Random periodic solutions of stochastic Burgers equation

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Random Periodic Solutions of Stochastic Burgers Equation

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Summary. The dynamics of randomly forced Burgers and Euler-Lagrange equations in $S^1 \times \mathbb{R}^{d-1}$ in the case when there is only one one-sided minimizer in a compact subset of $S^1 \times \mathbb{R}^{d-1}$ is studied. The existence of random invariant periodic minimizer orbits and periodicity of the stationary solution of the stochastic Burgers equations are obtained.

Keywords: randomly forced Burgers equation, Euler-Lagrange equation, one-sided minimizer orbit, random dynamical system, random periodic orbit.

1 Introduction

The randomly forced Burgers’ equation with zero viscosity on the space $S^1 \times \mathbb{R}^{d-1}$ discussed here is of the following form:

$$\frac{\partial}{\partial t} u + (u \cdot \nabla) u = f(x,t)$$

where $u(x,t) = (u_i(x,t), 1 \leq i \leq d)$ is a velocity field, $x = (x_i, 1 \leq i \leq d) \in S^1 \times \mathbb{R}^{d-1}$, and $f(x,t) = (f_i(x,t), 1 \leq i \leq d)$ is an external force. It is assumed that $f(x,t)$ takes the form $f(x,t) = -\nabla F^\omega(x,t)$, where $F^\omega$ is a random potential on a probability space $(\Omega, \mathcal{F}, P)$. The probability space is equipped with a $P$-preserving metric dynamical system $\theta : (-\infty, +\infty) \times \Omega \to \Omega$ such that $\theta_t \circ \theta_s = \theta_{t+s}$. In the past decade, this equation has been a subject of intensive studies in the physical and mathematical literature [3],[5],[9]-[11],[26],[30]. Although the Burgers equation arises naturally in many different physical problems e.g. in the large-scale structure of the universe, recent interest has been mostly motivated by investigation of hydrodynamics and turbulence. The existence of the stationary random point in the one-dimensional $S^1$ case was developed in the seminal works [7],[8]. A stationary random point of (1) is a solution $u$ satisfying $u^\omega(x,t) = u^\omega(x,0)$ for all $t \in (-\infty, +\infty)$, $x \in S^1 \times \mathbb{R}^{d-1}$ almost surely. The concept of the stationary random point of a random dynamical system is a natural extension of the equilibrium or fixed point in deterministic systems. Such a random fixed point for SPDEs consists of infinitely many random moving invariant surfaces on the configuration space. It is a more realistic model than many deterministic models as it demonstrates some complicated phenomena such as turbulence. Finding such stationary solutions for SPDEs is one of the basic problems. But the study of random cases is much more difficult and subtle, in contrast to deterministic problems. Unlike the usual search for invariant measures, this “one-force, one-solution” setting describes the pathwise invariance of the stationary solution over time along $\theta$ and the pathwise limit of the random dynamical system. Their existence and/or stability for various stochastic partial differential equations have been under active study recently ([6],[8],[16],[17],[22],[23],[27]). It was proven in [7],[8] that there exists a unique stationary distribution for the solutions of the random inviscid Burgers equation, and typical
solutions are piecewise smooth with finite numbers of jump discontinuities corresponding to shocks. It was shown that there exists a unique global minimizer for the corresponding stochastic Lagrangian system. Moreover, the global minimizer is a hyperbolic orbit of the Lagrangian flow, and its unstable manifold is closely connected with the solutions of the inviscid Burgers equation. The analysis in [7],[8] was based on the study of geometric and dynamical properties of minimizing orbits. The geometrical picture proved in [8] enables one not only to analyze the structure of singularities for typical stationary solutions, but also to make quantitative predictions for universal scaling exponents related to the probability distribution function for the velocity gradients (see [7] and [13]).

One of the main aims of the present paper is to study the d-dimensional non-compact space $S^1 \times R^{d-1}$, particularly in connection with the existence of random periodic minimizing orbits and periodicity of the stationary solutions to the Burgers’ equation (1). It is natural to regard the stochastic Lagrangian systems as random dynamical systems. In order to study Burgers equations, it is useful to study a variety of dynamical behaviours of the stochastic Lagrangian systems. It is well known in the deterministic dynamical system theory, apart from the equilibrium points, another kind of important limit set is a set of periodic orbits. As already mentioned, the pathwise stationary points, the concept corresponding to an equilibrium point are currently the subject of intensive study. On the other hand, the study of the periodic solutions is one of the major research problems in dynamical systems since Poincaré’s seminal work [21]. Therefore needless to say, it is fundamental in both mathematics and physics to extend the concept of periodic orbits to random cases and to study their existence, number, and local topological structure of the dynamical system near the periodic orbits. These results have deep implications for Burgers equations. They are also of independent interests in the area of random dynamical systems, especially in studying the global topological structure. In section 2, we will introduce the notion of the random periodic solutions of a random dynamical system. In section 3, we will study the random dynamical system generated by the stochastic Lagrange systems and obtained the existence and their numbers assuming a contraction condition using Lyapunov exponent near the attractor. In section 4, we will apply the periodicity of the Lagrange flow and the continuity equation to obtain a further new property about the stationary solution of the stochastic Burgers equations obtained in [8], [11].

2 The notion of random periodic orbits

The extension of the notion of a periodic orbit in a cylinder to the random case is given as follows (see Fig.1), where $\mathbb{T}$ is either $[0, \infty)$, or $(-\infty, 0)$, or $(-\infty, +\infty)$:

**Definition 2.1** Let $\varphi^\omega : R \rightarrow R^{d-1}$ be a continuous periodic function of period $\tau \in \mathbb{N}$ for each $\omega \in \Omega$. Define $L^\omega = \text{graph}(\varphi^\omega) = \{(s \mod 1, \varphi^\omega(s)) : s \in R^1\}$. If $L^\omega$ is invariant with respect to the random dynamical system $\Phi : \Omega \times \mathbb{T} \times S^1 \times R^{d-1} \rightarrow S^1 \times R^{d-1}$, i.e. $\Phi^\omega(t)L^\omega = L^{\theta t^\omega}$, and there exists a minimum $T > 0$ (or maximum $T < 0$) such that for any $s \in [0, \tau)$, $t \in \mathbb{T}$

$$
\Phi^\omega(t + T, (s \mod 1, \varphi^\omega(s))) = \Phi^{\theta t^\omega}(t, (s \mod 1, \varphi^{\theta t^\omega}(s))),
$$

for almost all $\omega$, then it is said that $\Phi$ has a random periodic orbit of period $T$ and winding number $\tau$. 


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It is easy to see that $\delta_{L^\omega}(dx)P(d\omega)$ is an invariant measure of the skew-product $(\Phi, \theta)$. In this paper, the pathwise property of the one-sided minimizer is studied and random periodic solutions are obtained and the number is proved to be finite. This result implies that the stationary solution $u^\omega(x, t)$ of the Burgers equation (1) is random periodic when $x$ moves along the random periodic orbits of the minimizer.

The random periodic orbit is a new concept in the literature. We believe it has some importance in random dynamical systems, for example, it can be studied systematically to establish the Hopf bifurcation theory of stochastic dynamical systems. This is not the objective of this paper, we will study this problem in future publications. But here in order to illustrate the concept, as a simple example, we consider the random dynamical system generated by a perturbation to the following deterministic ordinary differential equation in $\mathbb{R}^2$, although it is not immediately relevant to Burgers equations:

$$
\begin{align*}
\frac{dx(t)}{dt} &= x(t) - y(t) - x(t)(x^2(t) + y^2(t)), \\
\frac{dy(t)}{dt} &= x(t) + y(t) - y(t)(x^2(t) + y^2(t)).
\end{align*}
$$

(3)

It is well-known that above equation has a limit cycle

$x^2(t) + y^2(t) = 1$.

Consider a random perturbation

$$
\begin{align*}
\frac{dx}{dt} &= (x - y - x(x^2 + y^2))dt + x dW(t), \\
\frac{dy}{dt} &= (x + y - y(x^2 + y^2))dt + y dW(t).
\end{align*}
$$

(4)

Here $W(t)$ is a one-dimensional motion on the canonical probability space $(\Omega, \mathcal{F}, P)$ with the $P$-preserving map $\theta$ being taken to the shift operator $\theta_{t}^{-1}\omega(s) = W(t + s) - W(t)$. Using polar coordinates

$$
x = \rho \cos 2\pi \alpha, \quad y = \rho \sin 2\pi \alpha,$$

Fig.1 Random periodic orbit of period $T$ and winding number $\tau = 2$.
then we can write
\[
\begin{align*}
\{ d\rho(t) & = (\rho(t) - \rho^3(t))dt + \rho(t)dW(t), \\
\frac{d\alpha}{d\tau} & = 1. \nonumber
\end{align*}
\] (5)

This equation has a unique close form solution as follows:
\[
\rho(t, \alpha_0, \rho_0, \omega) = \rho_0 e^{\int^t_0 (\rho_0 e^{t+W_\omega(s)} - \rho(s))ds} e^{\int^t_0 e^{2s+W_\omega(s)}ds}, \quad \alpha(t, \alpha_0, \rho_0, \omega) = \alpha_0 + \frac{t}{2\pi}.
\]

It is easy to check that
\[
\rho^*(\omega) = (2\int_{-\infty}^0 e^{2s+2W_\omega(s)}ds)^{-\frac{1}{2}}
\]
is the stationary solution of the first equation of (5) i.e.
\[
\rho(t, \alpha_0, \rho^*(\omega), \omega) = \rho^*(\theta t \omega)
\]
and
\[
\Phi(t, \omega)(\alpha_0, \rho_0) = (\alpha_0 + \frac{t}{2\pi} \mod 1, \rho(t, \alpha_0, \rho_0, \omega))
\]
defines a random dynamical system \( \Phi(t, \omega) = (\Phi_1(t, \omega), \Phi_2(t, \omega)) : [0, 1] \times R^1 \to [0, 1] \times R^1 \).

Define
\[
L^\omega = \{ (\alpha, \rho^*(\omega)) : 0 \leq \alpha \leq 1 \},
\]
then
\[
L^{\theta_t \omega} = \{ (\alpha, \rho^*(\theta t \omega)) : 0 \leq \alpha \leq 1 \}.
\]

It it noticed that
\[
\Phi(t, \omega)L^\omega = \{ (\alpha + \frac{t}{2\pi} \mod 1, \rho^*(\theta t \omega)) : 0 \leq \alpha \leq 1 \}
\] \[
= \{ (\alpha, \rho^*(\theta t \omega)) : 0 \leq \alpha \leq 1 \}.
\]

Therefore
\[
\Phi(t, \omega)L^\omega = L^{\theta_t \omega},
\]
i.e. \( L^\omega \) is invariant under \( \Phi \). Moreover
\[
\Phi(2\pi, \omega)(\alpha, \rho^*(\omega)) = (\alpha, \rho^*(\theta_{2\pi} \omega)).
\]

Define for \((x, y) \in R^2, \ x = \rho \cos 2\pi\alpha, \ y = \rho \sin 2\pi\alpha \)
\[
\tilde{\Phi}(t, \omega)(x, y)
\] \[
= (\Phi_2(t, \omega)(\alpha, \rho) \cos(\Phi_1(t, \omega)(\alpha, \rho)), \ \Phi_2(t, \omega)(\alpha, \rho) \sin(\Phi_1(t, \omega)(\alpha, \rho))),
\]
and
\[
\tilde{L}^\omega = \{ (\rho^*(\omega) \cos 2\pi\alpha, \rho^*(\omega) \sin 2\pi\alpha) : 0 \leq \alpha \leq 1 \}
\] \[
= \{ (x(\omega), y(\omega)) : x^2(\omega) + y^2(\omega) = \rho^*(\omega)^2 \}.\]
It is obvious that
\[ \Phi(2\pi, \omega)(\rho^*(\omega) \cos 2\pi \alpha, \rho^*(\omega) \sin 2\pi \alpha) = (\rho^*(\theta_{2\pi \omega}) \cos 2\pi \alpha, \rho^*(\theta_{2\pi \omega}) \sin 2\pi \alpha), \]
and
\[ \Phi(t, \omega)\hat{L} = \hat{L}^\gamma, \quad \text{for all } t \geq 0. \]

From this we can tell that the random dynamical system generated by the stochastic differential equation (4) has a random periodic solution. Moreover if \( x(0)^2 + y^2(0) \neq 0 \), then
\[ x^2(t, \theta(-t, \omega)) + y^2(t, \theta(-t, \omega)) \to \rho^*(\omega)^2 \]
as \( t \to \infty \).

3 Random periodic minimizer

It is well known that due to the existence of shocks the inviscid Burgers equation has no smooth solutions. This can be viewed as the inviscid limit of the viscous Burgers equations when the viscous coefficient tends to zero ([12],[15]). However, there exists a unique viscosity solution \( u(x, t) \), a “physical” weak solution for the Cauchy problem. One of the most popular models for random potentials is the time white noise. It is noted that the vector field \( u(x, t) \) is a potential one, i.e. there is a scalar function \( S(x, t) \) such that \( u(x, t) = \nabla S(x, t) \). It is then easy to see that the function \( S(x, t) \) satisfies the stochastic Hamilton-Jacobi equation:

\[ dS(x, t) + \frac{1}{2} \left( \nabla S(x, t) \right)^2 dt + \sum_{k=0}^{N} F_k(x) dW_k(t) = 0, \tag{6} \]

where \( F_k \) are smooth non-random potentials and \( W_k(t) \) are independent Wiener processes on the canonical probability space \((\Omega, \mathcal{F}, P)\). Denote \( W(t) = (W_0(t), W_1(t), \ldots, W_k(t)) \).

As in the case of the Burgers’ equation, there are many weak solutions to the Hamilton-Jacobi equation, but there exists a unique viscosity solution. The exact expression for the viscosity solution is given by the Hopf-Lax-Oleinik variational principle [12],[14],[15],[18],[20].

Given an initial condition \( S_0 \) at time \( t' \), then for \( t \in [t', T) \):
\[ S(x, t, \omega) = S(S_0, t, \omega)(x) \]
\[ = \inf \{ S_0(\gamma(t'), t') + \int_{t'}^{t} \frac{1}{2} \theta^2(d\tau) - \sum_{k=0}^{N} \int_{t'}^{t} F_k(\gamma(\tau)) dW_k(\tau) \}, \tag{7} \]

where the infimum is taken over all absolutely continuous curves \( \gamma : [t', t] \to S^1 \times \mathbb{R}^{d-1} \) such that \( \gamma(t) = x \) (see [11],[13],[25],[26],[4],[19]). The Lagrangian action is:
\[ A_{t', t}^y(\gamma) = \int_{t'}^{t} \frac{1}{2} \theta^2(d\tau) - \sum_{k=0}^{N} \int_{t'}^{t} F_k(\gamma(\tau)) dW_k(\tau). \tag{8} \]

For \( t' \leq t \), denote by \( AC(y, t'; x, t) \) the set of absolutely continuous curves \( \gamma : [t', t] \to S^1 \times \mathbb{R}^{d-1} \) such that \( \gamma(t') = y \) and \( \gamma(t) = x \); by \( AC(t'; x, t) \) the set of absolutely continuous curves \( \gamma : [t', t] \to S^1 \times \mathbb{R}^{d-1} \) such that \( \gamma(t) = x \); by \( AC(x, t) \) the set of absolutely continuous curves \( \gamma : (\infty, t] \to S^1 \times \mathbb{R}^{d-1} \) such that \( \gamma(t) = x \); by \( AC \) the set of absolutely continuous curves \( \gamma : (-\infty, \infty) \to S^1 \times \mathbb{R}^{d-1} \). The Lax operator is:
Obviously, a function $S(x,t), x \in S^1 \times \mathbb{R}^{d-1}, t \in [t_1, t_2]$ is a “viscosity” solution of the Hamilton-Jacobi equation if for all $t_1 \leq s < t \leq t_2$:

$$K_{t',t}^\omega S(\cdot, t', \omega) = S(\cdot, t, \omega).$$

The Lax operator $K_{t',t}^\omega$ defines a semi-flow satisfying $K_{t',t}^\omega \circ K_{t,t'}^\omega = K_{t',t}^\omega$ for any $t' \leq t \leq r$ for each $\omega$.

A curve $\gamma$ in $AC(y,t';x,t)$ is called a minimizer over $[t', t]$ if it minimizes the action $H^\omega$ among all the curves in $AC(y,t';x,t)$, denote by $\gamma_{y,t',t,x}^\omega$; a curve $\gamma$ in $AC(t';x,t)$ is called a $S_0$ minimizer over $[t', t]$ if it minimizes the action $S_0(\gamma(t'), t') + H^\omega$ among all the curves $\sigma$ in $AC(t';x,t)$, denote this curve by $\gamma_{S_0, t',t,x}^\omega$; a curve $\gamma$ in $AC(x,t)$ is called a one-sided minimizer if it is a minimizer over all the time intervals $[t', t]$ for all $t' \leq t$, denote this curve by $\gamma_{x,t}^\omega$; a curve $\gamma$ in $AC$ is called a global minimizer if it is a minimizer over all the time intervals $[t_1, t_2]$, denote this curve by $\gamma_{x,t}^\omega$.

In the limit $t' \to -\infty$, a stationary regime was obtained in [11]. The solution is independent of $S_0$ and determined by the one-sided minimizer. It is easily seen from the equation (7) that all the minimizers are the solutions of the stochastic Euler-Lagrange equation:

$$\frac{dx}{dt} = v, \quad dv = -\sum_{k=0}^{N} \nabla F_k(x(t))dW_k(t).$$

In the following we always assume the following conditions posed in [11]:

**Maxima-Minima Condition.** Let $F_0$ have a maxima at $x_{\max}^\omega$, $F_0(x_{\max}^\omega) > L > 0$, and a minima at $x_{\min}^\omega$, $F_0(x_{\min}^\omega) < -L, \max\{|x_{\max}^\omega|, |x_{\min}^\omega|\} \leq a$ for a positive constant $a$. When $|x| < b$, ($b > a$ is a constant) $|\nabla F_j(x)| < K$ for all $j$ and $K^2 < cL^3$ for some constant $c > 0$. When $|x| > b$, assume that $|\nabla F_j(x)| < K_1$ and $|F_j(x)| < L_1$ for all $j = 0, 1, \ldots, N$. We also suppose that $|\nabla^2 F_j(x)|$ are bounded. The restriction of $L, K, K_1, L_1$ are given later. Let $W(t)$ be a standard Brownian motion starting at 0. Let $E_1 = E|W(1)|, E_2 = E\{\max_{0 \leq s \leq 1} |W(s)| \}, E_3 = E\{\max_{0 \leq s \leq 1} |W(s)|^2 \}$: We require that

$$\frac{128a^2E_2K^2(N+1)^2}{E_1^2} < L^3, \quad 8a^2 < LE_1, \quad 16(N+1)L_1 < L$$

and

$$8(N+1)^2K^2E_3 < LE_1.$$
It follows from the compactness of \( \partial B(b) \) that there exist \( b_2 > b \) and \( T(\omega) > 0 \) such that for any \( x \in clB(b) \)
\[
\gamma_{x,t_0}^\omega(t) \in B(b_2), \quad \text{for all } t \in (-\infty, t_1).
\]
Therefore there exists \( T(\omega, b_2) > 0 \) such that for any \( x \in clB(b) \)
\[
\gamma_{x,t_0}^\omega(t) \in B(b_2), \quad \text{for all } t \in (-\infty, t_0 - T(\omega, b_2)].
\]

Then by compactness argument, we know there is an attractor \( X^\omega \).

In the following, denote \( \gamma_{x,0}(t) \) by \( \gamma_x(t) \) and the \( P \)-preserving map \( \theta \) is taken to be the shift operator i.e. \( \theta(\omega)(s) = W(t+s) - W(t) \). Noting that \( \gamma \) solves the stochastic differential equation (11), so by [2], it is easy to see that \( \gamma \) is a perfect cocycle, i.e. there exists a version of \( \gamma \) such that
\[
\gamma_x(t_1 + t_2) = \gamma_x(t_1) \circ \theta(t_2), \quad \text{for all } t_1, t_2 \leq 0 \text{ for each } \omega \in \Omega.
\]

The random winding system is used here to characterize the random attractor \( X^\omega \) further and to study the existence of the periodic minimizer orbits for the corresponding random Lagrangian systems. Recently, a variety of novel phenomena have been observed in numerical experiments with simple deterministic quasi-periodically forced systems, including the widespread existence of strange non-chaotic attractors [24],[29]. Consider the following discrete time random winding system
\[
\begin{cases}
  s_{n+1} = s_n + \beta \mod 1, & s_n \in S^1 \\
  y_{n+1} = g^\omega(s_n, y_n), & y_n \in R^{d-1},
\end{cases}
\]
(13)

where \( S^1 \) is the unit circle and \( \beta \) is irrational, \( g : S^1 \times R^{d-1} \to R^{d-1} \) is \( \mathcal{F} \otimes B(S^1) \otimes B(R^{d-1}), B(R^{d-1}) \) measurable and \( g^\omega : S^1 \times R^{d-1} \to R^{d-1} \) is jointly continuous for each \( \omega \) and \( g^\omega(s, \cdot) : R^{d-1} \to R^{d-1} \) is differentiable for each \( \omega \in \Omega, s \in S^1 \). To be convenient, denote \( h(s) = s + \beta \mod 1 \) and
\[
H^\omega(s, y) = (h(s), g^\omega(s, y)),
\]
(14)

for the skew product map on \( S^1 \times R^{d-1} \). We shall also define \( g^{(n)} \) iteratively by \( H^{(n)}\omega(s, y) = (h^{(n)}(s), g^{(n)}(s, y)) \). Assume there exists a \( P \)-preserving map \( \theta : \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} \times \Omega \to \Omega \) such that for all \( m, n \in \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} \)
\[
H^{(m+n)}\omega(s, y) = H^{(m)}\theta^n\omega(H^{(n)}\omega(s, y)),
\]
(15)

for all \((s, y) \in S^1 \times R^{d-1} \) almost surely.

Let \( \pi : R \times R^{d-1} \to S^1 \times R^{d-1} \) be the natural covering \( \pi(a, y) = (a \mod 1, y) \) and \( \omega^\varphi \) a random periodic continuous function. If \( \pi(\text{graph } \varphi) \) is invariant under \((H, \theta)\), that is to say that \( H^{\theta^{-1}}\omega \pi(\text{graph } \varphi^{\theta^{-1}}\omega) = \pi(\text{graph } \varphi^\omega) \) a.s., it is said that \( \varphi \) is an invariant curve for the skew product (14).

The following two conditions are also needed.

**Condition (i)** Assume that there exists a random compact subset \( Y \) of \( R^{d-1} \) and \( t_1 < 0 \) such that for any \( x_1 \in S^1 \) and \( y = (x_2, x_3, \cdots, x_d) \in Y^\omega \) (denote \( z_0 = (x_1, x_2, \cdots, x_d) \)), there exists a unique one-sided minimizer orbit \( \gamma_{z_0}^\omega : (-\infty, 0] \to S^1 \times R^{d-1} \), the first component of the orbit is non-random for all \( t < 0 \) and \( \gamma_{z_0}^\omega(t_1) = (x'_1, x'_2, \cdots, x'_d) \) with
\[
x'_1 = x_1 + \beta \mod 1,
\]
and
\[ y' = (x'_2, x'_3, \cdots, x'_d) \in Y^{\theta_1, \omega}, \]
where \( \beta \) is irrational, \( z_0 = (x_1, y) \in S^1 \times Y^\omega. \)

The local unique one-sided minimizer assumption says that the Burgers equation has no shocks in \( S^1 \times Y. \) Although in general, it is not known whether or not with probability one, for any \( t, \) the shock waves are not dense in \( S^1 \times \mathbb{R}^d, \) but there should be many examples that the local assumption is valid. It was proved that the stochastic Burgers equation has at most countable shock waves in \( S^1 \) for each \( t \) with probability one in the one dimensional compact \( S^1 \) case ([8]).

In the following we denote \( \hat{\theta} \omega = \theta_1, \omega. \) Under the above assumption, a random winding system can be defined from the one-sided minimizer orbits by letting
\[ h(x_1) = x'_1, \]
and
\[ g^\omega(x_1, x_2, \cdots, x_d) = (x'_2, x'_3, \cdots, x'_d). \]

**Condition (ii)** Suppose that \( X^\omega = S^1 \times Y^\omega \) is a random compact invariant set of \( H^{\hat{\theta}^{-1}, \omega} : S^1 \times Y^{\hat{\theta}^{-1}, \omega} \rightarrow S^1 \times Y^\omega \) (that is \( H^{\hat{\theta}^{-1}X^{\hat{\theta}^{-1}, \omega} = X^\omega} \)) such that:- (a). there exists \( \delta > 0 \) such that for any \( (s, y) \in X^{\omega}, \) there exists a continuous function \( f : S^1 \rightarrow \mathbb{R}^d \) with \( f(s) = y \) such that \( (s^*, f(s^*)) \in X^{\omega} \) when \( s^* \in [s - \delta, s + \delta]; \) (b). there exists a \( \lambda < 1, \varepsilon > 0 \) and an \( n_0 \in \mathbb{N} \) such that for all \( \omega \in \Omega, \)
\[ ||D_y g_{x_1}^{(n_0)\omega}|| \leq \lambda \] for all \( (x_1, y) \in B(X^\omega, \delta, \varepsilon), \]
(16)
and
\[ c = \text{ess sup}_{\omega} \sup_{(x_1, y) \in B(X^\omega, \delta, \varepsilon)} \| \partial_{y} g_{x_1}^{(n_0)\omega}(s, y) \| < +\infty, \]
(17)
where
\[ B((s, y), \delta, \varepsilon) = \{(s', y') : s' \in [s - \delta, s + \delta], ||y' - f(s')|| \leq \varepsilon\}, \]
and
\[ B(X^\omega, \delta, \varepsilon) = \bigcup_{(s, y) \in X^\omega} B(s, y, \delta, \varepsilon). \]

This assumption can be understood as a condition on the amplitude of Lyapunov exponent of the random map. It is not difficult to prove that for all \( m \in \mathbb{N} \)
\[ H^{m\delta - mn_0\omega}(B(X^{\hat{\theta}^{-m\delta - mn_0}\omega}, \delta, \varepsilon)) \subset B(X^\omega, \delta, \varepsilon). \]
That is to say that there exists a random invariant compact set \( B(X^\omega, \delta, \varepsilon). \) By the chain rule, for all \( (s, y) \in B(X^{\hat{\theta}^{-m\delta - mn_0} \omega}, \delta, \varepsilon) \) and \( m \in \mathbb{N}, \)
\[ ||D_y g_{x}^{(mn_0)\hat{\theta}^{-m\delta - mn_0} \omega}(y) || \leq \lambda^m. \]
(18)

In the following, we will use the pullback of random maps ([2]), the Poincaré map, and some of the ideas in [24] to prove that \( X^\omega \) is a union of finite number of \( C^1 \)-periodic curves. That is to say that there exists \( r \) continuous periodic functions \( \varphi_1, \varphi_2, \cdots, \varphi_r \) on \( \mathbb{R}^d \) with periods \( \tau_1, \tau_2, \cdots, \tau_r \) such that \( X^\omega = L_{\varphi_1}^\omega \cup L_{\varphi_2}^\omega \cup \cdots \cup L_{\varphi_r}^\omega, \) where \( L_{\varphi_i}^\omega = \text{graph}(\varphi_i^\omega) = \{(s \mod 1, \varphi_i(s)) : s \in [0, \tau_i]\}, \ i = 1, 2, \cdots, r, \) are invariant under \( (H, \hat{\theta}) \). The estimates in
the proof of the following lemmas (Lemma 3.1-Proposition 3.8) are the extension of the results in [24] to the stochastic case. This is not trivial and pullback technique has to be used to make the estimates work. The fact that the periodic orbits are not the trajectories of the random dynamical system makes it difficult to follow the trajectories of the random dynamical systems.

To prove the above claim, for \((s, y) \in B(X^\theta, \delta, \epsilon)\), denote
\[
\begin{align*}
  h_1(s) &= h_{n_0}(s), \\
  g_1(s, y) &= g^{(n_0)}(s, y) = g(h_{n_0}^{-1}(s), g_{n_0}^{-1}(s, y)).
\end{align*}
\]

For any \((x_1^*, y^*) \in S_1 \times Y^{\theta, \delta, \epsilon}\), define
\[
\xi_m^\theta: \{h_1^m(x_1^*) - \delta_1, h_1^m(x_1^*) + \delta_1\} \to Y^\omega
\]
by induction:
\[
\begin{align*}
  \xi_0^\theta(s) &= y^* \in Y^{\theta, \delta, \epsilon}, \forall s \in [x_1^* - \delta_1, x_1^* + \delta_1], \\
  \xi_1^\theta(s) &= \xi_1^\theta(h_1^{-1}(s), \xi_0^\theta(h_1^{-1}(s))), \\
  \forall s \in [h_1(x_1^*) - \delta_1, h_1(x_1^*) + \delta_1],
\end{align*}
\]
and
\[
\begin{align*}
  \xi_m^\theta(s) &= \xi_1^\theta(h_1^{-1}(s), \xi_{m-1}^\theta(h_1^{-1}(s))) \in Y^\omega, \\
  \forall s \in [h_1^m(x_1^*) - \delta_1, h_1^m(x_1^*) + \delta_1].
\end{align*}
\]

Denote
\[
L = \frac{c}{1 - \lambda}.
\]

**Lemma 3.1** Under Condition (ii), the function \(\xi_m^\theta\) is Lipschitz with Lipschitz constant \(L\) for all \(m \in \mathbb{N}\), that is
\[
\|\xi_i^\theta(s) - \xi_i^\theta(s')\| \leq L|s - s'|, \quad i = 1, 2, \ldots, m
\]
\[
\forall s, s' \in [h_1^m(x_1^*) - \delta_1, h_1^m(x_1^*) + \delta_1].
\]

**Proof** We prove this by induction on \(i\), when \(i = 1\)
\[
\begin{align*}
  \|\xi_1^\theta(s) - \xi_1^\theta(s')\| &= \|\xi_1^\theta(h_1^{-1}(s), y^*) - \xi_1^\theta(h_1^{-1}(s'), y^*)\| \\
  &\leq c|h_1^{-1}(s) - h_1^{-1}(s')| \\
  &< L|s - s'|.
\end{align*}
\]
Now suppose the required result holds for \(i - 1 \geq 1, i \leq m\), then
\[
\begin{align*}
  \|\xi_i^\theta(s) - \xi_i^\theta(s')\| &= \|\xi_i^\theta(h_1^{i-1}(s), \xi_{i-1}^\theta(s)) - \xi_i^\theta(h_1^{i-1}(s'), \xi_{i-1}^\theta(s'))\| \\
  &= \|\xi_{i-1}^\theta(s) - \xi_{i-1}^\theta(s')\| \\
  &\leq L|s - s'|.
\end{align*}
\]
For any \( (N_{\text{subcover}}, \omega) \), note that \( B_\omega \) ment below. It is therefore to merge such boxes and work with the connected components of \( p \) Under Condition (ii), for any \( t \) between any two of them be

\[
\leq c|h_1^{-1}(s) - h_1^{-1}(s')| + \lambda \| \xi_{i-1} - \xi_i \| \leq (c + \lambda L) |s - s'| \leq L|s' - s|.
\]

\[+ \| g_1^{\delta - m_0 \omega}(h_1^{-1}(s'), \xi_{i-1} - \xi_i) \| \leq c|h_1^{-1}(s) - h_1^{-1}(s')| + \lambda \| \xi_{i-1} - \xi_i \| \leq (c + \lambda L) |s - s'| \leq L|s' - s|.
\]

For any \( s \in S \), define

\[
X_s^\omega = X^\omega \cap \{ (s, y) \times Y^\omega \}.
\]

For any \( (s, y) \in X_s^\omega \), let \( N(s, y, \delta_1, \varepsilon_1) \) be the interior of \( B^\omega(s, y, \delta_1, \varepsilon_1) \). Then for any \( s^* \in S \), \( \{ N(s^*, y, \delta_1, \varepsilon_1) \} \) is an open covering of \( X_s^\omega \). By compactness of \( X_s^\omega \), a finite subcover, \( N(s^*, y^{(1)}_\omega, \delta_1, \varepsilon_1), N(s^*, y^{(2)}_\omega, \delta_1, \varepsilon_1), \ldots, N(s^*, y^{(p_\omega)}_\omega, \delta_1, \varepsilon_1) \), could be found. Define

\[
\tilde{N}^\omega(s^*, \delta_1, \varepsilon_1) = \bigcup_{i=1}^{p_\omega} N(s^*, y^{(i)}_\omega, \delta_1, \varepsilon_1),
\]

\[
\tilde{B}^\omega(s^*, \delta_1, \varepsilon_1) = \bigcup_{i=1}^{p_\omega} B(s^*, y^{(i)}_\omega, \delta_1, \varepsilon_1).
\]

Note that \( \tilde{B}^\omega(s^*, \delta_1, \varepsilon_1) \) is the closure of \( \tilde{N}^\omega(s^*, \delta_1, \varepsilon_1) \). It is easy to see:

**Lemma 3.2** **Under Condition (ii), there exists a** \( \delta^*_2 \in (0, \delta_1] \) **such that**

\[
X^\omega \cap \{ (s^* - \delta_2, s^* + \delta_2) \times Y^\omega \} \subset \tilde{N}^\omega(s^*, \delta_1, \varepsilon_1).
\]

It is possible for \( B(s^*, y^{(i)}_\omega, \delta_1, \varepsilon_1) \) to overlap, which leads to inconvenience in the argument below. It is therefore to merge such boxes and work with the connected components of \( \tilde{B}^\omega(s^*, \delta_1, \varepsilon_1) \). Denote them by \( \tilde{B}^\omega_1(\delta_1), \tilde{B}^\omega_2(\delta_1), \ldots, \tilde{B}^\omega_{r^*}(\delta_1) \) and let the minimal distance between any two of them be \( \Delta^\omega > 0 \). Note that the diameter of any \( \tilde{B}^\omega_j(\delta_1) \) in the \( y \)-direction is at most \( 2p^\omega \varepsilon_1 \). It is noted here \( r^* \) does not depend on \( \omega \). This can be seen from the continuity of \( H_1 \) and the inverse \( H_1^{-1} \).

**Lemma 3.3** **Under Condition (ii), for any** \( j \in \{ 1, 2, \ldots, r^* \} \) **and any** \( m \in \mathbb{N} \),

\[
\| y - y' \| \leq L|s - s'| + 4\lambda m p^{\delta - n_0 \omega} \varepsilon_1,
\]

\[
\forall (s, y), (s', y') \in H_1^{m, \tilde{B}^\omega_1}(\tilde{B}^\omega_j^{\delta - m_0 \omega}(\delta_1)).
\]

**Proof** Choose \( (h_1^{-m}(s), \tilde{g}), (h_1^{-m}(s'), \tilde{g}') \in \tilde{B}^\omega_1^{\delta - m_0 \omega}(\delta_1) \) such that

\[
H_1^{m, \tilde{B}^\omega_1}(h_1^{-m}(s), \tilde{g}) = (s, y),
\]

\[
H_1^{m, \tilde{B}^\omega_1}(h_1^{-m}(s'), \tilde{g}') = (s', y').
\]

Then

\[
y = g_1^{(m), \tilde{B}^\omega_1}(h_1^{-m}(s), \tilde{g}),
\]

\[
y' = g_1^{(m), \tilde{B}^\omega_1}(h_1^{-m}(s'), \tilde{g}').
\]
Let \((s^*, y^*) \in X^{\theta - \alpha_0 \omega} \cap \bar{B}_s^{\theta - \alpha_0 \omega}(\delta_2)\), then from (18) and Lemma 3.1,

\[
\|y - y\| = \|g_1^{(m)}(\theta - \alpha_0 \omega) (h_1^{m}(s), \hat{y}) - g_1^{(m)}(\theta - \alpha_0 \omega) (h_1^{m}(s'), \hat{y}')\|
\leq \|g_1^{(m)}(\theta - \alpha_0 \omega) (h_1^{m}(s), \hat{y}) - g_1^{(m)}(\theta - \alpha_0 \omega) (h_1^{m}(s), y^*)\|
+ \|g_1^{(m)}(\theta - \alpha_0 \omega) (h_1^{m}(s), y^*) - g_1^{(m)}(\theta - \alpha_0 \omega) (h_1^{m}(s'), y^*)\|
+ \|g_1^{(m)}(\theta - \alpha_0 \omega) (h_1^{m}(s'), y^*) - g_1^{(m)}(\theta - \alpha_0 \omega) (h_1^{m}(s'), \hat{y}')\|
\leq 4\lambda^{m} p^{\theta - \alpha_0 \omega} \epsilon_1 + L|h_1^{m}(s) - h_1^{m}(s')|
\leq 4\lambda^{m} p^{\theta - \alpha_0 \omega} \epsilon_1 + L|s - s'|.
\]

Choose \(N \in \mathbb{N}\) such that

\[
N^\omega > \frac{\Delta_\omega}{4p^{\theta - \alpha_0 \omega} \epsilon_1}. 
\]

This implies

\[
4\lambda^{N^\omega} p^{\theta - \alpha_0 \omega} \epsilon_1 < \Delta_\omega.
\]

Choose \(\delta_3 \in (0, \delta_2)\) satisfying

\[
\delta_3^\omega < \frac{\Delta_\omega - 4\lambda^{N^\omega} p^{\theta - \alpha_0 \omega} \epsilon_1}{L}.
\]

Denote

\[
R^\omega(\delta) = \{m \geq N : |h_1^{m}(s^*) - s^*| < \delta\}
\]

for all \(\delta \in \left(0, \frac{1}{2} \delta_3\right]\).

**Lemma 3.4** Under Condition (ii), there exists a \(\delta_3^\omega \in \left(0, \frac{1}{2} \delta_3\right]\) such that

\[
X^\omega \cap ([s^* - \delta_3^\omega, s^* + \delta_3^\omega] \times Y^\omega) \subset \bigcup_{i=1}^{r} H_1^{i, \theta - \alpha_0 \omega} \left(\bar{B}_s^{\theta - \alpha_0 \omega}(\delta_3)\right)
\]

\(\forall m \in R^\omega(\delta_3^\omega).

**Proof** By the definition of \(\bar{B}_s^{\theta - \alpha_0 \omega}\) we know that

\[
\left\{ \text{Int} \bar{B}_s^{\theta - \alpha_0 \omega}(\delta_3) : i = 1, 2, \ldots, r \right\}
\]

is an open converging of \(X_s^{\theta - \alpha_0 \omega}\). It is easy to know from the definition of \(\bar{B}_s\) that

\[
X_s^{\theta - \alpha_0 \omega} \cap ([s^* - \delta_3, s^* + \delta_3] \times Y^{\theta - \alpha_0 \omega}) \subset \bigcup_{i=1}^{r} \text{Int}(\bar{B}_s^{\theta - \alpha_0 \omega}(\delta_3)).
\]

Let \(\delta_3 = \frac{1}{2} \delta_3\). Then \(|h_1^{m}(s^*) - s^*| < \delta_4\) and \([s^* - \delta_4, s^* + \delta_4] \subset [h_1^{m}(s^*) - \delta_3, h_1^{m}(s^*) + \delta_3] \) \(\forall m \in R^\omega(\delta_4)\). Hence

\[
X^\omega \cap [s^* - \delta_4, s^* + \delta_4] \times Y^\omega
\subset \{(s'', y'') : (s'', y'') \in X^\omega \text{ and } s' \in [h_1^{m}(s^*) - \delta_3, h_1^{m}(s^*) + \delta_3]\}.
\]
As $X$ is invariant with respect to $H_1$, hence

$$\{(s'', y'') : (s'', y'') \in X^\omega, s'' \in [h^m_1(s^*) - \delta_3, h^m_1(s^*) + \delta_3]\}$$

$$= \{(s'', y'') : (s'', y'') \in H_1^m, \bar{\delta} - m - n - \omega(X^\delta - m - n - \omega), s'' \in [h^m_1(s^*) - \delta_3, h^m_1(s^*) + \delta_3]\}$$

$$= \{H_1^m, \bar{\delta} - m - n - \omega(\delta, \bar{y}) : (\delta, \bar{y}) \in \bar{X}^\delta - m - n - \omega, \delta \subset [s^* - \delta_3, s^* + \delta_3]\}.$$

Note (19), then it is easy to know that

$$X^\omega \cap [s^* - \delta_4, s^* + \delta_4], xY^\omega \subset \bigcup_{i=1}^{r^*} H_1^m, \bar{\delta} - m - n - \omega(\bar{B}_i^\delta - m - n - \omega(\delta)) = \emptyset.$$

**Lemma 3.5** Under Condition (ii), for any $m \in \mathbb{R}^\omega(\delta_2^\omega)$ and any $j \in \{1, 2, \ldots, r^*\}$, there exists a unique $i \in \{1, 2, \ldots, r^*\}$ such that

$$\bar{B}_i^\omega(\delta_3) \cap H_1^m, \bar{\delta} - m - n - \omega\left(\bar{B}_j^\delta - m - n - \omega(\delta_3)\right) \neq \emptyset.$$

**Proof** By definition of $\bar{B}_j^\delta - m - n - \omega(\delta_3)$, we know that there exists a $(s^*, y^*) \in \bar{X}^\delta - m - n - \omega \cap \bar{B}_j^\delta - m - n - \omega(\delta_3)$. Because of the invariance of $X$ with respect $H_1$, we get $H_1^m, \bar{\delta} - m - n - \omega(s^*, y^*) \in X^\omega$.

For $|h^m_1(s^*) - s^*| \leq \delta_4$, so $H_1^m, \bar{\delta} - m - n - \omega(s^*, y^*) \in B^\omega(s^*, \delta_3, \epsilon_1)$. Hence there exists an $i \in \{1, 2, \ldots, r^*\}$ such that $H_1^m, \bar{\delta} - m - n - \omega(s^*, y^*) \in \bar{B}_i^\omega(\delta)$. So

$$\bar{B}_i^\omega(\delta_3) \cap H_1^m, \bar{\delta} - m - n - \omega\left(\bar{B}_j^\delta - m - n - \omega(\delta_3)\right) \neq \emptyset.$$

Now we prove the uniqueness of $i$. For any $(s, y) \in \bar{B}_j^\delta - m - n - \omega(\delta_3)$, $(h^m_1(s), g^m_1(s, y)) \in H_1^m, \bar{\delta} - m - n - \omega(\bar{B}_j^\delta - m - n - \omega(\delta_3))$. From Lemma 3.3 we know that

$$||g^m_1(s^*, y^*) - g^m_1(s, y)|| \leq L|s - s^*| + 4\lambda^m p \bar{\delta} - m - n - \omega \epsilon_1$$

$$< L\delta_3^\omega + 4\lambda^m p \bar{\delta} - m - n - \omega \epsilon_1$$

$$< \Delta_\omega.$$

So for any $i' \in \{1, 2, \ldots, r\} \setminus \{i\}$, $(h^m_1(s), g^m_1(s, y)) \notin \bar{B}_i^\omega(\delta_3)$. Thus

$$H_1^m, \bar{\delta} - m - n - \omega\left(\bar{B}_j^\delta - m - n - \omega(\delta_3)\right) \cap \bar{B}_i^\omega(\delta_3) = \emptyset,$$

and the uniqueness of $i$ follows. 

**Definition 3.6** Given any $m \in \mathbb{R}^\omega(\delta_4)$ and $j \in \{1, 2, \ldots, r^*\}$, denote by $\sigma^\omega_m(j)$ the unique $i \in \{1, 2, \ldots, r^*\}$ such that $\bar{B}_i^\omega(\delta_3) \cap H_1^m, \bar{\delta} - m - n - \omega\left(\bar{B}_j^\delta - m - n - \omega(\delta_3)\right) \neq \emptyset$.

**Lemma 3.7** Under Condition (ii), for any $m \in \mathbb{R}^\omega(\delta_4)$, the function $\sigma^\omega_m : \{1, 2, \ldots, r^*\} \rightarrow \{1, 2, \ldots, r^*\}$ is a permutation. Hence, in particular, $\sigma^\omega_m$ is invertible and given any $m \in \mathbb{R}^\omega(\delta_4)$, $i \in \{1, 2, \ldots, r^*\}$, there exists a unique $j = (\sigma^\omega_m)^{-1}(i) = r^\omega_m(\omega(i))$ such that $\bar{B}_i^\omega(\delta_3) \cap H_1^m, \bar{\delta} - m - n - \omega\left(\bar{B}_j^\delta - m - n - \omega(\delta)\right) \neq \emptyset$. 


Proof. Clearly, for any \( i \in \{1, 2, \ldots, r^*\} \), \( B_i^*(\delta_4) \cap X_{i}^* \neq \emptyset \). Hence

\[
B_i^*(\delta_4) \cap \left( \bigcup_{j=1}^{r^*} H_j^m, \delta_{m-n}^n (B_j^m, \delta_{m-n}^n (\delta_4)) \right) \neq \emptyset.
\]

Thus \( \sigma_m \) is onto. Because \( \{1, 2, \ldots, r^*\} \) is finite, \( \sigma_m \) is one-to-one. Therefore \( \sigma_m \) is a permutation. \( \blacksquare \)

Proposition 3.8 Under Condition (ii), there exist \( r^* \) Lipschitz functions \( \psi_i^\omega : [s^* - \delta_4, s^* + \delta_4] \to X^\omega \) such that \( X^\omega \cap \left( [s^* - \delta_4, s^* + \delta_4] \times Y^\omega \right) \subset \bigcup_{i=1}^{r^*} \text{graph } \psi_i^\omega \) and for each \( i \in \{1, 2, \ldots, r^*\} \), we have graph \( \psi_i^\omega \subset B_i^*(\delta_4^2) \).

Proof: Let \( \tau_m^\omega \) be the inverse of \( \sigma_m^\omega \) and for each \( i \in \{1, 2, \ldots, r^*\} \), define

\[
W_i^\omega = \bigcap_{m \in R^\omega(\delta_4)} H_1^m, \delta_{m-n}^n \left( B_m^\omega, \delta_{m-n}^n (\delta_4) \right). \tag{20}
\]

For any \((s, y), (s', y') \in W_i^\omega\), by Lemma 3.3, we have

\[
||y - y'|| \leq L|s - s'| + 4\lambda m p \delta_{m-n}^n \varepsilon_1,
\]

for any \( m \in R^\omega(\delta_4) \). Let \( m \to \infty \), we get \( ||y - y'|| \leq L|s - s'| \). That is, each \( W_i^\omega \) is contained in the graph of a Lipschitz function with Lipschitz constant \( L \). Let

\[
X_i^\omega = X^\omega \cap B_i^*(\delta_4)
\]

\[
= X^\omega \cap \left( [s^* - \delta_4, s^* + \delta_4] \times Y^\omega \right) \cap B_i^*(\delta_4). \tag{21}
\]

By the definition of \( \delta_4 \), it is easy to see that

\[
X^\omega \cap \left( [s^* - \delta_4, s^* + \delta_4] \times Y^\omega \right) \subset \bigcup_{i=1}^{r^*} B_i^*(\delta_3).
\]

Thus it follows from Lemma 3.4 that

\[
X^\omega \cap \left( [s^* - \delta_4, s^* + \delta_4] \times Y^\omega \right) \subset \bigcap_{m \in R^\omega(\delta_4)} \bigcup_{i=1}^{r^*} H_1^m, \delta_{m-n}^n \left( B_i^*(\delta_3) \right). \tag{22}
\]

By Lemma 3.7, for any \( m \in R^\omega(\delta_4) \),

\[
\bar{B}_i^*(\delta_3) \cap H_1^m, \delta_{m-n}^n \left( B_m^\omega, \delta_{m-n}^n (\delta_3) \right) \neq \emptyset, \tag{23}
\]

\[
\bar{B}_i^*(\delta_3) \cap \left( \bigcup_{j \neq \tau_m^\omega(i)} H_j^m, \delta_{m-n}^n \left( B_j^\omega, \delta_{m-n}^n (\delta_3) \right) \right) = \emptyset. \tag{24}
\]

So it follows from (20)-(24) that

\[
X_i^\omega \subset W_i^\omega.
\]
Next we will show that

\[ X_i^\omega \cap X_s^\omega \neq \emptyset, \]

for any \( s \in [s^* - \delta_4, s^* + \delta_4], \) \( i = 1, 2, \cdots, r^*. \) Clearly

\[ X_i^\omega \cap X_s^\omega \neq \emptyset, \]

for any \( i = \{1, 2, \cdots, r^*\}. \) Choose \( \{m_k\} \subset R^2(\delta_k) \) such that \( \lim_{k \to \infty} m_k = \infty \) and \( \lim_{k \to \infty} h_{1m_k}(s^*) = s. \) Fix \( i = \{1, 2, \cdots, r^*\} \) and for any \( k \in N, \) choose \( (s^*, y_k) \in X_{m_k}^\omega \cap X_{s^* - m_k \omega}^\omega. \) Since \( X \) is invariant with respect to \( H_1, \) so

\[ H_{1m_k}^{\gamma \omega}(s^*, y_k) \in X^\omega \cap \left([s^* - \delta_4, s^* + \delta_4] \times Y^\omega\right). \]

Thus there is an \( i_k \in \{1, 2, \cdots, r^*\} \) such that

\[ H_{1m_k}^{\gamma \omega}(s^*, y_k) \in X_{i_k}^\omega \subset B_i^\omega(\delta_3). \]

By

\[ H_{1m_k}^{\gamma \omega}(s^*, y_k) \in H_{1m_k}^{\gamma \omega}(X_{i_k}^\omega) \subset H_{1m_k}^{\gamma \omega}(B_{i_k}^\omega(\delta_3)) \]

and

\[ H_{1m_k}^{\gamma \omega}(B_{i_k}^\omega(\delta_3)) \cap \left( \bigcup_{i' \neq i} B_i^\omega(\delta_3) \right) = \emptyset, \]

so we have \( i_k = i, \) for any \( k \in N \) and \( H_{1m_k}^{\gamma \omega}(s^*, y_k) \in X_i^\omega. \) That is to say \( H_{1m_k}^{\gamma \omega}(s^*, y_k) \in X_{i_k}^\omega. \) For \( X_i^\omega \) is closed subset of \( X^\omega \) and \( X^\omega \) is compact, so \( H_{1m_k}^{\gamma \omega}(s^*, y_k) \) has a convergent sequence. Without loss of generality, suppose

\[ \lim_{k \to \infty} H_{1m_k}^{\gamma \omega}(s^*, y_k) = (\bar{s}, \bar{y}), \quad (\bar{s}, \bar{y}) \in X^\omega. \]

Because \( \lim_{k \to \infty} h_{1m_k}(s^*) = s, \) we have \( \bar{s} = s. \) Hence \( (\bar{s}, \bar{y}) \in X_{i_k}^\omega \cap X_{i_k}^\omega \neq \emptyset. \)

Therefore for each \( s \in [s^* - \delta_4, s^* + \delta_4], \) \( X_i^\omega \cap X_s^\omega \) contains exactly one point by considering that \( ||y - y'|| \leq L|s - s'|, \forall (s, y), (s', y') \in X_{i_k}^\omega. \) Denote this point by \( (s, \psi_i^\omega(s)). \) The graph \( \psi_i^\omega = X_i^\omega \subset B_i^\omega(\delta_3), \psi_i^\omega \) is a Lipschitz function with constant \( L. \)

**Theorem 3.9 Under Condition (ii), \( X^\omega \) is a union of a finite number of Lipschitz periodic curves.**

**Proof:** By the compactness of \( S^1, \) we can choose a \( \delta_4 > 0 \) independent of \( s^* \in S^1. \) Let \( M \in N \) such that \( \frac{1}{M} \leq \delta_4. \) Define \( s_m = \frac{m}{M}, \) \( m = 1, 2, \cdots, M. \) Then \( \{s_{m-1}, s_{m+1} : m = 1, 2, \cdots, M\} \) (in which \( s_{M+1} = s_1, x_0 = s_M \)) covers \( S^1. \) By Proposition 3.8, we know that

\[ X^\omega \cap \left([s_{m-1}, s_{m+1}] \times Y^\omega\right) \]

contains a finite number of Lipschitz curves, denote their number by \( r^*(m). \) Since \( [s_{m-1}, s_m] \subset [s_{m-2}, s_{m+1}] \cap [s_{m-1}, s_{m+1}], \) so we have \( r^*(m_1) = r^*(m_2) \) when \( m_1 \neq m_2. \) So \( r \) is independent of \( m \) and define all of them by \( r^*. \) Thus Lipschitz curves on
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$X^\omega \cap \left([s_{m-1}, s_{m+1}] \times Y^\omega\right)$ could be expanded to $S^1$ and we have the following random Poincare map

$$H^{\delta - n_0 \omega}: G^{\delta - n_0 \omega} \to G^\omega,$$

in which $G^\omega$ is a finite set containing $r^*$ elements:

$$G^\omega = \{(s \mod 1, \psi_i^\omega(s)) : i = 1, 2, \cdots, r^*\},$$

for a fixed $s \in R^1$. By the finiteness of $G^\omega$, we know

$$\psi_i^\omega(s + 1) = \psi_i^\omega(s),$$
$$\psi_i^\omega(s + 2) = \psi_i^\omega(s),$$
$$\cdots,$$
$$\psi_i^\omega(s + r^*) = \psi_i^\omega(s).$$

Actually above is true for any $s$ due to the continuity of $\psi_i^\omega$’s. Therefore there are three cases:-

(i). Exact one of $i_1, i_2, \cdots, i_r^*$ = $i$. Say $i_{r_1} = i$. Then

$$\psi_i^\omega(s + \tau_i) = \psi_i^\omega(s),$$

for any $s \in R$. So $\psi_i^\omega$ is a periodic function of period $\tau_i^*$.

(ii). More than one of $i_1, i_2, \cdots, i_r^*$ = $i$. Denote $\tau_i$ the smallest number $j$ such that $i_j = i$ and $\tilde{\tau}_i > \tau_i$ such that $i_{\tilde{\tau}_i} = i$. Then

$$\psi_i(s + \tau_i) = \psi_i(s),$$
$$\psi_i(s + \tilde{\tau}_i) = \psi_i(s).$$

But

$$\psi_i(s + \tau_i) = \psi_i(s + \tilde{\tau}_i - \tilde{\tau}_i + \tau_i)$$
$$= \psi_i(s + \tilde{\tau}_i - \tau_i)$$
$$= \cdots$$
$$= \psi_i(s + \tilde{\tau}_i - k\tau_i),$$

where $k$ is the smallest integer such that $\tilde{\tau}_i - (k + 1)\tau_i \leq 0$. Then by definition of $\tau_i$,

$$\tilde{\tau}_i - k\tau_i = \tau_i,$$

so

$$\tilde{\tau}_i = (k + 1)\tau_i$$

and therefore $\psi_i$ is a periodic function of period $\tau_i$.

(iii). None of $i_1, i_2, \cdots, i_r^*$ is equal to $i$. In this case, at least two of $i_1, i_2, \cdots, i_r^*$ must be equal. Say $\tau_2 > \tau_1$ are the two such integers such that $i_{\tau_1} = i_{\tau_2}$ with smallest difference $\tau_2 - \tau_1$. Then

$$\psi_i(s + \tau_1) = \psi_i(s + \tau_2).$$

Denote $s + \tau_1$ by $s_1$, then

$$\psi_i(s) = \psi_i(s + \tau_2 - \tau_1), \forall s \in R^1.$$

Same as (ii) we can see for all other possible $\tilde{\tau}_2$ and $\tilde{\tau}_1$, $\tilde{\tau}_2 > \tilde{\tau}_1$ and $i_{\tilde{\tau}_2} = i_{\tilde{\tau}_1}$, $\tilde{\tau}_2 - \tilde{\tau}_1$ must be an integer multiple of $\tau_2 - \tau_1$. That is to say $\psi_i$ is a periodic curve with period $\tau_2 - \tau_1$. ■
Theorem 3.9 says there exist a finite number of continuous periodic functions \( \phi_1, \phi_2, \cdots, \phi_r \) on \( \mathbb{R}^1 \) with periods \( \tau_1, \tau_2, \cdots, \tau_r \in \mathbb{N} \) respectively such that

\[
X^\omega = L^{\omega_1}_1 \cup L^{\omega_2}_2 \cup \cdots \cup L^{\omega_r}_r
\]

where

\[
L^{\omega_i}_i = \text{graph}(\phi^{\omega_i}_i) = \{(s \mod \phi^{\omega_i}_i) : s \in [0, \tau_i)\}
\]

are invariant under \( (H, \theta), \ i = 1, 2, \cdots, r \). Moreover we can prove they are in fact invariant under the continuous dynamical system \((\gamma(t), \theta(t))\) in the following proposition.

**Proposition 3.10** Under the Maxima-Minima condition and Conditions (i), (ii), for any \( t < 0 \) and \( s \in [0, \tau_i) \)

\[
\gamma_{(s \mod 1, \phi_i^{\theta - t \omega}(s))}^{\theta - t \omega}(t) \in L_i^{\omega}, \text{ for each } \omega.
\]

**Proof** First from condition (ii) note for any \( \epsilon_1 > 0 \), there exists a \( T_1 < 0 \) such that for any \( t \in (-\infty, T_1) \),

\[
\gamma_{(s \mod 1, \phi_i^{\theta - s \omega}(s))}^{\theta - s \omega}(t) \in O(L_i^{\omega}, \epsilon_1), \text{ for all } 0 \leq s < \tau_i.
\]

That is to say that

\[
\lim_{t \to \infty} \rho \left( \gamma_{(s \mod 1, \phi_i^{\theta - t \omega}(s))}^{\theta - t \omega}(t), L_i^{\omega} \right) = 0.
\]

Assume the claim of the proposition is not true, i.e. there exist \( t^* < 0 \) and \( 1 \leq s_1 < \tau_i \) such that

\[
\rho \left( \gamma_{(s_1 \mod 1, \phi_i^{\theta - t^* \omega}(s_1))}^{\theta - t^* \omega}(t^*), L_i^{\omega} \right) = d > 0.
\]

From (25), we know there exists \( \tau_2 < 0 \) such that for any \( t \in (-\infty, \tau_2] \)

\[
\rho \left( \gamma_{(s \mod 1, \phi_i^{\theta - t \omega}(s))}^{\theta - t \omega}(t), L_i^{\omega} \right) < \frac{d}{4}, \text{ for all } s \in S^2.
\]

By continuity argument, there exists \( \epsilon_2 \in (0, \frac{d}{4}) \) such that for any \( (s, y) \in O((s_1 \mod 1, \phi_i^{\theta - t^* \omega}(s_1)), \epsilon_2) \), then

\[
\rho \left( \gamma_{(s \mod 1, y)}^{\theta - t^*}(t^*), \gamma_{(s_1 \mod 1, \phi_i^{\theta - t^* \omega}(s_1))}^{\theta - t^* \omega}(t^*) \right) < \frac{d}{4}.
\]

Moreover, from (25), we know that there exists \( t_2 < \tau_2 \) and \( 1 \leq s_2 < \tau_i \) such that

\[
\rho \left( \gamma_{(s_2 \mod 1, \phi_i^{\theta - t_2 \omega}(s_2))}^{\theta - t_2 \omega}(t_2), (s_1 \mod 1, \phi_i^{\theta - t^* \omega}(s_1)) \right) < \epsilon_2.
\]

So from (26), then

\[
\rho \left( \gamma_{(s_2 \mod 1, \phi_i^{\theta - t_2 \omega}(s_2))}^{\theta - t_2 \omega}(t_2 + t^*), \gamma_{(s_1 \mod 1, \phi_i^{\theta - t^* \omega}(s_1))}^{\theta - t^* \omega}(t^*) \right) < \frac{d}{4}.
\]

Therefore from the triangle inequality that
\[
\rho \left( \gamma_{(s_2 \mod 1, \phi_{\theta^{-t^*} \omega}^{(s_2)})}^{\theta_{-t_2-t^* \omega}} (t_2 + t^*), L_1^{\omega} \right) \\
\geq \rho \left( \gamma_{(s_1 \mod 1, \phi_{\theta^{-t^*} \omega}^{(s_1)})}^{\theta_{-t^*} \omega} (t^*), L_1^{\omega} \right) \\
- \rho \left( \gamma_{(s_2 \mod 1, \phi_{\theta_{-t^*} \omega}^{(s_2)})}^{\theta_{-t_2-t^* \omega}} (t_2 + t^*), \gamma_{(s_1 \mod 1, \phi_{\theta^{-t^*} \omega}^{(s_1)})}^{\theta_{-t^*} \omega} (t^*) \right) \\
\geq \frac{3}{4} d.
\]

Since \( t_2 + t^* < \tau_2 \), so this is a contradiction. The claim is asserted.

**Theorem 3.11** Under the Maxima-Minima condition and Conditions (i), (ii), the Lagrange flow has \( r \) random periodic solutions.

**Proof.** From Proposition 3.10, it is clear that once \( L_i^\omega \) is known for one \( \omega \), then \( L_i^{\theta^{-t^*} \omega} \) is determined for any \( t < 0 \). In the following \( \phi \) is used to represent any \( \phi_i \) and \( \tau \) the corresponding \( \tau_i \). One can find a smallest integer \( N \) such that

\[
(N - 1) \beta < \tau < N \beta.
\]

Then

\[
\gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^{\theta_{-N t_2 \omega}} (N t_1) = (s + \delta_1 \mod 1, \phi^\omega(s + \delta_1)), \text{ for } \delta_1 > 0
\]

and

\[
\gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^{\theta_{-(N - 1) t_1 \omega}} ((N - 1) t_1) = (s - \delta_2 \mod 1, \phi^\omega(s - \delta_2)), \text{ for } \delta_2 > 0.
\]

Then applying the middle value theorem, there exists a \( t^* \in ((N - 1) t_1, N t_1) \) such that

\[
\gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^{\theta_{-t^*} \omega} (t^*) = (s \mod 1, \phi^\omega(s)). \tag{27}
\]

Now notice for any \( t \), by the cocycle property of \( \gamma \),

\[
\gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^{\theta_{-t^*} \omega} (t^* + t) = \gamma_{(s_1 \mod 1, \phi_{\theta^{-t^*} \omega}^{(s_1)})}^{\theta_{-t^*} \omega} (t^*)
\]

where \( (s_1 \mod 1, \phi_{\theta^{-t^*} \omega}^{(s_1)}) = \gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^{\theta_{-t^*} \omega} (t) \in L_{t^*}^{\theta^{-t^*} \omega} \). Therefore from (27) it follows that

\[
\gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^{\theta_{-t^*} \omega} (t + t^*) = (s_1 \mod 1, \phi^\omega(s_1)) = \gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^{\theta_{-t} \omega} (t).
\]

This gives that

\[
\gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^{\theta_{-t^*} \omega} (t + t^*) = \gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^\omega \gamma_{(s \mod 1, \phi_{\theta^{-t^*} \omega}^{(s)})}^{\theta_{-t^*} \omega} (t)
\]

for any \( t \). That is to say \( \gamma \) has a random periodic orbit with period \( t^* \). There are \( r \) such \( \phi \). That is to say \( \gamma \) has \( r \) random periodic orbits.

**4 Random periodic stationary solution of Stochastic Burgers equations**

Recall that the main result of [11] says there is a stationary solution to the stochastic Burgers equation associated with the one-sided minimizer: \( u^{\omega}(x, t) = u^{\theta \omega}(x, 0) \) for each \( t \) and \( x \in S^1 \times \mathbb{R}^{d-1} \).
Now we would like to apply the result we proved in section 3 to obtain some new result to the stationary solution of the stochastic Burgers equations. As
\[ \gamma_{(s \mod 1, \omega^s)}^{\theta, t, \omega} (t^* + t) = \gamma_{(s \mod 1, \omega^s)}^{\omega} (t) \]
for all \( t \). So differentiating above with respect to \( t \) we have
\[ \gamma_{(s \mod 1, \omega^s)}^{\theta, t, \omega} (t^* + t) = \gamma_{(s \mod 1, \omega^s)}^{\omega} (t). \]
But \( \gamma_{(s \mod 1, \omega^s)}^{\omega} (t) \) is the solution of Euler-Lagrange equation (11) with initial position \((s, \omega^s(s))\) and initial velocity \( \gamma_{(s \mod 1, \omega^s)}^{\omega} (0) \), so from [25],
\[ \gamma_{(s \mod 1, \omega^s)}^{\omega} (t) = u^s(t, \gamma_{(s \mod 1, \omega^s)}^{\omega} (t)). \]
Similarly
\[ \gamma_{(s \mod 1, \omega^s)}^{\theta, t, \omega} (t^* + t) = u^s(t, \gamma_{(s \mod 1, \omega^s)}^{\omega} (t^* + t)). \]
Therefore for all \( s \in [0, \tau] \)
\[ u^s(t, \gamma_{(s \mod 1, \omega^s)}^{\omega} (t)) = u^s(t, \gamma_{(s \mod 1, \omega^s)}^{\omega} (t^* + t)). \]
Define
\[ Q^s(t, s) = u^s(t, \gamma_{(s \mod 1, \omega^s)}^{\omega} (t)). \]
then
\[ Q^s(t, s) = Q^s(t + t^*, s), \forall s \in [0, \tau). \]
Thus we have obtained finally

**Theorem 4.1** Under the Maxima-Minimum condition and Conditions (i), (ii), for all \( t \leq 0 \)
\[ Q^s(t + t^*, s) = Q^s(t, s), \forall s \in [0, \tau) \]
almost surely.

That is to say the stationary solution of the Burgers equation is random periodic along the minimizer \( x = \gamma_{(s \mod 1, \omega^s)}^{\omega} (t) \). This result is new in the literature.

There are many examples of Euler-Lagrange equations with random periodic orbits. They include the following simple example by adding one dimension \( ds(t) = 0 \) to an Euler-Lagrange equation with a stationary random point \( \omega^s \). Then \( L^s = \{(s, \omega^s) : s \in S^1\} \) is an invariant random closed curve. The stationary solution of Euler-Lagrange equation is under active investigation in the past ten years e.g. see [1] and references therein. But random periodic orbits in general are very complex and deserve further serious study in the future.

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