Aspects of feedback and a local approach for linear systems

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To
The memory of my Father
To
my Mother
my Wife Nahid
my Children Amir, Azin, Amin
Aspects of Feedback, and a Local Approach, for Linear Systems.

by

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A doctoral thesis submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of the Loughborough University of Technology

Aspects of Feedback, and a Local Approach, for Linear Systems

Synopsis:

The effect of the implementation of constant output feedback on a general rational transfer function matrix has long been of interest. More recently, interest has been shown in the properness of closed-loop systems when such constant output feedback is applied to a general open-loop $G(s)$ which is given either in terms of a state space realisation or as a matrix fraction description. In the first part of this work the effect of constant output feedback on a general composite system is considered and a simple sufficient condition for properness of such a system is derived.

Conventionally the matrix fraction description plays an essential role in the study of linear systems. Frequently however the conditions on the matrix fractions appear excessive and the relative primeness requirement may be difficult to verify. Further since the investigation is often only concerned with one specific frequency (as for instance in infinite frequency investigations where after bilinear transformation the origin of the transformed plane is considered) much of the information carried by the matrix fraction is not utilised. In the second part of the work this objection is considered and a highly localised approach to linear systems theory is developed. Specifically an analysis of the system structure at a single frequency $s_0$ to the exclusion of all other frequencies in the closed complex plane is carried out. The necessary local matrix results and local systems theory are then presented. Of particular interest is a useful local state space form realisation of a given transfer function matrix. This local approach not only enables a known result concerning the invariance of the infinite zero structure of proper systems to be generalised to the non-proper case, but also reveals a property concerning the invariance of the infinite pole structure of the system under state feedback. Consequent to this the concept of local state feedback is introduced and certain invariants are noted.
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Introduction
Introduction

It has long been known that the effect of the implementation of constant output feedback for a general rational transfer function matrix does not necessarily result in an overall closed-loop transfer function matrix which is proper. However, for a strictly proper rational transfer function matrix, Rosenbrock and Pugh (33) have shown that the closed-loop system will always be strictly proper, and in the case of a general rational transfer function matrix Scott and Anderson (34) have shown that the closed-loop system is generically proper.

More recently some necessary and sufficient condition for closed-loop properness have been established by Pugh and Ratcliffe (24), Pugh (26) and Krishnarao and Chen (10). In the first part of this thesis, applications of such results to composite systems are studied and a simple sufficient condition for properness of a general composite system is established. Despite the apparent limitations of our result, it produces surprisingly good results when applied to the two specific composite systems.

The idea of proper and non-proper system in the conventional sense refers to there being no poles at infinity. This leads us naturally to investigate about the behaviour of the system at a specific finite point $s_0$ in the complex plane. However, in order to be able to use the idea of properness at $s_0$ and study the effect of the state variable feedback on the pole/zero structure at $s_0$ of a general transfer function matrix, it was necessary to have a local matrix theory which in turn led to the development of a local system theory and in particular a local (rational) state-space realisation. Based on this rational state-space realisation it is then possible to look at the idea of the local state feedback, and investigate the problem of the general pole assignment through the invariant polynomials.
The basic definitions and results concerning polynomial and rational functions, polynomial and rational matrices, system matrices and their properties which are used in the subsequent chapters are given in the second chapter.

Chapter three is centred on the properness of a composite system. In this regard after reviewing some known results in the sections two and three, a representation of the general composite system is given in the fourth section. Starting from this representation of the composite system, some sufficient conditions for properness of a general composite system and then application of these results to two specific composite systems are given in sections five and six. Chapter four concentrates on the local matrix theory and parallels the conventional results, the most appropriate and useful local versions of these results such as elementary operations and equivalence, valuation and Smith-McMillan form for polynomial and rational matrices are discussed and presented.

Having provided the necessary background in the previous chapters, a highly localised approach to linear system theory is developed in chapter five. In this regard the ideas of system equivalence at \( s_0 \) and also decoupling zeros at \( s_0 \) are studied in sections two and three and then using the results given in these two sections an interesting local state-space realisation and some local standard forms for system matrices are presented in section four. In the last section of chapter five systems of least order at \( s_0 \) are studied and some relevant results are given.

Finally the infinite frequency structure of the transfer function matrix under state variable feedback which reveals the necessity of the local system theory is considered in chapter six. Also in this chapter aspects of local
state feedback and the general problem of pole assignment are examined and
certain invariants are established.
CHAPTER II

Preliminaries
Section II-1: §Introduction

This chapter presents the mathematical background relevant to linear system theory which is necessary for subsequent chapters. In this regard the basic results and definitions concerning the polynomials, rational functions and polynomial and rational matrices are given in sections two and three. In section four after defining the infinite poles and zeros of a rational matrix, some related results are presented. Finally system matrices and their properties including decoupling zeros, systems of least order and some relevant results are given in sections five and six.

Section II-2: §Polynomials and Rational Functions

Let $F$ denote the field of real or complex numbers ($\mathbb{R}$ or $\mathbb{C}$) then a polynomial in the indeterminate $s$ with coefficients in $F$ is an algebraic expression of the form

$$\sum_{i=0}^{n} p_i s^i = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n$$  \hspace{1cm} (2.1)

The degree of a polynomial as given by (2.1) is defined as $n$ and $p_n$ is called the Leading Coefficient.

**Definition 2.2.** A polynomial whose leading coefficient is unity is said to be MONIC.

The operations of addition and multiplication are as defined in a ring with identity one. i.e. if

$$P(s) = \sum_{i=0}^{n} p_i s^i \quad \left\{ \begin{array}{l} n' \geq n \end{array} \right.$$  \hspace{1cm} (2.3)

$$P'(s) = \sum_{i=0}^{n'} p'_i s^i$$

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Then their sum and product are given by

\[
P(s) + P'(s) = \sum_{i=0}^{n} (p_i + p_i') s^i + \sum_{i=n+1}^{n'} p_i' s^i
\]

\[
P(s) P'(s) = \sum_{i=1}^{n+n'} p_i'' s^i
\]

where \( p_i'' = \sum_{j+k=i} p_j p_k' \)

Now let \( \mathcal{F}[s] \) denote the ring of polynomials as given by (2.1), then an important property of the polynomials which is related to ordinary long division, is called the "DIVISION ALGORITHM" and can be stated in the following theorem.

Theorem 2.5. Let \( P_1(s), P_2(s) \in \mathcal{F}[s] \) and be non zero. Then there exist unique polynomials \( Q(s) \) and \( R(s) \) in \( \mathcal{F}[s] \) such that

\[
P_1(s) = P_2(s) Q(s) + R(s)
\]

with \( \deg R(s) < \deg P_2(s) \) or \( R(s) = 0 \). The polynomial \( Q(s) \) is called the QUOTIENT of \( P_1(s) \) by \( P_2(s) \), and \( R(s) \) is the REMAINDER. If \( R(s) = 0 \), then \( P_2(s) \) clearly divides \( P_1(s) \). Now if \( P_1(s), P_2(s) \in \mathcal{F}[s] \) (both non zero), then the GREATEST COMMON DIVISOR of \( P_1(s), P_2(s) \) denoted by (g.c.d) is \( P'(s) \in \mathcal{F}[s] \) such that \( P'(s) \) divides \( P_1(s) \) and \( P_2(s) \), and also any other polynomial in \( \mathcal{F}[s] \) which divides \( P_1(s) \) and \( P_2(s) \) also divides \( P'(s) \).

Definition 2.7. Let \( P_1(s), P_2(s) \in \mathcal{F}[s] \), then if the degree of the (g.c.d) of \( P_1(s) \) and \( P_2(s) \) is zero, they are said to be RELATIVELY PRIME denoted by (r.p).

Now a rational function is defined as \( f(s) = \frac{\alpha(s)}{\beta(s)} \), where \( \alpha(s), \beta(s) \in \mathcal{F}[s] \) and \( \beta(s) \neq 0 \), then \( \alpha(s) \) is the NUMERATOR and \( \beta(s) \) is the
DENOMINATOR polynomial. It is well known that in complex variable theory the finite poles and zeros of a rational function \( f(s) \) are defined via the above factorisation.

\[ f(s) = \frac{\alpha(s)}{\beta(s)} \]

where \( \alpha(s) \) and \( \beta(s) \) are \((r.p)\) polynomials. Then

**Definition 2.8.** ZEROS of \( f(s) \) are the zeros of \( \alpha(s) \) and POLES of \( f(s) \) are the zeros of \( \beta(s) \).

**Definition 2.9.** The rational function \( f(s) \) is said to be PROPER if \( f(\infty) \) exists and it is NON-PROPER if \( f(\infty) \) does not exist.

Now let \( F(s) \) be the field of rational functions \( f(s) = \frac{\alpha(s)}{\beta(s)} \), where \( \alpha(s), \beta(s) \in F[s] \) and \( \beta \neq 0 \). Then a discrete valuation on \( F(s) \) is a mapping \( V : F(s) \rightarrow Z \cup \{+\infty\} \) (\( Z : \) The ring of integers) such that

\[
\begin{align*}
V(f_1(s)f_2(s)) &= V(f_1(s)) + V(f_2(s)) \\
V(f_1(s) + f_2(s)) &\geq \min \{V(f_1(s)), V(f_2(s))\}
\end{align*}
\]

(2.10)

for all \( f_1(s), f_2(s) \in F(s) \) and \( V(0) \triangleq +\infty \).

Taking \( f(s) = \frac{\alpha(s)}{\beta(s)} \in F(s) \) and defining

\[ V(f(s)) = \deg \beta(s) - \deg \alpha(s) \]

(2.11)

It is clear that (2.11) satisfies (2.10) and therefore it is a valuation on \( F(s) \). Hence

**Lemma 2.12.** \( f(s) \in F(s) \) is a PROPER RATIONAL FUNCTION in case \( V(f(s)) \geq 0 \). i.e. \( \deg \beta(s) \geq \deg \alpha(s) \).

**Definition 2.13.** Any rational function \( f(s) \) is said to be BIPROPER in case \( f(\infty) \) exists and is not zero. Therefore \( f(s) \) is biproper in case \( V(f(s)) = 0 \). i.e. \( \deg \alpha(s) = \deg \beta(s) \).
Now as before let \( f(s) = \frac{\alpha(s)}{\beta(s)} \in F(s) \) and write

\[
f(s) = \frac{\alpha(s)}{\beta(s)} \cdot s^{V(f(s))} \cdot s^{-V(f(s))}
\]  

(2.14)

where \( V(f(s)) \) is the valuation of \( f(s) \). Then (2.14) becomes

\[
f(s) = f'(s) \cdot s^{-V(f(s))}
\]

(2.15)

where \( f'(s) \) is a biproper rational function. It can be seen that any rational function \( f(s) \) can be written as a product of a biproper rational function and a factor \( s^{-V(f(s))} \) where \( V(f(s)) \) is the valuation of \( f(s) \).

**Lemma 2.16.** Let \( f_1(s), f_2(s) \in F(s) \) and \( f_2(s) \neq 0 \), then there exists a proper rational function \( q(s) \in F(s) \) and a rational function \( r(s) \in F(s) \) such that

\[
f_1(s) = f_2(s)q(s) + r(s)
\]

(2.17)

where

\[
V(r(s)) < V(f_2(s)) \text{ or } r(s) \equiv 0
\]

(2.18)

As in the subsequent chapters a comparison between Pernebo's work on generalised polynomials and \( \Lambda \)-generalised polynomials and our local approach to linear system theory is made, it will be appropriate that a brief discussion of his work be presented in the final part of this section.

According to Pernebo, a \( \Lambda \)-generalised polynomial is defined as a rational function with all its poles outside a given subset \( \Lambda \) of the complex plane \( C \). The reason that they are called generalised polynomials rather than generalised rational functions is simply because algebraically they have more properties in common with the polynomials than with the rational functions.
They are a ring and in fact a euclidean domain just as the polynomials are, but not a field like the rational functions.

The following definitions and results are also given by Pernebo.

**Definition 2.19.** The set of \( \Lambda \)-generalised polynomials, denoted \( F_\Lambda[s] \), is defined as the set of rational functions with no poles in \( \Lambda \).

Addition and multiplication in \( F_\Lambda[s] \) are defined as in \( F[s] \). It then follows directly from the definition that \( F_\Lambda[s] \) is a ring. The invertible elements (units) in the ring \( F_\Lambda[s] \) are characterised by the following result.

**Lemma 2.20.** The units in \( F_\Lambda[s] \) are all non-zero rational functions with no poles and zeros in \( \Lambda \). i.e. with the previous results in mind, it can be seen that the units in \( F_\Lambda[s] \) are biproper rational functions in \( \Lambda \). Hence

**Lemma 2.21.** Every \( g(s) \in F_\Lambda[s] \) can be uniquely factorised as

\[
g(s) = g'(s)r(s) \tag{2.22}
\]

where \( g'(s) \) is a monic polynomial with all its zeros in \( \Lambda \) and \( r(s) \) biproper in \( \Lambda \).

**Definition 2.23.** Let \( g(s) \in F_\Lambda[s] \) be non-zero. Then the \( \Lambda \)-degree of \( g(s) \), denoted \( \deg_\Lambda g(s) \), is the degree of the polynomial \( g'(s) \), defined in (2.22).

**Section II-3: Polynomial and Rational Matrices**

In this section after defining polynomial and rational matrices some of their important and useful properties will be presented, (Macduffee, 1956; Lancaster, 1969; Lancaster and Tismenetsky, 1985).
A most useful class of matrices in linear systems theory, are those whose elements are polynomials of finite degree with coefficients in the field of real or complex numbers.

Since the elementary operations play an essential role in the theory of the polynomial matrices, it is appropriate to define them as follows,

**Definition 3.1.** The following three elementary row (column) operations on a polynomial matrix $A(s)$ with coefficients in $F$ are defined;

(i) Interchanging any two rows (columns).

(ii) Adding to any row (column), any other row (column) multiplied by a polynomial.

(iii) Multiplying any row (column) by a non-zero scalar in $R$.

**Definition 3.2.** The polynomial matrix $U(s)$ is called a UNIMODULAR matrix if and only if its determinant is a non-zero scalar in $R$.

From the above definition, it can be seen that any sequence of elementary row (column) operations on $A(s)$ is equivalent to premultiplication (postmultiplication) from left (right) of $A(s)$ by an appropriate unimodular matrix $U_L(s)$ ($U_R(s)$).

**Definition 3.3.** Two polynomial matrices $A(s)$ and $A_1(s)$ are called UNIMODULAR EQUIVALENT if and only if one can be obtained from the other by a sequence of elementary operations i.e. if and only if

$$A(s) = U_L(s) A_1(s) U_R(s)$$

where $U_L(s)$ and $U_R(s)$ are unimodular matrices.
**Definition 3.5.** The degree of a polynomial matrix $A(s)$ is equal to the degree of the polynomial element of highest degree in $A(s)$. Also the degree of the $i^{th}$ column (row) of $A(s)$, is the degree of the polynomial element of highest degree in the $i^{th}$ column (row) of $A(s)$.

The high-order coefficient matrix for the columns (rows) of $A(s)$ is a scalar matrix with elements in $R$ consisting of the coefficients of the highest degree $s$ terms in each column (row) of $A(s)$.

**Definition 3.6.** An $(m \times n)$ polynomial matrix $A(s)$ is said to be COLUMN (ROW) PROPER if and only if the high-order coefficient matrix for its columns (rows) has full rank ($= \min (m, n)$).

**Definition 3.7.** Let $M(s)$ be an $(l \times l)$ nonsingular polynomial matrix, then $M(s)$ is said to be ROW-COLUMN-REDUCED (r.c.r) if and only if there exist integers $r_i \geq 0$, $i \in \{1, 2, \ldots, l\}$ and $k_j \geq 0$, $j \in \{1, 2, \ldots, l\}$ such that

$$
\lim_{s \to \infty} \text{diag} \left[ s^{-r_i} \right]_{i=1}^{l} M(s) \text{ diag} \left[ s^{-k_j} \right]_{j=1}^{l} = M_h
$$

(3.8)

with the $(l \times l)$ constant matrix $M_h$ nonsingular. The integers $r_i$ and $k_j$ are called ROW POWERS, respectively, COLUMN POWERS, and $M_h$ is the high-order coefficient matrix.

The notion of "Relatively Prime Polynomials" is a fundamental one in linear algebra, which can be extended to the case of polynomial matrices. But first the following definitions.

**Definition 3.9.** Let the $(m \times n)$ polynomial matrix $A(s)$ be given as $A(s) = B(s)C(s)$, then the $(m \times m)$ polynomial matrix $B(s)$ is called a LEFT DIVISOR of $A(s)$, and $A(s)$ is called a LEFT MULTIPLE of $C(s)$. 

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A GREATEST COMMON LEFT DIVISOR denoted by (g.c.l.d) of two polynomial matrices $A(s)$ and $A_1(s)$ is a common left divisor which is a left multiple of every common left divisor of $A(s)$ and $A_1(s)$. Similar definition can be given for RIGHT DIVISOR.

**Definition 3.10.** The polynomial matrices $A(s)$ and $A_1(s)$ of ($m \times n$) and ($m \times l$) dimensions respectively are said to be RELATIVELY LEFT PRIME denoted by (r.l.p) if and only if their (g.c.l.d) are unimodular matrices. RELATIVELY RIGHT PRIME may be defined similarly.

**Definition 3.11.** Let $G(s)$ be an ($m \times l$) rational matrix, then the triple of polynomial matrices ($V(s)$, $T(s)$, $U(s)$) is said to be RIGHT-LEFT-COPRIM FRACTION (r.l.c.f) of $G(s)$ if and only if

(i) $\det (T(s)) \neq 0$,

(ii) $G(s) = V(s) T^{-1}(s) U(s)$,

(iii) $(V(s), T(s))$ is right coprime (r.c) and $(T(s), U(s))$ is left coprime (l.c).

If (iii) is not required, then $(V(s), T(s), U(s))$ is said to be a RIGHT-LEFT FRACTION (r.l.f) of $G(s)$.

**Definition 3.12.** Two polynomial matrices $A(s)$ and $A_1(s)$ of ($m \times n$) and ($m \times l$) dimensions respectively are said to form a minimal basis (Forney, 1975) if and only if

(i) $A(s)$ and $A_1(s)$ are relatively left prime.

(ii) The high-order coefficient matrix for the rows of $A(s)$, $A_1(s)$ has full row rank.
Now note that any \((m \times n)\) rational matrix \(W(s)\) can be factored in either of two (non-unique) ways (Rosenbrock, 1970; and Wolovich, 1973), i.e.

\[
W(s) = D^{-1}(s) N(s) = N_1(s) D_1^{-1}(s)
\]  

(3.13)

where \(D(s)\) and \(N(s)\) are \((m \times m)\) and \((m \times n)\) and \(N_1(s)\) and \(D_1(s)\) are \((m \times n)\) and \((n \times n)\) polynomial matrices. If additionally \(D(s)\) and \(N(s)\) are \((r.l.p)\) and \(N_1(s)\) and \(D_1(s)\) are \((r.r.p)\), then (3.13) is called a LEFT (RIGHT) COPRIME MATRIX FRACTION DESCRIPTION (MFD).

**Definition 3.14.** The left (right) matrix fraction description (3.13) is said to be MINIMAL in case the matrix \((D(s) N(s)) ((D_1^T(s) N_1^T(s))^T)\) forms a minimal basis.

As we have seen, unimodular matrices play an essential role in the various matrix equivalence transformations, one of these important transformations for a polynomial matrix is the Smith form given by the following theorem.

**Theorem 3.15.** Let \(A(s)\) be an \((m \times n)\) polynomial matrix with rank \(r = \min (m,n)\). Then there exist unimodular matrices \(M(s)\) and \(N(s)\) such that

\[
S(s) = M(s) A(s) N(s)
\]  

(3.16)

where

\[
S(s) = \begin{pmatrix}
\varepsilon_1(s) \\
\vdots \\
\varepsilon_r(s)
\end{pmatrix}
\begin{pmatrix}
O_{r,n-r} \\
O_{m-r,r} \\
O_{m-r,n-r}
\end{pmatrix}
\]

(3.17)
where $\varepsilon_i(s), \ i = 1, \cdots, r$ are monic polynomials with divisibility property

$$\varepsilon_i(s) / \varepsilon_{i+1}(s) \quad (3.18)$$

Then $S(s)$ is called the SMITH FORM of $A(s)$ and $\varepsilon_i(s)$ are called the IN Variant POLYNOMIALS of $A(s)$.

Alternatively the Smith form may be found by defining the (g.c.d) of all the minors of order $i = 1, \cdots, r$ where $D_0(s) \geq 1$, then the $\varepsilon_i(s)$ are given by

$$\varepsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)}, \ i = 1, \cdots, r \quad (3.19)$$

The $D_i(s)$ are then called DETERMINANTAL DIVISORS of $A(s)$.

Now suppose that $\varepsilon_i(s)$ of $A(s)$ are decomposed into their monic irreducible factors to give

$$\varepsilon_i(s) = [\phi_1(s)]^{K_{i1}} [\phi_2(s)]^{K_{i2}} \cdots \cdots [\phi_j(s)]^{K_{ij}} \quad (3.20)$$

Then those factors $\phi^K$ of (3.20) with non-zero exponents are called the ELEMENTARY DIVISORS of $A(s)$.

The following is a well known result (Rosenbrock, 1970) concerning relatively prime polynomial matrices.

**Theorem 3.21.** The polynomial matrices $A(s) \ (m \times m), \ A'(s) \ (m \times n)$ are relatively left prime if and only if one of the following equivalent statements is satisfied;

(i) The rank of $(A(s), A'(s))$ is $m$ for all $s \in F$.

(ii) The Smith form of $(A(s), A'(s))$ is $(I_m \ 0)$. 

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Similar results can be given for relatively right prime polynomial matrices.

Now the set of rational functions $f(s)$ constitute a field called the FIELD OF RATIONAL FUNCTIONS denoted by the $F(s)$. A matrix $W(s)$ whose elements are from the field of rational functions is called a RATIONAL MATRIX. As in the case of the polynomial matrices the following elementary row (column) operations on an $(m \times n)$ rational matrix $W(s)$ are defined,

(i) Interchanging any two rows (columns).

(ii) Adding to any row (column), any other row (column) multiplied by a rational function.

(iii) Multiplying any row (column) by a non-zero rational function.

The Smith-McMillan form of a rational matrix $W(s)$ can be determined by the following:

Let $W(s) = \frac{N(s)}{d(s)}$ where $d(s)$ is the monic least common multiple of the denominators of the entries of $W(s)$ and $N(s)$ is a polynomial matrix. Then $N(s)$ can be written as

$$N(s) = d(s) \frac{W(s)}{W(s)} = U_1(s) S(s) U_2(s)$$

(3.22)

where $U_1(s)$, $U_2(s)$ are unimodular matrices and $S(s)$ is in Smith form. Now from (3.22)

$$U_1^{-1}(s) W(s) U_2^{-1}(s) = \frac{S(s)}{d(s)} = \text{diag} \left\{ \frac{\phi_i(s)}{d(s)} \right\}$$

(3.23)

$i = 1, \cdots, r$ where $r = \text{rank of } W(s)$, and reduce the elements of the rational matrix $\frac{S(s)}{d(s)}$ to lowest terms. Therefore

$$\frac{\phi_i(s)}{d(s)} = \frac{\epsilon_i(s)}{\psi_i(s)}$$

(3.24)
where \( \varepsilon_i(s) \) and \( \psi_i(s) \) are relatively prime. Then \( W(s) \) can be written as

\[
W(s) = U_1(s) \, M(s) \, U_2(s) \quad (3.25)
\]

where

\[
M(s) = \begin{pmatrix}
\begin{pmatrix}
\varepsilon_1(s) \\
\psi_1(s)
\end{pmatrix} & \cdots & \begin{pmatrix}
O_{r,n-r} \\
O_{m-r,r} & O_{m-r,n-r}
\end{pmatrix}
\end{pmatrix}
\]

with the properties that

\[
\varepsilon_i(s) / \varepsilon_{i+1}(s) \quad i = 1, \ldots, r - 1 \quad (3.27)
\]

\( M(s) \) is called the SMITH-McMILLAN FORM of \( W(s) \).

Finally in this section some definitions and results concerning the finite zeros and poles of rational functions, polynomial and rational matrices are presented.

**Definition 3.28.** \( s_0 \in C \) is a FINITE ZERO OF DEGREE \( k \) of a polynomial matrix \( A(s) \) in case \( (s - s_0)^k \) is an elementary divisor of \( A(s) \).

(In fact \( s_0 \) is a zero of some invariant polynomial in the Smith form of \( A(s) \)).

The set of zeros of \( A(s) \) is the set of all such numbers \( s_0 \), a zero of degree \( k \) being included \( k \) times.

A simple test to determine whether \( s_0 \) is a zero of \( A(s) \) or not is through the following lemma;
**Lemma 3.29.** \( s_0 \in C \) is a zero of the polynomial matrix \( A(s) \) if and only if
\[
\text{Rank } A(s_0) < \text{Normal rank } A(s) \tag{3.30}
\]

Now from (3.13), it can be seen that any \((m \times n)\) rational matrix \( W(s) \) can be written in a non unique \((r.l.p)\) or \((r.r.p)\) decomposition, then

**Definition 3.31.** Any \((m \times n)\) polynomial matrix such as \( N(s) \) or \( N_1(s) \) satisfying (3.13) is called a NUMERATOR of \( W(s) \) and any \((m \times m)\) polynomial matrix such as \( D(s) \) or \((n \times n)\) polynomial matrix such as \( D_1(s) \) satisfying (3.13) is called a DENOMINATOR of \( W(s) \).

In case of \((r.l.p)\) or \((r.r.p)\) decomposition it can be shown (Wolovich, 1974) that any two numerators of a \((m \times n)\) rational matrix are unimodular equivalent and any two denominators are extended unimodular equivalent (Pugh and Ratcliffe, 1979). Now based on the above definitions, the poles and zeros of a rational matrix \( W(s) \) can be defined as,

**Definition 3.32.** \( s_0 \in C \) is said to be a ZERO (POLE) of degree \( k \) of \( W(s) \) in case it is a zero of degree \( k \) of any numerator (denominator).

An alternative equivalent definition for finite poles and zeros of a rational matrix \( W(s) \) has been provided by Rosenbrock (1970) by means of the McMillan standard form in the following way,

**Definition 3.33.** The zeros of the \( \varepsilon_i(s) \) in (3.26) are the FINITE ZEROS of \( W(s) \) and the zeros of the \( \psi_i(s) \) in (3.26) are the FINITE POLES of \( W(s) \) (counted according to their multiplicity and degree).
Section II-4: §Infinite Poles and Zeros of a Rational Matrix and Some Related Results

Having defined and discussed the finite zeros and poles of a rational matrix, we are now in a position to define and discuss the infinite poles and zeros of a rational matrix $G(s)$. In this regard some known results and definitions are presented and for completeness a few proofs are also provided.

To define the infinite poles and zeros, the standard technique of complex variable theory is used, and the simple bilinear transformation $s = \frac{1}{w}$ is performed. This transformation takes the point $s = 0$ to the point $w = \infty$ and the point $s = \infty$ to the point $w = 0$, while all other points in the complex $s$-plane are carried onto finite points in the complex $w$-plane in a one-to-one manner. Hence,

**Definition 4.1.** $G(s)$ is said to have an INFINITE ZERO (POLE) OF DEGREE $k$ in case $w = 0$ is a finite zero (pole) of degree $k$ of the rational matrix $G(\frac{1}{w})$.

Definition (4.1) together with those of the previous section permit many of the results of complex variable theory concerning rational functions to be generalised to rational matrices. The following is one such result.

**Theorem 4.2.** The rational matrix $G(s)$ is polynomial if and only if it has no finite poles.

**Proof.** If $G(s)$ is polynomial then it has a relatively prime factorisation

$$G(s) = I_m^{-1} G(s) \quad (4.3)$$

Hence,
$I_m$ is a denominator of $G(s)$, and clearly $G(s)$ has no finite poles. Conversely suppose that

$$G(s) = D^{-1}(s) N(s) \quad (4.4)$$

is a relatively prime factorisation of $G(s)$. If $G(s)$ has no finite poles then all denominators of $G(s)$ have no finite zeros. Thus the Smith form of $D(s)$ is $I_m$, and so $D(s)$ is unimodular. Consequently $D^{-1}(s)$ is a polynomial matrix as is $G(s)$.

Two immediate deductions from the above theorem are;

**Corollary 4.5.** A polynomial matrix has all its poles at infinity.

**Corollary 4.6.** If $G(s)$ is an $(m \times m)$ rational matrix with rank $m$, then it is a unimodular polynomial matrix if and only if it has no finite poles and no finite zeros.

**Proof.** If $G(s)$ has no finite poles, then by theorem (4.2) it is polynomial. Since the polynomial matrix $G(s)$ has no finite zeros, then its Smith form is $I_m$. i.e. $G(s)$ is unimodular.

Conversely if $G(s)$ is polynomial, then it has no finite poles. Also since $G(s)$ is unimodular,

$$G(s) = [G(s)^{-1}]^{-1} I_m \quad (4.7)$$

is a prime factorisation of $G(s)$ from which it is obvious that $G(s)$ has no finite zeros.

**Theorem 4.8.** A rational matrix $G(s)$ is proper if and only if it has no infinite poles.
**Proof.** See Pugh and Ratcliffe (1979)

**Corollary 4.9.** A rational matrix $G(s)$ has infinite poles if and only if it is non-proper.

**Proof.** Follows immediately from theorem (4.8)

**Corollary 4.10.** $G(s)$ has a pole at infinity if and only if for some $i$ and $j$

$$\lim_{s \to \infty} g_{ij}(s) = \infty$$  \hspace{1cm} (4.11)

**Proof.** $G(s)$ has a pole at infinity if and only if it is non-proper by corollary (4.9). But by definition $G(s)$ is non-proper if and only if for some $i$ and $j$

$$\lim_{s \to \infty} g_{ij}(s) = \infty$$

and the corollary follows immediately.

The following theorem due to Ratcliffe (1982) is also concerning the finite and infinite poles of a rational matrix.

**Theorem 4.12.** Let $G(s)$ be an $(m \times l)$ rational matrix. Then $G(s)$ may be written as

$$G(s) = G_s(s) + D(s)$$  \hspace{1cm} (4.13)

where $G_s(s)$ is strictly proper and $D(s)$ is polynomial. Then the finite poles of $G(s)$ are the finite poles of $G_s(s)$ and the infinite poles of $G(s)$ are the infinite poles of $D(s)$.

Finally this section is concluded by the following result.

**Theorem 4.14.** $s_0 \in C$ is a pole of the rational matrix $G(s)$ if and only if, for some $i$ and $j$,

$$\lim_{s \to s_0} g_{ij}(s) = \infty$$  \hspace{1cm} (4.15)
Proof. If $s_0 \in C$ is a pole of $G(s)$ then $(s - s_0)$ is a factor of some invariant polynomial of the denominators of $G(s)$. In particular, if the Smith form of the denominator of $G(s)$ is

$$S(D) = \text{diag} \ (\phi_r, \ \phi_{r-1}, \cdots, \phi_1) \quad (4.16)$$

where $r = l$ or $m$ depending on the factorisation of $G(s)$, then $(s - s_0)$ divides $\phi_i(s)$ for some $i$. By the divisibility properties of the $\phi_i(s)$, $(s - s_0)$ must be a factor of $\phi_1(s)$. But $\phi_1(s)$ is the least common denominator of elements of $G(s)$ and hence $(s - s_0)$ occurs in the denominator of at least one element $g_{i,j}(s)$ of $G(s)$. Thus by the corresponding theorem of complex variable theory,

$$\lim_{s \to s_0} g_{ij}(s) = \infty$$

Conversely, since $g_{ij}(s)$ is a rational function, its only singularities are poles (finite or infinite). Thus if (4.15) holds for finite $s_0$, then $s_0$ is a finite pole of $g_{i,j}(s)$. Consequently $(s - s_0)$ is a factor of $\phi_1(s)$ and hence $s_0$ is a pole of $G(s)$.

Section II-5: §System Matrices

In the theory of linear multivariable systems (Rosenbrock, 1970; Wolovich, 1974), a system may be described by a set of linear differential and algebraic equations of the form

$$\begin{align*}
T(E) \xi(t) &= U(E) u(t) \\
y(t) &= V(E) \xi(t) + W(E) u(t)
\end{align*} \quad (5.1)$$

where

$$E \equiv \frac{d}{dt} \quad (5.2)$$
The matrices $T(r \times r)$, $U(r \times l)$, $V(m \times r)$ and $W(m \times l)$ are matrices of polynomials in the differential operator $E$ with coefficients from $C$, the field of complex numbers. $\xi(t)$, $u(t)$ and $y(t)$ are respectively the $r$, $l$ and $m$ dimensional vectors of system, input and output variables, and it is assumed that $|T(E)| \neq 0$, otherwise the first equation in (5.1) is indeterminate, in the sense that $\xi(t)$ will not be determined uniquely by the initial conditions.

In order to have a purely algebraic investigation it is more convenient to work in the frequency domain than the time domain. Accordingly, after Laplace Transformation with zero initial conditions, the equations (5.1) become the algebraic equations

$$
\begin{align*}
T(s)\bar{\xi} &= U(s)\bar{u} \\
\bar{y} &= V(s)\bar{\xi} + W(s)\bar{u}
\end{align*}
$$

(5.3)

where the matrices $T(s)$, $U(s)$, $V(s)$ and $W(s)$ are now matrices of polynomials in the Laplace Transform variable $s$, and $\bar{\xi}$, $\bar{u}$ and $\bar{y}$ are the Laplace Transforms of the system variable vector $\xi(t)$, the input vector $u(t)$ and the output vector $y(t)$ respectively.

Now the equations (5.3) can be written as a single matrix equation

$$
\begin{pmatrix}
T(s) & U(s) \\
-V(s) & W(s)
\end{pmatrix}
\begin{pmatrix}
\bar{\xi} \\
\bar{u}
\end{pmatrix} =
\begin{pmatrix}
0 \\
\bar{y}
\end{pmatrix}
$$

(5.4)

The $(r + m) \times (r + l)$ polynomial matrix,

$$
P(s) = \begin{pmatrix}
T(s) & U(s) \\
-V(s) & W(s)
\end{pmatrix}, \quad r \geq n
$$

(5.5)
which contains all the mathematical information required to study the system is called the POLYNOMIAL SYSTEM MATRIX of (5.3). Now the ORDER, denoted by \( n \), of \( P(s) \) is defined as

\[
 n = \delta (|T(s)|) \quad (5.6)
\]

where \( \delta(\cdot) \) denotes the degree of the indicated polynomial, and \( |\cdot| \) denotes the determinant of the indicated matrix.

Many systems can also be described (after taking Laplace Transforms and assuming zero initial conditions) by linear equations of the form

\[
\begin{align*}
(sI - A) \bar{x} &= B \bar{u} \\
\bar{y} &= C \bar{x} + D(s) \bar{u} 
\end{align*} \quad (5.7)
\]

where the \((n \times n)\) plant matrix \( A \), \((n \times l)\) input matrix \( B \) and \((m \times n)\) output matrix \( C \) are constants, and the state vector \( \bar{x} \), input vector \( \bar{u} \) and output vector \( \bar{y} \) are of \( n \), \( l \) and \( m \) dimensions (vector) respectively. \( D(s) \) is generally a polynomial matrix in \( s \) with coefficients in \( C \). The associated system matrix of (5.7) is of the form

\[
P_1(s) = \left( \begin{array}{c|c}
  sI_n - A & B \\
  \hline
  -C & D(s)
\end{array} \right) \quad (5.8)
\]

which is a special case of the polynomial system matrix (5.5) and is said to be in STATE - SPACE FORM.

Another representation useful in the study of a linear multivariable system is its input-output behaviour which is described in the frequency domain by a transfer function matrix \( G(s) \). The transfer function matrix of the system (5.3) associated with the polynomial system matrix \( P(s) \) is given by;
\[ G(s) = V(s) T^{-1}(s) U(s) + W(s) \]  

where \( G(s) \) is clearly a rational matrix, and the transfer function matrix of the system (5.7) associated with the system matrix \( P_1(s) \) in state-space form is given by;

\[ G(s) = C(s I - A)^{-1} B + D(s) \]  

The following definition concerning the transfer function matrix \( G(s) \) is important and used frequently hereafter.

**Definition 5.11.** The transfer function matrix \( G(s) \) is said to be **proper** if \( G(\infty) \) is finite, and it is **non-proper** if \( G(\infty) \) does not exist. If additionally \( G(\infty) = 0 \), then \( G(s) \) is said to be **strictly proper**.

Now looking at (5.10), it can be seen that the first term on the right hand side of (5.10) is strictly proper and the second term is polynomial, hence \( G(s) \) is proper if and only if \( D \) is constant and strictly proper if and only if \( D \) is zero.

**Section II-6: §Properties of System Matrices**

In this section some of the important properties of the system matrices will be presented.

Starting by considering the system matrix \( P(s) \) as given in (5.5) whose associated transfer function matrix \( G(s) \) is

\[ G(s) = V(s) T^{-1}(s) U(s) + W(s) \]  

Then the decoupling zeros of the system may be defined as;
Definition 6.2. If \( \beta \in C \) is a zero of the matrix \((T(s) U(s))\), then \( \beta \) is said to be an INPUT-DECOUPLING ZERO of \( P(s) \) denoted by \((i.d.z)\). The set \( \{\beta_1, \cdots, \beta_b\} \) of all such zeros is called the set of INPUT-DECOUPLING ZEROS of \( P(s) \). If \( \gamma \in C \) is a zero of the matrix \( \begin{pmatrix} T(s) \\ -V(s) \end{pmatrix} \), then \( \gamma \) is said to be an OUTPUT-DECOUPLING ZERO of \( P(s) \) denoted by \((o.d.z)\), and the set \( \{\gamma_1, \cdots, \gamma_c\} \) of all such zeros is called the set of OUTPUT-DECOUPLING ZEROS of \( P(s) \).

The set of input-decoupling zeros which are at the same time output-decoupling zeros is called the set of INPUT-OUTPUT-DECOUPLING ZEROS of \( P(s) \) denoted by \((i.o.d.z)\) and is given by \( \{\delta_1, \cdots, \delta_d\} \).

Then the set \( \{\beta_1, \cdots, \beta_b, \gamma_1, \cdots, \gamma_c\} - \{\delta_1, \cdots, \delta_d\} \) is called the set of DECOUPLING ZEROS of \( P(s) \).

The relationship between the decoupling zeros and the relative primeness for a system matrix \( P(s) \) can be seen through the following important theorem given by Rosenbrock (1970).

Theorem 6.3. The system matrix \( P(s) \) has no \((i.d.z)\) if and only if \( T(s) \) and \( U(s) \) are \((r.l.p)\) and has no \((o.d.z)\) if and only if \( T(s) \) and \( V(s) \) are \((r.r.p)\).

In state space terms it can be seen that the lack of \((i.d.z)\) corresponds exactly to the system being controllable (Rosenbrock, 1970). Similarly the lack of \((o.d.z)\) corresponds exactly to the system being observable.

It is well known that for a given transfer function matrix \( G(s) \) there are many system matrices, called REALISATIONS of \( G(s) \) which give rise to it. Now the addition of more decoupling zeros has no effect on \( G(s) \), but increases the order of a realisation of it, therefore there is no upper bound on the order of such realisation of a given transfer function matrix \( G(s) \). There
is, however, a lower bound on the order of a realisation of $G(s)$, denoted by $\gamma(G)$. Any realisation with $n = \gamma(G)$ is said to have least order, where $n$ the order of the system is given by the degree of $|T(s)|$.

The following theorem (Rosenbrock, 1970) gives a characterisation of such realisations.

**Theorem 6.4.** A system matrix $P(s)$ has least order if and only if one of the following equivalent conditions holds.

(i) $P(s)$ has no decoupling zeros.

(ii) $T(s)$, $U(s)$ are (r.l.p) and $T(s)$, $V(s)$ are (r.r.p)

One important concept associated with a system is the McMillan Degree, denoted by $\delta(G)$ of the associated transfer function matrix $G(s)$. Suppose $G(s)$ is written as

$$G(s) = G_s(s) + D(s) \quad (6.5)$$

where $G_s(s)$ is strictly proper and $D(s)$ is a polynomial matrix, then the McMillan degree of $G(s)$ is defined as (Kalman, 1965; Rosenbrock, 1970),

$$\delta(G(s)) = \gamma(G_s(s) + \gamma(D(s^{-1}))) \quad (6.6)$$

where $\gamma(\cdot)$ is the least order of the indicated matrix.

It has been seen that any system may be described by many different sets of equations of the form (5.1). In order to be able to analyse the system, it is often necessary to reduce these equations to a simpler form. The state-space form of equations (5.6) can be one of these simpler forms. One advantage of using the system matrix idea is that such transformations of the system equations may be represented as transformations on the system matrix $P(s)$ which are more easily understood. It is also important that
the transformations do not change the transfer function matrix $G(s)$ or the order of the system $P(s)$, since then such transformations do not effectively change the system or its important characteristics. Strict system equivalence (Rosenbrock, 1970) is one such transformation, defined as follows;

Let

$$P(s) = \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix} \quad \text{and} \quad P'(s) = \begin{pmatrix} T'(s) & U'(s) \\ -V'(s) & W'(s) \end{pmatrix}$$  \hspace{1cm} (6.7)

be two $(r + m) \times (r + l)$ polynomial system matrices, then,

**Definition 6.8.** $P(s)$ and $P'(s)$ are said to be STRICTLY SYSTEM EQUIVALENT (s.s.e) if and only if there exist unimodular matrices $M(s), (r \times r)$ and $N(s), (r \times r)$ and polynomial matrices $X(s), (m \times r)$ and $Y(s), (r \times l)$ such that

$$\begin{pmatrix} M(s) & 0 \\ X(s) & I_m \end{pmatrix} P(s) \begin{pmatrix} N(s) & Y(s) \\ 0 & I_l \end{pmatrix} = P'(s)$$ \hspace{1cm} (6.9)

where the order of the system $P(s)$ is $n$. Now some consequences of the above definition are

**Theorem 6.10.** Under (s.s.e) the following are invariant.

(i) The transfer function matrix $G(s)$ and therefore the set of finite poles of $G(s)$ and the order of the system.

(ii) The set of finite (i.d.z), (o.d.z) and (i.o.d.z) of $P(s)$.

(iii) The set of decoupling zeros of $P(s)$.

(iv) The set of zeros of $|T(s)|$.  


Generally, the operations of (s.s.e) and system matrix in state-space form, do not preserve the state-space form. If however $H$ is a non-singular constant matrix, then the transformation

$$
\begin{pmatrix}
H^{-1} & 0 \\
0 & I_m
\end{pmatrix}
\begin{pmatrix}
sI_n - A & B \\
-C & D(s)
\end{pmatrix}
\begin{pmatrix}
H & 0 \\
0 & I_l
\end{pmatrix}
= \begin{pmatrix}
sI_n - A_1 & B_1 \\
-C_1 & D_1(s)
\end{pmatrix}
$$

(6.11)

clearly preserves the state-space form. This transformation is called SYSTEM SIMILARITY denoted by (s.s), which is a special case of (s.s.e) and therefore the results of theorem (6.10) also hold for system similarity. The relationship between the above definitions is given by the following theorem (Rosenbrock, 1970).

**Theorem 6.12.** Two system matrices in state-space form are system similar if and only if they are strictly system equivalent.

Another form of system matrix $P(s)$ is when $P(s)$ is in rational form i.e. $T(s), U(s), V(s)$ and $W(s)$ are matrices over the rational functions. Clearly the class of rational system matrices includes the class of polynomial system matrices, and therefore includes system matrices in state-space form. It can be shown (Rosenbrock, 1970) that not only the transfer function matrix $G(s)$ is invariant under system equivalence, (for definition of system equivalence see Rosenbrock, 1970) but also

**Theorem 6.13.** The transfer function matrix $G(s)$ is a standard form for system matrices under system equivalence.

And
Theorem 6.14. Two system matrices are system equivalent if and only if they give rise to the same transfer function matrix.

The following results concerning the least order polynomial system matrices are used in the subsequent chapters, therefore they are stated here.

Theorem 6.15. Two \((r+m)\times(r+l)\) least order polynomial system matrices \(P(s), \ P_1(s)\) are strictly system equivalent if and only if they give the same transfer function matrix \(G(s)\).

Proof. See Rosenbrock (1970). \(\square\)

An immediate deduction from theorem (6.15) is

Corollary 6.16. If \(P(s)\) and \(P_1(s)\) are two system matrices in state-space form with least order, then they give the same transfer function matrix if and only if they are (s.s).

Theorem 6.17. Let

\[
P(s) = \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix}
\]  

be an \((r+m)\times(r+l)\) polynomial system matrix of least order giving rise to \(G(s)\). Let also the McMillan form of \(G(s)\) be \(M(s)\), the Smith form of \(P(s)\) be \(S_p(s)\) and the Smith form of \(T(s)\) be \(S_T(s)\). Then

\[
S_p(s) = \begin{pmatrix} I_r & 0_{r,m} \\ 0_{m-r} & \text{diag}(\varepsilon_1, \ldots, \varepsilon_m) \end{pmatrix} \quad l = m
\]  

with appropriate changes when \(l \neq m\). Also

\[
S_T(s) = \begin{pmatrix} I_{r-m} & 0_{r-m,m} \\ 0_{m,r-m} & \text{diag}(\psi_m, \ldots, \psi_1) \end{pmatrix} \quad l = m
\]  

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with the appropriate changes when \( l \neq m \). Here \( \psi_i(s) \) are in reverse order.

**Proof.** See Rosenbrock (1970).

Section II-7: §Conclusion and Discussions

The central idea of this chapter has been to provide sufficient mathematical background to the subsequent chapters and cover the most useful and essential results and definitions. In this regard it has been tried to be as brief as possible and only very few proofs are given. The polynomial and rational functions and some of their important properties have been described in section two and then the extension of the results and definitions given in section two (plus some more useful material) to the polynomial and rational matrices has been the subject of section three. The structure of infinite poles and zeros of a rational matrix and related results have been studied in section four. In section five after describing the system equations and system matrices, different representations of them and some definitions concerning the transfer function matrix were given. Finally in section six some of the most important properties of system matrices including the system of least order have been presented.
CHAPTER III

Some Results for Properness of a Composite System.
Section III-1: §Introduction

The implementation of constant output feedback and the consequent properness or non-properness of the resulting closed-loop transfer function matrix have long been of interest. Rosenbrock and Pugh (1974) have shown that for a strictly proper rational transfer function matrix $G(s)$ the closed-loop system will always be strictly proper (whenever it is defined). But for a general rational function matrix $G(s)$ the implementation of constant output feedback does not necessarily result in a proper closed-loop transfer function matrix, although Scott and Anderson (1976) have shown the closed-loop system to be generically proper. In the case of a general open-loop $G(s)$ that is given either in terms of a state-space realisation or as a matrix fraction description, necessary and sufficient conditions for closed-loop properness have been provided respectively by Pugh and Ratcliffe (1981) and Pugh (1984). For a proper $G(s)$ Krishnarao and Chen (1984) have provided some sufficient conditions for the properness and non-properness of the transfer function matrix of a state-estimator type feedback system. Application of these results to composite systems have also been of interest (Pugh, 1984). Our interest and contribution is based on the latter case. Sections 2, 3 and 4 present a review of known results mentioned above. In this regard, conditions for closed-loop system properness, both for the state-space approach and matrix fraction description are considered. In section five the feedback and open-loop compensator is described. A representation of the composite system is discussed in section six. Then using this representation a simple sufficient condition for properness of a general composite system is derived in section seven. Application of our results given in section seven to two specific composite systems is considered in section eight. Also in this section
non-properness of the two composite systems and application of the minimal factorisation to the two same systems are studied and some interesting results are derived. Finally the properness and non-properness of the two composite systems described in section eight from an infinite frequency point of view is considered in section nine.

Section III-2: §Well-Formedness and Properness of Systems

The study of the time domain properties of a polynomial system matrix and related algebraic properties of transfer function matrices, requires an investigation of the quality of the behavior at \( t = 0 \) and a description of the properties of the transfer function matrices generated by such a system. Therefore in this section a brief discussion on the various concepts of interval properness and well-formedness of a system is presented (Callier and Desoer, 1982; Kucera, 1984). Subsequently the problem of the overall properness and non-properness of closed-loop systems is discussed.

Consider a system described by the equations of (5.1), (5.2) and (5.3) of the chapter two with the polynomial system matrix \( P(s) \) and transfer function matrix \( G(s) \) given by

\[
P(s) = \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix}, \quad r \geq n
\]

and

\[
G(s) = V(s)T^{-1}(s)U(s) + W(s)
\]

where \( n \) is the order of the system. Then we have,
**Definition 2.3** The system matrix described by (5.1) of the chapter II is said to be WELL-FORMED if and only if for every initial value of $\xi(\cdot)$ and its derivatives at $t = 0^-$, and for every input $u(\cdot)$ such that

(i) $u^j(0^-) = \theta_t$ for all $j = 0, 1, 2, \ldots$, and

(ii) $\bar{u}$ is strictly proper rational vector (1-dimensions) we have that the state trajectory $\xi(\cdot)$ and response $y(\cdot)$ of the system satisfy that $\bar{\xi}$ and $\bar{y}$ are strictly proper rational vectors of $r$ and $m$ dimensions respectively.

**Theorem 2.4** Consider a system matrix given by the equations (5.1) of the Chapter II with transfer function matrix $G(s)$ and polynomial system matrix $P(s)$ given by (2.1). Then the system is well formed if and only if the rational matrices $T^{-1}(s), V(s)T^{-1}(s), T^{-1}(s)U(s)$ and $G(s) = V(s)T^{-1}(s)U(s) + W(s)$ are proper.

**Proof** See Callier and Desoer (1982).

Now the following result can be used as a test for a well formed system matrix.

**Corollary 2.5** Consider a system matrix described by the equations (5.1) of the Chapter II. If

(i) $T(s)$ is (r.c.r) with row powers $m_i \geq 0$, $i \in \{1, 2, \ldots, r\}$ and column powers $l_j \geq 0$, $j \in \{1, 2, \ldots r\}$,

(ii) $\text{Deg}_{ri}(U(s)) \leq m_i$ for all $i \in \{1, 2, \ldots r\}$, $\text{deg}_{cj}(V(s)) \leq l_j$ for all $j \in \{1, 2, \ldots, r\}$,
(iii) $W(s)$ is an $(m \times l)$ constant matrix, where $\deg_{ri}(\cdot)$ and $\deg_{cj}(\cdot)$ denote the degree of $i^{th}$ row and $j^{th}$ column of the indicated matrix, then the system is well formed.

**Proof** See Callier and Dosoer (1982).

Now consider a system matrix with the transfer function matrix $G(s) = V(s)T^{-1}(s)U(s)$ which is a (r.l.f). If the system is least order, then this (r.l.f) has to be coprime. If the system is well formed, then according to theorem (2.4) it is not sufficient for $G(s)$ to be proper. Hence the following definition, which makes the association of a (r.l.f) with a well-formed system matrix automatic.

**Definition 2.6** The (r.l.f) triple of polynomial matrices $(V(s), T(s), U(s))$ of $G(s)$ is said to be INTERNALLY PROPER if and only if $T^{-1}(s), V(s)T^{-1}(s), T^{-1}(s)U(s)$ and $G(s) = V(s)T^{-1}(s)U(s)$ are proper rational matrices.

The above theorem has the following corollary:

**Corollary 2.7** Let $G(s)$ be an $(m \times l)$ rational matrix and have a left-fraction $(T_L(s), N_L(s))$. The left-fraction $(T_L(s), N_L(s))$ of $G(s)$ is internally proper or equivalently the system matrix $(T_L(s), N_L(s), I, 0)$ is well formed if and only if $T(s) = T_1(s)T_2(s)$ where $T_2(s)$ is an $(r \times r)$ biproper rational matrix and $T_1(s)$ is an $(r \times r)$ column-reduced polynomial matrix such that $T_1(s)N_L(s)$ is proper.

Similar results can be given for the transfer function matrix $G(s)$ with a right-fraction $(N_R(s), T_R(s))$.  

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Another interesting result concerning the internal properness of $G(s)$ is given by the following lemma.

**Lemma 2.8** A left (right)-fraction $(T_L(s), N_L(s))((N_R(s), T_R(s)))$ of $G(s)$ (an $(m \times l)$ rational matrix) is internally proper if and only if $T_L^{-1}(s)(T_R^{-1}(s))$ and $G(s) = T_L^{-1}(s)N_L(s)(G(s) = N_R(s)T_R^{-1}(s))$ are proper rational matrices.

**Proof** See Callier and Desoer (1982).

Now depending on the particular realisation of the transfer function matrix $G(s)$ available, two types of conditions for those constant output feedback matrices $F$ which produce a non-proper closed-loop system $G_F(s)$ are presented. The first type of condition arises when a state-space realisation of the transfer function matrix $G(s)$ is known in which case the interrelation of $F$ and the polynomial part of $G(s)$ is seen to be crucial. The second type of condition arises when the transfer function matrix $G(s)$ is written as a minimal matrix fraction, in which case the high order-coefficient matrix of the matrix fraction turns out to play a fundamental role.

The following results show how the poles of an open-loop system represented by a general transfer function matrix $G(s)$ are effected by application of a constant output feedback.

**Lemma 2.9.** Under constant output feedback:

(i) The McMillan degree of the open-loop transfer function matrix $G(s)$, denoted $\delta(G(s))$ is invariant.

(ii) The least order of the open-loop transfer function matrix $G(s)$, denoted $\gamma(G(s))$, may change.
Proof.


(ii) See the following example.

Example 2.10. Let

\[
G(s) = \begin{pmatrix}
\frac{1}{s} & 1 \\
1 & 0 \\
\end{pmatrix}
\]  

(2.11)

Then a (r.l.p) factorisation of the rational matrix \(G(s)\) is given by

\[
G(s) = \begin{pmatrix}
0 & 1 \\
s & 0 \\
\end{pmatrix}^{-1} \begin{pmatrix}
1 & 0 \\
1 & s \\
\end{pmatrix}
\]  

(2.12)

Now the least order of \(G(s)\) is simply the degree of the determinant of the denominator matrix. Clearly in the case of (2.12)

\[
\gamma(G(s)) = 1
\]  

(2.13)

Next consider the closed-loop system, when the output feedback matrix \(F\) given by the following is applied.

\[
F = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\]  

(2.14)

Then

\[
G_F(s) = (I + G(s) F)^{-1} G(s) = \begin{pmatrix}
1 - \frac{s}{s} & \frac{s}{s} \\
\frac{s}{s} & -\frac{s}{s} \\
\end{pmatrix}
\]  

(2.15)

A (r.l.p) factorisation of (2.15) is then given by

\[
G_F(s) = \begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}^{-1} \begin{pmatrix}
1 & 0 \\
\frac{s}{s} & -\frac{s}{s} \\
\end{pmatrix}
\]  

(2.16)

It is then obvious from (2.16) that

\[
\gamma(G_F(s)) = 0
\]  

(2.17)
Comparing (2.13) and (2.17) clearly shows that the least order of $G(s)$ has not been preserved under the constant output feedback of (2.14). Now from a poles and zeros point of view lemma (2.9) can be interpreted by use of the following characterisations.

**Lemma 2.18.**

(i) The total number of finite and infinite poles of $G(s)$ (counted according to their degree and multiplicity) is the McMillan degree of $G(s)$.

(ii) The total number of finite poles of $G(s)$ (counted according to their degree and multiplicity) is the least order of $G(s)$.

**Proof.** See Rosenbrock (1970) or Pugh and Ratcliffe (1979).

The above results simply show that the poles of the open-loop system move around the complex plane when a constant output feedback is applied. The difference between the least order and the McMillan degree occurs when some of the poles are moved to the point at infinity. But if the application of constant output feedback does not place any poles at infinity, then the least order and McMillan degree are the same and that is precisely the definition of properness for $G_F(s)$, Rosenbrock and Pugh (1974), Verghese (1978) and Pugh and Ratcliffe (1979). Thus the properness or non-properness of a closed-loop system $G_F(s)$ is of interest, since it shows whether or not $G_F(s)$ has any infinite poles. This section is finally concluded with the following result.

**Lemma 2.19.** For a finite constant feedback matrix $F$;

(i) $G_F(s)$ is strictly proper for all $F$, if and only if $G(s)$ is strictly proper.

(ii) $G_F(s)$ is almost always proper if $G(s)$ is not strictly proper.
Proof.

(i) See Rosenbrock and Pugh (1974).


From part (ii) of lemma (2.19) it can be seen that non-properness of the closed-loop system occurs only if the open-loop system is not strictly proper. As a consequence of this result there remains just one interesting question and that is to characterise those feedback matrices $F$ which produce a non-proper closed-loop system $G_F(s)$. It is this problem which is studied in the next sections and in this regard some necessary and sufficient conditions are derived.

Section III-3: §State-space Approach

It has been seen that the question of whether or not a closed-loop system is proper only arises in case the open-loop system is not strictly proper. So assuming that a least order state-space realisation of the transfer function matrix $G(s)$ is known and $G(s)$ can be written as

$$G(s) = G_s(s) + D(s) \quad (3.1)$$

where $G_s(s)$ is the strictly proper part and $D(s)$ is the polynomial part of $G(s)$ and also

$$D(s) \neq 0 \quad (3.2).$$

Those feedback matrices $F$ which produce a non-proper closed-loop system $G_F(s)$ are characterised as follows.
Let
\[
\begin{pmatrix}
  sI - A & B \\
  -C & D(s)
\end{pmatrix}
\]  
be the system matrix corresponding to the known least order state-space realisation of \( G(s) \), where \( \gamma = \gamma (G(s)) \). The transfer function matrix \( G(s) \) then is given by;
\[
G(s) = C(sI - A)^{-1} B + D(s)
\]  
where \( A, B, C \) are constant matrices and \( D(s) \) is as in (3.2).

The following theorem due to Pugh and Ratcliffe (1981) is of great importance to the rest of this section, therefore its proof is also provided.

**Theorem 3.5.** If the least order system matrix (3.3) has the transfer function matrix \( G(s) \) as in (3.4) with \( D(s) \neq 0 \), then the closed-loop transfer function matrix \( G_F(s) \) is proper if and only if
\[
\delta(|I_m + D(s)F|) = \delta(D(s))
\]  
where \( | \cdot | \) denotes the determinant of the indicated matrix.

**Proof.** Following Rosenbrock (1970) after Laplace transformation, the above system has the equations,
\[
\begin{cases}
  (sI - A) \bar{x} = B \bar{u} \\
  \bar{y} = C \bar{x} + D(s) \bar{u}
\end{cases}
\]  

Now let the feedback \( F \) be applied according to the equation
\[
\bar{u} = \bar{u}_C - F \bar{y}
\]  
and then
\[
\bar{y} = \bar{y}_C
\]  

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where $\tilde{u}_C$ and $\tilde{y}_C$ are the new input and output. Equations (3.7), (3.8) and (3.9) can be written as a single matrix equation,

$$
\begin{pmatrix}
  sI - A & B & 0 & 0 \\
  -C & D(s) & I & 0 \\
  0 & -I & F & I \\
  0 & 0 & -I & 0 \\
\end{pmatrix}
\begin{pmatrix}
  \tilde{x} \\
  -\tilde{u} \\
  \tilde{y} \\
  -\tilde{u}_C \\
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
\end{pmatrix}.
$$

(3.10)

Now since $P(s)$ in (3.3) has least order, a least order realisation of the closed-loop transfer function matrix $G_F(s)$ is given by

$$
P_F(s) = \begin{pmatrix}
  sI - A & B & 0 & 0 \\
  -C & D(s) & I & 0 \\
  0 & -I & F & I \\
  0 & 0 & -I & 0 \\
\end{pmatrix}
$$

(3.11)

By (s.s.e) (Rosenbrock, 1970) $P_F(s)$ can be reduced to the following least order form;

$$
P'_F(s) = \begin{pmatrix}
  sI - A & BF & B \\
  -C & I + D(s)F & D(s) \\
  0 & -I & 0 \\
\end{pmatrix}
$$

(3.12)

Since $P_F(s)$ and $P'_F(s)$ are least order realisations of $G_F(s)$, then

$$
\gamma(G_F(s)) = \delta \begin{pmatrix}
  sI - A & BF \\
  -C & I + D(s)F \\
\end{pmatrix}
$$

(3.13)

The Laplace expansion of the first $\gamma$ rows of the determinant

$$
\begin{vmatrix}
  sI - A & BF \\
  -C & I + D(s)F \\
\end{vmatrix}
$$

(3.14)

shows that the highest degree for (3.14) generated from the first $\gamma$ rows is $\gamma$, and since $C$ is constant the highest degree among minors of all orders of $(-C \ I + D(s)F)$ is the McMillan degree of $(I + D(s)F)$. Now

$$
\delta (I + D(s)F) = \delta (D(s)F) \leq \delta (D(s))
$$

(3.15)
From (3.15) it can be seen that $\delta(D(s))$ is an upper bound for the degree of minors of all orders of (3.14). Hence if

$$\delta([I + D(s) F]) = \delta(D(s)) \tag{3.16}$$

then a term of degree $(\gamma + \delta(D(s)))$ is contained in the above Laplace expansion of (3.14) and in fact from the form of the first $\gamma$ rows of (3.14) it can be seen that this is the only term with degree $(\gamma + \delta(D(s)))$ and all other terms have degree strictly less than $(\gamma + \delta(D(s)))$. Hence (3.14) has degree $(\gamma + \delta(D(s)))$ if (3.16) is true i.e.

$$\delta \left( \begin{vmatrix} s I_{\gamma} - A & BF \\ -C & I + D(s) F \end{vmatrix} \right) = \delta(D(s)) + \gamma. \tag{3.17}$$

But since $P_{F}^{'t}(s)$ has least order, we have

$$\gamma(G_F(s)) = \delta(D(s)) + \gamma. \tag{3.18}$$

Now by definition (Kalman, 1965; Rosenbrock, 1970),

$$\delta(G(s)) \triangleq \gamma(G_s(s)) + \gamma(D(s^{-1})) \tag{3.19}$$

$$\gamma(D(s^{-1})) \triangleq \delta(D(s)) \tag{3.20}$$

$$\gamma(G_s(s)) \triangleq \gamma \tag{3.21}$$

where $G_s(s)$ and $D(s)$ are defined in (3.1). Therefore

$$\gamma(G_F(s)) = \delta(G(s)). \tag{3.22}$$

Now by lemma (2.1), $\delta(G(s)) = \delta(G_F(s))$ and hence

$$\gamma(G_F(s)) = \delta(G(s)) = \delta(G_F(s)). \tag{3.23}$$
i.e. $G_F(s)$ is proper.

Conversely if $G_F(s)$ is proper, then $\gamma(G_F(s)) = \delta(G_F(s))$. But we have

$$\delta(G_F(s)) = \delta(G(s)) = \gamma + \delta(D(s))$$

and

$$\gamma(G_F(s)) = \delta \left( \begin{bmatrix} sI - A & BF \\ -C & I + D(s)F \end{bmatrix} \right)$$

i.e.

$$\gamma + \delta(D(s)) = \delta \left( \begin{bmatrix} sI - A & BF \\ -C & I + D(s)F \end{bmatrix} \right)$$

But as proved in the first part (see (3.16) and (3.17)) this relation is true only if $\delta(|I + D(s)F|) = \delta(D(s))$. Thus if $G_F(s)$ is proper, then (3.16) is satisfied and that completes the proof.

Theorem (3.5) has the following corollaries.

**Corollary 3.26.** The closed-loop system transfer function matrix $G_F(s)$ is non-proper if and only if

$$\delta(|I + D(s)F|) < \delta(D(s))$$

**Proof.** As noted in theorem (3.5), $\delta(D(s))$ is an upper bound for $\delta(|I + D(s)F|)$ and so (3.6) does not hold if and only if (3.27) is true.

**Corollary 3.28.** If $G(s)$ is proper (i.e. $D(s) \equiv D$ a constant matrix), then the closed-loop system transfer function matrix $G_F(s)$ is non-proper if and only if

$$|I + DF| = 0$$

(3.29)
Proof. If $D(s) = 0$ a constant matrix, then $\delta(D) = 0$ and so from (3.27) $G_F(s)$ is non-proper if and only if $\delta(|I + DF|) < 0$ and this is true if and only if $|I + DF| = 0$. The following example illustrates the above result. □

Example 3.30. Reconsider example (2.2) with $G(s) = \begin{pmatrix} \frac{1}{s} & 1 \\ 1 & 0 \end{pmatrix}$, then according to (3.1), $G(s)$ can be written as

$$\begin{pmatrix} \frac{1}{s} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(3.31)

where

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(3.32)

Then for $F$ of (2.6) we have

$$|I + DF| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

(3.33)

which by corollary (3.28) indicates that the closed-loop system transfer function matrix $G_F(s)$ will be non-proper and this is verified directly by (2.7).

The condition of theorem (3.5) provides a complete characterisation of those feedback matrices producing non-proper closed-loop systems. Although this result was proved by reference to a state-space realisation of $G(s)$, it is not necessary to generate such a realisation, if only $G(s)$ is available, in order to check the conditions (3.6). All that is required is the direct feed through matrix $D(s)$ of $G(s)$ and this can be more simply generated via (3.1).

Despite this, the condition (3.6) of theorem (3.5) may be rather difficult to check. For if the size of the matrix $D(s)$ is large, although is not too difficult to compute the $\delta(D(s))$ using for example the algorithms (5.1) of Rosenbrock (1970) or other methods, it is almost impossible to determine
the δ(|I + D(s) F|) for a large matrix D(s), simply because of difficulty of computing the determinant of matrix (I + D(s) F). How this difficulty can be overcome is the subject of the next section.

Section III-4: §Matrix Fraction Description

It has been seen that the condition (3.6) of theorem (3.5) may prove difficult to check. If however a matrix fraction description of G(s) is available, then a more readily verifiable condition may be generated as follows.

Definition 4.1. Let

\[ G(s) = T_L^{-1}(s) N_L(s) \]  

(4.2)

be a left matrix fraction description such that the matrix

\[
\begin{pmatrix}
T_L(s) & N_L(s)
\end{pmatrix}
\]  

(4.3)

forms a minimal basis, then (4.2) is said to be MINIMAL. Similarly let

\[ G(s) = N_R(s) T_R^{-1}(s) \]  

(4.4)

be a right matrix fraction description such that the matrix

\[
\begin{pmatrix}
T_R(s) \\
N_R(s)
\end{pmatrix}
\]  

(4.5)

forms a minimal basis, then (4.4) is said to be MINIMAL.

Now the importance of minimal factorisation from a feedback point of view is that a minimal factorisation of the closed-loop system transfer function matrix \( G_F(s) \) can be generated from a minimal factorisation of the open-loop system transfer function matrix \( G(s) \) in a rather simple manner. The following result explains.
Lemma 4.6. If the minimal factorisation (4.2) gives rise to $G(s)$, then a minimal factorisation of $G_F(s)$ is given by

$$G_F(s) = (T_L(s) + N_L(s) F)^{-1} N_L(s)$$

(4.7)

further more,

$$(T_L(s) + N_L(s) F, N_L(s))$$

(4.8)

and (4.3) have the same row degrees. Dually for the right matrix fraction description of (4.4) a similar result can be obtained.

Proof. See Pugh and Ratcliffe (1980)

The second type of condition for those output feedback matrices which produce proper closed-loop system transfer function matrices can be seen through the following theorem due to Pugh (1984).

Theorem 4.9. Let the minimal factorisation (4.2) of the $(m \times 1)$ open-loop transfer function matrix $G(s)$ be given. Then the closed-loop transfer function matrix $G_F(s)$ is proper if and only if the constant $(m \times m)$ matrix

$$(T_L(s) \ N_L(s))_{hr} \left( \begin{array}{c} I_m \\ F \end{array} \right)$$

(4.10)

is non-singular, where $(\cdot)_{hr}$ denotes the high-order coefficient matrix for the rows of the indicated matrix.


Similar results can be given for the right matrix fraction description of (4.4).

Corollary 4.11. With the condition of the theorem (4.9), $G_F(s)$ is non-proper if and only if (4.10) is singular.
Proof. Obvious

The following example illustrates the above results.

Example 4.12. Reconsider example (2.10) with \( G(s) = \begin{pmatrix} \frac{1}{s} & 1 \\ 1 & 0 \end{pmatrix} \). It has been seen that \( G(s) = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & s \end{pmatrix} \) is a relatively prime factorisation of \( G(s) \). Note further that this factorisation is minimal since

\[
(T_L(s) \, N_L(s))_{hr} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}
\]

has full rank. Now

\[
(T_L(s) \, N_L(s))_{hr} \left( \begin{array}{c} I_m \\ F \end{array} \right) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]

where \( F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Clearly (4.14) is singular and hence by corollary (4.11) the closed-loop transfer function matrix \( G_F(s) \) is non-proper a fact which is easily verified from (2.15).

Section III-5: §Feedback and Open-Loop Compensator

The main objective in the design of feedback control systems is that the system be well-posed. For different cases and configurations it has been shown (Desoer and Vidyasagar, 1975; Vidyasagar, 1975; Scott and Anderson, 1976) that the properness of the transfer function matrices of the resulting systems is required in order that the systems be well-posed. Based on the feedback implementation of open-loop compensators, Wolovich (1974), Krishnarao and Chen (1984) studied the properness of the transfer function...
matrix of the following system which arises in the design of state feedback and state estimator.

![Figure 1](image-url)

Since their results are of interest in the subsequent sections a review of them is necessary and given in the following.

**Lemma 5.1.** Consider a unique decomposition of a square rational matrix $G(s)$ as

$$ G(s) = G_s(s) + D(s) \quad (5.2) $$

where $G_s(s)$ is a strictly proper rational matrix and $D(s)$ is a polynomial matrix. Then $G^{-1}(s)$ is proper if and only if $D^{-1}(s)$ exists and is proper.

**Proof.** See Krishnarao and Chen (1984).

Closely related and similar results can be found in Callier and Desoer (1982), Rosenbrock and Hayton (1978).

The following definition is an extension of the concepts of column-reduceness and column degree of polynomial matrices to rational matrices. (Verghese and Kailath, 1979; Kailath, 1980).
**Definition 5.3.** Let \( g_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)} \) be a rational function, then the degree of \( g_{ij}(s) \) is defined as

\[
\deg (g_{ij}(s)) = \delta (g_{ij}(s)) = \deg (n_{ij}(s)) - \deg (d_{ij}(s))
\]  

(5.4)

If \( g_{ij}(s) = 0 \), then \( \deg (g_{ij}(s)) = -\infty \). Also degree of the \( j^{th} \) column of \( G(s) \) is defined as

\[
\deg_{cj}(G(s)) = \delta_{cj}(G(s)) = \max \{\deg (g_{ij}(s))\}.
\]  

(5.5)

The following lemma stated (without proof) by Krishnarao and Chen (1984) plays a fundamental role in the subsequent results, hence it will be stated and then illustrated by some examples.

**Lemma 5.6.** Let

\[
G(s) = N(s) D^{-1}(s)
\]  

(5.7)

where \( G(s), N(s) \) and \( D(s) \) are respectively \((m \times l)\), \((m \times l)\) and \((l \times l)\) rational matrices. If \( \delta_{ck}(D(s)) < \delta_{ck}(N(s)) \) for some \( k \), then \( G(s) \) is not proper. If \( D(s) \) is column-reduced, then \( \delta_{ck}(D(s)) \geq \delta_{ck}(N(s)) \) if and only if \( G(s) \) is proper.

**Proof.** See Kailath (1980).

**Example 5.8.** Let

\[
G(s) = \begin{pmatrix} s^2 - s + 1 & 1 \\ \frac{1}{s} & \frac{s - 1}{s^2} \end{pmatrix}.
\]  

(5.9)

Then by (5.7) we have

\[
G(s) = \begin{pmatrix} \frac{s^2 + 1}{s} & 1 \\ \frac{1}{s} & \frac{s - 1}{s^2} \end{pmatrix} \begin{pmatrix} \frac{1}{s} & 0 \\ 1 & 1 \end{pmatrix}^{-1}
\]  

(5.10)
where
\[ N(s) = \begin{pmatrix} \frac{s^2 + 1}{s} & 1 \\ \frac{1}{s} & \frac{s - 1}{s^2} \end{pmatrix} \] (5.11)
\[ D(s) = \begin{pmatrix} \frac{1}{s} & 0 \\ 1 & 1 \end{pmatrix} \] (5.12)

Now
\[ \delta_{c1}(D(s)) = \max \{-1, 0\} = 0 \]
\[ \delta_{c2}(D(s)) = \max \{-\infty, 0\} = 0 \] (5.13)

and
\[ \delta_{c1}(N(s)) = \max \{1, -1\} = 1 \]
\[ \delta_{c2}(N(s)) = \max \{0, -1\} = 0 \] (5.14)

From (5.13) and (5.14) it is then clear that \( \delta_{c1}(D(s)) = 0 < \delta_{c1}(N(s)) = 1 \). Therefore by lemma (5.6), \( G(s) \) is not proper a fact which can be verified directly from (5.9).

**Example 5.15** Let
\[ G(s) = \begin{pmatrix} \frac{1}{s} & -1 \\ \frac{1}{s} & 0 \end{pmatrix} \] (5.16)

Then by (5.7) we have
\[ G(s) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \] (5.17)

where
\[ N(s) = \begin{pmatrix} 0 & -1 \\ \frac{1}{s} & 0 \end{pmatrix} \] (5.18)
\[ D(s) = \begin{pmatrix} 1 & 0 \\ \frac{1}{s} & 1 \end{pmatrix} \] (5.19)
It can be seen that $D(s)$ is column reduced. Now

\[
\begin{align*}
\delta_{c1}(D(s)) &= \max \{0, -1\} = 0 \\
\delta_{c2}(D(s)) &= \max \{-\infty, 0\} = 0
\end{align*}
\]  
(5.20)

and

\[
\begin{align*}
\delta_{c1}(N(s)) &= \max \{-\infty, -1\} = -1 \\
\delta_{c2}(N(s)) &= \max \{0, -\infty\} = 0
\end{align*}
\]  
(5.21)

From (5.20) and (5.21) it is clear that $\delta_{c1}(D(s)) = 0 > \delta_{c1}(N(s)) = -1$. Therefore by lemma (5.6), $G(s)$ is proper a fact which can be verified easily from (5.16).

The following result due to Krishnarao and Chen (1984) is used in the subsequent sections.

**Theorem 5.22.** Consider the feedback system of fig (1), where $P(s)$ is an $(l \times l)$ non-singular polynomial matrix and $G(s)$, $C_0(s)$ and $C_1(s)$ are $(m \times l), (l \times l)$ and $(l \times m)$ proper rational matrices. Then

(i) $G_F(s)$ is proper if $[P(s) + C_0(\infty) + C_1(\infty) G(\infty)]^{-1}$ exists and is proper.

(ii) $G_F(s)$ is non-proper if $[P(s) + C_0(\infty) + C_1(\infty) G(\infty)]^{-1}$ is non-proper or does not exist, and

(a) $[P(s) + C_0(\infty)]^{-1}$ is proper, or

(b) $G(\infty)$ is non-singular.

**Proof.** See Krishnarao and Chen (1984).

**Section III-6: Composite System Representation**

In the previous sections it has been discussed that if the open-loop transfer function matrix is written in terms of its strictly proper and
polynomial parts, then the conditions for closed-loop transfer function matrix properness generated may prove difficult to check. But if a minimal matrix fraction description of the open-loop transfer function matrix is available, then a more readily verifiable condition for properness of the closed-loop transfer function matrix results. In order to be able to investigate the implications of these observations for composite systems, a composite system representation (Rosenbrock and Pugh, 1974) which is described in the following is needed.

Let $S$ be a system formed by the interconnection of subsystems $S_i$ $(i = 1, \ldots, N)$. The polynomial system matrices corresponding to $S$ and $S_i$ are then respectively given by

\begin{equation}
P(s) = \begin{pmatrix}
  T(s) & U(s) \\
  -V(s) & W(s)
\end{pmatrix}
\end{equation}

and

\begin{equation}
P_i(s) = \begin{pmatrix}
  T_i(s) & U_i(s) \\
  -V_i(s) & W_i(s)
\end{pmatrix}
\end{equation}

Now the equations of subsystems $S_i$ $(i = 1, \ldots, N)$ can be written in the form

\begin{equation}
\begin{aligned}
T_S(s) \xi_s &= U_S(s) u_S \\
y &= V_S(s) \xi_s + W_S(s) u_S
\end{aligned}
\end{equation}

where

\begin{equation}
\xi_s = \begin{pmatrix}
  \xi_1 \\
  \xi_2 \\
  \vdots \\
  \xi_N
\end{pmatrix},
\quad u_S = \begin{pmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_N
\end{pmatrix}
\end{equation}
and

\[
\begin{align*}
T_S(s) &= \text{diag} \left( T_1(s), T_2(s), \ldots, T_N(s) \right) \\
U_S(s) &= \text{diag} \left( U_1(s), U_2(s), \ldots, U_N(s) \right) \\
V_S(s) &= \text{diag} \left( V_1(s), V_2(s), \ldots, V_N(s) \right) \\
W_S(s) &= \text{diag} \left( W_1(s), W_2(s), \ldots, W_N(s) \right)
\end{align*}
\] (6.5)

The polynomial subsystem matrix corresponding to (6.3) is then

\[
P_S(s) = \begin{pmatrix} T_S(s) & U_S(s) \\ -V_S(s) & W_S(s) \end{pmatrix}
\] (6.7)

Similarly the most general form of a composite system \(S_C\) is then obtained by interconnecting the \(S_i\) \((i = 1, \ldots, N)\) according to the interconnection relations

\[
\begin{align*}
\begin{cases}
    u_S = -F y_S + k u_C \\
    y_C = L y_S
\end{cases}
\end{align*}
\] (6.8)

where

\[
F = \begin{pmatrix} F_1^{(1)} & F_1^{(2)} & \ldots & F_1^{(N)} \\ F_2^{(1)} & F_2^{(2)} & \ldots & F_2^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ F_N^{(1)} & F_N^{(2)} & \ldots & F_N^{(N)} \end{pmatrix}
\] (6.9)

\[
L = (L_1, L_2, \ldots, L_N)
\] (6.10)

\[
K = \begin{pmatrix} K_1 \\
K_2 \\
\vdots \\
K_N \end{pmatrix}
\] (6.11)
$u_S$ is defined as in (6.4) and

$$y_S = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} \quad (6.12)$$

$u_i$ and $u_C$ (resp. $y_i$ and $y_C$) are the inputs (resp. outputs) of $S_i$ and $S_C$. A polynomial system matrix representation for the composite system $S_C$ is then

$$P_C(s) = \begin{pmatrix} T_C(s) \\ -V_C(s) \end{pmatrix} \begin{pmatrix} U_C(s) \\ W_C(s) \end{pmatrix} \triangleq \begin{pmatrix} T_S(s) & U_S(s) & 0 & 0 \\ -V_S(s) & W_S(s) & I & 0 \\ 0 & -I & F & K \\ 0 & 0 & -L & 0 \end{pmatrix} \quad (6.13)$$

where $I$ denotes a unit matrix of appropriate dimension and it is assumed that

$$\begin{vmatrix} T_S(s) & U_S(s) & 0 \\ -V_S(s) & W_S(s) & I \\ 0 & -I & F \end{vmatrix} \neq 0 \quad (6.14)$$

The system matrix $P_C(s)$ of (6.13) yields a natural description of the composite system directly in terms of the subsystems and interconnection structure. (6.13) also demonstrates the limitations of the state-space approach in composite system studies, for if the subsystems are in state-space form, it can be seen that (6.13) is not in conventional state-space form and if we try to reduce it to state-space form, that will destroy the interconnection structure. As indicated by Verghese (1978) the generalised state-space form is a better framework for the study of such a problem. The input-output behaviour of $S_C$ may be described in similar terms. Specifically
if $G_S(s)$ denotes the transfer function matrix corresponding to $P_S(s)$, and $G_i(s)$, $G_C(s)$ those corresponding to $S_i$ and $S_C$ respectively then

$$G_S(s) = V_S(s) T_S^{-1}(s) U_S(s) + W_S(s)$$

and it may be shown that (Rosenbrock and Pugh, 1974)

$$G_C(s) = L G_S(s) [I + F G_S(s)]^{-1} K$$

It is clear from these relationships that any composite system may be viewed merely as a closed-loop system constructed from the open-loop system $G_S(s)$ under constant output feedback and described by the matrix $F$, together with closed-loop system pre-and post-compensation as described by $K$ and $L$ respectively. The composite system may thus be viewed in the manner of figure (2) below.

**Figure 2.**

With $L$ and $K$ as unit matrices Callier and Desoer (1982) considered similar composite systems and defined the concept of well-posed systems as follows,
Definition (6.17). Consider the composite system of the figure (2) with $L$ and $K$ as unit matrices, then it is said to be WELL-POSED i.e. $\det (I + FG_S)(\infty) \neq 0$ if and only if all closed-loop transfer function matrices $G_S(I + FG_S)^{-1}$, $(I + FG_S)^{-1}$ and $[(I + FG_S)^{-1} - I]$ are proper.

Section III-7: A Simple Condition for Composite System Properness

As a direct result of the above observations and as a contribution of this thesis, we now consider the properness of a composite system. First let the closed-loop system transfer function matrix formed by applying the constant output feedback $F$ to the open-loop transfer function matrix $G_S(s)$ be denoted by $G_F(s)$, then

$$G_F(s) = G_S(s)[I + FG_S(s)]^{-1} = [I + G_S(s)F]^{-1}G_S(s) \quad (7.1)$$

and so from (6.16) the composite system transfer function matrix is given by

$$G_C(s) = LG_F(s)K \quad (7.2)$$

It can be seen from (7.2) that the output feedback case is a special case of composite system where $L$ and $K$ are unit matrices. Now we can state the main result of this section.

Theorem 7.3. A sufficient condition for the composite system transfer function matrix $G_C(s)$ to be proper is that its associated closed-loop transfer function matrix $G_F(s)$ be proper.

Proof. Since in (7.2) the matrices $L$, $K$ are constant it follows that the elements of $G_C(s)$ are simply linear combinations with constant coefficients of the elements of $G_F(s)$. If these latter elements are proper rational functions,
then such linear combinations can only result in expressions which are proper rational matrices. Thus the properness of $G_F(s)$ ensures the properness of $G_C(s)$.

An obvious and immediate result from theorem (7.3) is;

**Corollary 7.4.** A necessary condition for the composite system transfer function matrix $G_C(s)$ to be non-proper is that its associated closed-loop transfer function matrix $G_F(s)$ be non-proper.

It thus follows from the above results that to deduce the properness of any composite system, it will be sufficient to determine the properness of its associated closed-loop system. To test the closed-loop system for properness, there are a number of necessary and sufficient conditions one may use (see III - 2) depending on the description of the open-loop transfer function matrix $G_S(s)$ that is available.

In view of theorem (7.3) it thus follows that a rough estimate of the properness of the composite system when the subsystem transfer function matrix is given in the form (3.1), may be obtained by testing the associated closed-loop system transfer function matrix in the manner of theorem (3.5).

**Section III-8: §Two specific Composite Systems**

**part one: Properness and Non-Properness Consideration**

In this part the implications of the above results for two configurations which commonly occur in the design of feedback control systems are considered. Although it is apparent that using the associated closed-loop system to assess the properness (non-properness) of the original composite system is likely to give rise to highly conservative estimates, we nevertheless
perform this exercise in two common cases. It will be seen that our results are some what surprising.

Case 1: Two subsystems in feedback connection

Consider two subsystems with transfer function matrices $G_1(s)$, $G_2(s)$ connected in feedback as in figure (3).

**Figure 3.**

In the notation of the previous section, the interconnection relations (6.8) become

$$
\begin{pmatrix}
  u_1 \\
  u_2 
\end{pmatrix} = - \begin{pmatrix}
  0 & I \\
  -I & 0 
\end{pmatrix} \begin{pmatrix}
  y_1 \\
  y_2 
\end{pmatrix} + \begin{pmatrix}
  I \\
  0 
\end{pmatrix} u_c \\

y_c = \begin{pmatrix}
  I & 0 
\end{pmatrix} \begin{pmatrix}
  y_1 \\
  y_2 
\end{pmatrix}
$$

Then we have;

**Theorem 8.2.** The feedback system of figure (3) is proper if

$$\delta([I + D_1(s) \ D_2(s)]) = \delta(D(s))$$

where $D_1(s), D_2(s)$ denote the polynomial parts of $G_1(s)$, $G_2(s)$ and $D(s)$ is a diagonal matrix having $D_1(s)$ and $D_2(s)$ as its principal diagonal.

**Proof.** If $D_1(s), D_2(s)$ denote the polynomial parts of $G_1(s), G_2(s)$ respectively, then from (3.1)


\[ D(s) = \begin{pmatrix} D_1(s) & 0 \\ 0 & D_2(s) \end{pmatrix} \]  

(8.4)

Now from (6.8) and (8.1)

\[ F = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \]  

(8.5)

and so

\[ |I + F D(s)| = \begin{vmatrix} I & D_2(s) \\ -D_1(s) & I \end{vmatrix} = |I + D_1(s) D_2(s)| \]  

(8.6)

Then in view of (8.6), theorem (7.3) together with theorem (3.5) give the result. \( \Box \)

**Corollary 8.7.** If the subsystem transfer function matrices \( G_1(s), G_2(s) \) are proper, then a sufficient condition for the feedback system of figure (3) to be proper is that,

\[ |I + D_1 D_2| \neq 0 \]  

(8.8)

**Proof.** If \( G_1(s), G_2(s) \) are proper, then \( D_1, D_2 \) are constants.

thus

\[ \delta(D) = \begin{cases} -\infty \text{ if } D_1 = D_2 = 0 \\ 0 \text{ otherwise} \end{cases} \]  

(8.9)

Now if \( D_1 = D_2 = 0 \), then (8.8) is trivially satisfied. In this case \( G_1(s), G_2(s) \) are strictly proper and it is known (Rosenbrock and Pugh, 1974) that the feedback composite system is always strictly proper. On the other hand if one of \( D_1, D_2 \) is non-zero, then (8.6) reduces to

\[ \delta(|I + D_1 D_2|) = 0 \]  

(8.10)

which is precisely the requirement (8.8). \( \Box \)
Theorem 8.11. The closed-loop system transfer function matrix $G_F(s)$ of figure (3) is non-proper if and only if

$$\delta(|I + D_1(s)D_2(s)|) < \delta(D(s)) \quad (8.12)$$

Proof. With $F$ as in (8.5) and $D(s)$ as in (8.4) by corollary (3.26) the closed-loop system transfer function matrix $G_F(s)$ is non-proper if and only if $\delta(|I + D(s)F|) < \delta(D(s))$. But by (8.6), $|I + D(s)F| = |I + D_1(s)D_2(s)|$.

The result then follows from theorem (3.5) and corollary (3.26).

Now applying corollary (7.4), having established the theorem (8.11), the following result is immediate.

Corollary 8.13. A necessary condition for the composite system of figure (3) to be non-proper is that its associated closed-loop transfer function matrix $G_F(s)$ be non-proper. i.e. that (8.12) holds.

Case 2: Feedback implementation of open-loop compensators

Consider the configuration of figure (4) which is basic in the feedback implementation of the open-loop compensations (Wolovich, 1974; Krishnarao and Chen, 1984).

Figure 4.
In the figure (4), \( G(s), C_0(s), C_1(s) \) are respectively \((m \times l), (l \times l), (l \times m)\) proper rational matrices and \( P(s) \) is an \((l \times l)\) polynomial matrix whose determinant is not the trivial polynomial. To apply the above results label the component systems as follows

\[
G_1(s) \triangleq P^{-1}(s); \ G_2(s) \triangleq G(s); \ G_3(s) \triangleq C_1(s), \ G_4(s) \triangleq C_0(s) \quad (8.14)
\]

Then in this case the interconnection relations (6.8) become,

\[
\begin{align*}
    u_S &= \begin{pmatrix}
        0 & 0 & I & I \\
        -I & 0 & 0 & 0 \\
        0 & -I & 0 & 0 \\
        -I & 0 & 0 & 0
    \end{pmatrix} \ y_S + \begin{pmatrix}
        I \\
        0 \\
        0 \\
        0
    \end{pmatrix} \ u_C \\
    y_C &= \begin{pmatrix}
        0 & I & 0 & 0
    \end{pmatrix} \ y_S
\end{align*}
\]

(8.15)

where from (8.15) it is clear that

\[
F = \begin{pmatrix}
    0 & 0 & I & I \\
    -I & 0 & 0 & 0 \\
    0 & -I & 0 & 0 \\
    -I & 0 & 0 & 0
\end{pmatrix}.
\]

(8.16)

Then we have;

**Theorem 8.17.** The composite system of figure (4) is proper if

\[
\delta(|I + (D_4 + D_3 D_2) D_1(s)|) = \delta(D_1(s))
\]

(8.18)

where \( D_i(s), i = 1, \ldots, 4 \) denote the polynomial parts of \( G_i(s), i = 1, \ldots, 4 \).

**Proof.** Let \( D_i(s) \) denote the polynomial parts of \( G_i(s), i = 1, \ldots, 4 \), then \( D_1(s) \) may be polynomial while \( D_2(s), D_3(s) \) and \( D_4(s) \) are at most constants.

Now from (6.8) and (8.16) it follows that

\[
|I + F D(s)| = \begin{vmatrix}
    I & 0 & D_3 & D_4 \\
    -D_1(s) & I & 0 & 0 \\
    0 & -D_2 & I & 0 \\
    -D_1(s) & 0 & 0 & I
\end{vmatrix}
\]

(8.19)
where

\[ D(s) = \begin{pmatrix}
D_1(s) & 0 & 0 & 0 \\
0 & D_2 & 0 & 0 \\
0 & 0 & D_3 & 0 \\
0 & 0 & 0 & D_4
\end{pmatrix} \quad (8.20) \]

Then by elementary row and column operations the determinant (8.19) may be reduced to the successive forms of

\[
\begin{vmatrix}
I & 0 & D_3 & D_4 \\
0 & I & 0 & 0 \\
-D_2D_1(s) & 0 & I & 0 \\
-D_1(s) & 0 & 0 & I
\end{vmatrix} \rightarrow \begin{vmatrix}
I + D_3D_2D_1(s) & 0 & 0 & D_4 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{vmatrix} \quad (8.21)
\]

Thus

\[
\delta(|I + F D(s)|) = \delta(|I + (D_4 + D_3 D_2) D_1(s)|). \quad (8.22)
\]

It is then clear from the diagonal form of \( D(s) \) in (8.20) and the fact that \( D_2, D_3, D_4 \) are constants that

\[
\delta(D(s)) = \delta(D_1(s)) \quad (8.23)
\]

Then in view of (8.22) and (8.23), theorem (7.3) together with theorem (3.5) give the result.

Two interesting special cases of the above result are:

**Corollary 8.24.** If \( G_1(s) = P^{-1}(s) \) is proper so that \( D_1(s) \equiv D_1 \), then the composite system of figure (4) is proper if

\[
|I + (D_4 + D_3 D_2) D_1| \neq 0 \quad (8.25)
\]
Proof. If $D_1(s) \equiv D_1$ is constant, then (8.1) reduces to

$$\delta(|I + (D_4 + D_3D_2)D_1|) = 0$$

which is precisely the requirement (8.25). \hfill \Box

Corollary 8.27. If $G_1(s) = P^{-1}(s)$ is strictly proper (so that $D_1 = 0$), then the composite system of figure (4) is always proper.

Proof. Obvious \hfill \Box

The condition (8.18) of theorem (8.17) has an alternative characterisation given as follows:

**Theorem 8.28.** The composite system of figure (4) is proper if

$$\left(P(s) + D_4 + D_3D_2\right)^{-1}$$

is proper.

Proof. Consider the closed-loop system $G_F(s)$ obtained by employing constant output feedback as described by the matrix $F \triangleq D_4 + D_3D_2$, around the open-loop system $G_1(s) = P^{-1}(s)$. Thus from theorem (3.5), the condition (8.18) is the exact requirement that $G_F(s)$ be proper. Now

$$G_F(s) = G_1(s) \left[I + FG_1(s)\right]^{-1} = P^{-1}(s)\left[I + (D_4 + D_3D_2)P^{-1}(s)\right]^{-1} = \left(P(s) + D_4 + D_3D_2\right)^{-1}$$

Hence the result. \hfill \Box

**Theorem 8.31.** The closed-loop system transfer function matrix $G_F(s)$ of figure (4) is non-proper if and only if

$$\delta(|I + (D_4 + D_3D_2)D_1(s)|) < \delta(D_1(s)).$$
Proof. With \( F \) as in (8.16) and \( D(s) \) as in (8.20) by corollary (3.26) the closed-loop system transfer function matrix \( G_F(s) \) is non-proper if and only if \( \delta(|I + D(s)F|) < \delta(D(s)) = \delta(D_1(s)) \). But by (8.22), \(|I + D(s)F| = |I + (D_4 + D_3 D_2)D_1(s)|\). The result then follows from theorem (3.5) and corollary (3.26).

Having established the theorem (8.31), by applying corollary (7.4), the following result is immediate.

**Corollary 8.33.** A necessary condition for the composite system of figure (4) to be non-proper is that its associated closed-loop transfer function matrix \( G_F(s) \) be non-proper. i.e. that (8.32) holds.

**Part two: Minimal Factorisation Consideration**

In the final part of this section, application of the results concerning the minimal factorisation of \( G_F(s) \) (given in section four) to our two specific composite systems described above will be considered in the following manner.

Consider the composite system of figure (3) where the constant output feedback \( F \) is applied as shown by figure (5) below.

**Figure 5.**
Let
\[ G_1(s) = D_1^{-1}(s)N_1(s) \]
\[ G_2(s) = D_2^{-1}(s)N_2(s) \] (8.34)
be the minimal factorisation of \( G_1(s) \) and \( G_2(s) \), then a minimal factorisation of the above system is given by
\[ \begin{pmatrix} D_1(s) & N_1(s) \\ 0 & D_2(s) \end{pmatrix}^{-1} \begin{pmatrix} N_1(s) & 0 \\ 0 & N_2(s) \end{pmatrix} \] (8.35)
Now by lemma (4.6) a minimal factorisation of \( G_F(s) \) is given by
\[ \begin{pmatrix} D_1(s) & 0 \\ 0 & D_2(s) \end{pmatrix} + \begin{pmatrix} N_1(s) & 0 \\ 0 & N_2(s) \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^{-1} \begin{pmatrix} N_1(s) & 0 \\ 0 & N_2(s) \end{pmatrix} \] (8.36)

Then we have,

**Theorem 8.37.** If (8.34) is a minimal factorisation of the composite system of figure (5), then the closed-loop system matrix \( G_F(s) \) of the composite system is proper if and only if
\[ \begin{pmatrix} (D_1(s) & N_1(s))_{hr} \\ (-N_2(s) & D_2(s))_{hr} \end{pmatrix} \] (8.38)
is non-singular, where \([ \cdot ]_{hr}\) denotes the high-order coefficient matrix for the rows of the indicated matrix.

**Proof.** From (8.36) it is clear that the constant output feedback matrix \( F \) is given by,
\[ F = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \] (8.39)
Now by theorem (4.9), the closed-loop system matrix \( G_F(s) \) is proper if and only if
\[ \begin{pmatrix} D_1(s) & 0 | N_1(s) & 0 \\ 0 & D_2(s) | 0 & N_2(s) \end{pmatrix}_{hr} \begin{pmatrix} I & 0 \\ 0 & I \\ 0 & I \\ -I & 0 \end{pmatrix} \] (8.40)
is non-singular. But (8.40) can be also written as

\[
\begin{pmatrix}
(D_1(s))_{hr} & (N_1(s))_{hr} \\
(-N_2(s))_{hr} & (D_2(s))_{hr}
\end{pmatrix}
= \begin{pmatrix}
(D_1(s))_{hr} & (N_1(s))_{hr} \\
(-N_2(s))_{hr} & (D_2(s))_{hr}
\end{pmatrix}
\]  

(8.41)

which completes the proof.

Notice that in this case the non-singularity condition of the constant high-order coefficient matrix of (8.41) is easy to check.

Consider now the composite system of figure (4) where the constant output feedback $F$ is applied as shown by figure (6).

**Figure 6.**

where $G_i(s), i = 1, \ldots, 4$ are as described in (8.14). Let

\[
G_i(s) = D_i^{-1}(s)N_i(s), \quad i = 1, 2, 3, 4
\]

(8.42)

be the minimal factorisation of $G_i(s), i = 1, \ldots, 4$, then a minimal factorisation of the above system is given by
\begin{align*}
\begin{pmatrix}
D_1(s) & 0 & 0 & 0 \\
0 & D_2(s) & 0 & 0 \\
0 & 0 & D_3(s) & 0 \\
0 & 0 & 0 & D_4(s)
\end{pmatrix}^{-1} & \begin{pmatrix}
N_1(s) & 0 & 0 & 0 \\
0 & N_2(s) & 0 & 0 \\
0 & 0 & N_3(s) & 0 \\
0 & 0 & 0 & N_4(s)
\end{pmatrix} \\
& \text{(8.43)}
\end{align*}

Again by lemma (4.6) a minimal factorisation of \(G_F(s)\) is given by
\begin{align*}
\begin{pmatrix}
D_1(s) & 0 & 0 & 0 \\
0 & D_2(s) & 0 & 0 \\
0 & 0 & D_3(s) & 0 \\
0 & 0 & 0 & D_4(s)
\end{pmatrix} & + \begin{pmatrix}
N_1(s) & 0 & 0 & 0 \\
0 & N_2(s) & 0 & 0 \\
0 & 0 & N_3(s) & 0 \\
0 & 0 & 0 & N_4(s)
\end{pmatrix} \\
& \text{(8.44)}
\end{align*}

Then we have

**Theorem 8.45.** If (8.42) is a minimal factorisation of the composite system of figure (6), then the closed-loop system matrix \(G_F(s)\) of the composite system is proper if and only if
\begin{align*}
\begin{pmatrix}
0 & 0 & I & I \\
-I & 0 & 0 & 0 \\
0 & -I & 0 & 0 \\
-I & 0 & 0 & 0
\end{pmatrix} & \begin{pmatrix}
N_1(s) & 0 & 0 & 0 \\
0 & N_2(s) & 0 & 0 \\
0 & 0 & N_3(s) & 0 \\
0 & 0 & 0 & N_4(s)
\end{pmatrix}^{-1} \\
& \text{(8.46)}
\end{align*}

is non-singular, where \([\cdot]_{hr}\) denotes the high-order coefficient matrix for the rows of the indicated matrix.
Proof. From (8.44) it is clear that the constant output feedback matrix $F$ is given by,

$$F = \begin{pmatrix} 0 & 0 & I & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (8.47)

Now by theorem (4.9), the closed-loop system matrix $G_F(s)$ is proper if and only if

$$\begin{pmatrix} D_1(s) & 0 & 0 & 0 & N_1(s) & 0 & 0 & 0 \\ 0 & D_2(s) & 0 & 0 & 0 & N_2(s) & 0 & 0 \\ 0 & 0 & D_3(s) & 0 & 0 & 0 & N_3(s) & 0 \\ 0 & 0 & 0 & D_4(s) & 0 & 0 & 0 & N_4(s) \end{pmatrix}_{hr}$$

is non-singular. But (8.48) can be also written as

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (8.48)

which completes the proof. \hfill \Box
Section III-9: §The Two Composite Systems and Infinite Frequency Point of View

In section eight the properness and non-properness of the closed-loop system matrix $G_F(s)$ and the composite systems of the figures (3) and (4) have been studied and in this regard different approaches have been considered and some important results presented. In this section the subject of study will be the same systems of the figures (3) and (4), but our approach will be in terms of infinite poles and zeros. Bearing in mind the result of section four of chapter two concerning the infinite poles and zeros of a rational matrix, the following considerations are given to the composite systems of the figures (3) and (4).

Consider the feedback system of figure (3), then we can state an equivalent result to the corollary (8.7) but from an infinite frequency point of view.

**Theorem 9.1.** The composite system of figure (3) is proper if the matrices $G_1(s)$ and $G_2(s)$ have no infinite poles.

**Proof.** If $G_1(s)$ and $G_2(s)$ have no infinite poles, then by theorem (4.8) of chapter two, they are proper and therefore the polynomial part of them, $D_1(s)$ and $D_2(s)$ are constant. The rest of the proof follows from corollary (8.7).

Consider now the feedback system of figure (4), then we have,

**Theorem 9.2.**

(i) The composite system of figure (4) is proper if

$$(P(s) + D_4 + D_3D_2)$$

(9.3)
has no infinite zeros.

(ii) The composite system of figure (4) is non-proper if (9.3) has infinite zeros and

\[(P(s) + D_4)\]  \hspace{1cm} (9.4)

has no infinite zeros.

**Proof.** Suppose that (9.3) has no infinite zeros then clearly

\[(P(s) + D_4 + D_3D_2)^{-1}\]  \hspace{1cm} (9.5)

has no infinite poles and therefore by theorem (4.8) of chapter two it is proper. But from (8.30), \(G_F(s) = (P(s) + D_4 + D_3D_2)^{-1}\) and hence \(G_F(s)\) must be proper. Then by theorem (8.28) the composite system is proper. This establishes (i).

Suppose now that (9.3) has infinite zeros, then clearly (9.5) has infinite poles, therefore by corollary (4.9) of chapter two it is non-proper. Now if \((P(s) + D_4)\) has no infinite zeros, then clearly \((P(s) + D_4)^{-1}\) has no infinite poles and so by theorem (4.8) of chapter two \((P(s) + D_4)^{-1}\) is proper. The rest of the proof follows from part (ii) of the theorem (5.22).

**Section III-10: §Conclusion and Discussions**

The properness and non-properness of the transfer function matrix \(G_F(s)\) of the closed-loop system have been investigated and various characterisations of those constant output feedback matrices \(F\) which produce non-proper closed-loop transfer function matrix system due to Pugh (1984) were reviewed in sections two, three and four. In section five problem of feedback and open-loop compensator has been studied and in this regard the work due to Krishnarao and Chen (1984) was reviewed. As our contribution application
of these results for properness of a general composite system and then to two specific composite systems were considered in sections 6, 7 and 8 and some interesting results have been established.

In theorem (7.3) a sufficient condition for the properness of a general composite system $G_C(s)$ has been given. It arises from the simple observation that such a composite system will always be proper if its associated closed-loop system $G_F(s)$ is proper. In fact it is also true that $G_C(s)$ can only be non-proper when $G_F(s)$ has this property.

From the manner in which theorem (7.3) has been obtained, it is obvious that the sufficient condition it yields is likely to be highly conservative. However it must be said that the condition (8.18) is extremely simple to derive and to employ, and the nature of its limitations are clear and obvious. Thus for example when theorem (7.3) is applied to the case of the composite system of figure (4), the extent of the conservatism of the sufficient condition of Krishnarao and Chen (1984) (described in the theorem (8.28)) is clearly revealed.

In applying theorem (7.3) to two specific composite systems (cases 1 and 2) certain interesting results have been obtained. Specifically in the open-loop compensation scheme of figure (4), theorem (7.3) in conjunction with the fact that the closed-loop transfer function matrix $G_F(s)$ is proper if and only if $\delta(|I + F D(s)|) = \delta(D(s))$ gives rise to a sufficient condition which is equivalent to that previously obtained by Krishnarao and Chen (1984). Additionally it is noted that for the simple feedback scheme of case 1, the condition (8.8) produced by theorem (7.3) in the case of proper subsystems is actually known to be necessary and sufficient (Desoer and Vidyasagar, 1975). Thus despite the apparent limitations of theorem (7.3), it
produces in case 2 a sufficient condition as good as any previously obtained, which in case 1 the resulting condition cannot be better since it is both necessary and sufficient. Also the non-properness of the two specific cases has been considered and some results have been established. The application of the minimal factorisation of $G_F(s)$ to the two above cases have also been considered and some interesting results have been given. Finally in section nine the same cases have been studied, but this time from an infinite frequency point of view, and in this regard some equivalent results have been provided.
CHAPTER IV

Matrices and Local Results
Section IV-1: §Introduction

As explained in chapter three the idea of proper and non-proper systems (there being no poles or being poles at infinity) leads naturally to study the behaviour of the system at a finite point $s_0$ in the complex plane and therefore to define the idea of proper and non-proper at $s_0$ systems. i.e. there being no poles at $s_0$ or being poles at $s_0$. Having defined the pole/zero at $s_0$ of a system, in order to be able to investigate the effect of the state variable feedback on the pole/zero structure at $s_0$ of a general transfer function matrix, it is necessary to have a local matrix theory serving our purpose which in turn leads to the development of a local system theory.

As in the case of the conventional study of linear systems in which polynomial and rational matrices play a fundamental role, in the local theory of linear systems a local approach to the study of rational matrices is similarly essential. Accordingly in the following sections a brief discussion of local results for rational matrices is presented. The basic and preliminary results concerning the rational functions at $s_0$ are given in sections two. The elementary operations and equivalence at $s_0$ are described in section three and valuations of rational matrices at $s_0$ are presented in section four. Finally the invariant rational functions of rational matrices and the Smith-McMillan form of a rational matrix at $s_0$ are given in section five. Also in section five an alternative way of finding the Smith-McMillan form of a rational matrix at $s_0$ is considered.
Section IV-2:  §Rational Functions and their Properties at $s_0$.

Let $F$ be the field of real or complex numbers ($R$ or $C$), $F[s]$ the ring of polynomials in indeterminate $s$ with coefficients in $F$ and $F(s)$ the field of rational functions $g(s) = \frac{\alpha(s)}{\beta(s)}$, $\alpha(s), \beta(s) \in F[s], \beta(s) \neq 0$, then

**Definition 2.1.** The rational function $g(s) \in F(s)$ is said to be **PROPER AT** $s_0$ in case

$$\lim_{s \to s_0} g(s)$$

exists. Additionally $g(s)$ is said to be **STRICTLY PROPER AT** $s_0$ if this limit is zero, and **BIPROPER AT** $s_0$ if it is not.

It is clear from this definition therefore that $g(s)$ is proper at $s_0$ if and only if it is a $\Lambda$-generalised polynomial (Pernebo, 1978, 1981a, 1981b, see Chapter II for a brief discussion on the $\Lambda$-generalised polynomial) with $\Lambda = \{s_0\}$. Thus $g(s)$ is proper at $s_0$ if and only if it has no poles at $s_0$, while it is biproper at $s_0$ in case it has no poles and no zeros at $s_0$.

An obvious result for any rational function, which has also been noted by Pernebo (ibid) in the case of $\Lambda$-generalised polynomial is the following.

**Lemma 2.2.** If $g(s) \in F(s)$ is non zero, then it may be uniquely factorised as

$$g(s) = (s - s_0)^k g'(s)$$

(2.3)

where $g'(s)$ is biproper at $s_0$ and $k$ is an integer. Then $k$ is called a **DISCRETE VALUATION OF** $g(s)$.

**Definition 2.4.** If $g(s) \in F(s)$ is factorised as in (2.3), then $k$ is called the **DEGREE OF** $g(s)$ AT $s_0$ for $k \geq 0$ denoted by $d_{s_0}(g(s))$. 
It is clear from the above that a given rational function \( g(s) \) will be proper (resp. biproper) at \( s_0 \) in case its degree at \( s_0 \) as defined by lemma (2.2) is non-negative (resp. zero).

Now if \( g(s) = \frac{\alpha(s)}{\beta(s)} \in F(s); \ \alpha(s), \ \beta(s) \in F[s], \ \beta(s) \neq 0 \), then the valuation at \( s_0 \) of \( g(s) \) is defined via

\[
V_{s_0}(g(s)) = \begin{cases} 
\{ d_{s_0}(\alpha(s)) - d_{s_0}(\beta(s)), g(s) \neq 0 \\
+\infty, g(s) \equiv 0 
\} 
\]

(2.5)

It can be seen from above that the rational function \( g(s) \) is proper (resp. biproper) at \( s_0 \) in case its valuation at \( s_0 \) is greater than or equal to zero (resp. zero), and it is non-proper at \( s_0 \) in case its valuation at \( s_0 \) is less than zero.

From (2.3) it can be seen that for \( k \geq 0 \), \( g(s) \) has no poles at \( s_0 \) and therefore by definition \( g(s) \) is proper at \( s_0 \), hence we define the ring of proper at \( s_0 \) rational functions denoted by \( F_{s_0}(s) \) which has common algebraic properties with the polynomials. It is also clear that the operations of addition and multiplication in \( F_{s_0}(s) \) are defined as in \( F(s) \). Therefore not only \( F_{s_0}(s) \) is a ring under the standard operations, but in fact it is a commutative ring with identity (real number 1) and no zero divisors. Hence by definition \( F_{s_0}(s) \) is an integral domain.

Also for every \( g_1(s), g_2(s) \in F_{s_0}(s), g_2(s) \neq 0 \) it can be shown (see lemma (2.16) of the Chapter II) that there exists a proper at \( s_0 \) rational function \( q(s) \) and a rational function \( r(s) \) such that

\[
g_1(s) = g_2(s) q(s) + r(s) \quad (2.6)
\]
where \( r(s) \neq 0 \) or \( V_{s_0}(r(s)) < V_{s_0}(g_2(s)) \). So baring in mind the definition (2.4), equation (2.6) describes a euclidean division algorithm. Thus by definition \( F_{s_0}(s) \) is a euclidean ring and therefore a principle ideal domain.

The units in \( F_{s_0}(s) \) are those proper at \( s_0 \) rational functions \( g(s) \in F_{s_0}(s) \) for which there exists a \( g'(s) \in F_{s_0}(s) \) such that \( g(s)g'(s) = 1 \). We call these rational functions (units) biproper at \( s_0 \).

Section IV-3: Elementary Operations and Equivalence at \( s_0 \)

The objective of this and next sections is the reduction of a rational matrix \( M(s) \) to a simpler and subsequently to the Smith-McMillan form at \( s_0 \) by means of an equivalence transformation based at \( s_0 \). i.e. by using the elementary row and column operations at \( s_0 \) defined as follows.

E.i) Interchanging any two rows (columns). This operation is the same as multiplication of \( M(s) \) on the left (right) by the following square matrices,
E-ii) Adding to any row (column), any other row (column) multiplied by any proper at $s_0$ rational function $d(s)$. This operation is the same as multiplication of $M(s)$ on the left (right) by the following square matrices,

$$L_1 = \begin{pmatrix} 1 & & & & 1 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & \vdots \\ 0 & & \cdots & \cdots & 1 \\ & & & & \end{pmatrix}_{\text{col } i \text{ col } j}$$

$$[R_1] = \begin{pmatrix} 1 & & & & 1 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & \vdots \\ 0 & & \cdots & \cdots & 1 \\ & & & & \end{pmatrix}_{\text{row } i \text{ row } j}$$
Multiplying any row (column) by a biproper at $s_0$ rational function $e(s)$. This operation is the same as multiplication of $M(s)$ on the left (right) by the following square matrices,
The above operations are called LEFT (RIGHT) ELEMENTARY OPERATIONS AT $s_0$ and the matrices $L_1, L_2, L_3$ ($R_1, R_2, R_3$) are called ELEMENTARY MATRICES AT $s_0$. It can be seen that by definition every unimodular matrix is also biproper at $s_0$.

Now we can use these elementary operations at $s_0$ and make a formal definition of equivalence transformation at $s_0$.

**Definition 3.4.** Two $(m \times n)$ rational matrices, $M(s)$ and $M_1(s)$ are called LEFT (RIGHT) EQUIVALENT AT $s_0$ if one can be obtained from the other by a sequence of left (right) elementary operations at $s_0$.

i.e.

$$M_1(s) = L(s)M(s) \quad (M_1(s) = M(s)R(s)) \quad (3.5)$$
where \( L(s) \) and \( R(s) \) are respectively \((m \times m)\) and \((n \times n)\) biproper at \( s_0 \) rational matrices.

**Definition 3.6.** Two \((m \times n)\) rational matrices \( M(s) \) and \( M_1(s) \) are called EQUIVALENT AT \( s_0 \) if one can be obtained from the other by a sequence of left and right elementary operations at \( s_0 \), i.e.

\[
M_1(s) = L(s) M(s) R(s)
\]  

(3.7)

where again \( L(s) \) and \( R(s) \) are respectively \((m \times m)\) and \((n \times n)\) biproper at \( s_0 \) rational matrices.

Having defined the equivalence transformations at \( s_0 \) we are now in a position to introduce the two most important standard forms at \( s_0 \) of any rational matrix by using the elementary operations at \( s_0 \). These standard forms are the Hermite form and the Smith-McMillan form at \( s_0 \). Hermite form is considered in this section and Smith-McMillan form at \( s_0 \) in the next.

**Theorem 3.8** Any \((m \times n)\) rational matrix \( M(s) \) of rank \( r \leq \min(m,n) \) can be brought to the following form by means of left elementary operations at \( s_0 \).

\[
H_M = 
\begin{pmatrix}
    m_1 & m_2 & \cdots & m_r \\
    1 & 2 & \cdots & r \\
    r & & & m - r \\
\end{pmatrix}
\]  

(3.9)
where $H$ is called HERMITE ROW FORM OF $M(s)$ with the following properties:

1. For all $i \in \{1, 2, \ldots, r\}$ the $i^{th}$ row has a leading non zero monic element $h_i m_i$ called leading entry such that $1 \leq m_1 < m_2 < \ldots < m_r \leq n$.

2. For all $i \in \{1, 2, \ldots, r\}$, if $h_i m_i = 1$, then $h_j m_i = 0$ for all $j < i$, if $h_i m_i \neq 1$, then $\deg (h_j m_i) < \deg (h_i m_i)$ for all $j < i$ such that $h_j m_i \neq 0$.

3. For all $j^{th}$ column such that $j < m_1$, $j^{th}$ column is zero; for all $j^{th}$ column such that $m_i \leq j < m_{i+1}$ with $i \in \{1, 2, \ldots, r-1\}$, then the last $(m - i)$ entries of $j^{th}$ column are zero; for all $j^{th}$ column such that $j \geq m_r$, the last $(m - r)$ entries of $j^{th}$ column are zero.

**Proof** See Callier and Desoer (1982).

Similar result can be given for Hermit column form of $M(s)$.

**Corollary 3.10** If in theorem (3.8), $M(s)$ has full column rank, i.e. rank $M(s) = n$ with $m \geq n$, then the Hermite row form of $M(s)$ is upper right triangular given by the following

\[ H_M = \begin{pmatrix}
1 & m_1 & m_2 & \ldots & m_n \\
2 & & & & \\
\vdots & & & & \\
n & & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
0 & & & & \\
\end{pmatrix} \]

(3.11)
Similar results can be stated for lower left triangular form of $M(s)$.

Section IV-4: §Valuations of Rational Matrices at $s_0$.

The structure of rational matrices and their applications have been widely investigated in the linear system theory area. In this section the extension of the previous result to the case of rational matrices is studied and some results are presented.

Valuations of rational matrices and in particular valuation of a rational matrix at infinity and also applications of valuation theory to control systems have been the subject of investigation for many authors (Kung and Kailath, 1978; Verghese, 1978; Kailath, 1980; Vardulakis and Karcarias, 1983).

Following the valuation of rational matrices at infinity (Vardulakis and Karcarias, 1983) valuations of rational matrices at $s_0$ and their applications are presented. This is of particular interest since it relates the number of zeros and poles of the rational matrices locally. Hence by considering valuations of a rational matrix at $s_0$, it will be possible to study the pole/zero structure of the rational matrix at $s_0$. But first the following definitions:

**Definition 4.1.** Let $W(s)$ be an $(m \times n)$ rational matrix, then it can be written as

$$ W(s) = D^{-1}(s) \ N(s) $$

(4.2)

where $D(s)$ is an $(m \times m)$ and $N(s)$ is an $(m \times n)$ polynomial matrix. Now if $D(s)$ and $N(s)$ are relatively prime at $s_0$, then rank $W(s)$ at $s_0$ is defined as rank $(N(s_0))$ and is denoted by $\rho_{s_0}(W(s))$.

Since $D(s)$ and $N(s)$ are polynomial matrices, then they have no finite poles and in particular they have no poles at $s_0$, hence by definition they
are proper at $s_0$. Therefore there exists proper at $s_0$ matrices $D(s)$ and $N(s)$ such that $W(s)$ an $(m \times n)$ rational matrix can be written as in (4.2) which is a standard decomposition for $W(s)$. However in general case $D(s)$ and $N(s)$ are proper at $s_0$ rational matrices as the following example shows.

**Example 4.3.** Let

$$W(s) = \begin{pmatrix}
\frac{1}{s-1} & \frac{s+1}{s} \\
-1 & \frac{1}{s}
\end{pmatrix}$$

and $s_0 = 1$, then $W(s)$ can be written as in (4.2) where $D(s)$ and $N(s)$ are proper at $s_0 = 1$ rational matrices. i.e.

$$W(s) = \begin{pmatrix}
\frac{1}{s-1} & \frac{s+1}{s} \\
-1 & \frac{1}{s}
\end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{s} & 1 \\
0 & \frac{1}{s} \end{pmatrix}$$

where

$$D(s) = \begin{pmatrix} 1 & 1 \\
1 & s \end{pmatrix} \quad \text{and} \quad N(s) = \begin{pmatrix} \frac{1}{s} & 1 \\
0 & \frac{1}{s} \end{pmatrix}$$

are relatively left prime at $s_0$, then rank $W(s)$ at $s_0$ is two which is the rank at $s_0$ of $N(s)$.

Now let $W(s)$ be an $(m \times n)$ rational matrix, then the $i^{th}$ valuation of $W(s)$ at $s_0$ denoted by $V_{s_0}^{(i)}(W(s))$ is defined as

$$V_{s_0}^{(i)}(W(s)) = \min \left\{ V_{s_0}(\cdot) \text{ among the } V_{s_0}(\cdot) \text{ of all } (i \times i) \text{ minors of } W(s) \right\}$$

(4.6)
As in the case of rational functions (section 2) the relationship between the valuations at \( s_0 \) of \( W(s) \) and the valuations at \( s_0 \) of \( D(s) \) and \( N(s) \) can be given by the following:

\[
\begin{align*}
V_{s_0}^{(m)}(W(s)) &= V_{s_0}^{(m)}(N(s)) - V_{s_0}^{(m)}(D(s)), \ m \leq n \\
V_{s_0}^{(n)}(W(s)) &= V_{s_0}^{(n)}(N(s)) - V_{s_0}^{(n)}(D(s)), \ m > n
\end{align*}
\] (4.7)

**Example 4.8.** Reconsider example (4.3) where \( W(s) \) is given as in (4.4), \( N(s) \) and \( D(s) \) as in (4.5), \( s_0 = 1 \) and \( m = n = 2 \), then by (4.6),

\[
\begin{align*}
V_1^{(1)}(W(s)) &= \min \{-1, 0, -1, 0\} = -1 \\
V_1^{(2)}(W(s)) &= -1 \\
V_1^{(1)}(D(s)) &= \min \{0, 0, +\infty, 1\} = 0 \\
V_1^{(2)}(D(s)) &= 1 \\
V_1^{(1)}(N(s)) &= \min \{0, 0, +\infty, 0\} = 0 \\
V_1^{(2)}(N(s)) &= 0
\end{align*}
\]

Therefore

\[
V_1^{(2)}(W(s)) = -1 = V_1^{(2)}(N(s)) - V_1^{(2)}(D(s)) = 0 - (1) = -1
\]

a fact shown by (4.7).

It is also clear from above that if \( s_0 \) is not a pole or zero of \( W(s) \), then all the valuations at \( s_0 \) of \( W(s) \) will be zero, and further

**Lemma 4.8.** Let \( W(s) \) be an \( (m \times n) \) rational matrix, then it is proper at \( s_0 \) if and only if

\[
V_{s_0}^{(1)}(W(s)) \geq 0
\] (4.9)
Proof. Suppose $W(s)$ is proper at $s_0$, then every $(1 \times 1)$ minor of $W(s)$ is proper at $s_0$ and hence $s_0$ is not a pole of $W(s)$, then (4.9) follows from (2.5). Conversely suppose that $V_{s_0}^{(1)}(W(s)) \geq 0$, then every $(1 \times 1)$ minor of $W(s)$ is proper at $s_0$ and hence $W(s)$ is proper at $s_0$. \hfill \Box

Lemma 4.10. Let $W(s)$ be an $(m \times m)$ proper at $s_0$ rational matrix, then it is biproper at $s_0$ if and only if

$$V_{s_0}^{(m)}(W(s)) = 0$$

(4.11)

Proof. Suppose $W(s)$ is biproper at $s_0$, then by definition $s_0$ is not a pole or zero of $W(s)$, therefore $W(s)$ is of full rank at $s_0$ and hence $V_{s_0}^{(m)}(W(s)) = 0$. Conversely assume that $W(s)$ is proper at $s_0$ and $V_{s_0}^{(m)}(W(s)) = 0$, then $W(s)$ is of full rank at $s_0$ and hence $s_0$ can not be a pole or zero of $W(s)$, therefore $W(s)$ is biproper at $s_0$. \hfill \Box

The above results show the close relationship between the definition of proper and biproper at $s_0$ of a rational matrix given in section two and its valuations at $s_0$.

The fundamental role of proper and biproper at $s_0$ rational matrices can be seen through the process of reduction of rational matrices using the elementary operations at $s_0$ described in section three on an $(m \times n)$ rational matrix $W(s)$ which also led us to the notion of equivalence at $s_0$ defined in the same section.
Our final step in reducing a rational matrix to a special canonical form is the reduction of such a matrix to the so-called SMITH-MCMILLAN FORM AT $s_0$.

The notion of equivalence at $s_0$ of the rational matrices leads us to a special canonical form for a given rational matrix similar to the conventional Smith-McMillan form. This canonical form which we call a local version of the Smith-McMillan form based at $s_0$ then enables us to study the structure of the finite poles and zeros of a rational matrix at $s_0$.

**Theorem 5.1.** Any $(m \times n)$ rational matrix $W(s)$ of $\rho_{s_0}(W(s)) = r = \min (m, n)$ is equivalent at $s_0$ to a matrix $M_{s_0}(W)$ such that

$$M_{s_0}(W) = R(s) W(s) C(s)$$

for some rational matrices $R(s), C(s)$ biproper at $s_0$, where

$$M_{s_0}(W) = \begin{cases} (Q(s), 0_{m, n-m}) , & m < n \\ Q(s) , & m = n \\ \left( \begin{array}{c} Q(s) \\ 0_{m-n, n} \end{array} \right) , & m > n \end{cases}$$

and

$$Q(s) = \text{diag} [(s - s_0)^{q_1}, \ldots, (s - s_0)^{q_r}, 0, \ldots, 0]$$

with

$$q_1 \leq q_2 \leq \ldots \leq q_k \leq 0 \leq q_{k+1} \leq \ldots \leq q_r$$
$M_{s_0}(W)$ is called the SMITH-MCMILLAN FORM OF $W(s)$ AT $s_0$ (Gohberg, Lancaster and Rodman, 1982; Pandolfi, 1982) which can be obtained by the following easy procedure.

By operations of unimodular equivalence which are (as stated before) a subset of the operations of equivalence at $s_0$ on $W(s)$, the unique conventional Smith-McMillan form of $W(s)$ can be obtained as

$$M(W) = \left( \begin{array}{c|c} \text{diag } \left( \frac{\phi_i(s)}{\psi_i(s)} \right) & 0_{r,n-r} \\ \hline 0_{m-r,r} & 0_{m-r,n-r} \end{array} \right), \quad i = 1, \ldots, r$$

(5.6)

Then repeated application of the elementary operations (E.iii) of equivalence at $s_0$ on the first $r$ rows of $M(W)$ will give

$$M_{s_0}(W) = \left( \begin{array}{c|c} \text{diag } (s-s_0)^{q_i} & 0_{r,n-r} \\ \hline 0_{m-r,r} & 0_{m-r,n-r} \end{array} \right), \quad i = 1, \ldots, r$$

(5.7)

where from the divisibility properties of the Smith McMillan form in $M(W)$ we have the relationship (5.5).

It is clear from the above result that the operations (E.i - iii) preserve the pole/zero structure of $W(s)$ at the frequency at $s_0$ to the exclusion of all other frequencies (including the point at infinity), since those $q_i$ in (5.5) for which $q_i < 0$ determine the pole structure of $W(s)$ while those $q_i$ for which $q_i > 0$ determine the zero structure.

Now the illustration of the above process of the reduction of a rational matrix at $s_0$ can be seen through the following example.

**Example 5.8.** Let

$$W(s) = \begin{pmatrix} \frac{1}{s-1} & 1 & \frac{s}{s+1} \\ s & \frac{1}{s} & 1 \\ 0 & 1 & \frac{s-1}{s} \end{pmatrix}$$

(5.9)
and \(s_0 = 0\). Choose an element with least \(V_0^{(1)}(W(s))\) and bring it to the position (1,1) by the elementary operations at \(s_0\). i.e.

\[
W(s) \rightarrow \begin{pmatrix}
\frac{1}{s} & s & 1 \\
1 & \frac{1}{s-1} & \frac{s}{s+1} \\
1 & 0 & \frac{s-1}{s}
\end{pmatrix}
\] (5.10)

Now, \(W_{11}(s) = \frac{1}{s}\) and \(V_0^{(1)}(W_{11}(s)) = -1\). It can be seen that

\[
\begin{align*}
V_0^{(1)}(W_{11}(s)) < V_0^{(1)}(W_{i1}(s)) & \text{ for } i = 2, 3 \\
V_0^{(1)}(W_{11}(s)) < V_0^{(1)}(W_{1j}(s)) & \text{ for } j = 2, 3
\end{align*}
\] (5.11)

We can write then

\[
\begin{align*}
W_{21}(s) &= \frac{1}{s} \times s \\
W_{31}(s) &= \frac{1}{s} \times s \\
W_{12}(s) &= \frac{1}{s} \times s^2 \\
W_{13}(s) &= \frac{1}{s} \times s \\
\end{align*}
\] (5.12)

where \(q_{21}, q_{31}, q_{12}\) and \(q_{13}\) are all proper at \(s_0\). Now subtract \(q_{21}\) times the 1st row from the 2nd and 3rd and also \(q_{12}, q_{13}\) times the 1st column from the 2nd and 3rd column to reduce (5.10) to the following form;

\[
(5.10) \rightarrow \begin{pmatrix}
\frac{1}{s} & 0 & 0 \\
0 & -s^3 + s^2 + 1 & -s^2 \\
0 & s^2 & -s^2 + s - 1
\end{pmatrix}
\] (5.13)
Now consider the \((2 \times 2)\) matrix and choose an element with least \(V_o^{(1)}(\cdot)\) and bring it to the position \((2,2)\) by the elementary operations at \(s_0\), i.e.

\[
(5.13) \rightarrow \begin{pmatrix} \frac{1}{s} & 0 & 0 \\ 0 & \frac{-s^2 + s - 1}{s} & s^2 \\ 0 & \frac{-s^2}{s - 1} & \frac{-s^3 + s^2 + 1}{s - 1} \end{pmatrix}
\]  

(5.14)

From (5.14), \(W_{22}(s) = \frac{-s^2 + s - 1}{s}\) and \(V_o^{(1)}(W_{22}(s)) = -1\). It is clear that

\[V_o^{(1)}(W_{22}(s)) = -1 < V_o^{(1)}(W_{23}(s)) = 2\]

and

\[V_o^{(1)}(W_{22}(s)) = -1 < V_o^{(1)}(W_{32}(s)) = 2.\]

Then we can write as in (5.12)

\[
\begin{align*}
W_{23}(s) &= \frac{-s^2 + s - 1}{s} \times \frac{s^3}{-s^2 + s - 1} \\
W_{32}(s) &= \frac{-s^2 + s - 1}{s} \times \frac{-s^3}{(-s^2 + s - 1)(s + 1)}
\end{align*}
\]

where

\[
\begin{align*}
q_{23}(s) &= \frac{s^3}{-s^2 + s - 1} \\
q_{32}(s) &= \frac{-s^3}{(-s^2 + s - 1)(s + 1)}
\end{align*}
\]

(5.15)

where \(q_{23}\) and \(q_{32}\) are proper at \(s_0\). Now subtract \(q_{23}\) times 2\(^{nd}\) row from 3\(^{rd}\) row and \(q_{32}\) times 2\(^{nd}\) column from 3\(^{rd}\) column to reduce (5.14) to the following form;

\[
(5.14) \rightarrow \begin{pmatrix} \frac{1}{s} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{s^6 - s^5 + s^4 - s^3 - s^2 - 1}{(-s^2 + s - 1)(s^2 - 1)} \end{pmatrix}
\]  

(5.16)

from which the Smith-McMillan form at \(s_0 = 0\) can be derived as follows;
Multiplied row three by the biproper at $s_0$ rational function
\[
\frac{s^6 - s^5 + s^4 - s^3 - s^2 - 1}{(-s^2 + s - 1)(s^2 - 1)}
\] to get;

\[
\begin{pmatrix}
\frac{1}{s} & 0 & 0 \\
0 & \frac{1}{s} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
(5.16) $\rightarrow$

(5.17)

An alternative way to determine the Smith-McMillan form at $s_0$ of a rational matrix is via the invariant rational functions and its valuations at $s_0$. Let $W(s)$ be an $(m \times n)$ rational matrix whose rank at $s_0$ is $r = \min (m, n)$, and let $V_{s_0}^{(i)}(W(s))$ be the least valuation at $s_0$ among the valuations at $s_0$ of all $(i \times i)$ minors of $W(s)$, then,

**Definition 5.18.** Define

\[
\begin{align*}
\Phi_1(s_0) &= (s - s_0)^{V_{s_0}^{(1)}(W(s))} \\
\Phi_2(s_0) &= (s - s_0)^{V_{s_0}^{(2)}(W(s)) - V_{s_0}^{(1)}(W(s))} \\
&\vdots \\
\Phi_r(s_0) &= (s - s_0)^{V_{s_0}^{(r)}(W(s)) - V_{s_0}^{(r-1)}(W(s))}
\end{align*}
\]

(5.19)

Then $\Phi_i(s_0)$'s are called the INVARIANT RATIONAL FUNCTIONS AT $s_0$ of $W(s)$ which can also be found by the following method.

Let $W(s)$ be an $(m \times n)$ proper at $s_0$ rational matrix of rank $r$ ($r \leq \min (m, n)$) and let $d_k(s)$ be the monic (g.c.d) of all $(k \times k)$ minors of $W(s)$, where $k = 1, \ldots, r$. Then for $k \geq 2$ any $(k \times k)$ minor of $W(s)$ can be expressed as a linear combination of $(k - 1) \times (k - 1)$ minors of $W(s)$, so if $d_0(s) \leq 1$, then in $(d_0(s), d_1(s), \ldots, d_r(s))$, $d_k(s)$ is divisible by $d_{k-1}(s)$ where $k = 1, \ldots, r$. Using the method of (2.15) of chapter two each $d_k(s)$
can be written as a product of \((s - s_0)^{n_k}\) and a biproper at \(s_0\) part, i.e. \(d_k(s) = d'_k(s)\cdot (s - s_0)^{n_k}, \ k = 1, \ldots, r\) \((n)\) is an integer). Then dividing out the factor \(d'_k(s)\), \((s - s_0)^{n_k}, \ k = 1, \ldots, r\) are called the DETERMINANTAL DIVISORS AT \(s_0\) of \(W(s)\).

From \(d_k(s) = d'_k(s)\cdot (s - s_0)^{n_k}, \ k = 1, \ldots, r, \ d_0(s) \neq 1\) implies that \((s - s_0)^{n_0} = 1. \) Let \((s - s_0)^{n_k}\) be ordered, i.e. \(n_0 \leq n_1 \leq \ldots \leq n_r\), then,

\[
i_1(s_0) = (s - s_0)^{n_1-n_0}, \ i_2(s_0) = (s - s_0)^{n_2-n_1}, \ldots, \ i_k(s_0) = (s - s_0)^{n_k-n_{k-1}}
\]

are the invariant rational functions at \(s_0\) of \(W(s)\) as defined by (5.19).

Now to determine the Smith-McMillan form at \(s_0\) of \(W(s)\) (whether it is proper at \(s_0\) or not) is simply to compute the \(\Phi_i(s_0)'s\) or the \(i(s_0)'s\) and hence,

\[
M_{s_0}(W) = \begin{cases} 
  \text{diag} (\Phi_1(s_0), \Phi_1(s_0), \Phi_2(s_0), \ldots, \Phi_r(s_0), 0, \ldots, 0) \\
or \\
  \text{diag} (i_1(s_0), i_2(s_0), \ldots, i_k(s_0), 0, \ldots, 0)
\end{cases}
\]

for \(k = 1, \ldots, r\).

The above process of finding the Smith-McMillan form at \(s_0\) can be illustrated by the following example.

**Example 5.22.** Reconsider example (5.8) where \(s_0 = 0\) and

\[
W(s) = \begin{pmatrix} 
  \frac{1}{s - 1} & 1 & \frac{s}{s + 1} \\
  \frac{s}{s - 1} & 1 & \frac{s}{s + 1} \\
  s & \frac{1}{s} & 1 \\
  0 & 1 & \frac{s - 1}{s}
\end{pmatrix}
\]
Then
\[ V_0^{(1)}(W) = \min \{0, 0, +1, +1, -1, 0, +\infty, 0, -1\} = -1 \]
\[ V_0^{(2)}(W) = \min \{+1, 0, -1, -1, -1, 0, -2, 0, +1\} = -2 \]
\[ V_0^{(3)}(W) = -2 \]

So using (5.19) gives
\[ \Phi_1(0) = (s)V_0^{(1)}(W) = s^{-1} = \frac{1}{s} \]
\[ \Phi_2(0) = (s)V_0^{(2)}(W) - V_0^{(1)}(W) = s^{-2} - (-1) = \frac{1}{s} \]
\[ \Phi_3(0) = (s)V_0^{(3)}(W) - V_0^{(2)}(W) = s^{-2} - (-2) = 1 \]

Therefore
\[ M_0(W) = \begin{pmatrix} 1 \frac{1}{s} 0 \\ 0 \frac{1}{s} 0 \\ 0 0 1 \end{pmatrix} \]

as has been found earlier in (5.17).

IV-6: §Conclusion and Discussions

The local study of rational matrices has been the major objective in this chapter, and plays an essential role in the local theory of linear systems. Accordingly in our new approach, the most useful and fundamental results at \( s_0 \) for rational functions has been presented in sections two. Also a local version for valuation of rational functions using the definition of degree of a rational function at \( s_0 \) has been given in section two. The elementary row and column operations at \( s_0 \) and also the notion of equivalence at \( s_0 \) have been presented in section three. In section four the valuations of rational matrices at \( s_0 \) have been considered and some related results have been given. Invraient rational functions and Smith-McMillan form of a rational matrix at \( s_0 \) have been the subject of the section five.
Also an alternative way of finding the Smith-McMillan form at $s_0$ of a rational matrix using the local valuations of the rational matrix has been discussed and illustrated by an example in this section.
CHAPTER V

Local Study of Linear Systems
Section V-1: §Introduction

The conventional study of linear multivariable systems is based on the transformation of equivalence by unimodular matrix multiplication (Rosenbrock, 1970; Kailath, 1980). This transformation preserves the structure at all frequencies in the finite complex plane \( \mathbb{C} \) of any matrix to which it is applied. If however the infinite frequency behaviour of a multivariable linear system is of interest then the requirements are in a sense reversed and it is now one specific frequency that is of concern to the exclusion of all others. (Vardulakis et al, 1982). For example the matrix fraction \( G(\frac{1}{w}) = D^{-1}(w)N(w) \) is used to study the infinite frequency structure of \( G(s) \). The point at infinity is of course represented in this matrix fraction by the point \( w = 0 \) and only this point is of interest. Insisting that the matrix fraction is (r.p) will ensure that the pole/zero structure of \( G(\frac{1}{w}) \) is given by the zero structure of \( D(w) \) and \( N(w) \) respectively at \( w = 0 \), however the same requirement also ensures that \( D(w) \) and \( N(w) \) describe the structure of \( G(\frac{1}{w}) \) at all other points \( w \) in the finite complex plane. Clearly the condition of (r.p) is excessive for the single requirement of isomorphic structures at \( w = 0 \), it will therefore be of interest to know the lesser conditions which will ensure the matrix fraction simply carries the pole/zero structure of \( G(\frac{1}{w}) \) at \( w = 0 \), and no other frequency.

In this chapter this latter point of view has been taken up and a highly localised approach to systems theory, by considering the behaviour of a system only at a specific finite frequency has been developed. Direct application of this local theory will be seen in the next chapter. In section two after defining system equivalence at \( s_0 \), some interesting related results are given. The idea of decoupling zeros at \( s_0 \) is studied in section three and then using
the definition of system equivalence at \( s_0 \) and decoupling zeros at \( s_0 \) a result concerning a system equivalent at \( s_0 \) to \( P(s) \), where \( s_0 \) is not a decoupling zero of \( P(s) \) is given. One of our important results which is a local state-space realisation and some further standard forms for system matrices at \( s_0 \) are given in section four. Finally in section five systems of least order at \( s_0 \) are studied and in this regard some relevant results are presented.

Section V-2: §System Matrices and Local Results

In the case of the conventional study of linear systems the equations describing the system may be rather complicated, and so it may be necessary to simplify them in some way. Similarly the local study of linear systems requires the same kind of facility, but now it is necessary to accomplish such transformations of the system locally and in a way which is more easily understood. In particular we are interested in such transformations of the system which do not change its associated transfer function matrix and its basic features at a specific frequency \( s_0 \).

The specification of the local matrix results of the previous chapter to the case of system matrices yields local versions of some well known results (Rosenbrock, 1970). The basis of these results is the notion of equivalence at \( s_0 \) (described in the previous chapter), suitably adapted to the system theory situation as follows;

Definition 2.1. Let

\[
P_i(s) = \begin{pmatrix} T_i(s) & U_i(s) \\ -V_i(s) & W_i(s) \end{pmatrix}, \quad i = 1, 2
\]  (2.2)

be two \( (m + n) \times (m + l) \) rational system matrices. \( P_1(s) \) and \( P_2(s) \) are said to be SYSTEM EQUIVALENT AT \( s_0 \) if there exist rational matrices
$M(s), N(s)$ biproper at $s_0$ and rational matrices $X(s), Y(s)$ proper at $s_0$, such that

$$
\begin{pmatrix}
M(s) & 0 \\
X(s) & I_n
\end{pmatrix}
\begin{pmatrix}
T_1(s) & U_1(s) \\
-V_1(s) & W_1(s)
\end{pmatrix}
\begin{pmatrix}
N(s) & Y(s) \\
0 & I_l
\end{pmatrix}
= 
\begin{pmatrix}
T_2(s) & U_2(s) \\
-V_2(s) & W_2(s)
\end{pmatrix}
$$

(2.3)

Alternatively the above definition could have been phrased in terms of a subset of the elementary operations at $s_0$ for rational matrices. These operations arise from elementary operations at $s_0$ in much the same way as those of strict system equivalence arise from the elementary operations of unimodular equivalence, in other words the relation (2.3) can be generated by the following elementary operations at $s_0$;

(i) Interchanging any two among the first $m$ rows (columns).

(ii) Adding to any row (column) a multiple of any one of the first $m$ rows (columns) by any proper at $s_0$ rational function $w(s)$.

(iii) Multiplying any one of the first $m$ rows (columns) by a non-zero biproper at $s_0$ rational function $f(s)$.

This clearly indicates the relationship between system equivalence at $s_0$ and other notions of equivalence for linear systems as the following theorem shows;

**Theorem 2.4.**

(i) System equivalence at $s_0$ (finite) includes strict system equivalence (Rosenbrock, 1970) as a special case.

(ii) System equivalence at $s_0$ (finite) is a special case of system equivalence (Rosenbrock, 1970).
Proof. The definition of strict system equivalence takes the form (2.3) with $M(s), N(s)$ unimodular and $X(s), Y(s)$ polynomial. However the unimodularity of $M(s), N(s)$ implies that they possess no poles or zeros at finite frequencies and specifically at $s_0$. Thus $M(s), N(s)$ are biproper at $s_0$. Additionally $X(s), Y(s)$ have no poles at $s_0$ (since they are polynomials) and so are proper at $s_0$ and hence (i). To prove (ii) note that only the following three elementary operations of four operations of the Rosenbrock’s system equivalence are needed in order to give rise to a matrix statement of the form (2.3) in which $M(s), N(s), X(s)$ and $Y(s)$ are general rational matrices with $M(s)$ and $N(s)$ invertible. These three elementary operations are;

(i) Interchanging any two among the first $m$ rows (columns).

(ii) Adding a multiple, by a rational function of any one of the first $m$ rows (columns) to any other row (column).

(iii) Multiplying any one of the first $m$ rows (columns) by a rational function, not identically zero.

Now a direct comparison between these elementary operations and elementary operations at $s_0$ gives part (ii) of the theorem.

The above results give some indication of the likely essential invariants of the transformation of system equivalence at $s_0$. In order to consider this problem we first require the following terminology.

Definition 2.5. If

$$P(s) = \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix}$$

(2.6)

then the degree at $s_0$ of the rational function $|T(s)|$ is called the ORDER AT $s_0$ OF $P(s)$ where $|\cdot|$ indicates the determinant of the indicated matrix.
Now we are in a position to give the following result concerning some of the invariants of the transformation of system equivalence at \( s_0 \).

**Theorem 2.7.** Two \((m + n) \times (m + l)\) rational system matrices which are system equivalent at \( s_0 \), have the same order at \( s_0 \) and give rise to the same transfer function matrix.

**Proof.** If \( P_1(s) \), \( P_2(s) \) are two rational system matrices related by system equivalence at \( s_0 \), then (2.3) holds, and in particular we have

\[
\begin{pmatrix}
M(s) & 0 \\
X(s) & I_n
\end{pmatrix}
\begin{pmatrix}
T_1(s) & U_1(s) \\
-V_1(s) & W_1(s)
\end{pmatrix}
\begin{pmatrix}
N(s) & Y(s) \\
0 & I_l
\end{pmatrix}
= 
\begin{pmatrix}
T_2(s) & U_2(s) \\
-V_2(s) & W_2(s)
\end{pmatrix}
\]

which gives

\[
M(s) T_1(s) N(s) = T_2(s) \tag{2.9}
\]

Hence

\[
|M(s)| \cdot |T_1(s)| \cdot |N(s)| = |T_2(s)| \tag{2.10}
\]

Since \( M(s), N(s) \) are biproper at \( s_0 \), the rational functions \( |M(s)|, |N(s)| \) have degree zero at \( s_0 \) and so themselves are biproper at \( s_0 \). Thus writing \( |T_1(s)| \) in the form of lemma (2.2) of chapter IV, it follows then from (2.10) that the degree at \( s_0 \) of \(|T_2(s)|\) and \(|T_1(s)|\) are identical. Thus \( P_1(s), P_2(s) \) have the same order at \( s_0 \) by definition (2.5). Now from (2.8) we have;

\[
\begin{pmatrix}
T_2(s) & U_2(s) \\
-V_2(s) & W_2(s)
\end{pmatrix} =
\begin{pmatrix}
M(s)T_1(s)N(s) \\
-(V_1(s) - X(s)T_1(s))N(s)
\end{pmatrix}
\begin{pmatrix}
M(s)(T_1(s)Y(s) + U_1(s)) \\
X(s)T_1(s)Y(s) - V_1(s)Y(s) + X(s)U_1(s) + W_1(s)
\end{pmatrix}
\]

Therefore
\[ G_2(s) = V_2(s)T_2^{-1}(s)U_2(s) + W_2(s) = (V_1(s) - X(s)T_1(s)) \]
\[ \times N(s)N^{-1}(s)T_1^{-1}(s)M^{-1}(s) \]
\[ + X(s)T_1(s)Y(s) \]
\[ - V_1(s)Y(s) + X(s)U_1(s) + W_1(s) \]
\[ = V_1(s)T^{-1}(s)U_1(s) + W_1(s) \]
\[ = G_1(s) \quad (2.12) \]

and that establishes directly the invariance of the associated transfer function matrix.

The invariance of the associated transfer function matrix which has been established by the above theorem can also be seen indirectly from part (ii) of the theorem (2.4). It is worth noting that a special case of system equivalence at \( s_0 \) which therefore preserves the order at \( s_0 \) and the associated transfer function matrix is system similarity which has been defined in the preliminary chapter.

**Section V-3: §Decoupling Zeros at \( s_0 \)**

One of the most important notions in the study of the structure of a linear system is that of the decoupling zero, and this is the subject of this section.

Returning to the system matrix \( P(s) \) where

\[ P(s) = \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix} \quad (3.1) \]
with its associated transfer function matrix $G(s)$ given by

$$G(s) = V(s)T^{-1}(s)U(s) + W(s) \quad (3.2)$$

Then the decoupling zeros at $s_0$ of $P(s)$ are defined as follows;

**Definition 3.3.** $s_0 \in C$ is called an **INPUT-DECOUPLING ZERO** of $P(s)$ denoted by $(i.d.z)_{s_0}$ in case it is a zero of the Smith form at $s_0$ of the matrix $(T(s)U(s))$. Similarly $s_0 \in C$ is called an **OUTPUT-DECOUPLING ZERO** of $P(s)$ denoted by $(o.d.z)_{s_0}$ in case it is a zero of the Smith form at $s_0$ of the matrix $\begin{pmatrix} T(s) \\ -V(s) \end{pmatrix}$.

Now let the set of $(i.d.z)_{s_0}$ of $P(s)$ be denoted by $\{s_0\}^P_i$ and the set of $(o.d.z)_{s_0}$ of $P(s)$ by $\{s_0\}^P_o$. Further let all $\{s_0\}^P_i$ be removed by system equivalence and call the resulting system matrix $P'(s)$. Let then the set of $(o.d.z)_{s_0}$ of $P'(s)$ be denoted by $\{s_0\}^{P'}_o$, then the set $\{s_0\}^P_i = \{s_0\}^P_o - \{s_0\}^{P'}_o$ is called the **INPUT-OUTPUT-DECOUPLING ZEROS** of $P(s)$ at $s_0$.

Having defined the decoupling zeros at $s_0$, we can now give the following result.

**Theorem 3.4.** Let $P(s)$ be a proper at $s_0$ system matrix, then any system matrix obtained from $P(s)$ by the removal of decoupling zeros is system equivalent at $s_0$ to $P(s)$, provided that these decoupling zeros are not located at $s_0$.

**Proof.** Consider the definition of system equivalence at $s_0$ and the relationship (2.3), then the order reduction procedure of Rosenbrock (1970) may be written in the form (2.3) in which $N(s) = I, X(s) = Y(s) = (0)$ and $M(s)$ is an invertible rational matrix having a single pole at $s = \beta$ where $\beta$ is the input decoupling zero to be removed. In order that (2.3) be satisfied,
$M(s)$ must be biproper at $s_0$, since $N(s) = I$ is biproper at $s_0$ and $X(s)$, $Y(s)$ are proper at $s_0$. It is clear that $\beta \neq s_0$, then $M(s)$ will be biproper at $s_0$ and vice versa, and this proves the theorem in the case of input decoupling zeros. In an analogous way the case of output decoupling zeros may be established.

\[\square\]

Notice that in the above theorem $P(s)$ is assumed to be proper at $s_0$. The necessity of the properness at $s_0$ of $P(s)$ can be illustrated by the following example.

**Example 3.5.** Consider the proper at $s_0 = 1$ system matrix $P_1(s)$ given by

$$P_1(s) = \begin{pmatrix} s(s + 1) & 0 & s \\ 0 & s(s + 2) & s \\ -1 & -1 & 0 \end{pmatrix}$$  \hspace{1cm} (3.6)

It is clear that $P_1(s)$ has two input-decoupling zeros and one output-decoupling zero at $s_0 = 0$. When these input-decoupling zeros are removed by elementary operations, the result is

$$P_2(s) = \begin{pmatrix} s + 1 & 0 & 1 \\ 0 & s + 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$ \hspace{1cm} (3.7)

Which additionally has no output-decoupling zeros at $s_0 = 0$. Now we can investigate that whether $P_1(s)$ and $P_2(s)$ are system equivalent at $s_0 = 1$ or not? Clearly the decoupling zeros of $P_1(s)$ are not located at $s_0 = 1$, so the matrices

$$M(s) = \begin{pmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{pmatrix}$$ \hspace{1cm} (3.8)
are biproper at \( s_0 = 1 \), while

\[
\begin{align*}
X(s) &= (0 \quad 0) \\
Y(s) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*}
\] (3.9)

are proper at \( s_0 = 1 \), therefore \( P_1(s) \) and \( P_2(s) \) are related by

\[
\begin{pmatrix} M(s) & 0 \\ X(s) & I \end{pmatrix} P_1(s) \begin{pmatrix} N(s) & Y(s) \\ 0 & I \end{pmatrix} = P_2(s)
\] (3.10)

which is the definition of system equivalence at \( s_0 = 1 \).

Notice that if our investigation of system equivalence is based at \( s_0 = 0 \),
then not only \( M(s) \) is no longer biproper at \( s_0 = 0 \), but also the removal
of the decoupling zeros at \( s_0 = 0 \) can not be achieved via the elementary
operations and hence \( P_1(s) \) and \( P_2(s) \) are not system equivalent at \( s_0 = 0 \).

An important feature of the equivalence transformation at \( s_0 \) with regard
to the decoupling zeros at \( s_0 \) is;

**Theorem 3.11.** Under system equivalence at \( s_0 \) the following are invariant:

(i) The set of (i.d.z)\( s_0 \) of \( P(s) \).

(ii) The set of (o.d.z)\( s_0 \) of \( P(s) \).

(iii) The set of (i.o.d.z)\( s_0 \) of \( P(s) \).

**Proof.** By (2.3) we have

\[
\begin{pmatrix} M(s) & 0 \\ X(s) & I_n \end{pmatrix} \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix} \begin{pmatrix} N(s) & Y(s) \\ 0 & I \end{pmatrix} = \begin{pmatrix} T'(s) & U'(s) \\ -V'(s) & W'(s) \end{pmatrix}
\] (3.12)
Then
\[
M(s) \begin{pmatrix} T(s) & U(s) \end{pmatrix} \begin{pmatrix} N(s) & Y(s) \\ 0 & I_l \end{pmatrix} = (T'(s) & U(s)) \quad (3.13)
\]

Now \(M(s)\) and \(N(s)\) are biproper at \(s_0\) and \(Y(s)\) is proper at \(s_0\), hence the pre- and post-multiplication of \((T(s) \ U(s))\) by the matrices which have no poles and zeros at \(s_0\) leave the Smith form at \(s_0\) of \((T(s) \ U(s))\) unchanged. Then (i) follows from definition (3.3). Similarly from (3.12);

\[
\begin{pmatrix} M(s) & 0 \\ X(s) & I_n \end{pmatrix} \begin{pmatrix} T(s) \\ -V(s) \end{pmatrix} N(s) = \begin{pmatrix} T'(s) \\ -V'(s) \end{pmatrix} \quad (3.14)
\]

Again pre and post-multiplication of \(\begin{pmatrix} T(s) \\ -V(s) \end{pmatrix}\) by the matrices which have no poles and zeros at \(s_0\) leave the Smith form at \(s_0\) of \(\begin{pmatrix} T(s) \\ -V(s) \end{pmatrix}\) unchanged and then (ii) follows from definition (3.3).

Now let the (i.d.z)\(_{s_0}\) of \((T(s) \ U(s))\) be removed by pre-multiplying \((T(s) \ U(s))\) by a rational matrix \(R(s)\). i.e. by system equivalence of Rosenbrock. Then from (3.13) we have,

\[
M^{-1}(s) \begin{pmatrix} M(s) & (T(s) & U(s)) \end{pmatrix} \begin{pmatrix} N(s) & Y(s) \\ 0 & I_l \end{pmatrix} = M^{-1}(s) \begin{pmatrix} T'(s) & U'(s) \end{pmatrix} \quad (3.15)
\]

and then since \(M^{-1}(s) \ M(s) = I\), by pre-multiplying (3.15) in both sides by \(M(s) \ R(s)\), it becomes,

\[
M(s) \ R(s) \begin{pmatrix} M(s) & (T(s) & U(s)) \end{pmatrix} \begin{pmatrix} N(s) & Y(s) \\ 0 & I_l \end{pmatrix} = M(s) \ R(s) \ M^{-1}(s) \begin{pmatrix} T'(s) & U'(s) \end{pmatrix} \quad (3.16)
\]
By similar argument as above, the matrices $R(s) (T(s) \ U(s))$ and $M(s) R(s) M^{-1}(s) (T'(s) \ U'(s))$ have the same Smith form at $s_0$. Now from (3.14),

$$-V'(s) = X(s) T(s) N(s) - V(s) N(s)$$

(3.17)

and from (3.16),

$$M(s) R(s) T(s) N(s) = M(s) R(s) M^{-1}(s) T'(s)$$

(3.18)

Then (3.17) and (3.18) can be written as a single matrix equation given by

$$
\begin{pmatrix}
M(s) R(s) M^{-1}(s) T'(s) \\
-V'(s)
\end{pmatrix} =
\begin{pmatrix}
M(s) & 0 \\
X(s) R^{-1}(s) & I_n
\end{pmatrix}
\begin{pmatrix}
R(s) T(s) \\
-V(s)
\end{pmatrix}
N(s)

(3.19)

It can be seen from (3.19) that

$$
\begin{pmatrix}
M(s) R(s) M^{-1}(s) T'(s) \\
-V'(s)
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
R(s) T(s) \\
-V(s)
\end{pmatrix}
$$

have the same Smith form at $s_0$. Then (iii) follows from definition (3.3).

\square

Section V-4: §Standard Forms and Local State-space Realisation

The existence of some standard forms and a local state-space realisation of a system matrix is the subject of this section. Although the definitions and results of the previous sections hold for general rational system matrices, the main context in which they will be applied will be that of rational system matrices which are proper at $s_0$. As in the conventional case it will be seen that any system can be described by many different system matrices, all of which contain the important mathematical information required about the system. In particular it is first shown that any system can be alternatively described by a system matrix in state-space form at $s_0$ which preserves the important characteristics of the system at $s_0$. 

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Our fundamental result on which much of the following depends, concerns the existence and structure of a local state-space realisation of a system matrix. This realisation is not entirely polynomial as the statement indicates. But first the following terminology.

According to Laurent's Theorem (Laurent expansion) any function \( G(s) \) can be written as

\[
G(s) = \sum_{n=0}^{\infty} a_n (s-s_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(s-s_0)^n}
\]

where \( \sum_{n=0}^{\infty} a_n (s-s_0)^n \) is called the analytic part and \( \sum_{n=1}^{\infty} \frac{b_n}{(s-s_0)^n} \) is called the principal part of the Laurent expansion of \( G(s) \) about \( s_0 \). Now if the principal part has only a finite number of terms given by \( \sum_{n=1}^{m} \frac{b_n}{(s-s_0)^n} \) where \( b_n \neq 0 \), then \( s = s_0 \) is called a pole of order \( n \). For example if \( G(s) = \frac{s}{(s+1)(s+2)} \) and \( s_0 = -2 \), then from above \( G(s) \) can be written as

\[
G(s) = \frac{2}{s+2} - \frac{1}{s+1}
\]

which clearly shows that \( G(s) \) has a pole of order one at \( s_0 = 2 \). Now we can state the main theorem.

**Theorem 4.1.** Let

\[
P(s) = \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix}
\]

be an \((m+n) \times (m+l)\) rational system matrix which is proper at \( s_0 \) and of order \( r \) at \( s_0 \) \((m \geq r)\). If the associated transfer function matrix is written as

\[
G(s) = G_{s_0}(s) + Q(s)
\]
where \( G_{s_0}(s) \) is the Principle Part of the Laurent expansion of \( G(s) \) about \( s_0 \), then \( P(s) \) is system equivalent at \( s_0 \) to a rational system matrix of the form

\[
P'(s) = \begin{pmatrix} I_{m-r} & 0 & 0 \\ 0 & sI_r - A & B \\ 0 & -C & Q(s) \end{pmatrix}
\]

(4.4)

where \( A, B, C \) are constant matrices, and \( A \) has eigenvalues only at \( s_0 \).

**Proof.** Without loss of generality \( P(s) \) may be taken to be such that \( T(s), U(s), V(s) \) are polynomials. This may be achieved by operations of system equivalence at \( s_0 \), by multiplying each row in the first block row of \( P(s) \) by its least common denominator and operating similarly on the columns from the first block column. This procedure does not change \( W(s) \) which is therefore still proper at \( s_0 \) and does not introduce any new decoupling zeros at \( s_0 \). With \( P(s) \) in this form it may now be brought by strict system equivalence to the extended state-space form (Rosenbrock, 1970) given by

\[
P_1(s) = \begin{pmatrix} I_{m-r_1} & 0 & 0 & 0 \\ 0 & sI_{r_1} - A_1 & 0 & B_1 \\ 0 & 0 & sI_r - A_2 & B_2 \\ 0 & -C_1 & -C_2 & D(s) \end{pmatrix}
\]

(4.5)

where \( \text{diag} (A_1, A_2) \) is in, say, first natural form with \( A_1 \) having no eigenvalues at \( s_0 \), while \( A_2 \) has only eigenvalues at \( s_0 \). Clearly then \( (sI_{r_1} - A_1) \) is biproper at \( s_0 \), and \( D(s) \) is proper at \( s_0 \) and so by operations of system equivalence at \( s_0 \), \( P_1(s) \) may be reduced to the successive forms,

\[
\begin{pmatrix} I_{m-r_1} & 0 & 0 & 0 \\ 0 & I_{r_1} - r & 0 & (sI_{r_1} - A_1)^{-1}B_1 \\ 0 & 0 & sI_r - A_2 & B_2 \\ 0 & -C_1 & -C_2 & D(s) \end{pmatrix} \rightarrow
\]
where

\[ R(s) \triangleq D(s) + C_1(sI_{n_1} - A_1)^{-1}B_1 \] (4.7)

is proper at \( s_0 \).

Now by theorem (2.7) \( P(s) \) and (4.6) give rise to the same transfer function matrix and so

\[ G(s) = C_2(sI_r - A_2)^{-1}B_2 + R(s) \] (4.8)

Since the first matrix on the RHS of (4.8) only has poles at \( s_0 \) and \( R(s) \) is proper at \( s_0 \), it follows from the uniqueness of the Laurent expansion that \( C_2(sI_r - A_2)^{-1}B_2 \) is the principle part of this expansion and that from (4.3) and (4.7)

\[ R(s) = Q(s) \] (4.9)

which establishes the theorem.

The following example illustrates the above result.

**Example 4.10.** Consider the following system matrix and let the point of investigation be \( s_0 = 1 \).

\[ P(s) = \begin{pmatrix}
1 & \frac{1}{s} & 0 & 0 \\
0 & s & 1 & 1 \\
s & \frac{1}{s} & 0 & s \\
1 & 1 + s & 0 & \frac{1 - 2s}{s}
\end{pmatrix} \] (4.11)
Then clearly $P(s)$ is a proper at $s_0 = 1$ rational system matrix. Now from (4.11),

$$|T(s)| = \begin{vmatrix}
1 & \frac{1}{s} & 0 \\
0 & s & 1 \\
\frac{1}{s} & s & 0
\end{vmatrix} = \frac{s - 1}{s}$$

(4.12)

Now by lemma (2.2) of previous chapter (4.12) can be uniquely written as

$$|T(s)| = \frac{s - 1}{s} = (s - 1)^\frac{1}{s}$$

(4.13)

where $\frac{1}{s}$ is biproper at $s_0 = 1$. The degree at $s_0$ of $|T(s)|$ is therefore one. Then using definition (2.5) the degree at $s_0$ of $|T(s)|$ is the order at $s_0$ of $P(s)$ which is in this case one. Also from (4.11)

$$G(s) = V(s) T^{-1}(s) U(s) + W(s)$$

$$= (-1 \ -1 - s \ 0) \begin{pmatrix}
-1 & 0 & \frac{1}{s - 1} \\
\frac{s^2}{s - 1} & 0 & \frac{-s}{s - 1} \\
\frac{-s^3}{s - 1} & 1 & \frac{s^2}{s - 1}
\end{pmatrix} \begin{pmatrix}
0 \\
1 \\
\frac{s}{s - 1}
\end{pmatrix} + \frac{1 - 2s}{s}$$

i.e. $G(s) = \frac{s^3 + s^2 - s}{s - 1} + \frac{1 - 2s}{s}$.

Hence by (4.3), $G(s) = \frac{1}{s - 1} + \frac{s^3 + 2s^2 - s + 1}{s}$

where $G_{s_0}(s) = \frac{1}{s - 1}$, $Q(s) = \frac{s^3 + 2s^2 - s + 1}{s}$.

Now $P(s)$ as given by (4.11) can be reduced to the following form by system equivalent at $s_0$ (elementary operations at $s_0$).

Multiplying column two by $s$ and then adding column two to the column one times -1 gives,
Adding \(-5\) times row one of (4.14) to the row three and then adding \(-1\) times row one of the resulting matrix to the row four gives,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & s^2 & 1 & 1 \\
s & 1 - s & 0 & s \\
1 & s^2 - 1 + s & 0 & 1 - 2s/s
\end{pmatrix}
\]  

(4.15)

Adding \(-s^2\) times column three of (4.15) to the column two and then adding \((1 + s)\) times row three of the resulting matrix to the row four gives,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 - s & 0 & s \\
0 & s & 0 & s(1 + s) + (1 - 2s/s)
\end{pmatrix}
\]  

(4.16)

Adding \(-1\) times column three of (4.16) to the column four and then interchanging column two and three of the resulting matrix gives,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 - s & 0 & s \\
0 & 0 & s & s^3 + s^2 - 2s + 1/s
\end{pmatrix}
\]  

(4.17)

Adding column three of (4.17) to the column four and then adding row three of the resulting matrix to the row four gives,
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 - s \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
s^3 + 2s^2 - s + 1 \\
s
\end{pmatrix}
\]

Clearly (4.18) is the required form and that is a fact shown by the theorem (4.1).

Now consider the following example, when the system matrix has a decoupling zero at the point of investigation.

**Example 4.19.** Let the point of investigation be \( s_0 = 1 \) and \( P(s) \) be given by,

\[
P(s) = \begin{pmatrix}
\frac{s + 1}{s} & 0 & \frac{1}{s} \\
\frac{s}{s + 1} & 0 & \frac{1}{s - 1} \\
0 & s - 1 & 0 \\
-1 & -1 & 0
\end{pmatrix}
\]  

(4.20)

As required \( P(s) \) is proper at \( s_0 = 1 \) and also has an (i.d.z)1. Then

\[
G(s) = V(s) T^{-1}(s) U(s) + W(s)
\]

\[
= (1 \quad 1) \begin{pmatrix}
\frac{s}{s + 1} & 0 \\
0 & \frac{1}{s - 1}
\end{pmatrix} \begin{pmatrix}
\frac{1}{s} \\
\frac{0}{s}
\end{pmatrix} + 0
\]

\[
= \left( \begin{array}{cc}
\frac{s}{s + 1} & 1 \\
\frac{1}{s - 1} & 0
\end{array} \right) \begin{pmatrix}
\frac{1}{s} \\
\frac{0}{s}
\end{pmatrix} = \frac{1}{s + 1}
\]

Hence by (4.3), \( G(s) = 0 + \frac{1}{s + 1} \) where \( G_{s_0}(s) = 0 \), \( Q(s) = \frac{1}{s + 1} \). The process of reduction can be achieved as before using the elementary operations at \( s_0 \) as follows;
Multiplying row one of $P(s)$ by $s$ and then multiplying column one of the resulting matrix by $\frac{1}{s + 1}$ gives,

$$
\begin{pmatrix}
1 & 0 & 1 \\
0 & s - 1 & 0 \\
-1 & s + 1 & 0 \\
\end{pmatrix}
$$

(4.21)

Adding column three of (4.21) to the column one times $-1$ and then adding row three of the resulting matrix to the row one times $\frac{1}{s + 1}$ gives,

$$
\begin{pmatrix}
1 & 0 & 1 \\
0 & s - 1 & 0 \\
0 & -1 & 1 \\
\end{pmatrix}
$$

(4.22)

which is the required realisation of $P(s)$.

It has been seen that although the system matrix can have decoupling zeros at $s_0$, it must however be proper at $s_0$, since if any of $T(s)$, $U(s)$ or $V(s)$ is non-proper at $s_0$, then we can not multiply or divide any rows (columns) of $P(s)$ by $(s - s_0)$. In other words we can not use elementary operations at $s_0$ to clear the elements which are not proper at $s_0$ and hence the reduction of $P(s)$ stops. However if $T(s)$, $U(s)$ and $V(s)$ are proper at $s_0$ but $W(s)$ is not proper at $s_0$, then the elementary operations at $s_0$ can be performed and the system matrix $P(s)$ can be reduced to the form,

$$
P''(s) = \begin{pmatrix}
I_{n-r} & 0 & 0 \\
0 & sI_r - A & B \\
0 & -C & E(s)
\end{pmatrix}
$$

(4.23)

where $E(s)$ contains some of the poles at $s_0$ of $P(s)$, therefore $E(s)$ is a part of $G(s)$ with poles at $s_0$ but not all poles at $s_0$. Hence if $W(s)$ is not proper at $s_0$, this reduction process under system equivalence at $s_0$ does not provide
a complete realisation of the pole structure of $G(s)$ at $s_0$, but only gives a partial realisation of the pole structure of $G(s)$ at $s_0$. This argument can be seen more clearly by the following example.

**Example 4.24.** Reconsider example (4.10) but with $W(s) = \frac{1}{s - 1}$, then clearly $P(s)$ is not proper at $s_0 = 1$. Since there is no change in $|T(s)|$, hence the order at $s_0$ of $P(s)$ is still one, but $G(s)$ is changed to,

$$G(s) = \frac{s^3 + s^2 - s}{s - 1} + \frac{1}{s - 1} = \frac{s^3 + s^2 - s + 1}{s - 1} = (s^2 + 2s + 1) + \frac{2}{s - 1}$$

using the same elementary operations at $s_0$ with different $W(s)$, $P(s)$ can be brought to the following form,

$$P''(s) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 - s & 1 \\
0 & 0 & 1 & \frac{s^2 + s^2 - s}{s - 1}
\end{pmatrix} \quad (4.25)$$

It is clear then that $\frac{s^3 + s^2 - s}{s - 1}$ is neither $G(s)$ nor its principle part at $s_0$ and hence it will not serve our objective and interest stated by theorem (4.1).

Now based on the idea of system equivalence at $s_0$ a standard form can be obtained for proper at $s_0$ system matrices as the following theorem shows;

**Theorem 4.26.** The transfer function matrix $G(s)$ is a standard form for proper at $s_0$ system matrix $P(s)$ under system equivalence at $s_0$ if and only if the matrix $T(s)$ is biproper at $s_0$.

**Proof.** Consider the proper at $s_0$ system matrix $P(s)$ given by (4.2) and suppose that $T(s)$ is biproper at $s_0$ (i.e. $T^{-1}(s)$ is also biproper at $s_0$). Then we can use the following elementary operations at $s_0$ to reduce $P(s)$ to the required form,
\[ P(s) = \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix} \rightarrow \begin{pmatrix} I & T^{-1}(s)U(s) \\ -V(s) & W(s) \end{pmatrix} \]
\[ \rightarrow \begin{pmatrix} I & T^{-1}(s)U(s) \\ 0 & V(s)T^{-1}(s)U(s) + W(s) \end{pmatrix} = \begin{pmatrix} I & T^{-1}(s)U(s) \\ 0 & G(s) \end{pmatrix} \]

(4.27)

since by definition \( G(s) = V(s)T^{-1}(s)U(s) + W(s) \). (4.27) can be further reduced to the following form,

\[ \begin{pmatrix} I & T^{-1}(s)U(s) \\ 0 & G(s) \end{pmatrix} \rightarrow \begin{pmatrix} I & 0 \\ 0 & G(s) \end{pmatrix} \rightarrow G(s) \]

(4.28)

conversely suppose that \( G(s) \) is a standard form for proper at \( s_0 \) system matrix \( P(s) \). Then the reverse operations are as follows;

\[ G(s) \rightarrow \begin{pmatrix} I & 0 \\ 0 & G(s) \end{pmatrix} \rightarrow \begin{pmatrix} I & 0 \\ 0 & V(s)T^{-1}(s)U(s) + W(s) \end{pmatrix} \]
\[ \rightarrow \begin{pmatrix} I & 0 \\ -V(s) & V(s)T^{-1}(s)U(s) + W(s) \end{pmatrix} \]

(4.29)

The last operation of (4.29) can be performed since \( V(s) \) is proper at \( s_0 \). Then (4.29) can be reduced to the following form, using the fact that \( T^{-1}(s)U(s) \) is proper at \( s_0 \).

\[ \begin{pmatrix} I & 0 \\ -V(s) & V(s)T^{-1}(s)U(s) + W(s) \end{pmatrix} \rightarrow \begin{pmatrix} I & T^{-1}(s)U(s) \\ -V(s) & W(s) \end{pmatrix} \]
\[ \rightarrow \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix} = P(s) \]

(4.30)
The last operation shows that $T(s)$ must be biproper at $s_0$, otherwise $G(s)$ is not a standard form for $P(s)$ and that completes the proof.

It can be seen that the theorem (4.26) is a special case of the theorem (4.1) where apart from the properness at $s_0$ of the system matrix $P(s)$, it also has no decoupling zeros at $s_0$ since $T(s)$ is biproper at $s_0$.

The above theorem has the following corollary which shows the case, when $s_0$ is a decoupling zero.

**Corollary 4.31.** If $s_0$ is a decoupling zero of the proper at $s_0$ system matrix $P(s)$, then $G(s)$ is not a standard form for $P(s)$.

**Proof.** Suppose that the point of investigation $s_0$ is also a decoupling zero of the proper at $s_0$ system matrix $P(s)$. Then $T(s)$ has a zero at $s_0$ and hence by definition, $T(s)$ can not be biproper at $s_0$ and if this is the case, the elementary operations at $s_0$ can not be carried out to reduce the system matrix to the required standard form of $G(s)$.

In the most general case when $P(s)$ is not proper at $s_0$ and may have some decoupling zeros at $s_0$, then a standard form for $P(s)$ can be obtained by system similarity given by the following form (Rosenbrock, 1970).

$$
\begin{align*}
P(s) &= \begin{pmatrix} sI_n - A & B \\ -C & D(s) \end{pmatrix} \\
\rightarrow \begin{pmatrix}
sI_d - A_{11} & -A_{12} & 0 & 0 & 0 \\
0 & sI_{b-d} - A_{22} & 0 & 0 & 0 \\
-A_{31} & -A_{32} & sI_{c-d} - A_{33} & -A_{34} & B_3 \\
0 & -A_{42} & 0 & sI_a - A_{44} & B_4 \\
0 & -C_2 & 0 & -C_4 & D(s) \end{pmatrix}
\end{align*}

(4.32)
where \(a\) is the number of poles of \(G(s)\) at \(s_0\) which are the eigenvalues at \(s_0\) of \(A_{44}\), \(b\) is the number of (i.d.z)\(_{s_0}\) of \(P(s)\) which are the eigenvalues at \(s_0\) of \(A_{11}\) and \(A_{22}\) together, \(C\) is the number of (o.d.z)\(_{s_0}\) of \(P(s)\) when (i.d.z)\(_{s_0}\) are removed and \(d\) is the number of (i.o.d.z)\(_{s_0}\) of \(P(s)\) which are the eigenvalues at \(s_0\) of \(A_{11}\).

The process of the theorem (4.26) for different \(s_0\) can be seen through the following example.

**Example 4.33.** Let

\[
P(s) = \begin{pmatrix} 1 & \frac{1}{s} & 0 & 0 \\ 0 & s & 1 & 1 \\ \frac{1}{s} & 0 & s \\ 1 & s & 0 & \frac{1}{s} \end{pmatrix}
\] (4.34)

and consider the following three cases:

(i) Let point of investigation be \(s_0 = 0\), then clearly \(P(s)\) is not proper at \(s_0\) and therefore we can not use elementary operations at \(s_0\) to reduce \(P(s)\) to the required form of (4.28).

(ii) Let point of investigation be \(s_0 = 1\), then \(P(s)\) is proper at \(s_0\) and elementary operations at \(s_0\) can be used at this stage to reduce \(P(s)\) as follows;

Interchanging columns two and three and then multiplying column three of the resulting matrix by \(s\) gives,

\[
\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & s^2 & 1 \\ s & 0 & 1 & s \\ 1 & 0 & s^2 & \frac{1}{s} \end{pmatrix}
\] (4.35)
Adding $-s$ times row one of (4.35) to the row three and then adding $-1$ times row one of the resulting matrix to the row four gives,

$$
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & s^2 & 1 \\
0 & 0 & 1 - s & \frac{1}{s} \\
0 & 0 & s^2 - 1 & \frac{1}{s}
\end{pmatrix}
$$

Adding $-1$ times column one of (4.36) to the column three and then adding $-s^2$ times column two of the resulting matrix to the column three gives,

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 - s & \frac{1}{s} \\
0 & 0 & s^2 - 1 & \frac{1}{s}
\end{pmatrix}
$$

Finally adding $-1$ times column two of (4.37) to the column four and then adding column three of the resulting matrix to the column four gives,

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 - s & 1 \\
0 & 0 & s^2 - 1 & \frac{1}{s} + s^2 - 1
\end{pmatrix}
$$

It can be seen from (4.38) that there is an output-decoupling zero at $s_0 = 1$ and since the point of interest is also $s_0 = 1$, then the factor $(s - 1)$ can not be eliminated and (4.34) can not be brought to the required form of (4.28).
(iii) Let the point of investigation be \( s_0 = 2 \), then \( P(s) \) is proper at \( s_0 \) and elementary operations at \( s_0 = 2 \) can be used to bring \( P(s) \) to the form (4.28), but now since there are no decoupling zeros at \( s_0 = 2 \) the following elementary operations at \( s_0 \) can be performed.

Multiplying column three of (4.38) by \( \frac{1}{1-s} \) and then adding \(-1\) times column three of the resulting matrix to the column four gives,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -(1 + s) & \frac{s^3 + s^2 + 1}{s}
\end{pmatrix}
\] (4.39)

And then adding \((1 + s)\) times row three of (4.39) to the row four gives,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \frac{s^3 + s^2 + 1}{s}
\end{pmatrix}
\] (4.40)

which is clearly the required standard form.

Section V-5: §Local Results for Systems of Least Order at \( s_0 \)

Another important characteristic of a polynomial system matrix \( P(s) \) is a lower bound on its order at \( s_0 \). This lower bound is called the least order at \( s_0 \) of \( P(s) \).

Now if \( P(s) \) is a rational system matrix with \((i.d.z)_{s_0}\) or \((o.d.z)_{s_0}\), then to find a lower bound on its order at \( s_0 \), these decoupling zeros at \( s_0 \) must be removed by system equivalence. This process of reduction eventually makes
the rational matrices \((T(s), U(s))\) (resp. \((T(s), V(s))\)) relatively prime at \(s_0\).

Before stating the main results of this section, we need the following definition.

**Definition 5.1.** The \((m+n) \times (m+l)\) rational system matrix

\[
P(s) = \begin{pmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{pmatrix}
\]  

where \(T(s), U(s), V(s),\) and \(W(s)\) are proper at \(s_0\) is said to have LEAST ORDER AT \(s_0\) if and only if \((T(s), U(s))\) (resp. \((T(s), V(s))\)) are (r.l.p)\(s_0\) (resp. (r.r.p)\(s_0\)).

An important result concerning least order at \(s_0\) realisations of a given transfer function matrix is;

**Theorem 5.3.** Let \(P_1(s), P_2(s)\) be two \((m+n) \times (m+l)\) rational system matrices which are proper at \(s_0\) and have least order at \(s_0\). Then \(P_1(s), P_2(s)\) give rise to the same transfer function matrix \(G(s)\) if and only if they are system equivalent at \(s_0\).

**Proof.** If \(P_1(s)\) and \(P_2(s)\) are system equivalent at \(s_0\), then by theorem (2.17) they give rise to the same transfer function matrix \(G(s)\). Conversely suppose that \(P_1(s), P_2(s)\) both give rise to the same \(G(s)\). By theorem (4.1) the given system matrices are system equivalent at \(s_0\) to \(P'_1(s), P'_2(s)\) respectively given by,

\[
P'_i(s) = \begin{pmatrix} I & 0 \\ 0 & sI - A_i \\ 0 & -C_i \\ 0 & Q_i(s) \end{pmatrix} B_i \quad i = 1,2
\]  

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where $Q_i(s)$ are proper at $s_0$ and $A_i$ only have eigenvalues at $s_0$.

Since $P_1(s)$, $P_2(s)$ give rise to the same transfer function matrix $G(s)$, then $P_1'(s)$, $P_2'(s)$ also give rise to the same $G(s)$. Thus

$$C_1(sI - A_1)^{-1}B_1 + Q_1(s) = C_2(sI - A_2)^{-1}B_2 + Q_2(s) \quad (5.5)$$

By the properties of $A_i$, $Q_i(s)$ and the uniqueness of the Laurent expansion of $G(s)$ it follows that

$$C_1(sI - A_1)^{-1}B_1 \equiv C_2(sI - A_2)^{-1}B_2 \quad (5.6)$$

and

$$Q_1(s) \equiv Q_2(s) \quad (5.7)$$

Now the system matrices

$$P_i''(s) \triangleq \begin{pmatrix} sI - A_i & B_i \\ -C_i & 0 \end{pmatrix} \quad (5.8)$$

have least order at $s_0$ and hence have least order in the usual sense since $A_i$ only have eigenvalues at $s_0$. Hence by conventional theory $P_1''(s)$ and $P_2''(s)$ are system similar. But system similarity is a special case of system equivalence at $s_0$ and so $P_1''(s)$, $P_2''(s)$ and therefore $P_1'(s)$, $P_2'(s)$ are system equivalent at $s_0$. This establishes the result.

The following result shows as expected that system matrices with least order at $s_0$ describe the structure of the transfer function matrix $G(s)$ at $s_0$ in a completely irredundant manner. We have seen (Section two, Chapter four) that $F_{s_0}(s)$ has more common algebraic properties with the polynomials than
with the rational functions and we have defined it as the ring of proper at \( s_0 \) rational functions. In the following theorem since \( P(s) \) is proper at \( s_0 \) rational system matrix, strictly speaking we are talking about Smith form of \( P(s) \), but since \( P(s) \) is also rational we abuse the term and use Smith-McMillan form instead.

**Theorem 5.9.** Let

\[
P(s) = \begin{pmatrix}
T(s) & U(s) \\
-V(s) & W(s)
\end{pmatrix}
\]  

be an \((m+n) \times (m+l)\) rational system matrix which is proper at \( s_0 \). Suppose \( P(s) \) has least order at \( s_0 \) and gives rise to \( G(s) \), and let the Smith-McMillan form of \( G(s) \) at \( s_0 \) be as in theorem (5.1) of chapter IV. Then the Smith-McMillan form of \( P(s) \) at \( s_0 \) is

\[
M_{s_0}(P) = \begin{pmatrix}
I_m & 0_{m,n} \\
0_{n,m} & \text{diag} (\Phi_i)
\end{pmatrix}, \quad l = n
\]  

where

\[
\Phi_i(s) = \begin{cases}
1 & \text{if } q_i \leq 0, \ i \leq \rho \\
(s - s_0)^{q_i} & \text{if } q_i > 0, \ i \leq \rho \\
0 & \text{if } i > \rho
\end{cases}
\]

(5.12)

(with necessary changes when \( l \neq n \)), while the Smith-McMillan form at \( s_0 \) of \( T(s) \) is

\[
M_{s_0}(T) = \begin{pmatrix}
I_{m-n} & 0_{m-n,n} \\
0_{n,m-n} & \text{diag} (\Psi_{n-i+1})
\end{pmatrix}
\]  

(5.13)

where
$$\Psi_i(s) = \begin{cases} (s - s_0)^{|q_i|}, & \text{if } q_i < 0, \ i \leq \rho \\ 1, & \text{if } q_i \geq 0, \ i > \rho \end{cases} \quad (5.14)$$

**Proof.** Assume that \( l = n \) (The proof for \( l \neq n \) then follows with only minor changes).

Let

$$\Phi(s) \triangleq \text{diag } (\Phi_i(s))$$

$$\Psi(s) \triangleq \text{diag } (\Psi_i(s)) \quad (5.15)$$

Then if \( M_{s_0}(G) \) is the Smith-McMillan form at \( s_0 \) of \( G(s) \) we have

$$M_{s_0}(G) = \Psi^{-1}(s) \Phi(s) \quad (5.16)$$

Hence

$$G(s) = L(s) \Psi^{-1}(s) \Phi(s) R(s) \quad (5.17)$$

for some rational matrices \( L(s), R(s) \) biproper at \( s_0 \). It thus follows that a rational system matrix which is proper at \( s_0 \) and gives rise to \( G(s) \) is

$$P_1(s) = \begin{pmatrix} I_{m-n} & 0 \\ 0 & \Phi(s) \\ 0 & -L(s) \end{pmatrix} \quad (5.18)$$

Now \( L(s), R(s) \) are biproper at \( s_0 \) and \( \Psi(s), \Phi(s) \) are \((r.l.p)_{s_0}\). It thus follows that \( P_1(s) \) of (5.18) has least order at \( s_0 \) and so by theorem (5.3), the given \( P(s) \) is system equivalent at \( s_0 \) to \( P_1(s) \). Hence

$$M_{s_0}(P) = M_{s_0}(P_1) \quad (5.19)$$
and

\[
M_{s_0}(T) = M_{s_0} \begin{pmatrix} I_{m-n} & 0 \\ 0 & \Psi(s) \end{pmatrix}
\]  \hspace{1cm} (5.20)

Since \( L(s) \), \( R(s) \) are biproper at \( s_0 \), it follows from (5.18) that \( M_{s_0}(P_1) \) is the Smith-McMillan form at \( s_0 \) of

\[
\begin{pmatrix} I_{m-n} & 0 & 0 \\ 0 & \Psi(s) & \Phi(s) \\ 0 & -I_n & 0 \end{pmatrix}
\]  \hspace{1cm} (5.21)

Now \( \Psi(s) \) is a polynomial matrix and hence is proper at \( s_0 \), hence by operations of system equivalence at \( s_0 \), (5.21) may be reduced to

\[
\begin{pmatrix} I_{m-n} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & \Phi(s) \end{pmatrix}
\]  \hspace{1cm} (5.22)

which establishes (5.11). On the other hand (5.13) follows immediately from (5.20) and the theorem is proved.

The theorem thus indicates that any proper at \( s_0 \) rational system matrix with least order at \( s_0 \) determines the zero structure at \( s_0 \) of \( G(s) \) via its own zero structure, while it determines the pole structure at \( s_0 \) of \( G(s) \) via the zero structure of \( T(s) \).

Section V-6: §Conclusion and Discussions

Study and behaviour of a linear system at specific finite frequency \( s_0 \) has been the main objective of this chapter, and in this regard a highly localised approach to system theory has been developed.
After giving the important definition of system equivalence at \( s_0 \), the relationship between strict system equivalence and system equivalence of Rosenbrock and system equivalence at \( s_0 \) has been established via the theorem (2.4) of section two. In section three the basic definition of decoupling zeros at \( s_0 \) and some relevant results have been given. Based on these definitions and results an important local state-space realisation for proper at \( s_0 \) system matrices has been presented and necessity of properness at \( s_0 \) of these systems has been discussed and established. Also a standard form for proper at \( s_0 \) system matrices under system equivalent at \( s_0 \) has been provided and several possibilities for different \( s_0 \) have been investigated in the section four.

Finally in the section five systems of least order at \( s_0 \) have been defined and some interesting results concerning these types of systems have been presented.
CHAPTER VI

Aspects of State and Local State Feedback
Section VI-1: §Introduction

In this chapter the invariance of the infinite frequency structure of the transfer function matrix $G(s)$ under state variable feedback is considered. This consideration reveals the necessity of the local system theory developed in the previous chapters.

Based on the local matrix results and local systems theory a known result concerning the invariance of the infinite zero structure of proper systems under state variable feedback is then generalised to the case of non-proper systems. This local approach also reveals a result concerning the invariance of the infinite pole structure of the system under state variable feedback.

Aspects of local state feedback (in the context of the local state-space realisation presented in the previous chapter) are also examined and certain invariants are established.

In this regard after giving some preliminary results in the second section, properties of state feedback are presented in the section three. Section four is concerned with local state feedback, and in section five the general problem of pole assignment is considered and some results are given.

Section VI-2: §Preliminaries and Background

The infinite frequency behaviour of linear multivariable systems has received considerable attention in the literature (for example, Rosenbrock 1970, 1974; Verghese 1978; Pugh and Ratcliffe 1979, 1980; Ferreira 1980;...). The infinite zero structure of a square, strictly proper and invertible transfer function matrix $T(s)$ and its relations to a set of feedback structural invariants originally studied by Morse (1973) and Thorpe (1973) was also examined by Owens (1978) and Vardulakis (1980). However the invariance of the
infinite zero structure under state feedback was first established by Vardulakis (1980), but only in the case of proper transfer function matrices. Ferreira (1982) considered the extension of this result to the non-proper case, but with an added assumption that the transfer function matrix should have full normal rank. In the next section it will be shown how these conditions of properness of Vardulakis and description of full normal rank of Ferreira for the transfer function matrices may be lifted and the chosen method provides a direct application of the local system theory of the previous chapter. Since Vardulakis's result is the basis of our generalisation, the statement of his result is most relevant and given as follows;

Consider a (completely controllable and observable) multivariable system in state-space form,

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}$$

(2.1)

where $A$, $B$, $C$ and $D$ are constant matrices with appropriate dimensions, giving rise to a proper transfer function matrix $T(s)$. Then the effect of state feedback defined by

$$u = Fx + Gv$$

(2.2)

where $F$ and $G$ are constant matrices and $\det G \neq 0$, on the infinite zero structure of the transfer function matrix of a closed-loop system can be seen through the following theorem,

**Theorem 2.3.** The infinite zero structure of the transfer function matrix of the closed-loop system (2.1), (2.2) is invariant under any state variable feedback.
Proof. See Vardulakis (1980).

Having stated the above result concerning the infinite zero structure of a proper transfer function matrix, we are now in a position to present a complete extension of this result to the case of general rational transfer function matrices and see the necessity of the local system theory described previously.

Section VI-3: §Some Properties of State Feedback

Consider the general \((m \times l)\) rational matrix \(G(s)\) and let \(P(s)\) be a conventional state-space matrix realisation of \(G(s)\), thus

\[
P(s) = \begin{pmatrix} sI_n - A & B \\ -C & D(s) \end{pmatrix}
\]  

(3.1)

where \(D(s)\) is an \((m \times l)\) polynomial matrix and \(A, B, C\) are constant matrices of the appropriate dimensions, and

\[
G(s) = C(sI_n - A)^{-1}B + D(s)
\]  

(3.2)

Consider now the state feedback control law given by

\[
u = Fx + H\nu
\]  

(3.3)

where \(F\) is an \((l \times n)\) constant matrix and \(H\) an \((l \times l)\) constant non-singular matrix. Let \(G_F(s)\) denote the closed-loop transfer function matrix corresponding to (3.3) and (3.1), then

\[
G_F(s) = (C + D(s)F)(sI - A - BF)^{-1}BH + D(s)H
\]  

(3.4)
We are interested in determining the effect that the state feedback controller (3.3) has on the infinite frequency structure of \( G(s) \). In this respect, recall that the infinite zero (resp. pole) structure of \( G(s) \) and \( G_F(s) \) are defined as the infinite zero (resp. pole) structure of \( G \left( \frac{1}{w} \right) \) and \( G_F \left( \frac{1}{w} \right) \) at \( w = 0 \) (Verghese, 1978; Pugh and Ratcliffe, 1979). Now

\[
G \left( \frac{1}{w} \right) = C \left( \frac{1}{w} I_n - A \right)^{-1} B + D \left( \frac{1}{w} \right) \tag{3.5}
\]

where \( D \left( \frac{1}{w} \right) \) is a rational matrix in \( w \) which possesses only poles at \( w = 0 \) (since \( D(s) \) is polynomial). Let therefore

\[
D \left( \frac{1}{w} \right) = C_1 (wI - A_1)^{-1} B_1 + D_1 \tag{3.6}
\]

be a conventional state-space realisation of \( D \left( \frac{1}{w} \right) \) having least order at \( w = 0 \). Note that \( D_1 \) is a constant matrix. It then follows from (3.6) and (3.5) that

\[
G \left( \frac{1}{w} \right) = C (I - Aw)^{-1} Bw + C_1 (wI - A_1)^{-1} B_1 + D_1 \tag{3.7}
\]

Notice from (3.7) that \( G \left( \frac{1}{w} \right) \) may be viewed as the parallel connection of the two transfer function matrices \( C (I - Aw)^{-1} Bw \) and \( D \left( \frac{1}{w} \right) \), thus a polynomial system matrix realisation of \( G \left( \frac{1}{w} \right) \) may be written as

\[
P'(w) = \begin{pmatrix} I - Aw & 0 & Bw \\ 0 & wI - A_1 & B_1 \\ -C & -C_1 & D_1 \end{pmatrix} \tag{3.8}
\]

Now since the realisation (3.6) has least order at \( w = 0 \) it is clear that \( P'(w) \) is a realisation of \( G \left( \frac{1}{w} \right) \) with least order at \( w = 0 \). It will not be
necessary to establish that (3.8) has least order in the conventional sense in view of the local theory established previously.

Consider now the infinite frequency structure of the feedback system (3.4) where

\[ G_F \left( \frac{1}{w} \right) = (C + D \left( \frac{1}{w} \right) F)(I - Aw - BFw)^{-1} BHw + D \left( \frac{1}{w} \right) H \]  

(3.9)

Then using (3.6) this becomes

\[
G_F \left( \frac{1}{w} \right) = (C + C_1(wI - A_1)^{-1} B_1 F + D_1 F)(I - Aw - BFw)^{-1} BHw \\
+ (C_1(wI - A_1)^{-1} B_1 + D_1)H \\
= C(I - Aw - BFw)^{-1} BHw + C_1(wI - A_1)^{-1} B_1 F \\
\times (I - Aw - BFw)^{-1} BHw + D_1 F(I - Aw - BFw)^{-1} BHw \\
+ C_1(wI - A_1)^{-1} B_1 H + D_1 H \\
\]  

(3.10)

Therefore

\[
G_F \left( \frac{1}{w} \right) = (C_1(wI - A_1)^{-1} B_1 + D_1)\{F(I - Aw - BFw)^{-1} BHw + H\} \\
+ C(I - Aw - BFw)^{-1} BHw \\
\]  

(3.11)

From (3.11) it can be seen that \( G_F \left( \frac{1}{w} \right) \) may be viewed as the parallel connection of two transfer functions, one being \( C(I - Aw - BFw)^{-1} BHw \), the second being the serial connection of two other transfer function matrices. This observation is an aid in the construction of the following realisation of
\( G_F \left( \frac{1}{w} \right) \). From (3.11), \( G_F \left( \frac{1}{w} \right) \) can be written in the following matrix form.

\[
G_F \left( \frac{1}{w} \right) = (C \ C_1 \ D_1 F)
\]

\[
\begin{pmatrix}
(I-Aw-BF-w)^{-1} & 0 & 0 \\
0 & (wI-A_1)^{-1} & (wI-A_1)^{-1}B_1 F (I-Aw-BFw)^{-1} \\
0 & 0 & (I-Aw-BFw)^{-1}
\end{pmatrix}
\times
\begin{pmatrix}
B H_w \\
B_1 H \\
B H_w
\end{pmatrix}
+ D_1 H
\]

i.e.

\[
G_F \left( \frac{1}{w} \right) = (C \ C_1 \ D_1 F) \begin{pmatrix}
(I-Aw-BFw) & 0 & 0 \\
0 & wI-A_1 & -B_1 F \\
0 & 0 & I-Aw-BFw
\end{pmatrix}^{-1}
\begin{pmatrix}
B H_w \\
B_1 H \\
B H_w
\end{pmatrix}
\]

(3.12)

which has a polynomial system matrix realisation of the form

\[
P_F(w) = \begin{pmatrix}
I - Aw - BFw & 0 & 0 & BH_w \\
0 & wI-A_1 & -B_1 F & B_1 H \\
0 & 0 & I - Aw - BFw & BH_w \\
-C & -C_1 & -D_1 F & D_1 H
\end{pmatrix}
\]

(3.13)
It can be seen that because of the complicated nature of (3.13) it is not possible to establish that $P'_F(w)$ has least order in the usual sense. Recall however that is only the infinite frequency structure of $G_F(s)$ or what is equivalent, the structure of $G_F \left( \frac{1}{w} \right)$ at $w = 0$, that is of interest. In this respect and in view of the local theory of linear systems developed in the previous chapter, it will be sufficient to verify that (3.13) has least order at $w = 0$. In fact this follows quite readily from the form of (3.13) and the least order property of (3.6). It thus follows from theorem (5.9) of chapter V that the pole/zero structure of $G_F \left( \frac{1}{w} \right)$ at $w = 0$ is described precisely by the zero structure at $w = 0$ of the relevant part of $P'_F(w)$. We are now able to establish the following results.

**Theorem 3.14.** The infinite zero structure of $G(s)$ is invariant under state variable feedback.

**Proof.** Since (3.13) has least order at $w = 0$, it follows from theorem (5.9) of chapter V that the infinite zero structure of $G_F(s)$ is the zero structure of $P'_F(w)$ at $w = 0$. Now $P'_F(w)$ may be written as

$$P'_F(w) = L(w)P_E(w)R(w) \quad (3.15)$$

where

$$L(w) = \begin{pmatrix}
I - Aw - BFw & 1 & 0 & 0 \\
0 & 0 & I & 0 \\
0 & I & 0 & 0 \\
-C & 0 & 0 & I
\end{pmatrix} \quad (3.16)$$
\[ R(w) = \begin{pmatrix} I & 0 & -I & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -F & H \end{pmatrix} \] (3.17)

And

\[ P_E(w) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I - Aw & 0 & Bw \\ 0 & 0 & wI - A_1 & B_1 \\ 0 & -C & -C_1 & D_1 \end{pmatrix} \] (3.18)

clearly \( L(w) \) and \( R(w) \) are biproper at \( w = 0 \) and so from (3.15) it can be seen that \( P'_F(w) \) is equivalent at \( w = 0 \) to \( P_E(w) \). However note that \( P_E(w) \) is merely a trivial expansion of \( P'(w) \). Hence \( P_F(w) \) and \( P'(w) \) have the same Smith-McMillan form at \( w = 0 \) modulo a trivial expansion. Now \( P'(w) \) has least order at \( w = 0 \) and gives rise to \( G_F \left( \frac{1}{w} \right) \), hence from theorem (5.9) of chapter V the zero structure of \( P'(w) \) at \( w = 0 \) is the zero structure of \( G_F \left( \frac{1}{w} \right) \) at \( w = 0 \). Thus the zero structures of \( G_F \left( \frac{1}{w} \right) \) and \( G \left( \frac{1}{w} \right) \) at \( w = 0 \) are identical. i.e. the infinite zero structures of \( G_F(s) \) and \( G(s) \) are identical.

\[ \square \]

**Theorem 3.19.** The infinite pole structure of \( G(s) \) is invariant under state variable feedback.

**Proof.** Since \( P'_F(w) \) is a realisation of \( G_F \left( \frac{1}{w} \right) \) with least order at \( w = 0 \), it follows from theorem (5.9) of chapter V and the least order at \( w = 0 \) property of (3.13) that the infinite pole structure of \( G_F(s) \) or, what is the same thing, the pole structure of \( G_F \left( \frac{1}{w} \right) \) at \( w = 0 \) is identical to the zero structure at \( w = 0 \) of

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This can be written as

\[ T_F(w) = L'(w)T_E(w)R'(w) \]  

where

\[ L'(w) = \begin{pmatrix} I - Aw - BFw & 0 & 0 \\ 0 & I & -B_1F \\ 0 & 0 & I - Aw - BFw \end{pmatrix} \]  

\[ R'(w) = \begin{pmatrix} (I - Aw)^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \]  

And

\[ T_E(w) = \begin{pmatrix} I - Aw & 0 & 0 \\ 0 & wI - A_1 & 0 \\ 0 & 0 & I \end{pmatrix} \]

Clearly \( L'(w) \) and \( R'(w) \) are biproper at \( w = 0 \) and so from (3.21), \( T_F(w) \) and \( T_E(w) \) are equivalent at \( w = 0 \). However from (3.24) and (3.8) \( T_E(w) \) is simply a trivial expansion of

\[ T(w) \triangleq \begin{pmatrix} I - Aw & 0 \\ 0 & wI - A_1 \end{pmatrix} \]
Hence $T_F(w)$ and $T(w)$ have the same Smith McMillan form at $w = 0$ which establishes the invariance of the infinite pole structure of $G(s)$.

The invariance of the infinite zero structure under state feedback was first established by Vardulakis (1980), but only in the case of proper transfer function matrices. Ferreira (1982) considered the extension of this result to the non-proper case, however there was an added assumption that the transfer function matrix should have full normal rank. The above result thus represents the complete extension of the Vardulakis result to the case of general rational transfer function matrices. The invariance of the infinite pole structure under state feedback is altogether more surprising, since it is well-known that this structure is not invariant under constant output feedback. This occurrence can be however be explained to some extent by the fact that the state as used in the control scheme (3.3) of the previous section is determined from entirely finite frequency considerations and as such can only be expected to change that part of the system to which it relates.

The process of the above results can be seen through the following example.

Example 3.26. Let $P(s)$ be given by

$$P(s) = \begin{pmatrix} s & 1 & 1 \\ 1 & s & 0 \\ -1 & -1 & s + 1 \end{pmatrix}$$

Then from (3.1) and (3.27) it is clear that

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = (1 \quad 1), \quad D(s) = (s + 1)$$

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Now \( D \left( \frac{1}{w} \right) = \left( \frac{1}{w} + 1 \right) \) which can be realised as in (3.6). i.e.

\[
D \left( \frac{1}{w} \right) = (1)(wI - 0)^{-1}(1) + (1)
\]  

(3.28)

where

\[
A_1 = (0), \quad B_1 = (1), \quad C_1 = (1), \quad D_1 = (1)
\]

Therefore by (3.8) and above;

\[
P'(w) = \begin{pmatrix}
1 & +w & 0 & w \\
+1 & 0 & 0 \\
0 & 0 & w & 1 \\
-1 & -1 & -1 & 1
\end{pmatrix}
\]  

(3.29)

So from (3.27),

\[
G(s) = (1 \quad 1) \begin{pmatrix}
s & 1 \\
1 & s
\end{pmatrix}^{-1} \begin{pmatrix}1 \\ 0\end{pmatrix} + (s + 1) = \frac{1}{s + 1} + (s + 1)
\]

And when the feedback control law given by (3.3) is applied to the above system with \( F = (1 \quad 1) \) and \( H = (1) \), then

\[
G_F(s) = (s + 2 \quad s + 2) \begin{pmatrix}
s - 1 & 0 \\
1 & s
\end{pmatrix}^{-1} \begin{pmatrix}1 \\ 0\end{pmatrix} + (s + 1)
\]

And hence

\[
G_F \left( \frac{1}{w} \right) = \left( \frac{1}{w} + 2 \quad \frac{1}{w} + 2 \right) \begin{pmatrix}
\frac{1}{w} - 1 & 0 \\
1 & \frac{1}{w}
\end{pmatrix}^{-1} \begin{pmatrix}1 \\ 0\end{pmatrix} + \left( \frac{1}{w} + 1 \right)
\]
But \( D \left( \frac{1}{w} \right) = \left( \frac{1}{w} + 1 \right) \) which is given by (3.28), therefore

\[
G_F \left( \frac{1}{w} \right) = \left( \frac{1}{w} + 2 \right) \left( \frac{1}{w} - 1 \right) \begin{pmatrix} \frac{1}{w} & 0 \\ 1 & \frac{1}{w} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1)(wI - 0)^{-1}(1) + (1)
\]

which by (3.13) has a polynomial system matrix given by

\[
P'_F(w) = \begin{pmatrix}
1 - w & 0 & 0 & 0 & 0 & 0 & w \\
+ w & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & w & -1 & -1 & 1 & \\
0 & 0 & 0 & 1 - w & 0 & w & \\
0 & 0 & 0 & + w & 1 & 0 & \\
-1 & -1 & -1 & -1 & -1 & 1 & 1
\end{pmatrix}
\]

Now by (3.15), \( P'_F(w) \) may be written as

\[
P'_F(w) = L(w)P_E(w)R(w)
\]

where

\[
L(w) = \begin{pmatrix}
1 - w & 0 & 1 & 0 & 0 & 0 & 0 \\
+ w & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\]

\[
P_E(w) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & + w & 0 & w & 0 \\
0 & 0 & + w & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & w & 1 & 0 \\
0 & 0 & -1 & -1 & -1 & 1 & 1
\end{pmatrix}
\]
\[ R(w) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \end{pmatrix} \]  

(3.34)

It is clear from (3.32) and (3.34) that \( L(w) \) and \( R(w) \) are biproper at \( w = 0 \) and hence \( P'_E(w) \) and \( P_E(w) \) are equivalent at \( w = 0 \). Also comparing (3.29) and (3.33) it can be seen that \( P_E(w) \) is a trivial expansion of \( P'(w) \) and therefore they have the same Smith McMillan form at \( w = 0 \). Now from (3.29), \( P'(w) \) has least order at \( w = 0 \) and gives rise to \( G \left( \frac{1}{w} \right) \), so by theorem (5.9) of the chapter V the zero structure of \( P'(w) \) at \( w = 0 \) is the zero structure of \( G \left( \frac{1}{w} \right) \) at \( w = 0 \) and hence the result of the theorem (3.14).

Similarly from (3.31) \( T_F(w) \) can be written as

\[ T_F(w) = \begin{pmatrix} 1 - w & 0 & 0 & 0 & 0 \\ +w & 1 & 0 & 0 & 0 \\ 0 & 0 & w & -1 & -1 \\ 0 & 0 & 0 & 1 - w & 0 \\ 0 & 0 & 0 & +w & 1 \end{pmatrix} \]  

(3.35)

Now since \( P'_E(w) \) given by (3.31) is a least order at \( w = 0 \) realisation of \( G_F \left( \frac{1}{w} \right) \), it follows from theorem (5.9) of the chapter V that the infinite pole structure of \( G_F \left( \frac{1}{w} \right) \) at \( w = 0 \) is identical to the zero structure at \( w = 0 \) of \( T_F(w) \) given by (3.35). But \( T_F(w) \) can be written as

\[ T_F(w) = L'(w)T_E(w)R'(w) \]

where
\[ L'(w) = \begin{pmatrix} 1 - w & 0 & 0 & 0 & 0 \\ +w & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 - w & 0 \\ 0 & 0 & 0 & +w & 1 \end{pmatrix} \] (3.36)

\[ T_E(w) = \begin{pmatrix} 1 & +w & 0 & 0 & 0 \\ +w & 1 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \] (3.37)

And

\[ R'(w) = \begin{pmatrix} \frac{1}{1 - w^2} & -w & 0 & 0 & 0 \\ \frac{-w}{1 - w^2} & \frac{1}{1 - w^2} & 0 & 0 & 0 \\ \frac{1}{1 - w^2} & \frac{-w}{1 - w^2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \] (3.38)

It can be seen that \( L'(w) \) and \( R'(w) \) are biproper at \( w = 0 \) and so \( T_F(w) \) and \( T_E(w) \) are equivalent at \( w = 0 \). But from (3.37) and (3.29), \( T_E(w) \) is simply a trivial expansion of \( T(w) \) given by

\[ T(w) = \begin{pmatrix} 1 & +w & 0 \\ +w & 1 & 0 \\ 0 & 0 & w \end{pmatrix} \] (3.39)

Hence \( T_F(w) \) and \( T(w) \) have the same Smith-McMillan form at \( w = 0 \) which gives the result of the theorem (3.19). From the form of \( G(s) \) and \( G_F(s) \) given
in the above example, it can be seen that the \((s + 1)\) part remains unchanged under state feedback and that explains the invariance of the pole structure of the given system.

**Section VI-4: §Local State Feedback**

The above observation leads naturally to the interesting question of what remains invariant if a state feedback controller is implemented on the local state-space realisation (section 4 of the chapter V). In this case the "state" contains only information relating to the frequency \(s_0\) and if the above observation is correct, then only the structure of \(G(s)\) at \(s_0\) should change under this action. Consider therefore the local state-space realisation of \(G(s)\) given by

\[
P(s) = \begin{pmatrix} sI - A & B \\ -C & Q(s) \end{pmatrix}
\]

(4.1)

where \(A\) has eigenvalues only at \(s_0\) while \(Q(s)\) has no poles at \(s_0\). Assume that \(P(s)\) has least order at \(s_0\) so that the pole structure of \(G(s)\) at \(s_0\) is realised in a non-redundant way by (4.1). Consider now the "local" state feedback controller

\[
u = Fx + H\nu
\]

(4.2)

where the term "state" is used in the context of (4.1). Thus this notion of state contains information relating to \(s_0\) alone and no other frequency. Applying (4.2) give rise to the transfer function matrix,

\[
G_F(s) = (C + Q(s)F)(sI - A - BF)^{-1}BH + Q(s)H
\]

(4.3)

i.e.
\[ G_F(s) = C(sI - A - BF)^{-1}BH + Q(s) \left\{ F(sI - A - BF)^{-1}BH + H \right\} \] (4.4)

In order to quantify the effect of the feedback (4.2) on the structure of \( G(s) \) it is necessary to realise \( Q(s) \) in the least order form

\[ Q(s) = C_1(sI - A_1)^{-1}B_1 + D_1(s) \] (4.5)

where \( D_1(s) \) is polynomial, and \( A_1 \) has no eigenvalues at \( s_0 \). The reason for using this form is that it brings the remaining finite frequency pole structure of \( G(s) \) into view. What is hidden by (4.5) and the subsequent analysis is the infinite pole structure, (as described by \( D_1(s) \)) which is known to be invariant from the previous section. Now we can state and prove the following theorem.

**Theorem 4.6.** The local state feedback (4.2) can only move those poles of the system which are located at \( s_0 \) (but may not move them all) and leaves all other poles unchanged.

**Proof.** Substituting (4.5) into (4.4) gives

\[ G_F(s) = C(sI - A - BF)^{-1}BH + \{ C_1(sI - A_1)^{-1}B_1 + D_1(s) \} \]

\[ \{ F(sI - A - BF)^{-1}BH + H \} \]

which in matrix form can be written as
\[ G_F(s) = (C \quad C_1 \quad D_1(s)F) \times \]
\[
\begin{pmatrix}
(sI - A - BF)^{-1} & 0 & 0 \\
0 & (sI - A_1)^{-1} & (sI - A_1)^{-1}B_1F(sI - A - BF)^{-1} \\
0 & 0 & (sI - A - BF)^{-1}
\end{pmatrix}
\times
\begin{pmatrix}
BH \\
B_1H \\
BH
\end{pmatrix}
+ D_1(s)H
\]
\[ (4.7) \]

Hence a polynomial system matrix realisation (of usual kind) of \( G_F(s) \) is,

\[
P_F(s) = \begin{pmatrix}
(sI - A - BF) & 0 & 0 & BH \\
0 & sI - A_1 & -B_1F & B_1H \\
0 & 0 & sI - A - BF & BH \\
-C & -C_1 & -D_1(s)F & D_1(s)H
\end{pmatrix}
\]
\[ (4.8) \]

This clearly does not have least order, so the input decoupling zeros of (4.8) can be removed by system equivalence in the following way;

Subtracting row three from row one of (4.8) gives

\[
\begin{pmatrix}
(sI - A - BF) & 0 & -(sI - A - BF) & 0 \\
0 & sI - A_1 & -B_1F & B_1H \\
0 & 0 & sI - A - BF & BH \\
-C & -C_1 & -D_1(s)F & D_1(s)H
\end{pmatrix}
\]
\[ (4.8a) \]

Then adding column one to column three of (4.8a) yields

\[
\begin{pmatrix}
(sI - A - BF) & 0 & 0 & 0 \\
0 & sI - A_1 & -B_1F & B_1H \\
0 & 0 & sI - A - BF & BH \\
-C & -C_1 & -C - D_1(s)F & D_1(s)H
\end{pmatrix}
\]
\[ (4.8b) \]
Finally multiplying row one of (4.8b) by \((sI - A - BF)^{-1}\) gives

\[
P'_F(s) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & sI - A_1 & -B_1F & B_1H \\
0 & 0 & sI - A - BF & BH \\
-C & -C_1 & -C - D_1(s)F & D_1(s)H
\end{pmatrix}
\] (4.9)

Substituting (4.5) into (4.1) yields the following state-space realisation of the open loop \(G(s)\)

\[
P'(s) = \begin{pmatrix}
sI - A & 0 & B \\
0 & sI - A_1 & B_1 \\
-C & -C_1 & D_1(s)
\end{pmatrix}
\] (4.10)

Now \(P'_F(s)\) has no input-decoupling zeros but may possess output-decoupling zeros (which is a well-known feature of state feedback). In fact this only occurs if the feedback is such as to produce zeros in \((sI - A - BF)\) which coincide with those of \((sI - A_1)\). In any case (4.9) represents a polynomial realisation (having no input-decoupling zeros) of \(G_F(s)\) which has the same (conventional) order as the least order realisation (4.10) of \(G(s)\). Thus the zeros of

\[
T'_F(s) = \begin{pmatrix}
1 & 0 & 0 \\
0 & sI - A_1 & -B_1F \\
0 & 0 & sI - A - BF
\end{pmatrix}
\] (4.11)

may be taken as the closed-loop locations of the open-loop poles under the local state feedback (4.2). It is immediately clear from (4.11) that the zeros of \((sI - A_1)\) (representing the finite pole structure of \(G(s)\) away from \(s_0\)) will always be present as zeros of (4.11), i.e. as poles of \(G_F(s)\). In this respect it can be seen therefore that the local feedback (4.2) does not move any of the poles of \(G(s)\) which are not located at \(s_0\). The true effect of such feedback is
thus to move only those poles of \( G(s) \) which are located at \( s_0 \) and hence the theorem (4.6).

In fact the local state is determined entirely from information about the system gathered only at the frequency \( s_0 \), the result represents a restatement of the classical result of pole assignment (Porter and Crossley, 1972). That is, in order to change a single pole (of a system with distinct poles) and have all other poles invariant, then a feedback based on the left eigenvector corresponding to this eigenvalue of the plant matrix should be used.

The following example shows the movement of those poles of the system which are located at \( s_0 \), while the other poles remain unchanged.

**Example 4.13.** Let \( s_0 = 0 \) and the system matrix \( P(s) \) be given by

\[
P(s) = \begin{pmatrix}
s & 1 \\
0 & s & 1 \\
-1 & 0 & \frac{1}{s-1}
\end{pmatrix}
\]  

(4.14)

Then

\[
G(s) = (1 \ 0) \begin{pmatrix}
s & 1 \\
0 & s
\end{pmatrix}^{-1} \begin{pmatrix}
0 \\
1
\end{pmatrix} + \frac{1}{s-1} = \frac{s^2 - s + 1}{s^2(s-1)}
\]

Now \( Q(s) = \frac{1}{s-1} \) which has no poles at \( s_0 = 0 \), while the system has a pole of order two at \( s_0 = 0 \) and a pole of order one at \( s_0 = 1 \). Consider now the effect of the local feedback \( F = (f_1 \ f_2) \) on the pole structure of the above system. Application of \( F \) to the \( P(s) \) in (4.14) produces the following system matrix \( P_F(s) \) given by
\[ P_F(s) = \begin{pmatrix} s & 1 & 0 \\ -f_1 & s - f_2 & 1 \\ -1 - \frac{f_1}{s - 1} & -f_2 & 1 \\ \end{pmatrix} \] \hspace{1cm} (4.15)

The new transfer function matrix corresponding to \( P_F(s) \) is then

\[ G_F(s) = \]

\[ (1 + \frac{f_1}{s - 1} \cdot \frac{f_2}{s - 1}) \begin{pmatrix} s & 1 \\ -f_1 & s - f_2 \\ \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ \end{pmatrix} + \frac{1}{s - 1} = \]

\[ (1 + \frac{f_1}{s - 1} \cdot \frac{f_2}{s - 1}) \begin{pmatrix} s - f_2 \\ \frac{s - f_2}{s(s - f_2) + f_1} \\ \frac{-1}{s(s - f_2) + f_1} \\ \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \end{pmatrix} + \frac{1}{s - 1} = \]

\[ \begin{pmatrix} s^2 - sf_2 - s + f_2 + sf_1 \\ \frac{s^2 - sf_2 - s + f_2 + sf_1}{(s - 1)(s - f_2) + f_1} \\ \frac{-s + 1 - f_1 + sf_2}{(s - 1)(s - f_2) + f_1} \\ \end{pmatrix} + \frac{1}{s - 1} = \]

\[ \frac{sf_2 - s + 1 - f_1}{(s - 1)(s - f_2) + f_1} + \frac{1}{s - 1} = \frac{s^2 - s + 1}{(s - 1)(s^2 - sf_2 + f_1)} \]

It is clear from \( G_F(s) \) that application of the local feedback \( F \) moved the poles of the system \( P(s) \) located at \( s_0 = 0 \) except for particular values of \( F \). For example if \( f_1 = 0 \), then application of the local feedback \( F = (0 \quad f_2) \) will move one pole located at \( s_0 = 0 \) and leaves the other poles unchanged, a fact which has been established by the theorem (4.6).

Section VI-5: §The General Problem of Pole Assignment

Having established the above result concerning the effect of the local feedback on the poles structure of the system matrix, we now consider the general pole assignment problem under local state feedback. Conventionally
the problem of pole assignment was considered (Rosenbrock, 1970) through the monic polynomials $\Psi_i(s)$ in the McMillan form of the closed-loop transfer function matrix satisfying certain necessary conditions. In this section a local version of the above pole assignment problem where the invariant polynomials are polynomial in $(s - s_0)$ will be considered and some necessary and sufficient conditions relating to these invariant polynomials will also be established.

Consider now a general $(m \times l)$ rational transfer function matrix $G(s)$ and in the context of the theorem (4.1) of the chapter V, a least order at $s_0$ local state-space realisation given by

$$P(s) = \begin{pmatrix} sI - A & B \\ -C & Q(s) \end{pmatrix}$$

(5.1)

where $G(s)$ is given as

$$G(s) = G_{s_0}(s) + Q(s)$$

(5.2)

Now let $Q(s)$ be given as in (4.5) and consider the local state feedback controller (4.2). Then the closed-loop transfer function matrix resulting from the application of (4.2) to $G(s)$ is given by (4.4), and a polynomial system matrix realisation of the closed-loop transfer function matrix $G_F(s)$ is given by (4.8).

By means of system equivalence at $s_0$ and unimodular equivalence (involving elementary row operations and removing the decoupling zeros which are not at $s_0$), (4.8) can be reduced to the following form

$$P_C(s) = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & sI_{n_1} - A_1 & -B_1F & B_1H \\ 0 & 0 & sI_n - A - BF & BH \\ 0 & -C_1 & -C - D_1(s)F & D_1(s)H \end{pmatrix}$$

(5.3)
Clearly the pole structure of the closed-loop system is the same as the pole structure of $T_C(s)$, where $T_C(s)$ is given by,

$$T_C(s) = \begin{pmatrix} sI_{n_1} - A_1 & -B_1 F \\ 0 & sI_n - A - BF \end{pmatrix} \quad (5.4)$$

Now by means of unimodular equivalence $T_C(s)$ can be brought to the form

$$\begin{pmatrix} \phi'_1 & \cdots & \phi'_{n_1} & E(s) \\ & & & \\ O & & \phi_1 & \cdots & \phi_n \end{pmatrix} \quad (5.5)$$

where $\phi'_i(s)$ are the invariant polynomials of $(sI_{n_1} - A_1)$, $i = 1, \ldots, n_1$, $\phi_i(s)$ are the invariant polynomials of $(sI_n - A - BF)$, $i = 1, \ldots, n$ and $E(s)$ is a polynomial matrix. Let also $\phi''_i(s)$ be the invariant polynomials of $T_C(s)$, $i = 1, \ldots, n_1 + n$, then we have;

**Theorem 5.6.** Let $\phi''_i(s)$, $i = 1, \ldots, n_1 + n$ denote the invariant polynomials of $T_C(s)$ and $n \leq n_1$. If $(sI_{n_1} - A_1, -B_1 F)$ and $(sI_n - A - BF)$ have no zeros in common then,

$$\phi''_i(s) = \begin{cases} 1, & i = 1, \ldots, n \\ \phi_{i-n}(s), & i = n + 1, \ldots, n_1 \end{cases} \quad (5.7)$$

with necessary changes when $n_1 < n$.

**Proof.** Suppose that $(sI_{n_1} - A_1, -B_1 F)$ and $(sI_n - A - BF)$ have no zeros in common. Let $d''_i(s)$ be the (g.c.d) of all $(i \times i)$ minors of $T_C(s)$ ($i \leq n \leq n_1$), $d'_i(s)$ be the (g.c.d) of all $(i \times i)$ minors of $(sI_{n_1} - A_1, -B_1 F)$ and $d_i(s)$ be the (g.c.d) of all $(i \times i)$ minors of $(sI_n - A - BF)$.
Now if \( d_i''(s) \neq 1 \), then it is a polynomial and every \((i \times i)\) minor of \( T_C(s)\) is divisible by \( d_i''(s) \). Since \( i \leq n \leq n_1 \), the set of \((i \times i)\) minors of \( T_C(s)\) contain all \((i \times i)\) minors of \((sI_n - A - BF)\) and \((sI_{n_1} - A_1, -B_1 F)\) with \((\text{g.c.d})\) of \( d_i(s) \) and \( d_i'(s) \) respectively. Hence \( d_i''(s) \) divides both \( d_i(s) \) and \( d_i'(s) \). But this can not be true since \((sI_{n_1} - A_1, -B_1 F)\) and \((sI_n - A - BF)\) have no zeros in common. Therefore \( d_i''(s) = 1 \), for all \( i = 1, \ldots, n \) and hence \( \phi_i''(s) = 1, \ i = 1, \ldots, n \).

Consider now the \((n + i) \times (n + i)\) minors of \( T_C(s)\), \( i = 1, \ldots, n_1 - n \). Every such minor must include at least \( i \) rows and \( i \) columns from the first \( n_1 \) rows and first \( n_1 \) columns of (5.5). It thus follows that

\[
\phi_1' \ldots \phi_i' / d_{n+i}''(s) = \phi_1' \ldots \phi_i' / d_{n+i}'(s) \tag{5.8}
\]

Therefore

\[
d_{n+i}''(s) = \phi_1' \ldots \phi_i' \alpha_i(s) \tag{5.9}
\]

for some polynomial \( \alpha_i(s) \) and \( i = 1, \ldots, n_1 - n \).

Assuming that \( \deg \alpha_i(s) \geq 1 \), then

\[
d_{n+i}''(s) / d_{n+i}'(s) \Rightarrow \alpha_i(s) / d_{n+i}'(s) \tag{5.10}
\]

Also \( \phi_1' \ldots \phi_i' \phi_1 \ldots \phi_n \) is one \((n + i) \times (n + i)\) minor of (5.5), so it follows from (5.9) that

\[
\phi_1' \ldots \phi_i' \alpha_i(s) / \phi_1' \ldots \phi_i' \phi_1 \ldots \phi_n. \tag{5.11}
\]

i.e.
\[ \alpha_i(s)/\phi_1 \ldots \phi_n. \] 
\[ (5.12) \]

Therefore by the divisibility property of \( \phi_i(i = 1, \ldots, n) \) it is clear that

\[ \alpha_i(s)/\phi_n \] 
\[ (5.13) \]

Hence if \( \deg \alpha_i(s) \geq 1 \), then it follows from (5.10) and (5.13) that \( \phi_n \) and \( d'_{n+i}(s) \) have a factor in common which is a contradiction to the assumption, hence \( \deg \alpha_i(s) = 0 \) and therefore \( \alpha_i(s) \) is constant (unit), and so

\[ d''_{n+i}(s) = \phi'_1 \ldots \phi'_i \] 
\[ (5.14) \]
i.e.

\[ \phi''_{n+i}(s) = \phi'_i(s) \quad i = 1, \ldots, n_1 - n \] 
\[ (5.15) \]

It can be seen from the above theorem that no general statement concerning \( \phi''_{n_1+1}, \ldots, \phi''_{n_1+n} \) is possible. So to obtain a complete result concerning the invariant polynomials of \( T_C(s), (sI_n - A - BF) \) and \( (sI_{n_1} - A_1) \) it remains to consider the remaining terms of the invariant polynomials of \( T_C(s) \). How they relate to the other invariant polynomials may be seen through the following theorem.

**Theorem 5.16.** Let \( \phi''_i(s), i = 1, \ldots, n_1 + n \) denote the invariant polynomials of \( T_C(s) \) and \( n_1 \geq n \). If \( (sI_{n_1} - A_1) \) and \( (sI_n - A - BF) \) have no zeros in common, then
\[
\phi_i''(s) = \begin{cases} 
1, & i = 1, \ldots, n \\
\phi_{i-n}(s), & i = n + 1, \ldots, n_1 \\
\phi_{i-n_1}(s)\phi_{i-n}(s), & i = n_1 + 1, \ldots, n_1 + n
\end{cases}
\] (5.17)

Again with necessary changes when \( n_1 < n \).

**Proof.** The case of \( \phi_i''(s) = 1, i = 1, \ldots, n \) and \( \phi_i''(s) = \phi_{i-n}(s), i = n + 1, \ldots, n_1 \) has been proven in the previous theorem. Now in the case of \( i = n_1 + 1, \ldots, n_1 + n \), suppose that \((sI_{n_1} - A_1)\) and \((sI_n - A - BF)\) have no zeros in common which means that \( \phi_{n_1}(s) \) and \( \phi_n(s) \) are relatively prime and consider the \((n_1 + i) \times (n_1 + i)\) minors of (5.5) for \( i = 1, \ldots, n \). These contain at least \( i \) rows and \( i \) columns from the last \( n \) rows and columns of (5.5), therefore

\[
\phi_1 \ldots \phi_i / d_{n_1+i}''(s)
\] (5.18)

On the other hand such minors of (5.5) contain at least \( n_1 - n + i \) rows and columns from the first \( n_1 \) rows and columns and so

\[
\phi_1' \ldots \phi_{n_1-n+i}' / d_{n_1+i}''(s)
\] (5.19)

Therefore from (5.18) and (5.19) we have

\[
\phi_1 \ldots \phi_i.\phi'_1 \ldots \phi'_{n_1-n+i} / d_{n_1+i}''(s)
\] (5.20)

Hence

\[
d_{n_1+i}''(s) = \phi_1 \ldots \phi_i.\phi'_1 \ldots \phi'_{n_1-n+i} \cdot c_i(s)
\] (5.21)
For some polynomial \( \alpha_i(s) \).

Now \( \phi_1 \ldots \phi_n \phi'_1 \ldots \phi'_{n-1} \) is one such \((n_1 + i) \times (n_1 + i)\) minor of (5.5) and therefore from above,

\[
\alpha_i(s)/\phi_n \quad (5.22)
\]

Similarly \( \phi_1 \ldots \phi_i \phi'_1 \ldots \phi'_{n_1} \) is one such minor, therefore

\[
\alpha_i(s)/\phi'_{n_1} \quad (5.23)
\]

But \( \phi_n \) and \( \phi'_{n_1} \) are relatively prime. Hence (5.22) and (5.23) imply that

\[
\alpha_i(s) = 1 \quad (5.24)
\]

and therefore from (5.23)

\[
d''_{n_1+i}(s) = \phi_1 \ldots \phi_i \phi'_1 \ldots \phi'_{n_1-n+i} \quad (5.25)
\]

Hence

\[
\phi''_{n_1+i}(s) = \phi_i(s)\phi'_{n_1-n+i}(s) \quad (5.26)
\]

for \( i = 1, \ldots, n \).

It has been shown that the pole structure of the closed-loop system given by the (5.3) is the same as the pole structure of \( T_C(s) \) given by the (5.4). Then by considering the invariant polynomials of \( T_C(s) \), \((sI_{n_1} - A_1)\) and \((sI_n - A - BF)\) a local version of the general pole assignment has been studied and established that under the condition of relative primeness of \( \phi_n \)

and \( d''_{n_1}(s) \), \( \phi''_i(s) = 1, \ i = 1, \ldots, n \) and \( \phi''_i(s) = \phi'_i(s), \ i = n + 1, \ldots, n_1 \).
But under the stronger condition of relative primeness of $\phi_n(s)$, $\phi'_{n_1}(s)$ a relationship between the remaining terms of the invariant polynomials of $T_C(s)$, $(sI_n - A - BF)$ and $(sI_{n_1} - A_1)$ has been derived. Therefore under the conditions stated in the above theorems a complete description of the invariant polynomials of $T_C(s)$ and hence the pole structure of the closed-loop system is possible.

The above results can be illustrated by the following examples.

Example 5.27. Consider the following state-space realisation where $s_0 = 1$ and

\[
P(s) = \begin{pmatrix}
    s & 1 & 0 \\
    0 & s & 1 \\
    -1 & 0 & \frac{1+s}{s}
\end{pmatrix}
\]

(5.28)

Then from (5.28) we have,

\[
A = \begin{pmatrix}
    0 & -1 \\
    0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
    0 \\
    1
\end{pmatrix}, \quad C = (1 \ 0)
\]

Now using the procedure of reduction of (4.8) to (5.3), with feedback $F = (1 \ 1)$ and $H = (1)$, (5.28) can be brought to the form

\[
P_C(s) = \begin{pmatrix}
    1 & 0 & 0 & 0 & 0 \\
    0 & s & -1 & -1 & 1 \\
    0 & 0 & s & 1 & 0 \\
    0 & 0 & -1 & s - 1 & 1 \\
    0 & -1 & -2 & -1 & 1
\end{pmatrix}
\]

(5.29)

where

\[
A_1 = (0), \quad B_1 = (1), \quad C_1 = (1) \text{ and } D_1 = (1)
\]

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Then

\[
T_C(s) = \begin{pmatrix}
  s & -1 & -1 \\
  0 & s & 1 \\
  0 & -1 & s - 1
\end{pmatrix}
\]  \hspace{1cm} (5.30)

Now \( T_C(s) \) can be reduced to the form (5.5) given by

\[
\begin{pmatrix}
  s & -1 & 1 - s \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]  \hspace{1cm} (5.31)

which has the Smith form given by

\[
\begin{pmatrix}
  s & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]  \hspace{1cm} (5.32)

From (5.31), \( \phi_1' = s, \ \phi_1 = 1, \ \phi_2 = 1 \). On the other hand from (5.32) the invariant polynomials of \( T_C(s) \) are \( \phi_1'' = 1, \ \phi_2'' = 1 \) and \( \phi_3'' = s \). Therefore since in this example \( n_1 < n \), then the only change we need to do is that of changing the role of \( n \) and \( n_1 \), \( \phi \) and \( \phi' \). Hence by theorem (5.6) we have \( \phi_1'' = 1 \) and \( \phi_2'' = \phi_1 = 1 \) a fact given by (5.31) and (5.32).

Example 5.33. Let \( s_0 = 0 \) and \( P(s) \) be given as

\[
P(s) = \begin{pmatrix}
  s & 0 & 0 \\
  0 & s & 2 \\
  -1 & 0 & \frac{1}{s - 1}
\end{pmatrix}
\]  \hspace{1cm} (5.34)

Then from (5.34),

\[
A = \begin{pmatrix}
  0 & 0 \\
  0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
  0 \\
  2
\end{pmatrix}, \quad C = (1 \quad 0)
\]

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Again by using the procedure of reduction of (4.8) to (5.3) with $F = (1 \quad 1)$ and $H = (1)$, $P(s)$ can be brought to the form

$$P_C(s) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & s-1 & -1 & -1 & 1 \\
0 & 0 & s & 0 & 0 \\
0 & 0 & -2 & s-2 & 2 \\
0 & -1 & -1 & 0 & 0
\end{pmatrix}$$

(5.35)

where

$$A_1 = (1), \quad B_1 = (1), \quad C_1 = (1), \quad D_1 = (0)$$

Then

$$T_C(s) = \begin{pmatrix}
s-1 & -1 & -1 \\
0 & s & 0 \\
0 & -2 & s-2
\end{pmatrix}$$

(5.36)

which can be reduced to the form given by (5.5), i.e.

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & s
\end{pmatrix}$$

(5.37)

Now from (5.37), $\phi'_1 = 1$, $\phi_1 = 1$, $\phi_2 = s$. Also $\phi''_1 = 1$, $\phi''_2 = 1$ and $\phi'_3 = s$. Therefore $\phi''_1 = 1$, $\phi''_2 = \phi_1 = 1$ and $\phi''_3 = \phi'_1 \phi_2 = 1 \times s = s$, a fact proven in the theorem (5.16).

Section IV-6: §Conclusions and Discussion

The effect of the state variable feedback on a general system and the invariance of the infinite zero and pole structure of such a system has been considered in detail and some important results have been derived. In
The effect of the state variable feedback on a general system and the invariance of the infinite zero and pole structure of such a system has been considered in detail and some important results have been derived. In this regard after stating the Vardulakis theorem which has established the invariance of the infinite zero structure of a proper transfer function matrix under any state variable feedback in section two, a generalisation of this result (for a general transfer function matrix) has been considered and not only the invariance of the infinite zero structure of such transfer function matrix has been established, but also it has been shown in the section three and four (by a direct application of the local system theory of the previous chapter) that the infinite pole structure of such general transfer function matrix remains unchanged. This can to some extent be explained by the fact that the state is determined from entirely finite frequency consideration and that the feeding back of such information can only be expected to affect that part of the system structure to which it relates. In other words if consideration is given to feedback of the slow and fast trajectories $x_s$ and $x_f$ separately and only the slow part of the trajectory $x_s$ is involved the resulting fast subsystem is essentially unchanged (Cobb, 1981). Therefore the infinite pole structure of the system under state variable feedback is unaffected when the feedback which is used, involves only the slow dynamics. This naturally led to consider the effect of the local state feedback on a general transfer function matrix in section five. Finally also in this section the idea of general pole assignment problem has been examined and some interesting results given.
CHAPTER VII

Conclusions
Conclusions

This thesis has been concentrated on the study of a general rational transfer function matrix and the properness or non-properness of the closed-loop system on applying constant output feedback or under state variable feedback. The definition of proper or non-proper system in conventional sense refers to there being no poles at infinity, which led naturally to look at the behaviour of the system at a specific finite point $s_0$ in the complex plane.

To study the effect of the state variable feedback on the pole/zero structure at $s_0$ of a general transfer function matrix, it has been required to have a local matrix theory and the development of a local system theory and in particular a local (rational) state-space realisation. This rational state-space realisation has been in turn the basis for the idea of the local state feedback and investigations into the problem of the general pole assignment through the invariant polynomials.

In considering the implementation of constant output feedback for a general rational transfer function matrix and its effect on the closed-loop transfer function matrix properness or non-properness some progress has been made in the chapter three and a sufficient condition for properness of a general composite system has been established that, despite its obvious limitations, gives surprisingly good results when applied to two specific composite systems as described in the chapter three.

In the chapter four substantial progress in the development of a local linear system theory has been made. In this regard the local study of polynomial and rational matrices which play an essential role in the development of the local theory of linear systems has been the main part of the fourth chapter. Also in this chapter, the local theory of valuation
for rational functions has been extended to rational matrices and the notion of equivalence at $s_0$ has been presented. One interesting outcome of the local valuation of the rational matrices was an alternative way of finding the Smith-McMillan form at $s_0$ of a rational matrix.

The important definition of system equivalence at $s_0$ and its relationship to strict system equivalence and system equivalence of Rosenbrock were given in the fifth chapter. Based on this definition and definition of decoupling zeros at $s_0$ a new and interesting local (rational) state-space realisation for system matrices has been proposed. Also in this chapter as in the conventional case, systems of least order at $s_0$ and some relevant results concerning these type of systems were presented.

Starting from the proposed local (rational) state-space realisation of the chapter five, the invariance of the infinite frequency structure of the transfer function matrix $G(s)$ under state variable feedback has been established in the chapter six. Consideration of this problem not only revealed the necessity of the local system theory, but also revealed a result concerning the invariance of the infinite pole structure of the system under state variable feedback. That linear state feedback should leave invariant the infinite pole structure of the open-loop system is not immediately obvious. However this phenomenon can to some extent be explained by the fact that the state is determined from entirely finite frequency consideration and that the feeding back of such information can only reasonably be expected to alter that part of the system structure to which it relates. In such circumstances the infinite frequency structure could well be imagined to be immune.

Finally, in the context of the local state-space realisation, the general problem of pole assignment has been discussed and a local version of the
conventional problem of pole assignment which was considered (Rosenbrock, 1970) through the monic polynomials $\Psi_i(s)$ in the McMillan form of the closed-loop transfer function matrix has been considered and some necessary and sufficient conditions relating to these invariant polynomials have been established.
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PUBLISHED PAPERS
Simple condition for composite-system properness

A. C. PUGH† and A. KAFAI†

A simple sufficient condition for a general composite system to be proper is derived. Although apparently highly conservative, it produces surprisingly good results with two commonly considered composite systems.

1. Introduction

For a general rational transfer-function matrix $G(s)$ it has long been known that the implementation of constant output feedback does not necessarily result in an overall closed-loop transfer function matrix which is proper. In the case of a strictly proper $G(s)$, Rosenbrock and Pugh (1974) have shown that the closed-loop system will always be strictly proper (whenever it is defined), while for a general $G(s)$ Scott and Anderson (1976) have shown the closed-loop system to be generically proper. More recently, Pugh and Ratcliffe (1981) and Pugh (1984) have provided necessary and sufficient conditions for closed-loop properness in the case of a general open-loop $G(s)$ that is given either in terms of a state-space realization or as a matrix-fraction description.

In this paper the case of a general composite system is considered. To some extent this represents a generalization of the closed-loop situation just described, as is apparent from the composite-system study of Rosenbrock and Pugh (1974). In this case a simple sufficient condition for composite-system properness is derived that, despite its obvious limitations, gives surprisingly good results when applied to two specific composite systems.

2. Simple condition for composite-system properness

Consider $N$ systems $S_1, ..., S_N$, each of which admits the polynomial system-matrix representation of Rosenbrock (1970). Following Rosenbrock and Pugh (1974), these $N$ system matrices may be written in the more concise form

$$ P_S(s) = \begin{bmatrix} T_s(s) & U_s(s) \\ -V_s(s) & W_s(s) \end{bmatrix} $$

(1)

where $T_s(s)$, $U_s(s)$, $V_s(s)$, $W_s(s)$ are block-diagonal matrices, the non-zero diagonal entry in the $i$th block row of any of these matrices being the corresponding matrix from the system matrix for $S_i$. $P_S(s)$ will be referred to as the subsystem matrix.

The most general form of a composite system $S_C$ is then obtained by interconnecting the $S_i$ ($i = 1, 2, ..., N$) according to the interconnection relations

$$ u_s = -F_{ys} + K_{uc} $$

$$ y_c = L_{ys} $$

(2)

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where $F$, $K$, $L$ are constant block matrices with partitioning consistent with that in $u_s$, $y_s$ given by

$$u_s = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad y_s = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

(3)

where $u_i$ and $u_c$ (respectively, $y_i$ and $y_c$) are the inputs (respectively, outputs) of $S_i$ and $S_c$.

A polynomial system-matrix representation for the composite system $S_c$ is then

$$P_c(s) = \begin{bmatrix} T_c(s) & U_c(s) \\ -V_c(s) & W_c(s) \end{bmatrix} = \begin{bmatrix} T_s(s) & U_s(s) & 0 & 0 \\ -V_s(s) & W_s(s) & I & 0 \\ 0 & -I & F & K \\ 0 & 0 & -L & 0 \end{bmatrix}$$

(4)

where $I$ denotes a unit matrix of appropriate dimension and where it is assumed that

$$\begin{bmatrix} T_s(s) & U_s(s) & 0 \\ -V_s(s) & W_s(s) & I \end{bmatrix} \neq 0$$

(5)

The system matrix $P_c(s)$ of (4) thus yields a natural description of the composite system directly in terms of the subsystems and the interconnection structure. The input–output behaviour of $S_c$ may be described in similar terms. Specifically, if $G_s(s)$ denotes the transfer-function matrix corresponding to $P_s(s)$, and $G_i(s)$, $G_c(s)$ those corresponding to $S_i$ and $S_c$ respectively, then

$$G_s(s) = V_s T_s^{-1} U_s + W_s = \text{diag}(G_1(s), \ldots, G_N(s))$$

(6)

and it may be shown (Rosenbrock and Pugh 1974) that

$$G_c(s) = L G(s) [I + FG_s(s)]^{-1} K = L [I + G_s(s)F]^{-1} G_s(s)K$$

(7)

It is clear from these relationships that any composite system may be viewed merely as a closed-loop system constructed from the open-loop system $G_s(s)$ under constant output feedback as described by the matrix $F$, together with closed-loop system pre- and post-compensation as described by $K$ and $L$ respectively. The composite system may thus be viewed in the manner of Fig. 1.

As a result of these observations, a simple sufficient condition for a composite system to be proper may be given as follows. Let the closed-loop system transfer-function matrix formed by applying the feedback $F$ to the open-loop transfer-function

![Figure 1.](image-url)
Composite-system properness

matrix $G_S(s)$ be denoted by $G_F(s)$ then

$$G_F(s) = G_S(s) [I + FG_S(s)]^{-1} = [I + G_S(s)F]^{-1} G_S(s)$$

and so

$$G_C(s) = LG_F(s)K$$

(7)

(8)

Theorem 1

A sufficient condition for the composite-system transfer-function matrix $G_C(s)$ to be proper is that its associated closed-loop transfer-function matrix $G_F(s)$ be proper.

Proof

Since in (8) the matrices $L, K$ are constant, it follows that the elements of $G_C(s)$ are simply linear combinations with constant coefficients of the elements of $G_F(s)$. If these latter elements are proper rational functions then such linear combinations can only result in expressions that are proper rational functions. Thus the properness of $G_F(s)$ ensures the properness of $G_C(s)$.

It thus follows from the above result that, to deduce the properness of any composite system, it will be sufficient to determine the properness of its associated closed-loop system. To test the closed-loop system for properness, there are a number of necessary and sufficient conditions one may use (Pugh 1984), depending on the description of the (open-loop) transfer-function matrix $G_S(s)$ that is available. For example, if $G_S(s)$ is given as

$$G_S(s) = G_{sp}(s) + D(s)$$

(9)

where $D(s)$ is polynomial and

$$\lim_{s \to \infty} G_{sp}(s) = 0$$

(10)

which is essentially the case when state-space descriptions of the subsystems are known, then the following holds.

Lemma 1

The closed-loop transfer-function matrix $G_F(s)$ is proper if and only if

$$\delta(I + FD(s)) = \delta(D(s))$$

(11)

where $\delta(\cdot)$ denotes the McMillan degree (Rosenbrock 1970) of the indicated matrix.

Proof

This is originally due to Pugh and Ratcliffe (1981), but for a complete proof see Pugh (1984).

In view of Theorem 1, it thus follows that a rough estimate of the properness of the composite system when the subsystem transfer-function matrix is given in the form (9) may be obtained by testing the associated closed-loop system in the manner of Lemma 1. In the next section we consider the results obtained by applying this procedure to two configurations that commonly occur in the design of feedback control systems.
3. Application to specific composite systems

It is apparent that using the associated closed-loop system to assess the properness of the original composite system is likely to give rise to highly conservative estimates. Nevertheless, we shall perform this exercise in two well-used examples.

Case 1. Two subsystems in feedback connection

Consider two subsystems with transfer-function matrices $G_1(s), G_2(s)$ connected in feedback as in Fig. 2. In the notation of the previous section, the interconnection relations (2) become

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = - \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} u_c$$

$$y_c = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Figure 2.

If $D_1(s), D_2(s)$ denote the polynomial parts of $G_1(s), G_2(s)$ respectively then, from (9),

$$D(s) = \begin{bmatrix} D_1(s) & 0 \\ 0 & D_2(s) \end{bmatrix}$$

(13)

Now from (2) and (12),

$$F = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

(14)

and so

$$|I + FD(s)| = \begin{vmatrix} I & D_2(s) \\ -D_1(s) & 1 \end{vmatrix} = |I + D_1(s)D_2(s)|$$

(15)

In view of (15), Theorem 1 together with Lemma 1 immediately gives the following.

Theorem 2

The feedback system of Fig. 2 is proper if

$$\delta(|I + D_1(s)D_2(s)|) = \delta(D(s))$$

(16)
Corollary 1

If the subsystem transfer-function matrices \( G_1(s), G_2(s) \) are proper then a sufficient condition for the feedback system of Fig. 2 to be proper is that

\[
|I + D_1D_2| \neq 0
\]  

(17)

Proof

If \( G_1(s), G_2(s) \) are proper then \( D_1, D_2 \) are constants. Thus

\[
\delta(D) = \begin{cases} 
-\infty & \text{if } D_1 = D_2 = 0 \\
0 & \text{otherwise}
\end{cases}
\]  

(18)

Now if \( D_1 = D_2 = 0 \) then (17) is trivially satisfied. In this case \( G_1(s), G_2(s) \) are strictly proper and it is known (Rosenbrock and Pugh 1974) that the feedback composite system is always strictly proper. On the other hand, if one of \( D_1, D_2 \) is non-zero then (16) reduces to

\[
\delta(|I + D_1D_2|) = 0
\]  

(19)

which is precisely the requirement (17).

Case 2. Feedback implementation of open-loop compensators

Consider the configuration of Fig. 3, which is basic in the feedback implementation of open-loop compensations (Wolovich 1974, Krishnarao and Chen 1984). In the figure \( G(s), C_0(s), C_1(s) \) are respectively \( m \times 1, 1 \times 1, 1 \times m \) proper rational matrices and \( P(s) \) is an \( I \times I \) polynomial matrix whose determinant is not the trivial polynomial. To apply the above results, label the component systems as follows:

\[
G_1(s) \triangleq P(s)^{-1}, \quad G_2(s) \triangleq G(s), \quad G_3(s) \triangleq C_1(s), \quad G_4(s) \triangleq C_0(s)
\]  

(20)

The interconnection relations (2) thus become

\[
\begin{bmatrix}
0 & 0 & I & I \\
-I & 0 & 0 & 0 \\
0 & -I & 0 & 0 \\
-I & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
y_s \\
0 \\
0 \\
0
\end{bmatrix}
+
\begin{bmatrix}
I \\
0 \\
0 \\
0
\end{bmatrix}u_c
= \begin{bmatrix}
y_c \\
0 \\
0 \\
0
\end{bmatrix}
\]  

(21)

Let \( D_i(s) \) denote the polynomial part of \( G_i(s) \) (i = 1, ..., 4); then \( D_1(s) \) may be
polynomial while \( D_2(s), D_3(s), D_4(s) \) are at most constant. Now from (2) and (21) it follows that

\[
|I + FD(s)| = \begin{vmatrix}
I & 0 & D_3 & D_4 \\
-D_1(s) & I & 0 & 0 \\
0 & -D_2 & I & 0 \\
-D_1(s) & 0 & 0 & I \\
\end{vmatrix}
\]  

(22)

By elementary row and column operations the determinant (22) may be reduced to the successive forms

\[
\begin{vmatrix}
I & 0 & D_3 & D_4 \\
0 & I & 0 & 0 \\
-D_2D_1(s) & 0 & I & 0 \\
-D_1(s) & 0 & 0 & I \\
\end{vmatrix} \rightarrow \begin{vmatrix}
I + D_3D_2D_1(s) & 0 & 0 & D_4 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
-D_1(s) & 0 & 0 & I \\
\end{vmatrix} \rightarrow \begin{vmatrix}
I + D_3D_2D_1(s) + D_4D_1(s) & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
\end{vmatrix}
\]

Thus

\[
\delta(I + FD(s)) = \delta(I + (D_4 + D_3D_2)D_1(s))
\]  

(23)

while it is clear from the diagonal form of \( D(s) \) and the fact that \( D_2, D_3, D_4 \) are constant that

\[
\delta(D(s)) = \delta(D_1(s))
\]  

(24)

**Theorem 3**

The composite system of Fig. 3 is proper if

\[
\delta(I + (D_4 + D_3D_2)D_1(s)) = \delta(D_1(s))
\]  

(25)

**Proof**

This follows directly from (23) and (24) on using Theorem 1 and Lemma 1.

Two interesting special cases of this result are as follows.

**Corollary 1**

If \( G_1(s) = P(s)^{-1} \) is proper so that \( D_1(s) \equiv D_1 \) (constant) then the composite system of Fig. 3 is proper if

\[
|I + (D_4 + D_3D_2)D_1| \neq 0
\]  

(26)

**Proof**

This follows in a similar way as Corollary 1 of Theorem 2.
Corollary 2

If $G_1(s) = P(s)^{-1}$ is strictly proper (so that $D_1 = 0$) then the composite system of Fig. 3 is always proper.

The condition (25) of Theorem 3 has an alternative characterization. For consider the closed-loop system $G_F(s)$ obtained by employing constant output feedback as described by the matrix $F \triangleq D_4 + D_3 D_2$ around the open-loop system $G_1(s) = P(s)^{-1}$. Thus from Lemma 1 the condition (25) is the exact requirement that $G_F(s)$ be proper. Now

$$G_F(s) = G_1(s)(I + FG_1(s))^{-1} = P(s)^{-1}(I + (D_4 + D_3 D_2)P^{-1}(s))^{-1} = (P(s) + D_4 + D_3 D_2)^{-1}$$

Consequently we have an alternative but equivalent statement of Theorem 3 as follows.

Theorem 4

The composite system of Fig. 3 is proper if

$$(P(s) + D_4 + D_3 D_2)^{-1}$$

is proper.

It should be noted that Theorem 4 is precisely the sufficient condition derived by Krishnarao and Chen (1984).

4. Discussion and conclusions

In Theorem 1 a sufficient condition for the properness of a general composite system $G(s)$ has been given. It arises from the simple observation that such a composite system will always be proper if its associated closed-loop system $G_F(s)$ is proper. In fact it is also true that $G(s)$ can only be non-proper when $G_F(s)$ has this property so that a necessary condition for composite-system non-properness could also have been given.

From the manner in which Theorem 1 has been obtained, it is obvious that the sufficient condition it yields is likely to be highly conservative. However, it must be said that the condition (25) is extremely simple to derive and to employ, and the nature of its limitations are clear and obvious. Thus for example when Theorem 1 is applied to the case of the composite system of Fig. 3, the extent of the conservatism of the sufficient condition of Krishnarao and Chen (1984) (and described in Theorem 4) is clearly revealed.

In applying Theorem 1 to two specific composite systems (Cases 1 and 2) certain interesting results have been obtained. Specifically, in the open-loop compensation scheme of Fig. 3, Theorem 1 in conjunction with Lemma 1 gives rise to a sufficient condition that is equivalent to that previously obtained by Krishnarao and Chen (1984). Additionally, it is noted that for the simple feedback scheme of Case 1, the condition (17) produced by Theorem 1 in the case of proper subsystems is actually known to be necessary and sufficient (Desoer and Vidyasagar 1975). Thus, despite the apparent limitations of Theorem 1, it produces in Case 2 a sufficient condition as good as any previously obtained, while in Case 1 the resulting condition cannot be better, since it is both necessary and sufficient.
Composite-system properness

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Some local results for linear systems and implications for state variable feedback

A. C. PUGH†, A. KAFAI† and G. E. HAYTON‡

A result concerning the invariance of the infinite-zero structure of a general transfer function matrix under state feedback is obtained. The approach indicates the need for a localized theory of linear systems and some basic results in this area are derived. A surprising property of the infinite-pole structure is also revealed.

1. Introduction

The conventional study of linear multivariable systems is based on the transformation of equivalence by unimodular matrix multiplication (Rosenbrock 1970, Kailath 1980). This transformation preserves the structure at all frequencies in the finite complex plane $C$ of any matrix to which it is applied. If, however, the infinite frequency behaviour of a multivariable linear system is of interest then the requirements are in a sense reversed, and it is now one specific frequency ($s = \infty$) that is of concern to the exclusion of all others (Vardulakis et al. 1982).

This paper takes up on this latter point of view and develops a highly localized approach to systems theory, by considering the behaviour only at a specific frequency. The basic results of this theory are obtained in § 2 and § 3, and the need for such an approach is displayed in § 4, where a result originally due to Vardulakis (1980), concerning the invariance of the infinite-zero structure of a proper transfer function matrix under state feedback, is extended to non-proper rational matrices. The approach also reveals an unexpected property of the infinite-pole structure.

2. Preliminaries

The central ideas of this section are originally due to Van Dooren et al. (1979) although the brief development given here is more in the spirit of Vardulakis et al. (1982). Pernebo (1978, 1981 a, b) has adopted a rather more general approach as is indicated in the sequel.

Let $F[s]$ denote the ring of polynomials, and $F(s)$ the field of rational functions in the indeterminate $s$ with coefficients taken from the field $F$ (usually assumed to be $R$ or $C$).

Definition 1

The rational function $g(s) \in F(s)$ is said to be proper at $s_0$ if $\lim_{s \rightarrow s_0} g(s)$ exists. Additionally $g(s)$ is said to be strictly proper at $s_0$ if this limit is zero, and hyper at $s_0$ if it is not.

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It is clear from this definition, therefore, that $g(s)$ is proper at $s_0$ if and only if it is a $\Lambda$-generalized polynomial (Pernebo 1978, 1981a, b) with $\Lambda = \{s_0\}$. Thus, $g(s)$ is proper at $s_0$ if and only if it has no poles at $s_0$, while it is biproper at $s_0$ in case it has no poles and no zeros at $s_0$. An obvious result that has been noted by Pernebo in the case of $\Lambda$-generalized polynomials is the following.

**Lemma 1**

If $g(s) \in F(s)$ is non-zero then it may be uniquely factorized as

$$g(s) = (s - s_0)^k g^*(s)$$

where $g^*(s)$ is biproper at $s_0$ and $k$ is an integer.

**Definition 2**

If $g(s) \in F(s)$ is factorized as in (1) then $k$ is called the degree of $g(s)$ at $s_0$.

It is clear from the above that a given $g(s) \in F(s)$ will be proper (or biproper) at $s_0$ in case its degree at $s_0$ is non-negative (or zero). We extend the above terminology to rational matrices as follows.

**Definition 3**

$G(s) \in F^{m \times l}(s)$ is said to be proper at $s_0$ if and only if $\lim_{s \to s_0} G(s)$ exists. Additionally, $G(s)$ is said to be strictly proper at $s_0$ if this matrix is zero, and biproper at $s_0$ if it is non-singular.

It turns out that matrices that are biproper at $s_0$ play a fundamental role in the study of matrices whose structure at a specified frequency is of interest. We define the following elementary row and column operations:

1. interchange any two rows (or columns);
2. multiply any row (or column) by a non-zero rational function that is biproper at $s_0$;
3. add to any row (or column) a multiple by any rational function, proper at $s_0$, of any other row (or column).

The effect that each of these operations has on any given matrix may be represented by pre- (post-)multiplication of the given matrix by an appropriate rational matrix, biproper at $s_0$. Thus, we say the following.

**Definition 4**

Two $m \times l$ rational matrices $G_1(s)$, $G_2(s)$ are said to be equivalent at $s_0$ if and only if $G_2(s)$ can be obtained from $G_1(s)$ by a sequence of the operations (EO) (i)–(iii). Equivalently $G_1(s)$, $G_2(s)$ are equivalent at $s_0$ if there exist rational matrices $M(s)$, $N(s)$, both biproper at $s_0$, such that

$$G_2(s) = M(s)G_1(s)N(s)$$

It is noted that elementary operations (EO) (i)–(iii) are of fairly general nature and include as a special case the usual elementary row and column operations of
equivalence for polynomial matrices. In these terms the following result, originally noted by Verghese (1978) (and hence not proved here), is not surprising.

**Theorem 1**

Any \( m \times 1 \) rational matrix \( G(s) \) of rank \( p \) is equivalent at \( s_0 \) to a matrix \( M_{s_0}(G) \), i.e.

\[
M_{s_0}(G) = M(s)G(s)N(s)
\]

for some rational matrices \( M(s) \), \( N(s) \) biproper at \( s_0 \), where

\[
M_{s_0}(G) = \begin{cases} 
(Q(s), Q_{m-l-m}) & m < l \\
Q(s) & m = l \\
[Q(s)]^T & m > l
\end{cases}
\]

and

\[
Q(s) = \text{diag } [(s - s_0)^{q_1}, ..., (s - s_0)^{q_l}, 0, ..., 0]
\]

with

\[
q_1 \leq ... \leq q_k \leq 0 \leq q_{k+1} \leq ... \leq q_p
\]

\( M_{s_0}(G) \) is called the Smith–McMillan form of \( G(s) \) at \( s_0 \).

It is clear from the above result that the operations (ER) (i)–(iii) preserve the pole/zero structure of \( G(s) \) at the frequency \( s_0 \) to the exclusion of all other frequencies (including the point at infinity), since those \( q_i \) in (6) for which \( q_i < 0 \) determine the pole structure of \( G(s) \) while those \( q_i \) for which \( q_i > 0 \) determine the zero structure.

3. Local results for system matrices

The specialization of the previous theory to the case of system matrices yields local versions of some well-known results (Rosenbrock 1970). The basis of these results is the notion of equivalence at \( s_0 \), suitably adapted to the systems theory situation.

**Definition 5**

Let

\[
P_i(s) = \begin{bmatrix} T_i(s) & U_i(s) \\
- \Gamma_i(s) & W_i(s) \end{bmatrix}
\]

be two \( (r + m)^+ (r + l) \) rational system matrices. \( P_1(s) \) and \( P_2(s) \) are said to be system equivalent at \( s_0 \) if there exist rational matrices \( M(s), N(s) \) biproper at \( s_0 \), and rational matrices \( X(s), Y(s) \) that are proper at \( s_0 \), such that

\[
\begin{bmatrix} M(s) & 0 \\
- \Gamma(s) & I_m \end{bmatrix} \begin{bmatrix} T_1(s) & U_1(s) \\
- \Gamma_1(s) & W_1(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\
0 & I_l \end{bmatrix} = \begin{bmatrix} T_2(s) & U_2(s) \\
- \Gamma_2(s) & W_2(s) \end{bmatrix}
\]

Alternatively, the above definition could have been phrased in terms of a subset of the elementary operations (EO) (i)–(iii) of the previous section. These operations arise from (EO) (i)–(iii) in much the same way as those of strict system equivalence.
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arise from the elementary operations of unimodular equivalence. This indicates the relationship between system equivalence at $s_0$ and other notions of equivalence for linear systems.

Theorem 2

(i) System equivalence at $s_0$ includes strict system equivalence (Rosenbrock 1970, p. 52) as a special case.

(ii) System equivalence at $s_0$ is a special case of system equivalence (Rosenbrock 1970, p. 58).

(iii) If $P(s)$ is a polynomial (rather than rational) system matrix then any system matrix obtained from $P(s)$ by the removal of decoupling zeros is system equivalent at $s_0$ to $P(s)$, provided these decoupling zeros are not located at $s_0$.

Proof

For (i) note that the definition of strict system equivalence takes the form (8) with $M(s)$, $N(s)$ unimodular and $X(s)$, $Y(s)$ polynomial. However, the unimodularity of $M(s)$, $N(s)$ implies that they possess no poles or zeros at finite frequencies and specifically at $s_0$. Thus $M(s)$, $N(s)$ are biproper at $s_0$. Additionally $X(s)$ and $Y(s)$ have no poles at $s_0$ (since they are polynomial) and so are proper at $s_0$, and (i) is proved.

In the case of (ii) note that the first three of the four operations of system equivalence (Rosenbrock 1970, p. 58) give rise to a matrix statement of the form (8) in which $M(s)$, $N(s)$, $X(s)$, $Y(s)$ are general rational matrices with $M(s)$, $N(s)$ invertible. This clearly contains Definition 5 as a special case.

For part (iii) note that in the case of input decoupling zeros, the order reduction procedure (Rosenbrock 1970, p. 60) may be written in the form (8) in which $N(s) = I$, $X(s) = 0$, $Y(s) = 0$ and $M(s)$ is an invertible rational matrix having a single pole at $s = \beta$, where $\beta$ is the input decoupling zero to be removed. Thus, provided $\beta \neq s_0$, $M(s)$ will be biproper at $s_0$ and (iii) follows in the case of input decoupling zeros. The case of output decoupling zeros may be established in an analogous way which completes the proof of (iii) and hence the theorem.

The above results give some indication of the essential invariants of the transformation of Definition 5. We first require the following terminology.

Definition 6

If

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}$$

the degree at $s_0$ of the rational function $|T(s)|$ is called the order at $s_0$ of $P(s)$.

Theorem 3

Two $(r + m) \times (r + l)$ rational system matrices that are system equivalent at $s_0$, have the same order at $s_0$ and give rise to the same transfer function matrix.

/
Proof

If $P_1(s), P_2(s)$ are two rational system matrices related by system equivalence at $s_0$ then (8) holds, and in particular we have

$$M(s)T_1(s)N(s) = T_2(s)$$

(10)

Hence

$$|T_2(s)| = |M(s)||T_1(s)||N(s)|$$

(11)

Since $M(s), N(s)$ are biproper at $s_0$, the rational functions $|M(s)|, |N(s)|$ have degree zero at $s_0$ and so are biproper at $s_0$. Thus, writing $|T_1(s)|$ in the form (1), it follows from (11) that the degrees at $s_0$ of $|T_2(s)|$ and $|T_1(s)|$ are identical. Thus $P_1(s), P_2(s)$ have the same order at $s_0$.

The invariance of the associated transfer function follows from the fact that it is invariant under system equivalence and hence (by (ii) of Theorem 2) under system equivalent at $s_0$.

The above definitions and results hold for general rational system matrices but the main context in which they will be applied will be that of rational system matrices that are proper at $s_0$. A fundamental result to much of what follows concerns the existence and structure of a local state-space realization of such a system matrix. This realization is not entirely polynomial as the statement indicates.

**Theorem 4**

Let

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ V(s) & W(s) \end{bmatrix}$$

(12)

be a $(p+m) \times (p+n)$ rational system matrix that is proper at $s_0$ and of order $n$ at $s_0$ $(p \geq n)$. If the associated transfer function matrix is written as

$$G(s) = G_{\infty}(s) + Q(s)$$

(13)

where $G_{\infty}(s)$ is the principal part of the Laurent expansion of $G(s)$ about $s_0$, then $P(s)$ is system equivalent at $s_0$ to a rational system matrix of the form

$$P'(s) = \begin{bmatrix} I_{p-m} & 0 \\ sI_n - A & B \\ 0 & -C \end{bmatrix}$$

(14)

where $A, B, C$ are constant matrices and $A$ has eigenvalues only at $s_0$.

**Proof**

Without loss of generality, $P(s)$ may be taken to be such that $T(s), U(s), V(s)$ are polynomial. This may be achieved by operations of system equivalence at $s_0$, by multiplying each row in the first block row of $P(s)$ by its least common denominator and operating similarly on the columns from the first block column. This procedure does not change $W(s)$ which is therefore still proper at $s_0$. With $P(s)$ in this form it may now be brought by strict system equivalence to the extended state-space form.
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(Rosenbrock 1970)

\[
P_1(s) = \begin{bmatrix}
I_{p-n}, & 0 & 0 & 0 \\
0 & sI_{n-n} - A_1 & 0 & B_1 \\
0 & 0 & sI_n - A_2 & B_2 \\
0 & -C_1 & -C_2 & D(s)
\end{bmatrix}
\]  \hspace{1cm} (15)

where \( \text{diag}(A_1, A_2) \) is in, say, first natural normal form with \( A_1 \) having no eigenvalues at \( s_0 \), while \( A_2 \) has only eigenvalues at \( s_0 \). Clearly then \( sI_{n-n} - A_1 \) is biproper at \( s_0 \), and \( D(s) \) is proper at \( s_0 \) and so by operations of system equivalence at \( s_0 \), \( P_1(s) \) may be reduced to the successive forms

\[
\begin{bmatrix}
I_{p-n}, & 0 & 0 & 0 \\
0 & I_{n-n} & 0 & (sI_{n-n} - A_1)^{-1} B_1 \\
0 & 0 & sI_n - A_2 & B_2 \\
0 & -C_1 & -C_2 & D(s)
\end{bmatrix} \rightarrow \begin{bmatrix}
I_{p-n}, & 0 & 0 \\
0 & sI_n - A_2 & B_2 \\
0 & -C_2 & R(s)
\end{bmatrix}
\]  \hspace{1cm} (16)

where

\[
R(s) \triangleq D(s) + C_1 (sI_{n-n} - A_1)^{-1} B_1
\]  \hspace{1cm} (17)

is proper at \( s_0 \).

Now by Theorem 3, \( P(s) \) and (16) give rise to the same transfer function matrix and so

\[
G(s) = C_2 (sI - A_2)^{-1} B_2 + R(s)
\]  \hspace{1cm} (18)

Since the first matrix on the right-hand side of (18) only has poles at \( s_0 \) and \( R(s) \) is proper at \( s_0 \), it follows from the uniqueness of the Laurent expansion that \( C_2 (sI - A_2)^{-1} B_2 \) is the principal part of this expansion and that from (13) and (17)

\[
R(s) = Q(s)
\]

which establishes the theorem.

In conventional linear systems theory an important property of a system matrix is the least order property. The local version of this is defined as follows.

**Definition 7**

Two rational matrices \( T(s), U(s) \) of respective dimensions \( r \times r \) and \( r \times l \) and which are proper at \( s_0 \) are said to be *relatively left prime at \( s_0 \)* in case

\[
\text{rank } [T(s_0), U(s_0)] = r
\]  \hspace{1cm} (19)

Relative right primeness at \( s_0 \) is defined similarly.

There are alternative characterizations of the condition (19), for example that the Smith–McMillan form at \( s_0 \) of \([T(s), U(s)]\) be \([I_r, O_r,l]\). The condition (19), however, will suffice for our present requirements.
Definition 8

The \((r + m) \times (r + l)\) rational system matrix

\[ P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \]

in which \(T(s), U(s), V(s), W(s)\) are proper at \(s_0\) is said to have least order at \(s_0\) if and only if \(T(s), U(s)\) (or \(T(s), -V(s)\)) are relatively left (or right) prime at \(s_0\).

An important result concerning least order at \(s_0\) realizations of a given transfer function matrix is the following.

Theorem 5

Let \(P_1(s), P_2(s)\) be two \((r + m) \times (r + l)\) rational system matrices that are proper at \(s_0\) and have least order at \(s_0\). Then \(P_1(s), P_2(s)\) give rise to the same transfer function matrix \(G(s)\) if and only if they are system equivalent at \(s_0\).

Proof

If \(P_1(s)\) and \(P_2(s)\) are system equivalent at \(s_0\) then by Theorem 3 they give rise to the same transfer function matrix \(G(s)\).

Conversely suppose that \(P_1(s), P_2(s)\) both give rise to the same \(G(s)\). By Theorem 4 the given system matrices are system equivalent at \(s_0\) to \(P_1'(s), P_2'(s)\), respectively, where

\[ P_i'(s) = \begin{bmatrix} I & 0 & 0 \\ 0 & sl - A_i & B_i \\ 0 & -C_i & Q_i(s) \end{bmatrix} \] (20)

where \(Q_i(s)\) is proper at \(s_0\) and \(A_i\) only has eigenvalues at \(s_0\).

Since \(P_1(s), P_2(s)\) give rise to the same transfer function matrix \(G(s)\), then \(P_1(s), P_2(s)\) also give rise to \(G(s)\). Thus

\[ C_1(sl - A_1)^{-1}B_1 + Q_1(s) = C_2(sl - A_2)^{-1}B_2 + Q_2(s) \]

By the properties of \(A_i, Q_i(s)\) \((i = 1, 2)\) and the uniqueness of the Laurent expansion of \(G(s)\) it follows that

\[ C_1(sl - A_1)^{-1}B_1 = C_2(sl - A_2)B_2 \]

or

\[ Q_1(s) = Q_2(s) \] (21) (22)

Now the system matrices

\[ P_i'(s) = \begin{bmatrix} sl - A_i & B_i \\ -C_i & 0 \end{bmatrix} \] (23)

have least order at \(s_0\) and hence have least order in the usual sense since \(A_i\) \((i = 1, 2)\) only have eigenvalues at \(s_0\). Hence, by conventional theory \(P_1'(s), P_2'(s)\) are system similar. However, system similarity is a special case of system equivalence at \(s_0\), and so \(P_1'(s), P_2'(s)\), and therefore \(P_1'(s), P_2'(s)\) are system equivalent at \(s_0\). This establishes the result.

\[ \Box \]
The following result shows as expected that system matrices with least order at \( s_0 \) describe the structure of the transfer function matrix \( G(s) \) at \( s_0 \) in a completely irredundant manner.

**Theorem 6**

Let

\[
P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix}
\]

be a \((r + m) \times (r + l)\) rational system matrix that is proper at \( s_0 \). Suppose \( P(s) \) has least order at \( s_0 \) and gives rise to \( G(s) \), and let the Smith–McMillan form of \( G(s) \) at \( s_0 \) be as in Theorem 1. Then the Smith–McMillan form of \( P(s) \) at \( s_0 \) is

\[
M_{s_0}(P) = \begin{bmatrix} I_r & O_{r,m} \\ O_{m,r} & \text{diag}(\phi_i) \end{bmatrix}, \quad l = m
\]

where

\[
\phi_i(s) = \begin{cases} 
1 & \text{if } q_i \leq 0, \quad i \leq \rho \\
(s - s_0)^{q_i} & \text{if } q_i > 0, \quad i \leq \rho \\
0 & \text{if } i > \rho
\end{cases}
\]

(with necessary changes when \( l \neq m \)), while the Smith–McMillan form at \( s_0 \) of \( T(s) \) is

\[
M_{s_0}(T) = \begin{bmatrix} I_{r-m} & O_{r-m,m} \\ O_{m,r-m} & \text{diag}(\psi_i) \end{bmatrix}
\]

where

\[
\psi_i(s) = \begin{cases} 
(s - s_0)^{q_i} & \text{if } q_i < 0, \quad i \leq \rho \\
1 & \text{if } q_i \geq 0, \quad \text{or } i > \rho
\end{cases}
\]

**Proof**

Assume that \( l \neq m \) (the proof for \( l \neq m \) then follows with only minor changes).

\[
\Phi(s) \triangleq \text{diag}[\phi_i(s)]
\]

\[
\Psi(s) \triangleq \text{diag}[\psi_i(s)]
\]

then if \( M_{s_0}(G) \) is the Smith–McMillan form at \( s_0 \) of \( G(s) \)

\[
M_{s_0}(G) = \Psi^{-1}(s)\Phi(s)
\]

Hence

\[
G(s) = L(s)\Psi^{-1}(s)\Phi(s)R(s)
\]

for some rational matrices \( L(s), R(s) \) biproper at \( s_0 \). It thus follows that a rational system
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matrix which is proper at \( s_0 \) and gives rise to \( G(s) \) is

\[
P_1(s) = \begin{bmatrix} I_{r-m} & 0 & 0 \\ 0 & \Psi(s) & \Phi(s) R(s) \\ 0 & -L(s) & 0 \end{bmatrix}
\]  

(31)

Now \( L(s), R(s) \) are biproper at \( s_0 \) and \( \Psi(s), \Phi(s) \) are relatively left prime at \( s_0 \). It thus follows that \( P_1(s) \) of (31) has least order at \( s_0 \), and so by Theorem 5, the given \( P(s) \) is system equivalent at \( s_0 \) to \( P_1(s) \). Hence

\[
M_{s_0}(P) = M_{s_0}(P_1)
\]

(32)

and

\[
M_{s_0}(T) = M_{s_0}\left( \begin{bmatrix} I_{r-m} & 0 \\ 0 & \Psi(s) \end{bmatrix} \right)
\]

(33)

Since \( L(s), R(s) \) are biproper at \( s_0 \), it follows from (31) that \( M_{s_0}(P_1) \) is the Smith–McMillan form at \( s_0 \) of

\[
\begin{bmatrix} I_{r-m} & 0 & 0 \\ 0 & \Psi(s) & \Phi(s) \\ 0 & -I_m & 0 \end{bmatrix}
\]

(34)

Now \( \Psi(s) \) is a polynomial matrix and hence is proper at \( s_0 \), hence by operations of system equivalence at \( s_0 \), (34) may be reduced to

\[
\begin{bmatrix} I_{r-m} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & \Phi(s) \end{bmatrix}
\]

which establishes (24). On the other hand (26) follows immediately from (33) and the theorem is proved.

The theorem thus indicates that any proper at \( s_0 \) rational system matrix with least order at \( s_0 \) determines the zero structure at \( s_0 \) of \( G(s) \) via its own zero structure, while it determines the pole structure at \( s_0 \) of \( G(s) \) via the zero structure of \( T(s) \).

4. **Infinite frequency structure and state variable feedback**

The invariance of the infinite-zero structure of a proper transfer function matrix \( G(s) \) under the application of linear state variable feedback has been established by Vardulakis (1980). Ferreira (1982) has considered the extension of this result to the case of non-proper \( G(s) \) but under the added assumption that \( G(s) \) has full row or column rank. It will now be shown how these latter assumptions may be lifted and the chosen method provides a direct application of the previous theory.

Consider therefore the general \( m \times l \) rational matrix \( G(s) \) and the least order state-space realization given by

\[
P(s) = \begin{bmatrix} sI_m - A | B \\ -C | D(s) \end{bmatrix}
\]

(35)
where $D(s)$ is an $m \times 1$ polynomial matrix, and $A, B, C$ are constant matrices of the appropriate dimensions. Then

$$G(s) = C(sI - A)^{-1}B + D(s)$$

(36)

Consider now the state feedback control law

$$u = Fx + Hv$$

(37)

where $F$ is an $l \times n$ constant matrix and $H$ is an $l \times l$ constant non-singular matrix. If $G_F(s)$ denotes the closed-loop transfer function matrix resulting from the application of (37) to $G(s)$ then

$$G_F(s) = (C + D(s)F)(sI - A - BF)^{-1}BH + D(s)H$$

(38)

Recall that the infinite-zero structure of $G(s)$ and $G_F(s)$ is given by the finite-zero structure of $G(1/w)$ and $G_F(1/w)$, respectively, at $w = 0$ (Verghese 1978, Pugh and Ratcliffe 1979). Now

$$G\left(\frac{1}{w}\right) = C\left(\frac{1}{w}I - A\right)^{-1}B + D\left(\frac{1}{w}\right)$$

(39)

Let,

$$D\left(\frac{1}{w}\right) = C_1(wI - A_1)^{-1}B_1 + D_1$$

(40)

be a least order at $s_0$ realization of $D(1/w)$ then

$$G\left(\frac{1}{w}\right) = C(I - Aw)^{-1}Bw + C_1(wI - A_1)^{-1}B_1 + D_1$$

(41)

and so a polynomial system matrix realization of $G(1/w)$ is

$$P(w) = \begin{bmatrix}
I - Aw & 0 & Bw \\
0 & wI - A_1 & B_1 \\
-C & -C_1 & D_1
\end{bmatrix}$$

(42)

Since the realization (40) has least order at $w = 0$ it follows readily that the system matrix (42) has, in particular, least order at $w = 0$. It is not necessary to deduce that (42) has least order in the usual sense in view of the theory developed above.

Recall now from (38) that

$$G_F(s) = C(sI - A - BF)^{-1}BH + D(s)(F(sI - A - BF)^{-1}BH + H)$$

and so using (40) gives

$$G_F\left(\frac{1}{w}\right) = C(I - Aw - BFw)^{-1}BHw$$

$$+(C_1(wI - A_1)^{-1}B_1 + D_1)[F(I - Aw - BFw)^{-1}BHw + H]$$
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\[
= [C \ C_1 \ D_1 F] \\
\times \left[ (I - Aw - BFw)^{-1} \ 0 \ 0 \right] \\
\times \left[ \begin{array}{ccc}
(wI - A_1)^{-1} & (wI - A_1)^{-1}B_1F(I - Aw - BFw)^{-1} \\
0 & 0 & (I - Aw - BFw)^{-1}
\end{array} \right] \\
\times \left[ \begin{array}{c}
BHw \\
B_1H \\
BHw
\end{array} \right] + D_1H
\]

Thus

\[
G_F\left(\frac{1}{w}\right) = [C \ C_1 \ D_1 F] \\
\times \left[ \begin{array}{ccc}
I - Aw - BFw & 0 & 0 \\
0 & wI - A_1 & -B_1F \\
0 & 0 & I - Aw - BFw
\end{array} \right]^{-1} \left[ \begin{array}{c}
BHw \\
B_1H \\
BHw
\end{array} \right]
\]

Hence a polynomial system matrix realization of \( G_F(1/w) \) is

\[
P_F(w) = \left[ \begin{array}{ccc}
I - Aw - BFw & 0 & 0 \\
0 & wI - A_1 & -B_1F \\
0 & 0 & I - Aw - BFw
\end{array} \right]^{-1} \left[ \begin{array}{c}
BHw \\
B_1H \\
BHw
\end{array} \right]
\]

It is not possible to deduce that (44) has least order in the usual sense, nor necessary in view of the local theory developed above. It is sufficient, and indeed quite simple, to note that (44) has least order at \( w = 0 \). It thus follows from Theorem 6 that the pole/zero structure of \( G_F(1/w) \) at \( w = 0 \) is described precisely by the zero structure at \( w = 0 \) of the relevant parts of \( P_F(w) \).

In the case of the zero structure of \( G_F(1/w) \) at \( w = 0 \), consider the zero structure of \( P_F(w) \) at \( w = 0 \). Note that

\[
P_F(w) = L(w)P_G(w)R(w)
\]

where

\[
L(w) = \left[ \begin{array}{ccc}
I - Aw - BFw & 1 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
-C & 0 & 0 \\
0 & 1 & 0
\end{array} \right], \quad R(w) = \left[ \begin{array}{ccc}
1 & 0 & -I \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & -F \\
0 & 0 & H
\end{array} \right]
\]

(46)

\[
P_G(w) = \left[ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & I - Aw & 0 & Bw \\
0 & 0 & wI - A_1 & B_1 \\
0 & -C & -C_1 & D_1
\end{array} \right]
\]

(47)
Now \( L(w), R(w) \) of (46) are biproper at \( w = 0 \) and so from (45), \( P_F(w) \) is equivalent at \( w = 0 \) to \( P_E(w) \). However, note from (47) and (42) that \( P_E(w) \) is simply a trivial expansion of \( P(w) \). Hence, \( P_F(w) \) and \( P(w) \) are equivalent at \( w = 0 \) and so have the same Smith–McMillan form at \( w = 0 \). Hence, \( G(1/w) \) and \( G_F(1/w) \) have identical zero structure at \( w = 0 \). Thus the following result has been established.

**Theorem 7**

The infinite zero structure of \( G(s) \) is invariant under state variable feedback.

The above theorem provides the required extension of the Vardulakis/Ferreira result to the case of general (non-proper) rational transfer function matrices. However, a closer examination of (44) reveals a much more surprising result. Consider the infinite-pole structure of \( G_F(s) \) or, what is the same thing, the pole structure of \( G_F(1/w) \) at \( w = 0 \). By Theorem 6 and the least order at \( w = 0 \) property of (44), this is identical to the zero structure at \( w = 0 \) of

\[
T_F(w) \triangleq \begin{bmatrix}
1 - Aw - Bw & 0 & 0 \\
0 & wI - A_1 & -B_1 F \\
0 & 0 & I - Aw - BF w
\end{bmatrix}
\]  

(48)

Note now that

\[
T_F(w) = L'(w)T_E(w)R'(w)
\]

(49)

where

\[
L'(w) = \begin{bmatrix}
1 - Aw - BF w & 0 & 0 \\
0 & I & -B_1 F \\
0 & 0 & I - Aw - BF w
\end{bmatrix}
\]

(50)

\[
R'(w) = \begin{bmatrix}
(I - A w)^{-1} & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\]

(51)

\[
T_E(w) = \begin{bmatrix}
1 - Aw & 0 & 0 \\
0 & wI - A_1 & 0 \\
0 & 0 & I
\end{bmatrix}
\]

(52)

Now \( L'(w) \) and \( R'(w) \) of (50) are biproper at \( w = 0 \), and so from (49), \( T_F(w) \) and \( T_E(w) \) are equivalent at \( w = 0 \). However, note from (51) and (42) that \( T_E(w) \) is merely a trivial expansion of \( T(w) \) where

\[
T(w) \triangleq \begin{bmatrix}
1 - Aw & 0 \\
0 & wI - A_1
\end{bmatrix}
\]

Hence, \( T_F(w) \) and \( T(w) \) are equivalent at \( w = 0 \) and so from Theorem 6, \( G(1/w) \) and \( G_F(1/w) \) have the same pole structure at \( w = 0 \). Thus the following result is proved.
The infinite-pole structure of $G(s)$ is invariant under state variable feedback.

5. Conclusions

A localized theory of linear systems has been briefly developed in §2 and §3. The results are largely obvious restrictions of the corresponding results of the conventional theory although the local state-space form presented in Theorem 4 is interesting for its clear representation of the Laurent expansion of the transfer function matrix.

The need for these local results is revealed in §5 where a result originally established by Vardulakis (1980) concerning the invariance under state feedback of the infinite-zero structure of a proper transfer function matrix, has been extended to the case of general (non-proper) transfer function matrices. It is noted that the proof hinges on the polynomial system matrix realization (44) of $G_t(1/w)$ and while it is not possible to determine that this has least order in the usual sense, it is quite easy to see that it has least order at $w = 0$. Thus, the local theory at $w = 0$ can be utilized to obtain the desired result.

The final result (Theorem 8) is quite surprising. That linear state feedback should leave invariant the infinite-pole structure of the open-loop system is not immediately obvious. However, this phenomenon can to some extent be explained by the fact that the state is determined from entirely finite frequency considerations and that the feeding back of such information can only reasonably be expected to alter that part of the system structure to which it relates. In such circumstances the infinite frequency structure could well be imagined to be immune.

REFERENCES
