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Local Time as a Rough Path

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Summary. In this paper, we prove that a semimartingale local time is a rough path of roughness $p$ for any $p \in [2,3)$ and establish the integral $\int_{-\infty}^{t} g(x) dL_t(x)$ for any finite $q$-variation function $g$ $(2 \leq q < 3)$ using Lyons’ rough path integration theory. We therefore obtain the Meyer-Tanaka’s formula for continuous function $f$ if $\nabla^{-1} f$ exists and is of finite $q$-variation when $2 \leq q < 3$. The case when $1 \leq q < 2$ was established in [1] using Young integral.

Keywords: Semimartingale local time; rough path.

In [1], Lemma 2.1 says that the semimartingale local time $L_t^\xi$ is of bounded $p$-variation in $x$ for any $t \geq 0$, $p > 2$ a.s., i.e. for almost all $\omega \in \Omega$,

$$\sup_D \sum_{i=0}^{m-1} |L^\xi_{t_i} - L^\xi_{t_{i+1}}|^p < \infty,$$

where $D = \{-N = x_0 < x_1 < \cdots < x_m = N\}$ is an arbitrary partition of interval $[-N,N]$ covering the support of $L_t$. So in Theorem 2.2, we gave a new condition for Tanaka-Meyer’s formula and the integral $\int_{-\infty}^{\infty} \nabla^{-1} f(x) d_x L_t^\xi$ as a Young integral, when $\nabla^{-1} f(x)$ is of bounded $q$-variation $(1 \leq q < 2)$. But how is about if $q \geq 2$? Can we still define such an integral $\int_{-\infty}^{\infty} \nabla^{-1} f(x) d_x L_t^\xi$ pathwise? If we can, we will get Tanaka-Meyer’s formula for wider class of functions. So let’s first try to define the integral $\int_{-\infty}^{\infty} g(x) d_x L_t^\xi$ pathwise for a continuous $g(x)$ with bounded $q$-variation $(2 \leq q < 3)$. And we also take $2 < p < 3$. We decompose local time

$$L_t^\xi = \hat{L}_t^\xi + \sum_{x_k^\xi < x} \hat{L}_t^\xi_k, \text{ where } \hat{L}_t^\xi := L_t^\xi - L_{t^-}^\xi,$$

Here $\hat{L}_t^\xi$ is continuous in $x$, and $x_k^\xi, k = 1, 2 \cdots$ are the discontinuous points (countable) of $L_t^\xi$. From Lemma 2.2 in [1], we know $h(t,x) := \sum_{x_k^\xi < x} \hat{L}_t^\xi_k$ is of bounded variation in $x$ for each $t$. So the key point is to define $\int_{-\infty}^{\infty} g(x) d_x \hat{L}_t^\xi$ pathwise for $g(x)$ with bounded $q$-variation $(2 \leq q < 3)$. For this, we will use Lyons’ rough path theory.

In fact, $g(x)$ and $\hat{L}_t^\xi$ can be regarded as rough paths. From [2], generally, we cannot expect to have an integration theory for defining integrals such as $\int_{-\infty}^{\infty} g(x) d_x \hat{L}_t^\xi$. But using the method in Chapter 6 in [2], we can treat $Z_x := (\hat{L}_t^\xi, g(x))$ together as a rough path and define $\hat{f}(x,y)(v,w) := (v, yv)$, so the integral will be the second element of $\int_{-\infty}^{\infty} \hat{f}(Z) dZ^1$. It’s easy to know that $Z_x$ is of bounded $\hat{q}$-variation in $x$, where $\hat{q} = q$, if $q > 2$, and $\hat{q} > 2$ can be taken as any number when $q = 2$. Most of the analysis in this paper works for $2 \leq q < 4$, especially we will establish the convergence of smooth rough path in $\theta$-variation topology for any $\theta \in (q,4)$, so to obtain $Z_{x,\theta}$ and $Z_{x,\theta,b}$. In particular, when $2 \leq q < 3$, we obtain the existence of the geometric rough path $X = (1, X^1, X^2)$ associated with $Z$. In the following we consider $2 \leq q < 4$ otherwise we will explicitly say so.

Let $[x', x'']$ be any interval in $R$. From the proof of Lemma 2.1 in [1], for any $p \geq 2$, we know there exists a constant $c > 0$ such that
\[ E|\hat{L}_t(b) - \hat{L}_t(a)|^p \leq c(b - a)^{\frac{p}{2}} \]  

(3)

i.e. \( \hat{L}_t(x) \) satisfies Hölder condition in the sense of [2] with exponent \( \frac{1}{2} \). Denote by \( w \) the control of \( g(x) \), i.e.

\[ |g(b) - g(a)|^q \leq w(a, b), \]

for any \( (a, b) \in \Delta := \{(a, b) : x' \leq a < b \leq x''\} \). It is obvious that \( w_1(a, b) := w(a, b) + (b - a) \) is also a control of \( g \). Denote \( h = \frac{1}{q} \). It is trivial to see for any \( \theta > q \) i.e. \( h\theta > 1 \) that,

\[ |g(b) - g(a)|^{\theta} \leq w_1(a, b)^{h\theta}, \text{ for any } (a, b) \in \Delta. \]  

(4)

Considering (3), we can see for such \( h = \frac{1}{q} \), and any \( \theta > q \) there exists \( c > 0 \) such that

\[ E|Z_b - Z_a|^\theta \leq cw_1(a, b)^{h\theta}, \text{ for any } (a, b) \in \Delta. \]  

(5)

For any \( m \in \mathbb{N} \), define a continuous and bounded variation path \( Z(m) \) by

\[ Z(m)_x := Z_{x_0} + \frac{w_1(x) - w_1(x_{l-1}^m)}{w_1(x_{l-1}^m) - w_1(x_{l-1})^m} \Delta^m_x, \]  

(6)

if \( x_{l-1}^m \leq x < x_l^m \), for \( l = 1, \ldots, 2^m \), and \( \Delta^m_x = Z_{x_l^m} - Z_{x_{l-1}^m} \). Here \( D_m := \{x' = x_0 < x_1 < \cdots < x_{2^m} = x''\} \) is a partition of \( [x', x''] \) such that \( w_1(x_{n}^m) - w_1(x_{n-1}^m) = \frac{1}{2^n}w_1(x', x'') \), where \( w_1(x) := w_1(x', x) \). It is obvious that \( x_l^m - x_{l-1}^m \leq \frac{1}{2^n}w_1(x', x'') \). The corresponding smooth rough path \( X(m) \) is built by taking its iterated path integrals, i.e.

\[ X(m)_{\theta, \gamma} = \int_{a < x_1 < \cdots < x_j < b} dZ(m)_{x_1} \otimes \cdots \otimes dZ(m)_{x_j}. \]  

(7)

Let’s first look at the first level path \( X(m)_{\theta, \gamma} \). The method and results are similar to Chapter 4 in [2]. Similar to Proposition 4.2.1 in [2], we can prove that for all \( n \in \mathbb{N} \), \( m \mapsto \sum_{k=1}^{2^n} |X(m)_{x_{k-1}^m, x_k^m}|^\theta \) is increasing. Let \( X_{\theta, \gamma} = Z_b - Z_a \). Inequality (5) implies \( E|X_{\theta, \gamma}|^\theta \leq cw_1(a, b)^{h\theta} \) for such points \( \{x_k^n\}, k = 1, \ldots, 2^n, n = 1, 2, \cdots \), defined above we still have the same inequality as in Proposition 4.1.1, [2], so for any \( \gamma > \theta - 1 \), there exists a constant \( C_1(\theta, \gamma, c) > 0 \) such that

\[ E\sup_{D} \sum_{l} |X_{\theta, \gamma}^{1}_{x_{l-1}, x_l}|^\theta \leq C(\theta, \gamma)E\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X_{\theta, \gamma}^{1}_{x_{k-1}^m, x_k^m}|^\theta \]

\[ \leq C_1(\theta, \gamma, c)\sum_{n=1}^{\infty} n^\gamma \left( \frac{1}{2^n}\right)^{h\theta-1}w_1(x', x'')^{h\theta}. \]  

(8)

Since \( h\theta - 1 > 0 \), the series on the right-hand side of (8) is convergent, so \( \sup_{D} \sum_{l} |X_{\theta, \gamma}^{1}_{x_{l-1}, x_l}|^\theta < \infty \) almost surely. This shows that \( X^1 \) has finite \( \theta \)-variation almost surely. Furthermore,

\[ E\sup_{m, D} \sum_{l} |X(m)_{\theta, \gamma}^{1}_{x_{l-1}, x_l}|^\theta \leq C(\theta, \gamma)E\sup_{m, D} \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X(m)_{\theta, \gamma}^{1}_{x_{k-1}^m, x_k^m}|^\theta \]

\[ \leq C(\theta, \gamma)E\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |X_{\theta, \gamma}^{1}_{x_{k-1}^m, x_k^m}|^\theta \]

\[ \leq C_1(\theta, \gamma, c)\sum_{n=1}^{\infty} n^\gamma \left( \frac{1}{2^n}\right)^{h\theta-1}w_1(x', x'')^{h\theta} < \infty, \]

(9)
where $C$ depends on $\theta$, $h$, $w_1(x', x'')$, and $c$ in (5). By the Hölder inequality,

$$E \sum_{m=1}^{\infty} \sup_D \left( \sum_l |X(m)^1_{x_{l-1}, x_l} - X_{x_{l-1}, x_l}^1|^\theta \right)^{\frac{1}{\theta}} \leq C \sum_{m=1}^{\infty} \left( \frac{1}{2^m} \right)^{\frac{k h - 1}{2^m}} < \infty,$$

(9)
as $h \theta > 1$. We immediately have

**Theorem 1** For a continuous path $Z_x$ with (5), we have

$$\sum_{m=1}^{\infty} \sup_D \left( \sum_l |X(m)^1_{x_{l-1}, x_l} - X_{x_{l-1}, x_l}^1|^\theta \right)^{\frac{1}{\theta}} < \infty \text{ a.s.}$$

(10)

In particular, $(X(m)^1_{a,b})$ converges to $(X^1_{a,b})$ in $\theta$-variation distance a.s. for any $(a, b) \in \Delta$.

We next consider the second level path $X(m)^2_{a,b}$. From [2], we know if $n \geq m$, $X(m)^2_{x_k, x_k} = 2^{(m-n)} \Delta^m Z \otimes \Delta^m Z$; if $n < m$,

$$X(m)^2_{x_k, x_k} = \frac{1}{2} \Delta^m Z \otimes \Delta^m Z + \frac{1}{2} \sum_{r < l = 2^{m-n}(k-1)+1} (\Delta^m Z \otimes \Delta^m Z),$$

so

$$X(m+1)^2_{x_k, x_k} - X(m)^2_{x_k, x_k} = \frac{1}{2} \sum_{l=2^{m-n}(k-1)+1}^{2^{m-n} k} (\Delta^m Z \otimes \Delta^m Z - \Delta^m Z \otimes \Delta^m Z),$$

$k = 1, \cdots, 2^n$. Similar to the proof of Proposition 4.3.3 in [2], we have

**Proposition 1** Suppose $Z_x$ is continuous in $x$ and satisfies (5). Then for $m \leq n$,

$$\sum_{k=1}^{2^n} E |X(m+1)^2_{x_k, x_k} - X(m)^2_{x_k, x_k}|^2 \leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2}} \frac{1}{2^m} \left( \frac{1}{2^m} \right)^{\frac{1}{2} h - \frac{1}{2}},$$

where $C$ depends on $\theta$, $h$, $w_1(x', x'')$, and $c$ in (5).

**Proposition 2** Assume $2 \leq q < 4$ and $q < \theta < 4$. Then for $m > n$, we have

$$E |X(m+1)^2_{x_k, x_k} - X(m)^2_{x_k, x_k}|^q \leq C \left[ \left( \frac{1}{2^n} \right)^{\frac{1}{2}} \left( \frac{1}{2^m} \right)^{\frac{1}{2} h - \frac{1}{2}} + \left( \frac{1}{2^n} \right)^{\frac{1}{2}} \left( \frac{1}{2^m} \right)^{\frac{1}{2} h - \frac{1}{2} q} \right],$$

(11)

where $C$ is a generic constant and also depends on $\theta$, $h(= \frac{1}{\theta})$, $w_1(x', x'')$, and $c$ in (5).

**Proof:** For $m > n$, we have

$$E |X(m+1)^2_{x_k, x_k} - X(m)^2_{x_k, x_k}|^2 \leq \frac{1}{4} E \left| \sum_{l=2^{m-n}(k-1)+1}^{2^{m-n} k} (\Delta^m Z \otimes \Delta^m Z - \Delta^m Z \otimes \Delta^m Z) \right|^2,$$
\[
= \frac{1}{4} \sum_{i,j=1 \atop i \neq j}^{2m-n-k} \sum_{l,r=2m-n-k}^{2m-n-k} \left( \Delta_{2l-1}^{m+1} Z^i \Delta_{2l+1}^{m+1} Z^i - \Delta_{2l}^{m+1} Z^i \Delta_{2l-1}^{m+1} Z^i \right) \\
\cdot \left( \Delta_{2r-1}^{m+1} Z^r \Delta_{2r+1}^{m+1} Z^r - \Delta_{2r}^{m+1} Z^r \Delta_{2r-1}^{m+1} Z^r \right)
\]

\[
= \frac{1}{4} \sum_{l,r} \left[ E(\Delta_{2l-1}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l})(\Delta_{2l-1}^{m+1} g(x)\Delta_{2l-1}^{m+1} g(x)) \\
+ E(\Delta_{2l-1}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l})(\Delta_{2l-1}^{m+1} g(x)\Delta_{2l-1}^{m+1} g(x)) \right] \\
- \frac{1}{4} \sum_{l,r} \left[ E(\Delta_{2l}^{m+1} \tilde{L}_{l} \Delta_{2l}^{m+1} \tilde{L}_{l})(\Delta_{2l}^{m+1} g(x)\Delta_{2l}^{m+1} g(x)) \\
+ (\Delta_{2l}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l})(\Delta_{2l}^{m+1} g(x)\Delta_{2l-1}^{m+1} g(x)) \right] \\
+ \frac{1}{4} \sum_{l,r} \left[ (\Delta_{2l}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l})(\Delta_{2l}^{m+1} g(x)\Delta_{2l-1}^{m+1} g(x)) \right] \\
- \frac{1}{4} \sum_{l,r} \left[ (\Delta_{2l-1}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l})(\Delta_{2l-1}^{m+1} g(x)\Delta_{2l-1}^{m+1} g(x)) \right] \\
+ (\Delta_{2l-1}^{m+1} \tilde{L}_{l} \Delta_{2l}^{m+1} \tilde{L}_{l})(\Delta_{2l-1}^{m+1} g(x)\Delta_{2l}^{m+1} g(x)) \right].
\]

(12)

Let \(X_t = M_t + V_t\), where \(M_t\) is a continuous local martingale, \(V_t\) is a continuous process of finite variation. So from [3] and [4], it’s easy to know that

\[
\tilde{L}_t = (X_t - x)^+ - (X_0 - x)^+ - \int_0^t 1_{\{X_s > x\}} dM_s := \phi(x) - \int_0^t 1_{\{X_s > x\}} dM_s,
\]

and using some estimate in the proof of Lemma 2.1 in [1], we have

\[
E \left[ \Delta_{2l-1}^{m+1} \tilde{L}_{l} \Delta_{2l-1}^{m+1} \tilde{L}_{l} \right] \\
= E \left[ (\tilde{L}_t(x_{2l-1}^{m+1}) - \tilde{L}_t(x_{2l-2}^{m+1})) (\tilde{L}_t(x_{2l-1}^{m+1}) - \tilde{L}_t(x_{2l}^{m+1})) \right] \\
= E \left[ (\phi(x_{2l-1}^{m+1}) - \phi(x_{2l-2}^{m+1})) - \int_0^t 1_{\{x_{2l-2}^{m+1} \leq X_s \leq x_{2l-1}^{m+1}\}} dM_s \right] \\
\cdot \left[ (\phi(x_{2l-1}^{m+1}) - \phi(x_{2l-2}^{m+1})) - \int_0^t 1_{\{x_{2l-2}^{m+1} \leq X_s \leq x_{2l-1}^{m+1}\}} dM_s \right] \\
\leq E |\phi(x_{2l-1}^{m+1}) - \phi(x_{2l-2}^{m+1})| \cdot |\phi(x_{2l-1}^{m+1}) - \phi(x_{2l-2}^{m+1})| \\
+ E \int_0^t 1_{\{x_{2l-2}^{m+1} \leq X_s \leq x_{2l-1}^{m+1}\}} dM_s \\
+ E \int_0^t 1_{\{x_{2l-2}^{m+1} \leq X_s \leq x_{2l-1}^{m+1}\}} dM_s \\
+ E \left[ \int_0^t 1_{\{x_{2l-2}^{m+1} \leq X_s \leq x_{2l-1}^{m+1}\}} dM_s \right] \\
\leq C \left[ (x_{2l-1}^{m+1} - x_{2l-2}^{m+1})(x_{2l-1}^{m+1} - x_{2l-2}^{m+1}) + (x_{2l-1}^{m+1} - x_{2l-2}^{m+1})(x_{2l-1}^{m+1} - x_{2l-2}^{m+1}) \right] \\
+ (x_{2l-1}^{m+1} - x_{2l-2}^{m+1})(x_{2l-1}^{m+1} - x_{2l-2}^{m+1}) \frac{1}{2}.$
\begin{align*}
+ E & \int_0^t 1_{\{x_{2r-2} < X_s < x_{2r-1} \}} 1_{\{x_{2r-2} < X_s < x_{2r-1} \}} d < M >_s \\
& \leq C \left[ \frac{1}{2m+1} \right]^2 w_1(x', x'')^2 + 2 \left( \frac{1}{2m+1} \right)^2 w_1(x', x'')^2 \\
+ E & \int_0^t 1_{\{x_{2r-2} < X_s < x_{2r-1} \}} 1_{\{x_{2r-2} < X_s < x_{2r-1} \}} d < M >_s \\
& \leq \left\{ \begin{array}{ll}
C \left( \frac{1}{2m+1} \right)^2, & \text{if } r \neq l, \\
C \frac{1}{2m+1}, & \text{if } r = l.
\end{array} \right.
\end{align*}

Here $C$ is a generic constant and also depends on $w_1(x', x'')$. So

\[
E |X(m+1)_{x_{k-1}^n} - X(m)_{x_{k-1}^n} |^2 \leq C \left[ 2^{m-n}(\frac{1}{2m+1})^{1+2h} + 2^{2(m-n)}(\frac{1}{2m+1})^{2+2h} \right].
\]

The other terms in (12) can be treated similarly, therefore

\[
E |X(m+1)_{x_{k-1}^n} - X(m)_{x_{k-1}^n} |^2 \leq C \left[ 2^{m-n}(\frac{1}{2m+1})^{1+2h} + 2^{2(m-n)}(\frac{1}{2m+1})^{2+2h} \right].
\]

Hence, as $\theta < 4$, by Jensen’s inequality,

\[
E |X(m+1)_{x_{k-1}^n} - X(m)_{x_{k-1}^n} |^2 \leq \left( E |X(m+1)_{x_{k-1}^n} - X(m)_{x_{k-1}^n} |^2 \right)^{\frac{1}{2}}
\]

\[
\leq C \left[ 2^{m-n}(\frac{1}{2m+1})^{1+2h} + 2^{2(m-n)}(\frac{1}{2m+1})^{2+2h} \right]^{\frac{1}{2}}
\]

\[
\leq C \left[ \left( \frac{2^{m-n}}{2m+1} \right)^{1/2} + \left( \frac{2^{2(m-n)}}{2m+1} \right)^{1/2} \right]^{\frac{1}{2}}
\]

\[
\leq C \left[ \left( \frac{2^{m-n}}{2m+1} \right)^{1/2} + \left( \frac{2^{2(m-n)}}{2m+1} \right)^{1/2} \right]^{\frac{1}{2}}
\]

where $C$ is a generic constant and also depends on $\theta, h, w_1(x', x''),$ and $c.$

\textbf{Theorem 2} Assume $2 \leq q < 4$ and $q < \theta < 4$. Then there exists a unique $X = (1, X^1, X^2)$ such that

\[
\sum_{i=1}^{2} \sup_D \left( \sum_{i} |X(m)_{x_{i-1}, x_i} - X_{x_{i-1}, x_i} |^\theta \right)^{\frac{1}{\theta}} \to 0,
\]

both almost surely and in $L^1(\Omega, \mathcal{F}, \mathcal{P})$ as $m \to \infty$, where $X(m)$ are smooth rough paths. In particular, when $2 \leq q < 3$, $X$ is the canonical geometric rough path associated to $Z$.

\textbf{Proof.} The convergence of $X(m)^1$ to $X^1$ is Theorem 1. In the following we will prove $X(m)^1_{a,b}$ converges in $\theta$-variation topology. By Proposition 4.1.2 in [2],

\[
E \sup_D \sum_{i} |X(m+1)_{x_{i-1}, x_i} - X(m)^1_{x_{i-1}, x_i} |^\theta
\]

\[
\leq C(\theta, \gamma) E \left( \sum_{n=1}^{\infty} n^2 \sum_{k=1}^{2^n} |X(m+1)_{x_{k-1}^n, x_k} - X(m)^1_{x_{k-1}^n, x_k} |^\theta \right)^{\frac{1}{\theta}}
\]
to compute that the two form \( \hat{a} \) of the one form \( \hat{b} \). Similar to the proof of Theorem 1, we can easily deduce that
\[
\frac{X(m+1)}{x_{k-1}^n} - X(m) - X_{x_{k-1}^n}^n |^\theta
\]
 Secondly from Proposition 1 and Proposition 2, we know
\[
\theta \leq \frac{X(m+1)}{x_{k-1}^n} - X(m) - X_{x_{k-1}^n}^n |^\theta
\]
\[
\leq C \left( \sum_{n=1}^{\infty} n^7 \sum_{k=1}^{2^n} |X(m+1)|_x^\theta - X(m) - X_{x_{k-1}^n}^n |^\theta \right)^\frac{1}{2}
\]

We will estimate part A, B respectively. First from (9), we know
\[
A \leq C \left( \sum_{n=1}^{\infty} n^7 \sum_{k=1}^{2^n} |X(m+1)|_x^\theta - X(m) - X_{x_{k-1}^n}^n |^\theta \right)^\frac{1}{2}
\]
\[
B \leq C \left[ \left( \sum_{n=1}^{\infty} n^7 \sum_{k=1}^{2^n} |X(m+1)|_x^\theta - X(m) - X_{x_{k-1}^n}^n |^\theta \right)^\frac{1}{2} \right]
\]

as \( q < \theta < 4 \), so \( h\theta > 1 \). Thus
\[
E \sup_{D} \sum_{i} |X(m+1)|_{x_{i-1}^n}^\theta - X(m) - X_{x_{i-1}^n}^n |^\theta \leq C \left[ \left( \frac{1}{2} \right)^{h\theta-1} + \left( \frac{1}{2} \right)^{h\theta-1} \right].
\]

Similar to the proof of Theorem 1, we can easily deduce that \( (X(m)^2)_{m \in \mathbb{N}} \) is a Cauchy sequence in \( \theta \)-variation distance. So when \( m \to \infty \), it has a limit, denote it by \( X^2 \), and from completeness under \( \theta \)-variation distance (Lemma 3.3.3 in [2]), \( X^2 \) is also of finite \( \theta \)-variation. The theorem is asserted.

**Remark 1.** We would like to point out that the above method does not seem to work for two arbitrary functions \( f \) of \( p \)-variation and \( g \) of \( q \)-variation \((2 < p, q < 3)\) to define a rough path \( Z_x = (f(x), g(x)) \). But the special property (13) of local times makes our analysis work. Similar method was used in [2] for fractional Brownian motion with the help of long-time memory. Here (13) serves a similar role of the long-time memory as in [2].

In the following, we will only consider the case that \( 2 \leq q < 3 \) and take \( q < \theta < 3 \).

As local time \( L_t^x \) has a compact support in \( x \) for each \( \omega \) and \( t \), so we can define integral of local time directly in \( R \). For this, we take \([x', x'']\) covering the support of \( L_t^x \). Recall the definition of the one form \( \tilde{f} : R^2 \to L(R^2, R^2) \), \( \tilde{f}(z) = (v, w) \), where \( z = (x, y) \) and \( \xi = (v, w) \). It is easy to compute that the two form \( \tilde{f} : R^2 \to L(R^2 \otimes R^2, R^2) \) is given by \( \tilde{f}(z) \xi = (0, 1) \).

Define
\[
Y_{a,b}^1 = \tilde{f}(Z_a)Z_{a,b}^1, Y_{a,b}^2 = \tilde{f}(Z_a)Z_{a,b}^2.
\]
From Chapter 5 in [2], we know that $Y = (1, Y_{a,b}^{1}, Y_{a,b}^{2})$ is an almost multiplicative functional of degree 2 and therefore one can use the almost rough path to construct the unique rough path $\int_{-\infty}^{\infty} \hat{f}(Z) dZ$ with roughness $\theta$ in $T^{(2)}(R^2)$. In particular,

$$\int_{-\infty}^{\infty} \hat{f}(Z) dZ^1 = \lim_{m(D) \to 0} \sum_{i} \left[ \hat{f}(Z_{x_{i-1}})(Z_{x_{i-1},x_{i}}^1) + \hat{f}^2(Z_{x_{i-1}})(Z_{x_{i-1},x_{i}}^2) \right],$$

where the limit exists so the integral is well-defined. Note

$$\hat{f}(Z(a))(Z_{a,b}^1) + \hat{f}^2(Z(a))(Z_{a,b}^2) = \left( \hat{L}_t^b - \hat{L}_t^a, g(a)\hat{L}_t^b - \hat{L}_t^a \right) + (0, (Z_{a,b}^2)_{1,2}),$$

where $(Z_{a,b}^2)_{1,2}$ means the upper-right element of $2 \times 2$ matrix $Z_{a,b}^2$. Note in our case the rough path

$$\int_{-\infty}^{\infty} \hat{f}(Z) dZ^1 = \left( \int_{-\infty}^{\infty} d\hat{L}_t^x, \int_{-\infty}^{\infty} g(x) d\hat{L}_t^x \right).$$

In particular,

$$\left( \int_{-\infty}^{\infty} g(x) d\hat{L}_t^x \right)^1 = \lim_{m(D) \to 0} \left[ \sum_{i} g(x_{i-1})(L_t^{x_i} - L_t^{x_{i-1}}) + (Z_{x_{i-1},x_{i}}^2)_{1,2} \right],$$

where the limit exists. Note it is clear to us that the Riemann sum $\sum g(x_{i-1})(L_t^{x_i} - L_t^{x_{i-1}})$ itself does not have a limit as $m(D) \to 0$. This is the very reason we use Lyons’ rough path integration theory. Still denote the integral by $\int_{-\infty}^{\infty} g(x) d\hat{L}_t^x$.

Note from Theorem 5.2.2 in [2], $\int \hat{f}$ is a continuous map from $\Omega_{\theta}(R^2)$ (the set of rough path in $R^2$ with roughness $\theta$) to $\Omega_{\theta}(R^2)$ in $\theta$-variation topology. Let $Z_n(x) := (\hat{L}_t^x, g_n(x))$, $Z(x) := (\hat{L}_t^x, g(x))$, where $g_n(x)$ is of bounded $q$-variation uniformly in $n$, $2 \leq q < 3$, and when $n \to \infty$, $g_n(x) \to g(x)$. What we should prove is that rough path $Z_n(x) \to Z(x)$ in $\theta$-variation distance.

Repeating the above argument, we can find the canonical geometric rough path associated with $Z_n$ is $X_n = (1, X_{a,b}^{1}, X_{a,b}^{2})$, the smooth rough path is $X_n(m) = (1, X_{n}(m)^1, X_{n}(m)^2)$. Actually, $(X_n)^{1}_{a,b} = (L_t^{1} - L_t^{a}, g_n(b) - g_n(a))$, $X^{1}_{a,b} = (L_t^{1} - L_t^{a}, g(b) - g(a))$, so $(X_n)^{1}_{a,b} \to X^{1}_{a,b}$ in the sense of uniformly topology, and also in the sense of $\theta$-variation topology. As for $(X_n)^{2}_{a,b}$, we can easily see that

$$|(X_n)^{2}_{a,b} - X^{2}_{a,b}| \leq |(X_n)^{2}_{a,b} - (X_n(m))^{2}_{a,b}| + |(X_n(m))^{2}_{a,b} - (X(m))^{2}_{a,b}| + |(X(m))^{2}_{a,b} - X^{2}_{a,b}|.$$

From Theorem 2, we know that the first and the third term on the right hand side is smaller than $\varepsilon w_1(a,b)$, for any small $\varepsilon > 0$. The second term can be easily dealt with from the definition of $(X_n(m))^{2}_{a,b}$ and $X(m)^{2}_{a,b}$. In fact, it is convergent in the uniformly topology. So $\int \hat{f}(Z_n) dZ_n \to \int \hat{f}(Z) dZ$ in $\theta$-variation distance a.s. Therefore $\int \hat{f}(Z_n) dZ_n \to \int \hat{f}(Z) dZ^1$ a.s., i.e. $\int_{-\infty}^{\infty} g_n(x) d\hat{L}_t^x \to \int_{-\infty}^{\infty} g(x) d\hat{L}_t^x$ a.s., when $n \to \infty$. As for the jump part, from Lebesgue’s dominated convergence theorem, $\int_{-\infty}^{\infty} g_n(x) dh(t,x) \to \int_{-\infty}^{\infty} g(x) dh(t,x)$, when $n \to \infty$. So we can get $\int_{-\infty}^{\infty} g_n(x) d\hat{L}_t^x \to \int_{-\infty}^{\infty} g(x) d\hat{L}_t^x$, when $n \to \infty$. If $g(x)$ has discontinuities, we can use the method of [5]. Finally, we deduce an extension of Tanaka-Meyer’s formula. A similar smoothing procedure with [1] can be used and the above convergence is enough to make our proof work. So we have the following theorem. The case $1 \leq q < 2$ was considered in [1].

**Theorem 3** Let $X = (X_t)_{t \geq 0}$ be a continuous semimartingale and $f : R \to R$ be an absolutely continuous function and have left derivative $\nabla^- f(x)$ being left continuous and locally bounded. Assume $\nabla^- f(x)$ is of bounded $q$-variation, $1 \leq q < 3$, then
\[ f(t) = f(0) + \int_0^t \nabla f(x) \, dx - \int_{-\infty}^{\infty} \nabla f(x) \, dL^x_t. \] (14)

Here the integral \( \int_{-\infty}^{\infty} \nabla f(x) \, dL^x_t \) is a Lebesgue-Stieltjes integral when \( q = 1 \), a Young integral when \( 1 < q < 2 \) and a Lyons’ rough path integral when \( 2 \leq q < 3 \) respectively.

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**References**