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Local Time as a Rough Path

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Summary. In this paper, we prove that a semimartingale local time is a rough path of roughness $p$ for any $p \in [2, 3)$ and establish the integral $\int_{-\infty}^{\infty} g(x) dZ_t(x)$ for any finite $q$-variation function $g$ ($2 \leq q < 3$) using Lyons’ rough path integration theory. We therefore obtain the Meyer-Tanaka’s formula for continuous function $f$ if $\nabla^{-1} f$ exists and is of finite $q$-variation when $2 \leq q < 3$. The case when $1 \leq q < 2$ was established in [1] using Young integral.

Keywords: Semimartingale local time; rough path.

In this paper, we prove that a semimartingale local time $L_t^x$ is of bounded $p$-variation in $x$ for any $t \geq 0$, $p > 2$ a.s., i.e. for almost all $\omega \in \Omega$,

$$\sup_D \sum_{i=0}^{m-1} |L_{t_{i+1}}^{x_{i+1}} - L_{t_i}^{x_i}|^p < \infty,$$

where $D = \{-N = x_0 < x_1 < \cdots < x_m = N\}$ is an arbitrary partition of interval $[-N, N]$ covering the support of $L_t$. So in Theorem 2.2, we gave a new condition for Tanaka-Meyer’s formula and the integral $\int_{-\infty}^{\infty} \nabla^{-1} f(x) d_x L_t^x$ as a Young integral, when $\nabla^{-1} f(x)$ is of bounded $q$-variation ($1 \leq q < 2$). But how is about if $q \geq 2$? Can we still define such an integral $\int_{-\infty}^{\infty} \nabla^{-1} f(x) d_x L_t^x$ pathwise? If we can, we will get Tanaka-Meyer’s formula for wider class of functions. So let’s first try to define the integral $\int_{-\infty}^{\infty} g(x) d_x L_t^x$ pathwise for a continuous $g(x)$ with bounded $q$-variation ($2 \leq q < 3$). And we also take $2 < p < 3$. We decompose local time

$$L_t^x = \hat{L}_t^x + \sum_{x_k^+ < x} \hat{\hat{L}}_{t_k}^{x_k}, \text{ where } \hat{L}_t^x := L_t^x - L_t^x^\pm.$$  

Here $\hat{L}_t^x$ is continuous in $x$, and $x_k^+, k = 1, 2, \cdots$ are the discontinuous points (countable) of $L_t^x$. From Lemma 2.2 in [1], we know $h(t, x) := \sum_{x_k^+_< x} \hat{\hat{L}}_{t_k}^{x_k}$ is of bounded variation in $x$ for each $t$. So the key point is to define $\int_{-\infty}^{\infty} g(x) d_x \hat{L}_t^x$ pathwise for $g(x)$ with bounded $q$-variation ($2 \leq q < 3$). For this, we will use Lyons’ rough path theory.

In fact, $g(x)$ and $\hat{L}_t^x$ can be regarded as rough paths. From [2], generally, we cannot expect to have an integration theory for defining integrals such as $\int_{-\infty}^{\infty} g(x) d_x \hat{L}_t^x$. But using the method in Chapter 6 in [2], we can treat $Z_x := (\hat{L}_t^x, g(x))$ together as a rough path and define $\hat{f}(x, y)(v, w) := (v, yv)$, so the integral will be the second element of $\int_{-\infty}^{\infty} \hat{f}(Z) dZ^1$. It’s easy to know that $Z_x$ is of bounded $\hat{q}$-variation in $x$, where $\hat{q} = q$, if $q > 2$, and $\hat{q} = 2$ can be taken as any number when $q = 2$. Most of the analysis in this paper works for $2 \leq q < 4$, especially we will establish the convergence of smooth rough path in $\theta$-variation topology for any $\theta \in (q, 4)$, so to obtain $Z_{a,b}^x$ and $Z_{a,b}^x$. In particular, when $2 \leq q < 3$, we obtain the existence of the geometric rough path $X = \langle 1, X^1, X^2 \rangle$ associated with $Z$. In the following we consider $2 \leq q < 4$ otherwise we will explicitly say so.

Let $[x', x'']$ be any interval in $R$. From the proof of Lemma 2.1 in [1], for any $p \geq 2$, we know there exists a constant $c > 0$ such that
i.e. $\hat{L}_t(x)$ satisfies Hölder condition in the sense of [2] with exponent $\frac{1}{2}$. Denote by $w$ the control of $g(x)$, i.e.

$$|g(b) - g(a)|^\theta \leq w(a, b),$$

for any $(a, b) \in \Delta := \{(a, b) : a' \leq a < b \leq x''\}$. It is obvious that $w_1(a, b) := w(a, b) + (b - a)$ is also a control of $g$. Denote $h = \frac{1}{\theta}$. It is trivial to see for any $\theta > q$ i.e. $h\theta > 1$ that,

$$|g(b) - g(a)|^\theta \leq w_1(a, b)^{h\theta}, \text{ for any } (a, b) \in \Delta.$$  

(4)

Considering (3), we can see for such $h = \frac{1}{\theta}$, and any $\theta > q$ there exists $c > 0$ such that

$$E|Z_b - Z_a|^{\theta} \leq cw_1(a, b)^{h\theta}, \text{ for any } (a, b) \in \Delta.$$  

(5)

For any $m \in N$, define a continuous and bounded variation path $Z(m)$ by

$$Z(m)_x := Z(x^m_1) + \sum_{l=1}^{m-1} \frac{w_l(x) - w_l(x^m_{l-1})}{w_l(x^m_{l-1}) - w_l(x^m_{l-1})} \Delta_l Z,$$

(6)

if $x^m_{l-1} \leq x < x^m_l$, for $l = 1, \cdots, 2^m$, and $\Delta_l Z = Z(x^m_l) - Z(x^m_{l-1})$. Here $D_m := \{x' = x^m_0 < x^m_1 < \cdots < x^m_{2^m} = x''\}$ is a partition of $[x', x'']$ such that $w_1(x^m_l) - w_1(x^m_{l-1}) = \frac{1}{2^l} w_1(x', x'')$, where $w_1(x) := w_1(x', x)$. It is obvious that $x^m_l - x^m_{l-1} \leq \frac{1}{2^l} w_1(x', x'')$. The corresponding smooth rough path $X(m)$ is built by taking its iterated path integrals, i.e.

$$X(m)_{a,b} := \int_{a < x_1 < \cdots < x_j < b} dZ(m)_{x_1} \otimes \cdots \otimes dZ(m)_{x_j}.$$  

(7)

Let’s first look at the first level path $X(m)_{a,b}$. The method and results are similar to Chapter 4 in [2]. Similar to Proposition 4.2.1 in [2], we can prove that for all $n \in \mathbb{N}$, $m \mapsto 2^n \sum_{k=1}^{2^n} |X(m)_{x_{k-1}, x_k}^1|^\theta$ is increasing. Let $X_{a,b}^1 = Z_b - Z_a$. Inequality (5) implies $E|X_{a,b}^1|^{\theta} \leq cw_1(a, b)^{h\theta}$. For such points $\{x_k^n\}$, $k = 1, \cdots, 2^n$, $n = 1, 2, \cdots$, defined above we still have the same inequality as in Proposition 4.1.1, [2], so for any $\gamma > \theta - 1$, there exists a constant $C_1(\theta, \gamma, c) > 0$ such that

$$E \sup_D \sum_{l} \left| X^1_{x_{l-1}, x_l} \right|^\theta \leq C(\theta, \gamma) E \sup_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left| X^1_{x_{k-1}, x_k} \right|^\theta$$

$$\leq C_1(\theta, \gamma, c) \sum_{n=1}^{\infty} n^\gamma (\frac{1}{2^n})^{h\theta-1} w_1(x', x'')^{h\theta}.$$  

(8)

Since $h\theta - 1 > 0$, the series on the right-hand side of (8) is convergent, so $\sup_D \sum_{l} \left| X^1_{x_{l-1}, x_l} \right|^\theta < \infty$ almost surely. This shows that $X^1$ has finite $\theta$-variation almost surely. Furthermore,

$$E \sup_m \sup_{D} \sum_{l} \left| X(m)_{x_{l-1}, x_l}^1 \right|^\theta \leq C(\theta, \gamma) E \sup_m \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left| X(m)_{x_{k-1}, x_k}^1 \right|^\theta$$

$$\leq C(\theta, \gamma) E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left| X^1_{x_{k-1}, x_k} \right|^\theta$$

$$\leq C_1(\theta, \gamma, c) \sum_{n=1}^{\infty} n^\gamma (\frac{1}{2^n})^{h\theta-1} w_1(x', x'')^{h\theta} < \infty,$$
where

\[ E \sup_D \sum_l |X(m)_{x_{i-1},x_i}^1 - X_{x_{i-1},x_i}^1|^\theta \leq C \left( \frac{1}{2^m} \right)^{\frac{\theta h - 1}{\theta}}, \]

where \( C \) depends on \( \theta, h, w_1(x', x'') \), and \( c \) in (5). By the Hölder inequality,

\[ E \sum_{m=1}^\infty \sup_D \left( \sum_l |X(m)_{x_{i-1},x_i}^1 - X_{x_{i-1},x_i}^1|^{\theta} \right)^{\frac{1}{\theta}} \leq C \sum_{m=1}^\infty \left( \frac{1}{2^m} \right)^{\frac{\theta h - 1}{\theta}} < \infty, \tag{9} \]

as \( h\theta > 1 \). We immediately have

**Theorem 1** For a continuous path \( Z_x \) with (5), we have

\[ \sum_{m=1}^\infty \sup_D \left( \sum_l |X(m)_{x_{i-1},x_i}^1 - X_{x_{i-1},x_i}^1|^{\theta} \right)^{\frac{1}{\theta}} < \infty \text{ a.s..} \tag{10} \]

In particular, \((X(m)_{a,b}^1)\) converges to \((X_{a,b}^1)\) in \( \theta \)-variation distance a.s. for any \((a, b) \in \Delta \).

We next consider the second level path \( X(m)^2_{a,b} \). From [2], we know if \( n \geq m \), \( X(m)^2_{x_{k-1},x_k} = 2^{2(m-n)}(\Delta_m^m Z)^{\otimes 2} \); if \( n < m \),

\[
X(m)^2_{x_{k-1},x_k} = \frac{1}{2} \Delta^m_m Z \otimes \Delta^m_m Z + \frac{1}{2} \sum_{r < l = 2^{m-n} (k-1) + 1}^{2^{m-n} k} (\Delta^m_m Z \otimes \Delta^m_m Z - \Delta^m_l Z - \Delta^m_m Z),
\]

so

\[
X(m+1)^2_{x_{k-1},x_k} - X(m)^2_{x_{k-1},x_k} = \frac{1}{2} \sum_{l=2^{m-n} (k-1) + 1}^{2^{m-n} k} (\Delta^m_{l+1} Z \otimes \Delta^m_{l+1} Z - \Delta^m_{l} Z - \Delta^m_{l+1} Z),
\]

\( k = 1, \cdots, 2^n \). Similar to the proof of Proposition 4.3.3 in [2], we have

**Proposition 1** Suppose \( Z_x \) is continuous in \( x \) and satisfies (5). Then for \( m \leq n \),

\[
\sum_{k=1}^{2^n} E|X(m+1)^2_{x_{k-1},x_k} - X(m)^2_{x_{k-1},x_k}|^\theta \leq C \left( \frac{1}{2^{n+m}} \right)^{\frac{\theta h - 1}{\theta}},
\]

where \( C \) depends on \( \theta, h, w_1(x', x'') \), and \( c \) in (5).

**Proposition 2** Assume \( 2 \leq q < 4 \) and \( q < \theta < 4 \). Then for \( m > n \), we have

\[
E|X(m+1)^2_{x_{k-1},x_k} - X(m)^2_{x_{k-1},x_k}|^\theta \leq C \left[ \left( \frac{1}{2^n} \right)^{\frac{\theta}{2}} \left( \frac{1}{2^m} \right)^{\frac{\theta h}{2}} \right],
\]

where \( C \) is a generic constant and also depends on \( \theta, h := \frac{1}{\theta}, w_1(x', x'') \), and \( c \) in (5).

**Proof:** For \( m > n \), we have

\[
E|X(m+1)^2_{x_{k-1},x_k} - X(m)^2_{x_{k-1},x_k}|^\theta \leq \frac{1}{4} E \left| \sum_{l=2^{m-n} (k-1) + 1}^{2^{m-n} k} (\Delta^m_{l+1} Z \otimes \Delta^m_{l+1} Z - \Delta^m_{l} Z - \Delta^m_{l+1} Z \otimes \Delta^m_{l+1} Z) \right|^\theta.
\]
\[
\begin{align*}
\frac{1}{4} & \sum_{l,r=2^{m-n}(k-1)+1}^{2^{m-n}k} \left( \Delta_{2l-1}^{m+1} Z^i \Delta_{2l}^{m+1} Z^i - \Delta_{2l}^{m+1} Z^i \Delta_{2l-1}^{m+1} Z^i \right) \\
\times & \left( \Delta_{2r-1}^{m+1} Z^r \Delta_{2r}^{m+1} Z^r - \Delta_{2r}^{m+1} Z^r \Delta_{2r-1}^{m+1} Z^r \right) \\
= & \frac{1}{4} \sum_{l,r} \left[ E(\Delta_{2l-1}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l}')(\Delta_{2l}^{m+1} g(x) \Delta_{2l}^{m+1} g(x)) \\
- & E(\Delta_{2l-1}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l}')(\Delta_{2l}^{m+1} g(x) \Delta_{2l}^{m+1} g(x)) \right] \\
+ & \frac{1}{4} \sum_{l,r} \left[ (\Delta_{2l}^{m+1} g(x) \Delta_{2l-1}^{m+1} g(x)) E(\Delta_{2l}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l}) \\
- & E(\Delta_{2l}^{m+1} g(x) \Delta_{2l-1}^{m+1} g(x)) E(\Delta_{2l}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l}) \right]. \\
\end{align*}
\]

Let \( X_t = M_t + V_t \), where \( M_t \) is a continuous local martingale, \( V_t \) is a continuous process of finite variation. So from [3] and [4], it’s easy to know that

\[
\tilde{L}_t^x = (X_t - x)^+ - (X_0 - x)^+ - \int_0^t 1_{(X_s > x)} dM_s := \phi(x) - \int_0^t 1_{(X_s > x)} dM_s,
\]

and using some estimate in the proof of Lemma 2.1 in [1], we have

\[
E \left[ \Delta_{2l-1}^{m+1} \tilde{L}_l \Delta_{2l-1}^{m+1} \tilde{L}_l' \right] = E \left[ (\tilde{L}_l(x_{2r-1}) - \tilde{L}_l(x_{2r-2})) (\tilde{L}_l(x_{2r-1}) - \tilde{L}_l(x_{2r-2})) \right] \\
= E \left[ (\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)) - (\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)) \right] \\
\leq E |\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)| + E |\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)| + E |\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)| + E |\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)| + E |\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)| + E |\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)| + E |\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)| + E |\phi(x_{2r-1}^m) - \phi(x_{2r-2}^m)| \\
\leq C \left[ (x_{2r-1}^m - x_{2r-2}^m)(x_{2r-1}^m - x_{2r-2}^m) + (x_{2r-1}^m - x_{2r-2}^m)(x_{2r-1}^m - x_{2r-2}^m) \right] \leq C \left[ (x_{2r-1}^m - x_{2r-2}^m)(x_{2r-1}^m - x_{2r-2}^m) \right].
\]
\[ \begin{align*}
+ E \left| \int_0^t 1_{\{x_{2r-2} < x_{r-1} < x_{r+1}^m \}} 1_{\{x_{2r-2} < x_{r+1} < x_{r+1}^m \}} d < M >s \right| \\
\leq C \left[ \left( \frac{1}{2^m+1} \right)^2 w_1(x', x'') + 2 \left( \frac{1}{2^m+1} \right)^2 w_1(x', x'') \right] \\
+ E \left| \int_0^t 1_{\{x_{2r-2} < x_{r-1} < x_{r+1}^m \}} 1_{\{x_{2r-2} < x_{r+1} < x_{r+1}^m \}} d < M >s \right| \\
\leq \left\{ \begin{array}{ll}
C \left( \frac{1}{2^m+1} \right)^{\frac{3}{2}}, & \text{if } r \neq l, \\
C \left( \frac{1}{2^m+1} \right), & \text{if } r = l.
\end{array} \right.
\end{align*} \]

Here \( C \) is a generic constant and also depends on \( w_1(x', x'') \). So

\[ \sum_{l, r = 2m-n(k-1)+1}^{2m-n} E(\Delta_{2r-1}^{m+1} \hat{L}_l \Delta_{2l-1}^{m+1} \hat{L}_l) E(\Delta_{2r-1}^{m+1} g(x) \Delta_{2l-1}^{m+1} g(x)) \leq C \left[ 2^{m-n} \left( \frac{1}{2^m+1} \right)^{1+2h} + 2^{2(m-n)} \left( \frac{1}{2^m+1} \right)^{\frac{3}{2}+2h} \right]. \]

The other terms in (12) can be treated similarly, therefore

\[ E \left| X(m+1)^2_{x_{k-1}, x_k} - X(m)^2_{x_{k-1}, x_k} \right|^2 \leq C \left[ 2^{m-n} \left( \frac{1}{2^m+1} \right)^{1+2h} + 2^{2(m-n)} \left( \frac{1}{2^m+1} \right)^{\frac{3}{2}+2h} \right]. \]

Hence, as \( \theta < 4 \), by Jensen’s inequality,

\[ \begin{align*}
E \left| X(m+1)^2_{x_{k-1}, x_k} - X(m)^2_{x_{k-1}, x_k} \right|^2 \\
\leq \left( E \left| X(m+1)^2_{x_{k-1}, x_k} - X(m)^2_{x_{k-1}, x_k} \right|^2 \right)^{\frac{\theta}{2}} \\
\leq C \left[ 2^{m-n} \left( \frac{1}{2^m+1} \right)^{1+2h} + 2^{2(m-n)} \left( \frac{1}{2^m+1} \right)^{\frac{3}{2}+2h} \right]^{\frac{\theta}{2}} \\
\leq C \left[ 2^{m-n} \left( \frac{1}{2^m+1} \right)^{\frac{3}{2}+2h} + 2^{2(m-n)} \left( \frac{1}{2^m+1} \right)^{\frac{3}{2}+2h} \right]^{\frac{\theta}{2}} \\
\leq C \left[ \left( \frac{1}{2^m+1} \right)^{\frac{3}{2}+2h} + \left( \frac{1}{2^m+1} \right)^{\frac{3}{2}+2h} \right]^{\frac{\theta}{2}},
\end{align*} \]

where \( C \) is a generic constant and also depends on \( \theta, h, w_1(x', x''), \) and \( c \).

\[ \diamond \]

**Theorem 2** Assume \( 2 \leq q < 4 \) and \( q < \theta < 4 \). Then there exists a unique \( X = (1, X^1, X^2) \) such that

\[ \sum_{i=1}^2 \sup_D \left( \sum_{i} \left| X(m)^i_{x_{i-1}, x_i} - X_{x_{i-1}, x_i} \right|^\frac{\theta}{2} \right)^{\frac{1}{\theta}} \to 0, \]

both almost surely and in \( L^1(\Omega, \mathcal{F}, \mathbb{P}) \) as \( m \to \infty \), where \( X(m) \) are smooth rough paths. In particular, when \( 2 \leq q < 3 \), \( X \) is the canonical geometric rough path associated to \( Z \).

**Proof.** The convergence of \( X(m)^1 \) to \( X^1 \) is Theorem 1. In the following we will prove \( X(m)^2_{a,b} \) converges in \( \theta \)-variation topology. By Proposition 4.1.2 in [2],

\[ \begin{align*}
E \sup_D \sum_{i} \left| X(m+1)^2_{x_{i-1}, x_i} - X(m)^2_{x_{i-1}, x_i} \right|^\frac{\theta}{2} \\
\leq C(\theta, \gamma) E \left( \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \left| X(m+1)^1_{x_{n-1}^k, x_n^k} - X(m)^1_{x_{n-1}^k, x_n^k} \right|^\theta \right)^{\frac{1}{\theta}}
\end{align*} \]
to compute that the two form $\hat{\theta}$ of the one form $\hat{\theta}$

Similar to the proof of Theorem 1, we can easily deduce that

$\sum_{n=1}^{\infty} n\gamma \left( \left( \sum_{k=1}^{2^n} \left( |X(m+1)|_{x_k}^{1/2} - |X(m)|_{x_k}^{1/2} \right)^{\theta} + |X(m)|_{x_k}^{1/2} \right) \right)^{1/2}$

$+ C(\theta, \gamma) \sum_{n=1}^{\infty} n\gamma \sum_{k=1}^{2^n} |X(m+1)|_{x_k}^{1/2} - |X(m)|_{x_k}^{1/2}|^{\theta}$

$=: A + B.$

We will estimate part A, B respectively. First from (9), we know

$A \leq C \left( \sum_{n=1}^{\infty} n\gamma \sum_{k=1}^{2^n} |X(m+1)|_{x_k}^{1/2} - |X(m)|_{x_k}^{1/2}|^{\theta} + |X(m)|_{x_k}^{1/2} \right)^{1/2}$

$\cdot \left( \sum_{n=1}^{\infty} n\gamma \sum_{k=1}^{2^n} |X(m+1)|_{x_k}^{1/2} - |X(m)|_{x_k}^{1/2}|^{\theta} + |X(m)|_{x_k}^{1/2} \right)^{1/2}$

$\leq C \left( \frac{1}{2^m} \right)^{\frac{h\theta - 1}{2}} \sum_{n=1}^{\infty} n\gamma \sum_{k=1}^{2^n} |X(m+1)|_{x_k}^{1/2} - |X(m)|_{x_k}^{1/2}|^{\theta} + |X(m)|_{x_k}^{1/2} \right)^{1/2}.$

Secondly from Proposition 1 and Proposition 2, we know

$B \leq C \left( \sum_{n=1}^{\infty} n\gamma \left( \frac{1}{2^m} \right)^{\frac{h\theta - 1}{2}} \sum_{n=1}^{\infty} n\gamma \left( \frac{1}{2^m} \right)^{\frac{h\theta - 1}{2}} + \sum_{n=1}^{\infty} n\gamma \left( \frac{1}{2^m} \right)^{\frac{h\theta - 1}{2}} \sum_{n=1}^{\infty} n\gamma \left( \frac{1}{2^m} \right)^{\frac{h\theta - 1}{2}} \right)$

$\leq C \left[ \left( \frac{1}{2^m} \right)^{\frac{h\theta - 1}{2}} + \left( \frac{1}{2^m} \right)^{\frac{h\theta - 1}{2}} \right],$

as $q < \theta < 4$, so $h\theta > 1$. Thus

$E \sup_{D} \sum_{i} |X(m+1)|_{x_{i-1}, x_{i}} - |X(m)|_{x_{i-1}, x_{i}}|^{\theta} \leq C \left[ \left( \frac{1}{2^m} \right)^{\frac{h\theta - 1}{2}} + \left( \frac{1}{2^m} \right)^{\frac{h\theta - 1}{2}} \right].$

Similar to the proof of Theorem 1, we can easily deduce that $(X(m))^2$ is a Cauchy sequence in $\theta$-variation distance. So when $m \to \infty$, it has a limit, denote it by $X^2$, and from completeness under $\theta$-variation distance (Lemma 3.3.3 in [2]), $X^2$ is also of finite $\theta$-variation. The theorem is asserted.

Remark 1 We would like to point out that the above method does not seem to work for two arbitrary functions $f$ of $p$-variation and $g$ of $q$-variation ($2 < p, q < 3$) to define a rough path $Z_x = (f(x), g(x))$. But the special property (13) of local times makes our analysis work. Similar method was used in [2] for fractional Brownian motion with the help of long-time memory. Here (13) serves a similar role of the long-time memory as in [2].

In the following, we will only consider the case that $2 \leq q < 3$ and take $q < \theta < 3$.

As local time $L^2_t$ has a compact support in $x$ for each $\omega$ and $t$, so we can define integral of local time directly in $R$. For this, we take $[x', x'']$ covering the support of $L^2_t$. Recall the definition of the one form $\bar{f} : \mathbb{R}^2 \to L(R^2, \mathbb{R}^2)$, $\bar{f}(z) = (v, yv)$, where $z = (x, y)$ and $\xi = (v, w)$. It is easy to compute that the two form $\hat{f} : \mathbb{R}^2 \to L(R^2 \otimes R^2, \mathbb{R}^2)$ is given by $\hat{f}(z)(\xi_1 \otimes \xi_2) = \begin{pmatrix} 0 \\ v_1 w_2 \end{pmatrix}$, where $\xi_1 = (v_1, w_1)$, $\xi_2 = (v_2, w_2)$. Define

$Y_{a,b}^1 = \bar{f}(Z_a) Z_{a,b}^1 + \hat{f}(Z_a) Z_{a,b}^2, Y_{a,b}^2 = (\hat{f}(Z_a) \otimes \bar{f}(Z_a)) Z_{a,b}^2.$
From Chapter 5 in [2], we know that \( Y = (1, Y_{a,b}^1, Y_{a,b}^2) \) is an almost multiplicative functional of degree 2 and therefore one can use the almost rough path to construct the unique rough path \( \int_{-\infty}^{\infty} \hat{f}(Z) dZ \) with roughness \( \theta \) in \( T^{(2)}(R^2) \). In particular,

\[
\int_{-\infty}^{\infty} \hat{f}(Z) dZ = \lim_{m(D) \to 0} \sum \left[ \hat{f}(Z_{x_{i-1}})(Z_{x_{i-1}, x_i}) + \hat{f}^2(Z_{x_{i-1}})(Z_{x_{i-1}, x_i}^2) \right],
\]

where the limit exists so the integral is well-defined. Note

\[
\hat{f}(Z(a))(Z_{a,b}^1) + \hat{f}^2(Z(a))(Z_{a,b}^2) = \left( \hat{L}_t^b - \hat{L}_t^a, g(a)(\hat{L}_t^b - \hat{L}_t^a) \right) + (0, (Z_{a,b}^{2,1,2}),
\]

where \((Z_{a,b}^{2,1,2})\) means the upper-right element of \( 2 \times 2 \) matrix \( Z_{a,b}^2 \). Note in our case the rough path

\[
\int_{-\infty}^{\infty} \hat{f}(Z) dZ = \left( \int_{-\infty}^{\infty} d\hat{L}_t^x, \int_{-\infty}^{\infty} g(x) d\hat{L}_t^x \right).
\]

In particular,

\[
\left( \int_{-\infty}^{\infty} g(x) d\hat{L}_t^x \right)^1 = \lim_{m(D) \to 0} \left[ \sum g(x_{i-1}) (L_t^x_i - L_t^{x_{i-1}}) + (Z_{x_{i-1}, x_i})^{2,1,2} \right],
\]

where the limit exists. Note it is clear to us that the Riemann sum \( \sum g(x_{i-1}) (L_t^x_i - L_t^{x_{i-1}}) \) itself does not have a limit as \( m(D) \to 0 \). This is the very reason we use Lyons’ rough path integration theory. Still denote the integral by \( \int_{-\infty}^{\infty} g(x) d\hat{L}_t^x \).

Note from Theorem 5.2.2 in [2], \( \hat{f} \) is a continuous map from \( \Omega_{\theta}(R^2) \) (the set of rough path in \( R^2 \) with roughness \( \theta \)) to \( \Omega_{\theta}(R^2) \) in \( \theta \)-variation topology. Let \( Z_n(x) := (\hat{L}_t^x, g_n(x)), Z(x) := (\hat{L}_t^x, g(x)) \), where \( g_n(x) \) is of bounded \( q \)-variation uniformly in \( n, 2 \leq q < 3 \), and when \( n \to \infty, g_n(x) \to g(x) \). What we should prove is that rough path \( Z_n(x) \to Z(x) \) in \( \theta \)-variation distance. Repeating the above argument, we can find the canonical geometric rough path associated with \( Z_n \), is \( X_n = (1, X_n^1, X_n^2) \), the smooth rough path is \( X_n(m) = (1, X_n(m)^1, X_n(m)^2) \). Actually, \((X_n)^1_{a,b} = (\hat{L}_t^1 - \hat{L}_t^a, g_n(b) - g_n(a)), (X_n)^1_{a,b} = (\hat{L}_t^1 - \hat{L}_t^a, g(b) - g(a))\), so \((X_n)^1_{a,b} \to X^1_{a,b} \) in the sense of uniformly topology, and also in the sense of \( \theta \)-variation topology. As for \((X_n)^2_{a,b} \), we can easily see that

\[
||X_n^2_{a,b} - X^2_{a,b}|| \leq ||X_n^2_{a,b} - (X_n(m)^2_{a,b})|| + ||(X_n(m)^2_{a,b}) - (X(m)^2_{a,b})|| + ||(X(m)^2_{a,b}) + X^2_{a,b}||.
\]

From Theorem 2, we know that the first and the third term on the right-hand side is smaller than \( \varepsilon \omega_1(a, b) \), for any small \( \varepsilon > 0 \). The second term can be easily dealt with from the definition of \((X_n(m)^2_{a,b})\) and \( X(m)^2_{a,b} \). In fact, it is convergent in the uniformly topology. So \( \int f(Z_n) dZ_n \to \int f(Z) dZ \) in \( \theta \)-variation distance a.s.. Therefore \( \int f(Z_n) dZ_n \to \int f(Z) dZ \) a.s., i.e. \( \int_{-\infty}^{\infty} g_n(x) d\hat{L}_t^x \to \int_{-\infty}^{\infty} g(x) d\hat{L}_t^x \) a.s., when \( n \to \infty \). As for the jump part, from Lebesgue’s dominated convergence theorem, \( \int_{-\infty}^{\infty} g_n(x) dh(t, x) \to \int_{-\infty}^{\infty} g(x) dh(t, x) \), when \( n \to \infty \). So we can get \( \int_{-\infty}^{\infty} g_n(x) d\hat{L}_t^x \to \int_{-\infty}^{\infty} g(x) d\hat{L}_t^x \), when \( n \to \infty \). If \( g(x) \) has discontinuities, we can use the method of [5]. Finally, we deduce an extension of Tanaka-Meyer’s formula. A similar smoothing procedure with [1] can be used and the above convergence is enough to make our proof work. So we have the following theorem. The case \( 1 \leq q < 2 \) was considered in [1].

**Theorem 3** Let \( X = (X_t)_{t \geq 0} \) be a continuous semimartingale and \( f : R \to R \) be an absolutely continuous function and have left derivative \( \nabla^- f(x) \) being left continuous and locally bounded. Assume \( \nabla^- f(x) \) is of bounded \( q \)-variation, \( 1 \leq q < 3 \), then
\[ f(X_t) = f(X_0) + \int_0^t \nabla^- f(X_s) dX_s - \int_{-\infty}^{\infty} \nabla^- f(x) d\mu_t^x. \] (14)

Here the integral \( \int_{-\infty}^{\infty} \nabla^- f(x) d\mu_t^x \) is a Lebesgue-Stieltjes integral when \( q = 1 \), a Young integral when \( 1 < q < 2 \) and a Lyons’ rough path integral when \( 2 \leq q < 3 \) respectively.

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**References**