at the same time), and the parting velocities. Through these results we can evaluate the validity of the coefficient of restitution model of the impact.

1.3 Outline of thesis

In chapter 2 we develop the first model of the system, and discuss some mathematical ideas concerning the impact oscillator. We complete the chapter with a discussion of the computer simulation of the system. In chapter 3, we view a numerical investigation into the dynamics of the overall system. The results in the chapter are assembled from extensive computer simulations. In chapter 4, we describe an experimental investigation into the system, and compare the theoretical and practical results. Letting the coefficient of restitution equal zero, in chapter 5 we investigate the dynamics of the inelastic impact oscillator. Chapter 6 begins the study of the nature of the impact itself. Initially we consider the impact between two bodies, and we introduce many of the wave propagation concepts used in the later chapters. In Chapters 7 and 8 we progress to modelling impacting beams. In Chapter 9, we investigate the impact in a single degree-of-freedom impact oscillator. Finally, in Chapter 10, we conclude this thesis and make some suggestions for future work.
Chapter 2

Mathematical Model

2.1 Introduction

In this chapter we first derive a mathematical model of the system, and then classify the type of motion which we can obtain. A computer simulation is used to produce the results, and some of the issues raised in its implementation are discussed.

The mathematical model of the system is essentially an extension of the model proposed by Kane et al [14], with the free motion of the delamination modified to include a linear damping coefficient. Some of the reference material can be found in many mechanical vibration textbooks, for example [28], [21], and initially we closely follow the model discussed in reference [14].

Figure 2.1: The model configuration
2.2 Derivation

2.2.1 The unconstrained motion of the beam

Initially, we consider the system without impacts, i.e. the unconstrained motion of a beam. The equation of motion is derived and the general solution to this equation is then found. Any assumptions are discussed as they become relevant during the development of the model.

For a beam material obeying Hooke's law there is a simple relationship between the stress and strain in the longitudinal direction of the beam, \( \sigma_x = E\varepsilon_x \). In order to include damping, the beam is modelled as a viscoelastic material, as discussed in [28]. In this case the following stress-strain relationship is defined, where \( \kappa \) is the damping coefficient.

\[
\sigma_x = E\left(\varepsilon_x + \kappa \frac{\partial \varepsilon_x}{\partial t}\right) \tag{2.1}
\]

We assume the beam is uniform and has a rectangular cross-section of area \( A_c \). Its dimensions are shown on the delamination in figure (2.1). The displacement of the neutral axis of any point \( Z \) along the beam \((0 < Z < l)\) is given by \( w(Z, \tilde{t}) \). Figure (2.2) shows an isolated section of the beam, in which the deflection is measured positive downward.

![Figure 2.2: Forces and bending moments acting on a small piece of a beam.](image)

The strain \( \varepsilon_x \) varies through the depth of the beam. For example in figure (2.2), compressive forces (negative strain) act along one side of the beam, and tensile forces (positive strain) act along the other side. At some point in the centre of the beam there is a neutral axis which is unstressed. The variable \( \bar{y} \) measures the distance from the neutral axis through the beam \((-h_s/2 < \bar{y} < h_s/2)\) and in a direction perpendicular to the neutral axis. The angle \( \phi \) indicated in figure (2.2) gives the angle between the neutral axis with the horizontal. At the right hand side of the element, the angle is \( \phi + \frac{\partial \phi}{\partial Z} dZ \), and the change of angle is \( d\phi \).

Therefore, the strain is given by

\[
\varepsilon_x = -\bar{y} \frac{\partial \phi}{\partial Z} \tag{2.2}
\]
The total bending moment $M$ of all the stresses throughout the depth of the beam is

$$M = \int \dot{y} \sigma_z \, dA_c$$

$$= E \left( \frac{\partial \phi}{\partial Z} + \kappa \frac{\partial^2 \phi}{\partial Z \partial \dot{t}} \right) \int \dot{y}^2 \, dA_c$$

$$= EI \left( \frac{\partial \phi}{\partial Z} + \kappa \frac{\partial^2 \phi}{\partial Z \partial \dot{t}} \right)$$

(2.3)

The parameter $I$ is called the second moment of area and for a beam of rectangular cross-section $I = \frac{1}{12} bh^3$.

Consider the free-body diagram in figure (2.2); $M(Z, \dot{t})$ is the bending moment and $S(Z, \dot{t})$ is the shear force, and these vary along the length of the section. Neglecting the second order terms, the moment equilibrium equation taken about the left hand side of the element results in the following expression

$$S = \frac{\partial M}{\partial Z}$$

(2.4)

Now, we resolve forces in the $\ddot{y}$ direction, and using the above expression for the shear force $S$, we obtain

$$\frac{\partial^2 M}{\partial Z^2} = \rho A_c \frac{\partial^2 \dot{\ddot{y}}}{\partial t^2}$$

(2.5)

We can substitute the expression for $M$ from equation (2.3) into the equation above. However, the angle $\phi$ satisfies $\tan(\phi) = \frac{\dot{y}}{\dot{Z}}$ and for small angles, $\tan(\phi) = \phi$. Therefore we can substitute for $\phi$ in equation (2.3) and using $A_c = bh$, we obtain

$$\frac{Eh^2}{12\rho} \left[ \frac{\partial^4 \ddot{y}}{\partial Z^4} + \kappa \frac{\partial^5 \ddot{y}}{\partial Z^5 \partial \dot{t}} \right] + \frac{\partial^2 \ddot{y}}{\partial \dot{t}^2} = 0$$

(2.6)

This is the equation for the damped vibration of a beam. To find a solution it must be constrained with the appropriate boundary and initial conditions. In this case the delamination is effectively a cantilever beam, i.e., clamped to the main beam at one end and free at the other. The movement of the main beam is also governed by equation (2.6). However, we assume at this stage that since the length of the delamination is small compared with the main beam, the curvature of the main beam is negligible along the length of the delamination, and can be ignored. We can therefore visualise the motion of the main beam as an oscillating straight beam with the frequency and amplitude of the oscillations variable.

The boundary conditions for the delamination equation are expressed as

$$\begin{align*}
\ddot{\ddot{y}} &= \ddot{F}(\dot{t}), \quad \frac{\partial \ddot{y}}{\partial Z} = 0 \quad \text{at } \dot{Z} = 0 \\
\frac{\partial^2 \ddot{y}}{\partial Z^2} &= 0, \quad \frac{\partial^3 \ddot{y}}{\partial Z^3} = 0 \quad \text{at } \dot{Z} = l
\end{align*}$$

boundary conditions

(2.7)
The forcing from the main beam is identified as the term $F(t)$, dictating the displacement at the clamped end of the cantilever. Note, as of yet we have not placed constraints on the motion of the delamination.

A simple, but useful change of variable occurs now, and introduces the forcing of the delamination into its actual equation of motion. Namely,

$$\ddot{x}(\tilde{Z}, t) = \ddot{F}(t) - \ddot{w}(\tilde{Z}, t)$$  \hspace{1cm} (2.8)

The beam equation and boundary conditions in terms of the new variable $x(\tilde{Z}, t)$ are as follows,

$$\frac{Eh^2}{12\rho} \left[ \frac{\partial^4 x}{\partial \tilde{Z}^4} + \kappa \frac{\partial^2 x}{\partial \tilde{Z}^2 \partial t^2} \right] + \frac{\partial^2 x}{\partial t^2} - \ddot{F}(t) = 0$$  \hspace{1cm} (2.9)

$$\begin{cases} \dot{x} = 0, \quad \frac{\partial \dot{x}}{\partial \tilde{Z}} = 0 & \text{at } \tilde{Z} = 0 \\ \frac{\partial^2 \dot{x}}{\partial \tilde{Z}^2} = 0, \quad \frac{\partial^3 \dot{x}}{\partial \tilde{Z}^3} = 0 & \text{at } \tilde{Z} = l \end{cases}$$  \hspace{1cm} (2.10) boundary conditions

The solution of equation (2.9) may be written in the form

$$\ddot{x}(\tilde{Z}, t) = \sum_{i=1}^{\infty} \ddot{\phi}_i(t) L_i(\tilde{Z})$$  \hspace{1cm} (2.11)

Hence, the solution is expressed as the sum of movements in each mode of vibration. The natural frequency of each mode is $\omega_i$, which has an associated mode shape function $L_i(\tilde{Z})$. The mode shapes and frequencies can be found from the undamped, unforced system. Therefore setting $\kappa = 0$ and $\ddot{F}(t) = 0$ in equation (2.9), we substitute a solution of the form

$$\ddot{x}(\tilde{Z}, t) = \sum_{i=1}^{\infty} B_i L_i(\tilde{Z}) \sin(\omega_i t + \phi_i)$$  \hspace{1cm} (2.12)

This results in the following equation which each shape function must satisfy,

$$\frac{Eh^2}{12\rho} \frac{d^4 L_i(\tilde{Z})}{d\tilde{Z}^4} - \omega_i^2 L_i = 0$$  \hspace{1cm} (2.13)

The shape function for each mode is obtained from the above equation, subject to the boundary conditions, equation (2.10). The functions $\ddot{\phi}_i(t)$ must now be found. Substituting equation (2.11) into equation (2.9), we obtain

$$\sum_{i=1}^{\infty} \left[ \frac{Eh^2}{12\rho} \left( \frac{\partial^4 L_i(\tilde{Z})}{\partial \tilde{Z}^4} + \kappa \frac{\partial \ddot{\phi}_i(t)}{\partial t} \frac{\partial^4 L_i(\tilde{Z})}{\partial \tilde{Z}^4} \right) + \frac{\partial^2 \ddot{\phi}_i(t)}{\partial t^2} L_i(\tilde{Z}) \right] - \ddot{F}(t) = 0$$  \hspace{1cm} (2.14)
An important property of the shape functions is their orthogonality over the range \((0, l)\). To use the orthogonality property, equation (2.14) is multiplied by the shape function \(L_j(\tilde{Z})\) and integrated over \((0, l)\) to obtain,

\[
\sum_{i=1}^{\infty} \frac{EH^2}{12\rho} \left[ \tilde{C}_i(t) \int_0^l L_j(\tilde{Z}) \frac{\partial^4 L_i(\tilde{Z})}{\partial \tilde{Z}^4} d\tilde{Z} + \kappa \frac{\partial \tilde{C}_i(t)}{\partial t} \int_0^l L_j(\tilde{Z}) \frac{\partial^4 L_i(\tilde{Z})}{\partial \tilde{Z}^4} d\tilde{Z} \right] + \frac{\partial^2 \tilde{C}_i(t)}{\partial t^2} \int_0^l L_j(\tilde{Z}) L_i(\tilde{Z}) d\tilde{Z} = \tilde{F}(t) \int_0^l L_j(\tilde{Z}) d\tilde{Z} \tag{2.15}
\]

The evaluation of the integrals is contained in Appendix A, together with the definitions of some parameters.

These details simplify equation (2.15) into the following form,

\[
\frac{d^2 \tilde{C}_i(t)}{dt^2} + \kappa \omega_i^2 \frac{d \tilde{C}_i(t)}{dt} + \omega_i^2 \tilde{C}_i(t) = \tilde{F}(t) \frac{2\theta_i}{\nu_i} \tag{2.16}
\]

The solution \(\tilde{C}_i(t)\) satisfying the above equation can be found for each mode of vibration. The definitions of \(\theta_i\) and \(\nu_i\) are found in Appendix A. The forcing term is a sine function of amplitude \(A\) and frequency \(\tilde{\omega}\); \(\tilde{F}(t) = -A \sin(\tilde{\omega}t)\). Thus

\[
\tilde{F}(t) = A\tilde{\omega}^2 \sin(\tilde{\omega}t) \tag{2.17}
\]

### 2.2.2 Introducing the impact

The solution of the equation of motion for the beam, equation (2.11), is now fully expressed, comprising of motion in each mode of vibration. However, since the delamination is attached to the main beam, its movement is constrained, and we now consider the implications.

The variables \(\tilde{\omega}(\tilde{Z}, \tilde{t})\) and \(\tilde{F}(\tilde{t})\) measures the absolute displacement of the delamination and the main beam respectively. The variable \(\tilde{x}(\tilde{Z}, \tilde{t})\) measures the relative displacement between the two components. Therefore in the impacting system it will be impossible to obtain a solution with both positive and negative values of \(\tilde{x}(\tilde{Z}, \tilde{t})\); an impact between the main beam and its delamination will be indicated by a zero value of \(\tilde{x}(\tilde{Z}, \tilde{t})\). At this stage we agree that a 'physical' solution is such that \(\tilde{x}(\tilde{Z}, \tilde{t})\) is positive.

At this stage we make another assumption. In a practical situation the forcing frequency, \(\tilde{\omega}\) is likely to be much less than fundamental frequency, \(\omega_1\) of the delamination; certainly as an upper limit the forcing frequency is unlikely to exceed \(2\omega_1\). Therefore in this range of frequencies the contribution in equation (2.11) from modes other than the fundamental mode would appear insignificant. This is a very important concession. If we assume that the delamination operates entirely in the fundamental mode, any contact between the main beam and the delamination always occurs along the entire length of the delamination. The impact can now be very simply modelled; a restitution relationship between the impacting
and rebounding velocities can be constructed. With these new assumptions, equation (2.16) becomes

\[
\frac{d^2 \dot{C}(t)}{dt^2} + \kappa \omega_1^2 \frac{d \ddot{C}(t)}{dt} + \omega_1^2 \dot{C}(t) = A \omega^2 \sin(\omega t) \frac{2 \theta_1}{\nu_1}
\]  

(2.18)

For the full impacting system, we are primarily interested in the character of the solutions. By non-dimensionalising equation (2.18) we eliminate the physical coefficients. We employ the following change of variables,

\[
t = \omega_1 \tilde{t}
\]

(2.19)

\[
\frac{2 \theta_1}{\nu_1} A \omega^2 C(t) = \ddot{C}(\tilde{t})
\]

(2.20)

\[
\omega = \frac{\omega}{\omega_1}
\]

(2.21)

\[
\gamma = \kappa \omega_1
\]

(2.22)

The parameter \( \gamma \) is the non-dimensional damping parameter. The frequency parameter is expressed in a useful form; a forcing frequency of \( \omega = 1 \) corresponds to a forcing at the fundamental frequency of the delamination. We can compute the non-dimensional forcing frequencies corresponding to the frequencies of the higher modes. Equation (A.9) states that the frequency of the \( i \)'th mode of vibration of the delamination is,

\[
\tilde{\omega}_i = (\beta_i l)^2 \frac{E}{2 h s} \sqrt{\frac{E}{3 \rho}}
\]

(2.23)

Hence, using \( \omega = \tilde{\omega}/\omega_1 \), the non-dimensional forcing frequencies for the first 4 modes of the delamination are shown in Table (2.1).

<table>
<thead>
<tr>
<th>mode</th>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>6.2669</td>
</tr>
<tr>
<td>3</td>
<td>17.5475</td>
</tr>
<tr>
<td>4</td>
<td>34.3861</td>
</tr>
</tbody>
</table>

Table 2.1: The non-dimensional frequencies of the modes of vibration of the delamination

After non-dimensionalising equation (2.18), the equation of motion for the delamination is governed by the following second-order linear differential equation,

\[
\frac{d^2 C(t)}{dt^2} + 2 \gamma \frac{dC(t)}{dt} + C(t) = \sin(\omega t)
\]

(2.24)

As mentioned previously, a physical solution is one where \( \bar{x}(\tilde{Z}, \tilde{t}) \) is positive. Since the shape function \( L_1(\tilde{Z}) \) is positive for \( 0 < \tilde{Z} \leq \tilde{t} \), the function \( C(t) \) must also be positive. When \( C(t_i) = 0 \) an impact has occurred. At this point we apply the following rule,

\[
\dot{C}^+(t_i) = -r \dot{C}^- (t_i)
\]

(2.25)
Thus, the delamination undergoes an instantaneous change in direction with its rebounding speed a fixed fraction of the impacting speed, and \( r \) is the coefficient of restitution.

As the system is invariant under the map \( t \rightarrow t + T \), where \( T = \frac{2\pi}{\omega} \) is the forcing period, we can define the phase, \( \phi \) by

\[
\phi = t \mod \left( \frac{2\pi}{\omega} \right) \tag{2.26}
\]

This is a useful definition; it is often beneficial to work with the phase as opposed to absolute time.

2.2.3 Solution of the differential equation

Assuming an impact occurs at \( t = t_0 \) with a velocity \( v_0 \), the rebounding velocity is \( -rv_0 \). The immediate task is to find the solution to equation (2.24), together with these initial conditions. Hence, we solve the following differential equation,

\[
\frac{d^2 C(t)}{dt^2} + 2\gamma \frac{dC(t)}{dt} + C(t) = \sin(\omega t) \tag{2.27}
\]

using the following initial conditions,

\[
C = 0 \quad \text{at} \quad t = t_0 \\
\frac{dC}{dt} = -rv_0 \quad \text{at} \quad t = t_0 \tag{2.28}
\]

The general solution of the differential equation is

\[
C = K_1 e^{-\gamma t} \sin(\omega_d t) + K_2 e^{-\gamma t} \cos(\omega_d t) + A \sin(\omega t - \psi) \tag{2.29}
\]

The functions \( A \) and \( \psi \) are the amplitude and phase shift of the particular integral, \( \omega_d = \sqrt{1 - \gamma^2} \) is the damped natural frequency and the constants \( K_1 \) and \( K_2 \) must be determined.

Inserting the initial conditions into the above expression and its derivative, gives the following equation system, from which the constants are found.

\[
\begin{bmatrix}
    e^{-\gamma t_0} \sin(\omega_d t_0) \\
    \omega_d e^{-\gamma t_0} \cos(\omega_d t_0) - \gamma e^{-\gamma t_0} \sin(\omega_d t_0) \\
    -\gamma e^{-\gamma t_0} \cos(\omega_d t_0) - \omega_d e^{-\gamma t_0} \sin(\omega_d t_0)
\end{bmatrix}
\begin{bmatrix}
    K_1 \\
    K_2
\end{bmatrix}
\]

\[=\begin{bmatrix}
    -A \sin(\omega t_0 - \psi) \\
    -rv_0 - A\omega \cos(\omega t_0 - \psi)
\end{bmatrix} \tag{2.30}\]

The solution to equation (2.27), the amplitude \( A \) and phase-shift \( \psi \), is found to be,

\[
C(t) = \left( \frac{-1}{\omega_d} \right) e^{-\gamma(t-t_0)} \left[ A\omega_d \sin(\omega t_0 - \psi) \cos(\omega_d(t - t_0)) + A\gamma \sin(\omega t_0 - \psi) \sin(\omega_d(t - t_0)) + A\omega \cos(\omega t_0 - \psi) \sin(\omega_d(t - t_0)) \right] + A \sin(\omega t - \psi) \tag{2.31}\]
where

\[
A = 1/\sqrt{(1 - \omega^2)^2 + (2\gamma \omega)^2} \\
\psi = \tan^{-1} \left( \frac{2\gamma \omega}{1 - \omega^2} \right)
\]

The velocity \( \dot{C}(t) \) can be found directly from the expression for \( C(t) \) by differentiation.

Whilst implementing a computer simulation for the system, it was found essential to work in terms of the phase \( \phi = t \mod (\frac{2\pi}{\omega}) \). As a simulation progresses, the time of each impact \( t_0, t_1, \ldots, t_n \) obviously increases. A problem arises since the computer implementation of the trigonometric functions becomes increasingly inaccurate for large arguments. We make adjustments to equation (2.31), thus

\[
C(t) = \left( \frac{-1}{\omega_d} \right) e^{-\gamma t_{rel}} \left[ A \omega_d \sin(\omega \phi_0 - \psi) \cos(\omega_d t_{rel}) + A \gamma \sin(\omega \phi_0 - \psi) \sin(\omega_d t_{rel}) + A \omega \cos(\omega \phi_0 - \psi) \sin(\omega_d t_{rel}) + r v_0 \sin(\omega_d t_{rel}) \right] + A \sin(\omega t_{act} - \psi)
\]

(2.32)

where we define,

\[
t_{rel} = t - t_0 \\
\phi_0 = t_0 \mod (\frac{2\pi}{\omega}) \\
t_{act} = (\phi_0 + t_{rel}) \mod (\frac{2\pi}{\omega})
\]

(2.33) (2.34)

This operates on the phase of impact times, and time relative to the impacts, rather than the absolute time. This eliminates many numerical problems which were previously encountered when using equation (2.31). Of course, a record of absolute time must be kept which is used mostly for output purposes.

The model derivation is now complete; we have two equations, one describing the motion between impacts (2.32), and the second modelling the impact itself, (2.25). The remaining sections of the chapter describe the finer points of using the model to find actual solutions, and computer implementation.

2.3 Discussion

As discussed in the introduction, the system has been reduced to a single degree of freedom impact oscillator; this type of system is studied as a mathematical system in its own right. We now review and apply some of the general terminology and language used when discussing this type of system.
We assume equation (2.24) is valid for $C(t) > 0$. Research literature into impact oscillators often includes another parameter, namely a clearance parameter. If a clearance parameter, say $\beta$, is included, equation (2.24) would be valid for $C(t) > \beta$, where impacts occur at $C(t) = \beta$. However in this situation a clearance parameter is not needed, since impacts occur when the relative displacement $C(t)$ is zero.

2.3.1 The phase space and the impact map

We can define the phase space $\Sigma$ for the system as follows,

$$\Sigma = [0, \frac{2\pi}{\omega}] \times (-\infty, 0]$$  \hspace{1cm} (2.35)

Any point $(\phi, v) \in \Sigma$ describes a unique impacting situation and satisfies,

$$C(\phi_0) = 0 \text{ and } \dot{C}^- (\phi_0) = v_0$$  \hspace{1cm} (2.36)

This represents an impact at phase $\phi_0$ with an impacting velocity $v_0$. In order to find the details of the next impact, two things must be done:

1. The rebounding velocity, $-rv_0$ must be found, by applying the relationship in equation (2.25).

2. Now, given the initial conditions,

$$C(\phi_0) = 0 \text{ and } \dot{C}^+ (\phi_0) = -rv_0$$  \hspace{1cm} (2.37)

the differential equation (2.24) describes the motion of the delamination until the next impact. We can find an analytical solution to (2.24), and then using a numerical scheme we can find the time, $t_1$, and hence the phase $\phi_1$, of the next impact. We can compute the impacting velocity, $\dot{C}^- (\phi_1) = v_1$; it is easy to find an analytical expression for the velocity, $\dot{C}(t)$.

This process is depicted graphically in figure (2.3). The impacts occur at times $t_0$ and $t_1$, corresponding respectively to phases $\phi_0$ and $\phi_1$.

We can define a mapping summarising the above process, $P = F \circ G$. The mapping $G$ performs the first step above, i.e., $G(\phi_0, v_0) = (\phi_0, -rv_0)$. The mapping $F$, performs the second step $F(\phi_0, -rv_0) = (\phi_1, v_1)$. Together, the action of $P$ maps one impact to the next.

For a given set of parameters (restitution $(r)$, frequency $(\omega)$ and damping $(\gamma)$), and initial conditions $(\phi_0, v_0)$, we can find an overall trajectory, for the movement of the delamination. The mapping can be repeatedly applied, with the details of each impact used as initial conditions to find the next impact. This trajectory is recorded as a sequence of impacts, $(\phi_0, v_0), (\phi_1, v_1), \ldots, (\phi_n, v_n)$, where $(\phi_n, v_n) = P^n(\phi_0, v_0)$. Hence, the sequence of impacts in a simulation can also be expressed as $(\phi_0, v_0), P(\phi_0, v_0), P^2(\phi_0, v_0), \ldots$. 

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If a zero velocity impact occurs and the acceleration $\dot{C}(t)$ is negative, the delamination 'sticks' to the main beam (the delamination travels in contact with the main beam), until the acceleration changes sign. From equation (2.24), at a zero velocity impact, $(C(t_i) = \dot{C}(t_i) = 0)$, stick occurs if

$$\frac{d^2 C(t_i)}{dt^2} = \sin(\omega t_i) < 0$$

i.e., if $\frac{\pi}{\omega} < \phi_i < \frac{2\pi}{\omega}$, where $\phi_i$ is the phase of the impact at time $t_i$. We can define the sticking region, thus

$$I = \left(\frac{\pi}{\omega}, \frac{2\pi}{\omega}\right) \times 0$$

(2.39)

If a particular impact $(\phi_0, v_0)$ maps under the action of $P$ to an element of $I$, there arises a slight failing of the definition of $P$. Since an impact in the sticking region remains stuck until the end of the sticking region, the definition of $P$ must be restated for this special case. Thus for points $(\phi_0, v_0)$ in phase-space which map to the sticking region $I$, the following is true,

$$P : (\phi_0, v_0) \rightarrow (\phi_1, 0) \rightarrow \left(\frac{2\pi}{\omega}, 0\right) \rightarrow (0, 0)$$

(2.40)

Due to this adjustment to the mapping $P$, it is not a one-one mapping, and also is not invertible at particular points in phase-space.

Although we cannot write an explicit expression for the mapping $P$, we may do so for the Jacobian matrix of the mapping. The Jacobian matrix gives a linearised version of the mapping, which is valid for a small region about the point in phase space at which it is evaluated [9]. The determinant of the Jacobian matrix measures the change in area of a small region of phase space. The expression for the Jacobian matrix is given by,

$$DP(\phi_0, v_0) = \begin{bmatrix}
\frac{\partial \phi_1}{\partial \phi_0} & \frac{\partial \phi_1}{\partial v_0} \\
\frac{\partial v_1}{\partial \phi_0} & \frac{\partial v_1}{\partial v_0}
\end{bmatrix}$$

(2.41)
The derivation is described by Budd et al in [2], where the determinant of the Jacobian is also calculated (assuming zero damping) thus,

$$\det(DP(\phi_0, v_0)) = \frac{r^2 v_0}{v_1}$$  \hspace{1cm} (2.42)

The computation of the Jacobian and its determinant can be repeated with the inclusion of a damping parameter. This was achieved symbolically with a computer program, and although the expression for the Jacobian is far too long to be written here, the determinant can be simplified to,

$$\det(DP(\phi_0, v_0)) = \frac{r^2 v_0 e^{-2\gamma(t_1-t_0)}}{v_1}$$  \hspace{1cm} (2.43)

where \(t_0\) and \(t_1\) are the absolute times of the two impacts with velocities \(v_0\) and \(v_1\) respectively.

One important type of motion exhibited by the impact oscillator is known as chatter which involves 'sticking'. This can be described as a series of progressively smaller 'bounces' of the delamination, rather like a ball bouncing on a table. Eventually the small bounces become negligible bounces and the ball comes to a stop. At this stage if the phase is in the sticking region (the impacting velocity is zero), the delamination 'sticks' to the main beam. This issue is addressed in further detail later in section 2.4.1.

Regarding the initial conditions \((\phi_0, v_0)\), it is important to note the following. Immediately before the forcing of the main beam is initiated the delamination and the main beam would be stationary with the relative amplitude zero. This would correspond to a starting condition of \((\phi_0, v_0) = (0, 0)\) at \(t = 0\). For many of the results presented later we assume this zero starting condition. However, sometimes we also consider non-zero starting conditions.

### 2.3.2 Attracting solutions

We now move on to introduce the possible motions of the system. Examples are shown which are obtained from computer simulations. Details of the computer implementation are discussed in the next section.

We concentrate on the case where the restitution \(r < 1\). If \(r = 1\) and \(\gamma = 0\) the system is energy preserving or Hamiltonian. This is a rather special case and will not receive any attention here. If \(r \neq 1\), the system is dissipative. From any initial condition the motion of the delamination will settle upon an attracting solution or an attractor. This type of process is well documented in many nonlinear dynamics books, see, for example, Moon [16]. Once this attracting state has been reached, the solution is in its steady state. By analogy, in a forced mass-spring-damper system, after the transient response has decayed, the steady state solution remains.

For a given choice of parameters there are a number of attracting solutions, the eventual choice of which is dependent on the initial conditions. The quantitative type of these
attractors for the impact oscillator can be classified into three types, which are detailed below.

Periodic solutions

![Figure 2.4: A periodic (1, 1) trajectory (w=1.4, r=0.3, γ=0)](image)

The motion of the delamination is periodic if it repeats after a time period \( T \) some multiple \( n \) of the forcing period. i.e.,

\[
T = n \frac{2\pi}{\omega}
\]

In every period of the motion there will be \( m \) impacts and therefore we can refer to the trajectory as a \((m, n)\) periodic trajectory or orbit. An example of a periodic \((1, 1)\) orbit is shown in figure (2.4). Occasionally in this text, we shall refer to period \( n \) motion, which is a notation for periodic motion with a period of \( n \) forcing periods.

Note that in the plots of figure (2.4) and the trajectory plots used for the remainder of the text, the top plot will generally be used to display a particular trajectory. The bottom-right plot is a simple plot of the relative amplitude \( C(t) \) against \( t \) (as defined previously, \( C(t) \) is always positive). The bottom-left plot, commonly known as a phase-plane plot, shows the relationship between the displacement, \( C(t) \) and velocity, \( \dot{C}(t) \). The top plot shows
the absolute displacement of the main beam and the delamination. The displacement of the main beam, is simply scaled to appear visible on the plot; it is incorrect to compare the relative amplitudes of the main beam and the delamination. These plots observe the general relationship expressed by the dimensional equation (2.8), i.e., relative displacement between the beam and delamination is absolute displacement of the main beam minus the absolute displacement of the delamination. The intention of showing the displacement of the main beam was to make the phase of each impact immediately obvious.

Chattering solutions

![Figure 2.5: A chattering \((\infty, 1)\) trajectory \((w=0.45, r=0.6, \gamma=0)\)](image)

A chattering motion is actually a periodic motion. However this type of motion is sufficiently interesting to justify its inclusion here. There are an infinite number of impacts per period. A chattering orbit can be classified thus \((\infty, n)\). An example trajectory is shown in figure (2.5).
Chaotic solutions

In a chaotic orbit no point in phase space is visited more than once; if this did occur the orbit would be periodic. The iterates \((\phi_t, v_t)\) of a chaotic orbit are confined to a strange attractor, which is a subset of the entire phase space. An example chaotic trajectory is shown in figure (2.6).

It is quite instructive to plot graphically the impact map on to the phase space and thus visualise the nature of the strange attractor, which often has a well-defined fractal shape as shown in figure (2.7).

![Figure 2.6: A chaotic trajectory \((w=2.8, r=0.8, \gamma=0)\)](image)

2.4 Implementation

It is an apparently simple task to iterate the map as a computer simulation and hence find the trajectory of the delamination for given parameters and initial conditions. Whilst this is true, it will be necessary to run the simulation for various sets of parameters and initial conditions to form an overall view of the performance of the system, and hence the program must be robust for a wide range of parameter and initial condition choices.

From equation (2.32), and a point in the phase space specifying the initial condition, a numerical scheme to implement the implicit function \(F\) can be constructed. This must find the smallest root, \(t_1 > t_0\) of (2.32). This is achieved using a simple bisection method. Sometimes the solution may pass through \(C = 0\) many times in a small time interval, and we must take care to find the first instance where \(C(t) = 0\). To help ensure this, firstly we can check that for any solution at time \(t_1\), the velocity evaluated at this point is negative. Secondly, we can check that for \(t < t_1\) and near \(t_1\) the solution \(C(t)\) is always positive.
2.4.1 Chattering motion

We have to specify some criteria by which a chattering orbit is to be recognised. This is absolutely essential; if a chattering trajectory is 'left to its own devices' numerical problems will almost certainly ensue. A chattering trajectory is a sequence of decreasingly low velocity impacts. One solution would be to count the number of consecutive low velocity impacts, once a specified low impact velocity threshold has been broken. Now, once the count has passed some other pre-determined value, we can strongly suspect chattering motion. The pre-determined value will be dependent on the restitution. For higher values of \( r \), a larger number of small 'bounces' occur before we can assume the impact is sufficiently close to a sticking solution. Therefore in the computer simulation the pre-determined value is dependent on the value of the restitution. Through experience of developing the program, we are able to predict the relationship between the coefficient of restitution and the number of low velocity impacts before sticking. This relationship is shown in figure (2.8).

Assuming we have now recognised a chattering trajectory we can approximate the sticking time; at this stage although the motion of the delamination consists of very small bounces, it is not 'stuck'. The technique for approximating the sticking time is described by Budd and Dux [3] and is repeated here with the addition of the damping parameter.

Suppose that the last impact was at \((\phi_0, v_0)\). From equation (2.27), we calculate the acceleration at the impact \( \ddot{C} = \sin(\omega t) - 2\gamma \dot{C} \), and call it \( g \). During the chattering region we
assume that the acceleration \( g \) is constant.

We expect a small 'bounce' before the next impact. From the equation of motion \( \ddot{C} = g \) we find the time to the next impact is \( \frac{2rv_0}{g} \), and the new impact velocity is \( v_1 = rv_0 \). Again, the acceleration at this impact is \( g \), and hence we find that the time of the second 'bounce' is \( \frac{2r^2v_0}{g} \), and the new impacting velocity is \( v_2 = r^2v_0 \).

An infinite sequence of such impacts accumulates in a zero velocity impact in a finite time. The phase at which this happens is \( \phi_\infty \), and this can be computed by summing a geometric series of the time of each bounce. Hence,

\[
\phi_\infty = \phi_0 + \frac{2rv_0}{g(1 - r)} \tag{2.44}
\]

A chattering orbit must be periodic, since after a region of chatter the delamination sticks to the main beam until the end of the sticking region where the acceleration becomes positive. This point in phase space is always precisely revisited after each chattering region, and thus must be part of a periodic orbit.

### 2.4.2 Identification of the various types of motion

For each individual simulation, the software must recognise some general properties of the orbit. Clearly it must distinguish between a periodic, chattering or a chaotic orbit. For the chattering or periodic orbit the period of the orbit as a multiple \( n \) of the forcing period must be found. In addition the number of impacts \( m \) per period must be found, or if chatter occurs, this must be separately recorded. As mentioned previously, after the transient response has decayed, the orbit settles on to some type of attracting orbit. An informal description of this process of recognition now follows.

- The simulation allows an arbitrary number of impacts to occur, typically 200, to allow the solution to settle onto an attracting orbit. During these initial impacts the program finds the maximum impacting velocity.

![Figure 2.8: Selecting the number of low velocity impacts before chatter](image-url)
For an \( m \) impact periodic orbit, \((\phi_i, v_i) = (\phi_{i+m}, v_{i+m})\). However in numerical simulations these numbers are never exactly the same and we have to specify a criterion by which, if they are close enough, we assume they are referring to the same point in phase space. Therefore we assume two points \((\phi_i, v_i)\) and \((\phi_j, v_j)\) in phase space are the same if the following is satisfied,

\[
|\phi_i - \phi_j| < \frac{(2\pi)}{10^7} \\
|v_i - v_j| < \frac{|v_{\text{max}}|}{10^7}
\]

where \(v_{\text{max}}\) is the maximum impacting velocity and the number \(10^7\) is a choice of 'large number', selected through experience of using the program. These conditions specify the required 'closeness' of two points in phase space so that they can be called equal. The computer works to a degree of numerical accuracy and hence if this condition is too strict, i.e., if we use a number larger than \(10^7\), periodic orbits may go undetected. This is because although analytically a periodic orbit revisits precisely the same point every \( m \) iterations of the map, in practice there is always some degree of numerical error, and if the 'closeness' condition is too strict, identification problems may arise. Conversely, if we use a number smaller than \(10^7\), separate but close points on a periodic orbit may incorrectly be assumed to be the same point. The choice of \(10^7\) achieves a happy balance when using double-precision (8 byte) floating-point variables and the Solaris Fortran 77 compiler.

At some point we must assume the transient response has disappeared, and then start the process of identifying the type of the attractor. We record some impact as a reference impact, and compare subsequent impacts with this reference. For a periodic orbit to be successfully recognised, we specify that an arbitrary number of 20 periods of the orbit must be recognised. This corresponds to 20 impacts identically matched with the reference impact. This is quite a strict condition. Sometimes there might be a few successful matches with the reference and then no further matches. A likely reason for this is that the transient has not yet fully disappeared; there is no way of predicting how many impacts occur before the transient has fully decayed. However, if the recognition condition is not fully satisfied, we reset the reference impact and the recognition process is restarted.

As described in section (2.4.1), it is a numerical necessity to recognise chatter. For recognition purposes it is simply a case of setting a 'chatter flag', when chatter is found.

At some stage if no impacts match the reference impact, and a large number of impacts have occurred, we suspect that the impact map is moving on a strange attractor and we define the orbit as chaotic.
The writing of the program was more complicated than initially expected. We discuss an example to illustrate the types of problem encountered. For parameters \( \omega = 0.5, r = 0, \gamma = 0 \) and zero initial conditions, figure (2.9) shows the steady state trajectory.

![Figure 2.9: A trajectory with completely inelastic impacts (\( w=0.5, r=0, \gamma=0 \))](image)

The case of inelastic impacts was discussed by Kane et al [14], and in chapter 5 of this thesis. In figure (2.9) the motion is periodic \((1, 1)\), where theoretically the impact should occur at a phase of exactly half the forcing period, which is just at the start of the sticking region. Since the rebounding velocity is zero, the delamination, should stick for the entire sticking region. However, problems arise since the program predicts numerically, the impact at a phase slightly less than half the forcing period, and thus inaccuracies then occur. The problem is rectified by checking if the impact phase is very close to the start of the sticking region with a low rebounding velocity, and if so, the phase is adjusted so that it is at the start of the sticking region. This is a necessary adjustment.

Figure (2.10) gives an example of a \((\infty, 3)\) chattering trajectory. For typical parameter choices this type of trajectory is relatively rare; period 1 chattering motion is far more common. However, for particular choices of parameters this type of motion is encountered, and it sometimes presents numerical difficulties. This happens because the first impact in the chattering progression is often at a low velocity. Thus, before the required number of impacts for chatter have occurred the bounces become so small that numerical problems ensue. This problem can be solved in part by testing for a very low velocity impact together with the condition that there must have been at least half the chattering impacts recognised. However, this type of numerical problem can still arise when initial conditions are specified very near the sticking region, i.e., \( v_0 \) small and \( \phi_0 \in (\pi, 2\pi/\omega) \), and this can be a problem when producing a basin of attraction plot (as we shall do in section 3.2).
In this chapter we developed a mathematical model of a laminated beam, with a single delamination at its free end. We made a number of assumptions, which we can summarise.

- The main beam has zero curvature along the length of the delamination.
- The delamination moves in the fundamental mode only, and the impacts occur at the same time along the length of the delamination. Also, after the impact the delamination moves in the fundamental mode.

We are particularly cautious about the second assumption, and through the experimental investigation (Chapter 4) and modelling the impact of a discrete model of a beam (Chapters 7, 8), we establish the validity of this assumption.

We expressed the essential dynamics of the system using a mapping which relates one impact to the next, \( \mathcal{P}(\phi_i, v_i) = (\phi_{i+1}, v_{i+1}) \). We discussed the computer implementation of the mapping, and through repeated applications of the mapping we performed simulations of the system. Some of the issues involved were discussed, including computer recognition of the distinct types of motion, and numerical problems arising due to floating-point precision.
In the next chapter we move on from the simulations for specific parameter choices, and investigate the overall behaviour of the system for a range of parameters.
Chapter 3

Computed Results

3.1 Introduction

In the previous chapter for specific choices of parameters and initial conditions, we observed different types of motion. In this chapter we use the mathematical model developed in the previous chapter and consider the global dynamics of the system, including the transitions between the different types of motion. For example, for fixed zero initial conditions we investigate the effect on the behaviour of the system of varying one or perhaps two parameters.

Since we have three parameters and initial conditions in the system, all of which can be varied, the scope for investigation is extensive. In this chapter, we discuss some graphical results obtained through large numerical simulations. At the end of the chapter we discuss the mapping $P$, highlighting some of its features and indicating some starting points for further investigation.

3.2 Basin of attractions

If we fix the frequency, restitution and damping, and vary the initial conditions we can study the various attractors present at this particular parameter choice. Often there will be two or more attractors and different initial conditions will tend towards one particular attracting orbit. We can split the phase space into a regular 'grid' of initial conditions, run the simulation for each initial condition, classify the eventual attracting orbit and indicate this on a plot of phase space. A colour scheme with an associated key is used to distinguish between the types of attractor. The plot produced is called a basin of attraction plot.

The entire phase space is defined by $\Sigma$ in equation (2.35). To produce a basin of attraction plot we consider a finite region of phase space. An example basin of attraction can be seen in figure (3.1), with a phase space $[0, \frac{2\pi}{\omega}] \times (-20, 0)$ and where the values of the parameters
are indicated in the figure. There are two competing periodic attractors, which are,

- The dark region. (4, 2) periodic motion.
- The light region. (10, 4) periodic motion.

The position in phase space of the attracting orbits are shown as circles and crosses for the dark and light regions, respectively. The pattern of the competing basins of attraction form an intricate and often fractal shape. This procedure can be repeated for any choice of parameters to discover the attractors which are present.

### 3.3 Bifurcation diagrams

A *bifurcation* is defined as a quantitative change of behaviour. For example, as a parameter is increased, the motion may change from periodic to chaotic motion. These changes may be indicated on a *bifurcation diagram*.

We can choose to vary either two parameters or just one when producing bifurcation diagrams, and examples of both are shown in the next sections. When there are two variable parameters, the two axes of the graph form the parameter space and every point in this space is coloured according to the type of motion at that point in the parameter space. However when only one parameter is variable, a single axis forms the parameter space and
hence we can use the second axis to display more information. This is shown in the following bifurcation diagrams with one parameter variable, we plot impact phases or velocities on the second axis.

In every bifurcation diagram, for each set of parameters we start the simulation with zero initial conditions (i.e. from rest).

3.3.1 Bifurcation plot with two parameters variable

In the first type of bifurcation diagram we fix one parameter and allow the other two to vary independently. For each choice of parameter in the parameter space we classify the nature of the attracting orbit, and colour a small area in the parameter space accordingly; a grid is used to separate the space into small areas and these areas are coloured according to the key of figure (3.2). Hence in the graphs of figures (3.3), (3.4) and (3.5), we fix the damping parameter (indicated on each plot) and plot a graph of the frequency-restitution space. The intention is to show the effect of the damping on the motions observed in the system.

![Figure 3.2: Key for the bifurcation plots with two parameters variable.](image)

We make the following notes about the key. Chattering \((\infty,1)\) motion is indicated by uncoloured regions of the parameter space and chattering \((\infty,n)\) motion where the period is greater than 1 is indicated by the very dark colour. Thus, the small dark region centred about \(1/\omega \approx 2.8, r \approx 0.9\) and \(\gamma = 0\) (figure (3.3)) indicates a region of chattering motion with a period greater than 1. Chaotic behaviour is shown by the grey colour. For \(\gamma = 0\) the chaotic regions appear as ‘tongues’ in the plot for values of restitution approximately greater than 0.6. The remainder of the key classifies the non-chattering periodic motion. As the colour changes from red through to blue, the number of impacts of the periodic motion increases. For each colour the key indicates the number of impacts per period. In the plots we note that as the damping parameter increases the chaos is eliminated and the region of chattering motion expands.

Note that in the plots shown in this chapter, we use either \(\omega\) or \(\frac{1}{\omega}\) as the variable frequency parameter. In the bifurcation plots of figures (3.3), (3.4) and (3.5) we plot the frequency variable in its reciprocal form. The reason for this is that at this stage we are more interested in a realistic frequency range, \(\omega < 1\) (i.e., we envisage that in a practical situation the forcing frequency will be less than the fundamental frequency of the delamination). However, in some cases it is useful to use the frequency parameter in its normal form.
Figure 3.3: Bifurcation Diagrams, where $\gamma = 0$ and $\gamma = 0.05$. 
Figure 3.4: Bifurcation Diagrams, where $\gamma = 0.075$, $\gamma = 0.1$, $\gamma = 0.2$ and $\gamma = 0.4$. 
Figure 3.5: Bifurcation Diagrams, where $\gamma = 0.8$, $\gamma = 0.999$, $\gamma = 1.1$ and $\gamma = 1.5$. 
In figure (3.6) we show the regions of frequency-restitution space where the motion is a sub-harmonic of the forcing frequency, indicated by the coloured regions. For this plot the key of figure (3.2) must be interpreted differently; period 1 motion is uncoloured and the chaotic regions are indicated by the very light grey shading, although there is of course no period for a chaotic motion. The rest of the plot (where the period \( n \geq 2 \)) is coloured using the key to indicate the period of the motion. These regions surround the chaotic regions, thus supporting the ‘period doubling route to chaos’ claim.

### 3.3.2 Bifurcation plot with one parameter variable

For the previous bifurcation plots with two parameters variable, the plots were only able to display a quantitative classification of the motion. If we fix two parameters and allow only one to vary and still assume zero initial conditions, we can produce another type of bifurcation plot containing more detailed information. We select a range for the variable parameter and divide this into a number of linearly spaced values. For each value of this variable, the phase and velocity of each impact is recorded as the map is iterated. This information can be plotted on a graph with the variable parameter on the x-axis, and the impact phase or velocity on the y-axis. An example is shown in figure (3.7). We also show in the bottom plot the maximum amplitude of each trajectory.

Often the forcing frequency parameter is the obvious choice as the variable; the damping
Figure 3.7: Bifurcation Diagrams ($r = 0.25, \gamma = 0.05$).
Figure 3.8: Example trajectories ($r=0.25$, $\gamma=0.05$, and various $\omega$).
and restitution are related to physical characteristics of the system, whereas the forcing frequency is an external influence and physically more likely to be variable. However, any one of the three parameters can be the variable, when producing these plots. Later we will discuss an experimental investigation of the system and for the purposes of comparing theoretical and experimental results, the fixed parameters $\epsilon = 0.25$ and $\gamma = 0.05$ are chosen so that the model is an approximate representation of the experimental apparatus. We use these parameters in the figure (3.7).

For the present, we concentrate on the theoretical results from a mathematical point of view, and concern ourselves with physical validity later. We can trace the plot from the left to the right and identify regions where particular motions exist, and where bifurcations occur. The frequency variable in figure (3.7) covers a range (from 0.1 to 6.5), including quite large (unrealistic) frequencies. The accompanying figure (3.8), shows some example trajectories at sample points in the bifurcation plot.

- For low frequencies (less than 1.1), we expect to see a transition from chattering to non-chattering periodic behaviour, all period 1. This effect can be clearly seen in the lower graph of figure (3.3), which has fixed damping of 0.05. The frequency parameter is in reciprocal form on the $x$-axis and viewing the plot from right to left, at a fixed restitution of 0.25, we clearly see a transition from chattering to non-chattering periodic motion. From numerical simulations this happens at a frequency of approximately 1.04. In figure (3.8) when $\omega = 0.5$ we show an example of a chattering trajectory. At $\omega = 1.02$ and $\omega = 1.11$ we observe example trajectories before and after the transition from chattering to non-chattering motion (it is not clear from the figure, but the reader is assured that the trajectory for $\omega = 1.11$ does not involve a period of stick). The region of chatter is indicated on the bifurcation plot, as a series of impacts with progressively lower impacting velocities (figure (3.7), top plot), or as a series of impacts with a progressively smaller time period between impacts (figure (3.7), middle plot). Notice too, in the chattering region the phase of the first impact increases as the frequency increases until the first impact occurs very near the end of the sticking region. The transition from chattering to non-chattering motion occurs because there simply is not enough time for stick.

- A graze is defined as a zero velocity impact, and research has shown that grazes are instrumental in determining the dynamics of the impact oscillator [2], [3], [5]. In the current example we can observe two grazes in the frequency range 1.1 to 1.5. Each time the number of impacts per period is reduced by one. When $\omega = 1.24$ and $\omega = 1.29$ we show trajectories immediately before and after a graze. The motion changes from (2, 1) to (1, 1) periodic motion at this grazing bifurcation.

- The sub-harmonic resonances in an impact oscillator are discussed in [26]. Referring to the bottom plot in figure (3.7), we observe local sub-harmonic resonances in the region of integer (even) values of the frequency. At frequencies around 2 and 4 the trajectory is periodic, (1, 1) and (1, 2) respectively. Hence, in both cases there is one impact per
period and this is quite clear in the figure. Between these frequencies there must be a sequence of bifurcations between the two resonances. The first bifurcation is known as a \textit{period doubling} bifurcation, and this occurs at an approximate frequency of 2.7. In this bifurcation the period \( n \) and the number of impacts per period \( m \), doubles. In figure (3.8), when \( \omega = 2.69 \), we observe a (1,1) trajectory before the period doubling bifurcation. At \( \omega = 2.73 \) the bifurcation has occurred and at this frequency we observe the (2,2) trajectory shown in the figure.

- For \( 3 < \omega < 3.25 \), there is a region of chatter, followed by non-chattering periodic motion, then after a graze the motion is periodic (1,2), with no sticking. This is similar to the transition from chatter to periodic (1,1) motion around the frequency of 1.

Another example bifurcation plot is shown in figure (3.9). The frequency variable is displayed in its reciprocal form, and we could have predicted some rough features of this plot by studying the lower plot in figure (3.3). We are allowing the frequency to vary, and fixing restitution at 0.9 allows us to study in detail the motion for parameters at a horizontal ‘slice’ across the plot of figure (3.3), for the given frequency range \( (0.65 < \frac{1}{\omega} < 1.2) \). Note, a coefficient of restitution of \( r = 0.9 \) indicates an almost elastic impact, which for the delamination and beam is unlikely. However, at this stage the aim is to illustrate the features of the system.

The slice intersects a region of chaos which appears in figure (3.9) as a chaotic distribution of impact phases and velocities. There is clearly a ‘period doubling cascade’ to chaos for \( 0.8 < \frac{1}{\omega} < 0.85 \), and small ‘windows’ of periodic behaviour in the chaotic region. There is a grazing bifurcation at \( \frac{1}{\omega} \approx 0.99 \) and figure (3.10) shows typical trajectories before and after the graze.

There is a distinct change in the system motion at \( \frac{1}{\omega} \approx 0.77 \). The (1,1) periodic motion changes to (3,2) periodic motion. This is due to the zero initial condition moving from one basin of attraction into another. This is confirmed by looking at a basin of attraction near the point where this change takes place, figure (3.11). It is clearly seen that there are two attracting orbits; a (1,1) and a (3,2) periodic attractor. The cross and circles indicate the position of the two attractors.

We have looked at some example bifurcation diagrams, and indicated some general features present throughout the system. An additional sample of bifurcation plots, with one parameter variable are contained in Appendix B. These plots are the product of a repetitive computer simulation, and of course the scope for this type of numerical investigation is endless.
Figure 3.9: Bifurcation Diagrams ($r=0.9$, $\gamma=0.05$).

Figure 3.10: Example trajectories before and after grazing ($r=0.9$, $\gamma=0.05$).
3.4 The mapping $P$

The mapping $P$ maps every point in phase space to another unique point in phase space. Actually, this is not strictly true since all points in the sticking region $I$, are assumed equivalent in terms of the mapping, since un-sticking always occurs at $\phi_1 = \frac{2\pi}{\omega}$. This is expressed in equation (2.40). We can define at this stage, a set $S$ of points in phase space which map to points with zero velocity (grazes) and a set $W$ of points mapped from zero velocity points in phase space. We note the following definitions of $S$ and $W$ taken from reference [3],

$$S = \{(\phi,v) : P(\phi,v) = (\phi_2,0)\} \quad (3.1)$$

$$W = \{(\phi,v) : (\phi,v) = P(\phi_2,0)\} \quad (3.2)$$

3.4.1 Graphically visualising the mapping

In this section we discuss some preliminary work. It is the author's opinion that viewing the mapping graphically assists understanding. It is interesting that apparently random chaotic motion can result in a system which is essentially defined by a well-behaved mapping. It is 'well-behaved' in as much as the image of any point in phase space will always be the same; there is no random characteristic of the map. One can view a graphic representation of the mapping, for a chaotic and non-chaotic parameter selection, and it is not immediately obvious what type of motion will emerge from iterating the map.

Figure (3.12) shows the phase space for a particular choice of parameters. Small regions are coloured according to the key on the right, depending on the velocity of the next impact...
Figure 3.12: Graph indicating the velocity of the next impact from any point in the phase space ($\omega = 2.8$, $r = 0.25$, $\gamma = 0.05$).

point in phase space. The discontinuous change in colour is the region where grazes occur. The set $S$ is along the line of the discontinuity, since points in this set map to a graze. The paper by Budd and Dux [3] examines the reason for this discontinuity in detail. The black lines on the plot show the set $W \cup S$. The sticking region is clearly visible. Points in or near the sticking region map to other points in the same area, indicated by the area of red colouring. For specified values of $r$ and $\gamma$ we can produce plots like figure (3.12) for a range of values of $\omega$. Animating these plots in a sequence, allows us to view the evolution of phase space.

A mapping $P(\phi_0, v_0) = (\phi_1, v_1)$ exists for all points in phase space. We can display this relationship graphically; for each point in a grid covering the phase space, we indicate by an arrow, the direction and distance to the image of the point. Hence, a short arrow indicates that the image point is close to the current point. Figure (3.13) is an example of this type of plot. The top plot shows the effect of a single mapping on representative points in phase space, the bottom plot shows the effect of two mappings.

A fixed point is defined as a point where the image point is the same point. Clearly, for a periodic $(1, n)$ orbit the image of the single impact point $(\phi_1, v_1)$ is the identical point,
Figure 3.13: Graphs showing the action of the maps \( P \) (top plot), and \( P^2 \) (bottom plot), where \( \omega = 2.8, r = 0.25 \) and \( \gamma = 0.05 \).
i.e., $P(\phi_i, v_i) = (\phi_i, v_i)$. This point is a stable fixed point of the mapping $P$. Similarly for a periodic $(m, n)$ orbit, $P^m(\phi_i, v_i) = (\phi_i, v_i)$ where $(\phi_i, v_i)$ is any point on the orbit. The parameter choice in figure (3.13) corresponds to a point in figure (3.7) just after the first period doubling bifurcation, where the attracting orbit for zero initial conditions is periodic $(2, 2)$. At the bifurcation the single impact orbit loses stability to a double impact orbit. Stability of fixed points of mappings is covered in many non-linear dynamics books, see [16], [9]. In figure (3.13), the fixed points are indicated by crosses. The fixed point in the top plot and the corresponding point in the bottom plot is unstable. The stable periodic $(2, 2)$ orbit has two stable fixed points, which make up the three fixed points in the bottom plot.

The location of fixed points can be found in one of two ways. Firstly, by iterating the map and waiting for the transient to fully decay, or secondly an optimising search of phase space, given an approximate starting point, could be used with an aim of minimising the distance in phase space between a point and its image. Once a fixed point is found, the Jacobian matrix gives a linearised version of the mapping close to the fixed point. A Fortran program was written to find the fixed points numerically and to evaluate the Jacobian matrix. An inspection of the eigenvalues of the Jacobian matrix evaluated at the fixed point gives an indication of the stability of the fixed point.

3.4.2 Animation

The resulting trajectories from the program lend themselves to being displayed as animations of the idealised system. A Matlab program was used to produce the graphical interface, and Fortran code was called from Matlab and used to quickly implement the impact mapping. Figure (3.14) shows a paused screen from the animation, which gives an impression of the output.

3.5 Chapter summary

The effect of the damping parameter on the motion of the system can be seen in the graphs of section 3.3.1. In the frequency-restitution space, for $\gamma = 0$ we observe that there is a large region of chattering motion, especially when $\omega$ is small. As the frequency variable increases, there is a change from $(\infty, 1)$ motion to periodic motion. Eventually, once the frequency has increased to $\omega = 2$ we observe $(1, 1)$ motion for all $r$. Between the chattering motion and the $(1, 1)$ motion we observe periodic motion, with 'tongues' of chaotic behaviour appearing for $r$ approximately greater than 0.55. These features can be seen in the graphs of section 3.3.1, and the bifurcation plots with one parameter variable. For example, in the bifurcation plot of figure (B.1) (Appendix B), for very small $\omega$ we first observe chattering behaviour and then as the frequency increases we observe periodic motion interspersed with 'bubbles' of chaotic motion; these are the 'tongues' of chaotic motion which appear in figure (3.3). As the damping parameter increases, we can observe in the plots of figures (3.3), (3.4) and (3.5) that the region of chatter in the frequency-restitution space increases, and also the
chaotic motion is gradually eliminated as $\gamma$ increases.

Choosing $\gamma = 0.05$ and $r = 0.25$, and using the bifurcation diagram of figure (3.7) we can describe the character of the motions of the system. As we 'sweep' through increasing forcing frequencies starting from $\omega = 0$, we can very briefly summarise the behaviour. Firstly we observe chattering motion, which becomes $(1,1)$ motion. This motion persists until the first period doubling bifurcation, and we note that at an approximate frequency of 2 the $(1,1)$ motion satisfies a resonance condition. There is a series of period doubling bifurcations, after the first, and then a small region of chaotic motion. Following this there is another region of chattering $(\infty, 2)$ motion, which eventually becomes $(1,2)$ motion. There is another resonance around $w \approx 4$.

When computing the bifurcation diagrams, we start each simulation with zero starting conditions, i.e., the system is at rest, and when the forcing is switched on the simulation begins. However, for particular choices of parameters there exist multiple attractors. In which case it is the initial conditions which determine the particular attractor the motion eventually settles on. In this chapter we produced some basin of attraction plots which indicate the co-existing attractors, and divides the phase space in to regions which move on each attractor.

By examining the results produced by computer simulations of the mathematical model, we can identify features of the system and transitions between the different types of motion. In the next chapter we use this information to predict the response of an experimental
investigation. Also in the experimental investigation we hope to evaluate the validity and inadequacies of the mathematical model examined in this chapter.
Chapter 4

Experimental Results

4.1 Introduction

To support the theoretical results, a practical investigation was proposed. It was hoped that at least some of the behaviour predicted by the model would be visible in the experimental results. It was also hoped that the experiment would investigate the validity of the major assumption that the flap moves in the fundamental mode of vibration only, or significantly, what is the range of parameters for which this is a good assumption? The model makes many other simplifying assumptions. For example, the use of a linear damping parameter and the extremely simple relationship used to model the impacts. Hence, the need for experimental verification and comparison is seen.

4.2 Experimental setup

The experimental apparatus and set up was not a strict representation of a beam with a small delamination at the free end. A point somewhere between the original physical system and its idealised model was chosen as a basis for the experiments. The model assumes that the main beam has no curvature along the delamination. The practical set up represents the main beam as a solid, flat, oscillating steel plate, to which a small 'flap' of material representing the delamination is clamped by a small clip, see figure (4.1). The joining mechanism is thus still effectively the same as the original system. A frequency generator was connected to a 'shaker' which drives the main plate at a given frequency. The forcing of the main plate can be tightly controlled and we are not concerned with resonances which occur if a flexible beam is used. In the experiment the dimension and material of the flap was varied. The size of the flap for the experiments was much larger than might be encountered in a real situation. This poses no problem as the model operates in terms of frequencies relative to the natural frequency of the flap; a larger flap has a lower natural frequency, and in fact eases many measuring complications.
Recording the experiment with limited measuring equipment was considered to be less tiresome with the proposed configuration. When conducting the experiments two pieces of measuring equipment were available. A laser device was used for all crucial measurements. This device produces a signal linearly proportional to the distance from the laser's source to where its beam is intersected by a solid object. An eddy current device is a crude, short range device, producing a signal non-linearly proportional to the distance of any object from its measuring 'tip'. This device was used to 'phase' measurements taken with the laser at different times.

Assuming movement in the fundamental mode only, it is sufficient to record the motion of the flap at one point only. The largest displacements occur at the free end, and this was therefore chosen. The movement of the forcing plate was also recorded, and hence the displacement at the end of the flap relative to the forcing can be found once these two time series are phased correctly. In addition, to gauge the contribution from higher modes, a measurement from elsewhere along the length of the flap was taken.

4.3 Experimental results

Figure (3.7) shows the effect of frequency variation on system motion, with fixed, realistic damping and restitution parameters. Ideally an experimental bifurcation diagram would have been produced. Indeed, frequency-phase plot (as in figure (3.7) middle plot) certainly would have been feasible as it is easier to record the phase rather than the velocity of an impact. In addition, computer monitoring could have handled the large number of measurements need. Unfortunately, the limitations of the experimental equipment are such that the best that could be achieved was to individually record a set of flap motions for
different forcing frequencies. Hence, taking a set of results required starting with forcing the plate at a fairly low frequency and recording the subsequent motion, and then repeating this process with increasingly higher forcing frequencies. Each separate trajectory could then be compared with sample trajectories from the model.

Therefore, we simply use the bifurcation plot of figure (3.7) to predict the experimental behaviour. This assumes that $r = 0.25$ and $\gamma = 0.05$ are reasonable choices of parameter. The choice of parameters is perhaps not as crucial as it would appear; bifurcation diagrams for parameter choices in the same region seem to display the same general features as figure (3.7). During the experiments it was noted that the effective coefficient of restitution is not constant, and that it appears to vary according to the impacting conditions. We conducted some experiments where the forcing plate was unforced and from an initial displacement away from the plate, we simply allowed the delamination to impact the plate. The resulting motion was recorded. These results are not included in this thesis, however they show that the effective coefficient of restitution approximately satisfies $0 < r < 0.3$. The later chapters of this thesis are concerned with a theoretical investigation into the effective coefficient of restitution.

In the experimental results that follow, no attempt was made to express the displacement in realistic units; the displacement is actually expressed in voltage form! However, the character of the motion is far more important than quantitative information at this stage.

Figures (4.2) and (4.3) show a set of experimental results, each plot is for a different forcing frequency. These plots show the variation with time (x-axis) of the position of the delamination and the forced plate. In this particular configuration a steel flap of length 10cm, width 12mm and thickness 0.16mm was used. The dashed line indicates the motion of the forcing plate, the solid line the motion of the free end of the flap and the dotted line the motion of a point approximately two-thirds along the flap. The location of the measurements are shown in figure (4.1). Figure (4.4) contains a set of theoretical trajectories obtained with the same parameter scheme as figure (3.7), and with a layout which enables direct comparison with figure (4.2).

In figure (4.2), at 13–23Hz the motion of the two points on the flap appear to be scaled versions of one another. We can only assume from this that the flap is operating almost entirely in the fundamental mode. However, this is not the case as the frequency is increased. A significant mode 2 component now appears to be visible, but in all the trajectories in figure (4.2), the impact occurs at almost the same phase for both points. We may conclude from this that the impact occurs at the same time along the entire length of the flap, and there are discrepancies between theory and practice only between impacts.

Therefore comparing the theoretical and experimental results at the lower frequencies we find there is an excellent correlation between the two. As the frequency is increased, the general pattern of chatter, period 1, period doubling, period 2, etc., predicted by the model does appear to be observed in the experimental results. However, as the frequency increases
the second mode of vibration of the delamination has a greater influence.

Thus at high forcing frequencies the two points on the flap no longer impact at the same phase, and the motion between the impacts is quite different. The forcing frequency is now closer to the natural frequency of the second mode, and it is expected that the motion of the flap will contain a large contribution from its second mode component.

In figure (4.3), at 65Hz and 70Hz, the movement of the flap has a degree of randomness, but it would still appear to be based at least around a period 2 motion. Due to the random nature of the trajectories the movement of the second point on the flap cannot be included in these plots. At 90Hz the forcing frequency is so extreme there are no longer any impacts; the flap acts as a simply clamped beam near its second resonant mode.

The model does predict a small region of chaotic behaviour soon after the first period doubling bifurcation, which was not positively identified in the experimental results. From the bifurcation diagram the impacts in this small chaotic region are not distributed throughout the whole of phase-space; the impacts in the periodic motion immediately before the chaotic region appear to have a strong influence on the location of the chaotically distributed impacts. So essentially, the small region of chaotic behaviour will not appear much different from the periodic motion immediately prior to the chaotic motion.

Experiments were also conducted for carbon fibre and wood flaps. The general behaviour was similar to the metal flap. Sometimes, at high frequencies, period 3 or 4 motion was observed for these different materials, instead of unconstrained mode 2 resonance. The properties of each material will affect the system performance in different ways, especially at high forcing frequencies. In particular, there was a distinct difference between the steel flap and the wood or carbon-fibre flap, with regards to the sticking region. Studying figure (4.2), at 13Hz and 19Hz the flap is stuck to the main plate for a period of time, but unsticking appears to happen at a later time than is predicted (compare figure(4.4) and figure (2.5)). This behaviour is peculiar to steel.

Appendix C, contains another set of results for the steel flap. There is one comment to be made here. There seems to be a 'false' period doubling bifurcation before the 'real' period doubling occurs. This can be seen in figure (C.1) at a forcing frequency of 36Hz. This is actually a periodic (4,4) motion and we perhaps would expect further transitions to the periodic (1,2) motion. However, at 39Hz, the motion has returned to a (1,1) periodic motion, and then the 'real' period doubling transition starts. A theoretical name for this type of behaviour is period bubbling, due to its bubble like appearance when observed in a bifurcation plot. Perhaps what we observe in these results is an experimental example of period bubbling, or maybe it is due to imperfections in the flap or forcing conditions, for example. These features do not appear in the idealised model.
4.4 Chapter summary

Referring to figure (3.7), for frequencies up to the second period doubling bifurcation frequency, we found experimental evidence of the features of the bifurcation diagram. Namely, as the forcing frequency increases, in the experimental results we observe chattering motion, followed by periodic (1,1) motion. We then observe a period doubling bifurcation to (2,2) motion. The next motion we reliably identify is (1,2) motion which persists until the model appears to breakdown; we did not obtain experimental evidence of the second period doubling bifurcation.

In table (2.1) we calculated that the non-dimensional forcing frequency of \( \omega = 6.2669 \) is equivalent to forcing the main beam at a frequency of the second mode of vibration of the delamination. Therefore, around \( \omega = 6.2669 \) the free movement of the delamination is dominated by the second mode. At such frequencies the mathematical model predicts (1,3) periodic motion, however from the experimental results we conclude that at these frequencies the model is not a reliable indicator of physical behaviour. Indeed, for forcing frequencies higher than the frequency of the first period doubling bifurcation, we can clearly see in the experimental results the increasing influence of the second mode of vibration on the movement of the delamination.

However, for \( 0 < \omega < 4 \) we observe excellent correlation between the experimental results and the predictions of the model. In an actual practical situation the delamination is likely to be much smaller than in the experimental set up, and also to have a higher fundamental frequency. Consequently, the forcing frequency is less likely to exceed the fundamental frequency, and hence the model is valid for a wider range of forcing frequencies.

During conducting the experiments it became clear that the effective coefficient of restitution is small; with the range \( 0 < r < 0.3 \). The damping parameter is also reasonably small. From examining bifurcation diagrams the features of the system persist for any realistic choice of \( r \) and \( \gamma \). Hence, if we let \( r = 0 \), the results are still helpful for predicting the physical behaviour (where \( r \neq 0 \) but small). The advantage of taking \( r = 0 \) is that we obtain a system which is easier to study analytically, and this is examined in the next chapter.
Figure 4.2: Experimental Results
Figure 4.3: Experimental Results
Figure 4.4: Theoretical Results ($r = 0.25$, $\gamma = 0.05$, and various $\omega$)
Chapter 5

The Inelastic \((r = 0)\) Case

5.1 Introduction

During the experimental investigation documented in chapter 4, we noted that the effective coefficient of restitution for the experimental apparatus is relatively small, certainly within the range \(0 < r < 0.3\). A typical bifurcation diagram is shown in figure (3.7) for \(r = 0.25\), \(\gamma = 0.05\) and \(0 < \omega < 6.5\).

In this chapter, by letting \(r = 0\), we consider the inelastic impact oscillator. Figures (5.1) and (5.2) show bifurcation diagrams for \(r = 0\), \(0 < \omega < 10\) and \(\gamma = 0\) and \(\gamma = 0.95\) respectively. If we compare figures (3.7) and (5.1) we find many similarities and consistent features, although when the damping parameter is very high as in figure (5.2) the comparison is not as valid.

We note the following points. When \(r = 0\) we do not observe chattering motion; any impact which occurs in the sticking region is immediately followed by sticking. Whereas, when the coefficient of restitution is non-zero we observe chatter followed by sticking. However, if we consider the large scale motion, for low damping and \(0 < r < 0.3\), the general characteristics are quite similar. If we take the coefficient of restitution \(r\) to be zero, we reduce the dimension of the impact map by one, i.e., \(P(\phi_i) = \phi_{i+1}\). Specifying an inelastic impact makes the analysis of the system easier, and importantly we can apply the results from this simpler system to predict the behaviour when the coefficient of restitution is small but non-zero.

We use an analytical and numerical approach to study the inelastic impact oscillator, taking advantage of the relative simplicities of the one-dimensional mapping. Firstly, in section 5.3 we find where the period doubling occurs from \((1,n)\) to \((2,2n)\) motion, and secondly, in section 5.4 we investigate the locations of grazes in the mapping and how they influence the dynamics of the system.
\( \gamma = 0, r = 0 \)

Figure 5.1: Bifurcation diagram \((\gamma = 0, r = 0)\)
Figure 5.2: Bifurcation diagram ($\gamma = 0.95, r = 0$)
5.2 Setting the coefficient of restitution to zero

In the bifurcation diagram of figure (5.1), for frequencies less than the frequency of the first period doubling bifurcation, we observe period 1 motion, which involves stick for \( \omega < 2 \). At \( \omega \approx 2.70 \), the first period doubling bifurcation to (2,2) motion occurs, and then at \( \omega = 3 \), we observe a (1,2) motion, firstly with stick, then for \( \omega > 4 \), without stick. At \( \omega \approx 4.66 \) the second period doubling bifurcation occurs from (1,2) to (2,4) motion. At \( \omega = 5 \), we observe (1,3) motion. The process is essentially the same at higher frequencies; we observe, through a sequence of bifurcations, transitions from (1,\(n\)) to (1,\(n+1\)) motion.

As stated in chapter 2, the behaviour of the delamination between impacts is defined in terms of the non-dimensional variable \( C(t) \), by a second order differential equation, the solution of which is:

\[
C(t) = \left( \frac{-1}{\omega_d} \right) e^{-\gamma(t-t_0)} \left[ A\omega_d \sin(\omega t_0 - \psi) \cos(\omega_d(t - t_0)) + A\gamma \sin(\omega t_0 - \psi) \sin(\omega_d(t - t_0)) + A\omega \cos(\omega t_0 - \psi) \sin(\omega_d(t - t_0)) + rv_0 \sin(\omega_d(t - t_0)) \right] + A \sin(\omega t - \psi) \tag{5.1}
\]

where

\[
A = \frac{1}{\sqrt{(1 - \omega^2)^2 + (2\gamma \omega)^2}} \tag{5.2}
\]

\[
\psi = \tan^{-1} \left( \frac{2\gamma \omega}{1 - \omega^2} \right) \tag{5.3}
\]

\[
\omega_d = \sqrt{1 - \gamma^2} \tag{5.4}
\]

When \( r=0 \) equation (5.1) can be written in the following form,

\[
C(t) = -e^{-\gamma(t-t_0)} \left\{ S A \cos(\omega_d(t - t_0)) + [\omega CA + \gamma SA] \frac{\sin(\omega_d(t - t_0))}{\omega_d} \right\} + S A \cos(\omega(t - t_0)) + CA \sin(\omega(t - t_0)) \tag{5.5}
\]

where we define the following,

\[
S = \sin(\omega t_0 - \psi) \tag{5.6}
\]

\[
C = \cos(\omega t_0 - \psi) \tag{5.7}
\]
5.3 Calculation of the period doubling points

The impact mapping $P$ is implemented numerically. Given $\phi_0$ (and hence $t_0$) we can find $t_1$ where $C(t)$ first satisfies $C(t) = 0$. We can learn more about the dynamics of the mapping by studying analytically, the $(1, n)$ orbits, i.e., orbits with period $n$ and one impact per period. This is done with the aim of determining when this orbit loses stability to a $(2, 2n)$ orbit at a period doubling bifurcation. On an $(1, n)$ orbit, $P(\bar{\phi}) = (\bar{\phi})$. If $t_0$ and $t_1$ are the times of two adjacent impacts then,

$$
\bar{\phi} = t_0 \mod \frac{2\pi}{\omega} \quad (5.8)
$$

$$
= t_1 \mod \frac{2\pi}{\omega} \quad (5.9)
$$

We refer to the time period $t_1 - t_0$ as the *excursion time*. For a $(1, n)$ orbit the excursion time is a multiple of the forcing period, thus $t_1 - t_0 = \frac{2\pi n}{\omega}$. Also, we can differentiate $C(t)$ to find the velocity, $\dot{C}(t)$, and we can place the following restrictions on both $C$ and $\dot{C}$, since every impact on the orbit has an equal phase and velocity. Thus,

$$
C(\bar{\phi} + \frac{2\pi n}{\omega}) = 0 \quad (5.10)
$$

$$
\dot{C}(\bar{\phi} + \frac{2\pi n}{\omega}) = \ddot{v} \quad (5.11)
$$

As a note of caution, we are only classifying $(1, n)$ motions which do not involve stick, since if the motion includes a period of stick the excursion time is not a multiple of the forcing period.

Now, from the conditions in equations (5.10) and (5.11) and with equation (5.5) we can compose the following matrix equation,

$$
\begin{pmatrix}
A - \frac{A}{E} \left( \dot{\phi} + \frac{\gamma S}{\omega_d} \right) & -\frac{A\omega \dot{S}}{E \omega_d} \\
\frac{A}{E} \left( \frac{\gamma^2 S}{\omega_d} + \omega_d \dot{S} \right) & A \omega + \frac{A}{E} \left( \frac{\gamma \omega \dot{S}}{\omega_d} - \omega \dot{C} \right)
\end{pmatrix}
\begin{pmatrix}
S \\
C
\end{pmatrix}
= \begin{pmatrix}
0 \\
\ddot{v}
\end{pmatrix}
\quad (5.12)
$$

In the equation above we used the following definitions,

$$
\dot{S} = \sin \left( \frac{2\pi n \omega_d}{\omega} \right) \quad (5.13)
$$

$$
\dot{C} = \cos \left( \frac{2\pi n \omega_d}{\omega} \right) \quad (5.14)
$$

$$
E = e^{\frac{2\pi n \omega_d}{\omega}} \quad (5.15)
$$

We can now solve the matrix system in equation (5.12) and find the following explicit expressions for $C$ and $S$,

$$
S = -\frac{v E (\dot{C} \omega_d + \gamma S - E \omega_d)}{\omega A \omega_d (\dot{C}^2 - 2 \dot{C} E + E^2 + S^2)} \quad (5.16)
$$

$$
C = \frac{v E \dot{S}}{A \omega_d (\dot{C}^2 - 2 \dot{C} E + E^2 + S^2)} \quad (5.17)
$$
Following [24], the stability of any orbit is determined by examining $|\frac{\partial P(\phi)}{\partial \phi}|$. If $|\frac{\partial P(\phi)}{\partial \phi}| < 1$ the orbit is stable; if $|\frac{\partial P(\phi)}{\partial \phi}| > 1$ the orbit is unstable. A bifurcation occurs when $|\frac{\partial P(\phi)}{\partial \phi}| = 1$, and in particular, a period doubling bifurcation occurs when $\frac{\partial P(\phi)}{\partial \phi} = -1$.

Assuming that $t_0$ is variable, $C$ is function of $t_0$ and $t_1$. We specify that $C(t_0, t_1) = 0$ or $C(t_0, P(t_0)) = 0$, and we can explicitly differentiate this expression. The differentiated expression can then be re-arranged to find $\frac{\partial P}{\partial \phi}$. Hence,

$$\frac{\partial P}{\partial \phi} = -\frac{\frac{\partial C(t_0, t_1)}{\partial t_0}}{\frac{\partial C(t_0, t_1)}{\partial t_1}}$$

(5.18)

Using equation (5.5) we can evaluate this equation enforcing, $t_1 - t_0 = \frac{2\pi}{\omega}$. Using the definitions for $C$ and $S$, and recalling that a period doubling bifurcation occurs when $\frac{\partial P(\phi)}{\partial \phi} = -1$ we can write the following bifurcation condition,

$$F_{pd}(\gamma, \omega_n(\gamma)) = \frac{\dot{S}}{\omega_d} \left\{ (1 - \omega^2) \frac{\dot{S}}{\omega_d} + 2\gamma \left[ \hat{E} - \hat{C} - \frac{\gamma S}{\omega_d} \right] \right\} + \hat{E}^2 - 2\hat{E}\hat{C} + 1 = 0$$

(5.19)

The bifurcation occurs from $(1,n)$ to $(2,2n)$ motion and therefore we must specify $n$. We numerically solve the above equation for $\omega$, given $n$ and $\gamma$, and hence we find the bifurcation frequency, which we label as $\omega_n(\gamma)$.

Thus, $\omega_n(0)$ is the period doubling frequency for the undamped system. Hence, setting the damping coefficient to zero we can write the following bifurcation condition for the undamped system (as also derived in [24]),

$$F_{pd}(0, \omega_n(0)) = \frac{\dot{S}}{\omega_d} (1 - \omega^2) + 2(1 - \hat{C}) = 0$$

(5.20)

We can solve this equation numerically to find the period doubling bifurcation frequencies $\omega_n(0)$. Hence, we find the frequencies at which period doubling bifurcations occur from $(1,n)$ to $(2,2n)$ motion in the undamped system. The first and second columns of table (5.1) show results for values of $n$ up to 6, and for $n \rightarrow \infty$.

We can approximate analytically the period doubling bifurcation values, for small damping, by expanding the function $\omega_n(\gamma)$ in a Taylor series:

$$\omega_n(\gamma) = \omega_n(0) + \frac{d\omega_n(0)}{d\gamma} \gamma + \frac{1}{2} \frac{d^2\omega_n(0)}{d\gamma^2} \gamma^2 + \ldots$$

(5.21)

We have already determined $\omega_n(0)$ numerically, and must now determine $\frac{d\omega_n(0)}{d\gamma}$ and $\frac{d^2\omega_n(0)}{d\gamma^2}$.

Differentiating implicitly the function

$$F_{pd}(\gamma, \omega_n(\gamma)) = 0,$$

(5.22)
with respect to $\gamma$, we obtain,

\[
\frac{\partial F_{pd}}{\partial \gamma} = \frac{\partial F_{pd}}{\partial \gamma} + \frac{\partial F_{pd}}{\partial \omega_n} \frac{d\omega_n}{d\gamma} = 0
\]

\[
\frac{\partial^2 F_{pd}}{\partial \gamma^2} = \frac{\partial^2 F_{pd}}{\partial \gamma^2} + 2 \frac{\partial^2 F_{pd}}{\partial \gamma \partial \omega_n} \frac{d\omega_n}{d\gamma} + \frac{\partial^2 F_{pd}}{\partial \omega^2_n} \left( \frac{d\omega_n}{d\gamma} \right)^2 + \frac{\partial F_{pd}}{\partial \omega_n} \frac{d^2 \omega_n}{d\gamma^2} = 0
\]

Rearranging, we can determine the remaining two coefficients in equation (5.21), which are,

\[
\frac{d\omega_n(0)}{d\gamma} = -\frac{\frac{\partial F_{pd}}{\partial \gamma}}{\frac{\partial F_{pd}}{\partial \omega_n}} \bigg|_{\omega_n = \omega_n(0)}
\]

(5.23)

\[
\frac{d^2\omega_n(0)}{d\gamma^2} = -\left\{ \frac{\partial^2 F_{pd}}{\partial \gamma^2} + 2 \frac{\partial^2 F_{pd}}{\partial \gamma \partial \omega_n} \frac{d\omega_n}{d\gamma} + \frac{\partial^2 F_{pd}}{\partial \omega^2_n} \left( \frac{d\omega_n}{d\gamma} \right)^2 \right\} \bigg|_{\omega_n = \omega_n(0)}
\]

(5.24)

From equations (5.23) and (5.24), we can find the numerical value for these coefficients of the Taylor expansion. These are listed in table (5.1) for $n = 1, 2, \ldots, 6$ and as $n \to \infty$. Now using the Taylor series approximation, equation (5.21), the period doubling frequencies can be found. Figure (5.3) shows the period doubling frequencies for $n = 1 \ldots 6$, and varying $\gamma$. The solid line indicates the numerical calculation and the dotted line the graph of the Taylor approximation. The approximation is good for small $\gamma$, as would be expected.
5.4 Finding where grazes occur

Figures (5.4) and (5.5) show the action of the mapping $P(\phi_i) = \phi_j$ for fixed values of damping ($\gamma = 0$ and $\gamma = 0.95$ respectively) and various choices of frequency. For convenience the forcing period is normalised between 0 and $2\pi$. Taking a large set of initial points distributed throughout the forcing period ($x$-axis), the $y$-axis shows the phase of the image points of the mapping. For starting points in the second half of the forcing period, it is clear that the image point is the constant $P(0)$, since the initial point is in the sticking region.

A *graze* is defined as a zero velocity impact. Referring to figures (5.4) and (5.5), a graze occurs when there is a discontinuity in the mapping. However, the discontinuity at a phase
Figure 5.4: Details of the mapping $\phi_1 = P(\phi_0)$ for $1.05 \leq \omega < 8$, $\gamma = 0$. 

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Figure 5.5: Details of the mapping $\phi_1 = P(\phi_0)$ for $1.05 < \omega < 8$, $\gamma = 0.95$. 
of exactly half the forcing period, is caused by the impact moving into the sticking region, and not by the influence of a graze.

As we previously commented, the grazes strongly influence the behaviour in the system. Therefore, given a particular choice of parameter we would like to determine where these grazes occur in the impact mapping. We could then obtain a global picture of the discontinuities if we fix the damping parameter and for varying frequencies, compute the location of the grazes. With this information we gain further insight into the dynamics of the system.

The conditions for a grazing impact at a time $t_1$ are, $C(t_1) = 0$ and $\dot{C}(t_1) = 0$. Referring to equation (5.5) we can express $C(t)$ and $\dot{C}(t)$ in the following form,

$$ C(t) = f_1(t-t_0)S(t_0) + f_2(t-t_0)C(t_0) \quad (5.25) $$

$$ \dot{C}(t) = f_3(t-t_0)S(t_0) + f_4(t-t_0)C(t_0) \quad (5.26) $$

Thus expanding the functions $f_1(t_1 - t_0)$ and equating $C(t_1)$ and $\dot{C}(t_1)$ both to zero (the amplitude variable $A$ is cancelled later), we obtain,

$$ \begin{pmatrix} A \cos(\omega(t_1 - t_0)) - \frac{A}{E} \left( \dot{C} + \frac{\gamma \dot{S}}{\omega_d} \right) & A \sin(\omega(t_1 - t_0)) - \frac{A \omega \dot{S}}{\omega_d} \\ \frac{A}{E} \left( \frac{\gamma^2 \dot{S}}{\omega_d} + \omega_d \dot{\dot{S}} \right) & \frac{A}{E} \left( \frac{\gamma \omega \dot{S}}{\omega_d} - \omega \dot{C} \right) + A \omega \sin(\omega(t_1 - t_0)) \end{pmatrix} \begin{pmatrix} S \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.27) $$

where the elements of the $2 \times 2$ matrix are labelled $f_1, f_2, f_3, f_4$ and we used the following new definitions for $\dot{S}, \dot{C}$ and $E$,

$$ \dot{S} = \sin(\omega_d(t_1 - t_0)) \quad (5.28) $$

$$ \dot{C} = \cos(\omega_d(t_1 - t_0)) \quad (5.29) $$

$$ E = e^{\gamma(t_1 - t_0)} \quad (5.30) $$

To ensure a non-zero solution to a matrix system of the form, $Ax = 0$, the determinant of the matrix must be zero, i.e., $f_1 f_4 - f_2 f_3 = 0$. The determinant is a function of $t_1 - t_0$, and must be solved numerically. We find a number of solutions for $t_1-t_0$ (excursion times).

Returning to the matrix system (5.27), we can evaluate the functions $f_i$ at the computed excursion times. From the matrix system, we define a new variable $T$, thus

$$ T = \frac{S}{C} = \tan(\omega t_0 - \psi) = -\frac{f_2}{f_1} = -\frac{f_4}{f_3} \quad (5.31) $$

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From this equation we can find an expression for \( t_0 \), which we shall refer to as the departure point. Hence,

\[
t_0 = \frac{(\tan^{-1} \left( -\frac{f_2}{f_1} \right) + \psi)}{\omega}
\]

(5.32)

\[
t_0 = \frac{(\tan^{-1} \left( -\frac{f_2}{f_3} \right) + \psi)}{\omega}
\]

(5.33)

Note, that from the determinant function \( \frac{f_2}{f_1} = \frac{f_2}{f_3} \).

Therefore given all possible excursion times \( t_1 - t_0 \), we can find the corresponding departure point, \( t_0 \) (or \( \phi_0 \)). However at this stage we must note that we have not paid any regard to any possible non-physical grazing pair \((\phi_0, t_1 - t_0)\); if \( \frac{\pi}{2} < \phi_0 < \frac{2\pi}{\omega} \), then the departure point is within the sticking region, and therefore not allowed. If we disregard this property of the system however, it is clear that if a zero velocity impact occurs in the first half of the forcing period, with a departure phase \( \phi_0 \) and excursion time \( t_1 - t_0 \), a zero velocity impact will also occur for departure phase of \( \phi_0 + \frac{\pi}{\omega} \) (i.e., within the second half of the forcing period) and with the same excursion time. We must therefore stipulate that, \( 0 < \phi_0 < \frac{\pi}{2} \).

We can do a little better than this however. Observing the impact map, as in figures (5.4) and (5.5), it appears that for any parameter selection, the grazes always appear with \( \phi_0 \) satisfying \( 0 < \phi_0 < \frac{\pi}{2\omega} \), i.e., in the first quarter of the forcing period. A similar observation was made in [23]. An interesting point to note is whether this can be proved.

The development of the mapping is as follows: for low frequencies, about \( \omega \approx 1 \) there are no discontinuities in the mapping (we don't recognise the apparent 'discontinuity' at the start of the sticking region \( \phi = \frac{\pi}{2} \)). At a specific point (in the undamped case, \( \omega = 3 \)), a graze appears with a departure point at \( \phi = 0 \), and as the frequency increases so does the phase at which the graze appears. As the frequency is increased further, another graze appears, again at \( \phi = 0 \), and a similar process of 'growth' occurs. These grazes are permanent features of the map once they are born; they can be seen to accumulate as small discontinuous 'pieces' of the mapping just before \( \phi = \frac{\pi}{2\omega} \). This is clear in figure (5.6), where the action of mapping with a large frequency parameter is shown. The x-axis in both plots show the forcing period normalised between 0 and \( 2\pi \). The top plot in the figure shows the absolute arrival point, \( t_1 \), on the y-axis, instead of the phase of the arrival point, \( \phi_1 \), as in figures (5.4) and (5.5). Also, the dot-dashed line indicates \( \phi = \frac{\pi}{2\omega} \). A crossing of the diagonal dotted lines by the mapping indicates fixed points of the mapping (unstable or stable), since at these points \( P(\phi_i) = \phi_i \). In the bottom plot of figure (5.6), the impacting velocity is shown. Obviously, the application of the impact model always sets the rebounding velocity to zero. However, it can be clearly seen that the discontinuities in the mapping are caused by grazes.

To compute the actual valid grazes we must conduct a number of tests on the departure phases, and excursion times. We can first test to ensure \( 0 < t_0 < \frac{\pi}{2\omega} \). We also find invalid
Figure 5.6: Graph showing the time $t_1$ of the next impact from a departure point at $\phi_0$ (where $0 < t_0 < \frac{2\pi}{\omega}$ and $\omega = 13.01$, $\gamma = 0.5$)
trajectories which cross the $C = 0$ axis, before the zero velocity 'impact' occurs. Furthermore, we can calculate $\dot{C}(t_1)$ and check if it is positive, indicating a minimum turning point at the graze. There are no grazes for $\omega < 1$, however for $\omega > 1$ the transient response has a longer time period than the forced response of equation (5.1), and therefore an excursion time much larger than $\frac{\pi}{\omega_d}$ is unlikely to result in a valid physical trajectory.

These are only guidelines. In an automated procedure to find the grazes we cannot be absolutely certain of finding all the valid grazes and none of the invalid grazes. However, it was noted that the 1st, 3rd, 5th, ... roots of the determinant function $\lambda_1 \lambda_4 - \lambda_2 \lambda_3 = 0$ produced the valid excursion times and departure phases. Assuming we have $n$ grazes, then for the first $2n$ roots of the determinant function we can make some general observations we believe to be true. Firstly, irrespective of any physical considerations, the zero velocity 'impact' at $t_1$ is alternately a local minimum/maximum turning point, starting with a minimum for the first root of the determinant function. Also the odd roots of the determinant function occur with a starting point $t_0$ in the first quarter of the forcing period and the resulting trajectories are valid. Conversely the even roots have a starting point in the second quarter of the forcing period and the corresponding trajectories are invalid since there are crossings of the $C = 0$ axis.

Figures (5.7) and (5.8) show the location of the grazes with the frequency parameter varying. In these plots, the damping parameter is fixed at $\gamma = 0$ and $\gamma = 0.95$. The x-axis is the frequency parameter, the y-axis is some measure of time; the first subplot shows the phase of the departure point of the graze, $\phi_0$, the second subplot shows the absolute excursion time $t_1 - t_0$ of the grazing trajectory and the third subplot shows the phase at the graze itself. As discussed previously, in all three of these subplots, the forcing period, $\frac{2\pi}{\omega}$, is normalised to $2\pi$.

As we have noted in both graphs the grazes appear at $\phi = 0$ and then, as the frequency increases, they accumulate at $\frac{\pi}{\omega}$. For small damping, the appearance of each new graze corresponds to the same point at which the transition to $(1, n + 1)$ motion occurs; for the undamped case the grazes appear at $\omega = 3, 5, 7, \ldots$ (see figure (5.1)), corresponding to the creation of the $(1, 2), (1, 3), (1, 4), \ldots$ orbits. This is because immediately after a new graze is 'born' the starting point of $\phi = 0$ is mapped to a point within the sticking region, therefore the next 'departure' on the orbit is from $\phi = 0$ again, and therefore we must have an orbit with a single impact.

Figure (5.9) shows the action of the mapping, with a starting point of $\phi_0 = 0$, for various values of $\gamma$. The x-axis shows the frequency variable. There are discontinuities in the graphs which correspond to the new grazes being created. For the undamped case (top plot) it can be clearly seen that immediately after the grazes emerge at $\phi = 0$, i.e., at $\omega = 3, 5, 7, \ldots$, the zero starting point maps to a point within the sticking region. However, as the damping parameter is increased this is not necessarily the case. Immediately after the creation of the grazes, the zero starting point does not always map to a point within the sticking region. Therefore, for higher values of damping the transition to $(1, n + 1)$ motion is clearly not so
easy to predict.

An example of the transition to (1,2) motion initiated by the mapping of the zero starting point to a point within the sticking region, is shown in figure (5.10) where \( \gamma = 0.95 \). The figure displays the mapping and example trajectories at sample frequencies in the range \( 3.3 < \omega < 3.63 \), where a transition occurs from (1,1) to (1,2) motion. The first two plots show simple (1,1) motion. In the next plot we have passed the period doubling bifurcation, and we observe (2,2) motion. In the last plot, the first graze has occurred and a zero starting point maps into the sticking region and the motion is now (1,2). At this value of damping, only the first transition, to (1,2) motion, works by this mechanism, and this can be seen in the plots of figure (5.11), where again \( \gamma = 0.95 \). There are four plots shown for sample frequencies in the range \( 5.05 < \omega < 5.25 \), where the transition to (1,3) motion occurs. In this case the transition is from (1,2) motion, through period doubling, then a small region of chaos, until eventually the (1,3) motion emerges. The transition to (1,3) motion is not caused by a mapping from the zero starting point into the sticking region. The dynamics of the mapping at this high value of damping requires further attention.

Figure 5.7: Location of the grazes, \( \gamma = 0 \).
Chapter summary

Firstly in this chapter we study the period doubling bifurcations from \((1, n)\) to \((2, 2n)\) motion. For specified \(n\) and \(\gamma\) we found an equation, which can be solved numerically, to find the frequency at which period doubling occurs. For given \(n\), we derived a Taylor series approximation of the period doubling frequency function. A graph of the results can be seen in figure (5.3), and we note that the approximation is good for small \(\gamma\).

We described a procedure whereby for a given choice of parameters we can determine the location of the grazes in the impact mapping. It is necessary to use some numerical techniques for this. The introduction of grazes at \(\phi = 0\) have a great influence on the dynamics of the system. Figure (5.9) shows the action of the mapping from a starting point of \(\phi_0 = 0\), and since \((1, n)\) motion is the most prevalent in the system (this can be observed in the bifurcation diagrams of figures (5.1) and (5.2)), there are large regions in the top and bottom plots of figure (5.9) which show similar behaviour to the bifurcation diagrams since both show the mapping from a starting phase of zero. In a practical situation the damping
is much smaller than 0.95 (recall that damping of $\gamma=1.0$ corresponds to critical damping). In which case it is true to say that it is the introduction of a graze at $\phi=0$ which initiates a new $(1,n)$ periodic motion.

We also noted that once the grazes appear at the start of the forcing period, as the frequency increases (and other new grazes emerge), the phase at which the graze is located also increases. The grazes in the impact mapping accumulate at $\frac{\pi}{\omega}$, and hence grazes always occur with starting phases in the first quarter of the forcing period. The graphs of figures (5.7) and (5.8), which show the details of grazes, confirm this statement.

In this chapter, we discussed an analytical investigation of the inelastic impact oscillator.
Figure 5.10: Graphs showing trajectories and details of mapping, for the specified choices of parameters
Figure 5.11: Graphs showing trajectories and details of mapping, for the specified choices of parameters.
Taking $r = 0$ we gained insight into the dynamics of this particular system, and noting similarities in bifurcation diagrams for $0 < r < 0.3$ with low damping, we appreciate that we can partially apply this understanding to more general impact oscillators with large dissipation at the impact.
Chapter 6

The Impact Problem

6.1 Introduction

This chapter marks the start of the second half of the thesis. In the first half we developed a model of a forced beam with a delamination at its free end, and in doing so we made some important assumptions. In particular, we made a number of assumptions regarding the impacting conditions, and how to model the impact. Firstly, we assumed that the main beam has zero curvature along the length of the delamination. This allows us to predict that for a reasonable range of forcing frequencies of the main beam (namely frequencies less than the frequency of the second mode of the delamination), the delamination moves in the fundamental mode of vibration, and therefore impact occurs at the same time along the length of the delamination. Secondly, we choose to model the impact by relating the impacting and rebounding speeds using a simple coefficient of restitution relationship. In the second half of this thesis we shall examine the impact in much greater detail. We wish to determine the extent to which the coefficient of restitution model is accurate and reasonable. This model is the simplest model of impact, and has well documented inadequacies, see [10], [7]. For example, it is assumed that the impact occurs instantaneously, although in practice this is never the case. We therefore study more sophisticated models of the impact, to give more accurate predictions of physical behaviour, and to determine the influence of material properties, impacting velocities and dimensions on this behaviour.

In chapter 2, the impacting delamination model is reduced to a single degree-of-freedom impacting system, also known as an impact oscillator. Typically this is a single forced spring-mass-damper system, with an amplitude limiting constraint. This system has been researched reasonably extensively [1], and in the majority of cases the impact is modelled with a coefficient of restitution relationship. The behaviour of the impact oscillator model is very complex, and the coefficient of restitution model is considered to be a sufficiently accurate model for the impact. Using a different model of the impact is unpopular, since it adds further complications to the analysis of the system.
We can obtain an improved model of impacting behaviour by considering the propagation of stress waves introduced by the impact.

In this chapter we study a single degree-of-freedom impacting system where a slug moving with a specified velocity impacts a stationary rod. During the impact there is contact between surfaces of the slug and rod, and we refer to the point of contact as the interface. The impact creates compressive waves in the bodies which we assume propagate in a direction perpendicular to the interface. These waves are reflected when they reach the boundaries of the bodies. We assume that these 'outer' boundaries are surfaces parallel with the interface. A transmitted wave and a reflected wave are set up when a wave meets the interface. To find these new waves we specify boundary conditions at the surfaces of the slug and rod, which must be maintained when waves are created.

When the impact occurs there is a compressive stress at the interface, but if the stress at the interface becomes tensile, the bodies part company. A change of stress is caused by the multiple reflections of the waves in the bodies. Over the course of the next four chapters, we construct models of various impacting systems, and using a number of different methods we determine the transient motion of the bodies. These solutions consist of expressions for the propagating waves. We specify the necessary parameters, and it is then possible to estimate the parting time and associated velocities of the slug and rod.

Initially, we shall study some single degree-of-freedom systems. These are interesting, and a detailed study allows us to develop some necessary mathematical techniques, which we can then apply to other models, namely inversion of Laplace transforms when analytical solutions are not available. We then move on to consider some multi-degree of freedom systems, in particular discretised models of beams where a set of slugs are joined together with shear springs.

Therefore, in this chapter we consider three single degree-of-freedom models,

- We first consider a single 'slug' impacting a rod of semi-infinite length in section 6.2. In this case there are no waves in the rod returning to the interface. It is the reflected waves in the slug which can cause the slug and rod to move apart. Specifically, we find that parting only occurs for a particular range of parameters, and this parting is always caused by the first wave in the slug returning to the interface.

- In section 6.3 we extend the model and consider a rod of finite length. There are wave reflections in both bodies and waves returning to the interface in either body can cause separation.

- Finally in section 6.4, we couple a single slug to two stationary neighbouring slugs with shear springs. This introduces the strength of coupling as another variable in the model. When the coupling is non-zero, we find that parting can occur at any time, and not only when reflected waves meet the interface.

In effect, we are examining the building blocks of a discretised beam model. Indeed, we
find close relationships between the single degree-of-freedom systems and the beam models, and we shall comment on these in later chapters.

In chapter 7 we join a line of slugs together with shear springs coupling each slug to its nearest neighbour and obtain a simple lumped model of a beam. Letting the number of slugs in the model approach infinity, we obtain a continuous model which behaves like a vibrating string when the slugs move as rigid bodies. In chapter 8 we couple each slug in a line to its nearest and next-nearest neighbour. In this case, the continuous structure behaves like a beam in bending when the slugs move as rigid bodies. In chapter 9 we return to another single degree of freedom system and examine in detail the impact in a conventional impact oscillator.

6.2 A slug hitting a semi-infinite length rod

6.2.1 Modelling

A slug travelling at speed \( V \), impacts an infinite length rod at time \( t = 0 \) and at \( \dot{x} = 0 \), where the \( \dot{\cdot} \) notation indicates dimensional variables. This is illustrated in figure (6.1). Let \( E \) and \( \bar{E} \) be the Young's modulus of the material of the slug and of the rod respectively, and \( \rho \) and \( \bar{\rho} \) be the densities of the slug and the rod respectively. Also, \( E = c^2 \rho \) and \( \bar{E} = \bar{c}^2 \bar{\rho} \), where \( c \) and \( \bar{c} \) are the wave speeds in the slug and rod, and the length of the slug is \( h_s \).

![Figure 6.1: Impact between the slug and the rod](image)

The positions at time \( t \) of the cross-section at location \( X_0 \) at \( t = 0 \) of the slug and rod are given by \( \ddot{x}(\dot{X}_0, t) \) and \( \ddot{x}(\dot{X}_0, t) \) respectively. Variation in stress throughout the slug or rod is caused by wave propagation from the impact. The functions \( \ddot{u}(\dot{X}_0, t) \) and \( \ddot{u}(\dot{X}_0, t) \) express the additional displacement following the impact. The position of the two bodies can be written as,

\[
\begin{align*}
\ddot{x}(\dot{X}_0, t) &= \dot{X}_0 + V t + \ddot{u}(\dot{X}_0, t) & \text{for } -h_s \leq \dot{X}_0 \leq 0 \quad (6.1) \\
\ddot{x}(\dot{X}_0, t) &= \dot{X}_0 + \ddot{u}(\dot{X}_0, t) & \text{for } \dot{X}_0 \geq 0 \quad (6.2)
\end{align*}
\]
We use the following non-dimensionalising scheme,
\[
\begin{align*}
\tilde{u} &= \frac{Vh_s u}{c} \\
\tilde{x} &= h_s x \\
\tilde{X}_0 &= h_s X_0 \\
\tilde{t} &= \frac{h_s t}{c} \\
\tilde{\tilde{u}} &= \frac{Vh_s \tilde{u}}{c} \\
\tilde{\tilde{x}} &= h_s \tilde{x}
\end{align*}
\]
Therefore, we can non-dimensionalise \( \tilde{x} \) and \( \tilde{\tilde{x}} \), (letting \( \alpha^2 = c^2/\tilde{c}^2 \))
\[
\begin{align*}
x &= X_0 + \frac{V}{c} \left( t + u(X_0, t) \right) \\
\tilde{x} &= X_0 + \frac{V}{c} \alpha \tilde{u}(X_0, t)
\end{align*}
\]

6.2.2 Propagating waves in the bodies

Figure (6.2) shows a representation of the slug or the rod. We focus on a small section of thickness \( \delta X_0 \), and at a cross-section \( X_0 \) of cross-sectional area \( A \). The stress in the body is denoted by \( \tilde{\sigma}(\tilde{X}_0, \tilde{t}) \). Therefore, considering the internal forces in this section,
\[
\begin{align*}
\rho A \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} \delta \tilde{X}_0 &= \left( \tilde{\sigma} + \frac{\partial \tilde{\sigma}}{\partial \tilde{X}_0} \delta \tilde{X}_0 \right) A - \tilde{\sigma} A \\
&= \frac{\partial \tilde{\sigma}}{\partial \tilde{X}_0} \delta \tilde{X}_0 A
\end{align*}
\]

The stress is related to \( \tilde{u} \) by \( \tilde{\sigma} = E \frac{\partial \tilde{u}}{\partial \tilde{X}_0} \), and hence
\[
\rho \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} = E \frac{\partial^2 \tilde{u}}{\partial \tilde{X}_0^2}
\]

Therefore, in the slug and rod, the functions \( \tilde{u}(\tilde{X}_0, \tilde{t}) \) and \( \tilde{\tilde{u}}(\tilde{X}_0, \tilde{t}) \) satisfy the following equations:
\[
\begin{align*}
\frac{E}{\rho} \frac{\partial^2 \tilde{u}}{\partial \tilde{X}_0^2} - \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} &= 0 \\
\tilde{E} \frac{\partial^2 \tilde{\tilde{u}}}{\partial \tilde{X}_0^2} - \frac{\partial^2 \tilde{\tilde{u}}}{\partial \tilde{t}^2} &= 0
\end{align*}
\]

Non-dimensionalising,
\[
\begin{align*}
\frac{\partial^2 u}{\partial X_0^2} - \frac{\partial^2 u}{\partial t^2} &= 0 \\
\frac{\partial^2 \tilde{u}}{\partial X_0^2} - \alpha^2 \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} &= 0
\end{align*}
\]

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These have the general solutions,

$$u(X_0, t) = f(X_0 - t) + g(X_0 + t)$$ \hspace{0.5cm} (6.7)

$$\bar{u}(X_0, t) = \bar{f}(X_0 - \frac{1}{a}t) + \bar{g}(X_0 + \frac{1}{a}t)$$ \hspace{0.5cm} (6.8)

where $f(...), g(...), \bar{f}(...)$ and $\bar{g}(...)$ are arbitrary functions.

Figure (6.3) represents the two bodies in contact after the impact has occurred, where the slug is shown by a darker shading. The horizontal axis indicates the cross-section in either body, and the vertical (downward) axis the non-dimensional time. At $t = 0$ a stress wave in each body propagates from the interface. After non-dimensionalising, the wave speed in the slug is 1, and in the rod it is $\frac{1}{a}$ or $\frac{t}{c}$. This is deliberately highlighted in the diagram; in this case the wave speed in the slug is greater than the wave speed in the rod.

Solution for $t < 1$

At impact the interface conditions state that the particle velocities and the stress must be equal. Therefore equating the particle velocities we obtain,

$$V + \frac{\partial u(\bar{X}_0, \bar{t})}{\partial \bar{t}} \bigg|_{\bar{X}_0=0} = \frac{\partial \bar{u}(\bar{X}_0, \bar{t})}{\partial \bar{t}} \bigg|_{\bar{X}_0=0}$$ \hspace{0.5cm} (6.9)

Equating the stress at the interface, we obtain,

$$E \frac{\partial \bar{u}(\bar{X}_0, \bar{t})}{\partial \bar{X}_0} \bigg|_{\bar{X}_0=0} = \bar{E} \frac{\partial \bar{u}(\bar{X}_0, \bar{t})}{\partial \bar{X}_0} \bigg|_{\bar{X}_0=0}$$ \hspace{0.5cm} (6.10)

Defining $z = \frac{c_0}{\bar{c}_0}$ and non-dimensionalising equations (6.9) and (6.10) we obtain,

$$1 + \frac{\partial u}{\partial t} \bigg|_{x_0=0} = \alpha \frac{\partial \bar{u}}{\partial \bar{t}} \bigg|_{x_0=0}$$ \hspace{0.5cm} (6.11)

$$z \frac{\partial u}{\partial X_0} \bigg|_{x_0=0} = \frac{\partial \bar{u}}{\partial \bar{X}_0} \bigg|_{x_0=0}$$ \hspace{0.5cm} (6.12)
A negative stress indicates compression, a positive stress indicates tension. Therefore, we examine the stress at the interface to determine if the two bodies part company, and at what time.

Substituting the general solutions (6.7) and (6.8) into the interface conditions,

\[ 1 - f'(-t) + g'(t) = -f'(-\frac{1}{\alpha} t) + \bar{g}'(\frac{1}{\alpha} t) \]  
\[ z[f'(-t) + g'(t)] = f'(-\frac{1}{\alpha} t) + \bar{g}'(\frac{1}{\alpha} t) \]

where \( t \) denotes differentiation with respect to the argument.

The function \( f(X_0 - t) \) represents a disturbance travelling in the positive \( X_0 \) direction in the slug, and since the impact occurs at \( X_0 = 0 \), no such disturbance will be set up immediately after the impact, so that \( f = 0 \) for \( t > 0 \). Similarly, \( \bar{g}(X_0 + \frac{1}{\alpha} t) \) is a wave travelling in the negative \( X_0 \) direction in the rod and we must have \( \bar{g} = 0 \) for \( t > 0 \). Referring to figure (6.3), the two waves set up at \( t = 0 \) are \( g_0 \) and \( \bar{f}_0 \), therefore with this change of notation, the
conditions at \( X_0 = 0 \) become,
\[
1 + g_0'(t) = -f_0'(-\frac{1}{\alpha}t) \quad (6.15)
\]
\[
z g_0'(t) = f_0'(-\frac{1}{\alpha}t) \quad (6.16)
\]
We integrate these two equations and solve to find the functions \( g_0 \) and \( f_0 \), where we use the initial conditions, \( g_0(0)=0 \) and \( f_0(0)=0 \). Therefore
\[
g_0(t) = -\frac{1}{1 + z} t
\]
\[
f_0(-\frac{1}{\alpha}t) = \frac{1}{\alpha} \frac{z}{1 + z} t
\]
Hence,
\[
g_0(X_0 + t) = -\frac{1}{1 + z} (t + X_0)H[t + X_0] \quad (6.17)
\]
\[
f_0(X_0 - \frac{1}{\alpha}t) = \frac{1}{\alpha} \frac{z}{1 + z} (t - \alpha X_0)H[t - \alpha X_0] \quad (6.18)
\]
The overall motion of the two bodies is defined by \( x \) and \( \bar{x} \) (equations (6.3) and (6.4)). The solutions \( u \) and \( \bar{u} \) are given by \( g_0 \) and \( f_0 \) respectively. The function \( H[...] \) is the Heaviside function. This is used to 'switch' on or off the presence of the wave in the slug or rod. This is dependent on whether the wave has propagated to the particular point, \( X_0 \), in the time, \( t \).

Therefore, substituting the solutions for the new waves into (equations (6.3) and (6.4)), we obtain,
\[
x = X_0 + \frac{V}{c} \left( t - \frac{1}{1 + z} (t + X_0)H[t + X_0] \right) \quad (6.19)
\]
\[
\bar{x} = X_0 + \frac{V}{c} \frac{z}{1 + z} (t - \alpha X_0)H[t - \alpha X_0]. \quad (6.20)
\]
The solution in the slug is valid for \( t < 1 \), and the solution in the rod is valid for \( t < 2 \).

Procedure for when a wave in the slug reaches the \( X_0 = -1 \) boundary

The solution is valid up to time \( t = 1 \) when the wave \( g_0(X_0 + t) \) reaches the boundary \( X_0 = -1 \) of the slug. At this time a wave travelling in the opposite (positive) direction must be set up in order to maintain zero stress at \( X_0 = -1 \); this wave is shown in figure (6.3) as \( f_1(X_0 - t) \). Similarly, other waves \( (g_2, g_4, g_6...) \) in the slug meet \( X_0 = -1 \) at times \( t = 3, 5, 7... \), and the boundary condition must also be satisfied at these times.

In the following we denote the incoming wave as \( g(X_0 + t) \), and the new wave as \( f_+(X_0 - t) \). Prior to the appearance of these waves, the existing waves maintain the boundary conditions at the interface. Therefore, we only need to impose the boundary conditions on the incident and departing wave.
The stress at the boundary $X_0 = -1$ is zero. Hence,

$$ V \rho \frac{\partial u}{\partial X_0} |_{X_0=-1} = 0 \quad (6.21) $$

Therefore,

$$ f^*_+(-1 - t) = -g'(-1 + t) $$

We consider the specific wave $g_0$, set up at $t = 0$. This is defined in equation (6.17), and we can find $g'_0(-1 + t) = -\frac{1}{1+z}$. Since the waves propagate with a constant profile the expression $g'(-1 + t)$ is always constant. Therefore, we can integrate $f^*_+$ simply (letting $A$ be a constant), giving

$$ f_+(s) = \left(-g'(-1 + t)\right)s + A \quad (6.22) $$

We assume that the new wave $f_+$ is initialised at $t = T$. Therefore at $X_0 = -1$ and $t = T$ the function $f_+$ is zero, i.e. $f_+(-1 - T) = 0$. From this we can find $A$, and hence we can find the full expression for the new wave. We use the Heaviside function to 'switch-on' the wave. Thus,

$$ f_+ (X_0 - t) = \left(g'\left(-1 + t\right)\right)(t - X_0 - (1 + T))H[t - X_0 - (1 + T)] \quad (6.23) $$

We find the solution in the slug for $1 < t < 2$ using the formula above to determine $f_1(X_0-t)$ and then adding this new wave to the expression in equation (6.19). The solution in the rod remains unchanged, and the solution in the slug is,

$$ x = X_0 + \frac{V}{c} \left(t - \frac{1}{1+z} \left( (t + X_0)H[t + X_0] + (t - X_0 - 2)H[t - X_0 - 2] \right) \right) \quad (6.24) $$

Procedure for when a wave in the slug reaches the interface, $X_0 = 0$

When the wave $f_1$ reaches the interface, a negative travelling wave $g_2$ is set up in the slug, and a positive travelling wave $\tilde{f}_2$ is set up in the rod. These waves maintain the boundary conditions at the interface, i.e. the particle velocities and the stresses must be equal. These conditions must be obeyed when any wave in the slug is incident on the interface. In figure (6.3) these waves are shown as $f_1,f_3,f_5$, etc.

As in the previous section, we now outline the general procedure for finding the two new waves created when a wave in the slug is incident on the interface at $t = T$. Before the wave arrives the boundary conditions at the interface are satisfied. Therefore, we only need examine the incoming wave, $f(X_0-t)$, to determine the new waves $g_+(X_0+t)$ and $\tilde{f}_+(X_0 - \frac{1}{\alpha}t)$. In the following equations the solutions in the slug and the rod (excluding these three waves) are denoted by $u_{\text{old}}(X_0,t)$ and $\tilde{u}_{\text{old}}(X_0,t)$ respectively.

$$ x = X_0 + \frac{V}{c} \left( t + u_{\text{old}}(X_0,t) + f(X_0-t) + g_+(X_0+t) \right) \quad (6.25) $$

$$ \tilde{x} = X_0 + \frac{V}{c} \alpha \left( \tilde{u}_{\text{old}}(X_0,t) + \tilde{f}_+(X_0 - \frac{1}{\alpha}t) \right) \quad (6.26) $$
Substituting the $u$ components of equations (6.25) and (6.26) into (6.11) and (6.12) we obtain,

$$1 + \frac{\partial u_{old}}{\partial t}_{x_0 = 0} - f'(-t) + g'_+ (t) = \alpha \frac{\partial \bar{u}_{old}}{\partial t}_{x_0 = 0} - \bar{f}'_+ (-\frac{1}{\alpha} t)$$

$$z \left[ \frac{\partial u_{old}}{\partial X_0}_{x_0 = 0} + f'(-t) + g'_+ (t) \right] = \frac{\partial \bar{u}_{old}}{\partial X_0}_{x_0 = 0} + \bar{f}'_+ (-\frac{1}{\alpha} t)$$

The solutions $u_{old}$ and $\bar{u}_{old}$ satisfy the boundary conditions at the interface, and therefore we can write,

$$-f'(-t) + g'_+ (t) = -\bar{f}'_+ (-\frac{1}{\alpha} t)$$

$$z \left[ f'(-t) + g'_+ (t) \right] = \bar{f}'_+ (-\frac{1}{\alpha} t)$$

We solve these equations to find,

$$\bar{f}'_+ (-\frac{1}{\alpha} t) = \frac{2z f'(-t)}{(1 + z)} \quad (6.27)$$

The function $\bar{f}_+(X_0 - \frac{1}{\alpha} t)$ is zero when $t = T$ at $X_0 = 0$. The function $f'(-t)$ is a constant expression. Integrating the equation above, and imposing the initial condition, we find

$$\bar{f}_+(s) = \left( \frac{2z f'(-t)}{(1 + z)} \right) \left( s + \frac{T}{\alpha} \right) \quad (6.28)$$

Also, solving for $g'_+$,

$$g'_+ (t) = f'(-t) \left( \frac{1 - z}{1 + z} \right) \quad (6.29)$$

As before, we integrate this expression and impose the initial condition, to obtain

$$g_+(s) = \left( f'(-t) \left( \frac{1 - z}{1 + z} \right) \right) \left( s - T \right) \quad (6.30)$$

Therefore, both of the new waves set up from the interface at time $T$ are,

$$g_+(X_0 + t) = \left( f'(-t) \left( \frac{1 - z}{1 + z} \right) \right) \left( t + X_0 - T \right) H[t + X_0 - T]$$

$$\bar{f}_+(X_0 - \frac{1}{\alpha} t) = \frac{1}{\alpha} \left( \frac{-2z f'(-t)}{(1 + z)} \right) \left( t - \alpha X_0 - T \right) H[t - \alpha X_0 - T]$$

We know $f_1$ and using the expressions above we can find $g_2$ and $\bar{f}_2$ and write the complete solution for both bodies, including the two new waves set up at $t = 2$ (for which $T = 2$). The solution in the slug is now valid for $t < 3$, i.e. until the wave $g_2$ reaches the $X_0 = -1$ boundary, and the solution in the rod is valid for $t < 4$, i.e. until another wave is incident on the interface again. Hence, we obtain

$$x = X_0 + \frac{V}{c} \left( t - \frac{1}{1 + z} \left( (t + X_0) H[t + X_0] + (t - X_0 - 2) H[t - X_0 - 2] \right) \right. \left. + \frac{(1 - z)}{(1 + z)^2} (t + X_0 - 2) H[t + X_0 - 2] \right) \quad (6.31)$$

$$\bar{x} = X_0 + \frac{V}{c} \left( \frac{z}{1 + z} (t - \alpha X_0) H[t - \alpha X_0] - \frac{2z}{(1 + z)^2} (t - \alpha X_0 - 2) H[t - \alpha X_0 - 2] \right) \quad (6.32)$$
6.2.3 General expression & results

We now possess techniques which allow us to propagate waves in the two bodies. When a wave is incident on either \( X_0 = 0 \) or \( X_0 = -1 \), we can maintain the correct boundary conditions by creating new waves. Therefore, following the progress of the waves in the two bodies and referring to figure (6.3), we can give the solution in the slug and the rod, for time \( t < 2(N+1) \).

\[
x = X_0 + \frac{V}{c} \left( t - \frac{1}{(z+1)} \right) \left\{ (t + X_0)H[t + X_0] + (t - X_0 - 2)H[t - X_0 - 2] \right\} \\
- \frac{(z-1)}{(z+1)^2} \left\{ (t + X_0 - 2)H[t + X_0 - 2] + (t - X_0 - 4)H[t - X_0 - 4] \right\} \\
- \frac{(z-1)}{(z+1)^3} \left\{ (t + X_0 - 4)H[t + X_0 - 4] + (t - X_0 - 6)H[t - X_0 - 6] \right\} \\
\vdots \\
- \frac{(z-1)^N}{(z+1)^{N+1}} \left\{ (t + X_0 - 2N)H[t + X_0 - 2N] + (t - X_0 - 2(N+1))H[t - X_0 - 2(N+1)] \right\} \quad (6.33)
\]

\[
\dot{x} = X_0 + \frac{V}{c} \left( t - \frac{z}{z+1} \right) \left( t - \alpha X_0 \right)H[t - \alpha X_0] \\
- \frac{2z}{(z+1)^2} \left( t - \alpha X_0 - 2 \right)H[t - \alpha X_0 - 2] \\
- \frac{2z(z-1)}{(z+1)^3} \left( t - \alpha X_0 - 4 \right)H[t - \alpha X_0 - 4] \\
\vdots \\
- \frac{2z(z-1)^{N-1}}{(z+1)^{N+1}} \left( t - \alpha X_0 - 2N \right)H[t - \alpha X_0 - 2N] \quad (6.34)
\]

In the expression for \( x \), it is seen that there is a sequence of pairs of Heaviside functions, with each pair multiplied by some function of \( z \). Each pair represent two waves \( g_n \) and \( f_{n+1} \) in the slugs, see figure (6.3).

From the boundary conditions the stresses at the interface are equal in both bodies (see equation (6.12)). A positive travelling wave in the slug, \( f_{2N-1} \), incident on the interface at \( t = 2N \) can cause the bodies to part company, and this is indicated by a positive stress at the interface. Computing the stress of the rod at the interface we obtain the following, (valid for \( t < 2(N + 1) \)),
\[
V c \rho \frac{\partial \bar{u}}{\partial x} \bigg|_{x_0 = o} = V c \rho \left( -\frac{1}{(z + 1)^r} H[t] + \frac{2}{(z + 1)^2} H[t - 2] + \frac{2(z - 1)}{(z + 1)^3} H[t - 4] + \cdots + \frac{2(z - 1)^{N-1}}{(z + 1)^{N+1}} H[t - 2N] \right)
\] (6.35)

From the expression above for the stress at the interface, is it clear that the wave propagation within the material causes discontinuities of stress and particle velocities. The additional stress caused by the wave \( f_{2N-1} \) can cause the bodies to part company. Thus, summing the right-hand side of (6.35) for \( 2N < t < 2(N + 1) \) we can state the following; the bodies part company at \( t = 2N \) if the following is satisfied,

\[
V c \rho \left( -\frac{1}{(z + 1)^r} + \frac{2}{(z + 1)^2} \sum_{r=0}^{N-1} \frac{(z - 1)^r}{(z + 1)^r} \right) > 0
\]

This reduces to

\[
-\frac{(z - 1)^N}{(z + 1)^{N+1}} > 0
\]

Therefore if \( z < 1 \) the slug and rod part company at \( t = 2 \) (when \( N = 1 \)). If \( z > 1 \) they never part company. In figure (6.4) and figure (6.5), for specified values of \( z \), we show the position and the stress at the interface. The dashed line indicates that the stress has become positive; beyond the time when the stress first becomes positive the solutions are physically incorrect.

On separation, we can find the velocity of the interface of each body. From equation (6.33) we compute the velocity using only the first two waves, \( g_0 \) and \( f_1 \), since the third wave \( g_2 \) in the slug never propagates as the bodies separate at the same instance this third wave is created. Similarly, to find the velocity of the rod, we consider the solution containing only the first wave \( f_1 \) in the rod.

\[
\frac{\partial \bar{x}}{\partial t} \bigg|_{x_0 = o} = V \frac{z - 1}{c (z + 1)}
\]
\[
\frac{\partial \bar{x}}{\partial t} \bigg|_{x_0 = o} = V \frac{z}{c (z + 1)}
\]

In the slug at \( t = 2 \) the velocity is constant throughout the body, and therefore the mean is equal to the velocity of the interface. In the semi-infinite rod the mean velocity is always zero, even when the velocity of the interface is non-zero. The coefficient of restitution is the
ratio of the relative mean speed of separation to the relative mean speed of approach. Hence, we can find the following expression for the coefficient of restitution (valid for $0 < z < 1$),

$$
    r = \frac{-V}{c} \left( \frac{0 - \frac{z - 1}{z + 1}}{0 - 1} \right)
    = \frac{z - 1}{z + 1}
$$

(6.36)

If $z > 1$ the condition for separation is never satisfied, and the slug and rod remain together. Referring to figure (6.5) where $z > 1$, the stress at the interface is always negative, but as $t \to \infty$, the stress approaches zero. In this case we could argue that the effective coefficient of restitution is zero. In the next section we extend the model by placing a restriction on the length of the rod, and we show that for $z > 1$ the two bodies now separate at some time $t > 2$, caused by a wave in the rod returning to the interface.
Figure 6.5: Graphs of the position and stress of the slug at $X_0 = 0$, for $0 < t < 8$ and $z > 1$.

6.3 A slug hitting a finite length rod

6.3.1 Implications

We now specify that the rod has a finite length, i.e. the location of a cross-section of the rod satisfies $0 < \bar{X}_0 < a\bar{h}_s$ where $a\bar{h}_s$ is the length of the rod. We measure the length of the rod relative to the length of the slug. As we expect the rod to be longer than the slug, we assume that $a > 1$. There is now a new boundary condition at $\bar{X}_0 = a\bar{h}_s$ (in non-dimensional variables, $X_0 = a$). The implication of this addition to the model is that there are now positive and negative travelling waves in both bodies. A wave returning to the interface creates new waves in both bodies. This is illustrated in figure (6.6), where we arbitrarily set $a = 2$. Therefore, in comparison with the model described in the previous section, as time progresses the number of individual waves in the system increases more rapidly.
6.3.2 Propagating waves in the bodies

With the same approach as in section 6.2, after the impact we allow waves to propagate in the two bodies, and we maintain the boundary conditions when necessary. As before we can study general procedures. Firstly when a wave is incident on the boundary $X_0 = a$, and secondly when a wave is incident on the interface.
Procedure when a wave in the rod reaches the \(X_0=a\) boundary

When a positive travelling wave \(\tilde{f}(X_0 - \frac{1}{\alpha} t)\) in the rod reaches the boundary \(X_0 = a\) at \(t = T\) it is reflected, i.e. it sets up a negative travelling wave \(\tilde{g}_+(X + \frac{1}{\alpha} t)\), in order to satisfy the boundary condition requiring the stress to be zero. Hence,

\[
V \tilde{p} \frac{\partial \tilde{u}}{\partial X_0} \bigg|_{X_0=a} = 0
\]

(6.37)

Substituting the new waves into the boundary condition, we obtain

\[
\tilde{g}_+(a + \frac{1}{\alpha} t) = -\tilde{f}(a - \frac{1}{\alpha} t)
\]

(6.38)

As before the right hand side of the expression is a constant. Therefore,

\[
\tilde{g}_+(s) = \left(-\tilde{f}(a - \frac{1}{\alpha} t)\right) s + A
\]

(6.39)

At time \(t = T\) and \(X_0 = a\), the function \(\tilde{g}_+(X_0 + \frac{1}{\alpha} t)\) is zero, i.e. \(\tilde{g}_+(a + \frac{1}{\alpha} T) = 0\), and we can find \(A\). Hence, we can find the full expression for the new wave, by again using the Heaviside function to 'switch on' the wave. Therefore,

\[
\tilde{g}_+(X_0 + \frac{1}{\alpha} t) = \frac{1}{\alpha} \left(-\tilde{f}(a - \frac{1}{\alpha} t)\right)(t - \alpha(a-X_0)-T) H[t - \alpha(a-X_0)-T]
\]

(6.40)

Therefore as an example, referring to figure (6.6), given the incident wave \(\tilde{f}_0\), we find the new wave \(\tilde{g}_{\text{ac}}\) set up at \(t = \alpha a\). From equation (6.34), with \(T = \alpha a\) we obtain

\[
\tilde{f}_0(X_0 - \frac{1}{\alpha} t) = \frac{1}{\alpha} \left(\frac{z}{z+1}\right) (t - \alpha X_0) H[t - \alpha X_0]
\]

(6.41)

Therefore, \(\tilde{f}_0'(a - \frac{1}{\alpha} t) = - \frac{z}{z+1}\), and thus,

\[
\tilde{g}_{\text{ac}}(X_0 + \frac{1}{\alpha} t) = \frac{1}{\alpha} \left(\frac{z}{z+1}\right) (t - \alpha(2a - X_0)) H[t - \alpha(2a - X_0)]
\]

(6.42)

Procedure when a wave in the rod reaches the interface

When a negative travelling wave, \(\tilde{g}(X_0 + \frac{1}{\alpha} t)\) reaches the interface at \(t = T\), two new waves \(g_+(X_0 + t)\) and \(\tilde{f}_+(X_0 - \frac{1}{\alpha} t)\) are set up in the slug and the rod respectively. The solutions in the slug and the rod become,

\[
x = X_0 + \frac{V}{c} \left(t + u_{\text{old}}(X_0, t) + g_+(X_0 + t)\right)
\]

\[
\tilde{x} = X_0 + \frac{V}{c} \alpha \left(u_{\text{old}}(X_0, t) + \tilde{g}(X_0 + \frac{1}{\alpha} t) + \tilde{f}_4(X_0 - \frac{1}{\alpha} t)\right)
\]

Substituting these expressions into (6.11) and (6.12), and noting that the solutions \(u_{\text{old}}\) and \(\tilde{u}_{\text{old}}\) already satisfy the boundary conditions, we find that,

\[
g'_+(t) = \tilde{g}'(\frac{1}{\alpha} t) - \tilde{f}_4'(-\frac{1}{\alpha} t)
\]

\[
z g'_+(t) = \tilde{g}'(\frac{1}{\alpha} t) + \tilde{f}_4'(-\frac{1}{\alpha} t)
\]
Also, at \( t = T \) and \( X_0 = 0 \), the functions \( g_+ \) and \( f_+ \) are zero. Solving the equations above we obtain,

\[
g_+(X_0 + t) = \left( \frac{2g'(\frac{1}{a}t)}{(1+z)} \right) (t + X_0 - T) H[t + X_0 - T] \tag{6.43}
\]

\[
f_+(X_0 - \frac{1}{a}t) = \frac{1}{a} \left( \frac{g'(\frac{1}{a}t)}{(1+z)} \right) (t - \alpha X_0 - T) H[t - \alpha X_0 - T] \tag{6.44}
\]

Again, as an example, we consider the wave \( \bar{g}_{oa} \) returning to the interface, and setting-up two waves in the slug and rod, \( g_{2oa} \) and \( f_{2oa} \). The expression for \( \bar{g}_{oa} \) is given in equation (6.42), and hence \( \bar{g}'_{oa} = \frac{z}{z+1} \). From equations (6.43) and (6.44) we can find expressions for the new waves,

\[
g_{2oa}(X_0 + t) = \frac{2z}{(z+1)^2} (t + X_0 - 2\alpha a) H[t + X_0 - 2\alpha a]
\]

\[
f_{2oa}(X_0 - \frac{1}{a}t) = -\frac{1}{\alpha} \frac{z(z-1)}{(z+1)^2} (t - \alpha(2\alpha + X_0)) H[t - \alpha(2\alpha + X_0)]
\]

### 6.3.3 General expression & results

In non-dimensional time, it takes 1 time unit for a wave to propagate through the slug and \( \alpha a \) time for a wave to propagate through the rod. From a practical point of view we would expect the materials of the slug and the rod to have comparable wave propagation speeds, and therefore \( \alpha \) to be approximately equal to one. In practice the length of the rod is greater than the length of the slug, i.e. \( \alpha > 1 \). Hence, we assume that \( \alpha a > 1 \). Referring to figure (6.6), this implies that of the two waves set up at \( t = 0 \), \( g_0 \) and \( f_0 \), the one in the slug is the first one to return back to the interface.

The wave in the slug could in fact bounce ‘back and forth’ a number of times before the wave in the rod returns to the interface. Indeed if \( z < 1 \) the slug and rod part company at \( t = 2 \) before a negative travelling wave in the rod reaches the interface. However, if \( z > 1 \), we can write the solution for \( t > 2 \).

We are interested primarily in the stress at the interface. So initially we will only reflect the first wave \( f_0 \) at the boundary \( X_0 = a \) of the rod. When this wave reaches the interface we then examine the stress at the interface, which allows us to determine whether the bodies part company. Any subsequent waves in the rod do not influence the interface at time \( t = 2\alpha a \). However, we compute the mean velocity of the slug and rod on parting, and these additional waves are important.

The following solutions for the slug and rod include all the waves which are active at \( t = 2\alpha a \).

Also, \( N \) is such that it satisfies \( 2N < 2\alpha a < 2(N + 1) \) (where \( N > 0 \)). The final terms in both solutions are new waves \( (g_{2oa} \text{ and } f_{2oa}) \) created when the first wave in the rod returns.
to the interface. We obtain the following solutions,

\[ x = X_0 + \frac{V}{c} \left( t - \frac{1}{(z+1)} \left\{ (t+X_0)H[t+X_0] + (t-X_0-2)H[t-X_0-2] \right\} \right. \]

\[ - \frac{(z-1)}{(z+1)^2} \left\{ (t+X_0-2)H[t+X_0-2] + (t-X_0-4)H[t-X_0-4] \right\} \]

\[ - \frac{(z-1)^2}{(z+1)^3} \left\{ (t+X_0-4)H[t+X_0-4] + (t-X_0-6)H[t-X_0-6] \right\} \]

\[ \vdots \]

\[ - \frac{(z-1)^N}{(z+1)^{N+1}} \left\{ (t+X_0-2N)H[t+X_0-2N] \right. \]

\[ + \left. (t-X_0-2(N+1))H[t-X_0-2(N+1)] \right\} \]

\[ + \frac{2z}{(z+1)^2} \left\{ (t+X_0-2\alpha)H[t+X_0-2\alpha] \right\} \] (6.45)

\[ \ddot{x} = X_0 + \frac{V}{c} \left( \frac{z}{z+1} \left\{ (t-\alpha X_0)H[t-\alpha X_0] + (t-\alpha(2\alpha-X_0))H[t-\alpha(2\alpha-X_0)] \right\} \right. \]

\[ - \frac{2z}{(z+1)^2} \left\{ (t-\alpha X_0-2)H[t-\alpha X_0-2] + (t-\alpha(2\alpha-X_0)-2)H[t-\alpha(2\alpha-X_0)-2] \right\} \]

\[ - \frac{2z(z-1)}{(z+1)^3} \left\{ (t-\alpha X_0-4)H[t-\alpha X_0-4] + (t-\alpha(2\alpha-X_0)-4)H[t-\alpha(2\alpha-X_0)-4] \right\} \]

\[ \vdots \]

\[ - \frac{2z(z-1)^{N-1}}{(z+1)^{N+1}} \left\{ (t-\alpha X_0-2N)H[t-\alpha X_0-2N] \right\} \]

\[ - \frac{z(z-1)}{(z+1)^2} \left\{ (t-\alpha(2\alpha+X_0))H[t-\alpha(2\alpha+X_0)] \right\} \] (6.46)

At the interface, the stress in the slug and the rod is given by \( Vc \rho \frac{\partial^2 u}{\partial X_0^2} \bigg|_{X_0=0} \) and \( V\partial \bar{\sigma} \frac{\partial \bar{\sigma}}{\partial X_0} \bigg|_{X_0=0} \) respectively. Both evaluate to the same expression, and we write as a condition for the parting of the two bodies,

\[ Vc \rho \left( \frac{2z}{(z+1)^2} - \frac{(z-1)^N}{(z+1)^{N+1}} \right) > 0 \] (6.47)

or,

\[ \frac{Vc \rho}{(z+1)^2} \left( 2z - (z+1) \left( \frac{z-1}{z+1} \right)^N \right) > 0 \] (6.48)

Therefore since \( z > 1 \) we know that \( 0 < \left( \frac{z-1}{z+1} \right)^N < 1 \) for all \( N > 0 \). Also \( 2z - (z+1) > 0 \), and therefore the above condition is satisfied for any \( N \) and we conclude that that the slug and the rod part company at \( t = 2\alpha \).

In both cases \((z < 1 \text{ and } z > 1)\), it is a wave returning to the interface in one of the bodies which causes the parting. This wave initiates new waves in both bodies, which travel away.
from the interface, and the last terms of equations (6.45) and (6.46) are these new waves. These new waves are important in determining if parting occurs, however if parting does occur they do not propagate since the bodies separate at the same instance as the new waves are created. In this study we do not calculate the solutions in the slug and rod after parting. However we do compute the mean velocities of the slug and the rod at the instance of parting, in which case we can fully specify the solutions in the slug and the rod. It is possible to find the correct solutions for the slug and the rod any time after parting has occurred. In which case the mathematical model must be re-defined; there are two separate models since there is no interaction between the bodies. The existing waves are still present in each body. However new waves must be created to maintain the new boundary conditions, which are, at both boundaries of each body the stress is zero.

To compute the mean velocities at parting we use equations (6.45) and (6.46), where the final terms in both expressions are not included. We calculate the definite integral of the velocity of the body with respect to $X_0$ over the entire height of the body. Hence, the mean velocity of the slug is,

$$\frac{V}{c} \int_{X_0=-1}^{X_0=0} \left(1 + \frac{\partial u}{\partial t}\right) dX_0$$

(6.49)

The mean velocity of the rod is given by,

$$\frac{1}{a} \frac{V}{c} \int_{X_0=0}^{X_0=\alpha a} a \frac{\partial u}{\partial t} dX_0$$

(6.50)

The expressions for the mean velocities are functions of $z$, $\alpha a$ and time $t$. We do not show the full expressions here, but we show the relationship between $z$ and the mean velocities, for fixed values of $\alpha a$ in figure (6.7). For $z < 1$ the mean velocity at the parting time of $t = 2$ is calculated and for $z > 1$ the mean velocity at $t = 2\alpha a$ is calculated. The coefficient of restitution is the ratio of the relative mean speed of separation to the relative mean speed of approach, and this is also shown in the figure.

If we let $\alpha a \to \infty$, we obtain the system studied in section 6.2. Recalling the results of the previous section, the mean velocity of the rod is always zero. Also, for $z > 1$ the slug and rod never actually part company, although as $t \to \infty$ the stress at the interface approaches zero, at which time the mean velocity of the slug is zero and hence we can perhaps consider the effective coefficient of restitution to be zero. In figure (6.7) we plot the results for $\alpha a = 25$, and it is quite clear that if we let $\alpha a \to \infty$, we would observe the same behaviour as predicted in the previous section.

It is worth considering the practical implications of the results shown in the graphs of figure (6.7). If the two bodies are constructed from the same material, then $z = 1$ (this is the case for a laminated beam of one material). Perhaps, it would be useful to focus our attention on the area surrounding the parameter choice $z = 1$. Although, we are aware that this is not a
Figure 6.7: Graph of the relationship between \( z \) and the coefficient of restitution, \( e \)
single degree-of-freedom system, in the case of a delamination and main beam constructed from the same material and given that the thickness of the beam is large compared with the thickness of the delamination, the quantity $\alpha a$ is large. For large $\alpha a$ ($\alpha a > 5$) the coefficient of restitution is within the range $0 < e < 0.3$, for $0.8 < z < 1.3$. As we have previously commented, the effective coefficient of restitution for the experimental apparatus was in the region $0 < e < 0.3$, particularly when the delamination was constructed from thin steel foil (where $z \approx 1$). It is encouraging that the theoretical model predicts similar behaviour.

6.4 A slug with side coupling hitting a semi-infinite length rod

6.4.1 Introduction

In this section we study a single slug and semi-infinite rod system as in section 6.2. However, we now couple the slug with continuously distributed shear springs attached to two stationary slugs on either side, as seen in figure (6.8). This is a first step in progressing from a one degree-of-freedom model, to a full discrete multi degree-of-freedom model of a beam. One of the implications of the side coupling is that the waves in the slug no longer propagate with constant profiles. Without coupling the stress at the interface is shown as a series of constant step functions. Waves in the slug arriving at the interface cause these step functions and they occur at time $t = 2$ and subsequent multiples of 2. With coupling, the time when the bodies part company is dependent on the strength of the coupling.

In this section we also introduce new techniques to implement the wave propagation in the system. In the first two sections of this chapter, sections 6.2.1 and 6.3, the lengthy analysis, where each wave was defined and computed individually, proved insightful. Through the ‘tracking’ of each wave we can appreciate the effect of each wave propagating from the impact. However, using the method discussed in this section, the process is more efficient.

We develop equations of motion for the slug and the rod, and then take Laplace transforms. Once in the frequency domain, we impose the boundary conditions, and then we find the inverse Laplace transform to give the time solution. We expect to observe a time solution involving step functions, which ‘switch-on’ new waves as time increases. Hence, the solution $u$ contains all the waves propagating from the impact, and through the expansions and subsequent inverse Laplace transform, the individual waves in the solution reveal themselves.

6.4.2 Modelling

Figure (6.8) shows the configuration of the slug, with side coupling. As in the previous two sections, the positions of the slug and the rod are given by,

$$\ddot{x}(\vec{x}_0, \vec{t}) = \vec{x}_0 + V\vec{t} + \vec{u}(\vec{x}_0, \vec{t})$$
$$\ddot{x}(\vec{x}_0, \vec{t}) = \vec{x}_0 + \vec{u}(\vec{x}_0, \vec{t})$$
Figure 6.8: Detail of a slug which is coupled to stationary slugs at either side. Also shown is the corresponding rod.

As in section 6.2.2, we consider the forces on the small section of the slug of thickness $\delta X_0$ (shown in the figure). In addition to the internal forces, the section is subject to a force from the coupling springs. This force is proportional to the relative displacement of the cross-section $X_0$ of both bodies, and this constant of proportionality, $k$, is defined per unit area of adjacent sides of the slug. The area of the adjacent sides of the slug is $b h$. Also we let $A$ be the area of the impacting surface. Therefore,

$$
\rho A \frac{\partial^2 \bar{u}}{\partial t^2} \delta \tilde{X}_0 = EA \frac{\partial^2 \bar{u}}{\partial X_0^2} \delta \bar{X}_0 - 2 k b (V \bar{t} + \bar{u}) \delta \bar{X}_0
$$

(6.51)

Letting $c^2 = \frac{E}{\rho}$ and $p^2 = \frac{2 kb}{A \rho}$ we can obtain

$$
\frac{\partial^2 \bar{u}}{\partial t^2} = c^2 \frac{\partial^2 \bar{u}}{\partial X_0^2} - p^2 (V \bar{t} + \bar{u})
$$

(6.52)

We use the same non-dimensionalising scheme as in section 6.2.1. After non-dimensionalising equation (6.52), we obtain the following equation and associated parameter $q^2$,

$$
\frac{\partial^2 u}{\partial \bar{t}^2} = \frac{\partial^2 u}{\partial \bar{X}_0^2} - q^2 (t + u)
$$

(6.53)

$$
q^2 = \frac{h^2 p^2}{c^2}
$$

(6.54)
We take Laplace transforms of the above equation. Therefore, we define,

\[ \hat{u}(X_0, s) = \int_0^\infty u(X_0, t)e^{-st}dt \]  

(6.55)

The initial conditions arising from the Laplace transform are zero since at time \( t = 0 \) the non-zero solution \( u \) is just at the point of being set up. Hence equation (6.53) becomes,

\[ \frac{\partial^2 \hat{u}}{\partial X_0^2} - (s^2 + q^2) \hat{u} = \frac{q^2}{s^2} \]  

(6.56)

Letting \( \lambda = \sqrt{s^2 + q^2} \) and solving the equation above we find,

\[ \hat{u}(X_0, s) = P e^{-\lambda X_0} + Q e^{\lambda X_0} - \frac{q^2}{s^2(s^2 + q^2)} \]  

(6.57)

where \( P \) and \( Q \) are arbitrary functions of \( s \).

We found the non-dimensional equation of motion for the rod in section 6.2.2, which is

\[ \frac{\partial^2 \bar{u}}{\partial X_0^2} - \alpha^2 \frac{\partial^2 \bar{u}}{\partial t^2} = 0 \]  

(6.58)

Taking Laplace transforms,

\[ \frac{\partial^2 \hat{u}}{\partial X_0^2} - s^2 \alpha^2 \hat{u} = 0 \]  

(6.59)

This has a solution,

\[ \hat{u}(X_0, s) = Fe^{-s \alpha X_0} + Ge^{s \alpha X_0} \]  

(6.60)

Note that \( \alpha^2 = \frac{c^2}{2} \), and \( F \) and \( G \) are both arbitrary functions of \( s \). Due to the requirement that the solution vanish at infinity, \( G = 0 \).

Boundary conditions

In order to find the solutions \( u(X_0, t) \) and \( \bar{u}(X_0, t) \) in the slug and the rod, we obtain expressions for \( P, Q \) and \( F \), and then find the inverse Laplace transforms of \( \hat{u}(X_0, s) \) and \( \bar{u}(X_0, s) \). Therefore, the first step is to impose the boundary conditions on the solutions and hence find the unknown expressions. The boundary conditions are:
1. The end of the slug \((\bar{X}_0 = -h, \bar{X}_0 = -1)\) is free of stress:

\[
\left. \frac{E \partial \bar{u}}{\partial \bar{X}_0} \right|_{\bar{X}_0 = -h} = 0
\]  

(6.61)

Non-dimensionalising,

\[
\left. V \rho_c \frac{\partial \bar{u}}{\partial \bar{X}_0} \right|_{\bar{X}_0 = -1} = 0
\]

(6.62)

Taking Laplace transforms,

\[
\left. \frac{\partial \tilde{u}}{\partial \bar{X}_0} \right|_{\bar{X}_0 = -1} = 0
\]

(6.63)

2. Continuity of particle velocities at interface,

\[
\left. \frac{\partial \bar{u}}{\partial \bar{t}} \right|_{\bar{X}_0 = 0} = \left. \frac{\partial \tilde{\bar{u}}}{\partial \bar{t}} \right|_{\bar{X}_0 = 0}
\]

Substituting for \(\bar{u}\) and \(\tilde{\bar{u}}\),

\[
\left. \left( V + \frac{\partial \bar{u}}{\partial \bar{t}} \right) \right|_{\bar{X}_0 = 0} = \left. \frac{\partial \tilde{\bar{u}}}{\partial \bar{t}} \right|_{\bar{X}_0 = 0}
\]

Non-dimensionalise and take Laplace transforms,

\[
\left. \frac{1}{s} + s \tilde{\bar{u}} \right|_{\bar{X}_0 = 0} = s \alpha \tilde{\bar{u}} \bigg|_{\bar{X}_0 = 0}
\]

Using the solution \(\tilde{\bar{u}} = Fe^{-s \alpha \bar{X}_0}\),

\[
\left. \frac{1}{s} + s \tilde{\bar{u}} \right|_{\bar{X}_0 = 0} = s \alpha F
\]

(6.64)

(6.65)

3. Continuity of stress at the interface,

\[
\rho c^2 \left. \frac{\partial \bar{u}}{\partial \bar{X}_0} \right|_{\bar{X}_0 = 0} = \rho c^2 \left. \frac{\partial \tilde{\bar{u}}}{\partial \bar{X}_0} \right|_{\bar{X}_0 = 0}
\]

Non-dimensionalise and take Laplace transforms, using \(z = \frac{\rho c}{\rho \bar{c}}\),

\[
\left. \frac{z}{z} \frac{\partial \tilde{\bar{u}}}{\partial \bar{X}_0} \right|_{\bar{X}_0 = 0} = \left. \frac{\partial \tilde{\bar{u}}}{\partial \bar{X}_0} \right|_{\bar{X}_0 = 0}
\]

(6.66)

Hence,

\[
\left. \frac{z}{z} \frac{\partial \tilde{\bar{u}}}{\partial \bar{X}_0} \right|_{\bar{X}_0 = 0} = -s \alpha F
\]

(6.67)
Substituting equation (6.57) into boundary condition 1, we obtain \(-PeA + QeA = 0\). From this we can write, \(Q = \frac{1}{2}KeA\) and \(P = \frac{1}{2}Ke^{-\lambda}\), where \(K\) is an arbitrary function of \(s\). Therefore, the solution in the slug is,

\[
\dot{u} = K \cosh(\lambda(1 + X_0)) - \frac{q^2}{s^2(s^2 + q^2)}
\]

(6.68)

Combining boundary conditions 2 and 3, and using the solution (above) in the slug,

\[
\frac{1}{s} + s(K \cosh(\lambda) - \frac{q^2}{s^2(s^2 + q^2)}) = -zK\lambda \sinh(\lambda),
\]

(6.69)

From this equation we can find an expression for \(K\),

\[
K = \frac{1}{s} - \frac{s q^2}{s^2(s^2 + q^2)} - \frac{s^2(1 + X_0)}{-z\lambda \sinh(\lambda) - s \cosh(\lambda)}.
\]

(6.70)

Substituting for \(K\) in equation (6.68)

\[
\dot{u}(X_0, s) = \left(\frac{1}{s} - \frac{s q^2}{s^2(s^2 + q^2)}\right) \cosh(\lambda(1 + X_0)) - \frac{q^2}{s^2(s^2 + q^2)} \frac{s \cosh(\lambda(1 + X_0))}{-z\lambda \sinh(\lambda) - s \cosh(\lambda)} - \frac{q^2}{s^2(s^2 + q^2)}
\]

(6.71)

Also in the \(s\)-domain we can find the stress at any point within the slug, and the mean velocity of the slug at any time. Thus, the stress is,

\[
Vcp \frac{\partial \dot{u}(X_0, s)}{\partial X_0} = Vcp \frac{s \sinh(\lambda(1 + X_0))}{\sqrt{s^2 + q^2}(-z\lambda \sinh(\lambda) - s \cosh(\lambda))}
\]

(6.72)

To find the mean velocity of the slug we use the expression for the velocity of any point in the slug, and integrate over the entire length of the slug to find the mean velocity. The displacement of the slug is given by, \(x(X_0, t) = X_0 + \frac{V}{c}(t + u(X_0, t))\), and the velocity is,

\[
\frac{\partial x(X_0, t)}{\partial t} = V \left(1 + \frac{\partial u(X_0, t)}{\partial t}\right)
\]

(6.73)

Therefore, the mean velocity is,

\[
V_m(t) = \frac{V}{c} \int_{X_0=-1}^{X_0=0} \left[1 + \frac{\partial u(X_0, t)}{\partial t}\right] dX_0
\]

\[
= \frac{V}{c} + \frac{V}{c} \int_{X_0=-1}^{X_0=0} \frac{\partial u(X_0, t)}{\partial t} dX_0
\]

(6.73)
Taking a Laplace transform of the above expression gives,

\[ V_m(s) = \frac{V}{cs} + \frac{V}{c} \int_{X_0=-1}^{X_0=0} s \tilde{u}(X_0, s) dX_0 \]

\[ = \frac{V}{cs} + \frac{V}{c} \int_{X_0=-1}^{X_0=0} \left[ s \left( \frac{s \cosh(\lambda(1+X_0))}{(s^2 + q^2)(-z\lambda \sinh(\lambda) - s \cosh(\lambda))} - \frac{q^2}{s^2(s^2 + q^2)} \right) \right] dX_0 \]

\[ = \frac{V}{cs} + \frac{V}{c} \left[ \frac{s^2 \sinh(\lambda)}{\lambda(s^2 + q^2)(-s \cosh(\lambda) - z\lambda \sinh(\lambda))} - \frac{q^2}{s(s^2 + q^2)} \right] \]

(6.74)

where \( \mathcal{L}^{-1} \left\{ \frac{q^2}{s(s^2 + q^2)} \right\} = 1 - \cos(q t) \).

Now we have three expressions, all in the s-domain, describing the position, stress and mean velocity of the slug. We now have to invert these expressions to find the time solution. Apart from the analytical inverse of the last term in the mean velocity and position expressions, the equations above do not have straightforward analytical inverse Laplace transforms. However, there are a number of other methods at our disposal and we consider some of them in the following sections.

6.4.3 Methods for finding the inverse Laplace transform

We can perform the inversion of the Laplace transform using three methods, namely

- A computer program implementing Crump's method [6], which estimates values of the inverse Laplace transform for a given function \( f(s) \) defined for complex values of \( s \). This is used by the NAG Fortran Library routine C06LAF. Using this routine is quite tedious however, and requires a trial-and-error approach for the choice of some parameters passed to the routine, and hence was only used to verify results produced using other methods, and to verify the algebra in the model derivation.

- We can find some of the complex poles of the function we wish to invert, and then use the theory of residues to find the inverse by summing the residues at these poles. This approach is considered first. The solution comprises of a summation of contributions from increasingly higher frequency oscillations; with a larger number of poles used, higher frequency components are contained in the solution. However, due to the discontinuities the solution sometimes overshoots. This is a disadvantage of this method since these overshoots can give false parting times.

- Finally, we can re-write the original function. We expand particular terms in the form of a series, and then express the whole function in the form of a series. We can find the inverse Laplace transform of each individual term in this series. We find that the solution comprises of components which 'switch-on' at points in time, and as we invert term-by-term we find components which 'switch on' at at increasingly larger
times. The disadvantage of this method is that the expansion is made about \( q = 0 \) and includes term up to \( O(q^6) \). Hence, the accuracy of the solution is dependent on the choice of \( q \), whereas for the summation of residues method this is not an issue.

In the next two sections, we first use the summation of residues method, and then the expansion method to perform the inverse Laplace transform. The computer implementation of Crump’s method is used to verify the results in both instances.

6.4.4 Inverting by summing the residues at the poles

We now consider finding the inverse Laplace transforms of the expressions above by summing the residues at the poles of the complex function. Initially we find the inverse when the system has zero coupling and then progress to non-zero coupling.

Zero coupling

Consider zero coupling between the slugs. In this case \( k = 0 \) and therefore \( q = 0 \) and \( \lambda = s \). Removing the coupling means that we have the same system as we studied in section 6.2. From equation (6.71), the solution at \( X_0 = 0 \) is,

\[
\hat{u}(0,s) = \frac{\cosh(s)}{s^2(-z \sinh(s) - \cosh(s))} \quad \text{at} \quad q = 0, \lambda = s
\]

We compute the inverse Laplace Transform by summing the residues at the poles of \( \hat{u}(0,s) \). Although there are an infinite number of poles, we sum the residues of a finite number of poles nearest to the real axis. As the number of poles used in the summation increases, the frequency of the highest frequency component increases. In results displayed in this chapter, 30 pairs of complex conjugate poles were used. This gives adequate accuracy for reasonable computational expense. We use the following summation to find the solution,

\[
\hat{u}(0,t) = \sum \text{residues of } e^{st}\hat{u}(0,s) \text{ at poles of } \hat{u}(0,s)
\] (6.75)

The poles of \( \hat{u}(0,s) \) are given by \( s^2(-z \tanh(s) - 1) = 0 \). There is a second order pole at \( s = 0 \). The other roots are given by,

\[
s_r = \begin{cases} 
  m + i(x \pi + \pi r) & \text{if } z < 1 \\
  m + i\pi r & \text{if } z > 1
\end{cases}
\] (6.76)

where \( m = \frac{1}{2} \log \left| \frac{z - 1}{z + 1} \right| \) and \( r = -\infty, \ldots, -1, 0, 1, \ldots, \infty \).
The residue at the second order pole at \( s = 0 \) is

\[
\lim_{s \to 0} \frac{d}{ds} \left\{ \frac{s^2 e^{st}}{s^2 (-z \tanh(s) - 1)} \right\} = \lim_{s \to 0} \left\{ \frac{-e^{st} s \text{sech}^2(s)}{(-1 - z \tanh(s))^2} + \frac{e^{st} t}{-1 - z \tanh(s)} \right\} = z - t
\]

The residue at \( s = s_r \) is

\[
\lim_{s \to s_r} \left\{ \frac{(s - s_r) e^{st}}{s^2 (-z \tanh(s) - 1)} \right\} = \lim_{s \to s_r} \left\{ \frac{s - s_r}{-z \tanh(s) - 1} \right\} \lim_{s \to s_r} \left\{ \frac{e^{st}}{s^2} \right\}
\]

Using l'Hôpital's rule,

\[
\lim_{s \to s_r} \left\{ \frac{\cosh^2(s)}{-z} \right\} \lim_{s \to s_r} \left\{ \frac{e^{st}}{s^2} \right\}
\]

We can evaluate the above limit using equation (6.76) for \( s_r \). We will find two expressions, one for \( z < 1 \) and one for \( z > 1 \). We can now use equation (6.75) and sum the residues to find a numerical solution for \( u(0, t) \). Graphical results are not included here, however we obtain results which look identical to those in figures (6.4) and (6.5), but with superimposed high frequency oscillations arising at the discontinuities, due to taking a finite number of poles in the summation.

**Non-zero coupling**

When \( k \neq 0 \) (and hence \( q \neq 0 \)), we have from equation (6.71) the following expression for \( \hat{u}(X_0, s) \)

\[
\hat{u}(X_0, s) = \frac{s \cosh(\lambda(1 + X_0))}{(s^2 + q^2)(-z \lambda \sinh(\lambda) - s \cosh(\lambda))} - \frac{q^2}{s^2(s^2 + q^2)} \hat{u}_1(X_0, s)
\]

We can find an analytical expression for the inverse Laplace Transform of the second term of \( \hat{u}(X_0, s) \),

\[
\mathcal{L}^{-1} \left\{ \frac{q^2}{s^2(s^2 + q^2)} \right\} = t - \frac{\sin(q t)}{q}
\]

We use the following to find the inverse of \( \hat{u}(X_0, s) \),

\[
u(X_0, t) = \sum \text{residues of } e^{st} \hat{u}_1(X_0, s) \text{ at poles of } \hat{u}_1(X_0, s) - \left( t - \frac{\sin(q t)}{q} \right)
\]
Two of the poles of $\hat{u}(X_0, s)$ are at $s = \pm iq$. The other poles can be found numerically. Figures (6.9) and (6.10) show two examples where we fix $z$ and vary $q$, and then plot the first few pairs of complex conjugate poles. If we plot a larger number of poles, these appear on the graph at positions further away from the real axis. The roots for $q=0$, are indicated by the small circles on the graphs. As the parameter $q$ increases, the positions of the roots change, and this movement is shown by the arrows in the figures. The numerical routine to find the roots is an implementation of the simplex method [11], which finds the roots to a specified accuracy.
In figure (6.9) for $q = 0$ there are three roots at $0 + i0$. As $q$ increases, two roots move in opposite directions along the imaginary axis, forming a complex conjugate pair. The third root from $0 + i0$ moves left along the real axis, and at some point meets another root coming from the other direction. At this point both roots move from the real axis, and form a complex conjugate pair. In figure (6.10) for $q = 0$, again there are three roots at $0 + i0$. As $q$ increases two of the roots form a complex conjugate pair moving on the imaginary axis, and the third root moves left along the real axis.

The residue at pole $s_r$ is computed,

$$
\lim_{s \to s_r} \left\{ \frac{e^{st} s \cosh(\lambda(1 + X_0))}{(s^2 + q^2)(-z\lambda \sinh(\lambda) - s \cosh(\lambda))} \right\}
$$

In order to compute the residues we must specify $z$, $q$ and $X_0$. Normally we consider $X_0 = 0$, i.e. the interface between the two bodies. As in equation (6.80), by summing the residue at each pole and then adding the analytic solution we find $u(X_0, t)$. Figures (6.11-6.18) show selected results, where $X_0 = 0$ and the position $x(0, t)$ and stress time history are shown for various values of $z$ and $q$. Beyond the time when the stress first becomes positive the solutions shown in the graphs are physically incorrect since parting has occurred, and this is indicated by the dashed line. As before, the presence of high frequency oscillations (a product of the residue summing process) can be seen imposed on the solutions in the graphs. When there is zero coupling and $z > 1$ the slug and the rod do not part company. However, the presence of the coupling causes the bodies to always part company and the time at which this occurs depends on the strength of the coupling. Perhaps this is to be expected; the force exerted by the coupling acts as if to pull the slug away from the rod.

6.4.5 Using series expansions and inverting term-by-term

Another method for finding the inverse Laplace transform is to re-write the expressions for the position, velocity and stress using series expansions for the component terms, and then invert analytically the resulting expression term-by-term using well known formulae. The details of this procedure are contained in Appendix D.

We now compute the parting times and mean velocities for a range of selections of $q$ and $z$. A bisection routine is used to automatically calculate the parting time, i.e. to find when the stress at the interface first becomes positive. Since the solutions produced by summing the residues at the poles give overshoots at the discontinuities, this can give false parting times. This is why we choose to use the expansion method to find the results shown in figure (6.19). The expansion is of order $O(q^6)$, and therefore for reasonably small values of $q$ we can expect to obtain reliable results.
6.4.6 Discussion

Figure (6.19) shows the parting times and the mean parting velocity of the slug, for varying values of \( z \) and some sample values of \( q \). When calculating the mean velocity we take \( V/c = 1 \). These are computed from the expressions derived in Appendix D. Note that since the rod is semi-infinite its mean velocity is always zero. We can obtain positive mean velocities of the slug, however on parting the pointwise velocity of the rod at the interface must be greater than that of the slug.

It is interesting to compare the results for \( q = 0.01 \) with the results for \( ax = 25.0 \) in figure (6.7). We notice that the mean velocity of the slug is comparable for both cases. This is to be expected as a coupling of \( q = 0.01 \) is very small, approaching the zero coupling of figure (6.7). In the absence of coupling to neighbouring slugs, and if the rod is infinite, the impacting bodies do not part company for \( z > 1 \). However, the presence of a small amount of coupling does cause the bodies to part company, with the slug moving at a low velocity. Conversely with zero coupling and a finite length rod, the first wave in the rod returning to the interface also causes the bodies to part company. If the time for a wave propagating...
in the rod is much larger than the time in the slug, the slug again moves at a low velocity after parting.

We examine the results for \( z < 1 \). For all \( q \), parting occurs at \( t = 2 \), and there is an increase in the mean slug speed as the coupling strength increases. This increase in speed however is relatively small. Referring to figures (6.11–6.18), it can be seen that when \( z \ll 1 \), the displacement of the interface at parting is small compared to when \( z > 1 \). Perhaps we can then say that the strength of the coupling has only a small influence on the parting velocity, since on parting the slug has not moved far from its rest position, and hence the wave propagation from the impact mainly determines the resulting motions.

For \( z > 1 \), the strength of the coupling has a greater influence on the mean velocity. As the coupling increases, the time of parting decreases. In figures (6.15–6.18), we observe that the time of parting increases as \( z \) increases. Similarly when \( q = 0 \), although parting never occurs, with increasing \( z \) the stress at the interface is increasingly slower to approach zero.

When \( q = 0 \) the time history of the stress at the interface is seen as a series of constant Heaviside functions which ‘switch on’ at time multiples of 2. The time of parting is when
the stress first becomes positive and for \( q = 0 \) this time is seen to be an even integer. For non-zero \( q \), the gradient of each region of the stress history is non-zero and moreover this gradient increases as \( q \) increases. Hence, we obtain non-integer parting times, and we define the region of non-integer parting times as the transition period. In figure (6.19) as \( q \) increases, the range of \( z \) for each transition period between even integer parting times increases. As \( z \) increases the gradient of the mean slug velocity alternates between positive and negative, with a change occurring at the start and finish of each transition period. During the transition itself the mean velocity gradient is always negative. This can be explained as follows. For a fixed parting time, and fixed \( q \), as \( z \) increases the mean velocity of the slug increases. However, during the transition period the contact period increases, and in this time a new negative travelling wave is set up in the slug, which reduces its positive velocity.

As we have already mentioned, when we calculate the mean velocity for figure (6.19) we take \( V/c = 1 \). The non-dimensional impacting velocity is \( V/c \). Therefore variation of \( V/c \) merely scales the mean velocity in the figure, hence importantly the ratio of impacting speed to the mean rebounding speed is the same. Therefore, as the mean velocity of the rod is always
zero, the mean velocity of the slug at separation is a measure of the effective coefficient of restitution. For any choice of $V/c$, as $z \to 0$ the impact approaches an elastic impact ($e = 1$) and for $z > 1$ a large proportion of the energy is lost during the impact, and therefore the effective coefficient of restitution is close to zero.

### 6.5 Chapter summary

In this chapter we looked at some single degree-of-freedom impacting systems. We observe results which when considering the limitations of the models, make intuitive sense.

Summarising, we first consider the uncoupled system. If $z < 1$, the parting occurs at $t = 2$ when the first wave in the slug returns back to the interface. For $z > 1$, separation is caused by the first wave in the rod returning back to the interface, and therefore we only observe parting if we make the rod of finite length, thus allowing reflections in the rod. As we have previously commented, for $z \approx 1$, and for $aa > 5$, we predict an effective coefficient of restitution in the range $0 < e < 0.3$. We then couple the slug to two stationary slugs, and
model the impact between the slug and a semi-infinite rod. Again parting occurs at $t = 2$ for $z < 1$. For $z \ll 1$, the impact approaches an elastic impact. For $z > 1$, the coupling causes separation to occur. The stronger the coupling, the shorter the time before parting, and the slug always has a small mean parting velocity.

In the next two chapters we find that we shall be reusing some of these results and applying them to more complicated impacting systems.
Figure 6.16: The position and stress of the slug at $X_0 = 0$ and $z = 1.36$. 
Figure 6.17: The position and stress of the slug at $X_0 = 0$ and $z = 1.63$. 
Figure 6.18: The position and stress of the slug at $X_0 = 0$ and $z = 1.9$. 
Figure 6.19: Slug departing times and mean velocities
Chapter 7

System of Connected Slugs with Nearest Neighbour Coupling

7.1 Introduction

In this chapter and the next, by considering the wave propagation after the impact, we model the impact between a delamination and its main beam. Since the beam equations take no account of displacement variations through the thickness of the beam which are introduced by wave propagation as a consequence of the impact, it is necessary to solve the full three-dimensional elastodynamic problem in order to determine the subsequent motion. This is a very complicated procedure and therefore we take an alternative approach where a discretised or lumped model is employed. The delamination is considered to be constructed from a series of slugs, which are connected by shear springs. This is done

Figure 7.1: A model of the delamination and the main beam, composed of blocks with shear spring coupling.
with the expectation that as the number of slugs approaches infinity, we obtain a closer approximation to the continuous model. The impacting region of the main beam is modelled as a series of rods, where impacts occur between corresponding pairs of slugs and rods in the models.

This discrete model is a significant simplification. When the delamination is moving freely, the slugs move as rigid bodies. The effect of the coupling is to introduce a finite number of natural modes, equal to the number of slugs in the discretisation.

When an impact occurs between the delamination and the main beam, we assume that the delamination is moving in the fundamental mode of vibration, and in accord with the assumption discussed in the first chapter, both the slugs and the rods are arranged in a straight line. During the impact we use similar techniques to those which were discussed in the previous chapter, to model the wave propagation in each slug/rod pair. We can find the solution at each slug in the discretisation.

In this chapter each slug is coupled by shear springs to its nearest neighbour. In the next chapter, we couple each slug to its nearest and next-nearest neighbour. When the slugs move as rigid bodies, for nearest neighbour coupling the continuous structure behaves like a vibrating string, and for nearest and next-nearest neighbour coupling the continuous structure behaves like a beam in bending.

We would like to determine whether an impact in the fundamental mode results in a rebound in the fundamental mode. We comment on the parting times, velocities, and in particular the effective coefficient of restitution.

7.2 Derivation of model

Consider a system of elastic slugs, connected in a 'train' by a series of shear springs as in figure (7.2). We number the slugs from 1 through to \( n \), where each slug is coupled to its neighbours. Slug 1 is connected to slug 0 which is not free to move, and also to slug 2. Slug \( n \) is at the free end of the beam and is only coupled to slug \( n-1 \). We can visualise the coupling between two slugs as a continuous distribution of springs connecting corresponding cross-sections \( X_0 \) of both bodies. Hence, the coupling between each slug is given by \( k \) which is defined as the force per unit extension of the spring, per unit area.

We assume that an impact between every slug and its respective rod occurs at \( t = 0 \), and at this time the velocity of slug \( i \) \( (i = 1, \ldots, n) \) is given by \( V_i \). If the bodies are elastic, the
position of the slug and the rod at time $t$, for cross-section $X_0$, are given by the following,

$$\ddot{x}_i(X_0, t) = \ddot{X}_0 + \ddot{V}_i + \ddot{u}_i(X_0, t)$$  \hspace{1cm} (7.1)
$$\ddot{x}_i(X_0, t) = \ddot{X}_i + \ddot{u}_i(X_0, t)$$  \hspace{1cm} (7.2)

As before the $^c$ notation indicates dimensional variables, and we use the same scheme as section 6.2.1 for non-dimensionalising. The $V$ parameter in the non-dimensionalising scheme is treated as a general velocity scaling parameter, and we define, $V^*_i = V_i/V$. As used previously, we also define $\alpha^2 = \frac{c^2}{v^2}$. Therefore, after non-dimensionalising the expressions for the positions of the slug and the rod, we obtain,

$$x_i(X_0, t) = X_0 + \frac{V}{c} \left( V^*_i t + u_i(X_0, t) \right)$$  \hspace{1cm} (7.3)
$$\ddot{x}_i(X_0, t) = X_0 + \frac{\alpha}{c} \ddot{u}_i(X_0, t)$$  \hspace{1cm} (7.4)

We consider a model of the beam composed of $n$ slugs connected in a 'train'. The more slugs we use to represent the beam, the closer the approximation will be to the continuous model of the beam. We let $l$ be the length of the beam, $b$ be the breadth of the beam, $A_T = lb$ be the total impacting surface area of the beam. We use the variable $\bar{Z}$ ($Z = Z/l$ in non-dimensional form) to measure along the length of the beam, and therefore the length of each slug is $\Delta \bar{Z} = l/n$. Also, the impacting surface area of each slug is $A = (\Delta \bar{Z} A_T)/l$. The inter-slug coupling is between two adjacent surfaces of area $A_e = bh_e$, and the coupling stiffness per unit area of adjacent sides of the slug is $k$.

In chapter 6 we derived the equation of motion for a slug without any external coupling. Later in the same chapter we considered a single slug coupled to two fixed neighbouring slugs. In this chapter the neighbouring slugs are not fixed, and hence considering all the internal and external forces acting on slug $i$, we obtain,

$$\rho \frac{\Delta \bar{Z} A_T}{l} \frac{\partial^2 \bar{x}_i}{\partial \bar{Z}^2} \delta \bar{X}_0 = \frac{\Delta \bar{Z} A_T}{l} \frac{\partial^2 \bar{x}_i}{\partial \bar{X}_0^2} \delta \bar{X}_0 - kb \left[ \bar{x}_i - \bar{x}_{i+1} \right] \delta \bar{X}_0 - kb \left[ \bar{x}_i - \bar{x}_{i-1} \right] \delta \bar{X}_0$$

Hence,

$$\rho \frac{\partial^2 \bar{x}_i}{\partial \bar{Z}^2} = \frac{\partial \bar{x}_i}{\partial \bar{X}_0^2} - \frac{k}{\Delta \bar{Z}} \left[ -\bar{x}_{i+1} + 2\bar{x}_i - \bar{x}_{i-1} \right]$$

Using the definition $\bar{\sigma}_i = E \frac{\partial \bar{u}_i}{\partial \bar{X}_0}$, we obtain

$$\frac{\partial^2 \bar{x}_i}{\partial \bar{Z}^2} = c^2 \frac{\partial^2 \bar{u}_i}{\partial \bar{X}_0^2} - \frac{k}{\rho \Delta \bar{Z}} \left[ -\bar{x}_{i+1} + 2\bar{x}_i - \bar{x}_{i-1} \right]$$  \hspace{1cm} (7.5)

In this chapter we are assuming that when the impact occurs, the beam is moving in its fundamental mode. When the slugs are moving as rigid bodies we can analyse the free movement of the beam and calculate the frequencies of the modes of vibration. Due to the absence of an impact there is no stress variation in the slug and we do not need to
include the term in the equation above which accounts for the stress distribution within the slug. Firstly, we derive a continuous model for the free movement of the slugs. Hence, from equation (7.5) we obtain,

$$\frac{\partial^2 \dot{x}_i}{\partial t^2} = \frac{k \Delta \tilde{Z}}{\rho} \frac{[\ddot{x}_{i+1} - 2\ddot{x}_i + \ddot{x}_{i-1}]}{(\Delta \tilde{Z})^2}$$ (7.6)

We let $k = K/\Delta \tilde{Z}$, where the parameter $K$ is known as the shear modulus of the beam. In the limit where $n \to \infty$ and $\Delta \tilde{Z} \to 0$, after non-dimensionalising the equation above becomes,

$$\frac{\partial^2 \bar{x}}{\partial t^2} = \frac{q^2}{\rho^2} \frac{\partial^2 \bar{x}}{\partial \bar{Z}^2}$$ (7.7)

where the parameters are defined as follows,

$$p^2 = \frac{K}{\rho}$$ (7.8)

$$\rho^2 = \frac{h_k^2 p^2}{c^2 l^2}$$ (7.9)
The solution \( x(Z, t) \) to this equation is a continuous function of both \( Z \) and \( t \). We can solve the partial differential equation using the method of separation of variables, where we impose the boundary conditions. We find the following expression for the frequencies of the infinite number of modes of the beam,

\[
\omega_j = \frac{(2j - 1)\pi q_\infty}{2} \quad (j = 1, \ldots, \infty)
\]  
(7.10)

Returning to the free movement of the discrete model, we can use the new definition for \( k \) and write equation (7.5) in the following form,

\[
\frac{\partial^2 x_i}{\partial t^2} + q_n^2[-x_{i+1} + 2x_i - x_{i-1}] = 0
\]  
(7.11)

In the above equation we use the following definitions for \( p_n \) and \( q_n \); the subscript \( n \) emphasises the dependence of the parameters on the number of slugs,

\[
p_n^2 = \frac{K n^2}{\rho l^2}
\]  
(7.12)

\[
q_n^2 = \frac{h^2 p_n^2}{c^2}
\]  
(7.13)

This equation is not valid for the first and the last slug. For these, the fictitious slug 0 has zero displacement, and the \( n \)'th slug is only coupled to the \( (n-1) \)'th slug. We can write the resulting system of equations for all the slugs in the following form,

\[
\frac{\partial^2 \vec{X}}{\partial t^2} - q_n^2 M^* \vec{X} = \vec{0}
\]  
(7.14)

where,

\[
M^* = \begin{pmatrix}
-2 & 1 & \cdots & 0 \\
1 & -2 & 1 & \cdots \\
1 & -2 & 1 & \cdots \\
1 & -2 & 1 & \cdots 
\end{pmatrix} \quad \text{and} \quad \vec{X} = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n 
\end{pmatrix}
\]  
(7.15)

We can find the solution \( \vec{X} \) to the system of equations above. The solution is given as a sum of contributions from each mode of vibration. We can find the mode frequencies and associated mode shapes, and by letting \( n \to \infty \) we can compare these frequencies with those given by equation (7.10) for the continuous model. For the discrete model the mode frequencies \( \omega_j \), are given by the following expression (\( \mu_j \) are the eigenvalues of \( M^* \), analytical expressions for which are given in Appendix E),

\[
\omega_j = \sqrt{-\mu_j q_n^2} \quad (j = 1, \ldots, n)
\]  
(7.16)
If we now include the term in equation (7.5) modelling the stress variation within the slug, after substituting for $\dot{x}$ and non-dimensionalising we obtain the following\(^1\),

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial^2 u_i}{\partial X_0^2} + q^2 \left[ u_{i+1} - 2u_i + u_{i-1} + \left( V_{i+1}^* - 2V_i^* + V_{i-1}^* \right) t \right]$$  \hspace{1cm} (7.17)

Each slug impacts its adjoining rod. Each rod is not coupled to its neighbour, and therefore $\dot{u}_i(X_0,t)$ satisfies the following equation which was first introduced in chapter 6,

$$\alpha^2 \frac{\partial^2 \ddot{u}_i}{\partial t^2} = \frac{\partial^2 \ddot{u}_i}{\partial X_0^2}$$ \hspace{1cm} (7.18)

We use the following standard definition for the Laplace transform of $u_i(X_0,t)$,

$$\tilde{u}_i(X_0,s) = \int_0^\infty u_i(X_0,t) e^{-st} \, dt$$ \hspace{1cm} (7.19)

The Laplace transform of $\ddot{u}_i(X_0,t)$ can be similarly defined. We take Laplace transforms of the equation (7.17) and obtain,

$$s^2 \ddot{u}_i - su_i(X_0,0) - \frac{\partial u_i}{\partial t}(X_0,0) = \frac{\partial^2 \ddot{u}_i}{\partial X_0^2} + q^2 \left[ \ddot{u}_{i+1} - 2\ddot{u}_i + \ddot{u}_{i-1} + \frac{1}{s^2} \left( V_{i+1}^* - 2V_i^* + V_{i-1}^* \right) \right]$$

The left-hand side of the equation above is the Laplace transform of $\dddot{u}_i$. The second and third terms are zero since at $t = 0$ the disturbance in the slug created by the impact has not yet begun to propagate.

We can follow a similar procedure to produce the equations for the 1st and the nth slug. We now have $n$ equations describing the motion of all the slugs, which are,

\begin{align*}
1 \frac{\partial^2 \ddot{u}_1}{\partial X_0^2} &+ (-2q_n^2 - s^2) \ddot{u}_1 + q_n^2 \ddot{u}_2 = -\frac{q_n^2}{s^2} \left( V_2^* - 2V_1^* \right) \\
\vdots & \\
\frac{\partial^2 \ddot{u}_i}{\partial X_0^2} &+ q_n^2 \ddot{u}_{i-1} + (-2q_n^2 - s^2) \ddot{u}_i + q_n^2 \ddot{u}_{i+1} = -\frac{q_n^2}{s^2} \left( V_{i+1}^* - 2V_i^* + V_{i-1}^* \right) \\
\vdots & \\
\frac{\partial^2 \ddot{u}_n}{\partial X_0^2} &+ q_n^2 \ddot{u}_{n-1} + (-2q_n^2 - s^2) \ddot{u}_n = -\frac{q_n^2}{s^2} \left( V_{n-1}^* - V_n^* \right)
\end{align*}

Denoting the vector of $\ddot{u}_i (i=1,\ldots,n)$ by $\vec{U}$, the system of equations above can be written in the following matrix form,

$$\frac{\partial^2 \vec{U}}{\partial X_0^2} + M(s) \vec{U} = \vec{W}(s)$$ \hspace{1cm} (7.20)

\(^1\)Again, at this stage, if we wanted to construct a continuous model of the system, we would consider the case where $n \to \infty$. Since $n^2 = 1/(\Delta Z)^2$, in the limit as $\Delta Z \to 0$ we obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial X_0^2} + q^2 \frac{\partial^2 u}{\partial Z^2} + q^4 \frac{\partial^4 V^*}{\partial Z^4}$$
Noting that \( M(s) = g_n^2 M^* - s^2 I \) and \( \tilde{W}(s) = -\frac{g_n^2}{s^2} M^* \tilde{V}^* \), where we let \( \tilde{V}^* \) represent the vector of \( V^*_i \) \( (i=1, \ldots, n) \), the equation above can also be given in the following form.

\[
\frac{\partial^2 \tilde{U}}{\partial X^2_0} + \left( g_n^2 M^* - s^2 I \right) \tilde{U} = -\frac{g_n^2}{s^2} M^* \tilde{V}^* \tag{7.21}
\]

7.3 Finding solutions

Define \( L \) to be the matrix whose columns are the normalised eigenvectors, \( \bar{L}_1, \bar{L}_2, \ldots, \bar{L}_n \), of \( M(s) \) and \( \Lambda \) to be the matrix whose diagonal elements are the eigenvalues \( \gamma^2_1, \gamma^2_2, \ldots, \gamma^2_n \) of \( M(s) \). If \( \mu_j \) are the eigenvalues of \( M^* \) then it can be shown that \( \gamma^2_j = \frac{g_n^2}{s^2} \mu_j - s^2 \), where \( M^* \) is defined in equation (7.15). Also, the eigenvectors of \( M(s) \) are also the eigenvectors of \( M^* \).

We can write \( M(s) = \Lambda \Lambda^T \), and find the solution to the matrix system (7.20),

\[
\tilde{U}(X_0,s) = M^{-1}(s) \tilde{W}(s) + \sum_{j=1}^{n} c_j \bar{L}_j e^{-i\gamma_j X_0} + \sum_{j=1}^{n} d_j \bar{L}_j e^{i\gamma_j X_0} \\
= \Lambda^{-1} \Lambda^{-T} \tilde{W}(s) + \Lambda E(X_0,s) \bar{D} + \Lambda^{-1}(X_0,s) \bar{C} \tag{7.22}
\]

The constants \( c_j \) and \( d_j \) \( (j = 1, 2, \ldots, n) \) must be determined. In addition we use the following definitions,

\[
\bar{C} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \quad \text{and} \quad E(X_0,s) = \begin{pmatrix} e^{i\gamma_1 X_0} & 0 \\ \vdots & \ddots \\ 0 & e^{i\gamma_n X_0} \end{pmatrix}. \tag{7.23}
\]

From equation (7.18) we take Laplace transforms, and find the transformed solutions. Thus,

\[
\tilde{u}_i(X_0,s) = f_i e^{-s \alpha X_0} + g_i e^{s \alpha X_0} \quad (i = 1, 2, \ldots, n) \tag{7.24}
\]

The \( f_i \) and \( g_i \) are arbitrary functions of \( s \). However \( g_i = 0 \) because the solution must be zero at \( X_0 = \infty \). Defining the vector \( \bar{F} \) to contain the functions \( f_i \), \( (i = 1, \ldots, n) \), we can express the solution in the rod as follows

\[
\bar{U} = \bar{F} e^{-s \alpha X_0} \tag{7.25}
\]
Boundary Conditions

For each slug/rod pair there are four boundary conditions; two conditions for the interface, and one each for the opposite ends of the slug and the rod respectively. The condition at the end of the rod, \( X_0=\infty \) has already been imposed.

1. The end \( (X_0 = -h_s) \) of each slug should be free of stress. Hence,

\[
E \frac{\partial \tilde{u}_i}{\partial X_0} \bigg|_{\tilde{X}_0 = -h_s} = 0
\]

Hence,

\[
\frac{\partial \tilde{U}}{\partial X_0} \bigg|_{X_0 = -1} = 0
\]

2. Continuity of particle velocities at the interface \( (X_0 = 0) \) gives,

\[
\frac{\partial \tilde{z}_i}{\partial \tilde{t}} \bigg|_{\tilde{X}_0 = 0} = \frac{\partial \tilde{z}_i}{\partial \tilde{t}} \bigg|_{\tilde{X}_0 = 0}
\]

Substituting for \( \tilde{z}_i \) and \( \tilde{\xi}_i \),

\[
V_i + \frac{\partial \tilde{u}_i}{\partial \tilde{t}} \bigg|_{\tilde{X}_0 = 0} = \frac{\partial \tilde{u}_i}{\partial \tilde{t}} \bigg|_{\tilde{X}_0 = 0}
\]

Hence, non-dimensionalising and taking Laplace transforms we obtain,

\[
\frac{1}{s} \tilde{U}^* + s\tilde{U} \bigg|_{X_0 = 0} = s \alpha \tilde{F}
\]

We use the solution \( \tilde{U} = \tilde{F} e^{-\alpha \tilde{X}_0} \) to obtain the above expression.

3. Continuity of stress at the interface, in any slug/rod pair, \( i \).

\[
\rho c^2 \frac{\partial \tilde{u}_i}{\partial \tilde{X}_0} \bigg|_{\tilde{X}_0 = 0} = \tilde{\rho c^2} \frac{\partial \tilde{u}_i}{\partial X_0} \bigg|_{X_0 = 0}
\]

Non-dimensionalising and taking Laplace transforms, letting \( z = \rho c / \tilde{\rho c} \),

\[
z \frac{\partial \tilde{u}_i}{\partial \tilde{X}_0} \bigg|_{\tilde{X}_0 = 0} = \frac{\partial \tilde{u}_i}{\partial X_0} \bigg|_{X_0 = 0}
\]

Thus,

\[
z \frac{\partial \tilde{U}}{\partial \tilde{X}_0} \bigg|_{\tilde{X}_0 = 0} = -s \alpha \tilde{F}
\]
These boundary conditions provide $3n$ equations which allow us to solve for the $3n$ arbitrary constants, $c_i$, $d_i$ and $f_i$, $i = 1, 2, \ldots, n$. The solution is then found by inverse Laplace transform of $u_i(X_0, s)$ and $\dot{u}_i(X_0, s)$ (equations (7.22) and (7.24)).

First, substituting $\tilde{U}$ into boundary condition 1,

$$
\begin{pmatrix}
-i\gamma_1 e^{-i\gamma X_0} & \cdots & 0 \\
0 & \cdots & \ddots \\
0 & \cdots & -i\gamma_n e^{-i\gamma X_0}
\end{pmatrix}
D + L
\begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & i\gamma_n e^{i\gamma X_0}
\end{pmatrix}
\left|_{X_0=1} \right. = 0 \\
\left|_{X_0=-1} \right. \bar{C} = 0
$$

(7.32)

Therefore we can specify $d_j$ and $c_j$, ($k_j$ are arbitrary functions of $s$). Hence,

$$
d_j = \frac{1}{2} k_j e^{i\gamma_j} \\
c_j = \frac{1}{2} k_j e^{-i\gamma_j}
$$

Hence from equation (7.22),

$$
\tilde{U} = L\Lambda^{-1}LT\tilde{W}(s) + L \begin{pmatrix}
\cosh(\lambda_1(X_0 + 1)) & 0 & \cdots \\
\vdots & \ddots & \ddots \\
0 & \cdots & \cosh(\lambda_n(X_0 + 1))
\end{pmatrix}
\begin{pmatrix}
k_1 \\
\vdots \\
k_n
\end{pmatrix}
\left|_{\Gamma(X_0, s)} \right. = \bar{K}
$$

(7.33)

We have written $\lambda_j^2 = -\gamma_j^2$ and equation (7.33) indicates the matrix $\Gamma(X_0, s)$ and the vector $\bar{K}$.

Combining boundary conditions 2 and 3,

$$
\frac{1}{s}\tilde{U}^* + s\tilde{U} \left|_{X_0=0} \right. = -z \frac{\partial \tilde{U}}{\partial X_0} \left|_{X_0=0} \right.
$$

(7.34)

Substituting (7.33) into the above equation and letting $\Sigma(X_0, s) = \frac{\partial \Gamma(X_0, s)}{\partial X_0}$ we obtain

$$
\frac{1}{s}\tilde{U}^* + s(L\Lambda^{-1}LT\tilde{W}(s) + L\Gamma(0, s)\bar{K}) = -z\Sigma(0, s)\bar{K}
$$

Pre-multiplying by $L^T$

$$
\frac{1}{s}L^T\tilde{U}^* + s(L\Lambda^{-1}LT\tilde{W}(s) + \Gamma(0, s)\bar{K}) = -z\Sigma(0, s)\bar{K}
$$

(7.35)

Solving the above equation for $\bar{K}$ gives

$$
\bar{K} = \left[-z\Sigma(0, s) - s\Gamma(0, s)\right]^{-1} \left(\frac{L^T\tilde{U}^*}{s} + sL^{-1}LT\tilde{W}(s)\right)
$$

(7.36)
We have now resolved all the unknown constants. The full transformed solution is
\[
\tilde{U}(X_0, s) = L\Lambda^{-1}LT\tilde{W}(s) + L\Gamma(X_0, s) \left[ -z\Sigma(0, s) - s\Gamma(0, s) \right]^{-1} \left( \frac{L^T\tilde{V}^*}{s} + s\Lambda^{-1}LT\tilde{W}(s) \right)
\]
\[
= L \left( \Gamma(X_0, s) \left[ -z\Sigma(0, s) - s\Gamma(0, s) \right]^{-1} \left( \frac{L^T\tilde{V}^*}{s} + s\Lambda^{-1}LT\tilde{W}(s) \right) \right) + \Lambda^{-1}LT\tilde{W}(s)
\]
\[
= \left( \begin{array}{ccc}
\tilde{L}_1 & \tilde{L}_2 & \cdots & \tilde{L}_n \\
\vdots & \vdots & & \vdots \\
\tilde{L}_1 & \tilde{L}_2 & \cdots & \tilde{L}_n
\end{array} \right) \left( \begin{array}{c}
\phi_1(X_0, s) \\
\vdots \\
\phi_n(X_0, s)
\end{array} \right)
\]
\[
= \phi_1(X_0, s) \tilde{L}_1 + \cdots + \phi_n(X_0, s) \tilde{L}_n
\] (7.37)

Hence the solution \( \tilde{u}_i \) at each slug is given in terms of contributions \( \phi_j \) \((j = 1 \ldots n)\) from each mode of vibration. Looking in more detail at the above equation, and its components,

\[
\Gamma(X_0, s) \left[ -z\Sigma(0, s) - s\Gamma(0, s) \right]^{-1} = \begin{pmatrix}
\frac{\cosh(\lambda_1(X_0 + 1))}{-z\lambda_1 \sinh(\lambda_1) - s \cosh(\lambda_1)} \\
\vdots \\
\frac{\cosh(\lambda_n(X_0 + 1))}{-z\lambda_n \sinh(\lambda_n) - s \cosh(\lambda_n)}
\end{pmatrix}
\] (7.38)

\[
\left( \frac{L^T\tilde{V}^*}{s} + s\Lambda^{-1}LT\tilde{W}(s) \right) = \begin{pmatrix}
\frac{1}{s} \sum_{i=1}^{n} l_{i1} V_i^* + \frac{s}{\gamma_1} \sum_{i=1}^{n} l_{i1} W_i(s) \\
\vdots \\
\frac{1}{s} \sum_{i=1}^{n} l_{in} V_i^* + \frac{s}{\gamma_n} \sum_{i=1}^{n} l_{in} W_i(s)
\end{pmatrix}
\]

We assume that the slugs are moving in the fundamental mode of vibration when the impact occurs. In this case the displacement of the impacting surface of the slugs is zero, and the velocity of the slugs \( \tilde{V}^* = \tilde{L}_1 \).

The \( \tilde{W}(s) \) vector is defined as
\[
\tilde{W}(s) = \begin{pmatrix}
-\frac{q_2}{s^2} (V_2^* - 2V_1^*) \\
\vdots \\
-\frac{q_n}{s^2} (V_{n+1}^* - 2V_{n-1}^* + V_{n-1}^*) \\
\vdots \\
-\frac{q_n}{s^2} (V_n^* - V_{n-1}^*)
\end{pmatrix}
\] (7.39)
Taking $\vec{V}^\ast = \vec{L}_1$, equation (7.39) becomes $\vec{W}(s) = -\frac{q_1^2}{s} M^\ast \vec{L}_1$ where $M^\ast \vec{L}_1$ is defined as follows,

$$
M^\ast \vec{L}_1 = \begin{pmatrix}
  l_{12} - 2l_{11} \\
  \vdots \\
  l_{1,n+1} - 2l_{1,n} + l_{1,n-1} \\
  \vdots \\
  l_{1,n-1} - l_{1n}
\end{pmatrix}
$$

(7.40)

It is also useful to note, $M^\ast \vec{L}_1 = \mu_1 \vec{L}_1$.

Using the vector $\vec{V}$ and $\vec{W}(s)$ defined above, we can now find $\phi_1, \phi_2, \ldots, \phi_n$ which were introduced in equation (7.37). Thus

$$
\begin{pmatrix}
  \phi_1 \\
  \vdots \\
  \phi_n
\end{pmatrix} = \begin{pmatrix}
  \frac{\cosh(\lambda_1(X_0+1))}{-z\lambda_1 \sinh(\lambda_1) - s \cosh(\lambda_1)} \left[ \frac{1}{s} \sum_{i=1}^{n} l_{ii} - \frac{\mu_1 q_n^2}{s^2 \gamma_1^2} \sum_{i=1}^{n} l_{ii} \right] - \frac{\mu_1 q_n^2}{s^2 \gamma_1^2} \sum_{i=1}^{n} l_{ii} \\
  \vdots \\
  \frac{\cosh(\lambda_n(X_0+1))}{-z\lambda_n \sinh(\lambda_n) - s \cosh(\lambda_n)} \left[ \frac{1}{s} \sum_{i=1}^{n} l_{ii} - \frac{\mu_1 q_n^2}{s^2 \gamma_1^2} \sum_{i=1}^{n} l_{ii} \right] - \frac{\mu_1 q_n^2}{s^2 \gamma_1^2} \sum_{i=1}^{n} l_{ii}
\end{pmatrix}
$$

(7.41)

Using the orthogonality property of the eigenvectors this can be simplified. Hence

$$
\begin{pmatrix}
  \phi_1 \\
  \vdots \\
  \phi_n
\end{pmatrix} = \begin{pmatrix}
  \frac{\cosh(\lambda_1(X_0+1))}{-z\lambda_1 \sinh(\lambda_1) - s \cosh(\lambda_1)} \left[ \frac{1}{s} - \frac{\mu_1 q_n^2}{s^2 \gamma_1^2} \right] - \frac{\mu_1 q_n^2}{s^2 \gamma_1^2} \\
  0 \\
  \vdots \\
  0
\end{pmatrix}
$$

(7.42)

Let $m_1^2 = -\mu_1 q_n^2$ and after simplification we obtain,

$$
\phi_1(X_0, s) = \frac{s \cosh(\lambda_1(X_0+1))}{(s^2 + m_1^2)(-z\lambda_1 \sinh(\lambda_1) - s \cosh(\lambda_1))} - \frac{m_1^2}{s^2(s^2 + m_1^2)}
$$

(7.43)

Also, $\lambda_1^2 = s^2 + m_1^2$.

The function $\phi_1$ is the only non-zero $\phi$ function, indicating a zero contribution from the non-fundamental beam modes to the rebounding motion of the delamination. This shows that after an impact in the fundamental mode, the rebound is also in the fundamental mode.

We can calculate the effective coefficient of restitution, by examining the solution $\vec{U}$ in more detail. We can compute $\phi_1(X_0, t)$ by finding the inverse Laplace transform of $\phi_1(X_0, s)$,
for which some techniques were investigated in chapter 6. However, if we compare the expression for \( \phi_1 \) with equation (6.71), we find that they are equivalent if \( \lambda \) is replaced by \( \lambda_1 \) and \( q \) by \( m_1 \). This is an interesting and on the face of it quite surprising result. It means that we do not have to find another long-winded inverse Laplace transform. Instead we can use the results from chapter 6. This is discussed in more detail in the next section.

7.4 An alternative approach

With the benefit of hindsight, the above procedure of imposing boundary conditions and finding the full solution for the model is unnecessary. At the modelling stage we can see the close association between the single slug with side coupling, and the beam model discussed in this chapter. However, we did not expect the beam model to reduce to the equation for a single slug with side coupling.

Let \( \Lambda^* \) be the diagonal matrix whose elements are the eigenvalues, \( \mu_1, \ldots, \mu_n \) of \( M^* \) (the * superscript on \( \Lambda^* \) distinguishes this matrix from \( \Lambda \) which contains the eigenvalues of \( M(s) \)). Therefore, \( M^* = \Lambda \Lambda^* L^T \), and the solution \( \vec{U} \) is governed by equation (7.21), which can be re-written as

\[
\frac{\partial^2 \dddot{U}}{\partial \lambda^2} + \Lambda (q_n^2 \Lambda^* - s^2) \Lambda^* L^T \dot{U} = \frac{-q_n^2}{s^2} \Lambda^* \Lambda^* L^T \dddot{V}^*
\]

(7.44)

Multiplying through by \( \Lambda^T \) gives,

\[
\frac{\partial^2 \dddot{U}}{\partial \lambda^2} (\Lambda^T \dot{U}) + \Lambda (q_n^2 \Lambda^* - s^2) \Lambda^* L^T \dot{U} = \frac{-q_n^2}{s^2} \Lambda^* \Lambda^* L^T \dddot{V}^*
\]

(7.45)

As introduced in equation (7.37), we can define \( \vec{U} = \Lambda \vec{\phi} \). The vector \( \vec{\phi} \) contains the functions \( \phi_1, \ldots, \phi_n \). Hence, we have given the solution in terms of the contribution from each mode of vibration. Similarly, the vector of initial velocities can be defined as \( \vec{V}^* = \Lambda \vec{\psi} \), where the elements of \( \vec{\psi} \) are \( \psi_1, \ldots, \psi_n \). Again, these functions give the contribution from each mode of vibration, to the initial velocity. Since \( \Lambda \) is orthogonal, in component form equation (7.45) becomes,

\[
\frac{\partial^2 \phi_i}{\partial \lambda^2} + (q_n^2 \mu_i - s^2) \phi_i = \frac{-q_n^2}{s^2} \mu_i \psi_i
\]

(7.46)

Letting \( m_i^2 = -\mu_i q_n^2 \), and dividing through the above equation by \( \psi_i \), we obtain,

\[
\frac{\partial^2 (\phi_i/\psi_i)}{\partial \lambda^2} - (m_i^2 + s^2) (\phi_i/\psi_i) = \frac{m_i^2}{s^2}
\]

(7.47)

Hence, the component of the solution in each mode is obtained independently of the other modes; through the process described above we have decoupled the modes. Equation (7.47) is the same as equation (6.56). However, the above equation is defined for the amplitude functions in each mode and not for the solution in each individual slug. If we prove that
the same boundary conditions apply for both sets of variables we can use equation (6.71) as the solution to equation (7.47).

The solution in the rod is given by equation (7.4). Expressing the solution in the slug as a summation of contributions from each mode, we can write,

\[ x_i = X_0 + \frac{V}{c} \left( t \sum_{j=1}^{n} \phi_j l_{ji} + \sum_{j=1}^{n} \phi_j(X_0, t) l_{ji} \right) \]  

(7.48)

The three boundary conditions given in section 6.4.2 can be expressed in terms of the modal variables. Hence, after taking Laplace transforms we obtain the following,

1. The end of the slug \((X_0 = -1)\) is free of stress :

\[ \sum_{j=1}^{n} \left( \frac{\partial \phi_j(X_0, s)}{\partial X_0} l_{ji} \right) \bigg|_{X_0=-1} = 0 \]  

(7.49)

2. Continuity of particle velocities at the interface,

\[ \sum_{j=1}^{n} \left( \frac{\psi_j}{s} + s \phi_j(X_0, s) \right) l_{ji} \bigg|_{X_0=0} = s \alpha \hat{u}_i \bigg|_{X_0=0} \]  

(7.50)

3. Continuity of stress at the interface,

\[ z \sum_{j=1}^{n} \left( \frac{\partial \phi_j(X_0, s)}{\partial X_0} l_{ji} \right) \bigg|_{X_0=0} = \frac{\partial \hat{\sigma}}{\partial X_0} \bigg|_{X_0=0} \]  

(7.51)

Recalling that we used the fourth boundary condition to find the expression for the solution in the rod, \(\hat{u} = Fe^{-s \alpha X_0}\). Using this solution in the rod we can combine boundary conditions 2 and 3, to obtain

\[ \sum_{j=1}^{n} \left( \frac{\psi_j}{s} + s \phi_j(X_0, s) \right) l_{ji} \bigg|_{X_0=0} = -z \sum_{j=1}^{n} \left( \frac{\partial \phi_j(X_0, s)}{\partial X_0} l_{ji} \right) \bigg|_{X_0=0} \]  

(7.52)

The solution is a combination of all the modes, and for fixed values of \(i\), equations (7.49) and (7.52) also involve linear combinations from the modes. Since the terms \(l_{ji}\) are non-zero, we can deduce that (for \(j=1, \ldots, n\)),

\[ \frac{\partial (\phi_j/\psi_j)}{\partial X_0} \bigg|_{X_0=-1} = 0 \]  

(7.53)

\[ \frac{1}{s} + s (\phi_j/\psi_j) \bigg|_{X_0=0} = -z \frac{\partial (\phi_j/\psi_j)}{\partial X_0} \bigg|_{X_0=0} \]  

(7.54)

Therefore, if we assume a solution expressed in terms of the modes of vibration, and impose the usual boundary conditions, we obtain the familiar form of the boundary conditions expressed in terms of \(\phi_i/\psi_i\). We can then correctly use equation (6.71) as the solution \(\phi_i/\psi_i\) to equation (7.47).
The initial velocity of each slug is given by \( \mathbf{V}' = \mathbf{L}_1 \). Hence, every element of \( \mathbf{V} \) is zero, except for the first element, \( \psi_1 = 1 \). Consequently, every element of \( \mathbf{L} \) is zero except for \( \phi_1 \). Replacing \( q \) by \( m_1 \) and \( \lambda \) by \( \lambda_1 \), the expression \( \phi_1(X_0, s) \) is identical to equation (6.71).

We now write the expressions for the stress and mean velocity of the slugs in the model. Hence, the following is a vector quantity, and gives the stress in each slug.

\[
E \frac{V}{c} \frac{\partial \phi_1}{\partial X_0} \mathbf{L}_1 = V \mathbf{c} \rho \left[ \frac{\psi_1 s \sinh(\lambda_1(1+X_0))}{\sqrt{s^2 + m_1^2(-z\lambda_1 \sinh(\lambda_1) - s \cosh(\lambda_1))}} \right] \mathbf{L}_1
\]  
(7.55)

The mean velocity of the slug is,

\[
\mathbf{V}_m(t) = \frac{V}{c} \int_{X_0=-1}^{X_0=0} \left( \psi_1 + \frac{\partial \phi_1}{\partial t} \right) dX_0 \mathbf{L}_1
\]

\[
= \frac{V}{c} \left[ \psi_1 + \int_{X_0=-1}^{X_0=0} \frac{\partial \phi_1}{\partial t} dX_0 \right] \mathbf{L}_1
\]  
(7.56)

Taking Laplace transforms to obtain,

\[
\mathbf{V}_m(s) = \frac{V}{s} \left[ \psi_1 + \int_{X_0=-1}^{X_0=0} s \phi_1(s) dX_0 \right] \mathbf{L}_1
\]

\[
= \frac{V}{c} \left[ \psi_1 + \int_{X_0=-1}^{X_0=0} s \phi_1(s) dX_0 \right] \mathbf{L}_1
\]

\[
= \frac{V}{c} \left[ \psi_1 + \frac{\psi_1 s^2 \sinh(\lambda_1)}{\lambda_1(s^2 + m_1^2)(-s \cosh(\lambda_1) - x\lambda_1 \sinh(\lambda_1))} \right] \mathbf{L}_1
\]  
(7.57)

In the above expressions for the stress and the mean velocity we set \( \psi_1 = 1 \). We can read the results from figure (6.19), where it is understood that the parameter \( q \) is replaced by \( m_1 \), and \( \lambda \) is replaced by \( \lambda_1 \). We can express \( m_1^2 \) in the following form,

\[
m_1^2 = -\mu_1 q_n^2
\]

\[
= 2 \left( 1 - \cos \left( \frac{0.5\pi}{n+0.5} \right) \frac{K h^2 n^2}{\rho c^2 l^2} \right)
\]

\[
= 2 \left( 1 - \left[ 1 - \frac{1}{2!} \left( \frac{0.5\pi}{n+0.5} \right)^2 + \ldots \right] \right) \frac{K h^2 n^2}{\rho c^2 l^2}
\]

Taking the limiting case as \( n \to \infty \),

\[
m_1^2 = \left( \frac{\pi}{2} \right)^2 \frac{K h^2}{\rho c^2 l^2}
\]

\[
= \left( \frac{\pi}{2} \right)^2 \frac{q_{\infty}^2}{\rho c^2 l^2}
\]  
(7.58)
7.5 Discussion & chapter summary

In sections 7.3 and 7.4 it was seen that the solution for the impacting and the rebounding motion can be expressed in terms of a contribution from each mode of vibration. It was shown in equation (7.46) that the solution for each mode is decoupled from the other modes. Therefore if we assume an impacting motion in the fundamental mode only, the rebounding motion is only in the fundamental mode too.

From equation (7.57), the mean impacting velocity of each slug is given by the vector \( \frac{V}{c} \hat{L}_1 \) and this expression can be factored from the expression for the rebounding velocity. As we discussed in chapter 6, if the rods are infinite they always have a mean velocity of zero. Also, since there is a common expression in the solutions for the mean velocity of the slugs before impact and after parting, the expression in the large square brackets of equation (7.57) is a measure of the effective coefficient of restitution. We note that this is independent of the impacting velocity \( V \). Hence, we can use the results of chapter 6 to find when parting occurs and the mean velocity of the slug at parting; we use figure (6.19), where it is understood that \( \lambda \) is replaced by \( \lambda_1 \) and \( q \) is replaced by \( m_1 \).

Relating this first model of a beam to the physical reality, the elasticity of the spring coupling between the slugs is provided by the elasticity of the beam material in shearing. In the discrete model, the strength of the coupling is given by \( k \), defined per unit area of the adjacent sides. The parameter \( k \) is related to the shear modulus of the material by, \( K = k \Delta \tilde{Z} \), and we can express the parameter \( q_n \) in the following form,

\[
q_n = \frac{\hat{c} h_s n}{c l} \tag{7.59}
\]

where \( \hat{c} = \sqrt{\frac{K}{\rho}} \) is the shear wave speed.

Thus \( q_n \) is directly proportional to the ratio \( \frac{h_s}{l} \) of the depth to the length of the delamination, and the number of slugs \( n \). In order to use the results from section 6.4, we noted that the parameter comparable to \( q \) in that section is \( m_1 = q_n \sqrt{-\mu_1} \). From appendix E we know that,

\[
-\mu_1 = 2 \left( 1 - \cos \left( \frac{\pi}{2n+1} \right) \right) \tag{7.60}
\]

For large \( n \), we can approximate \(-\mu_1 \). Thus,

\[
-\mu_1 \approx \frac{\pi^2}{(2n+1)^2} \tag{7.61}
\]

Hence, we can write the parameter \( m_1 \) in the following form,

\[
m_1 = \frac{\pi \hat{c}}{2 c} \frac{1}{\left( 1 + \frac{1}{2n} \right)} \frac{h_s}{l}
\approx \frac{\pi \hat{c}}{2 c} \left( 1 - \frac{1}{2n} \right) \frac{h_s}{l} \tag{7.62}
\]
Therefore, for large $n$, the parameter $m_1$ is directly proportional to $\frac{h}{T}$, and virtually independent of $n$. Since $\frac{5}{T^2}$ is of order 1, the coupling effect is controlled by $\frac{h}{T}$, and as $\frac{h}{T} \ll 1$ this becomes negligible. The use of beam theory to model the delamination is expected to be valid for $\frac{h}{T}$ less than $\frac{1}{10}$, therefore an upper limit to the value of $m_1$ which is appropriate to our study would be $m_1 \approx 0.1$.

Briefly, we can summarise the results which were discussed in more detail at the end of the previous chapter. When $z > 1$ the departing mean velocity of the beam is small, and for $z \ll 1$ the impact approaches an elastic impact. Further details can be seen in figure (6.19).
Chapter 8

System of Connected Slugs with Nearest and Next-Nearest Neighbour Coupling

8.1 Introduction

In this chapter we couple each slug to its nearest neighbour and also to its next-nearest neighbour. This is illustrated in figure (8.1). It is seen in this chapter that as the number of slugs in the discretisation approaches infinity, we obtain the beam bending equation for the rigid body movement of the slugs. Also, we can insist on a strict interpretation of the boundary conditions for a cantilever beam, and in doing so we obtain a non-symmetrical main system matrix. However, to make use of the results in Chapter 6, we use an alternative formulation, which results in a symmetrical matrix.

![Figure 8.1: An n slug beam model with nearest and next-nearest neighbour coupling.](image)

We first discuss the rigid body motion of the delamination, and derive an expression describing this motion. We then introduce the impact conditions into the equations. The resulting system of equations can be arranged into a matrix form where the main square matrix is symmetrical. Consequently, we can write the equation of motion in a form ready to use the results of chapter 6 to calculate the inverse Laplace transform. The implication
of insisting that the main system matrix is symmetrical is that we do not maintain the
correct boundary conditions for a clamped-free beam. We can examine the equations for
the first two slugs and the last two slugs, and hence interpret the new boundary conditions,
and their physical implications.

To conclude this chapter, we make some informal comments on the results from computer
simulations of two similar impacting systems. For these simulations we use a discrete model
of a beam and constrain the movement of some or all of the slugs by placing rods which
interrupt the free movement. We observe the motion between impacts, and draw parallels
with the results of chapter 3.

8.2 Rigid body motion

8.2.1 Discretising the beam equation & boundary conditions

In this section we consider first the continuous model and then the discrete model. In both
cases, we find expressions for the frequencies of the modes of vibration, and then for the
discrete model we find the full solution for the forced response of the beam with given initial
conditions.

The beam bending equation was derived in section 2.2.1. The cross-sectional area of the
beam is $A_c = bh_s$, the elastic modulus is $E$, the density $\rho$ and $I$ is the second moment of
area. For time $t$ the displacement of a point a distance $Z$ along the central axis of the beam
is given by $\dot{x}(Z, t)$, where $0 < Z < l$, and $\ddot{x}$ satisfies the following equation,

$$\frac{\partial^2 \ddot{x}}{\partial t^2} = -\frac{EI}{A_c \rho} \frac{\partial^4 \ddot{x}}{\partial Z^4}$$ (8.1)

We use the same non-dimensionalising scheme as in section 6.2.1. We introduce the parameter $h_s$ into the equation during non-dimensionalising. It is noted this is both the height of
a slug and also the depth of the beam. Non-dimensionalising equation (8.1), we obtain

$$\frac{\partial^2 \ddot{x}}{\partial t^2} + q_\infty \frac{\partial^4 \ddot{x}}{\partial Z^4} = 0$$ (8.2)

$$p_\infty^2 = \frac{EI}{A_c \rho}$$ (8.3)

$$q_\infty^2 = \frac{h_s^2 p_\infty^2}{c^2 l^4}$$ (8.4)

These definitions for $p_\infty^2$ and $q_\infty^2$ are distinct from the definitions with the same names in
chapter 7.

Following the procedure detailed in Appendix A for each mode of vibration we can find the
mode frequencies and shape functions. The frequencies can be calculated from the following
(where the quantities $\beta_1 = 1.875$, $\beta_2 = 4.694$, etc., are discussed in the Appendix)

$$\omega_i^2 = \beta_i^4 q_\infty^2$$ (8.5)
To obtain this solution we had to specify the boundary conditions at the ends of the beam. The correct boundary conditions for a clamped unforced cantilever beam are: at the free end of the beam the bending moment and the shearing force are both zero, and at the clamped end the position and the gradient of the beam are both zero. We can vibrate the beam by specifying a non-zero displacement but retaining the zero gradient condition at the clamped end of the beam. Mathematically these conditions are

\[ \ddot{x}(0, t) = A \sin(\omega t) \]  
(8.6)

\[ \frac{\partial \ddot{x}}{\partial Z} = 0 \]  
(8.7)

\[ \frac{\partial^2 \ddot{x}}{\partial Z^2} = 0 \]  
(8.8)

\[ \frac{\partial^3 \ddot{x}}{\partial Z^3} = 0 \]  
(8.9)

We develop a discrete model of the beam. Formulating a difference equation for the fourth derivative term, we obtain

\[ \frac{\partial^4 \ddot{x}}{\partial Z^4} \bigg|_{i=0, n=0} = \lim_{\Delta Z \to 0} \left[ \frac{\ddot{x}_{i-2} - 4\ddot{x}_{i-1} + 6\ddot{x}_i - 4\ddot{x}_{i+1} + \ddot{x}_{i+2}}{(\Delta Z)^4} \right] \]  
(8.10)

We are using the same variable notation as used in previous chapters. In particular, \( \dot{Z} = lZ \) and \( \Delta Z = l/n \). The approximation above is used to obtain a discrete version of equation (8.1),

\[ \frac{\partial^2 \ddot{x}_i}{\partial t^2} = -\frac{EI n^4}{A_c \rho l^4} \left( \ddot{x}_{i-2} - 4\ddot{x}_{i-1} + 6\ddot{x}_i - 4\ddot{x}_{i+1} + \ddot{x}_{i+2} \right) \]  
(8.11)

### 8.2.2 Solution of the discrete system

The displacement of the \( i \)'th slug is given by \( \ddot{x}_i(t) \). Using the same scheme as section 6.2.1 we non-dimensionalise equation (8.11) and obtain the following equation and associated parameters,

\[ \frac{\partial^2 \ddot{x}_i}{\partial t^2} = -q_i \left( \ddot{x}_{i-2} - 4\ddot{x}_{i-1} + 6\ddot{x}_i - 4\ddot{x}_{i+1} + \ddot{x}_{i+2} \right) \]  
(8.12)

\[ p_i = \frac{EI n^4}{A_c \rho l^4} \]  
(8.13)

\[ q_i = \frac{h^2 p_i^2}{c^2} \]  
(8.14)

We can insist that at the first and last slugs the boundary conditions are satisfied. We approximate the second and third derivatives with respect to \( Z \) as difference equations. Evaluating these at the free end of the beam we obtain

\[ \frac{\partial^2 x}{\partial Z^2} \bigg|_{Z=1} = \lim_{n \to \infty} \left[ \frac{x_{n+1} - 2x_n + x_{n-1}}{(\Delta Z)^2} \right] \]  
(8.15)

\[ \frac{\partial^3 x}{\partial Z^3} \bigg|_{Z=1} = \lim_{n \to \infty} \left[ \frac{x_{n+2} - 2x_{n+1} + 2x_{n-1} - x_{n-2}}{2(\Delta Z)^3} \right] \]  
(8.16)
Therefore we insist on the following solutions for the four 'outside' slugs in order for the boundary conditions to be satisfied

\[
\begin{align*}
    x_{-1} &= x_1 \\
    x_0 &= A \sin(\omega t) \\
    x_{n+1} &= 2x_n - x_{n-1} \\
    x_{n+2} &= 4x_n - 4x_{n-1} + x_{n-2}
\end{align*}
\]  
(8.15-8.18)

Using equation (8.12) with \(i = 1, \ldots, n\), we obtain \(n\) differential equations describing the motion of each slug in the beam. Using equations (8.15-8.18), the fictitious slugs are eliminated, and we obtain

\[
\begin{pmatrix}
    \ddot{x}_1 \\
    \ddot{x}_2 \\
    \ddot{x}_3 \\
    \vdots \\
    \ddot{x}_{n-2} \\
    \ddot{x}_{n-1} \\
    \ddot{x}_n
\end{pmatrix} =
\begin{pmatrix}
    -7 & 4 & -1 & \cdots & 0 \\
    4 & -6 & 4 & -1 & \cdots & 0 \\
    -1 & 4 & -6 & 4 & -1 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
    -1 & 4 & -6 & 4 & -1 & \cdots & 0 \\
    -1 & 4 & -5 & 2 & \cdots & 0 \\
    0 & \cdots & -2 & 4 & -2 & 0
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    \vdots \\
    x_{n-2} \\
    x_{n-1} \\
    x_n
\end{pmatrix} =
\begin{pmatrix}
    4q_n^2 A \sin(\omega t) \\
    -q_n^2 A \sin(\omega t) \\
    0 \\
    \vdots \\
    0 \\
\end{pmatrix}
\]  
(8.19)

In matrix form

\[
\frac{d^2 \vec{X}}{dt^2} - q_n^2 \mathbf{R}^* \vec{X} = \vec{W}
\]  
(8.20)

The solution to this equation comprises of a sum of displacements in each mode of vibration, plus the particular integral. Assuming the eigenvalues of \(R^*\) are given by \(\xi_i\), the \(n\) natural frequencies are given by

\[
\omega_i^2 = -\xi_i \cdot q_n^2
\]  
(8.21)

We cannot find the eigenvalues \(\xi_i\) analytically, and therefore we must resort to numerical techniques (in this case a standard Matlab routine was used). We can numerically compare the frequencies of the continuous and discrete model, and as we would expect for increasing \(n\) we find that the natural frequencies of the discrete model approach those of the continuous model given in equation (8.5). For various selections of \(n\), table (8.1) shows the natural frequencies of the lowest two modes of vibration, where we used equation (8.21) to compute the modes of vibration for the discrete system, noting that \(q_n^2 = n^4 \cdot q_{\infty}^2\).

The natural frequencies \(\omega_i^2\) \((i = 1, 2, \ldots, n)\) are the eigenvalues of \(q_n^2 \mathbf{R}^*\). Also, we define \(\tilde{L}\) as the matrix whose columns are the normalised eigenvectors, \(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_n\), of \(\mathbf{R}^*\). Since
Table 8.1: Natural frequencies of the n-slug beam model

<table>
<thead>
<tr>
<th>n</th>
<th>$\omega_1^2/q_{100}$</th>
<th>$\omega_2^2/q_{100}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>12.156356</td>
<td>446.62644</td>
</tr>
<tr>
<td>20</td>
<td>12.310264</td>
<td>475.27768</td>
</tr>
<tr>
<td>50</td>
<td>12.354000</td>
<td>483.85538</td>
</tr>
<tr>
<td>100</td>
<td>12.360272</td>
<td>485.10206</td>
</tr>
<tr>
<td>500</td>
<td>12.362267</td>
<td>485.50213</td>
</tr>
<tr>
<td>$\infty$</td>
<td>12.362363</td>
<td>485.518819</td>
</tr>
</tbody>
</table>

The matrix $R^*$ is not symmetric $LL^T \neq I$. Solving equation (8.20), we can find expressions for $\vec{X}(t)$ and $\vec{\dot{X}}(t)$. Hence

$$
\vec{X}(t) = \sum_{j=1}^{n} c_j \bar{L}_j e^{-i\omega_j t} + \sum_{j=1}^{n} d_j \bar{L}_j e^{i\omega_j t} + (q_n^2 R^* - \omega^2 I)^{-1}\bar{W}^* \sin(\omega t)
$$

$$
\vec{\dot{X}}(t) = \vec{L}(t)\bar{D} + \vec{L}^{-1}(t)\bar{C} + (q_n^2 R^* - \omega^2 I)^{-1}\bar{W}^* \sin(\omega t), \quad (8.22)
$$

$$
\vec{\dot{X}}(t) = \vec{L}(t)\bar{D} + \vec{L}^{-1}(t)\bar{C} + \omega(q_n^2 R^* - \omega^2 I)^{-1}\bar{W}^* \cos(\omega t), \quad (8.23)
$$

The vectors $\vec{C}$ and $\bar{D}$ contain the $n$ constants $c_j$ and $d_j$ respectively. We also define the following,

$$
E(t) = \begin{pmatrix} e^{i\omega_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{i\omega_n t} \end{pmatrix}, \quad \bar{W}^* = \begin{pmatrix} 4q_n^2 A \\ -q_n^2 A \\ \vdots \\ 0 \end{pmatrix}, \quad (8.24)
$$

Now, we can specify the initial conditions in the slug train from a starting time, $t = t_0$. Let, $\vec{X}_0$ be the initial displacement of the slugs, and $\vec{\dot{X}}_0$ the initial velocities. Hence,

$$
\vec{X}_0 = \vec{L}(t_0)\bar{D} + \vec{L}^{-1}(t_0)\vec{C} + (q_n^2 R^* - \omega^2 I)^{-1}\bar{W}^* \sin(\omega t_0) \quad (8.25)
$$

$$
\vec{\dot{X}}_0 = \vec{L}(t_0)\bar{D} + \vec{L}^{-1}(t_0)\vec{C} + \omega(q_n^2 R^* - \omega^2 I)^{-1}\bar{W}^* \cos(\omega t_0), \quad (8.26)
$$

It is noted $\dot{E}(t_0) = \frac{d}{dt}(E(t))_{t=t_0}$ and $E^{-1}(t_0) = \frac{d}{dt}(E^{-1}(t))_{t=t_0}$ and therefore $\dot{E}(t)E^{-1}(t) \neq I$.

From the expressions for $\vec{X}_0$ and $\vec{\dot{X}}_0$ we can find expressions for $\vec{C}$ and $\bar{D}$. Thus,

$$
\bar{D} = [\vec{L}(t_0) - \vec{L}^{-1}(t_0)E(t_0)^2]^{-1}[\vec{X}_0 - \omega(q_n^2 R^* - \omega^2 I)^{-1}\bar{W}^* \cos(\omega t_0) \\
- \vec{L}^{-1}(t_0)E(t_0)L^{-1}(\vec{X}_0 - (q_n^2 R^* - \omega^2 I)^{-1}\bar{W}^* \sin(\omega t_0))] \quad (8.27)
$$

$$
\bar{C} = E(t_0)[L^{-1}(\vec{X}_0 - (q_n^2 R^* - \omega^2 I)^{-1}\bar{W}^* \sin(\omega t_0)) - E(t_0)\bar{D}] \quad (8.28)
$$

Now with equations (8.22) and (8.23), given initial positions and velocities we have the expressions for the position and velocity of the each slug in the beam. We use these expressions in the computer simulations discussed at the end of the chapter.
8.3 Including elastic wave propagation

8.3.1 Modelling

Up to this point in this chapter we have assumed the slugs move as rigid bodies. We now include the elastic wave propagation within each slug. After we have described the model, we consider the impact between this beam and a row of uncoupled rods.

As in figure (8.1), each slug is coupled to its nearest and next-nearest neighbour. The coupling strength between a slug and its nearest neighbours is \( k_s \), and the coupling between a slug and its next-nearest neighbours is \( k_d \). Both stiffnesses are expressed as stiffness per unit cross-sectional area of the adjacent sides of the slug. The parameters \( k_s \) and \( k_d \) are constants. We derive the equation of motion for the \( i \)'th slug, where we follow a very similar procedure to that of section 7.2. However, due to nearest and next-nearest neighbour coupling, each slug is coupled by shear springs to four other slugs. Thus,

\[
\frac{\Delta \ddot{X}_i}{\rho \Delta Z} \frac{\partial^2 x_i}{\partial t^2} = \frac{\Delta \ddot{X}_i}{\rho \Delta Z} \frac{\partial^2 x_i}{\partial t^2} - k k_s b [-\dot{x}_{i-1} + 2\dot{x}_i - \dot{x}_{i+1}] \delta X_0 - k k_d b [-\dot{x}_{i-2} + 2\dot{x}_i - \dot{x}_{i+2}] \delta X_0
\]

Rearranging

\[
\frac{\partial^2 x_i}{\partial t^2} = -k \frac{\partial^2 x_i}{\partial X_0^2} \left(-k_d \dot{x}_{i-2} - k_s \dot{x}_{i-1} + 2(k_s + k_d) \dot{x}_i - k_s \dot{x}_{i+1} - k_d \dot{x}_{i+2}\right)
\]

Recalling the definitions for \( x_i \) and \( \ddot{x}_i \) in equations (7.1) and (7.2), and using \( \ddot{x}_i = E \dddot{u}_i \),

\[
\frac{\partial^2 x_i}{\partial t^2} = c^2 \frac{\partial^2 u_i}{\partial X_0^2} - \frac{k}{\rho \Delta Z} \left(-k_d \dot{x}_{i-2} - k_s \dot{x}_{i-1} + 2(k_s + k_d) \dot{x}_i - k_s \dot{x}_{i+1} - k_d \dot{x}_{i+2}\right) \quad (8.29)
\]

The first term on the right-hand side of the equation above is zero if we again assume rigid body motion. Comparing this equation for rigid body motion with equation (8.11), by choosing \( k = EI/(A_c(\Delta Z)^3) \) and \( k_s = 4, k_d = -1 \), then these two equations are identical. In which case, \( kA_c = EI/(\Delta Z)^3 \), where we recall that \( k \) is the stiffness per unit cross-sectional area of adjacent sides of the slug. Substituting for \( \ddot{x}_i \) into equation (8.29) above, and using the definitions for \( k, p_s^2 \) and \( q_s^2 \), and finally non-dimensionalising using the scheme of section 6.2.1, we can write the equation in the following form,

\[
\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial^2 u_i}{\partial X_0^2} - q_n \left[(-k_d u_{i-2} - k_s u_{i-1} + 2(k_s + k_d) u_i - k_s u_{i+1} - k_d u_{i+2})
+ (-k_d V_{i-2}^* - k_s V_{i-1}^* + 2(k_s + k_d) V_i^* - k_s V_{i+1}^* - k_d V_{i+2}^*) \right] \quad (8.30)
\]
Note, by letting $n \to \infty$ and $\Delta Z \to 0$ and provided that we take $k_s = 4$ and $k_d = -1$ we obtain the fourth derivative term in the beam bending equation. Thus from equation (8.30) we obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_0^2} - q_{\infty}^2 \frac{\partial^4 u}{\partial z^4} - q_{\infty}^2 \frac{\partial^4 V^*}{\partial z^4}$$  \hspace{1cm} (8.31)

Using (7.19) we take Laplace transforms of equation (8.30). This gives the transformed equation for the $i$'th slug. Assuming that there are no slugs for $i < 0$ and $i > n$ we can find equations for all $n$ slugs, and organise these into matrix form. Thus,

$$\frac{\partial^2 \tilde{U}}{\partial x_0^2} + \left( q_n^2 \left[ k_s M^* + k_d N^* \right] - s^2 \right) \tilde{U} = \frac{-q_n^2}{s^2} \left( k_s M^* + k_d N^* \right) \tilde{V}^*$$  \hspace{1cm} (8.32)

The matrix $M^*$ was defined in equation (7.15), and the matrix $N^*$ is defined thus,

$$N^* = \begin{pmatrix}
-1 & 0 & 1 & \cdots & 0 \\
0 & -2 & 0 & 1 \\
1 & 0 & -2 & 0 & 1 \\
\vdots & & & & \\
1 & 0 & -2 & 0 & 1 \\
1 & 0 & -1 & 0 \\
0 & \cdots & 1 & 0 & -1
\end{pmatrix}$$  \hspace{1cm} (8.33)

Letting $H^* = k_s M^* + k_d N^*$, and using $k_s$ and $k_d$ which give the correct fourth derivative term, the matrix $H^*$ is,

$$H^* = \begin{pmatrix}
-7 & 4 & -1 & \cdots & 0 \\
4 & -6 & 4 & -1 \\
-1 & 4 & -6 & 4 & -1 \\
\vdots & & & & \\
-1 & 4 & -6 & 4 & -1 \\
-1 & 4 & -7 & 4 \\
0 & \cdots & -1 & 4 & -3
\end{pmatrix}$$  \hspace{1cm} (8.34)

8.3.2 Finding the solution

In section 8.2 in order to obtain the correct boundary conditions given by equations (8.6)-(8.9) for a cantilever beam, we introduce 'fictitious' slugs in the modelling process. We specified the solution at these slugs specifically to give the correct boundary conditions.

In the previous section we did not introduce fictitious slugs, we simply considered the $n$ slugs and the connections between them. The resultant matrix $H^*$ is symmetrical, although with this approach we are not guaranteed the correct boundary conditions for a cantilever beam. We must examine the equations at the first and last slug in order to establish a physical interpretation of the boundary conditions.
However, the compensation for not obtaining the correct boundary conditions is that the matrix $H^*$ is symmetric, whereas the matrix $R^*$ of equation (8.20) is not. Hence, to find the solution to equation (8.32) and the subsequent inverse Laplace transform, we can follow a similar procedure to that employed in section 7.4, but using a more sophisticated model for the impacting beam.

Let $A^*$ be the matrix whose diagonal elements are the eigenvalues $\xi_i^* (i = 1, \ldots, n)$ of $H^*$ (the eigenvalues of $R^*$ are $\xi_i$). As usual the columns of the matrix $L$ are the normalised eigenvectors of $H^*$. Since the matrix $H^*$ is symmetric it can be written as $H^* = L A^* L^T$.

The solution is expressed as $\bar{U} = L \bar{\Phi}$, where the elements $\phi_i$ of $\bar{\Phi}$ measure the contribution to the solution from each mode of vibration. The vector of initial velocities is $\bar{V}^* = L \bar{\Psi}$, where $\bar{\Psi}$ contains the elements $\varphi_i$. We follow an identical procedure to section 7.4 and express equation (8.32) in terms of the modal variables. Hence,

$$\frac{\partial^2 (\phi_i / \varphi_i)}{\partial x_0^2} - (n_i^2 + \varepsilon^2)(\phi_i / \varphi_i) = \frac{n_i^2}{s^2}$$

(8.35)

We define the variables $n_i^2 = -\xi_i \varepsilon_i q_i^2$. Assuming the beam is moving in the fundamental mode only at the impact, $\bar{\Psi}$ is zero apart from $\varphi_1 = 1$. We can use the results from chapter 6 to find $\phi_i$. Again, it is seen that $\phi_1$ is the only non-zero solution.

As in section (7.4), we can read results from figure (6.19) where we note that $q$ in this figure is replaced by $n_1 = \sqrt{-\xi_1 \varphi_1 q_1^2}$. We commented on the characteristics of the solutions in chapters 6 and 7.

8.3.3 Interpreting the new boundary conditions

When deriving the equation (8.32) and the associated matrix $H^*$, we assume there are no slugs for $i > n$ and $i < 0$. The result of this is the matrix $H^*$ is symmetrical, and hence we are able to find the solution for the system using the results for a single slug/rod pair. However, the system we obtain does not have immediately recognisable boundary conditions. We can examine the matrix $H^*$ to evaluate the physical interpretation of the boundary conditions.

The first 2 rows of the matrix $H^*$ are identical to those of $R^*$. Therefore we obtain the correct boundary conditions, namely the displacement and gradient of the beam are zero at the fixed end.

The $(n-1)\text{th}$ equation gives

$$-u_{n-3} + 4u_{n-2} - 7u_{n-1} + 4u_n$$

(8.36)

instead of

$$-u_{n-3} + 4u_{n-2} - 6u_{n-1} + 4u_n - u_{n+1}$$

(8.37)
Comparing the coefficients of these two expressions, we find that \( u_{n+1} = u_{n-1} \). Hence, we conclude that \( \frac{\partial u_{n}}{\partial x} = 0 \), i.e. the gradient of the free end of the beam is zero.

The \( n \)'th equation gives

\[-u_{n-2} + 4u_{n-1} - 3u_{n} \quad (8.38)\]

instead of

\[-u_{n-2} + 4u_{n-1} - 6u_{n} + 4u_{n+1} - u_{n+2} \quad (8.39)\]

Using \( u_{n+1} = u_{n-1} \) and comparing the coefficients of these two expressions, the implication is that \( u_{n+2} = 4u_{n-1} - 3u_{n} \). We cannot attach any physical interpretation to this. However, if we make \( u_{n+2} = u_{n-2} \) this would imply that \( \frac{\partial^2 u_{n}}{\partial x^2} = 0 \), i.e. the shearing force is zero, but equation (8.39) then gives,

\[-2u_{n-2} + 8u_{n-1} - 6u_{n} \quad (8.40)\]

This is equation (8.38) multiplied by 2, and so there is some mismatch. The conclusion is that we cannot make any physical interpretations with regards to the boundary conditions at the free end of the beam, when the matrix is forced to be symmetric.

### 8.4 Simulations involving impact

We studied two additional examples of impacting systems, using computer simulations. We performed simulations of the whole system, i.e., the free movement of the beam interrupted by impacts. For both simulations we reverted back to the coefficient of restitution model for the impact, mainly because the simulations were computationally expensive, without the added computation required to model the impact with wave propagation considerations. The discretised model of the beam with nearest and next-nearest neighbour coupling was used to model the beam. The \( n \)-slug approximation of a continuous beam has \( n \) modes of vibration and the motion of the beam is expressed in terms of contributions from each mode of vibration. Equations (8.22), (8.27) and (8.28) give the displacement of the beam (for \( t > t_0 \)), for any given initial displacements and velocities at \( t = t_0 \). These solutions are used directly in the simulations for the free movement of the beams.

Firstly, we looked at a forced cantilever beam, subject to an amplitude limiting constraint in the form of a fixed rod placed at an arbitrary position along the length of the beam. In the simulations the stop was placed to correspond with the \((n-1)\)'th or \( n \)'th slug, see figure (8.2). This system has been studied for a continuous beam both theoretically and experimentally by Moon and Shaw [17], and Shaw [22].

Secondly, we looked at the full impacting system of a main beam and its delamination. This is a more complex problem than the beam and a single stop. In the rest of this thesis we
consider the main beam to have zero curvature. In this simulation both beams are flexible, with the physical dimensions influencing the choice of parameters in the model of the beam.

We can describe a typical simulation for the impacting beams. Initially, if we assume that there is no contact between the main beam and delamination, at some point in the future the free movement is constrained due to an impact condition. The simulation must find the time at which this first occurs, by monitoring the relative displacement between each slug/rod pair. We can compute the impacting velocities and then find the rebounding velocities. We can compute the velocities and displacement of each slug and rod, and using the newly computed initial velocities of the impacting slug and rod, we have a complete set of new initial conditions, and the simulation can be restarted.

Chattering behaviour is observed in the simulations. Recall, that at a certain point we assume that the small 'bounces' are negligible and hence sticking occurs. The relative acceleration of the slug and the rod must be acting such that the two bodies remain in contact after a period of chatter, and stick. When the acceleration changes sign, the bodies part company. During the period of stick, the other slugs in the simulation are free to move and hence the beam is no longer a simple cantilever beam, but instead it is a clamped-free beam which is also simply-supported at certain points along its length. In the case of a simple impact oscillator we can easily determine the acceleration acting on the mass, and once a low velocity impact occurs in the sticking region (the convention established in chapter 2 is that negative acceleration indicates the sticking region), a chattering sequence is observed until stick eventually occurs. For small velocities the acceleration can be found by evaluating a sine function. The only disruption of the chattering sequence occurs when the impacts leave the sticking region.

However, in the impacting beams system the acceleration acting on each slug during a sequence of low velocity impacts is determined largely by the position of the surrounding slugs. Many modes of vibration contribute to the solution for the position of these surrounding slugs. Therefore, we do not observe smooth sequences of decreasingly smaller impacts. In fact on a small scale, the nature of the chattering motion is quite erratic, and we were unable to successfully count the impacts of a chattering motion and conclude that sticking occurs. In particular if an impact occurs for a number of slug/rod pairs at the same time, the acceleration acting on each body can be unpredictable.
Hence, a significant proportion of computing time is spent calculating periods of small chattering type motions, and we conclude it is too time consuming to investigate the system motion with computer simulations of this type. It is essential to make some important simplifying assumptions to eliminate some of the complexity from the system.

In this section we discussed some computer simulations of impacting systems. In terms of detailed results the investigations were largely inconclusive, mainly due to the excessive amount of computing time required. However some interesting issues arose and we discussed those here.

8.5 Discussion & chapter summary

The results derived in this chapter allows us to make use of figure (6.19) to show the parting times and effective coefficient of restitution for our current model, where we replace $q$ by $n_1 = q_n \sqrt{-\xi'_1}$. In equation (8.14) we defined,

$$g_n^2 = \frac{EI}{c^2 \rho A_c} \left( \frac{n}{l} \right)^4 h_s^2$$  \hspace{1cm} (8.41)

Using the results that $I = \frac{bh^3}{12}$, $A_c = bh_s$, $E = c^2 \rho$, we obtain,

$$g_n^2 = \left( \frac{h_s}{l} \right)^4 \frac{n^4}{12}$$  \hspace{1cm} (8.42)

We can write $n_1$ as follows,

$$n_1 = \left( \frac{h_s}{l} \right)^2 n^2 \sqrt{\frac{1}{12} - \xi'_1}$$  \hspace{1cm} (8.43)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$-\mu_1 n^2$</th>
<th>$-\xi'_1 n^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.234</td>
<td>27.22</td>
</tr>
<tr>
<td>20</td>
<td>2.347</td>
<td>29.28</td>
</tr>
<tr>
<td>50</td>
<td>2.418</td>
<td>30.49</td>
</tr>
<tr>
<td>100</td>
<td>2.443</td>
<td>30.89</td>
</tr>
</tbody>
</table>

Table 8.2: Table of $-\mu_1 n^2$ and $-\xi'_1 n^4$, for various $n$.

Table 8.2 shows that $n^2 \sqrt{\frac{1}{12} - \xi'_1}$ tends to a finite limit as $n \to \infty$. We can therefore approximate the equivalent value of $q$, to be employed for making use of figure (6.19). Thus,

$$n_1 = \frac{\sqrt{31} (h_s)}{\sqrt{12} (l)^2}$$  \hspace{1cm} (8.44)

For the delamination to be modelled by the beam equation, we expect $\left( \frac{h_s}{l} \right) < \frac{1}{10}$, and hence an approximate maximum value for $n_1$ is 0.016.
In this chapter, we coupled each slug to its nearest and next-nearest neighbour, and we proved that as $n \to \infty$ the derived equation is equivalent to the Euler-Bernoulli beam equation. However, in order to again reuse figure (6.19) to evaluate the effective coefficient of restitution, the square matrix of equation (8.32) must be symmetric. By insisting this is the case, we forfeit obtaining the correct boundary conditions for a clamped-free beam.

However, the model studied in this chapter is still a closer approximation to the physical system, than the model of chapter 7. We note that the approximate maximum value of the parameter $n_1$ in this chapter is of magnitude 10 smaller than the parameter $m_1$ of chapter 7. Again, we can summarise the main characteristics of the impact: for $z > 1$ the effective coefficient of restitution is quite small, and for $z \ll 1$ the impact approaches an elastic impact. The effective coefficient of restitution is also independent of the impacting velocity $V$. 

Chapter 9

A Single DOF Impact Oscillator

9.1 Introduction

In the study of impacting systems, it is often the case that the impact itself is modelled using a simple coefficient of restitution relationship. Consequently, the speed of the impacting body changes instantaneously. It is possible to use more accurate models of the impact. However, for any chosen model, strongly nonlinear behaviour persists in the system and researchers often focus on these nonlinear aspects. In this chapter we consider in detail the impact in a single degree-of-freedom impact oscillator and identify the inaccuracies of the coefficient of restitution model.

![Figure 9.1: A typical trajectory of an impact oscillator.](image)

We develop a model of a single DOF impact oscillator, and intend to model the impact by considering the wave propagation from the impact. A slug of length $h_s$ is attached to the
end of a massless spring, with the opposite end of the spring forced harmonically. The free oscillations of the slug are constrained by a rod placed such that the slug impacts the rod at \( x=0 \). This is shown in figure (9.1) together with a typical trajectory of its movement.

In figure (9.1) the horizontal axis indicates time, and we can see the free-movement of the slug is interrupted by impacts with the stationary rod. We assume that when the system is unforced, the slug and rod are in contact with zero stress at the contact surface, and hence the spring is in equilibrium with natural length \( d_0 \). The spring in the figure is shown in this state. When the system is forced and an impact occurs, the force exerted by the spring on the slug is dependent on the change in length of the spring from its natural length. In the figure, the arrows at each impact give an impression of the direction and magnitude of the force exerted by the spring.

We assume that the position of the forced end of the spring is given by \(-h_s-d_0-A\sin(\omega \tilde{t})\). Given that the position of the slug is \( x(\tilde{x}_0, \tilde{t}) \) (where \(-h_s<\tilde{x}_0<0\) and \( \tilde{t}>0 \)), the length of the spring at any time during the motion of the slug is given by the following expression

\[
x(-h_s, \tilde{t}) = -h_s - d_0 - A\sin(\omega \tilde{t})
\]  

(9.1)

When \( \tilde{x}(0, \tilde{t}) = 0 \) there is contact between the slug and the rod. Since wave propagation in the slug only accounts for small deformations, when impact occurs we assume that \( \tilde{x}(-h_s, \tilde{t}) \approx -h_s \). At impact, the change in spring length (relative to the unstressed length) is \( A\sin(\omega \tilde{t}) \). In the mathematical model that follows we take \( \tilde{F} \) to mean the forcing exerted by the spring, where it is understood that given the phase of the forcing when an impact occurs we can calculate \( \tilde{F} \). Therefore, assuming that the spring obeys Hooke’s law, the force exerted by the spring is \( \tilde{F} = k A\sin(\omega \tilde{t}) \), where \( k \) is the spring modulus of elasticity. Also, relative to the period of the forcing, the duration of the impact is small, and therefore we assume that \( \tilde{F} \) is constant throughout the impact.

Between impacts, in a simulation of the system, the external forcing \( \tilde{F} \) varies continuously. Overall, the forcing is a piecewise continuous function, where the discontinuities result from the impacts. With the persistent influence of the external forcing, the slug becomes pre-stressed. In particular, immediately prior to an impact there is a stress variation throughout the slug.

In section 9.2 we begin by calculating the stress variation in the slug prior to the impact. Then, with the slug in a pre-stressed state, we consider the impact itself in section 9.2.2. In section 9.3 we calculate the solutions in the two bodies, and finally in section 9.4 we display some results.
9.2 Mathematical modelling

9.2.1 Pre-stressing the slug before impact

Before the impact occurs, the presence of the external forcing pre-stresses the slug, and also produces a uniform acceleration. Therefore, we must determine the deformation and the acceleration, \( \ddot{a} \). Referring to figure (6.2), we can write the equation of motion for any section (thickness \( \delta X_0 \)) of the slug, the result of which is,

\[
\frac{\partial \sigma}{\partial X_0} = \rho \ddot{a}
\]

(9.2)

We can state the boundary conditions at the end of the slug: at \( X_0 = 0 \) the stress is zero, and at \( X_0 = -h_s \) the stress in the slug is equal to the stress caused by the external forcing \( \ddot{F} \). Integrating the equation of motion for the stress and using the boundary condition at \( X_0 = 0 \), we obtain \( \ddot{\sigma} = \rho \ddot{a} \dot{X}_0 \).

A positive force \( \ddot{F} \) acting at \( \dot{X}_0 = -h_s \) results in compression in the slug, and hence \( \ddot{\sigma} \) is negative. From the boundary condition at \( \dot{X}_0 = -h_s \),

\[
\ddot{\sigma} \bigg|_{\dot{X}_0 = -h_s} = -\ddot{F} = -\rho \ddot{a} h_s A
\]

(9.3)

Rearranging the equation above, we find an expression for the acceleration of the slug,

\[
\ddot{a} = \frac{\ddot{F}}{h_s \rho A}
\]

(9.4)

The stress in the slug is related to the displacement \( u_0 \) by,

\[
\dot{\sigma} = E \dot{\eta} \frac{\partial u_0}{\partial \dot{X}_0} = \rho \ddot{a} \dot{X}_0
\]

(9.5)

The stress at \( X_0 = 0 \) is zero, and hence integrating the expression above we find,

\[
\ddot{u}_0 = \frac{\ddot{F} \dot{X}_0^2}{2EA h_s}
\]

(9.6)

This expression for \( \ddot{u}_0 \) gives the displacement throughout the slug prior to impact, due to the external forcing. The slug is now pre-stressed and it is in this state when the impact occurs. At the impact, the pre-stressed slug is moving with velocity \( V \), and is accelerated with a constant acceleration \( \ddot{a} \). Hence, the position of the slug is given by,

\[
\ddot{x}_0 = \ddot{X}_0 + V \ddot{\gamma} + \dddot{\ddot{F}} \frac{\dot{X}_0^2}{2A h_s \rho} + \dddot{\dddot{F}} \frac{\dot{X}_0^3}{2EA h_s}
\]

(9.7)

The third term is the displacement due to the constant acceleration, and the fourth term is due to the pre-stressing. This solution does not include waves which propagate from the impact, and we move on to consider this now.
9.2.2 The impact at \( t=0 \)

The displacements in the slug and rod due to the wave propagation after the impact is expressed by \( \ddot{u} \) and \( \ddot{u} \) respectively. After the impact, the positions of the slug and rod are

\[
\ddot{x} = \dot{X}_0 + V \ddot{t} + \frac{\ddot{F} \ddot{t}^2}{2 A \rho s} + \frac{\ddot{F} X_0^2}{2 E A h_s} + \dddot{u}(\ddot{X}_0, \ddot{t}) \tag{9.8}
\]

\[
\ddot{z} = \dot{Z}_0 + \dddot{u}(\ddot{X}_0, \ddot{t}) \tag{9.9}
\]

We use the same non-dimensionalising scheme introduced in chapter 6. Also, we non-dimensionalise the external forcing \( \ddot{F} \) thus

\[
F = \frac{\ddot{F}}{V A \rho c} \tag{9.10}
\]

Therefore, the non-dimensional solution in the slug and rod is,

\[
x = X_0 + \frac{V}{c} \left( t + \frac{F}{2} t^2 + \frac{F}{2} X_0^2 + u(X_0, t) \right) \tag{9.11}
\]

\[
z = X_0 + \frac{V}{c} \alpha \dddot{u}(X_0, t) \tag{9.12}
\]

As usual, solutions \( u \) and \( \dddot{u} \) satisfy the wave equation. We now follow a very similar procedure to section 6.4.2. However, the boundary conditions are changed due to the presence of the external forcing. In the \( s \)-domain the solutions for \( u \) and \( \dddot{u} \) are

\[
\dddot{u}(X_0, s) = P e^{-sX_0} + Q e^{sX_0} \tag{9.13}
\]

\[
\dddot{u}(X_0, s) = F e^{-s \alpha X_0} + G e^{s \alpha X_0} \tag{9.14}
\]

The functions \( P, Q, F \) and \( G \) are arbitrary functions of \( s \).

Boundary conditions

There are four boundary conditions. Briefly these are,

1. Continuity of stress at the end of the slug \( \dddot{X}_0 = -h_s \),

\[
E \left( \frac{\partial \dddot{z}}{\partial \dddot{X}_0} - 1 \right) \bigg|_{\dddot{X}_0 = -h_s} = -\frac{\dddot{F}}{A} \tag{9.15}
\]

After substitutions and non-dimensionalising, we find that the terms containing the forcing \( \dddot{F} \) cancel and the following expression remains,

\[
\frac{\partial \dddot{u}}{\partial X_0} \bigg|_{X_0 = -1} = 0 \tag{9.16}
\]
2. Continuity of particle velocities at the interface,
\[
\left(1 + \frac{F}{t} + \frac{\partial u}{\partial t}\right)_{x_0=0} = \alpha \frac{\partial \hat{u}}{\partial t} |_{x_0=0}
\]  
(9.17)

Taking Laplace transforms,
\[
\frac{1}{s} + \frac{F}{s^2} + s \hat{u} |_{x_0=0} = s \alpha \hat{u} |_{x_0=0}
\]  
(9.18)

3. In the s-domain the expression for the continuity of stress at the interface is,
\[
\frac{z}{X_0} \frac{\partial \hat{u}}{\partial X_0} |_{x_0=0} = \frac{\partial \hat{u}}{\partial X_0} |_{x_5=0}
\]  
(9.19)

4. The end of the rod is free of stress,
\[
\frac{\partial \hat{u}}{\partial X_0} |_{x_5=0} = 0
\]  
(9.20)

Substituting equations (9.13) and (9.14) into boundary conditions 1 and 4 we can write, 
\[Q = \frac{1}{2} \alpha K \xi, \quad P = \frac{1}{2} \alpha K \xi, \quad F = \frac{1}{2} \alpha K \xi a, \quad \text{and} \quad Q = \frac{1}{2} \alpha K \xi a, \]  
where \(K\) and \(\bar{K}\) are both arbitrary functions of \(s\). We can then write equations (9.13) and (9.14) as follows,
\[
\hat{u} = K \cosh(s(1+X_0))
\]  
(9.21)
\[
\hat{u} = K \cosh(s(a-X_0))
\]  
(9.22)

Using these solutions, we find the following expressions from boundary conditions 2 and 3,
\[
\frac{1}{s} + \frac{F}{s^2} + s \alpha \cosh(s) = s \alpha \bar{K} \cosh(s a a)
\]  
(9.23)
\[
s z \alpha \sinh(s) = -s \alpha \bar{K} \sinh(s a a)
\]  
(9.24)

From these we can find expressions for \(K\) and \(\bar{K}\). Letting \(T = \tanh(s a a)\), we now have the following solutions in the slug and the rod,
\[
\hat{u} = -\left(\frac{1}{s^2} + \frac{F}{s^2}\right) T \cosh(s(1 + X_0))
\]  
(9.25)
\[
\hat{u} = \frac{z}{s^2} \left(1 + \frac{F}{s^2}\right) T \cosh(s) \cosh(s a - X_0))
\]  
(9.26)

In the next section we find the inverse Laplace transforms of these solutions in the slug and rod.

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9.3 Finding the inverse Laplace transforms

9.3.1 Useful expansions

We first expand some smaller expressions which will be useful later.

\[ T = \frac{\sinh(s\alpha)}{\cosh(s\alpha)} = \frac{1 - e^{-2s\alpha}}{1 + e^{-2s\alpha}} \]  
\[ (9.27) \]

\[ \frac{1}{T + z} = \frac{e^{s\alpha}(1 + z) - e^{-s\alpha}(1 - z)}{1 + e^{-2s\alpha}} \]
\[ = \frac{(1 + z) \left[ 1 - e^{-2s\alpha} \left( \frac{1 - z}{1 + z} \right) \right]}{1 + e^{-2s\alpha}(1 - z)} \]
\[ (9.28) \]

\[ \left[ 1 + \left( \frac{T - z}{T + z} \right) e^{-2s} \right]^{-1} = \left[ 1 + \frac{(1 - z) \left[ 1 - e^{-2s\alpha} \left( \frac{1 + z}{1 - z} \right) \right]}{(1 + z) \left[ 1 - e^{-2s\alpha} \left( \frac{1 - z}{1 + z} \right) \right]} e^{-2s} \right]^{-1} \]
\[ (9.29) \]

We also use the following Taylor series expansions,

\[ (1 + x)^{-r} = \sum_{i=0}^{\infty} (\frac{-1)^i (r - 1 + i)!}{i!(r - 1)!} x^i \]  
\[ (9.30) \]

\[ (1 + x)^r = \sum_{i=0}^{r} \frac{r! x^i}{i!(r - i)!} \]  
\[ (9.31) \]
9.3.2 Finding the solutions in the slug

Using the formulae above, equation (9.25) is recast in the following form

\[
\dot{u} = -\left( \frac{1}{s^2} + \frac{P}{s^3} \right) T \left[ e^{s(1+X_0)} + e^{-s(1+X_0)} \right]
\]

\[
= -\left( \frac{1}{s^2} + \frac{P}{s^3} \right) T \left[ e^{sX_0} + e^{-s(2+X_0)} \right]
\]

\[
= \frac{-\left( \frac{1}{s^2} + \frac{P}{s^3} \right) \left[ 1 - e^{-2s\alpha a} \right] \left[ e^{sX_0} + e^{-s(2+X_0)} \right]}{(1+z) \left[ 1 - e^{-2s\alpha a} \left( \frac{1-z}{1+z} \right) \right]}
\]

\[
= \sum_{r=0}^{\infty} \frac{(z-1)^r}{(z+1)^{r+1}} \left[ 1 - e^{-2s\alpha a} \left( \frac{1-z}{1+z} \right) \right] \sum_{j=0}^{\infty} \frac{(-1)^j (r+j)!}{j! \cdot r!} \left( \frac{z-1}{z+1} \right)^j e^{-2s\alpha a r j}
\]

The expression \([1 - e^{-2s\alpha a}] T\) gives an infinite series, of terms \(e^{-2s\alpha a i}\) (where \(i = 0, \ldots, \infty\)). If we make the rod semi-infinite then \(a \rightarrow \infty\) and \([1 - e^{-2s\alpha a}] T = 1\). Therefore, for a semi-infinite rod, at this point we can find the solution in the slug. Thus,

\[
\dot{u} = \sum_{r=0}^{\infty} \left( \frac{1}{s^2} + \frac{P}{s^3} \right) \left( \frac{z-1}{z+1} \right)^r \left[ e^{-s(2r-X_0)} + e^{-s(2(r+1)+X_0)} \right] \left[ 1 - e^{-2s\alpha a} \right] T
\]

\[
= \sum_{r=0}^{\infty} \left( \frac{1}{s^2} + \frac{P}{s^3} \right) \left[ 1 - e^{-2s\alpha a} \right] \left[ e^{sX_0} + e^{-s(2+X_0)} \right] \left[ z+1 \right] \left[ 1 - e^{-2s\alpha a} \right] T
\]

This solution in the slug is the same as equation (6.33), but with an additional contribution due to the external forcing \(F\). We can still use figure (6.3) to visualise the propagation of the waves, although it is understood that each wave has an additional component due to
the forcing, and the origin of this is the interface, which can be verified from the boundary conditions. Hence, the solution consists of a single sequence of waves travelling back-and-forth in the slug, which also initiate positive travelling waves in the rod. For each value of \( r \), equation (9.34) consists of two waves: one negative travelling wave from the interface, and its reflection from the \( X_0 = -1 \) boundary.

In contrast, when the rod is finite, negative travelling waves in the rod are possible. Each wave in the rod returning to the interface sets up a new sequence of back-and-forth waves in the slug, which in turn introduce new waves. A cumulative effect on the number of waves in the two bodies is clearly seen. We must therefore find the inverse of equation (9.33). The term \([1-e^{-2\omega \sigma \alpha}] \gamma \) is responsible for the waves set up in the slug when a wave in the rod returns back to the interface.

Expanding \( \gamma \) we obtain,

\[
\gamma = \left[ \sum_{i=0}^{r} \frac{r!}{i!(r-i)!} \frac{(z+1)^i}{(z-1)^i} e^{-2\omega \sigma \alpha i} \right] \left[ \sum_{j=0}^{\infty} \frac{(-1)^j(r+j)!}{j!r!} \frac{(z-1)^j}{(z+1)^j} e^{-2\omega \sigma \alpha j} \right] \\
= \left[ 1 + \frac{r!}{(r-1)!} \frac{(z+1)e^{-2\omega \sigma \alpha}}{(z-1)} + \frac{1}{2} \frac{r!}{(r-2)!} \frac{(z+1)^2 e^{-4\omega \sigma \alpha}}{(z-1)^2} + \ldots + \frac{(z+1)^r e^{-2\omega \sigma \alpha r}}{(z-1)^r} \right] \\
\quad \left[ 1 - \frac{(r+1)!}{(z-1)} e^{-2\omega \sigma \alpha} + \frac{1}{2} \frac{(r+2)!}{(z+1)^2} e^{-4\omega \sigma \alpha} + \ldots \right] \\
= 1 - \frac{(r+1)(z-1)e^{-2\omega \sigma \alpha}}{(z+1)} + \frac{1}{2} \frac{(r+1)(r+2)(z-1)^2 e^{-4\omega \sigma \alpha}}{(z+1)^2} + \ldots \\
\quad + \frac{r(z+1)e^{-2\omega \sigma \alpha}}{(z-1)} - r(r+1)e^{-4\omega \sigma \alpha} + \frac{1}{2} \frac{r(r+1)(r+2)(z-1)e^{-6\omega \sigma \alpha}}{(z+1)} + \ldots \\
\quad + \frac{1}{2} \frac{r(r-1)(z+1)^2 e^{-4\omega \sigma \alpha}}{(z-1)^2} - \frac{1}{2} \frac{r(r-1)(r+1)(z+1)e^{-6\omega \sigma \alpha}}{(z-1)} + \ldots \\
\quad + \frac{(z+1)^r}{(z-1)} e^{-2\omega \sigma \alpha r} - \frac{(z+1)^{r-1}}{z-1} (r+1)e^{-2\omega \sigma \alpha (r+1)} + \ldots \\
\text{(9.35)}
\]

Note, the above expression consists of a summation of \( r+1 \) infinite series. If we collect the common exponential terms in the expression above, we obtain an infinite series, where each term in the series contains the general term, \( e^{-2\omega \sigma \alpha i} \) (where \( i = 0, \ldots, \infty \)). Note, a wave in the rod travelling from the interface takes time \( 2\alpha \) to return back to the interface.

The expression \( V c p \frac{\partial u}{\partial X_0} \) gives the stress in the slug at cross-section \( X_0 \) and time \( t \). This can be calculated from equation (9.34). We can also calculate the mean velocity of the slug. The velocity of the slug at cross-section \( X_0 \) and time \( t \) is given by

\[
\frac{\partial x}{\partial t} = \frac{V}{c} \left( 1 + Ft + \frac{\partial u}{\partial t} \right) \quad (9.36)
\]

We can find the mean velocity by calculating the definite integral of the expression for the velocity of the slug with respect to \( X_0 \) over the height of the slug. We then take Laplace
transforms, to obtain the following expression.

\[
V_m(s) = \frac{V}{cs} + \frac{VF}{cs^2} + \frac{c}{s} \int_{X_0=1}^{X_0=0} s \, \dot{u} \, dX_0
\]

\[
= \frac{V}{cs} + \frac{VF}{cs^2} - \frac{V}{c} \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \frac{T \sinh(s)}{[\sinh(s) + T \cosh(s)]}
\]

(9.37)

We evaluated the integral in the above expression using the definition for \( \dot{u} \) in equation (9.25). We can expand this expression for the mean velocity, into a suitable form for inversion. Hence,

\[
V_m(s) = \frac{V}{cs} + \frac{VF}{cs^2} - \frac{V}{c} \sum_{r=0}^{\infty} \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \frac{(z-1)^r}{(z+1)^{r+1}} \left[ e^{-2sr} - e^{-2s(r+1)} \right] [1 - e^{-2sa}] Y
\]

(9.38)

9.3.3 Finding the solutions in the rod

Equation (9.26) is recast in the following form

\[
\dot{u} = z \left( \frac{1}{s^2} + \frac{F}{s^3} \right) T \left[ e^s - e^{-s} \right] \left[ e^{sa(a-X_0)} + e^{-sa(a-X_0)} \right]
\]

\[
\alpha \left[ e^{sa} - e^{-sa} \right] \left[ (z+T) e^s - (z-T) e^{-s} \right]
\]

\[
= \frac{z \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \left[ 1 - e^{-2s} \right] \left[ e^{-saX_0} + e^{-sa(2a-X_0)} \right]}{\alpha (1+z) \left[ 1 - e^{-2sa} \left( \frac{1-z}{1+z} \right) \right] \left[ 1 + \left( \frac{T-z}{T+z} \right) e^{-2s} \right]}
\]

\[
= \frac{z \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \left[ 1 - e^{-2s} \right] \left[ e^{-saX_0} + e^{-sa(2a-X_0)} \right]}{\alpha \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \left[ 1 - e^{-2sa} \left( \frac{1-z}{1+z} \right) \right] \left[ 1 + \left( \frac{T-z}{T+z} \right) e^{-2s} \right]}
\]

(9.39)

We can find the inverse Laplace transform of the expression above, term by term.

The stress in the rod is given by \( V \varepsilon \frac{\partial \varepsilon}{\partial X_0} \). At any time \( t \) and cross-section \( X_0 \) the pointwise velocity is,

\[
\frac{\partial \varepsilon}{\partial t} = \frac{V}{c} \frac{\partial \dot{u}}{\partial t}
\]

(9.40)
In computing the mean velocity of the rod, we calculate the definite integral of the pointwise velocity with respect to $X_0$ over the height of the rod, and then take Laplace transforms to obtain,

$$\mathcal{V}_m(s) = \frac{1}{a} \int_{X_0=0}^{X_0=a} \frac{V}{c} \alpha s \hat{u} dX_0$$

$$= \frac{V}{c} \alpha \left\{ \frac{1}{z} \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \right\} \tau \sinh(s)$$

$$(9.41)$$

Comparing equation (9.41) with equation (9.37), we observe that there is a common expression in both equations, and hence we use equation (9.38) to obtain the following expression.

$$\mathcal{V}_m(s) = \frac{V}{c} \alpha \sum_{r=0}^{\infty} \frac{1}{s^2} \left[ 1 - e^{-2s} \right] \left[ 1 - e^{-2s r (r+1)} \right] [1 - e^{-2s \alpha}] \tau$$

$$(9.42)$$

### 9.3.4 The solutions in the slug and rod when $z=1$

If we assume that the slug and rod are constructed from the same material then $z=1$ and $\alpha = 1$. In which case it is possible to rewrite the expressions for the position, stress and mean velocity. Hence, we can write the expressions for $\hat{u}$ and $\hat{u}$ in the following forms,

$$\hat{u} = \sum_{r=0}^{\infty} \frac{1}{2} \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \left[ 1 - e^{-2s} \right] \left[ 1 - e^{-2s (2r+1)} \right] e^{-2s r (r+1)}$$

$$\hat{u} = \sum_{r=0}^{\infty} \frac{1}{2} \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \left[ 1 - e^{-2s} \right] \left[ 1 - e^{-2s (2r+1)} \right] e^{-2s r (r+1)}$$

Also, we can write the common expression in equations (9.37) and (9.41), in the following form,

$$\frac{1}{z} \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \tau \sinh(s)$$

$$= \sum_{r=0}^{\infty} \frac{1}{2} \left( \frac{1}{s^2} + \frac{F}{s^3} \right) \left[ 1 - e^{-2s} \right] \left[ 1 - e^{-2s r (r+1)} \right] e^{-2s r (r+1)}$$

$$(9.43)$$

### 9.4 Results

We can find the inverse Laplace transform of the $s$-domain solutions in the slug and rod. In figures (9.2), (9.3) and (9.4) we show graphs of the time history of the displacement and stress at the interface, for given choices of $z$ and $F$, and where we have set $\alpha = 5.5$. Also, given the expressions for the stress at the interface, we can calculate the parting time for any given choice of parameters. As previously, we can then use the parting times to calculate the mean velocities of the slug and rod at parting, and therefore we obtain the effective coefficient of restitution. This information is contained in the graphs of figures (9.5-9.9). These results are discussed in the final section of this chapter.
9.5 Discussion & chapter summary

From figures (9.2-9.4), in plots of the stress at the interface, the initialisation of new waves in the system when existing waves meet the boundaries can be seen by the presence of the Heaviside functions in the plots. Hence, at even integer times, we observe discontinuities in the stress, due to returning waves in the slug. At \( t = 2n \), there is a discontinuity due to the first wave in the rod returning to the interface. Also in these solutions, the effect of the external forcing is clearly seen. For example, when \( 0 < t < 2 \), according to the direction of the forcing, the stress at the interface either increases or decreases during this period. Intuitively, we would expect to obtain smaller parting times for when \( F \) is negative.

After parting has occurred, it is possible to continue computing the solutions in the slug and the rod. Both bodies become separate systems; the external forcing remains acting at one boundary of the slug, however the boundary condition at \( X_0 = 0 \) in both bodies changes, since the stress at these points must be equal to zero. However, in this work we consider the mean velocities of the slug and rod at the instance of parting, and therefore we do not calculate the solutions after parting occurs.

The dimensional forcing term \( \tilde{F} \) is a bounded expression, in as much as when the slug and the rod are in contact, due to the positioning of the forced end of the spring, the force exerted by the spring on the slug has a minimum and maximum value. Given the form of the non-dimensional forcing parameter \( F \), in equation (9.10), and noting that both \( \rho \) and \( c \) are both large numbers, it is clear that within normal operating conditions \( F \) is small. However, there is an important exception. If \( V \) is very small, for example during a sequence of chattering motion, the non-dimensional \( F \) will become very large.

Comparing figures (9.6) and (9.8), we observe that many of the characteristics of the coefficient of restitution persist for any given value of \( \alpha \). However, we note that for typical values of \( F \) (\( |F| < 0.1 \)), the effective coefficient of restitution is smaller when the relative length of the rod is long.

We observe in figures (9.5-9.9), for particular regions of the parameter space, negative values for the effective coefficient of restitution. In these cases, the stress at the interface becomes positive indicating parting has occurred, and examining the pointwise velocities at the interface, there is a positive relative pointwise velocity. However, the relative mean velocities of parting is negative, due to the slug having a greater mean velocity than the rod. We note that when this occurs, \( F \) is large and positive, i.e., when the force exerted by the spring is pushing the two objects together and the velocity \( V \) is small. We identify that these are the characteristics of chattering motion and sticking. Indeed, if the relative mean velocity is negative, it is likely that after parting a second impact is likely to occur almost immediately. Hence, for large values of \( F \), we conclude that the use of an effective coefficient of restitution is too crude to capture the complexities of the system.

In the likely situation where the slug and rod are constructed from the same material, the
graphs of figure (9.9) show the characteristics of the impact, where $\alpha \alpha = 25.0$. It can be seen that for typically small values of $F$ ($|F| < 0.5$), the effective coefficient of restitution is small, and steadily increases as $|F|$ increases. For low velocity impacts, $F$ is large and the graph shows a sudden change in the value of the effective coefficient of restitution. In practice we perhaps expect this change to happen more smoothly, although this is worthy of further investigation.

Hence, unlike the impact models of chapters 6, 7 and 8, the characteristics of the impact of this chapter is strongly dependant on the impacting velocity.
Figure 9.2: The position of the interface, and stress at the interface (for $z=0.36$, $\alpha a = 5.5$ and various $F$)
Figure 9.3: The position of the interface, and stress at the interface (for \( z=0.9 \), \( \alpha \alpha = 5.5 \) and various \( F \))
Figure 9.4: The position of the interface, and stress at the interface (for $z=1.9$, $\alpha a = 5.5$ and various $F$)
Figure 9.5: Graphs showing the parting times, mean velocities and coefficients of restitution, against $z$, for various values of $F$ and where $\alpha \delta = 5.5$. 
Figure 9.6: Graphs showing the parting times, mean velocities and coefficients of restitution, against $F$, for various values of $z$ and where $\alpha a = 5.5$. 
Figure 9.7: Graphs showing the parting times, mean velocities and coefficients of restitution, against $z$, for various values of $F$ and where $\alpha_0 = 25.0$. 
Figure 9.8: Graphs showing the parting times, mean velocities and coefficients of restitution, against $F$, for various values of $z$ and where $a = 25.0$. 
Figure 9.9: Graphs showing the parting times, mean velocities and coefficients of restitution, against $F$ and where $\alpha a = 25.0$ and $z = 1$
Chapter 10

Conclusions

The derivation of the first idealised model of the system in Chapter 2 hinges on three major assumptions, which are

- The main beam has zero curvature along the length of the delamination.
- The delamination moves in the fundamental mode of vibration only.
- A coefficient of restitution is used to model the impact.

With the assumptions above, we derived a relatively simple model, identical to the impact oscillator. This system is studied in detail in Chapters 3, 4 and 5. In the later parts of the thesis we examined the validity of the second and third assumptions in more detail.

In Chapter 3 we present some results illustrating the global behaviour of the system, and also how the transition between the different motions can occur. From the plots in section 3.3.1, we can make some general conclusions concerning the effect of introducing damping to the system. Considering at this stage the frequency range $0 < \omega < 1$, the damping gradually eliminates all chaotic motion and periodic motion with a period greater than one. The regions of chatter also enlarge with the introduction of damping.

The results of Chapter 3 are contrasted with the results of an experimental investigation presented in Chapter 4. We found an excellent comparison between theory and practice over a range of parameters. We found that a coefficient of restitution in the range $0 < r < 0.3$, and a low value of the damping parameter, was successful for the purposes of the comparison. In the later chapters of this thesis, we found strong evidence that we can be confident in this selection for the coefficient of restitution. For forcing frequencies less than the frequency of the second mode of the delamination, the model exhibits the characteristics of the physical system: chatter, period 1 and 2 motion, with a period doubling transition between the period 1 and 2 motions. Immediately after this first period doubling bifurcation, the presence of the higher modes of vibration is clear from the experimental results. However the overall features of the motion seem to be dictated by the contribution from first mode
of vibration, and the impact does appear to occur at the same time, along the length of the delamination. At even higher forcing frequencies, it would appear that the influence from the higher modes causes the lack of correspondence between theory and practice. A more sophisticated model would be needed for forcing frequencies in this range. Also, as would perhaps be expected, the experimental results are influenced by the material of the delamination. For example, with a carbon-fibre or wood delamination, period 3 and 4 motion was also observed, contrasting with steel, where the second bifurcation from period 2 to 3 was never observed. Using a steel delamination, at high forcing frequencies, a non-impacting motion often resulted, where the delamination moves as a simple, cantilever beam.

In an experimental situation there will obviously be a number of unavoidable complications. For example, in the equilibrium position, the delamination is imperfect in terms of its flatness and hence is not in contact with the main beam along the entire length of the delamination. Also the delamination sometimes suffers fatigue at the join to the main beam, and at rest appears in approximately a mode 1 shape, slightly displaced from the main beam. The model could be extended to include a clearance parameter. The effect of clearance variation on the general character of figure (3.7) is an interesting investigation, see [2]. However, the motion of the system is generally quite robust to small changes in the experimental details.

As mentioned in the introduction, numerical simulation is often the only available option when finding solutions to non-linear problems. Indeed, the theoretical results presented in this report are the result of large scale numerical simulations, which reveal much about the behaviour of the system. However, the mechanism of the bifurcations are better understood by approaching the problem analytically, and this is done in Chapter 5. Another investigation was documented by Budd et al [2], where it is shown that it is analytically possible to compute the regions of parameter space which contain possible period 1 orbits. Also at this stage some of the graphical techniques of section 3.4.1 may be useful, for example, when looking at fixed points of the mapping.

In Chapter 5, we set the coefficient of restitution to zero, and hence we examine the dynamics of the inelastic impact oscillator. Firstly, we compared bifurcation diagrams for low damping and \(0 < r < 0.3\), with bifurcation diagrams for low damping and \(r = 0\). We noted that many features are similar, and concluded that the observations of the inelastic impact oscillator are quite applicable to more general impact oscillators with large dissipation at the impact.

The consequence of setting \(r = 0\) is that we reduce the dimension of the impact mapping by one, resulting in a one-dimensional mapping. This mapping is easier to visualise graphically, and analytical investigations are simplified. For given values of damping, we found an analytical expression for when period doubling bifurcations occur. Numerically, we computed these frequencies and these can be shown graphically to indicate the influence of damping on the period doubling frequencies.

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Next, we computed the locations of the grazes in the impact mapping, for given choices of parameters. We noted that the grazes initially appear with a phase at the start of the forcing period, and as the frequency parameter increases, the phase of the graze also increases and eventually the phase of the grazes tends to \( \frac{\pi}{2} \). Therefore, in the impact mapping, grazes always appear at phases in the first quarter of the forcing period. In order to calculate the location of the grazes, we found an expression which when solved numerically for the roots, leads to a set of initial conditions in the phase-space \( [0, \frac{\pi}{2}] \), with trajectories which contain zero velocity 'impacts'. Some of these trajectories are invalid since only positive displacements are possible and there is negative displacement before the zero velocity impact occurs. We made some observations regarding the valid and invalid grazes, and as a suggestion for further work, we would like to derive a thorough mathematical procedure for specifying which roots (of the expression for possible grazes) correspond to the valid grazes.

The coefficient of restitution model of an impact is quite crude. For instance, it predicts an instantaneous reversal of velocity at the impact, and an infinitely small time when the bodies are actually in contact. The model developed in Chapter 2 also assumes that the delamination moves in the fundamental mode, and the impact does not introduce any other modes of vibration into the motion. Therefore, in order to determine if the impact introduces components from the higher modes of vibration into the motion of the delamination, and to be able to predict an effective coefficient of restitution, dependent on impacting materials, velocity, etc., we developed more sophisticated models of the impact.

In Chapter 6, we studied three closely related models of single degree-of-freedom impacting systems. Firstly, we examined a single slug impacting a semi-infinite rod. We found that whether parting occurs is dependent on the parameter \( z \), which is the ratio of the acoustic impedance of the slug to that of the rod. If \( z \leq 1 \), then parting occurs at time \( t = 2 \), when the first reflected wave in the slug returns to the interface. The effective coefficient of restitution is computed as,

\[
r = \frac{1 - z}{1 + z}
\]

Therefore, if \( z \ll 1 \) the impact approaches an elastic impact. If \( z > 1 \) then parting does not occur. The stress at the interface is asymptotic to zero as time approaches infinity, although it always indicates compression. This being the case, we could agree that the effective coefficient of restitution is zero.

Secondly in Chapter 6, we investigated a single slug impacting a finite rod. Figure (6.7) shows the characteristics of the impact in this system. Due to its finite length, we obtain non-zero mean parting velocities in the rod. For \( z < 1 \), the first reflected wave in the slug still causes parting, however the effective coefficient of restitution is dependent on the length of the rod, although it remains the case that if \( z \ll 1 \), the impact approaches an elastic impact. For \( z > 1 \) the reflected waves in the rod are important in causing parting to occur at time \( t = 2 \alpha a \) which corresponds to the first reflected wave in the rod reaching the interface.
Again, the effective coefficient of restitution is dependent on the length of the rod. For all \( z \), the longer the rod, the smaller the restitution. If we examine the solutions for \( z = 1 \), we find that \( r = \frac{1}{a} \). In a practical situation it is quite likely that \( z = 1 \), and if we also assume that the rod is long compared with the slug (for example, \( a > 5 \)), then the effective coefficient of restitution in such situations is in the range \( 0 < r < 0.3 \).

Thirdly in Chapter 6, we examine the impact of a slug on a semi-infinite rod, where the slug is coupled to two stationary neighbouring slugs. Results can be seen in figure (6.19). The presence of the coupling guarantees that parting always occurs, even though there are not any reflected waves in the rod. For \( z < 1 \) parting mostly occurs at \( t = 2 \), although when the coupling is extremely high the parting time can be smaller. When the coupling parameter \( q < 0.2 \), the effective coefficient of restitution is not greatly affected by the strength of the coupling. For \( z > 1 \), parting always occurs for \( q \neq 0 \). The stronger the coupling the shorter the time before parting. As we observe in figure (6.19), the mean parting velocity of the slug fluctuates about 0, although we conclude that generally the velocity is small.

Throughout Chapter 6, we found that the effective coefficient of restitution is independent of the impacting velocity of the slug. However, the parameter \( z \) and the relative lengths of the slug and rod have a strong influence on the characteristics of the impact.

In Chapters 7 and 8 we develop models of impacting beams. A number of slugs are assembled in a line and are coupled together with shear springs. In Chapter 7, each slug is coupled with its nearest neighbour, and in Chapter 8 each slug is coupled to its nearest and next-nearest neighbour. The effect of the coupling is to introduce a finite number of natural modes, equal to the number of the slugs in the discretisation. When the slugs move as rigid bodies, for nearest neighbour coupling the continuous structure behaves as a vibrating string, and for nearest and next-nearest coupling the continuous structure behaves as a beam in bending. Obviously, the model of Chapter 8 is a closer approximation to the physical system. The delamination impacts the main beam, and the main beam is represented by a row of stationary rods which are uncoupled. We can justify this modelling decision, since we assume the main beam is sufficiently massive for its motion not to be disturbed by the impact.

For the beam model with nearest neighbour coupling, we found a solution where the modes are uncoupled. However, for the model with nearest and next-nearest neighbour coupling, we found that it is not possible to implement the correct boundary conditions at the free end of the delamination and also find a solution where the modes are uncoupled. At the expense of the correct boundary conditions at the free end, we obtain a decoupled solution. In both models we then found that, assuming a impact only in the fundamental mode, the rebound is also only in the fundamental mode. We found that the expression for the effective coefficient of restitution is the same expression as in section 6.4, and this is independent of the impacting velocity. This allows us to use the same results (particularly figure (6.19)), where it is understood that the coupling \( q \) is replaced by \( m_1 \) (Chapter 7) and \( n_1 \) (Chapter 8). We computed maximum values of \( m_1 \) and \( n_1 \), which we can use to read results from
section 6.4. We found that \( m_1 \) has a maximum value of 0.1, and \( n_1 \) has a maximum value of 0.016. Hence, in terms of obtaining results from figure (6.19), the effect of the coupling is less pronounced for the model of Chapter 8. The conclusions of section 6.4 can be applied to summarise the behaviour of the impact of the delamination.

An extension to the work of Chapter 8 would be to model the slugs coupled to the nearest and next-nearest neighbours, but implementing the correct boundary conditions for a clamped-free beam. However, the implications of this are that the modes in the solution will not be uncoupled. In effect, in the procedure of Chapter 8, after taking Laplace transforms and applying the boundary conditions, the solution must be found at each slug by calculating the individual inverse Laplace transforms. The overall solution at all of the slugs will not be conveniently expressed in terms of contributions from each mode of vibration.

In Chapter 9 we examined, in detail, the impact in an impact oscillator. We developed a model where a slug impacts a rod at a particular velocity, and where the non-impacting end of the slug is connected to a spring. The force exerted by the spring onto the slug is dependent on the phase of the forcing cycle at which the impact occurs. We noted that the effective coefficient of restitution varies considerably according to the choice of \( z \), and the non-dimensional forcing parameter \( F \). It was also noted that the forcing parameter, is inversely proportional to the impacting velocity. Hence, when the impacting velocity is small, the forcing parameter is large. When \( F \) is large and positive, we identify the type of impact as that observed during chattering motion. In the results of Chapter 9, for large positive values of \( F \) and for certain ranges of the parameter \( z \), we observe a negative effective coefficient of restitution. We conclude that this measure of the impact is an inadequate measure of the impacting behaviour. However, we note that a negative coefficient of restitution, does suggest that another impact will follow almost immediately after parting, as in chattering motion.

We propose that further refinement of the model is necessary. Particularly the use of the effective coefficient of restitution to measure the impact behaviour is shown to be misleading. It would also be beneficial to calculate the solutions in the two separate bodies after parting, to gain further insight into the behaviour after parting. This would be particularly relevant for when the effective coefficient of restitution is negative.

The model derivation and subsequent simulation is part of a larger problem; the growth of the delamination. The dynamic stresses arising from repeated impacts, is an important destructive influence, and it it also thought that some types of motion are more likely to cause a growth of the delamination, an example being chattering motion. Looking at the likely causes of the growth of the delamination is a productive area which is worthy of further investigation, but is outside the scope of this particular work.
Chapter 11

References


Chapter 12

Bibliography


Appendix A

Shape Functions

To simplify equation (2.15) we need to evaluate the integrals. The first two integrals may be simplified by multiplying equation (2.13) by $L_j$ and integrating over the range $(0, l)$. After some manipulation and interchanging of variables, the orthogonality relationships are obtained:

$$\int_0^1 L_i(\tilde{Z})L_j(\tilde{Z}) \, d\tilde{Z} = 0 \quad \text{for } i \neq j, \quad (A.1)$$

$$\int_0^1 L_j(\tilde{Z}) \frac{d^4 L_i(\tilde{Z})}{d\tilde{Z}^4} \, d\tilde{Z} = \begin{cases} 0 & i \neq j \\ \frac{12\rho c_i^2}{E h_i^2} \int_0^1 L_i^2(\tilde{Z}) \, d\tilde{Z} & i = j \end{cases} \quad (A.2)$$

To evaluate the integrals of $L_i(\tilde{Z})$ and $L_i^2(\tilde{Z})$ the shape function has to be calculated explicitly from equation (2.13). The general solution of equation (2.13) is

$$L_i(\tilde{Z}) = A \cosh(\beta_i \tilde{Z}) + B \sinh(\beta_i \tilde{Z}) + C \cos(\beta_i \tilde{Z}) + D \sin(\beta_i \tilde{Z}) \quad (A.3)$$

where $A, B, C, D$ are constants of integration and $\beta_i$ is defined by

$$\beta_i^2 = \frac{12\rho c_i^2}{E h_i^2} \quad (A.4)$$

Applying the boundary conditions (2.10) for the fixed end $\tilde{Z} = 0$, the equation for the shape function reduces to

$$L_i(\tilde{Z}) = A \left( \cosh(\beta_i \tilde{Z}) - \cos(\beta_i \tilde{Z}) \right) + B \left( \sinh(\beta_i \tilde{Z}) - \sin(\beta_i \tilde{Z}) \right) \quad (A.5)$$

To calculate the characteristic frequency equation for the vibrating delaminated portion we apply the boundary conditions (2.10) at the free end ($\tilde{Z} = l$) to obtain constraints on the allowed values of $A, B,$ and $\beta_i$. Recasting these constraints in terms of a matrix we obtain

$$\begin{pmatrix} \cosh(\beta_i l) + \cos(\beta_i l) & \sinh(\beta_i l) + \sin(\beta_i l) \\ \sinh(\beta_i l) - \sin(\beta_i l) & \cosh(\beta_i l) + \cos(\beta_i l) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (A.6)$$
For this system to have a non-trivial solution we require
\[
\begin{vmatrix}
\cosh(\beta l) + \cos(\beta l) & \sinh(\beta l) + \sin(\beta l) \\
\sinh(\beta l) - \sin(\beta l) & \cosh(\beta l) + \cos(\beta l)
\end{vmatrix} = 0
\] (A.7)

Expansion and simplification of the determinant gives the characteristic frequency equation
\[
\cosh(\beta l) \cos(\beta l) = -1
\] (A.8)

The roots of the equation (A.8) can be calculated numerically by a suitable method such as Newton's method. The first few roots are \(\beta_1 l = 1.875\), \(\beta_2 l = 4.694\), and \(\beta_3 l = 7.855\). For large \(i\) the roots may be approximated by \(\beta l = (n - \frac{1}{2}) \pi\). By rearranging equation (A.4) we can define the natural frequencies of the partially delaminated portion of the beam as a function of the physical parameters. The natural frequencies are thereby defined by
\[
\tilde{\omega}_i = \frac{(\beta_i l)^2 h_s}{2l^2} \sqrt{\frac{E}{3\rho}}
\] (A.9)

We define \(\nu_i = \beta_i l\), and thus the shape functions become
\[
L_i(\tilde{Z}) = \cosh\left(\frac{\nu_i \tilde{Z}}{l}\right) - \cos\left(\frac{\nu_i \tilde{Z}}{l}\right) - \left[\cosh(\nu_i) + \cos(\nu_i)\right] \left[\sinh\left(\frac{\nu_i \tilde{Z}}{l}\right) - \sin\left(\frac{\nu_i \tilde{Z}}{l}\right)\right]
\] (A.10)

The amplitude function can now be used to evaluate the integrals of \(L_i(\tilde{Z})\) and \(L_i^2(\tilde{Z})\) over the range \((0, l)\):
\[
\int_0^l L_i(\tilde{Z}) \, d\tilde{Z} = \frac{2l \theta_i}{\nu_i}
\] (A.11)
\[
\int_0^l L_i^2(\tilde{Z}) \, d\tilde{Z} = l
\] (A.12)

where \(\theta_i\) is defined by
\[
\theta_i = \left(\frac{\cosh(\nu_i) + \cos(\nu_i)}{\sinh(\nu_i) + \sin(\nu_i)}\right)
\] (A.13)
Appendix B

Additional Bifurcation Diagrams
(with one parameter variable)
Figure B.1: Bifurcation Diagrams ($r=0.8$, $\gamma=0.05$).
Figure B.2: Bifurcation Diagrams ($r=0.86$, $\gamma=0$).
Appendix C

Additional Experimental Results

In figures (C.1), (C.2) and (C.3), a set of results is presented for a slightly different configuration from the results in chapter 4. This is a larger set of results which illustrate the developing motion of the system in greater detail.
Figure C.1: Experimental Results
Figure C.2: Experimental Results
Figure C.3: Experimental Results
Appendix D

Using Series Expansions and Inverting Term-by-Term

D.1 Some useful expansions

The expressions for the position, stress and displacement of the slug are given by equations (6.71), (6.72) and (6.74). We can break down these expressions and rewrite the sub-expressions in the form of a Taylor series. Hence in the following, we expand about $q = 0$ and include terms up to order $O(q^6)$.

Therefore approximating $A$,

$$A = \sqrt{s^2 + q^2} = s\sqrt{1 + \frac{q^2}{s^2}} = s \left(1 + \frac{q^2}{2s^2} - \frac{q^4}{8s^4} + O(q^6)\right)$$

We can expand a general exponential term, $e^{AD}$, where $D$ is a constant expression.

$$e^{AD} = 0 \left(1 + \frac{q^2}{2s^2} - \frac{q^4}{8s^4} \cdots \right)$$

$$= e^{sD} e^D \left(\frac{q^2}{2s^2} - \frac{q^4}{8s^4} \cdots \right)$$

$$= e^{sD} \left(1 + \frac{q^2D}{2s} - \frac{q^4D^2}{8s^3} + \frac{q^6D^3}{8s^5} \cdots \right)$$

$$= e^{sD} \left(1 + \frac{D}{2s} q^2 + \left(\frac{D^2}{8s^2} - \frac{D}{8s^3}\right) q^4 + O(q^6)\right)$$
We expand the expression \((s \pm z \lambda)\), which is needed in another expansion.

\[
s \pm z \lambda = s \pm sz \sqrt{1 + \frac{q^2}{s^2}}
\]

\[
= s(1 \pm z \left(1 + \frac{q^2}{2s^2} - \frac{q^4}{8s^4} + O(q^6)\right))
\]

\[
= s(1 \pm z) \left(1 \pm \frac{zq^2}{2s^2(1 \pm z)} \mp \frac{zq^4}{8s^4(1 \pm z)} + O(q^6)\right)
\]

Raising the series in the equation above to the power of \(j\), we obtain,

\[
\left(1 + \frac{zq^2}{2s^2(1 \pm z)} - \frac{zq^4}{8s^4(1 \pm z)} + O(q^6)\right)^j = 1 + \frac{jzq^2}{2s^2(1 \pm z)} - \frac{jzq^4}{8s^4(1 \pm z)} + \frac{j(j-1)z^2q^4}{8s^4(1 \pm z)^2} + O(q^6)
\]

\[
\left(1 - \frac{zq^2}{2s^2(1 \pm z)} + \frac{zq^4}{8s^4(1 \pm z)} + O(q^6)\right)^j = 1 - \frac{jzq^2}{2s^2(1 \pm z)} + \frac{jzq^4}{8s^4(1 \pm z)} + \frac{j(j-1)z^2q^4}{8s^4(1 \pm z)^2} + O(q^6)
\]

Hence, we can represent the following in the form of a series,

\[
\frac{(s - z \lambda)^r}{(s + z \lambda)^{r+1}} = \frac{(1-z)^r}{s(1+z)^{r+1}} \left[1 - \frac{r z q^2}{2s^2(1+z)} + \frac{r z q^4}{8s^4(1+z)^2} + \frac{r(r-1)z^2 q^4}{8s^4(1+z)^3} + O(q^6)\right]
\]

\[
\cdot \left[1 - \frac{(r+1)z q^2}{2s^2(1+z)} + \frac{(r+1)z q^4}{8s^4(1+z)^2} + \frac{(r+1)(r+2)z^2 q^4}{8s^4(1+z)^3} + O(q^6)\right]
\]

\[
= \frac{(1-z)^r}{s(1+z)^{r+1}} \left[1 + \left(\frac{(r+1)z}{2s^2(1+z)} - \frac{rz}{2s^2(1+z)}\right)q^2
\right.
\]

\[
+ \left(\frac{(r+1)(r+2)z^2}{8s^4(1+z)^2} + \frac{r(r-1)z^2}{8s^4(1+z)^2} + \frac{r(r+1)z^2}{8s^4(1+z)^3} + \frac{r z}{8s^4(1+z)} + \frac{(r+1)z}{8s^4(1+z)}\right)q^4
\]

\[
+ O(q^6)\right]
\]

where \(\Upsilon\) replaces the series in the equation above.

D.2 Finding \(u(X_0, t)\)

For convenience we write equation (6.71) as \(\dot{u} = \dot{u}_1 + \dot{u}_2\), where

\[
\dot{u}_2(X_0, s) = -\frac{q^2}{s^2(s^2 + q^2)}
\]

This has an analytical inverse. The definition of \(\dot{u}_1\) is given below. Firstly, we rewrite the expression in a more useful form and we can then expand it using the expansions from
section D.1.

\[ \hat{U}_1(X_0, s) = \frac{s \cosh(\lambda(1 + X_0))}{(s^2 + q^2)(-z\lambda \sinh(\lambda) - s \cosh(\lambda))} \]

\[ = \frac{-s(\lambda(1 + X_0) + e^{-\lambda(1 + X_0)})}{(s^2 + q^2)(\lambda(\lambda + e^{-\lambda}) + s(\lambda + e^{-\lambda}))} \]

\[ = \frac{-s(e^{\lambda X_0}(1 + e^{-2\lambda(1 + X_0)})}{(s^2 + q^2)(s + z\lambda)(1 + (s-z\lambda)e^{-2\lambda})} \]

\[ = \frac{-s(e^{\lambda X_0}(1 + e^{-2\lambda(1 + X_0)})}{(s^2 + q^2)(s + z\lambda)} \sum_{r=0}^{\infty} (-1)^r \left( \frac{s - z\lambda}{s + z\lambda} \right)^r (e^{-2\lambda})^r \]

\[ = \sum_{r=0}^{\infty} \left[ (-1)^r \frac{-se^{\lambda}e^{(X_0-2r)}}{(s^2 + q^2)(s + z\lambda)} \left( \frac{s - z\lambda}{s + z\lambda} \right)^r \right] \] (D.1)

Now, substituting the smaller expansions into equation (D.1) we obtain,

\[ \hat{U}_1(X_0, s) = -\sum_{r=0}^{\infty} \left[ \frac{(-1)^r}{(s^2 + q^2)(1 + z)^{r+1}} \right] \]

\[ \cdot \left( e^{s(X_0-2r)} \left[ 1 + \frac{(X_0-2r)^2}{2s} - \frac{(X_0-2r)^4}{8s^2} + \frac{(X_0-2r)^6}{8^2s^4} \ldots \right] \right. \]

\[ \left. + e^{-s(X_0+2(r+1))} \left[ 1 - \frac{(X_0+2(r+1))^2}{2s} + \frac{(X_0+2(r+1))^4}{8s^2} + \frac{(X_0+2(r+1))^6}{8^2s^4} \ldots \right] \right) \]

\[ = -\frac{e^{sX_0}}{(s^2 + q^2)(1 + z)} \left\{ 1 + \frac{X_0^2}{2s} - \frac{X_0^4}{2^2s^2} \ldots \right\} \]

\[ - e^{-s(X_0+2)} \left\{ 1 - \frac{X_0^2}{2s} - \frac{X_0^4}{2^2s^2} \ldots \right\} \]

\[ + \frac{e^{s(X_0-2)}(1 - z)}{(s^2 + q^2)(1 + z)^2} \left\{ 1 + \frac{(X_0-2)^2}{2s} + \frac{(X_0-2)^4}{2^2s^2} \ldots \right\} \]

\[ + e^{-s(X_0+4)}(1 - z)^2 \left\{ 1 - \frac{(X_0+4)^2}{2s} + \frac{(X_0+4)^4}{2^2s^2} \ldots \right\} \]

\[ - e^{s(X_0-4)}(1 - z)^2 \left\{ 1 + \frac{(X_0-4)^2}{2s} + \frac{(X_0-4)^4}{2^2s^2} \ldots \right\} \]

\[ - e^{-s(X_0+6)}(1 - z)^2 \left\{ 1 - \frac{(X_0+6)^2}{2s} + \frac{(X_0+6)^4}{2^2s^2} \ldots \right\} \]

\[ + \ldots \] (D.2)
D.3 Finding the stress in the slug

Re-writing the expression for the stress in the slug,

\[
\frac{\partial u(X_0, s)}{\partial X_0} = \frac{ss \sinh(\lambda(1 + X_0))}{\sqrt{s^2 + q^2(\lambda - s \cosh(\lambda))}} - s(e^{\lambda X_0}(1 - e^{-2\lambda(1+X_0)})}
\]

\[
= -\frac{(e^{\lambda X_0}(1 - e^{-2\lambda(1+X_0)}))(1 + s^2)\sum_{r=0}^{\infty} (-1)^r \left(\frac{s - z \lambda}{s + z \lambda}\right)^r(e^{-2\lambda})^r}{s(1+z)^{r+1}(1 + \frac{q^2}{(s+z)\lambda})}
\]

Substituting series approximations into the above expression,

\[
\frac{\partial u(X_0, s)}{\partial X_0} = -\sum_{r=0}^{\infty} \left[ (-1)^r \frac{(1-z)^r}{s(1+z)^{r+1}} \left[ 1 - \frac{q^2}{2s^2} + \frac{2q^2}{8s^4} \ldots \right] \right.
\]

\[
\cdot \left( (e^{s(X_0-2r)}(1 + \frac{(X_0-2r)q^2}{2s}) - \frac{(X_0-2r)q^2}{8s^3} + \frac{(X_0-2r)q^4}{8s^4} \ldots) \right)
\]

\[
- e^{-s(X_0+2(r+1))}\left[ 1 - \frac{(X_0+2(r+1))q^2}{2s} + \frac{(X_0+2(r+1))q^4}{8s^3} + \frac{(X_0+2(r+1))q^6}{8s^4} \ldots \right]
\]

\[
= -\frac{e^{sX_0}}{s(1+z)}\left( 1 + \frac{X_0q^2}{2s} - \frac{q^2}{2s^2} + \frac{2q^2}{2s^2(1+z)} \ldots \right) + \frac{e^{-sX_0+2}}{s(1+z)}\left( 1 - \frac{(X_0+2)q^2}{2s} - \frac{q^2}{2s^2} - \frac{2q^2}{2s^2(1+z)} \ldots \right)
\]

\[
+ \frac{e^{sX_0+4}}{s(1+z)^2}\left( 1 + \frac{(X_0+4)q^2}{2s} - \frac{q^2}{2s^2} + \frac{2q^2}{2s^2(1+z)} \ldots \right) - \frac{e^{-sX_0+4}}{s(1+z)^2}\left( 1 - \frac{(X_0+4)q^2}{2s} + \frac{2q^2}{2s^2(1+z)} \ldots \right)
\]

\[
+ \frac{e^{sX_0+6}}{s(1+z)^3}\left( 1 - \frac{(X_0+6)q^2}{2s} - \frac{q^2}{2s^2} - \frac{2q^2}{2s^2(1+z)} \ldots \right)
\]

\[
+ \ldots
\]

D.4 Finding the mean velocity of the slug

Returning to the expression for the mean velocity of the slug,

\[
\frac{V}{c} = \frac{V}{c} \left[ \frac{-q^2(1 - e^{-2\lambda})}{\lambda(s^2 + q^2)(s(1 + e^{-2\lambda}) + z\lambda(1 + e^{-2\lambda}))} - \frac{q^2}{s(s^2 + q^2)} \right]
\]

\[
= \frac{V}{c} \left[ \frac{-q^2(1 - e^{-2\lambda})}{\lambda(s^2 + q^2)} \sum_{r=0}^{\infty} (-1)^r \frac{(s - z \lambda)^r}{(s + z \lambda)^{r+1}(e^{-2\lambda})^r} \right] - \frac{V}{c} \frac{q^2}{s(s^2 + q^2)}
\]
Table D.1: Table of some useful inverse Laplace transforms.

<table>
<thead>
<tr>
<th>( f(s) )</th>
<th>( \mathcal{L}^{-1}(f(s)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{Be^{-As}}{s} )</td>
<td>( BH[t - A] )</td>
</tr>
<tr>
<td>( \frac{Be^{-As}}{s^2} )</td>
<td>( B(t - A)H[t - A] )</td>
</tr>
<tr>
<td>( \frac{Be^{-As}}{s^3} )</td>
<td>( \frac{1}{2} B(t - A)^2 H[t - A] )</td>
</tr>
<tr>
<td>( \frac{Be^{-As}}{s^2 + q^2} )</td>
<td>( \frac{B}{q} \sin(q(t - A)) H[t - A] )</td>
</tr>
<tr>
<td>( \frac{Be^{-As}}{s(s^2 + q^2)} )</td>
<td>( \frac{B}{q^2} \left( 1 - \cos(q(t - A)) \right) H[t - A] )</td>
</tr>
<tr>
<td>( \frac{Be^{-As}}{s^2(s^2 + q^2)} )</td>
<td>( \frac{B}{q^3} \left( q(t - A) - \sin(q(t - A)) \right) H[t - A] )</td>
</tr>
</tbody>
</table>

The expressions for the position, stress and mean velocity can now be inverted term-by-term to find the time solutions. Some of the analytical inverse Laplace transforms are shown in table D.1. The function \( H[\cdot] \) is the unit step function and \( A \) and \( B \) are constants; we assume that \( A \) is positive.

D.5 Useful analytical inverse Laplace Transforms

The expressions for the position, stress and mean velocity can now be inverted term-by-term to find the time solutions. Some of the analytical inverse Laplace transforms are shown in table D.1. The function \( H[\cdot] \) is the unit step function and \( A \) and \( B \) are constants; we assume that \( A \) is positive.
Appendix E

Finding the Eigenvalues of the Matrix $M^*$

We can find analytical expressions for the eigenvalues and eigenvectors of the $n \times n$ matrix first introduced in chapter 7. The $n$ eigenvalues and the corresponding eigenvectors are denoted by $\mu_i$ and $\vec{e}_i$ respectively, and satisfy the following equations.

$$M^* \vec{e}_i = \mu_i \vec{e}_i$$

In a longer form,

$$\begin{pmatrix}
-2 & 1 & \cdots \\
1 & -2 & 1 \\
\vdots & \ddots & \ddots \\
1 & -2 & 1 \\
1 & -1 & \cdots
\end{pmatrix}
\begin{bmatrix}
l_{i,1} \\
l_{i,2} \\
\vdots \\
l_{i,n}
\end{bmatrix}
= \mu_i
\begin{bmatrix}
l_{i,1} \\
l_{i,2} \\
\vdots \\
l_{i,n}
\end{bmatrix}$$

This system of equations can be expressed as the following set of difference equations (where we have dropped the $i$ subscript),

(i). $l_0 = 0$

(ii). $l_{j-1} - (2+\mu)l_j + l_{j+1} = 0$

(iii). $l_{n-1} - (1+\mu)l_n = 0$

Difference equation (ii) has a solution, $l_j = A m_1^j + B m_2^j$ where $m_1$ and $m_2$ are the roots of,

$$m^2 - (2+\mu)m + 1 = 0$$

Rewriting, $m_1 = re^{i\theta}$ and $m_2 = \frac{1}{r}e^{-i\theta}$. We use the following identities.

$$m_1 m_2 = 1$$

$$m_1 + m_2 = re^{i\theta} + \frac{1}{r}e^{-i\theta}
= (r + \frac{1}{r}) \cos(\theta) + i (r - \frac{1}{r}) \sin(\theta)
= 2 + \mu$$
However, a symmetric matrix has real eigenvalues and therefore, \( r = 1 \). Therefore, from the equation above we can write,

\[
\mu = -2 + 2\cos(\theta) \tag{E.1}
\]

Now,

\[
l_j = A(e^{i\theta})^j + B(e^{-i\theta})^j
\]

\[
= A(\cos(j\theta) + i\sin(j\theta)) + B(\cos(j\theta) - i\sin(j\theta))
\]

\[
= C\cos(j\theta) + iD\sin(j\theta)
\]

From difference equation (i), we find that \( C = 0 \), and hence,

\[
l_j = D\sin(j\theta) \tag{E.2}
\]

By substituting this solution into the third difference equation and simplifying, we obtain

\[
D\sin((n-1)\theta) - (1+\mu)D\sin(n\theta) = 0
\]

\[
\sin((n-1)\theta) - (-1+2\cos(\theta))\sin(n\theta) = 0
\]

\[
\sin(n\theta)(1-\cos(\theta)) - \sin(\theta)\cos(n\theta) = 0
\]

\[
2\sin(n\theta)\sin^2(\frac{\theta}{2}) - 2\cos(n\theta)\sin(\frac{\theta}{2})\cos(\frac{\theta}{2}) = 0
\]

\[
-2\sin(\frac{\theta}{2})\sin(n\theta + \frac{\theta}{2}) = 0
\]

For this final equation to be satisfied, \( \theta = 0 \) or \( \theta(\frac{1}{2} + n) = (\frac{1}{2} + i)\pi, \ (i = 0, \pm 1, ..) \). From these values of \( \theta \), we can use equation (E.1) to find the \( n \) distinct eigenvalues.

\[
\mu_i = -2 \left( 1 - \cos\left(\frac{i - 0.5}{n + 0.5}\pi\right) \right) \ (i = 1, \ldots, n)
\]

The corresponding eigenvector is \( \bar{L}_i = (l_{i,1}, \ldots, l_{i,n}) \). The equation for the \( j \)th component of the \( i \)th eigenvector is given by,

\[
l_{i,j} = D\sin\left(\frac{i - 0.5}{n + 0.5}\pi j\right)
\]