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UNIVERSAL CONSTRAINTS ON THE LOCATION OF EXTREMA OF EIGENFUNCTIONS OF NON-LOCAL SCHRÖDINGER OPERATORS

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Abstract. We derive a lower bound on the location of global extrema of eigenfunctions for a large class of non-local Schrödinger operators in convex domains under Dirichlet exterior conditions, featuring the symbol of the kinetic term, the strength of the potential, and the corresponding eigenvalue, and involving a new universal constant. We show a number of probabilistic and spectral geometric implications, and derive a Faber-Krahn type inequality for non-local operators. Our study also extends to potentials with compact support, and we establish bounds on the location of extrema relative to the boundary edge of the support or level sets around minima of the potential.

1. Introduction

Recently, in the paper [53] the remarkable bound
\[ \text{dist}(x^*, \partial D) \geq c \left\| \frac{\Delta \varphi}{\varphi} \right\|_{L^\infty(D)}^{-1/2} \] (1.1)
has been obtained on the distance between the location of an assumed global maximum \( x^* \) of any eigenfunction \( \varphi \) of the Schrödinger operator \( H = -\Delta + V \) with Dirichlet boundary condition set for a simply connected domain \( D \subset \mathbb{R}^2 \), and the boundary of \( D \). Here the potential \( V \) is bounded and includes the eigenvalue corresponding to \( \varphi \), and \( c > 0 \) is a constant, independent of \( D, \varphi \) and \( V \).

The paper [8] established a similar relationship for the fractional Schrödinger operator \( (-\Delta)^{\alpha/2} + V \), \( 0 < \alpha < 2 \), for arbitrary dimensions \( d \geq 2 \), and pointed out some interesting corollaries.

In this paper we present a counterpart of this inequality for a large class of non-local Schrödinger operators, and investigate its multiple implications involving an interplay of probabilistic and spectral geometric aspects. Our key results stated in Theorems 3.1-3.2 go beyond (1.1) on several counts. One is that our expressions feature the symbol of the kinetic part of the operator, allowing an understanding of the inequality from a probabilistic perspective in terms of mean survival times of paths of a related random process in the domain. Another is that we lift the limitation to bounded potentials, and allow a large class including local singularities. Finally, a third is that apart from bounded domains we also consider potentials with compact support in full space \( \mathbb{R}^d \), and derive predictions on the location of extrema relative to the edge of their supports (Theorem 4.5 below).

Non-local (i.e., integro-differential) equations are gaining increasing interest recently from the corners of both pure and applied mathematics. While PDE based on the Laplacian and related elliptic operators proved to be ubiquitous in virtually every fundamental model of dynamics for a long time, it is now recognized that a new range of effects is captured if one uses a class of non-local operators, in which the classical Laplacian is just one special case. Much work has been done...
recently on the well-posedness and regularity theory of such equations, see, e.g., [18, 54] and many related references. This interest is furthered to a great extent by the fact that non-local operators are generators of Lévy or Feller processes [14, 34], and their study is made possible by probabilistic and potential theory methods. On the other hand, applications to mathematical physics, such as anomalous transport [39, 51] and quantum theory [23, 33, 46], or more computationally, image reconstruction via denoising [16, 26], to name just a few, provide a continuing incentive to the development of these ideas and techniques.

The study of non-local Schrödinger equations is one aspect of this work, and some primary work on developing a related potential theory has been done in [10, 11]. In what follows we consider non-local Schrödinger operators of the form

$$H = \Psi(-\Delta) + V,$$

where $\Psi$ is a so called Bernstein function, and $V$ is a multiplication operator called potential (for details see the next section). Such operators have been considered from a combined perturbation theory and functional integration point of view in [31, 32]. When $\Psi$ is the identity function, we get back to classical Schrödinger operators. We will be interested in the properties of solutions of two eigenvalue problems. The first is a non-local Dirichlet-Schrödinger problem for a bounded convex domain $D \subset \mathbb{R}^d$, $d \geq 1$, given by

$$\begin{cases} H\varphi = \lambda\varphi & \text{in } D, \\ \varphi = 0 & \text{in } D^c, \end{cases}$$

in weak sense. In this case there is a countable set of eigenvalues

$$\lambda_1^{D,V} < \lambda_2^{D,V} \leq \lambda_3^{D,V} \leq \ldots$$

of finite multiplicities each, and a corresponding orthonormal set of eigenfunctions $\varphi_1, \varphi_2, \ldots \in \text{Dom}(H) \subset L^2(D)$. When $V \equiv 0$, the problem reduces to the non-local Dirichlet eigenvalue equation. In this case the spectrum is still discrete, and we use the notation $\lambda_1^D, \lambda_2^D, \ldots$ for the eigenvalues.

The other problem we consider is the eigenvalue equation

$$H\varphi = \lambda\varphi, \quad \text{supp } V = \mathcal{K},$$

in $L^2(\mathbb{R}^d)$, $d \geq 1$, with a bounded set $\mathcal{K} \subset L^2(\mathbb{R}^d)$. We will also be interested in the specific case of potential wells of depth $v > 0$, when $V = -v\mathbb{1}_\mathcal{K}$. In this case appropriate conditions need to be drawn on $V$ in order to have any $L^2$-eigenfunctions. The Dirichlet-Schrödinger problem can also be seen as a Schrödinger problem in full space, where the potential equals $V$ in $D$ and infinity elsewhere.

Explicit solutions of eigenvalue problems for non-local Schrödinger operators are few. In [49] this has been obtained for the operator $(-d^2/dx^2)^{1/2} + x^2$, and in [25] for $(-d^2/dx^2)^{1/2} + x^4$, both in $L^2(\mathbb{R})$. A detailed study of the asymptotic behaviour at infinity of the eigenfunctions for a large class of non-local Schrödinger operators has been made in [37, 38]. Bounds, monotonicity and continuity properties for Dirichlet eigenvalues for large classes of domains have been established in [21, 22], approximate solutions and detailed estimates for some non-local Dirichlet problems in intervals, half-spaces or boxes have been presented in [35, 36, 42, 43, 44]. Since an explicit computation of the principal Dirichlet eigenvalue or eigenfunction is not available even for the simplest cases, a study of the properties of the spectrum becomes important. Some results on the shape of eigenfunctions or solutions were obtained in [4, 41], and [50] investigates the local behaviour of eigenfunctions for potentials wells.

Our results in this paper contribute to a study of local properties of eigenfunctions of non-local Schrödinger operators. There are several reasons why information on the location of extrema of eigenfunctions is important. One closely related problem is the localization of the so called hot-spots, which has attracted much attention for classical domain Laplacians. A hot-spot is a point
in space where the solution of the heat equation in a bounded domain at a given time attains its maximum, and an object of study has been how hot-spots move in time. In the case of Neumann boundary conditions the solution approaches the second eigenfunction on the long run, and the so-called Rauch-conjecture states that the second eigenfunction attains its maximum on the boundary of the domain, thus the hot-spots in this case are expected to be located on the edge. This conjecture turned out to be false in general, but it has been proven to hold under specific assumptions on the domain, see [2, 17] and references therein. For Dirichlet boundary conditions the situation is different as now the solution of the heat equation evolves in a way to take the shape of the first (principal) eigenfunction, and the hot-spot becomes its maximizer, away from the boundary. While there are many classical results in this respect, we mention in particular [28], in which this problem is studied for bounded convex sets in $\mathbb{R}^2$, and the recent paper [15] which obtained a lower bound on the location of the maximum of the principal Dirichlet eigenfunction for the Laplacian. One implication of our results below is an improved bound, see a discussion in Remark 3.7 below.

Another special role to maximizers of principal eigenfunctions is given by their relationship with random processes generated by the kinetic part of the operator $H$. Since this eigenfunction can be chosen to be strictly positive, by its harmonicity it can be used as a Doob $h$-transform to construct a stochastic process obtained under the perturbation of $V$ of the subordinate Brownian motion generated by $\Psi(-\Delta)$. In case of a classical Schrödinger operator this is a diffusion, while for non-local cases it is a Lévy-type jump process with, in general, unbounded coefficients. In both cases the maximizer of the first eigenfunction gives the mode of the stationary probability density of this process. When the operators relate to a model of an atomic system or the flow of, e.g., contaminated groundwater in a porous soil, the maximizers acquire extra meanings related to the localization of the quantum particle or the highest-concentration point of the plume, respectively. This is further discussed in Remark 3.4 below.

As it will be seen in what follows, the localization of the extrema of Dirichlet-Schrödinger eigenfunctions is, roughly speaking, an isoperimetric-type property, determined by underlying geometric principles. In Corollary 3.4 we obtain a new Faber-Krahn type inequality for non-local Schrödinger operators as a direct consequence of the estimates on the location of extrema. This coincides up to a numerical factor with the classical Faber-Krahn inequality for Dirichlet eigenvalues and with the variant known for Schrödinger operators [19], in the sense that our constant is weaker than the known optimal value. We also obtain a variety of geometric and probabilistic bounds on the eigenvalues in the spirit of the discussions in [3, 5] and references therein and, conversely, derive a lower estimate on all moments of exit times of subordinate Brownian motion from convex domains. Applications in the direction of maximum principles are presented separately in [9].

Furthermore, we extend our study to locating the maxima of eigenfunctions for operators with potentials having compact support. We are not aware of any similar results even for classical Schrödinger operators. In this case it is natural to consider distances from the boundary edge of the support or from neighbourhoods (e.g., level sets) of the minima of the potential. We observe some interesting behaviours dependent on whether the potential is attracting or repelling.

The remainder of this paper is organized as follows. In Section 2 we discuss some properties of Bernstein functions $\Psi$ on which we rely throughout below when using the operators $\Psi(-\Delta)$ and related subordinate Brownian motions. In Section 3 first we establish some basic facts on the Dirichlet-Schrödinger eigenvalue problem, which do not seem to be available in the literature. Next we state and prove our main results in Theorems 3.1-3.2, and then discuss a number of consequences and implications in corollaries and a string of remarks. Section 4 is devoted to operators having potentials with compact support.
2. Bernstein functions of the Laplacian and subordinate Brownian motions

Now we turn to describe the above objects formally. Denote
\[ H_0 = \Psi(-\Delta), \]  
where \( \Psi \) is a Bernstein function given below. This operator can be defined via functional calculus by using the spectral decomposition of the Laplacian. It is a pseudo-differential operator whose symbol is given by the Fourier multiplier
\[ \hat{H}_0 f(y) = \Psi(|y|^2) \hat{f}(y), \quad y \in \mathbb{R}^d, \ f \in \text{Dom}(H_0), \]
with domain \( \text{Dom}(H_0) = \{ f \in L^2(\mathbb{R}^d) : \Psi(|\cdot|^2) \hat{f} \in L^2(\mathbb{R}^d) \} \). It follows by general arguments that \( H_0 \) is a positive, self-adjoint operator with core \( C_c^\infty(\mathbb{R}^d) \).

Recall that a Bernstein function is a non-negative completely monotone function, i.e., an element of the set
\[ \mathcal{B} = \left\{ f \in C^\infty((0, \infty)) : f \geq 0 \text{ and } (-1)^n \frac{d^n f}{dx^n} \leq 0, \text{ for all } n \in \mathbb{N} \right\}. \]
In particular, Bernstein functions are increasing and concave. We will make use below of the subset
\[ \mathcal{B}_0 = \left\{ f \in \mathcal{B} : \lim_{u \downarrow 0} f(u) = 0 \right\}. \]
Let \( \mathcal{M} \) be the set of Borel measures \( \mu \) on \( \mathbb{R} \setminus \{0\} \) with the property that
\[ \mu((-\infty, 0)) = 0 \text{ and } \int_{\mathbb{R} \setminus \{0\}} (y \wedge 1) \mu(dy) < \infty. \]
Notice that, in particular, \( \int_{\mathbb{R} \setminus \{0\}} (y^2 \wedge 1) \mu(dy) < \infty \) holds, thus \( \mu \) is a Lévy measure supported on the positive semi-axis. It is well-known then that every Bernstein function \( \Psi \in \mathcal{B}_0 \) can be represented in the form
\[ \Psi(u) = bu + \int_{(0, \infty)} (1 - e^{-yu}) \mu(dy) \]  
with \( b \geq 0 \), moreover, the map \( [0, \infty) \times \mathcal{M} \ni (b, \mu) \mapsto \Psi \in \mathcal{B}_0 \) is bijective. \( \Psi \) is said to be a complete Bernstein function if there exists a Bernstein function \( \overline{\Psi} \) such that
\[ \Psi(u) = u^2 \mathcal{L}(\overline{\Psi})(u), \quad u > 0, \]
where \( \mathcal{L} \) stands for the Laplace transformation. It is known that every complete Bernstein function is also a Bernstein function. Also, for a complete Bernstein function the Lévy measure \( \mu(dy) \) has a completely monotone density with respect to the Lebesgue measure. The class of complete Bernstein functions is large, including important cases such as (i) \( u^{\alpha/2}, \ \alpha \in (0, 2] \); (ii) \( (u + m^{2/\alpha})^{\alpha/2} - m, \ m \geq 0, \ \alpha \in (0, 2) \); (iii) \( u^{\alpha/2} + u^{\beta/2}, \ 0 < \beta < \alpha \in (0, 2) \); (iv) \( \log(1+u^{\alpha/2}), \ \alpha \in (0, 2) \); (v) \( u^{\alpha/2}(\log(1+u))^{\beta/2}, \ \alpha \in (0, 2), \ \beta \in (0, 2) \); (vi) \( u^{\alpha/2}(\log(1+u))^{-\beta/2}, \ \alpha \in (0, 2), \ \beta \in (0, \alpha) \). On the other hand, the Bernstein function \( 1 - e^{-u} \) is not a complete Bernstein function. For a detailed discussion of Bernstein functions we refer to the monograph [56].

Bernstein functions are closely related to subordinators, and we will use this relationship below. Recall that a one-dimensional Lévy process \( (S_t)_{t \geq 0} \) on a probability space \( (\Omega_S, \mathcal{F}_S, \mathbb{P}_S) \) is called a subordinator whenever it satisfies \( S_s \leq S_t \) for \( s \leq t \), \( \mathbb{P}_S \)-almost surely. A basic fact is that the Laplace transform of a subordinator is given by a Bernstein function, i.e.,
\[ \mathbb{E}_{\mathbb{P}_S}[e^{-uS_t}] = e^{-t\Psi(u)}, \quad t \geq 0, \]
holds, where \( \Psi \in \mathcal{B}_0 \). In particular, there is a bijection between the set of subordinators on a given probability space and Bernstein functions with vanishing right limits at zero; to emphasize this, we will occasionally write \( (S^\Psi_t)_{t \geq 0} \) for the unique subordinator associated with Bernstein function \( \Psi \). Corresponding to the examples of Bernstein functions above, the related processes are (i) \( \alpha/2 \)-stable
subordinator, (ii) relativistic $\alpha/2$-stable subordinator, (iii) sums of independent subordinators of different indices, (iv) geometric $\alpha/2$-stable subordinators (specifically, the Gamma-subordinator for $\alpha = 2$), etc. The non-complete Bernstein function mentioned above describes the Poisson subordinator.

Let $(B_t)_{t \geq 0}$ be $\mathbb{R}^d$-valued a Brownian motion on Wiener space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$, running twice as fast as standard $d$-dimensional Brownian motion, and let $(S_t^\Psi)_{t \geq 0}$ be an independent subordinator. The random process

$$\Omega_W \times \Omega_S \ni (\omega, \varpi) \mapsto B_{S_t(\varpi)}(\omega) \in \mathbb{R}^d$$

is called subordinate Brownian motion under $(S_t^\Psi)_{t \geq 0}$. For simplicity, we will denote a subordinate Brownian motion by $(X_t)_{t \geq 0}$, its probability measure for the process starting at $x \in \mathbb{R}^d$ by $\mathbb{P}^x$, and expectation with respect to this measure by $\mathbb{E}^x$. Every subordinate Brownian motion is a Lévy process, with infinitesimal generator $H_0 = \Psi(-\Delta)$. Subordination then gives the expression

$$\mathbb{P}(X_t \in E) = \int_0^\infty \mathbb{P}_W(B_s \in E) \mathbb{P}_S(S_t \in ds),$$

for every measurable set $E$.

Our main concern in what follows are some properties in the bulk of functions satisfying the eigenvalue equations (1.3-1.4) in weak sense. Specifically, we will focus on the location of extrema of eigenfunctions by using a stochastic representation of the solutions, featuring subordinate Brownian motion.

3. Constraints on the location of extrema

3.1. The Dirichlet-Schrödinger problem

In this section we assume $\mathcal{D} \subset \mathbb{R}^d$ to be a bounded open set. Consider a complete Bernstein function $\Psi$ and the operator $H_0 = \Psi(-\Delta)$ on $L^2(\mathbb{R}^d)$. The Dirichlet eigenvalue problem (1.3) for $V \equiv 0$ has been studied in various papers, including [21, 22, 36, 43, 44]. In particular, the following is known. Consider the space $C_c^\infty(\mathcal{D})$, and define the operator $H_0^\mathcal{D}$ given by the Friedrichs extension of $H_0|_{C_c^\infty(\mathcal{D})}$. It can be shown that the form-domain of $H_0^\mathcal{D}$ contains those functions that are in the form-domain of $H_0$ and are almost surely zero outside of $\mathcal{D}$. Furthermore, the operator $-H_0^\mathcal{D}$ generates the strongly continuous operator semigroup

$$T_t^\mathcal{D} = e^{-tH_0^\mathcal{D}}, \quad t \geq 0.$$  

Each operator $T_t^\mathcal{D}$ is a contraction on $L^p(\mathcal{D})$, for every $p \geq 1$, including $p = \infty$. When $\Psi$ is unbounded, $T_t^\mathcal{D}$ is a contraction also on $C_0(\mathcal{D})$. If $e^{-t\Psi(|x|^2)} \in L^1(\mathcal{D})$ for $t > 0$, then each $T_t^\mathcal{D}$ is a Hilbert-Schmidt operator, in particular, they are compact. Hence, by general theory, the equation

$$T_t^\mathcal{D} \varphi = e^{-t\lambda} \varphi, \quad t > 0,$$

is solved by a countable set of eigenvalues $\lambda_1^\mathcal{D} < \lambda_2^\mathcal{D} \leq \lambda_3^\mathcal{D} \leq \cdots \to \infty$, of finite multiplicity each, corresponding to an orthonormal set of eigenfunctions $\varphi_1^\mathcal{D}, \varphi_2^\mathcal{D}, \ldots \in L^2(\mathcal{D})$. The principal eigenvalue $\lambda_1^\mathcal{D}$ has multiplicity one, and the principal eigenfunction $\varphi_1^\mathcal{D}$ has a strictly positive version, which we will adopt throughout. Moreover, due to strong continuity of the semigroup, the spectrum independent of $t > 0$, in particular, since $-H_0^\mathcal{D}$ is the infinitesimal generator of $\{T_t^\mathcal{D} : t \geq 0\}$, the same eigenvalues and eigenfunctions also solve (1.3) for $V \equiv 0$. It is also known that $\{T_t^\mathcal{D} : t \geq 0\}$ is the Markov semigroup of killed subordinate Brownian motion, i.e., we have

$$T_t^\mathcal{D} f(x) = \mathbb{E}^x[f(X_t)\1_{\{\tau_\mathcal{D} > t\}}], \quad x \in \mathcal{D}, t > 0, f \in L^2(\mathcal{D}),$$

where

$$\tau_\mathcal{D} = \inf\{t > 0 : X_t \notin \mathcal{D}\}$$

is the first exit time of $(X_t)_{t \geq 0}$ from $\mathcal{D}$. 


In contrast to the pure Dirichlet problem, the Dirichlet-Schrödinger problem (1.3) with \( V \neq 0 \) has been much less studied and the counterparts of the above facts do not seem to be readily available in the literature. Let \( V \in L^\infty(\mathbb{R}^d) \) and consider \( H = \Psi(-\Delta) + V. \) This operator is bounded from below, and self-adjoint on the dense domain \( \text{Dom}(\Psi(-\Delta)) \subset L^2(\mathbb{R}^d), \) with core \( C^\infty_c(\mathbb{R}^d). \) For a bounded open set \( D \subset \mathbb{R}^d \) we define the non-local Schrödinger operator \( H^{D,V} \) as the Friedrichs extension of \( H|_{C^\infty_c(D)}. \) Also, define

\[
T^{D,V}_t f(x) = \mathbb{E}^x \left[ e^{-\int_0^t V(X_s) \, ds} f(X_t) \mathbf{1}_{\{\tau_D > t\}} \right], \quad x \in D, \ t > 0, \ f \in L^2(D).
\]  (3.3)

We denote \( L^p \) norm on \( D \) by \( \| \cdot \|_{p,D} \), whereas \( \| \cdot \|_p \) denotes the \( L^p \) norm on \( \mathbb{R}^d \). We show the following properties.

**Lemma 3.1.** Consider the operators \( H^{D,V} \) and \( T^{D,V}_t, \ t > 0, \) and let \( (S^{\Psi}_t)_{t \geq 0} \) be the subordinator corresponding to the Bernstein function \( \Psi \in \mathcal{B}_0. \) The following hold:

(i) Every \( T^{D,V}_t \) is an integral operator and we have the representation

\[
T^{D,V}_t f(x) = \int_D \mathbb{E}^{x}_{p_t} \left[ p_{S^{\Psi}_t}(x-y) \mathbb{E}_{0,S^{\Psi}_t}^{x,y} \left[ e^{-\int_0^t V(B^{S^{\Psi}_t}_s) \, ds} \mathbf{1}_{\{\tau_D > t\}} \right] \right] f(y) \, dy
\]  (3.4)

where \( p_t(x) = (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}}, \) and \( \mathbb{E}_{0,S^{\Psi}_t}^{x,y} \) denotes expectation with respect to the Brownian bridge measure from \( x \) at time 0 to \( y \) at time \( s, \) evaluated at random time \( s = S^{\Psi}_t. \) Furthermore, for every \( t > 0, \) we have \( T^{D,V}_t(x,y) = T^{D,V}_t(t,y,x) \) for all \( x, y \in \mathbb{R}^d. \)

(ii) \( \{T^{D,V}_t : t \geq 0\} \) is a strongly continuous semigroup on \( L^p(D), \ p \geq 1, \) with infinitesimal generator \(-H^{D,V}.\)

(iii) Suppose that \( \Psi \) satisfies the Hartman-Wintner condition

\[
\lim_{|u| \to \infty} \frac{\Psi(|u|^2)}{\log|u|} = \infty. \tag{3.5}
\]

Then every \( T^{D,V}_t \) is a Hilbert-Schmidt operator on \( L^2(D), \) for all \( t > 0.

(iv) The map \((0, \infty) \times D \times D \ni (t, x, y) \mapsto T^{D,V}_t(t, y, x) \in \mathbb{R} \) is continuous.

(v) If \( D \) is a convex bounded open set, then for every \( f \in L^\infty(D) \) we have that \( T^{D,V}_t f \) continuous in \( D \) with value 0 on the boundary, for every \( t > 0. \)

**Proof.** (i) (3.4) follows from a standard conditioning argument, see [31, Lem. 3.4]. We define

\[
T^{D,V}_t(t, x, y) = \mathbb{E}^{x}_{p_t} \left[ p_{S^{\Psi}_t}(x-y) \mathbb{E}_{0,S^{\Psi}_t}^{x,y} \left[ e^{-\int_0^t V(B^{S^{\Psi}_t}_s) \, ds} \mathbf{1}_{\{\tau_D > t\}} \right] \right],
\]

and show that

\[
T^{D,V}_t(t, x, y) = T^{D,V}_t(t, y, x), \quad \text{for } t > 0. \tag{3.6}
\]

Consider the Brownian bridge on the interval \([0, S^{\Psi}_t]\) starting with \( x \) and ending at \( y \) given by

\[
Z^{x,y}_{S^{\Psi}_t} = (1 - \frac{s}{S^{\Psi}_t}) x + \frac{s}{S^{\Psi}_t} y + B_s - \frac{s}{S^{\Psi}_t} B^{S^{\Psi}_t}_s,
\]

where \( (B_t)_{t \geq 0} \) is the Brownian motion running twice as fast as the standard Brownian motion, independent of the subordinator \( (S^{\Psi}_t)_{t \geq 0}. \) A change of variable gives

\[
\int_0^t V(Z^{x,y}_{S^{\Psi}_s}) \, ds = \int_0^t V(Z^{x,y}_{S^{\Psi}_{s+}}) \, ds,
\]

and we also have

\[
Z^{x,y}_{S^{\Psi}_s} \in D, \ \forall s \in [0, t] \iff Z^{x,y}_{S^{\Psi}_{s-}} \in D, \ \forall s \in [0, t].
\]
Therefore to show (3.6) we only need to show that
\[ (Z_{S_t}^{x,y} |_{[0,t]}, S_t^\Psi) \overset{d}{=} \left( Z_{S_t}^{x,y} |_{[0,t]}, S_t^\Psi \right). \] (3.7)  

This can be shown by using the fact that for any Lévy process \((L_t)_{t \geq 0}\) starting at zero we have
\[ (L_{t-}, L_t) \overset{d}{=} (L_t - L, L_t). \] (3.8)

First we show (3.7) using (3.8). Since the Brownian motion is independent of \((S_t^\Psi)_{t \geq 0}\), we get the following equalities in distribution
\[
Z_{S_t}^{x,y} \overset{d}{=} (1 - \frac{S_t^\Psi - S_s^\Psi}{S_t^\Psi})x + \frac{S_t^\Psi - S_s^\Psi}{S_t^\Psi} y + B_{S_t} - B_{S_s} - \frac{S_t^\Psi - S_s^\Psi}{S_t^\Psi} B_{S_t}.
\]

This proves (3.7). Next we come to (3.8). It suffices to show that the finite dimensional distributions coincide. Consider \(t > s_1 > s_2 > \cdots > s_k \geq 0\) and \(\xi_i \in \mathbb{R}\) for \(i = 1, \ldots, k+1\). Then it is seen that
\[
\xi_1 L_{t-s_1} + \cdots + \xi_k L_{t-s_k} + \xi_{k+1} L_t = \sum_{i \geq 1} \xi_i L_{t-s_1} + \sum_{i \geq 2} \xi_i (L_{t-s_2} - L_{t-s_1}) + \cdots + (\xi_k + \xi_{k+1})(L_{t-s_k} - L_{t-s_k-1}) + \xi_{k+1}(L_t - L_{t-s_k})
\]

and
\[
\xi_1 (L_t - s_1) + \cdots + \xi_k (L_t - L_{s_k}) + \xi_{k+1} L_t = \sum_{i \geq 1} \xi_i (L_t - s_1) + \sum_{i \geq 2} \xi_i (L_{s_1} - L_{s_2}) + \cdots + (\xi_k + \xi_{k+1})(L_{s_k+1} - L_{s_k}) + \xi_{k+1} L_{s_k}.
\]

On the other hand,
\[
(L_{t-s_1}, L_{t-s_2} - L_{t-s_1}, \cdots, L_{t} - L_{t-s_k}) \overset{d}{=} (L_t - L_{s_1}, L_{s_1} - L_{s_2}, \cdots, L_{s_k}).
\]

Thus \((L_{t-s_1}, \cdots, L_{t-s_k}, L_t)\) has the same characteristic function as \((L_t - L_{s_1}, \cdots, L_t - L_{s_k}, L_t)\), implying (3.8).

(ii) We establish the Chapman-Kolmogorov relation
\[ T_{D,V}^D(t+s,x,y) = \int_D T_{D,V}(t,x,u)T_{D,V}(s,u,y) \, du, \quad t,s > 0, \quad x,y \in \mathbb{R}^d. \] (3.9)

Denote
\[
\Xi(r,z,y) = E_{0,S^\Psi}^{z,y} \left[ e^{-\int_0^r V(Z_u) \, du} \mathbb{I}_{\{\tau_D > r\}} \right],
\]

where \((Z_t)_{t \geq 0}\) denotes the Brownian bridge as defined above. Let \((S_t^\Psi)_{t \geq 0}\) be a subordinator given by Bernstein function \(\Psi\), independent of \(S_t^\Psi, B, Z\). Then we have
\[
\begin{align*}
E_{0,S^\Psi}^{0} & \left[ p_{S_t}^{x,y}(x-y)E_{0,S^\Psi}^{x,y} \left[ e^{-\int_0^{\tau_D} V(Z_u) \, du} \mathbb{I}_{\{\tau_D > t+s\}} \right] \right] \\
& = E_{0,S^\Psi}^{0} \left[ p_{S_{t+s}}^{x,y} (x-y)E_{0,S^\Psi}^{x,y} \left[ \mathbb{I}_{\{\tau_D > t\}} e^{-\int_0^{\tau_D} V(Z_u) \, du} \mathbb{E}_{0,S^\Psi}^{S_{t+s}^{x,y} - S_t^{x,y}} \left[ e^{-\int_0^{\tau_D} V(Z_{u+t-s}) \, du} \mathbb{I}_{\{\tau_D > s\}} \right] \right] \right]
\end{align*}
\]
\begin{align*}
&= \mathbb{E}^0_{FS} \left[ p_{\tilde{S}^\Phi_t + \tilde{S}^\Phi_t}(x-y) \mathbb{E}^x_y \mathbb{E}^{x,y}_{0,S^\Phi_t + \tilde{S}^\Phi_t} \left[ \mathbb{I}_{\{\tau_D > t\}} e^{-\int_0^t V(Z_{S^\Phi_u}) du} \mathbb{E}^{Z_{S^\Phi_t}, y}_{0,S^\Phi_t} \left[ e^{-\int_0^t V(Z_{S^\Phi_u}) du} \mathbb{I}_{\{\tau_D > s\}} \right] \right] \right] \\
&= \mathbb{E}^0_{FS} \left[ p_{\tilde{S}^\Phi_t + \tilde{S}^\Phi_t}(x-y) \mathbb{E}^x_y \mathbb{E}^{x,y}_{0,S^\Phi_t + \tilde{S}^\Phi_t} \left[ \mathbb{I}_{\{\tau_D > t\}} e^{-\int_0^t V(Z_{S^\Phi_u}) du} \mathbb{E}^{Z_{\tilde{S}^\Phi_s}, Z_{S^\Phi_y}}_{S^\Phi_s} \left[ \mathbb{I}_{\{\tau_D > s\}} \right] \right] \right] \\
&= \mathbb{E}^0_{FS} \left[ p_{\tilde{S}^\Phi_t + \tilde{S}^\Phi_t}(x-y) \mathbb{E}^x_y \left[ \mathbb{I}_{\{\tau_D > t\}} e^{-\int_0^t V(B_{S^\Phi_u}) du} \Xi(\tilde{S}^\Phi_s, B_{S^\Phi_y}, y)p_{\tilde{S}^\Phi_s}(B_{S^\Phi_t} - y) \frac{1}{p_{\tilde{S}^\Phi_t + \tilde{S}^\Phi_t}(x-y)} \right] \right] \\
&= \mathbb{E}^0_{FS} \left[ \mathbb{E}^x \left[ \mathbb{I}_{\{\tau_D > t\}} e^{-\int_0^t V(B_{S^\Phi_u}) du} \Xi(\tilde{S}^\Phi_s, B_{S^\Phi_y}, y)p_{\tilde{S}^\Phi_s}(B_{S^\Phi_t} - y) \right] \right] \\
&= \mathbb{E}^0_{FS} \left[ \mathbb{E}^x \left[ \mathbb{I}_{\{\tau_D > t\}} e^{-\int_0^t V(B_{S^\Phi_u}) du} T^D,V(s, X_t, y) \right] \right] \\
&= \int_{\mathcal{D}} T^D,V(t, x, u)T^D,V(u, y, y) du,
\end{align*}

where the first equality follows from the Markov property of Brownian bridge, in the fourth line we used [57, Prop. A.1], and the sixth line follows by taking expectation with respect to \((\tilde{S}^\Phi_t)_{t \geq 0}\). Strong continuity follows along the line of [57, Prop. 3.3].

(iii) The symmetry of \(T^D,V(t, x, y)\) implies that \(T^D,V_t\) is a self-adjoint operator on \(L^2(\mathcal{D})\). Let \(q_t(x, y)\) be the transition density of \((X_t)_{t \geq 0}\). Then the Hartman-Wintner condition (3.5) implies that for every \(t > 0\), \(q_t(\cdot)\) is bounded and continuous [30, 40] and therefore, \(q_t(x, \cdot) \in L^2(\mathbb{R}^d)\). Indeed, for \(t > 0\),

\[
q_{2t}(x, x) = \int_{\mathbb{R}^d} q_t(x - y)q_t(y - x) dy = \int_{\mathbb{R}^d} q_t^2(x - y)dy < \infty.
\]

The transition density for the process \((X_t)_{t \geq 0}\) killed upon the first exit from \(\mathcal{D}\) is given by Hunt’s formula

\[
q_{t}^D(x, y) = q_t(x, y) - \mathbb{E}^x \left[ q_{t - \tau_D}(X_{\tau_D}, y) \mathbb{I}_{\{\tau_D > t\}} \right] \quad t > 0,\ x, y \in \mathbb{R}^d.
\]

In particular, \(q_{t}^D(x, y) \leq q_t(x, y)\). Since \(V\) is bounded, we obtain

\[
|T^D,V_t f(x)| \leq e^{t||V||_\infty} \int_{\mathcal{D}} |f(y)|q_{t}^D(x, y)dy \leq e^{t||V||_\infty} ||f||_{2,\mathcal{D}} ||q_t(x, \cdot)||_2.
\]

Note that \(||q_t(x, \cdot)||_2\) does not depend on \(x\). Therefore

\[
\int_{\mathcal{D} \times \mathcal{D}} (T^D,V(t, x, y))^2 dy dx \leq C_t,
\]

with a constant \(C_t > 0\), implying that \(T^D,V_t\) is a Hilbert-Schmidt operator.

(iv) We claim that for every \(t > 0\) and \(y \in \mathcal{D}\),

\[
x \mapsto T^D,V_t(t, x, y) \quad \text{is continuous in }\mathcal{D}.
\]

To show (3.10), write

\[
T^D,V_{\varepsilon}(t, x, y) = \mathbb{E}^0_{FS} \left[ \int_{\mathbb{R}^d} p_{\tilde{S}^\Phi_{t-\varepsilon}}(x-z)p_{\tilde{S}_{t-\varepsilon}}(z-y) \Xi(t-\varepsilon, z, y) \right] \\
= \mathbb{E}^0_{FS} \left[ \int_{\mathcal{D}} p_{\tilde{S}^\Phi_{t-\varepsilon}}(x-z)p_{\tilde{S}_{t-\varepsilon}}(z-y) \Xi(t-\varepsilon, z, y) \right] \\
= \mathbb{E}^0_{FS} \left[ \int_{\mathbb{R}^d} p_{\tilde{S}^\Phi_{t-\varepsilon}}(x-y)\mathbb{E}^{x,y}_{0,S^\Phi_t + \tilde{S}^\Phi_t} \left[ e^{-\int_{0}^{t-\varepsilon} V(Z_{S^\Phi_u}) du} \mathbb{I}_{\{\tau_D > t-\varepsilon\}} \right] \right] \\
= \mathbb{E}^0_{FS} \left[ \int_{\mathbb{R}^d} p_{\tilde{S}^\Phi_{t-\varepsilon}}(x-y)\mathbb{E}^{x,y}_{0,S^\Phi_t + \tilde{S}^\Phi_t} \left[ e^{-\int_{0}^{t-\varepsilon} V(Z_{S^\Phi_u}) du} \mathbb{I}_{\{\tau_D > t-\varepsilon\}} \right] \right].
\]
where $\sigma_\varepsilon$ denotes the $\varepsilon$-shift operator, and the third line above follows from [57, Cor. A.2]. It is straightforward to see that (3.10) holds for $T_{t}^{D,V}$. On the other hand, $T_{\varepsilon}^{D,V}(t,\cdot,y)$ converges to $T^{D,V}(t,\cdot,y)$ as $\varepsilon \to 0$, uniformly on the compact subsets of $D$, see for example, [57, eq. (3.21)]. This proves (3.10). The proof of (iv) can be completed employing a similar argument as in [57, Prop. 3.5] combining (3.9), (3.10) and (ii).

Finally we prove (v). Denote $\tilde{f}(t,x) = T_{t}^{D,V}f(x)$. In view of (iv) it is enough to show that for $x_n \to z \in \partial D$ we have

$$\lim_{n \to \infty} |\tilde{f}(t,x_n)| = 0. \quad \text{(3.11)}$$

Since $f$ and $V$ are bounded, we obtain from (3.3) that

$$|\tilde{f}(t,x_n)| \leq e^{\|V\|_{\infty} t}\|f\|_{\infty} \mathbb{P}^{x_n}(\tau_{D} > t).$$

Since $z \in D$ is regular, see Lemma 3.2 below, we have

$$\lim_{x_n \to z} \mathbb{P}^{x_n}(\tau_{D} > t) = 0.$$

By combining the above two equalities (3.11) follows. \hfill \Box

Now we prove the statement used in part (v) above.

### Lemma 3.2

Let $D \subset \mathbb{R}^{d}$ be a convex bounded open set, and assume that $(X_{t})_{t \geq 0}$ is not a compound Poisson process. Then all boundary points of $D$ are regular, i.e., for every $z \in \partial D$ it follows that $\mathbb{P}^{z}(\tau_{D} = 0) = 1$.

**Proof.** For a subset $I \subset \{1,2,\ldots,d\}$ we define the cone

$$C_{I} = \left\{ x \in (\mathbb{R}^{+})^{d} : x_{i} = 0 \text{ if } i \in I, \text{ and } x_{i} > 0 \text{ if } i \in I^{c} \right\}.$$

Let $C_{0}$ be the open cone corresponding to $I = \emptyset$. Denote

$$C = C_{0} \cup C_{[1]} \cup C_{[1,2]} \cup \cdots \cup C_{[1,\ldots,d]}.$$

It is easy to see that we can rotate $C \setminus \{0\}$ suitably to generate $2^{d}$ copies which are disjoint and form a partition of $\mathbb{R}^{d} \setminus \{0\}$. Let $z = 0 \in \partial D$. We may also assume that the cone $C$ defined above is in $D^{c}$. Define $U_{n} = C \cap B_{1/n}(0)$ and $\tau_{n} = \tau_{B_{1/n}(0)}$. Since $(X_{t})_{t \geq 0}$ is isotropic, we have

$$\mathbb{P}^{0}(X_{\tau_{n}} \in U_{n}) = \frac{1}{2^{d}}, \quad \forall n \geq 1.$$

Let $\tau_{D}^{z}$ be the first exit time from $D$ where $X_{0} = z$. Then

$$\cap_{m=1}^{\infty} \cup_{n \geq m} \{ X_{\tau_{n}} \in U_{n} \} \subset \{ \tau_{D}^{z} = 0 \}.$$

Hence

$$\mathbb{P}^{0}(\tau_{D}^{z} = 0) \geq \mathbb{P}^{0}(\cap_{m=1}^{\infty} \cup_{n \geq m} \{ X_{\tau_{n}} \in U_{n} \})$$

$$\geq \limsup_{n \to \infty} \mathbb{P}^{0}(X_{\tau_{n}} \in U_{n}) = \frac{1}{2^{d}}.$$

Due to the strong Markov property of $(X_{t})_{t \geq 0}$, Blumenthal’s zero-one law gives $\mathbb{P}^{0}(\tau_{D}^{z} = 0) = 1$. \hfill \Box

### Remark 3.1

We note that Lemma 3.1 can be obtained also for $\Psi$-Kato class potentials, which may have local singularities (see below). Also, further (such as contractivity, positivity improving etc) properties of $\{ T_{t}^{D,V} : t \geq 0 \}$ can be shown, which is left to the reader. The lemma can further be extended for other non-local Schrödinger operators, involving more general isotropic Lévy processes. We also note that a proof of Lemma 3.2 does not seem to be available in the literature, and it is of independent interest. A brief remark has been made in [12, Rem. 1.9] on a subset of boundary points.
From Lemma 3.1 it then follows that the Dirichlet-Schrödinger eigenvalue equation (1.3) is solved
by a countable set of eigenvalues $\lambda_1^{D,V} < \lambda_2^{D,V} \leq \lambda_3^{D,V} \leq \cdots \to \infty$ and a corresponding orthonormal set of $L^2(D)$-eigenfunctions, such that the principal eigenvalue is simple and the corresponding principal eigenfunction has a strictly positive version.

3.2. The location of extrema

In the following we will use a class of potentials, which are general enough to contain many interesting cases (such as Coulomb-type potentials), while being naturally suitable for defining Feynman-Kac semigroups. Consider the set of functions

$$
\mathcal{K}_\Psi = \left\{ f : \mathbb{R} \to \mathbb{R}^d : f \text{ is Borel measurable and } \limsup_{t \downarrow 0} \mathbb{E}^x \left[ \int_0^t |f(X_s)| ds \right] = 0 \right\}.
$$

We say that the potential $V : \mathbb{R}^d \to \mathbb{R}$ belongs to $\Psi$-Kato class whenever it satisfies

$$
V_-, V_+ \in \mathcal{K}_\Psi \quad \text{and} \quad V_+ \in \mathcal{K}_{\Psi, \text{loc}}, \quad \text{with} \quad V_+ = \max\{V, 0\}, \quad V_- = \min\{V, 0\},
$$

where $V_+ \in \mathcal{K}_{\Psi, \text{loc}}$ means that $V_+1_C \in \mathcal{K}_\Psi$ for all compact sets $C \subset \mathbb{R}^d$, and $(X_t)_{t \geq 0}$ is the Lévy process generated by $\Psi(-\Delta)$. It is direct to see that $L_{\text{loc}}^\infty(\mathbb{R}^d) \subset \mathcal{K}_\Psi$, moreover, by stochastic continuity of $(X_t)_{t \geq 0}$ also $\mathcal{K}_{\Psi, \text{loc}} \subset L_{\text{loc}}^\infty(\mathbb{R}^d)$. By standard arguments based on Khasminskii’s Lemma, for a $\Psi$-Kato class potential $V$ it follows that there exist suitable constants $C_1(\Psi, V), C_2(\Psi, V) > 0$ such that

$$
\sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ e^{-\int_0^t V(X_s) ds} \right] \leq \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ e^{\int_0^t V_-(X_s) ds} \right] \leq C_1 e^{C_2 t}, \quad t > 0.
$$

For further details we refer to [32, Sect. 4] and [48].

For Bernstein functions we will use the following property repeatedly below, which has been introduced in [13].

\textbf{Assumption 3.1.} The function is said to satisfy a weak local scaling (WLSC) property with parameters $\mu > 0$ and $\varepsilon \in (0, 1]$, if

$$
\Psi(\gamma u) \geq \varepsilon \gamma^\mu \Psi(u), \quad u > 0, \gamma \geq 1.
$$

We will show some typical examples of Bernstein functions satisfying Assumption 3.1 further below in this section.

Now we present two expressions of the main result of this section. The first uses $\Psi$-Kato class potentials $V$ and a restricted class of $\Psi$, the second uses a more general class of Bernstein functions $\Psi$ and bounded potentials.

\textbf{Theorem 3.1.} Let $\Psi \in \mathcal{B}_0$ satisfy Assumption 3.1 with $\mu > 0$ and $\varepsilon \in (0, 1]$. Let $V \in \mathcal{K}_\Psi$ be a $\Psi$-Kato class potential, with $V^- \in L^p(\mathbb{R}^d)$, $p > d/2\mu$. Also, let $\varphi$ be a non-zero solution of (1.3) at eigenvalue $\lambda^{V,D}$. Assume that $|\varphi|$ attains a global maximum at $x^* \in D$, and denote $r = \text{dist}(x^*, \partial D)$ and $\eta = 1 - \frac{d}{2\mu p}$. Then there exists a constant $\Theta_1 > 0$, dependent on $d, \mu, \varepsilon, \eta, \text{inrad} D$, and a constant $\Theta_2 > 0$, dependent on $\eta$ only, such that

$$
\Theta_1 \|V^-\|_p^{1/p} - \inf_D V^+ + \lambda^{V,D} \geq \Theta_2 \Psi(r^{-2}).
$$

The proof of Theorem 3.1 is simpler if the potential $V$ is bounded. Moreover, one can allow a larger class of $\Psi$, not necessarily satisfying WLSC, and the dependence of $\Theta_1$ on the domain parameters can be waived when $V \in L^\infty(D)$. This is obtained in the following theorem.
(T3.1) **Theorem 3.2.** Let $\Psi \in B_0$, $V \in L^\infty(\mathbb{R}^d)$, and $\varphi$ be a non-zero solution of (1.3) at eigenvalue $\lambda_{V,D}$. Assume that $|\varphi|$ attains a global maximum at $x^* \in D$, and denote $r = \text{dist}(x^*, \partial D)$. Then there exists a universal constant $\theta > 0$, independent of $D$, $x^*$, $V$, $\Psi$ and the dimension $d$, such that

$$
\|V^-\|_{\infty, D} - \inf_D V^+ + \lambda_{V,D} \geq \theta \Psi (r^{-2}),
$$

with

$$
\theta = - \min_{\kappa > 1} \frac{1}{\kappa} \log \left(1 - F(-1)(1 - e^{1-\kappa})\right) \approx 0.0833,
$$

where $F$ is the probability distribution function of a Gaussian random variable $N(0, 2)$. In particular, if $\Psi$ is strictly increasing, then

$$
\text{dist}(x^*, \partial D) \geq \frac{1}{\sqrt{\Psi^{-1} \left(\|V^-\|_{\infty, D} - \inf_D V^+ + \lambda_{V,D}\right)}}.
$$

Next we turn to proving these theorems. For technical reasons we start by showing first the latter theorem.

**Proof of Theorem 3.2.** Let $\tau_D$ be the first exit time of $(X_t)_{t \geq 0}$ from $D$, as defined in (3.2). Using the eigenvalue equation and the representation (3.3), we have that

$$
|\varphi(x^*)| \leq e^{\lambda_{V,D} t} \mathbb{E}^x [e^{-\int_0^t V(X_s) ds} |\varphi(X_t)| \mathbb{1}_{\{t < \tau_D\}}] \leq |\varphi(x^*)| e^{\lambda_{V,D} t (\|V^-\|_{\infty, D} - \inf_D V^+) t} \mathbb{P}^x (\tau_D > t),
$$

that is,

$$
e^t (\|V^-\|_{\infty, D} - \inf_D V^+ + \lambda_{V,D}) \mathbb{P}^x (\tau_D > t) \geq 1, \quad t \geq 0.
$$

We choose

$$
t = \frac{\kappa}{\Psi (r^{-2})}
$$

with a suitable $\kappa$, which will be justified below, and show that for this $t$ we have

$$
\mathbb{P}^x (\tau_D > t) < \delta < 1,
$$

where $\delta$ does not depend on $x^*$, $D$.

Let $z \in \partial D$ be such that $\text{dist}(x^*, z) = r$, and consider the half-space $\mathcal{H} \subset D^c$ intersecting $D$ at $z$. Note that this is made possible by the convexity of $D$, and

$$
\mathbb{P}^x (\tau_D \leq t) \geq \mathbb{P}^x (X_t \in \mathcal{H})
$$

holds. We assume with no loss of generality that $\mathcal{H}$ is perpendicular to the $x$-axis, $x^* = 0$ and $z = (r, 0, \ldots, 0)$. This is possible, since we can inscribe a ball of radius $r$ in $\bar{D}$ centered at $x^*$ and $\mathcal{H}$ would be a tangent plane to it at the point $z$. Therefore, we have for $s \geq r^2$ that

$$
\mathbb{P}_W (B_s \in \mathcal{H}) = \mathbb{P}_W (B_{s}^1 \geq r) = \frac{1}{\sqrt{4\pi}} \int_{\frac{r^2}{s}}^{\infty} e^{-\frac{y^2}{4}} dy \geq \frac{1}{\sqrt{4\pi}} \int_{1}^{\infty} e^{-\frac{y^2}{4}} dy = F(-1),
$$

where $(B_t^1)_{t \geq 0}$ denotes a one-dimensional Brownian motion running twice as fast as standard Brownian motion, and $F$ is the probability distribution function of a Gaussian random variable with mean 0 and variance 2. Using the subordination formula (2.4) and the uniform estimate (3.21), we have

$$
\mathbb{P}^x (X_t \in \mathcal{H}) = \int_0^\infty \mathbb{P}_W (B_s \in \mathcal{H}) \mathbb{P}^x (S^w_t \in ds) \geq \int_0^\infty \mathbb{P}_W (B_s \in \mathcal{H}) \mathbb{P}^x (S^w_t \in ds) \geq F(-1) \mathbb{P} (S^w_t \geq r^2).
$$
By (2.3) and (3.19) we have
\[ \mathbb{P}_S(S_t^\Psi \leq r^2) = \mathbb{P}_S(e^{-r S_t^\Psi} \geq e^{-1}) \leq e \mathbb{E}_\mathbb{P}_S[e^{-r S_t^\Psi}] = e^{1-t \Psi(r^2)} = e^{1-\kappa} . \]
Hence with \( \kappa > 1 \) we obtain \( \mathbb{P}_S(S_t^\Psi \leq r^2) < 1 \), and thus (3.20) holds with \( \delta = 1 - F(-1)(1 - e^{-1-\kappa}) \), independently on \( r \). This then implies (3.15) with constant prefactor
\[ \theta_\kappa = -\frac{1}{\kappa} \log \left( 1 - F(-1)(1 - e^{-1-\kappa}) \right) , \]
which on optimizing over \( \kappa \) gives the constant (3.16).

\[ \square \]

**Proof of Theorem 3.1.** The key estimate for the proof is the following improvement of (3.13): for any \( \kappa > 0 \) there exists a constant \( C_1 > 0 \), dependent on \( \kappa_1, d, \mu, \xi \), satisfying for \( t \in [0, \kappa_1] \) and \( \vartheta > 0 \)
\[ \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ e^{\int_0^t \vartheta V^-(X_s) \, ds} \right] \leq m_\vartheta e^{\left( C_1 \vartheta \| \eta \|_{\mu} \| \Gamma(\eta) \|_{\mu} \right)^{1/\delta} t} , \]
where \( \eta = 1 - \frac{d}{3\mu} \) and \( m_\vartheta \) depends only on \( \eta \). First we complete the proof of the theorem assuming (3.22).

Choose \( \kappa_1 = \frac{2}{\| \text{inrad} D \|^{-2}} \). Suppose that \( r = \text{dist}(x^*, D) \) and let \( t = \frac{2}{\| \text{inrad} D \|^2} \leq \kappa_1 \). Then using (3.3) and Hölder inequality, we obtain for \( \vartheta > 1 \) that
\[ 1 \leq e^{t \left( \lambda V : D - \inf_D V^+ \right)} \mathbb{E}^{x^*} \left[ e^{\int_0^t \vartheta V^-(X_s) \, ds} \right] \leq \left( \frac{m_\vartheta}{\vartheta} \right)^{1/\delta} \left( \mathbb{E}^{x^*} \left( \tau_D > t \right) \right)^{\frac{\delta - 1}{\delta}} . \]
Hence from (3.20), (3.22) and (3.23) we see that
\[ 1 \leq \delta^{\frac{\delta - 1}{\delta}} (m_\vartheta)^{1/\delta} \exp \left( t \left( \lambda V : D - \inf_D V^+ + \frac{1}{\vartheta} (C_1 \vartheta \| \eta \|_{\mu} \| \Gamma(\eta) \|_{\mu} \right)^{1/\delta} \right) \]
\[ = \delta \left( \frac{m_\vartheta}{\vartheta} \right)^{1/\delta} \exp \left( t \left( \lambda V : D - \inf_D V^+ + \frac{1}{\vartheta} (C_1 \vartheta \| \eta \|_{\mu} \| \Gamma(\eta) \|_{\mu} \right)^{1/\delta} \right) . \]
Since \( \delta < 1 \) and \( \lim_{\vartheta \to \infty} \left( \frac{m_\vartheta}{\vartheta} \right)^{1/\delta} = 1 \), we can choose \( \vartheta \) large enough such that
\[ \delta_1 = \delta \left( \frac{m_\vartheta}{\vartheta} \right)^{1/\delta} < 1 . \]
Thus we obtain
\[ \log \frac{1}{\delta_1} \leq t \left( \lambda V : D - \inf_D V^+ + \frac{1}{\vartheta} (C_1 \vartheta \| \eta \|_{\mu} \| \Gamma(\eta) \|_{\mu} \right)^{1/\delta} \right) , \]
implying
\[ \left( \frac{1}{2} \log \frac{1}{\delta_1} \right) \Psi(r^{-2}) \leq \lambda V : D - \inf_D V^+ + \frac{1}{\vartheta} (C_1 \vartheta \| \eta \|_{\mu} \| \Gamma(\eta) \|_{\mu} \right)^{1/\delta} \].
This gives (3.14) for
\[ \Theta_1 = \frac{1}{\vartheta} (C_1 \vartheta \| \eta \|_{\mu} \| \Gamma(\eta) \|_{\mu} \right)^{1/\delta} \text{ and } \Theta_2 = \frac{1}{2} \log \frac{1}{\delta_1} . \]

Now we proceed to establish (3.22). Since \( \Psi \) has the WLSC property, the characteristic exponent \( \Phi(r) = \Psi(r^2) \) also has the WLSC property, namely
\[ \Phi(\gamma u) \geq c_2 \gamma^{2\mu} \Phi(u), \quad \text{for all } u > 0 \text{ and } \gamma \geq 1 . \]
Thus by [13, Prop. 19] there exists a constant \( K_1 \), dependent on \( d, \mu, c_2 \), satisfying
\[ q_t(x, y) = q_t(|x - y|) \leq K_1 \left( \Phi^{-1} \left( \frac{1}{t} \right) \right)^d, \quad \forall t > 0 . \]
Here \( q_t(x, y) \) denotes the transition density function of \( (X_t)_{t \geq 0} \). On the other hand, from the WLSCore property of \( \Phi \) it follows that
\[
\Phi^{-1}(\lambda) \leq \lambda^{\frac{1}{2\mu}} \frac{u}{\Phi(u)^{\frac{1}{2\mu}}} \quad \text{for all } \lambda \geq \Phi(u), \ u > 0.
\]
Choose \( \nu > 0 \) and denote \( \nu_1 = \frac{\nu}{(\Phi(\nu))^{\frac{1}{2\mu}}} \). Then for \( s \geq \Phi(\nu) \) we obtain
\[
\Phi^{-1}(s) \leq \nu_1 s^{-2\mu}.
\]
Hence, using the above estimate in (3.24) we get that
\[
q_t(x, y) \leq K_2 t^{-\frac{d}{(\nu_1)^{-2}}}, \quad t \leq \frac{1}{\Psi(\nu)},
\]
where \( K_2 \) depends on \( d, \mu \) and \( \nu_1 \). Let \( \kappa_1 \) be positive and choose \( \nu = \sqrt{\Psi^{-1}(\frac{1}{\kappa_1})} \). With this choice of \( \nu \) we have from (3.26) that
\[
q_t(x, y) \leq K_2 t^{-\frac{d}{\nu}}, \quad t \leq \kappa_1.
\]
For every \( t \in (0, \kappa_1) \) and \( f \in L^p(\mathbb{R}^d) \) we have
\[
\mathbb{E}^x [f(X_t)] \leq \|f\|_p \left[ \int_{\mathbb{R}^d} (q_t(|x-y|))^{p'} \, dy \right]^{\frac{1}{p'}} \\
\leq K_2^{\frac{1}{p}} \|f\|_p t^{-\frac{d}{p'}} \left[ \int_{\mathbb{R}^d} q_t(|x-y|) \, dy \right]^{\frac{1}{p'}} = K_3 \|f\|_p t^{-\frac{d}{xp'}},
\]
where \( p' = \frac{p}{p-1} \), \( K_3 = K_2^{\frac{1}{p}} \), and in the second line above we used (3.27). Let now \( 0 \leq s_1 \leq \ldots \leq s_k, \ k \in \mathbb{N} \). Using the Markov property of \( (X_t)_{t \geq 0} \) with respect to its natural filtration \( (\mathcal{F}_t)_{t \geq 0} \), for \( f \geq 0 \) we obtain
\[
\mathbb{E}^x[f(X_{s_1}) \cdots f(X_{s_k})] = \mathbb{E}^x[f(X_{s_1}) \cdots f(X_{s_{k-1}})] \mathbb{E}^x[f(X_{s_k}) \mid \mathcal{F}_{s_{k-1}}] \\
= \mathbb{E}^x[f(X_{s_1}) \cdots f(X_{s_{k-1}})] \mathbb{E}^x[f(X_{s_k-s_{k-1}})] \\
\leq K_3 \|f\|_p (s_k - s_{k-1})^{-\frac{d}{xp'}} \mathbb{E}^x[f(X_{s_1}) \cdots f(X_{s_{k-1}})] \\
\leq \ldots \leq (K_3 \|f\|_p) (s_1 - s_1)^{-\frac{d}{xp'}} \cdots (s_k - s_{k-1})^{-\frac{d}{xp'}}.
\]
Hence (compare [48, Lem. 4.51] in the second edition)
\[
\mathbb{E}^x \left[ \frac{1}{k!} \left( \int_0^t f(X_s) \, ds \right)^k \right] \\
\leq \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_k} ds_k \mathbb{E}^x[f(X_{s_1}) f(X_{s_2}) \cdots f(X_{s_k})] \\
\leq K_3^k \|f\|_p^k \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_k} ds_k (s_2 - s_1)^{-\frac{d}{xp'}} \cdots (s_k - s_{k-1})^{-\frac{d}{xp'}} \\
= \frac{(K_3 \|f\|_p^k \Gamma(\eta))^k}{\Gamma(1 + k\eta)}, \quad t \leq \kappa_1,
\]
where \( \eta = 1 - \frac{d}{2xp'} > 0 \), by our choice of \( p \). Recall the Mittag-Leffler function
\[
\mathcal{M}_\beta(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(1 + \beta k)}
\]
Corollary 3.2. Suppose that $\lambda$ the conditions of Theorem 3.2. Suppose that $\lambda \in [0, \kappa_1]$, 

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ e^{\int_0^t f(X_s) \, ds} \right] \leq M_\eta(K_3 \|f\|_p \Gamma(\eta)).$$

(3.28) \[E^3.2G\]

It is also known that for some constant $m_\eta$, dependent only on $\eta$, 

$$M_\eta(x) \leq m_\eta e^{x^{1/\eta}}, \quad x \geq 0,$$

holds. Thus, using (3.28) we have for $t \leq \kappa_1$ that 

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ e^{\int_0^t f(X_s) \, ds} \right] \leq m_\eta e^{(K_3 \|f\|_p \Gamma(\eta))^{\frac{1}{\eta}} t}.$$

(3.29) \[E^3.2H\]

Putting $f = \partial V^-$ in (3.29), we obtain (3.22). \[\square\]

The dependence of $\Theta_1$ on $\nu_1$ is due to the factor 

$$\nu_1 = \frac{\nu}{(\Phi(\nu))^{\frac{1}{2p}}},$$

which appears in (3.25). For $\Psi(u) = u^{\alpha/2}$, however, $\nu_1$ does not depend on $\nu$. Thus we have the following improvement to Theorem 3.1.

Corollary 3.1. Suppose that $\Psi(u) = u^{\alpha/2}$. Moreover, assume that $V$ is a $\Psi$-Kato class function with $V^- \in L^p(\mathbb{R}^d), p > \frac{d}{\alpha}$. Let $\varphi$ be a non-zero solution of (1.3) at eigenvalue $\lambda^{V,D}$. Assume that $|\varphi|$ attains a global maximum at $x^* \in D$, and denote $r = \text{dist}(x^*, \partial D)$. Then there exist $\Theta_1, \Theta_2$, dependent on $d, \alpha, \gamma, \eta$, and $\Theta_2$, dependent on $\eta$, such that 

$$\Theta_1 \|V^-\|_p^{1/\eta} - \inf_{D} V^+ + \lambda^{V,D} \geq \Theta_2 \Psi(r^{-2})$$

holds.

Remark 3.2. For classical Schrödinger operators we have $\Psi(u) = u$, for which Theorem 3.2 implies (1.1), possibly with a different constant $c$. Also, for fractional Schrödinger operators we have $\Psi(u) = u^{\alpha/2}$, which reproduces the result obtained in [8]. Formulae (3.15)-(3.17) equally apply for $V \equiv 0$, in which case the statement refers to the Dirichlet eigenfunctions and eigenvalues.

Example 3.1. Some important examples of $\Psi$ satisfying Assumption 3.1 include:

(i) $\Psi(u) = u^{\alpha/2}, \alpha \in (0, 2], \mu = \frac{\alpha}{2}$.

(ii) $\Psi(u) = (u + m)^{\alpha/2} - m, m > 0, \alpha \in (0, 2], \mu = \frac{\alpha}{2}$.

(iii) $\Psi(u) = u^{\alpha/2} + u^{\beta/2}, \alpha, \beta \in (0, 2], \mu = \frac{\alpha}{2} + \frac{\beta}{2}$.

(iv) $\Psi(u) = u^{\alpha/2}(\log(1 + u))^{\beta/2}, \alpha \in (0, 2), \beta \in (0, 2 - \alpha), \mu = \frac{\alpha}{2} - \frac{\beta}{2}$.

(v) $\Psi(u) = u^{\alpha/2}(\log(1 + u))^{\gamma/2}, \alpha \in (0, 2], \beta \in (0, 2), \mu = \frac{\alpha}{2} - \frac{\beta}{2}$ (Since for $\gamma \geq 1, u > 0, (1 + u)^{\gamma} \geq (1 + \gamma u) \Rightarrow u^{\beta/2}(\log(1 + u))^{\beta/2} \leq (\log(1 + \gamma u))^{\beta/2}$.)

3.3. Consequences on the spectrum

The above theorems have a number of implications on the eigenvalues and related quantities. Here we discuss these implications involving an interplay of survival times of paths and geometric features.

Corollary 3.2. Let $\varphi$ be an eigenfunction corresponding to eigenvalue $\lambda^{V,D}$ of $\Psi(-\Delta) + V$ under the conditions of Theorem 3.2. Suppose that $\lambda^{V,D} > 0$. Then we have 

$$\int_0^\infty \mathbb{E}^{x^*} \left[ e^{-\int_0^t V(X_s) \, ds} \mathbb{1}_{\tau_D > t} \right] \, dt \geq \frac{1}{\lambda^{V,D}},$$

(3.30) \[C^3.1A\]

where $x^*$ is a maximizer of $|\varphi|$ in $D$. 

(3.1)
Proof. From the proof of Theorem 3.2 we have
\[ \mathbb{E}^{x^*} \left[ e^{-\int_0^t V(X_s) \, ds} \mathbb{1}_{\{\tau_D > t\}} \right] \geq e^{-\lambda t}, \quad t \geq 0. \] (3.31)

By integrating both sides in \( t \) on \((0, \infty)\) we obtain (3.30). \( \square \)

Note that the left hand side of (3.30) gives the mean survival time of the process \((X_t)_{t \geq 0}\) starting from \(x^*\), perturbed by the potential \(V\), thus the above result gives a probabilistic bound on the Dirichlet-Schrödinger eigenvalues.

Remark 3.3. Using the trivial bound
\[ \text{dist}(x^*, \partial D) \leq \text{inrad} D, \]
invoking the inradius of \(D\), we get the geometric constraint
\[ \lambda^{V,D} \geq \theta \Psi \left( \frac{1}{(\text{inrad} D)^2} \right) - \|V\|_{\infty} + \inf_D V^+ \]
on the bottom of the spectrum.

Remark 3.4. Since \(\Psi^{-1}\) is an increasing function, the bound (3.17) can be interpreted as saying that if the potential is not strong enough, the global extrema of \(\varphi\) cannot be too close to the boundary. Intuitively it is clear that one can decrease \(\text{dist}(x^*, \partial D)\), for instance, by a potential which has a hole close to the boundary, that is deep enough to make the process stay in that region with a sufficiently high probability, preventing it to hit the boundary too soon and get killed. It is seen that the condition only requires sufficient strength of the potential and no details on its local behaviour. There is also a probabilistic interpretation of relation (3.15). From [55, Rem. 4.8] we find that
\[ \frac{c_1}{\Psi^{1/2} \varphi_1} \leq \mathbb{E}^{x^*}[\tau_{B_r}(x)] \leq \frac{c_2}{\Psi^{1/2} \varphi_1}, \]
for some constants \(c_1, c_2 > 0\) depending only on \(d\). A combination with (3.30) then implies that the inequality makes a comparison of the mean survival time of the process starting from \(x^*\) perturbed by the potential with the mean survival time of the free (unperturbed) process, involving the proportionality constant \(\theta\). Note that since the principal eigenfunction \(\varphi_1\) is strictly positive, by the Doob \(h\)-transform \(f \mapsto \varphi_1 f\), \(f \in L^2(D)\), we can construct a random process generated by the operator \(\tilde{H} f = \frac{1}{\varphi_1} H(\varphi_1 f)\) whose stationary measure is \(\varphi_1^2 dx\). The location \(x^*\) of a global maximum of \(\varphi_1\) then corresponds to a mode of the stationary density of the process conditioned never to exit the domain \(D\).

Next we consider the principal Dirichlet eigenvalues in the absence of a potential.

Corollary 3.3. Let \(V = 0\) and consider the principal eigenvalue \(\lambda_1^D\) of the Dirichlet problem (1.3) for \(\Psi(-\Delta)\).

(i) We have
\[ \lambda_1^D \geq \left( \frac{\Gamma(p+1)}{\sup_{x \in D} \mathbb{E}^x[\tau_D]} \right)^{1/p}, \] (3.32)
for every \(p \geq 1\).

(ii) Let \(\Psi \in B_0\) be a complete Bernstein function. Then there exist positive universal constants \(C_1, C_2\), dependent only on \(d\), such that
\[ \frac{C_1}{\Psi([\text{inrad} D]^{-2})} \leq \sup_{x \in D} \mathbb{E}^x[\tau_D] \leq \frac{C_2}{\Psi([\text{inrad} D]^{-2})}. \] (3.33)
(iii) There exists a constant $C_3 > 0$, dependent on $d$, such that

$$\frac{1}{\sup_{x \in D} \mathbb{E}^x[\tau_D]} \leq \lambda_{1, \text{Lap}}^P \leq \frac{C_3}{\sup_{x \in D} \mathbb{E}^x[\tau_D]}.$$  \hfill (3.34)  

Proof. Let $p \geq 1$. To obtain (i) multiply both sides of (3.31) by $pt^{p-1}$ and integrate with respect to $t$ over $(0, \infty)$.

Next consider (ii). To prove (3.33) first note that by the domain monotonicity property we have

$$\lambda_{1, \text{Lap}}^P \leq \frac{\kappa_1}{|\text{inrad } D|^2},$$

where $\kappa_1 = \lambda_{1, \text{Lap}}^B$ is the Dirichlet principal eigenvalue in the unit ball, and $\lambda_{1, \text{Lap}}^P$ denotes the principal Dirichlet eigenvalue of the Laplacian in $D$. Therefore by [21] we obtain

$$\lambda_1^P \leq \Psi(\lambda_{1, \text{Lap}}^P) \leq \Psi\left(\frac{\kappa_1}{|\text{inrad } D|^2}\right).$$

Thus, using (3.32) for $p = 1$, we have

$$\frac{1}{\Psi(\kappa_1^{-1}|\text{inrad } D|^{-2})} \leq \sup_{x \in D} \mathbb{E}^x[\tau_D].$$

From the Laplace transform of $(S_t^\Psi)_{t \geq 0}$ and the monotonicity of $\Psi$ it is seen that for every $\delta \geq 1$ we have

$$\Psi(u) \leq \Psi(\delta u) \leq \delta \Psi(u), \quad \forall u \geq 0.$$ \hfill (3.35)  

Thus by (3.35) we get the left hand side of (3.33) with $\kappa_1^{-1} \vee 1 = C_1^{-1}$. To prove the converse implication we use a result from [52]. Note that since $\Psi$ is a complete Bernstein function, the process $(X_t)_{t \geq 0}$ has a transition density $q(t, x, y) = q(t, x - y)$. Moreover, $q(t, \cdot)$ is radially symmetric and decreasing. Denote by $r_D = \text{inrad } D$ and define

$$S_D = \left\{ x \in \mathbb{R}^d : x \in \mathbb{R}^{d-1} \times (-r_D, r_D) \right\}.$$  

Fix $t > 0$ and $z_0 \in D$. By $\tau_{S_D}$ we denote the first exit time from $S_D$. Then

$$\mathbb{P}^{z_0}(\tau_D > t) = \lim_{m \to \infty} \mathbb{P}^{z_0}(X_{\frac{t}{m}} \in D, X_{\frac{2t}{m}} \in D, \ldots, X_{\frac{mt}{m}} \in D)$$

$$= \lim_{m \to \infty} \int_{S_D} \cdots \int_{S_D} \prod_{j=1}^m q\left(\frac{t}{m}, z_j - z_{j-1}\right) dz_1 dz_2 \cdots dz_m$$

$$\leq \lim_{m \to \infty} \mathbb{E}^0(\mathbb{P}\left(\sum_{j=1}^m q\left(\frac{t}{m}, z_j - z_{j-1}\right) dz_1 dz_2 \cdots dz_m \right)$$

$$= \lim_{m \to \infty} \mathbb{P}^0\left(X_{\frac{t}{m}} \in S_D, X_{\frac{2t}{m}} \in S_D, \ldots, X_{\frac{mt}{m}} \in S_D \right) = \mathbb{P}^0(\tau_{S_D} > t),$$

where in the inequality above we used [52, Th. 1.2]. On the other hand, the first exit time $\tau_{S_D}$ starting from 0 is equal in distribution to the first exit time of a one-dimensional subordinate Brownian motion from the interval $B_{r_D} = (-r_D, r_D)$ starting from 0. Let $(B_{S_D}^t)_{t \geq 0}$ be a one-dimensional subordinate Brownian motion, and $\tau_{r_D}$ be its first exit time from $B_{r_D}$. The above estimate gives

$$\sup_{x \in D} \mathbb{E}^x[\tau_D] \leq \mathbb{E}^0[\tau_{r_D}].$$ \hfill (3.36)  

Since the Lévy exponent of $(B_{S_D}^t)_{t \geq 0}$ is given by $\Psi(u^2)$, we obtain from [55, Rem. 4.8] and (3.35) that

$$\mathbb{E}^0[\tau_{r_D}] \leq \frac{C_2}{\Psi(r_D)},$$

for some universal constant $C_2$. Hence using (3.36) and the above estimate we obtain the right hand side of (3.33).
Finally, consider (iii). In view of (3.32) we only need to show the right hand side of (3.34). Using (3.36) and the estimate above, we get that

\[\sup_{x \in D} \mathbb{E}^x[\tau_D] \leq \frac{C_2}{\Psi(r_D^{-2})} = \frac{C_2 \lambda_1^D}{\Psi(r_D^{-2}) / \lambda_1^D}.\]  (3.37)

On the other hand, using [21] and the domain monotonicity of the principal eigenvalue, we obtain

\[\lambda_1^D \leq \Psi(\lambda_{1,\text{Lap}}^D) \leq \Psi\left(\frac{\kappa_1}{r_D}\right),\]

where \(\kappa_1 = \lambda_{1,\text{Lap}}^D\). A combination with (3.37) gives

\[\sup_{x \in D} \mathbb{E}^x[\tau_D] \leq C_2 \sup_{s \in (0,\infty)} \frac{\Psi(\kappa_1 s)}{\Psi(s)} \frac{1}{\lambda_1^D}.\]  (3.38)

Hence using (3.38) and (3.35) we find

\[\sup_{x \in D} \mathbb{E}^x[\tau_D] \leq C_2 (1 \lor \kappa_1) \frac{1}{\lambda_1^D}.\]

This completes the proof of (3.34). 

\[\square\]

**Remark 3.5.** The bound in (3.32) implies for \(p = 1\) the well-known relation

\[\lambda_1^D \geq \frac{1}{\sup_{x \in D} \mathbb{E}^x[\tau_D]},\]

between the principal Dirichlet eigenvalue and the mean survival time in the domain, first obtained by Donsker and Varadhan for diffusion processes [24, eq. (1.2)]. The converse expression

\[\sup_{x \in D} \mathbb{E}^x[\tau_D^p] \geq \frac{\Gamma(p+1)}{(\lambda_1^D)^p}, \quad p \geq 1,\]

has an independent interest, giving an estimate of the (integer and fractional) moments of the mean exit time from \(D\) for subordinate Brownian motion, which has not been known before. Also, using the same (3.31), it follows that \(\tau_D\) has \(p\)-exponential moments of order \(p < \lambda_1^D\), and we have the bound

\[\mathbb{E}^x[e^p \tau_D] \geq \frac{p}{\lambda_1^D - p}, \quad p < \lambda_1^D.\]

**Remark 3.6.** There is much important work on estimates similar to (3.33)-(3.34) for cases when the reference domain is a simply connected set in \(\mathbb{R}^2\) and the stochastic process is Brownian motion. Much effort has been made on finding the best possible universal constants for these estimates; see, for instance, [1, 3] and the references therein. For similar estimates for symmetric stable processes we refer to [5, 52]. Corollary 3.3 extends the earlier results to subordinate Brownian motion, possibly with non-optimal constants.

**Remark 3.7 (Hot spots).** In the literature the location of the maximizer of the principal eigenfunction is referred to as a *hot spot*. Identifying possible hot spots in a convex domain is known to be quite challenging and there is an extensive literature in this direction, see for instance [15, 28, 53] and references therein. In [15, Th. 2.8] it is shown that there exists a constant \(c\), dependent only on \(d\), such that for any bounded convex set \(D\) one has

\[\text{dist}(x^*, \partial D) \geq c \text{ inrad } D \left(\frac{\text{inrad } D}{\text{diam } D}\right)^{d^2-1},\]  (3.39)
where $x^*$ denotes a hot spot of the Laplacian in $\mathcal{D}$ with Dirichlet boundary condition. Note that Theorem 3.2 improves this result substantially. For the principal Dirichlet eigenvalue $\lambda_{1,\text{Lap}}^D$ of the Laplacian, by the domain monotonicity property we have

$$\lambda_1^D \leq \frac{1}{(\text{inrad}\, \mathcal{D})^2} \lambda_1^B, \quad \lambda_{1,\text{Lap}}^D \leq \frac{\lambda_1^B}{\Psi^{-1}(\lambda_1^D)},$$

where $\mathcal{B}$ is the unit ball centered in the origin. Using the above in (3.17) we obtain

$$\text{dist}(x^*, \partial\mathcal{D}) \geq \theta \text{ inrad} \mathcal{D},$$

which improves (3.39).

**Remark 3.8 (Universal upper bound on the distance of maximizer).** An interesting question is whether a reverse inequality to (3.15) may also hold. The following example shows that this is not the case in general. Consider the domain $\mathcal{D} = [0, \pi]^2$ and the Laplace operator. Then $\varphi_n(x, y) = \sin((2n+1)x)\sin(y)$ is an eigenfunction with eigenvalue $\lambda_n = (2n+1)^2 + 1$. Note that $|\varphi_n(\frac{\pi}{2}, \frac{\pi}{2})| = 1$ and $\text{dist}((\frac{\pi}{2}, \frac{\pi}{2}), \partial\mathcal{D}) = \frac{\pi}{2}$. Thus there is no $c > 0$ such that

$$\frac{\pi}{2} = \text{dist}((\frac{\pi}{2}, \frac{\pi}{2}), \partial\mathcal{D}) \leq \left(\frac{c}{\lambda_n}\right)^{1/2}, \quad \text{for all } n > 0.$$

However, an affirmative answer is possible for principal eigenfunctions when $\Psi$ is a complete Bernstein function. Let $\mathcal{D}$ be a bounded convex domain and $\varphi_1$ be the principal eigenfunction and $\lambda_1^D$ is the principal eigenvalue for the Dirichlet problem. Since $\Psi$ is a complete Bernstein function, it is known from [21] that

$$\frac{1}{2} \Psi(\lambda_{1,\text{Lap}}^D) \leq \lambda_1^D \leq \Psi(\lambda_{1,\text{Lap}}^D), \quad \text{(3.40)}$$

where $\lambda_{1,\text{Lap}}^D$ denotes the Dirichlet principal eigenvalue of the Laplacian in $\mathcal{D}$. Let $x^*$ be a maximizer of $\varphi_1$ in $\mathcal{D}$. Then by domain monotonicity

$$\text{dist}(x^*, \partial\mathcal{D})^2 \leq (\text{inrad} \mathcal{D})^2 \leq \lambda_{1,\text{Lap}}^B \leq \frac{\lambda_{1,\text{Lap}}^B}{\Psi^{-1}(\lambda_1^D)},$$

using (3.40). Hence with $c = \sqrt{\lambda_{1,\text{Lap}}^B}$ we obtain

$$\text{dist}(x^*, \partial\mathcal{D}) \leq \frac{c}{\sqrt{\Psi^{-1}(\lambda_1^D)}}.$$

Inequality (3.17) has another important consequence, which we single out next. Assume that $\Psi$ is strictly increasing in $(0, \infty)$, denote the Lebesgue measure of $\mathcal{D}$ by $|\mathcal{D}|$, and $|\mathcal{B}| = \omega_d$.

**Corollary 3.4 (Faber-Krahn inequality).** Under the conditions of Theorem 3.2 we have

$$|\mathcal{D}| \left(\Psi^{-1}\left(\frac{\|V^-\|_\infty - \inf_{\mathcal{D}} V^+ + \lambda_{V,\mathcal{D}}^D}{\theta}\right)\right)^{4/2} \geq \omega_d. \quad \text{(3.41)}$$

**Proof.** Since $\mathcal{B}_r(x^*) \subset \mathcal{D}$ whenever $r = \text{dist}(x^*, \partial\mathcal{D})$, using (3.17) it is immediate that

$$|\mathcal{D}| \geq |\mathcal{B}_r(x^*)| \geq \omega_d \left(\Psi^{-1}\left(\frac{\|V^-\|_\infty - \inf_{\mathcal{D}} V^+ + \lambda_{V,\mathcal{D}}^D}{\theta}\right)\right)^{-4/2}.$$ 

\[\square\]
Remark 3.9. For \( \Psi(u) = u \) we obtain
\[
|\mathcal{D}| \left( \|V^-\|_\infty - \inf_{\mathcal{D}} V^+ + \lambda V,\mathcal{D} \right)^{d/2} \geq \theta^{d/2} \omega_d,
\]
which is [19, Th. 1.1]. Our constant differs from the known sharp constant which is, e.g. for \( d = 2 \) equal to 2.4048, however, our proof of the Faber-Krahn inequality uses a very different argument. Similarly, for \( \Psi(u) = u^{\alpha/2} \) we get
\[
|\mathcal{D}| \left( \|V^-\|_\infty - \inf_{\mathcal{D}} V^+ + \lambda V,\mathcal{D} \right)^{d/\alpha} \geq \theta^{d/\alpha} \omega_d,
\]
and in this sense we can retain the same terminology. For \( V \equiv 0 \), given \( \Psi \), and \( \mathcal{D} = \mathcal{B} \) the Faber-Krahn inequality (3.41) implies
\[
\lambda_1^B \geq \theta \Psi(1).
\]
This may be compared with the lower bound \( \lambda_1^B \geq \frac{1}{2} \Psi(\lambda_{\text{Lap}}^B) \) in (3.40), which shows that in our expression the constant is not optimal, however, our derivation of this type of bound is very different from [21].

Remark 3.10. The original Faber-Krahn inequality says that of all convex domains in the Euclidean plane it is the disk that maximizes the principal eigenvalue. The case of non-zero \( V \) extends this, and in this sense we can retain the same terminology. For \( V \equiv 0 \), given \( \Psi \), and \( \mathcal{D} = \mathcal{B} \) the Faber-Krahn inequality (3.41) implies
\[
\lambda_1^B \geq \theta \Psi(1).
\]
This may be compared with the lower bound \( \lambda_1^B \geq \frac{1}{2} \Psi(\lambda_{\text{Lap}}^B) \) in (3.40), which shows that in our expression the constant is not optimal, however, our derivation of this type of bound is very different from [21].

Remark 3.11 (Torsion). Recall the notation \( H_0 = \Psi(-\Delta) \), and consider the torsion function \( v \) satisfying
\[
\begin{cases}
H_0^D v = 1, & \text{in } \mathcal{D} \\
v = 0, & \text{in } \mathcal{D}^c.
\end{cases}
\]
Note that \( v(x) = \mathbb{E}^x[\tau_D] \). Then it is immediate from Corollaries 3.2-3.3 that with a constant \( C = C(d) > 0 \), we have
\[
\sup_{\mathcal{D}} v(x) \leq C v(x^*),
\]
where \( x^* \) is a maximizer of the principal eigenfunction of \( H_0^D \) in \( \mathcal{D} \). A similar result was obtained in [53, Cor. 2] for the case of Laplacian in dimension 2, see also [7] and references therein.

4. COMPACTLY SUPPORTED POTENTIALS

In this section we consider the eigenvalue problems (1.3)-(1.4) for the special choice of bounded potentials with compact support. In case \( V = -v 1_{\mathcal{K}} \) with a bounded set \( \mathcal{K} \subset \mathbb{R}^d \) with non-empty interior, we say that \( V \) is a potential well with coupling constant \( v > 0 \).

Concerning the eigenvalue problem in \( L^2(\mathbb{R}^d) \), recall that the non-local Schrödinger operator \( H = \Psi(-\Delta) + V \) admits a Feynman-Kac representation [32] of an eigenfunction \( \varphi \) in the form
\[
e^{-tH} \varphi(x) = e^{\lambda t} \mathbb{E}^x[e^{-\int_0^t V(X_s)ds} \varphi(X_t)], \quad x \in \mathbb{R}^d, \ t \geq 0.
\]
For a potential well \( -v 1_{\mathcal{K}} \) this becomes specifically
\[
e^{-tH} \varphi(x) = e^{\lambda t} \mathbb{E}^x[e^{v \int_0^t 1_{\mathcal{K}}(X_s) \varphi(X_t)}],
\]
where
\[
U_t^X(K) = \int_0^t 1_{\mathcal{K}}(X_s) ds
\]
is the occupation measure of the set \( \mathcal{K} \) by subordinate Brownian motion \( (X_t)_{t \geq 0} \).

For non-local Schrödinger operators \( H \) above the semigroup \( \{T_t : t \geq 0\} \), \( T_t = e^{-tH} \), is well-defined and strongly continuous. For all \( t > 0 \), every \( T_t \) is a bounded operator on every \( L^p(\mathbb{R}^d) \) space, \( 1 \leq p \leq \infty \). The operators \( T_{t_1} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) for \( 1 \leq p \leq \infty, \ t > 0 \), and \( T_{t_1} : L^p(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \) for \( 1 < p \leq \infty, \ t \geq t_b \), and \( T_{t_1} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \) for \( t \geq 2t_b \) are bounded, with some \( t_b \geq 0 \).
Theorem 4.1. Let $D$ be $\mathbb{R}^d$ or a bounded subset of $\mathbb{R}^d$, $V$ be a convex increasing function attaining a global minimum at $\hat{x} \in D$, and consider the respective eigenvalue equations with a pair $\lambda$ and $\varphi$ such that $\lim_{x \to z\in \partial D} \varphi(x) = 0$. Furthermore, let $x^*$ be the location of a global maximum of $|\varphi|$, and consider the set

$$U_\lambda = \{x \in D : V(x) \leq \lambda\} \cap D.$$ 

Then we have $x^* \in U_\lambda$ and hence,

$$\text{dist}(x^*, \hat{x}) \leq \max_{z \in \partial U_\lambda} \text{dist}(x^*, z).$$

Proof. Note that we only need to show that $x^* \in U_\lambda$. Assume, to the contrary, that $x^* \in D \cap \{x : V(x) > \lambda\}$. Clearly, we have $V > \lambda$ on $\mathcal{D} \cap U_\lambda^c$. Also, we may assume that $\varphi(x^*) > 0$. Using the strong Markov property in the Feynman-Kac representation, we find that

$$\varphi(x^*) = \mathbb{E}^{x^*}\left[e^{-\int_0^{\tau_{U_\lambda}} (V(X_s) - \lambda) \, ds} \varphi(X_{\tau_{U_\lambda}}) \mathbb{I}_{\{t \wedge \tau_{U_\lambda} < \tau_D\}}\right],$$

which implies,

$$1 \leq \mathbb{E}^{x^*}\left[e^{-\int_0^{\tau_{U_\lambda}} (V(X_s) - \lambda) \, ds} \mathbb{I}_{\{t \wedge \tau_{U_\lambda} < \tau_D\}}\right], \quad \forall \ t > 0.$$

However, the above is not possible and hence this is a contradiction. \hfill $\square$

We note that in [6] a related question has been addressed for classical Dirichlet-Schrödinger equations in convex planar domains.

Remark 4.1. It should noted that the convexity of $V$ is not used to find the location of the maximizer. For instance, if we have $V$ compactly supported inside $D$ and $\lambda < 0$, then the same proof above shows that $x^* \in \text{supp} V$.

Next we consider a situation in a bounded domain. The following result shows how far from the support of a not sufficiently negative potential inside a bounded domain a maximizer can move out. This may be compared with Theorem 3.2, in particular, it will be seen that the effect of the potential is exercised by the eigenvalue alone. We will use the following condition on $\Psi$ repeatedly, which we single out here.
**Assumption 4.1.** Let \( \Psi \in \mathcal{B}_0 \). We assume that for every \( \gamma_0 > 0 \)

\[
\lim_{s \to 0} \sup_{\gamma \in [\gamma_0, \infty)} \frac{\Psi(s \gamma)}{\Psi(\gamma)} = 0 \tag{4.1}
\]

holds.

We will comment and give some examples of Bernstein functions satisfying this assumption following the proof of our next main result.

**Theorem 4.2.** Let \( \Psi \) satisfy Assumption 4.1, and \( V \) be a potential with compact support \( \text{supp} V = \mathcal{K} \). Consider a convex bounded domain \( D \subset \mathbb{R}^d \), containing \( \mathcal{K} \), and let \( \text{dist}(\mathcal{K}, \partial D) = \kappa > 0 \). Also, let \( \varphi \) be an eigenfunction at eigenvalue \( \lambda > 0 \) solving (1.3), and suppose it is known about a global maximizer \( x^* \) of \( |\varphi| \) that \( \text{dist}(x^*, \partial D) \leq \kappa/2 \). Then there exists a constant \( \zeta > 0 \), dependent on \( d \), \( \kappa \) and \( \Psi \), but not on \( D \), \( \mathcal{K} \), \( \varphi \) or \( \lambda \), such that

\[
\text{dist}(x^*, \partial D) \geq \frac{1}{\sqrt{\Psi^{-1}\left(\frac{\lambda}{\zeta}\right)}}. \tag{4.2}
\]

**Proof.** Denote \( r = \text{dist}(x^*, \partial D) \), and without loss of generality assume that \( x^* = 0 \). Let \( t = \frac{c}{\Psi(r^{-2})} \), where the constant \( c \) will be chosen below. From the proof of Theorem 3.2 it follows that we can choose \( c \) large enough to satisfy

\[
\mathbb{P}_S(S_t^\Psi < r^2) < \frac{1}{4}, \quad \forall \, r > 0. \tag{4.3}
\]

Fix this choice of \( c \) and define \( T_c = \frac{c}{s_{\Psi(r^{-2})}} \). Since \( r \leq \frac{\kappa}{2} \), we have \( t \leq T_c \). Using (4.1) we show below that there exists \( T_0 > 1 \) such that

\[
\mathbb{P}_S(S_t^\Psi \leq r^2 T_0) \geq \frac{1}{2}, \quad \text{for all } 0 < r < \frac{\kappa}{2}.
\]

Define \( Y_r = \frac{1}{\sqrt{r}} S_t^\Psi \). Then the Laplace transform of \( Y_r \) is given by

\[
\hat{f}(s) = \mathbb{E} \left[ e^{-s Y_r} \right] = \mathbb{E} \left[ e^{-s \frac{1}{\sqrt{r}} S_t^\Psi} \right] = e^{-s \frac{1}{\sqrt{r}} \Psi^{\Psi^{-1}}(sr^{-2})}. \tag{4.4}
\]

Since \( r < \kappa/2 \), using (4.1) and (4.4) we see that \( \hat{f}(s) \to 1 \) as \( s \to 0 \), uniformly in \( r \in (0, \frac{\kappa}{2}) \). Thus by the uniform Tauberian theorem [45, Th. 3], we obtain

\[
\mathbb{P}_S(Y_r \leq y) \to 1 \quad \text{as } y \to \infty, \quad \text{uniformly in } r \in (0, \frac{\kappa}{2}).
\]

Hence we can find \( T_0 > 1 \) satisfying

\[
\mathbb{P}_S(S_t^\Psi \leq r^2 T_0) \geq \frac{1}{2}, \quad \text{for all } 0 < r < \frac{\kappa}{2}. \tag{4.5}
\]

Combining (4.3) and (4.5) we obtain that

\[
\mathbb{P}_S(S_t^\Psi \in [r^2, r^2 T_0]) > \frac{1}{4}, \quad \forall \, r \in (0, \frac{\kappa}{2}). \tag{4.6}
\]

Now we fix the above choice of \( T_0 \), which depends on \( c, \kappa \) and \( \Psi \). On the other hand, since \( D \) is convex, we may assume that there exists a point \( z_0 \in \partial D \) such that \( \text{dist}(0, z_0) = r \), \( z_0 \) lies on the \( x_1 \)-axis and \( D \) lies of the on the complement of the half-space \( \{ y \in \mathbb{R}^d : z_0 \cdot y \geq r^2 \} \). Define \( \chi : [0, T_0] \to \mathbb{R}^d \) by

\[
\chi(s) = 2 \sqrt{s} e_1,
\]
Lemma 4.1. Suppose that $\Psi$ is unbounded and regularly varying at infinity, i.e., with a slowly varying function $\ell$ and constant $\beta > 0$ we have

$$\Psi(u) \asymp u^\beta \ell(\beta), \text{ for all large } u.$$ 

Then Assumption 4.1 holds.

There is a large family of subordinate Brownian motions satisfying Assumption 4.1. First we show a general statement and then illustrate it by some important examples.
Proof. It suffices to show that for any sequence \((s_n, \gamma_n)\) with
\[
s_n \to 0, \quad \gamma_n \to \infty, \quad \text{and} \quad s_n \gamma_n \to \infty,
\]
we have
\[
\lim_{n \to \infty} \frac{\Psi(s_n \gamma_n)}{\Psi(\gamma_n)} = 0. \quad (4.11)
\]
Fix any \(\varepsilon > 0\). Then for large \(n\),
\[
\frac{\Psi(s_n \gamma_n)}{\Psi(\gamma_n)} \leq \frac{\Psi(\varepsilon \gamma_n)}{\Psi(\gamma_n)} \propto \varepsilon^\beta \frac{\ell(\varepsilon \gamma_n)}{\ell(\gamma_n)} \propto \varepsilon^\beta.
\]
Hence (4.11) follows.

\[\square\]

**Example 4.1.** By Lemma 4.1 the following Bernstein functions satisfy Assumption 4.1.

(i) \(\Psi(u) = u^{\alpha/2}, \alpha \in (0, 2]\).

(ii) \(\Psi(u) = (u + m^{2/\alpha})^{\alpha/2} - m, \ m > 0, \alpha \in (0, 2]\).

(iii) \(\Psi(u) = u^{\alpha/2} + u^{\beta/2}, \ 0 < \beta < \alpha \in (0, 2]\).

(iv) \(\Psi(u) = u^{\alpha/2}(\log(1 + u))^{\beta/2}, \alpha \in (0, 2], \beta \in (0, 2 - \alpha)\).

(v) \(\Psi(u) = u^{\alpha/2}(\log(1 + u))^{-\beta/2}, \alpha \in (0, 2], \beta \in (0, \alpha)\).

**Example 4.2.** On the other hand, \(\Psi(u) = \log(1 + u^{\alpha/2}), \alpha \in (0, 2]\), does not satisfy Assumption 4.1. To see this note that for \(s = \frac{1}{n}\) and \(\gamma = n^2\) we have
\[
\lim_{n \to \infty} \frac{\Psi(s \gamma)}{\Psi(\gamma)} = \lim_{n \to \infty} \frac{\log(1 + n^{\alpha/2})}{\log(1 + n^2)} \geq \frac{1}{2}.
\]

In the remaining part of this section we consider the eigenvalue problem in full space.

\[\text{T4.3} \quad \text{Theorem 4.3.} \quad \text{Consider the operator} \ H \ \text{given by (1.2),} \ \text{supp} \ V = K, \ \text{and let} \ \varphi \ \text{be a solution of the Schrödinger eigenvalue problem (1.4) for} \ H, \ \text{corresponding to eigenvalue} \ \lambda = -|\lambda| < 0. \ \text{If} \ |\varphi| \ \text{has a global maximum at} \ x^* \in \mathbb{R}^d, \ \text{then} \ x^* \in K. \]

**Proof.** We show that there is no maximizer in \(K^c\). Assume, to the contrary, that \(x^* \in K\). Therefore, for a suitable \(\delta > 0\) we have \(B_\delta(x^*) \subset K^c\). Let \(\tau_\delta\) be the exit time from the ball \(B_\delta(x^*)\). Since \(\mathbb{P}^{x^*}[\tau_\delta > t] > 0\), we find \(t > 0\) such that \(\mathbb{P}^{x^*}[\tau_\delta > t] > 0\). As before, we can also assume that \(\varphi(x^*) > 0\).

By the Feynman-Kac representation we have
\[
\varphi(x^*) = \mathbb{E}^{x^*}\left[e^{\lambda t \tau_\delta} \varphi(X_t I_{\tau_\delta > t})\right] \\
\leq e^{\lambda t} \mathbb{E}^{x^*}\left[\varphi(X_t) I_{\tau_\delta > t}\right] + e^{\lambda t} \mathbb{E}^{x^*}\left[\varphi(X_t) I_{\tau_\delta \leq t}\right] \\
\leq e^{\lambda t} \varphi(x^*) \mathbb{P}^{x^*}() \mathbb{P}^{x^*}(\tau_\delta > t) + \varphi(x^*) \mathbb{P}^{x^*}(\tau_\delta \leq t) .
\]

This would imply \(e^{\lambda t} > 1\), which is a contradiction as \(\lambda < 0\). Hence \(x^* \in K\). \[\square\]

**Remark 4.2.** Recall that the eigenfunctions are continuous, as mentioned earlier. Since \(V\) is bounded, from the Feynman-Kac representation we have for every \(t > 0\) that
\[
|\varphi(x)| \leq e^{(|V|_{-\lambda}) t} \mathbb{E}^{x}[|\varphi|^2(X_t)]^{1/2} \leq e^{(|V|_{-\lambda}) t} \left(\int_{\mathbb{R}^d} |\varphi(y)|^2 q_t(x - y) \, dy\right)^{1/2}, \quad (4.12)
\]
where \(q_t(x, y) = q_t(x - y)\) denotes the transition density of \((X_t)_{t \geq 0}\) starting at \(X_0 = x\). It follows by subordination, see (2.4), that
\[
q_t(x - y) = \int_0^\infty \frac{1}{(4\pi s)^d} e^{-|x - y|^2/4s} \mathbb{P}_S(S^w_t \in ds).
\]
Therefore, for every fixed \(y\) we have \(q_t(x - y) \to 0\) as \(|x| \to \infty\). Moreover, if \(\Psi\) satisfies the Hartman-Wintner condition (3.5), then \(q_t(x, y)\) is bounded and continuous. Hence by dominated
convergence we obtain from (4.12) that \( \lim_{|x| \to \infty} |\varphi(x)| = 0 \), thus every eigenfunction attains its maximum in \( \mathbb{R}^d \).

Finally, we show how deep inside the support the maximizer can be for a potential well. We denote by \( \text{Int} \mathcal{K} \) the interior of \( \mathcal{K} \).

**Theorem 4.4.** Let \( V = -\varphi \mathcal{K} \) with a bounded convex set \( \mathcal{K} \), and \( \varphi \) be an eigenfunction corresponding to eigenvalue \( \lambda = -|\lambda| < 0 \) solving the eigenvalue problem (1.4). Suppose that \( \Psi \) is unbounded and satisfies Assumption 4.1. Then there exist two constants \( \varrho_1, \varrho_2 > 0 \), dependent only on \( \Psi \) and \( \text{inrad} \mathcal{K} \), such that if

\[
\frac{v - |\lambda|}{|\lambda|} \leq \varrho_1, \quad (4.13)
\]

then \( x^* \in \text{Int} \mathcal{K} \) and

\[
\text{dist}(x^*, \partial \mathcal{K}) \geq \frac{1}{\sqrt{\Psi^{-1}(\frac{|\lambda|}{\varrho_2})}}. \quad (4.14)
\]

**Proof.** *Step 1:* First we prove (4.14) assuming that \( x^* \in \text{Int} \mathcal{K} \). By a shift we can assume that \( x^* = 0 \) with no loss of generality, and we denote \( r = \text{dist}(x^*, \partial \mathcal{K}) > 0 \). Let \( t = \frac{c}{\sqrt{|\lambda|}} \) where the constant \( c \) will be chosen later. From the proof of Theorem 4.2 we see that we can choose \( c \) large enough such that

\[
\mathbb{P}_S(S_{i,t}^\Psi \in [r^2, r^2 T_0]) = \delta_1 > \frac{1}{4}, \quad \forall \ r \in (0, \text{inrad} \mathcal{K}); \quad (4.15)
\]

see (4.6) above. Therefore, by the independence of increments we have from (4.15) that

\[
\mathbb{P}_S(S_{i,t}^\Psi \in [r^2, r^2 T_0], S_{i,t}^\Psi - S_{i,t}^\Psi \in [r^2, r^2 T_0]) = \delta_2^2, \quad \forall \ 0 < r \leq \text{inrad} \mathcal{K}. \quad (4.16)
\]

Now we fix the above choice of \( T_0 \) which depends on \( c \) and \( \Psi \) and \( \text{inrad} \mathcal{K} \) (recall that \( r \) and \( t \) are related). Since \( \mathcal{K} \) is convex, we may assume that the point \( z_0 = (r, 0, \ldots, 0) \in \partial \mathcal{K} \) is such that \( \text{dist}(0, z_0) = r \), \( \mathcal{K} \) lies on the on the complement of the half-space \( \{ y \in \mathbb{R}^d : z_0 \cdot y \geq r^2 \} \). Define \( \chi : [0, T_0] \to \mathbb{R}^d \) by

\[
\chi(s) = 2s,
\]

Note that \( \text{dist}(0, \chi(\frac{1}{2}r)) = r \). Define

\[
\mathcal{N} = \left\{ f \in C([0, 2T_0], \mathbb{R}) : f(0) = 0 \text{ and } \max_{s \in [0, 2T_0]} |f(s) - \chi(s)| < \frac{1}{2} \right\}.
\]

In a similar manner as in the proof of Theorem 4.2 we find that there is a \( \delta_2 > 0 \) such that

\[
\mathbb{P}_W^0 \left( \frac{1}{r} B_{r^2} \in \mathcal{N} \right) = \mathbb{P}_W^0 \left( B_1^1 \in \mathcal{N} \right) = \delta_2. \quad (4.17)
\]

Also, we have

\[
\left\{ \max_{s \in [0, 2T_0]} \left| \frac{1}{r} B_{r^2} \right| - \chi(s) \right| < \frac{1}{2} \right\} = \left\{ \max_{s \in [0, 2T_0]} \left| B_1^1 - \frac{1}{r} \chi(s) \right| < \frac{r}{2} \right\},
\]

using scaling and that \( r^2 \chi(s) = \frac{1}{r} \chi(r^2 s) \). Thus we obtain

\[
\mathbb{P}_W^0 \left( \max_{s \in [0, 2r^2 T_0]} \left| B_1^1 - \frac{1}{r} \chi(s) \right| < \frac{r}{2} \right) = \delta_2. \quad (4.18)
\]

Combining (4.16) and (4.18) gives

\[
\mathbb{P}_W^0 \left( (\omega, \varpi) \text{ s.t. } \max_{s \in [0, 2r^2 T_0]} \left| B_1^1(\varpi) \right| - \frac{1}{r} \chi(s) \right| < r/2, S_{i,t}^\Psi(\varpi) \in [r^2, r^2 T_0], \ S_{i,t}^\Psi(\varpi) - S_{i,t}^\Psi(\varpi) \in [r^2, r^2 T_0] \right) = \mathbb{P}_W^0 \left( \max_{s \in [0, 2T_0]} \left| B_1^1 - \chi(s) \right| < r/2 \right) \mathbb{P}_S(S_{i,t}^\Psi \in [r^2, r^2 T_0], S_{i,t}^\Psi - S_{i,t}^\Psi \in [r^2, r^2 T_0]) = \delta_2 \delta_1^2, \quad (4.19)
\]
where the third line follows from the independence of the two processes. Let
\[ \hat{\Omega} = \left\{ (\omega, \varpi) : \max_{s \in [0, 2t_0^2]} \left| B_{s}^{t}(\omega) - \frac{1}{r} \chi(s) \right| < \frac{r}{2}, S_{t}^{\Psi}(\varpi) \in [r^2, r^2T_0], S_{2t}^{\Psi}(\varpi) - S_{t}^{\Psi}(\varpi) \in [r^2, r^2T_0] \right\}. \]
We see that
\[ \hat{\Omega} \subset \left\{ (\omega, \varpi) : \sup_{s \in [0, t]} \left| B_{s}^{t}(\omega) - \frac{1}{r} \chi(s) \right| < \frac{r}{2}, S_{t}^{\Psi} \in [r^2, r^2T_0], S_{2t}^{\Psi} - S_{t}^{\Psi} \in [r^2, r^2T_0] \right\}. \]
By the construction of \( \chi \) it follows that for every \( (\omega, \varpi) \in \hat{\Omega} \), \( B_{s}^{t} \in \mathcal{K}^c \) and the paths of \( B_{S_{t}^{\Psi}} \) stay in \( \mathcal{K}^c \) for all \( s \in [t, 2t] \). This observation will play a key role in our analysis below.

Let \( \delta = \delta_2 \delta_1^2 \), and define
\[ 2\varrho_1 = \frac{\delta}{2 - \delta} \in (0, 1), \]
and a function \( \xi : \mathbb{R} \to \mathbb{R}^+ \) by
\[ \xi(y) = \delta e^{\frac{1}{2}(1 - \varrho_1)y} + (1 - \delta) e^{\varrho_1 y}. \]
It is direct to see that \( \xi'(\varepsilon_0) = 0 \) gives
\[ \varepsilon_0 = \frac{2}{1 + \varrho_1} \log \frac{\delta(1 - \varrho_1)}{2\varrho_1(1 - \delta)}. \]
Since
\[ \varrho_1 < \frac{\delta}{2 - \delta} \text{ implies } \frac{\delta(1 - \varrho_1)}{2\varrho_1(1 - \delta)} > 1, \]
we have \( \varepsilon_0 > 0 \). Again observe that \( \xi'(0) = 0 \), and therefore \( \xi(y) < 1 \) for \( y \in (0, \varepsilon_0) \).

Suppose now that \( \frac{\varepsilon_0 - |\lambda|}{\lambda} \leq \varrho_1 \). By the Feynman-Kac representation we have
\[ \varphi(0) = \mathbb{E}^0 \left[ e^{\int_{0}^{2t} (\lambda - V(X_s)) \text{d}s} \varphi(X_{2t}) \right], \]
which, in turn, implies
\[ 1 \leq \mathbb{E}^0 \left[ e^{\int_{0}^{2t} (\lambda - V(X_s)) \text{d}s} \right] = \mathbb{E}^0 \left[ e^{\int_{0}^{2t} (\lambda - V(X_s)) \text{d}s} \mathbb{1}_{\hat{\Omega}} \right] + \mathbb{E}^0 \left[ e^{\int_{0}^{2t} (\lambda - V(X_s)) \text{d}s} \mathbb{1}_{\hat{\Omega}^c} \right] \leq \mathbb{E}^0 \left[ e^{\int_{0}^{2t} (\lambda - V(X_s)) \text{d}s} + \lambda \mathbb{1}_{\hat{\Omega}} \right] + (1 - \delta)e^{(\varrho_1 + \lambda)2t} \leq \delta e^{(\varrho_1 + \lambda)2t} + (1 - \delta)e^{\varrho_1 \lambda 2t} \leq \delta e^{\varrho_1 |\lambda| - |\lambda| t} + (1 - \delta)e^{\varrho_1 \lambda 2t} = \xi(2t|\lambda|), \]
where in the fourth line we used (4.19). Since \( 2t|\lambda| > 0 \) and \( \xi(2t|\lambda|) \geq 1 \), we conclude that
\[ 2t|\lambda| \geq \varepsilon_0 \]
holds. Hence (4.14) follows with \( \varrho_2 = \frac{\varepsilon_0}{2\varrho_1} \).

**Step 2:** To conclude, we prove that under the condition (4.13) we have \( x^* \not\in \partial \mathcal{K} \). Like before, we may assume that \( x^* = 0 \) and \( \mathcal{K} \subset \{ x_1 \leq 0 \} \). Note that the estimate (4.19) holds uniformly in \( r \in (0, \text{rad} \mathcal{K}) \). Since 0 is on the boundary of \( \mathcal{K} \) and the function \( \chi \), defined above, lies in \( \{ x_1 \geq 0 \} \), we observe that for every \( r > 0 \) and every \( (\omega, \varpi) \in \hat{\Omega} = \hat{\Omega}_r \) we have \( B_{S_{t}^{\Psi}} \in \mathcal{K}^c \) and the paths \( B_{S_{t}^{\Psi}} \) stay \( \mathcal{K}^c \) for \( s \in [t, 2t] \), where \( t = \frac{c}{\Psi(r^2)} \) and \( c \) is chosen the same as before. Therefore, following a similar argument as in the proof of (4.20), we obtain
\[ 1 \leq \xi(2t|\lambda|), \]
for all \( r > 0 \). Since \( t \to 0 \) as \( r \to 0 \), and since \( \Psi \) is unbounded, the above estimate cannot hold for small \( t \). Thus we have a contradiction showing that \( 0 = x^* \in \text{Int} \mathcal{K} \). □
Remark 4.3. We note that for a potential well $V = -v \mathbb{1}_K$, $v > 0$, we have

$$v - |\lambda| \leq \lambda^K_1,$$

where $\lambda^K_1$ is the principal eigenvalue of $\Psi(-\Delta)$ in $K$ with Dirichlet exterior condition on $K^c$. Indeed, from the Feynman-Kac formula we get that

$$\varphi(x) \geq \mathbb{E}^x \left[ e_{t_0}^{(v_\delta}(X_{t_0} + \lambda) ds \varphi(X_s) \mathbb{1}_{\{t < \tau_K\}} \right]$$

$$\geq e^{(v-|\lambda|)} \min_{y \in K} \mathbb{E}^x(t < \tau_K), \quad x \in K.$$

By taking logarithms on both sides and dividing by $t > 0$, we get

$$v - |\lambda| \leq - \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}_x(t < \tau_K) \leq \lambda^K_1.$$

Thus the numerator at the left hand side of (4.13) is always bounded by $\lambda^K_1$, and so for $|\lambda|$ large enough (4.13) holds. Also, notice that the result in Theorem 4.4 continues to hold for more general potentials $V$ supported on $K$ and $\lambda < 0$. In this situation (4.13) will be replaced by

$$- \min_{x \in K} V(x) - |\lambda| \leq \rho_1.$$

Notice that the dependence of $\rho_1$ and $\rho_2$ on inrad $K$ comes from (4.1), which has been crucially used in (4.5). This dependence can be waived for a class of $\Psi$ for which (4.1) holds uniformly in $\gamma_0 > 0$, i.e., when

$$\lim_{s \to 0} \sup_{\gamma \in (0, \infty)} \frac{\Psi(s\gamma)}{\Psi(\gamma)} = 0. \quad (4.21)$$

Observe that if $\Psi$ satisfies Assumption 3.1, then (4.21) holds. Indeed, we have then

$$\lim_{s \to 0} \sup_{\gamma \in (0, \infty)} \frac{\Psi(s\gamma)}{\Psi(\gamma)} = \lim_{s \to 0} \sup_{\gamma \in (0, \infty)} \frac{\Psi(s\gamma)}{\Psi(s^{-1}s\gamma)} \leq \lim_{s \to 0} s^\mu = 0.$$

Moreover, (4.5)-(4.6) follow then uniformly in $r \in (0, \infty)$. Therefore, in this case $\rho_1$ and $\rho_2$ only depend on $\Psi$ and not on inrad $K$. This is recorded in the following result.

(T3.4) Theorem 4.5. Suppose that $\Psi$ satisfies Assumption 3.1, and let $\varphi$ and $\lambda = -|\lambda|$ solve the eigenvalue equation (1.4) for $H$ with a potential well $V = -v \mathbb{1}_K$. Then there exist positive $\rho_1, \rho_2$, dependent only on $\Psi$, such that if

$$\frac{v - |\lambda|}{|\lambda|} \leq \rho_1,$$

then $x^* \in \text{Int} K$ and

$$\text{dist}(x^*, \partial K) \geq \frac{1}{\sqrt{\Psi^{-1}(\frac{|\lambda|}{\rho_2})}}.$$

Theorems 4.4-4.5 have the following interesting “no-go” type consequence.

(nogo) Corollary 4.1. Under the conditions of Theorems 4.4-4.5 we have that whenever

$$v < \rho_2 \Psi \left( \frac{1}{(\text{inrad } K)^2} \right),$$

then either $|\lambda| < v/(1 + \rho_1)$ or the non-local Schrödinger operator $H$ has no $L^2$-eigenfunctions.
Proof. We have trivially $\text{dist}(x^*, \partial K) < \text{inrad} K$. Also, $|\lambda| < v$, and $\Psi^{-1}$ is an increasing function. Hence Theorems 4.4-4.5 give
\[
inrad K \geq \frac{1}{\sqrt{\Psi^{-1}(\frac{v}{\varrho})}},
\]
implying the result. \hfill \square

Remark 4.4.
(i) We note that, using direct techniques of differential equations, for usual Schrödinger operators $H = -\Delta - v1_{B_{\alpha}}$ in $L^2(\mathbb{R}^d)$, it is well-known that for $d \geq 3$, the smallness of the quantity $va^2$ implies that no $L^2$-eigenfunctions exist. Using the Birman-Schwinger principle, bounds on $va^2$ can also be derived ruling out $L^2$-eigenfunctions of $H = (-\Delta)^{\alpha/2} - v1_{B_{\alpha}}$ and further non-local operators [47]. Although the constants may in general differ, we have the same type of bounds resulting from Corollary 4.1 above.

(ii) We can also use Green functions to find a “no-go” type consequence, which does not involve (4.13). Suppose that $d \geq 3$ and the transition density probability function of $(X_t)_{t \geq 0}$ decays to 0 as $t \to \infty$. Then the ground state $\varphi_1$ of $H = \Psi(-\Delta) - v1_{K}$ has the representation
\[
\varphi_1(x) = \int_{\mathbb{R}^d} (\lambda_1 - V(y)) \varphi(y) G(x,y) \, dy,
\]
where $G(\cdot, \cdot)$ is the associated Green function. It is known [29, Th. 3] that there exists a constant $C_d$, dependent only on $d$, such that
\[
G(x,y) \leq \frac{C_d}{|x-y|^d \Psi(|x-y|^{-2})}.
\]
Let $R = \text{diam} K$. Since $x^* \in K$, by Theorem 4.3, and $\lambda_1 < 0$ we see from (4.22) that
\[
\varphi_1(x^*) \leq C_d (v - |\lambda_1|) \varphi_1(x^*) \int_K \frac{dy}{|x^*-y|^d \Psi(|x^*-y|^{-2})},
\]
which implies
\[
1 \leq C_d (v - |\lambda_1|) \int_{B_R(x^*)} \frac{dy}{|x^*-y|^d \Psi(|x^*-y|^{-2})} = C_d \omega_d v \int_0^R \frac{ds}{s^{d/2} \Psi(s^{-2})},
\]
where $\omega_d$ denotes the volume of the unit ball in $\mathbb{R}^d$. Therefore, if the right hand side is finite (for example, for $\Psi$ satisfying Assumption 3.1), then there is no ground state whenever
\[
v < \frac{1}{C_d \omega_d \int_0^R \frac{ds}{s^{d/2} \Psi(s^{-2})}}.
\]

Finally we note that our technique in proving Theorem 4.4 is also applicable to a more general class of potentials. Consider equation (1.4). For $V$ convex and increasing we have shown in Theorem 4.1 that the maximizer $x^* \in U_\lambda = \{x \in D : V(x) \leq \lambda\} \cap D$. For $\delta > 0$ we define the $\delta$-neighborhood of $U_\lambda$, i.e.
\[
U_\lambda^\delta = \{x \in \mathbb{R}^d : \text{dist}(x, U_\lambda) \leq \delta\}.
\]
The following result provides a sufficient condition for the maximizer to be strictly inside $U_\lambda$.

Theorem 4.6. Suppose that $\Psi$ satisfies Assumption 3.1. There exist positive constants $\varrho_1$ and $\varrho_2$, dependent only on $\Psi$, such that if for some $\delta \in (0, \text{inrad} U_\lambda)$
\[
\frac{\lambda - \min_{x \in \mathbb{R}^d} V(x)}{\min_{x \in \mathbb{R}^d \setminus U_\lambda^\delta} (V(x) - \lambda)} \leq \varrho_1,
\]

then
\[ \text{dist}(x^*, \partial \mathcal{U}_\lambda) \geq \frac{1}{\sqrt{\Psi - 1 \left( \min_{x \in \mathbb{R}^d \setminus B_\delta}(V(x) - \lambda) \right)^2}}. \]

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