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27 JUN 1997
26 JUN 1998
3 DEC 1997
A Nonlinear Control Design Methodology for Computer-Controlled Vehicle Suspension Systems.

C.Marsh

Submitted for the degree of Doctor of Philosophy in the Department of Transport Technology, Loughborough University, September 1992.
Dedication.

This thesis is like the channel tunnel: I have continually lied about its expected completion date, its cost and its global impact! - if I had known the truth of the matter, I would probably never have started.

*Le monde est dure.*

Joking aside, I would like to dedicate this thesis to Ruth and Kate for their love, guidance and motivation, thank you.
Summary

This study demonstrates a new nonlinear controller design methodology applicable to automotive suspension systems. It enhances the ability of the designer to tackle the complex design problem of a controller for a computer-controlled suspension. Based on the principles of optimal control, it permits the use of more general system models and cost functions than the standard linear optimal design techniques and hence, increases the freedom of the designer. It implements the control with an optimal, nonlinear feedback law and is shown to have the potential to improve vehicle performance.

The methodology is demonstrated here for the design of a controller for an ideal active suspension system employed in a quarter vehicle model, though it is potentially applicable for other hardware employed in other, more complex vehicle models. It is based on a simple fundamental control strategy which is in turn based on the principles of optimal control. The control strategy permits the use of non-quadratic cost functions, the benefit of which is illustrated by comparison of open loop optimal regulator performances.

To evaluate the optimal controls for any given conditions requires the numerical solution of a nonlinear two point boundary value problem. This is too computationally intensive to permit real-time implementation and a nonlinear feedback law is used for this purpose. The development of an appropriate law requires the fitting of parameters to give a good representation of the optimal behaviour defined by the control strategy. Having designed an appropriate feedback law, simulations of the quarter vehicle model employing an ideal active suspension with this nonlinear closed loop control can be performed.

The quarter vehicle model is the simplest that is able to provide meaningful assessments of the performance of a suspension system and appropriate simulation test conditions for the model are derived from studies of it employing a nominal passive suspension system. The model employing the nonlinear active system is also subjected to these simulated tests and the performance compared to that of the model employing the nominal passive system. The performance is found to be superior and though some of the improvement could be attributed to the greater capabilities of ideal active hardware, a significant proportion are directly attributable to the effectiveness of the controller.
Acknowledgements

This work was conducted in the Department of Transport Technology, Loughborough University and was financially supported by the Chassis Research section of the Ford Motor Company.

The author would like to thank the Ford Motor Company for the financial support and guidance of the project and in particular, Mr Ian Crawford of Chassis Research.

The work was mainly supervised at Loughborough by Dr T.J. Gordon and I would like to extend my thanks to him for his exceptional supervision, guidance and encouragement. The knowledge and insight he has passed to me during the course of this project has left me indebted to him for the rest of my life.

I also wish to thank Dr M.G.Milsted, formerly of Loughborough University, for his assistance and supervision during his presence at the start of the project and for his work in getting the project off the ground.

Finally, I wish to thank my colleague Mr. M.C.Best for his assistance and support during the writing of this thesis.
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# Chapter 1

## Introduction.

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Introduction

1 Introduction.

This study is devoted to the design of the controller for a computer-controlled suspension system and a new nonlinear design methodology is presented. The design of vehicle suspensions is quite complex and requires the compromise of several inter-related performance desires. For computer-controlled suspensions the performance is dictated by the controller within the constraints of the hardware and hence, an effective strategy is crucial. Standard techniques applied to this problem do not give satisfactory solutions and this methodology aims to increase the ability of the designer to realise the potential benefits of a computer-controlled suspension.

1.1 Design of Vehicle Suspension Systems.

It is recognised that the ride and handling characteristics are important attributes of a vehicle. They describe the conflicting desires for occupant comfort and vehicle maneuverability, and traditionally suspension system design has involved a compromise between these two requirements. Furthermore, the workspace available for a suspension system is usually constrained and this further compromises the design. These compromises still prevail for the design of computer controlled suspension systems and this nonlinear controller design methodology aims to improve the ability of the designer to tackle this problem.

Many types of suspension system have been considered by vehicle designers, but until recently they have consisted solely of passive components, commonly some combination of spring and damper. Currently, there is great interest in the use of active and semi-active suspension systems to improve vehicle performance.

In an active system, active force generators are used either in conjunction with or instead of the passive suspension components. These active force generators, typically electro-hydraulic devices, are capable of both supplying energy to and removing energy from the vehicle and hence require a power source. There are usually practical limits associated with the bandwidth of their operation and the maximum amplitude of forces that can be applied. The amount of power consumed can also be quite significant and the cost of the hardware is currently substantially greater than of conventional passive hardware; however, studies have shown (Sharp & Hassan, 1986) that such systems can give substantial performance improvements.

Semi-active systems contain force generators which are purely dissipative and hence do not require power input; typical devices are variable dampers, either continuously variable or switchable between different rates. This type of hardware is currently much cheaper than fully active hardware but still more expensive than conventional passive
Introduction

hardware. Studies have again shown (Sharp & Hassan, 1986) that whilst semi-active systems are unable to match the performance of active systems, they may have significant performance advantages over passive systems.

Both active and semi-active components are able to provide forces in response to external signals which can be supplied from a micro-processor programmed to implement closed loop control for the whole suspension system. An effective control strategy is essential to realise desirable performance from such systems and, since the dynamics and performance requirements are quite complex, any simple empirical strategy is unlikely to yield satisfactory performance. The extra production cost of an effective control strategy -however complex- is negligible, since it is realised by some software in a micro-processor and hence, it can give a very cost effective improvement to a vehicle.

1.2. Review of Previous Work on Computer-Controlled Suspensions.

Studies of the use of active suspension systems date back to the 1960s', Bender (1967,1968) regarded the application as a special case in the study of vibrations and studies have continued, notably Thompson (1970) and Goodall and Kortüm (1983). A good review of basic principles of such systems is given in the comprehensive work on the design of vehicle suspension systems by Sharp and Crolla (1987). Similar interest has been shown in the use of semi-active systems dating back to Karnopp and Crosby (1973) and it has been noted by Karnopp et al (1974), later by Margolis (1983) and Sharp and Hassan (1987) that they are able to offer similar improvements to active systems under certain circumstances.

The importance of the control strategy for these intelligent suspension systems has also been widely recognised. Karnopp et al (1974) developed a heuristic control scheme known as 'skyhook damping' which has since been commonly considered for the control of semi-active systems. Control laws for active systems developed using the principles of linear time-domain optimal control have been studied by Thompson (1976), Thompson (1984), Karnopp (1986) and Wilson, Sharp and Hassan (1986). These approaches have become known as 'First Generation' control strategies and are the basic standards used in most current practical system development.

The performance offered by these systems has been shown to have some fundamental limits and further developments are taking place, see Thompson and Davis (1992) amongst others, leading to 'Second Generation' control strategies. Chalasani (1986a) recognised the restrictions of 'First Generation' control strategies and suggested that 'Nonlinear control laws may further improve the performance of active suspensions and need to be explored'. Nonlinear designs were investigated with a bilinear
modelling approach by Kimbrough (1986) and some promising results from a nonlinear control strategy were presented by Gordon, Marsh and Milsted (1990).

Most authors have considered models of 'quarter vehicle' suspension systems when investigating the design and performance of control strategies and Sharp and Crolla (1987) in particular, give detailed justification for the use of this model. It is the simplest model that is adequately able to describe the interrelated problems of controlling body isolation for occupant comfort, dynamic tyre loads for handling ability and constraining relative wheel to body deflections to the available working space. 'Half vehicle' models have been considered, Sharp and Wilson (1990); these are additionally able to analyse pitch dynamics and the interconnection between front and rear axles. Other 'full vehicle' models have been used (Abdel Hady and Crolla, 1989 and Chalasani , 1986b) , which combine four quarter vehicle models using a common sprung mass into a seven degree of freedom model. These are further able to describe the problem of attitude control through roll, pitch and bounce dynamics.

The primary source of disturbance to the system considered by most authors is that due to road unevenness and this is again justified by Sharp and Crolla (1987) in their review. Other authors also consider disturbing forces applied directly to the body of the vehicle, e.g. Karnopp (1986) when examining the fundamental performance possibilities offered by active suspensions and Elmadany (1990) and Thompson and Davis (1992) who make deliberate efforts in their control law design to account for these. Such disturbances arise due to aerodynamic effects and can also be used to describe the coupling forces between the quarters of a vehicle; however, these disturbances have generally been accepted as less important than those due to road unevenness.

Models describing the relative motion of occupants or powertrains and the sprung mass have also been considered, (Kimbrough, 1986) but it is generally felt that these motions are of secondary interest in suspension design. The effects of other degrees of freedom such as bushing dynamics or chassis flexure are also usually neglected since they occur outside the frequency range of interest.

A further major area of interest is that of using 'previewed' information about the road surface to improve the control and hence the performance of computer-controlled suspension systems. The potential performance improvements are well recognised, (Bender, 1968) and Thompson, Davis and Pearce, (1980) and effective ways of obtaining and utilising the information are still being sought, (Hac, 1992). There are essentially two methods of obtaining such information; one is to scan the elevation of the road surface in front of the vehicle and the other, known as wheelbase preview, is to infer knowledge about forthcoming disturbances to the rear from events at the front and both these topics are discussed in the recent work by Louam et al (1992).
1.3 Proposed Approach.

In this work a fresh look is taken at the problem of the design of control laws for computer-controlled suspensions. A general nonlinear methodology is proposed which is based upon the principles of optimal control, and for ease of conception, this can be compared to the standard linear optimal approach; however, design freedom and performance potential are not sacrificed for mathematical convenience in this new nonlinear methodology.

The linear optimal approach, or LQG as it is commonly known, requires the use of a linear system model and assumes unconstrained actuation of the control. It aims to produce optimal responses for Gaussian inputs subject to performance assessments which must be expressed by a quadratic cost function. The method is well established and easily executed; however, the above requirements certainly restrict the freedom and flexibility of the designer.

One major restriction of the LQG approach is that it assumes suspension systems which are linear and hence vehicles which have fixed characteristics. This is likely to be undesirable, and ideally the suspension system would be able to change its characteristics to suit the prevailing road conditions. For example, under motorway driving conditions it may be desirable to control the suspension to give the vehicle a 'soft' characteristic that enhances the ride, whilst under country road driving conditions, a more 'sporting' characteristic may be more appropriate. The hardware may be capable of producing a whole range of characteristics but a quadratic cost function leading to a linear control law will constrain the designer to choosing a fixed constant one.

If a 'worst case' design philosophy is considered where parameters are chosen to produce a suspension system which avoids excessive suspension travel and contact of the hardware components for a given severe disturbance, then, in the case of a linear design methodology, this defines the characteristics of the system which will be prevalent for other less severe inputs. Conventional passive systems often incorporate nonlinearities in spring stiffness or shaped 'bump stops' to meet such 'worst case' design criteria and allow softer characteristics to be prevalent for less severe conditions.

A more general nonlinear methodology which does not restrict the designer to quadratic cost functions allows greater freedom for the expression of performance requirements, and is potentially able to produce controllers which enable inherent adaptation of the characteristics of the vehicle. This fundamental advantage of a nonlinear methodology is likely to be most effectively demonstrated for an ideal active suspension system since this offers the greatest freedom of actuation.
Introduction

There are further drawbacks of the LQG approach; often the system can not adequately be described by a linear model and, especially in the case of semi-active systems, the control actuation is not unconstrained. These effects must be ignored in the LQG approach but could be considered in a more general methodology and hence, in principle, a more effective control strategy could be produced. However, absolute performance improvements may not be so great for these conditions, since semi-active systems offer less actuation ability and hence less performance enhancement potential.

For this reason, the demonstration of the nonlinear control design methodology is confined to ideal active suspension systems in this study. The increase in design freedom and ability of the resulting controller to adapt the characteristics of the system in line with the disturbing inputs is to be illustrated. The performance goals and assessments used in this study are nominal artificial ones based on simulation results from a mathematical model. There are many other performance goals which the design methodology could accommodate and performance modifications can be effected by the choice of values for parameters in the design. This ability to modify the performance of the vehicle is also illustrated.

The quarter vehicle model is to be used to design and test the suspension system because of its simplicity and ability to model the most fundamental aspects of the performance of a suspension system. The analysis, simulation and performance assessment of a nominal quarter vehicle model is considered in detail in §2, where a model incorporating a reference passive suspension system is used. Subsequently, this passive system is to be replaced by an ideal active one within the same nominal quarter vehicle model and the performance of the model using the reference passive system is used as a benchmark for guidance and comparison.

The nonlinear controller design methodology is considered from fundamental principles in §3 and a basic control strategy developed. This strategy is based on the general principles of time domain optimal control and seeks to minimise undesirable performance effects as expressed by a cost function. It does not involve the use of any sort of preview, though, in principle, the ideas being developed elsewhere on this subject and documented in the literature could be incorporated to further enhance the performance.

The strategy is initially illustrated for the familiar special case when the restrictions of linear optimal control are imposed and the strategy can be realised with a linear feedback law designed by standard techniques. The restrictions placed on the design and the performance are illustrated and this motivates the increase of freedom in the design and leads to the development of a procedure for implementing the strategy under more general circumstances. This requires the solution of a more general nonlinear optimisation problem, this is examined in detail in §4.
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The results produced by the active system controlled by this nonlinear optimal strategy, relating to a chosen reference cost function, are evaluated and are found to compare favourably with both those produced by the linear technique and those of the passive system. However, the computational effort required for the application of the strategy is greatly increased and is too great to allow real-time implementation, hence, a practicable method of implementation must be sought and this is considered in §5.

Having successfully developed a method of implementation of the strategy using nonlinear feedback, comprehensive performance assessments are made of the resulting nonlinear closed loop active system and comparisons are made with the reference passive system in §6. This closed loop system again relates to the chosen reference cost function on which the system performance depends. An integral part of the design methodology is the ability to tailor the performance of the system to given requirements via a tuning procedure evoked by changing the cost function, and this is also demonstrated.
Chapter 2

Analysis of Quarter Vehicle Model Employing Passive Suspension System.

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2.2 Dynamic Analysis of Quarter Vehicle Model. 12

2.3 Modal Analysis of Quarter Vehicle Model. 14

2.4 Initial Disturbance Responses of Passive System. 17

2.5 Forced Response. 20
## Nomenclature

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<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
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<tr>
<td>A</td>
<td>System matrix</td>
<td></td>
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<tr>
<td>b_s</td>
<td>Passive Suspension damping rate</td>
<td>Ns/m</td>
</tr>
<tr>
<td>B</td>
<td>Input matrix</td>
<td></td>
</tr>
<tr>
<td>e</td>
<td>Equilibrium tyre compression</td>
<td>mm</td>
</tr>
<tr>
<td>F_t</td>
<td>Tyre force</td>
<td>N</td>
</tr>
<tr>
<td>F_s</td>
<td>Suspension force</td>
<td>N</td>
</tr>
<tr>
<td>g</td>
<td>Gravitational constant</td>
<td>m/s²</td>
</tr>
<tr>
<td>h</td>
<td>Road height</td>
<td>m</td>
</tr>
<tr>
<td>K_t</td>
<td>Tyre spring stiffness</td>
<td>N/m</td>
</tr>
<tr>
<td>K_s</td>
<td>Passive suspension spring stiffness</td>
<td>N/m</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Wave number (Cycles per meter)</td>
<td></td>
</tr>
<tr>
<td>m_b</td>
<td>Sprung mass</td>
<td>Kg</td>
</tr>
<tr>
<td>m_w</td>
<td>Unsprung mass</td>
<td>Kg</td>
</tr>
<tr>
<td>n</td>
<td>Index in road height model</td>
<td></td>
</tr>
<tr>
<td>p</td>
<td>Real part of Eigenvalue</td>
<td></td>
</tr>
<tr>
<td>q</td>
<td>Imaginary part of Eigenvalue</td>
<td></td>
</tr>
<tr>
<td>R_c</td>
<td>Roughness constant</td>
<td></td>
</tr>
<tr>
<td>s</td>
<td>Horizontal distance along road</td>
<td>m</td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
<td>s</td>
</tr>
<tr>
<td>v</td>
<td>Horizontal velocity of vehicle traversing road</td>
<td>m/s</td>
</tr>
<tr>
<td>V_0</td>
<td>Velocity of tyre contact point</td>
<td>m/s</td>
</tr>
<tr>
<td>( \mathbf{v}_1, \mathbf{v}_2 )</td>
<td>Eigenvectors</td>
<td></td>
</tr>
<tr>
<td>x_1</td>
<td>Tyre deformation</td>
<td>m</td>
</tr>
<tr>
<td>x_2</td>
<td>Suspension deflection</td>
<td>m</td>
</tr>
<tr>
<td>x_3</td>
<td>Velocity of unsprung mass</td>
<td>m/s</td>
</tr>
<tr>
<td>x_4</td>
<td>Velocity of sprung mass</td>
<td>m/s</td>
</tr>
<tr>
<td>x</td>
<td>State vector</td>
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<tr>
<td>w</td>
<td>Vector of disturbing inputs</td>
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In this chapter, the dynamics of a model of a quarter vehicle employing a conventional passive suspension system are studied. In later chapters, a similar quarter vehicle model will be used to design and test active suspension systems. The performance of the model employing this conventional passive system will be used to guide the design and performance analysis of these active systems.

The quarter vehicle model is used because of its simplicity and its ability to provide clear and meaningful comparisons and evaluations of different suspension systems. It gives an accurate description of the problem of controlling tyre contact forces, levels of vertical body acceleration and workspace usage. These quantities in turn give a meaningful assessment of the ride quality and handling ability of the vehicle operating within a constrained workspace.

The passive suspension system is typical of that used on a 'D' class family saloon and the parameters used throughout the model have been identified from such a vehicle. The system is known as passive since its components have no controlling inputs.

2.1 Model of Quarter Vehicle with a Passive Suspension System.

The model is depicted in Fig 2.1, there are two masses, sprung and unsprung, denoted $m_b$ and $m_w$ and these approximately represent the body mass and wheel mass of the quarter vehicle. The wheel is isolated from the road by the tyre which is modelled as a simple linear spring with constant $K_t$. The body is isolated from the wheel by the suspension system which is modelled as a linear spring, constant $K_s$, in parallel with a linear damper, rate $b_s$.

The values of the masses and spring constants are taken from system identification tests performed on a reference car described in Marsh, Gordon & Best (1992), and the linear damping rate is a nominal one. These values are:

\[
\begin{align*}
    m_b &= 259 \text{ Kg}, \\
    m_w &= 36.5 \text{ Kg}, \\
    K_t &= 2.02 \times 10^5 \text{ N/m}, \\
    K_s &= 17595 \text{ N/m}, \\
    b_s &= 1000 \text{ Ns/m}.
\end{align*}
\]
Analysis of Quarter Vehicle Model Employing Passive Suspension System.

Fig 2.1: Model of Quarter Vehicle with Passive Suspension.

The system has two degrees of freedom whose dynamics can be described by four state variables: $x_1$ - tyre deformation, $x_2$ - suspension deflection, $x_3$ - velocity of unsprung mass and $x_4$ - velocity of sprung mass. The forces applied to the masses by the tyre, $F_t$, and by the suspension, $F_s$ are given by

$$F_t = K_t x_1 \quad \text{and} \quad F_s = K_s x_2 + b_s (x_3 - x_4). \quad (2.1.1)$$

All motions and forces are purely vertical and the only disturbance considered is that due to road unevenness which can be described by the vertical velocity of the tyre contact point, $V_0$. The state differential equations of the system are:

$$x_1 = V_0 - x_3$$
$$x_2 = x_3 - x_4$$
$$x_3 = \frac{F_t - F_s}{m_w}$$
$$x_4 = \frac{F_s}{m_b}.$$  \quad (2.1.2)
2.2 Dynamic Analysis of the Quarter Vehicle model.

This model allows the behaviour of a quarter car to be predicted when it is subjected to disturbances caused by road irregularities. The performance is assessed from the analysis of the time histories of the responses. The ride quality offered by the vehicle and its handling ability are two important considerations; furthermore, the model must operate within physical limits placed on the suspension workspace. There are inherent trade offs between these three attributes of performance, and a reasonable compromise should be made under all operating conditions. The balance between these performance attributes defines the characteristics of the vehicle and can broadly be dictated by the suspension system. E.g. under certain conditions, a conventional 'hard' or 'sports' passive suspension system which has a relatively stiff spring and high level of damping will provide relatively good handling performance and relatively poor ride quality in comparison with a 'soft' system, which has a softer spring and lower damping ratio. There is also a trade-off between ride quality and workspace usage; a 'soft' system will again provide a relatively good ride but use more workspace and hence be more likely to violate the physical limits.

A good estimation of the ride comfort has been shown by Sharp and Crolla (1987) to be given by the rms level of body accelerations. In their publication they draw on experimental evidence from Aspinall and Oliver (1964) and Smith et al (1978). This measure shall be used to quantify the simulated ride comfort offered and good levels of ride comfort are desirable in all circumstances.

The quarter vehicle model is unable to describe the most important handling operations of a vehicle, i.e. cornering, braking and accelerating and to consider the behaviour of the vehicle during such operations is beyond the scope of this work. However, it is known from full vehicle models that the principal requirements of these operations are lateral and longitudinal force inputs via the tyres. It is also known that the ability of tyres to provide these forces is dependent upon their dynamic loading which can be effectively estimated by the quarter vehicle model. The dynamic loads are approximately proportional to the levels of vertical tyre deformation and hence the simulated handling ability of the vehicle can be based upon these.

One could simply take the rms of the tyre deformation to represent the simulated handling ability, however, this is felt to be a little simplistic. Consider two simulated time histories of tyre deformations with the same rms, one is of fairly constant amplitude and the other is predominantly of a lower amplitude but with some very high peaks. It is reasonable to assume that the handling of a vehicle characterised by the second of these time histories is worse than that by the first since the high peak deformations are likely to have dramatic and undesirable effects on tyre force generation capabilities. Hence, the rms alone is not a sufficient measure of handling ability and
attention must also be paid to the distribution of the deformations; many other statistics could be considered such as the 95th percentile or the amount of time spent above a given amplitude on a given simulation. However, for simplicity, the peak value of the tyre deformations is to be considered along with the rms to quantify the handling ability of the vehicle.

To quantify the expected levels of tyre deformation and to provide an approximate calibration of the tyre force generation capabilities to which these relate, consider the deflection of the tyre in equilibrium. From the model, it can be deduced that the weight of the vehicle is reacted by a tyre spring equilibrium compression, $e$, where

$$e = \frac{(m_b + m_w)g}{K_t} = 14.35 \text{ mm.}$$ \hspace{1cm} (2.2.1)

This forms the datum for the dynamic model and hence when $x_1 > 14.35\text{mm}$ the tyre has actually lifted off the road. The linear spring model for the tyre is then not valid for these values of $x_1$; however, for the sake of simplicity in the analysis, the linear spring model is to be used throughout this study. It is reasonable to assume that this will not seriously invalidate this fundamental analysis and comparison of suspension systems. To assess the handling ability of the vehicle under extreme circumstances, disturbances which produce peak tyre deformations of approximately 25mm will be considered.

The extremities of the suspension workspace of a vehicle are usually guarded by 'bump-stops' which are stiff rubber elements that prevent the contact of rigid hardware components in the suspension system. If these stops are contacted in operation then high suspension forces and hence high body accelerations will result. These stops and their effects are not included in detail in this model; however, artificial limits are placed on the magnitudes of suspension deflection that are permissible. These limits effectively impose the physical limits of the suspension workspace of the vehicle in the model.

The suspension workspace available on this type of vehicle is typically ±100mm. These limits will be considered when analysing the time histories of suspension deflection with the knowledge that an infringement of these limits represents a bump-stop contact which would have highly undesirable effects. Again, for simplicity, the peak value of suspension deflections shall be considered a measure of the ability to operate within the constrained workspace.
2.3 Modal analysis of Quarter Vehicle Model.

The model has two degrees of freedom and its dynamic behaviour can be characterised by its modes. To investigate the nature of these modes, the state differential equations (2.1.2) are re-expressed in the standard format for the analysis of linear time-invariant systems:

\[ \dot{x} = Ax + Bw \]  

(2.3.1)

where \( A \) is the system matrix, \( B \) the input matrix and \( w \) the vector of disturbing inputs. For the quarter vehicle model,

\[
A = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 \\
\frac{K_t}{m_w} & -\frac{K_s}{m_w} & -\frac{b_s}{m_w} & \frac{b_s}{m_w} \\
0 & \frac{K_s}{m_b} & \frac{b_s}{m_b} & -\frac{b_s}{m_b}
\end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

and since there is only a single disturbing input, \( w \), is the scalar \( V_0 \).

The modes can be described by the complex eigenvalues and eigenvectors of the system matrix \( A \); for the quarter vehicle model the eigenvalues are \(-1.65 \pm 7.79i\) and \(-14.0 \pm 75.7i\) and the corresponding eigenvectors can be represented by \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) where:

\[
\mathbf{v}_1 = \begin{pmatrix}
-0.002 + 0.010 i \\
0.028 + 0.113 i \\
0.075 + 0.034 i \\
1
\end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix}
0.002 + 0.013 i \\
-0.002 - 0.013 i \\
1 \\
-0.018 - 0.047 i
\end{pmatrix}.
\]

The detailed interpretation of the nature of the modes of a system described by complex eigenvalues and eigenvectors is given in Newland (1989) and a summary of this is given below using this quarter vehicle model as an example.

The 'shape' of a mode can be described by any eigenvector for that mode, the components of which can be depicted as phasors on an Argand diagram as in Fig 2.3. It should be noted that any eigenvector can be multiplied by any complex constant and still remain an eigenvector, hence the phasors representing the mode can be rotated and scaled and still represent the mode. A method of visualising a mode shape is to consider free undamped oscillations in that mode which can be represented by rotating the phasors. Such motion is not physically possible since the system is damped, however, the concept remains a useful way of visualising the mode shapes. At any
point in time, the value of each of the states of the system is given by the real part of the corresponding phasor, hence, the maximum amplitude of each state during the undamped motion is given by the magnitude of its phasor and the relative phases are given by the angles between the phasors.

The true dynamic behaviour of each mode is described by its complex conjugate pair of eigenvalues and is generally characterised by its natural frequency, damping ratio and its decay rate or settling time. Denoting the conjugate eigenvalues \( p \pm q \, i \), the natural frequency of the mode is given by \( \frac{\sqrt{p^2 + q^2}}{2\pi} \), its decay rate by \( e^{pt} \) and its 98% settling time by \( \frac{4}{p} \). The level of damping in a mode is commonly described by the damping ratio which is equal to \( \frac{p}{\sqrt{p^2 + q^2}} \). The damped free motion of the system in any mode can be represented by rotating the phasors at a rate of \( 2\pi \) times the natural frequency and allowing them to decay by a factor of \( e^{pt} \).

The shape of the first mode is described by eigenvector \( v_1 \), the magnitude of the largest component of which is scaled to 1 permitting relative comparisons of the maximum amplitudes of the states in the free undamped motion. It can be seen that the motion is dominated by the movement of the body with the peak body velocity chosen to be 1 m/s, and that the movement of the wheel is much smaller with a peak velocity of 0.082 m/s. The body motion causes large suspension deflections with a peak of 116 mm which occur approximately 90° out of phase with the body velocity. The tyre deformations are relatively small peaking at just 10 mm. This mode shape can be envisaged as the body oscillating with the wheel fixed and hence is known as the body bounce mode. The dynamics of this mode are described by the corresponding conjugate pair of eigenvalues which specify a natural frequency of 1.27 Hz, a damping ratio of 0.21 and a 98% settling time of 2.42 seconds.

In the second mode, described by \( v_2 \), the motion is dominated by oscillations of the wheel with the peak wheel velocity chosen to be 1 m/s. This causes tyre deformations and suspension deflections which are of the same magnitude, with a peak of 13.2 mm, and are 180° out of phase. The body movements are relatively small with a peak body velocity of just 0.05 m/s. This mode shape can be envisaged as the wheel oscillating with the body fixed causing equal and opposite tyre deformations and suspension deflections, hence, it is known as the wheel hop mode. Consideration of its eigenvalues shows that the natural frequency is 12.25 Hz with a damping ratio of 0.18 and a 98% settling time of 0.29 seconds, hence the natural frequency of this mode is much higher and the settling time is much shorter.
The shapes and dynamic characteristics of the modes for this quarter vehicle model are dependent on the nature of the suspension system used and hence will change when the application of active systems is considered. Modal analysis is also only strictly applicable to linear models and thus when nonlinear active suspension systems are studied there will not be well defined modes. However, these modes relate to behavioural conditions and remain a useful way of analysing the dynamic behaviour of all quarter vehicle models.

It is necessary to test the performance of the vehicle under varied conditions and to examine the balance between the various attributes of the performance. It is possible to examine the performance trade-offs independently by exciting the system in each of these behavioural conditions; the wheel hop condition enables the balance between ride quality and handling performance to be examined whereas the body bounce condition tests the balance between ride quality and workspace usage. Therefore, tests of the behaviour of the model in these two conditions can be used to identify the balance between the performance attributes.
2.4. Initial Disturbance Responses of Passive System.

In this section some simple initial disturbance tests are conducted to analyse the dynamics of the passive system and the behaviour is to be examined in both the body bounce condition and the wheel hop condition by choosing the initial disturbances such that responses are dominated by motion in each of the conditions. These initial disturbances are: a downwards wheel velocity of 2.5 m/s with all other states zero and a downwards body velocity of 1.25 m/s again with all other states zero. These can be expressed in state vector format by $[0 \ 0 \ 2.5 \ 0]^T$ and $[0 \ 0 \ 0 \ 1.25]^T$ respectively. The amplitudes of the initial disturbances have been chosen to test the system under extreme circumstances producing peak tyre deformations and suspension deflections which are approximately equal to 25mm and 100mm respectively.

The responses produced are shown in Fig 2.4.1 & 2.4.2 and summarised in Table 2.4. It can be seen that the extreme wheel hop condition test produces a peak tyre deformation of 25.3mm and the extreme body bounce condition test produces a peak suspension deflection of 110mm. The higher natural frequency and shorter settling time of the wheel hop condition are also apparent from the time history plots.

### Extreme Amplitude Wheel Hop Condition.

<table>
<thead>
<tr>
<th></th>
<th>R.M.S values for first 0.5 seconds</th>
<th></th>
<th>Peak Tyre Deformation (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tyre Deformation (mm)</td>
<td>6.16</td>
<td>6.35</td>
<td>1.92</td>
</tr>
<tr>
<td>Suspension Deflection (mm)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Body Acceleration (m/s²)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive Suspension System</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Extreme Amplitude Body Bounce Condition.

<table>
<thead>
<tr>
<th></th>
<th>R.M.S values for first second</th>
<th></th>
<th>Peak Suspension Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tyre Deformation (mm)</td>
<td>5.47</td>
<td>55.6</td>
<td>4.18</td>
</tr>
<tr>
<td>Suspension Deflection (mm)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Body Acceleration (m/s²)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive Suspension System</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.4: Response Summary for Initial Disturbance Simulations.
Fig 2.4.1: Extreme Amplitude Wheel Hop Condition

Tyre Deformation (m)

Suspension Deflection (m)

Body Acceleration (m/s²)

Time (secs)

0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5
Fig 2.4.2: Extreme Amplitude Body Bounce Condition

- Tyre Deformation (m)
- Suspension Deflection (m)
- Body Acceleration (m/s²)

Time (secs)
These tests are very simple but they do give useful indications of the balance between the different attributes of the performance of the model. In the wheel hop condition an example of the trade off between tyre deformations and body accelerations can be seen whilst in the body bounce condition that between suspension deflections and body accelerations is apparent. These 'worst case' results for the passive system are to be used as performance guidelines for the active systems. For similar initial condition disturbances, the peak tyre deformations and suspension deflections produced should be similar to those seen here. Hence, these tests yield a simple guide to the balance between handling ability and ride quality and between workspace usage and ride quality at extreme amplitudes of disturbance.

2.5. Forced Response.

A second and more comprehensive test of the performance of the quarter vehicle is to consider its behaviour in response to disturbing inputs. As previously stated, for the scope of this study disturbances are limited to those caused by road surface irregularities. In this section the response of the model employing the passive suspension system to disturbances generated by known stochastic models shall be examined, and also simulations of the vehicle traversing real roads will be considered.

Studies of road profiles, notably Robson (1979), have shown that their power spectral density can be approximated by:

\[ h(\lambda) = \frac{R_e}{\lambda^n} \]  

where \( h \) is the power spectral density of road height, \( \lambda \) is the wave number in cycles per metre and \( R_e \) is a parameter called the roughness constant. The index \( n \) is dependent upon the type of road taking values approximately between 2 and 3. The disturbance to the system must be presented as the vertical velocity of the tyre contact point the spectrum of which is related to the spectrum of the profile by

\[ V_d(\lambda) = \lambda^2 h(\lambda) . \]  

The performance of the model is to be tested when the motion is predominantly in each of the behavioural conditions. The relative amplitudes of excitation in the wheel hop condition and the body bounce condition are dependent upon the disturbance spectra and hence disturbance models are chosen which principally excite each of the conditions. The ratio of disturbance close to the resonant frequency of the wheel hop condition (=12Hz) to that close to the resonant frequency of the body bounce condition (=1Hz) is greater for roads whose profile has a relatively low value of the index, \( n \).
From (2.5.2), a model of a road whose tyre contact point vertical velocity spectrum is described by Gaussian white noise is equivalent to a road profile with $n=2$ and hence, will principally excite the wheel hop condition and provide a good test of the balance between ride quality and handling ability. The system is again to be tested up to extreme amplitudes and to do this the amplitude of the disturbance is adjusted such that the peak tyre deformation is approximately 25mm.

At this extreme amplitude of disturbance a 50 second simulation of the behaviour was performed and a summary of the results is given in Table 2.5.1. The extremity of the amplitude of the disturbance is measured by the peak tyre deformation observed during the simulation which was 26.6 mm.

This response gives an indication of the balance between handling ability and ride quality at this extreme disturbance amplitude. The ride quality is simply measured by the rms body acceleration and the handling ability by whole of the distribution of the tyre deformations. It should be noted that since the input to the system is Gaussian and the system is linear the distributions of the responses of the state variables are also Gaussian, hence, the distribution of the tyre deformations can be characterised simply by the rms level.

To test the performance of the model in the body bounce condition a disturbance spectrum with a higher value of the index, $n$, could be used. However, a simple but effective approximation to this is made by low pass filtering the Gaussian white noise disturbance signal at 3Hz. This disturbance is to be used to test the balance between workspace usage and ride quality of the system, extreme test conditions are again required and the amplitude has been adjusted such that the maximum suspension deflections observed are approximately 100mm.

Again a 50 second simulation was performed and a summary of the results is also given in Table 2.5.1. The peak suspension deflection of 99.9mm illustrates the extremity of the disturbance amplitude.

The balance between ride quality and workspace usage for extreme disturbances is examined here and again the ride quality is assessed by the rms body acceleration. The performance relating to workspace usage is only concerned with the ability to avoid the bump stops and is measured by the peak value of the suspension deflections.
Analysis of Quarter Vehicle Model Employing Passive Suspension System.

Gaussian White Noise Disturbance.

<table>
<thead>
<tr>
<th></th>
<th>Tyre Deformation (mm)</th>
<th>Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Tyre Deformation (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive Suspension System</td>
<td>6.36</td>
<td>19.3</td>
<td>2.30</td>
<td>26.6</td>
</tr>
</tbody>
</table>

Filtered Noise Disturbance.

<table>
<thead>
<tr>
<th></th>
<th>Tyre Deformation (mm)</th>
<th>Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Suspension Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive Suspension System</td>
<td>3.52</td>
<td>35.2</td>
<td>2.75</td>
<td>99.9</td>
</tr>
</tbody>
</table>

Table 2.5.1: Response Summary for Stochastic Disturbance Models.

These stochastic disturbance models give a comprehensive analysis of the performance capabilities of the system and now simulations of traversing roads will be considered. These form a more realistic test of the behaviour of the model which can more easily be related to experienced vehicle behaviour. Profile data of both 'A' and 'B' class roads has been made available by Chassis Research at Ford Motor Company. A 300m section of the A127 London-Southend 'arterial' and a 200m section of the 'B' class Lower Dunton Road have been considered and the profiles of these are shown in Fig 2.5.1. The traversing of the A127 is simulated at a constant horizontal velocity of 30m/s and the Lower Dunton Road at 20m/s.

The profiles are used to calculate the vertical tyre contact point disturbing inputs for these 10 second simulations. The profile data can be converted to a time history of the tyre contact point height by:

\[ h(t) = h(s) \bigg|_{s=vt} \]

(2.5.3)

where \( s \) is the horizontal distance and \( v \) the constant horizontal velocity of the vehicle. This tyre contact point height time history is then differentiated by central differences to give the tyre contact point vertical velocity time history which is the disturbance signal. The spectra of these resulting disturbances are also shown in Fig 2.5.1.
Analysis of Quarter Vehicle Model Employing Passive Suspension System.

The disturbance spectrum for the A127 simulation contains significant power up to approximately 9 Hz and can be expected to principally excite the wheel hop condition. Conversely the power in the disturbance spectrum for the Lower Dunton Road simulation is concentrated below 2 Hz and this can be expected to principally excite the body bounce condition.

The results for the 10 second simulations of traversing the roads are summarised in Table 2.5.2. For the A127 it can be seen that the principal excitations are in the wheel hop condition producing a peak tyre deformation of 12.3 mm, a peak suspension deflection of just 35.0 mm. Hence, this is principally a test of handling ability and ride quality with only small amounts of workspace being used. Conversely, the Lower Dunton Road principally excites the body bounce condition producing a peak suspension deflection of 80.9 mm, a peak tyre deformation of just 15.4 mm. This is quite a severe disturbance producing much higher accelerations than the A127 simulation and a large proportion of the available workspace is used. The level of peak tyre deformation is also quite significant, hence this disturbance is a further test of handling ability.

A127 @ 30 m/s.

<table>
<thead>
<tr>
<th>Passive Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>R.M.S.</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Tyre Deformation (mm)</td>
<td>R.M.S.</td>
<td>Suspension Deflection (mm)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.30</td>
<td>13.1</td>
<td>1.21</td>
</tr>
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</table>

Lower Dunton Road @ 20 m/s.

<table>
<thead>
<tr>
<th>Passive Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>R.M.S.</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Tyre Deformation (mm)</td>
<td>R.M.S.</td>
<td>Suspension Deflection (mm)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.73</td>
<td>26.4</td>
<td>2.18</td>
</tr>
</tbody>
</table>

Table 2.5.2 Response Summary for Simulations of Traversing Measured Roads.
Fig 2.5.1: Profiles and Disturbance Spectra for Measured Roads
Chapter 3

Introduction to Controller Design for an Ideal Active System.

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Nomenclature.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>Parameter in cost function</td>
<td></td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>Parameter in cost function</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>Parameter in cost function</td>
<td></td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>Parameter in cost function</td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td>Vector of disturbances</td>
<td></td>
</tr>
<tr>
<td>$f$</td>
<td>Vector function of state derivatives</td>
<td></td>
</tr>
<tr>
<td>$F_e$</td>
<td>Control force supplied by active force generator</td>
<td>N</td>
</tr>
<tr>
<td>$F_e^*$</td>
<td>Optimal controls for active system</td>
<td>N</td>
</tr>
<tr>
<td>$H$</td>
<td>Hamiltonian function</td>
<td></td>
</tr>
<tr>
<td>$J$</td>
<td>Dynamic cost</td>
<td></td>
</tr>
<tr>
<td>$K$</td>
<td>Ricatti gain matrix</td>
<td></td>
</tr>
<tr>
<td>$L$</td>
<td>Cost function</td>
<td></td>
</tr>
<tr>
<td>$L_1$</td>
<td>Tyre deformation term in non-quadratic cost function</td>
<td></td>
</tr>
<tr>
<td>$L_2$</td>
<td>Suspension deflection term in non-quadratic cost function</td>
<td></td>
</tr>
<tr>
<td>$n_1$</td>
<td>Index used in non-quadratic cost function</td>
<td></td>
</tr>
<tr>
<td>$n_2$</td>
<td>Index used in non-quadratic cost function</td>
<td></td>
</tr>
<tr>
<td>$P$</td>
<td>Ricatti state to costate transformation matrix</td>
<td></td>
</tr>
<tr>
<td>$p$</td>
<td>Costate vector</td>
<td></td>
</tr>
<tr>
<td>$p_1, p_2, p_3, p_4$</td>
<td>Individual costates</td>
<td></td>
</tr>
<tr>
<td>$t_0$</td>
<td>Time at start of optimisation</td>
<td></td>
</tr>
<tr>
<td>$t_f$</td>
<td>Time at end of optimisation (Horizon time)</td>
<td></td>
</tr>
<tr>
<td>$u$</td>
<td>Vector of control inputs</td>
<td></td>
</tr>
<tr>
<td>$u^*$</td>
<td>Vector of optimal controls</td>
<td></td>
</tr>
<tr>
<td>$x$</td>
<td>State vector for general system</td>
<td></td>
</tr>
</tbody>
</table>
3. Introduction to Controller Design for an Ideal Active System.

In §2, a model of a quarter vehicle employing a conventional passive suspension system was introduced and the following three chapters are concerned with the design of the controller for an active suspension system to be employed in this model. The design of the controller is to be considered from fundamental principles and a control strategy based on the concepts of optimal control is developed. By imposing certain constraints and making certain assumptions, the strategy can be implemented with using the standard LQR controller design technique and the performance of the quarter vehicle model employing an active suspension system controlled by such a controller is studied. It shows that the constraints placed upon the design to permit the use of this technique restrict the system performance and this motivates a more general implementation of the control strategy. This requires the numerical solution of a nonlinear optimisation problem and a method for choosing a suitable feedback law which are subsequently studied in the following chapters. Hence, the nonlinear controller design methodology is developed from first principles and is demonstrated for this particular quarter vehicle model employing this ideal active suspension system.

The precise performance requirements of the model of the quarter vehicle employing an active suspension system are not known and the main purpose of this study is to illustrate the controller design methodology. The quarter vehicle performance guidelines set by the model with the passive suspension system in the previous section provide a target and some potential performance improvements permitted by the use of an active system are also to be illustrated.

As stated in the introduction, this active system is assumed to be ideal allowing a fundamental analysis of the performance of active suspensions and a comparison of controllers to be carried out. In reality there would be limits on the amplitude of the force that could be applied and the force generator would also have internal dynamics, so it could not instantly supply the prescribed forces and would have limits on the frequency range over which it could operate. However, for this study, the hardware is assumed to be ideal and the force generator instantly supplies the prescribed force.

The suspension force, $F_s$, provided by the spring and damper of the passive suspension system in the quarter vehicle model of §2 is now to be replaced by a force, $F_c$, provided by an ideal active force generator (Fig 3.1). The force to be applied is prescribed by the control signal from the controller which is in turn based on the values of the states.
Replacing $F_s$ by $F_c$ in the state equations for the model (2.1.2) gives

$$
\begin{align*}
    x_1 &= V_0 - x_3 \\
    x_2 &= x_3 - x_4 \\
    x_3 &= \frac{F_t - F_c}{m_w} \\
    x_4 &= \frac{F_c}{m_b} 
\end{align*}
$$

The performance of the quarter vehicle model is to be optimised by the design of the controller for the active suspension system. The performance of the quarter vehicle model was studied in the previous section and the principal aims were conceived to be to provide good ride and handling whilst operating within the limits placed on the workspace. These qualities can be quantified by the levels of undesirable effects; i.e. body accelerations for ride, tyre deformations for handling and suspension deflections for workspace limitations. A suitable controller design methodology for optimising such system performance requirements is that of time domain optimal control. This produces a control strategy which seeks to minimise the time integrals of such undesirable effects. This control strategy requires the knowledge of all the states of the system and for the scope of this study, it is assumed that the controller is supplied with accurate information of the values of these states.
It has already been noted that there are inherent trade-offs between ride quality, handling ability and workspace usage. The balance between these attributes of the performance can be dictated by parameters in the control strategy and hence, this balance must be adjustable within the controller design methodology such that the relative importance placed on each of the undesirable effects dictates the design of the controller. An iterative process of varying these importances can be performed to achieve the desired balance, and this is known as 'tuning the controller'.

3.1 General Concepts of Time-Domain Optimal Control and Development of Control Strategy.

Let us review some of the general concepts of time-domain optimal control and develop a control strategy. Take a system model defined by the state differential equations:

\[ \dot{x}(t) = f[x(t), u(t), d(t)] \quad (3.1.1) \]

where \( x \) is the vector of state variables of the system, \( u \) is the vector of control inputs to the system, and \( d \) is the vector of disturbing inputs. Let us consider a period, \( t_0 \leq t \leq t_f \), where the initial state values, \( x(t_0) \), and the disturbances, \( d(t) \), are known. The performance of the system is assessed via a cost function, \( L[x(t), u(t)] \), which is a function of both the state variables and the controls. This is chosen to accumulate all the undesirable effects on the states caused by the disturbances and the costs of the control actions into a single function. This cost function is then integrated over the period to form the dynamic cost, \( J \), where

\[ J = \int_{t_0}^{t_f} L[x(t), u(t)] \, dt . \quad (3.1.2) \]

The optimal controls \( u^*(t) \), \( t_0 \leq t \leq t_f \), are the values of \( u(t) \) which minimise \( J \). These general concepts can be used to develop a control strategy to relate the control inputs to the states of the system.

For a given state of the quarter vehicle system model, \( x(t) \), a control strategy specifies the control input, \( u(t) \), to be applied. The state, \( x(t) \), can be considered to be the initial condition, \( x(t_0) \), for an optimisation of \( J \) for a given cost function. Then, given knowledge or predictions of the forthcoming disturbances, \( d(t) \), \( t_0 \leq t \leq t_f \), the optimal controls, \( u^*(t) \), can be evaluated. The initial optimal control value, \( u^*(t_0) \), relates to this initial state defining the control strategy at this point.

This procedure requires information about forthcoming road disturbances and this can
be gained from preview techniques. However, as discussed in the introduction, the use of such techniques is beyond the scope of this work and here, the choice of controls for unknown forthcoming road disturbances must be considered.

It would be possible to make predictions about these disturbances using a stochastic model of the road profile and to minimise the expected value of the dynamic cost, \( E(J) \), for \( t_0 \leq t \leq t_f \). However, this approach would require knowledge of parameters in a statistical model of the prevailing road and would produce a control strategy particularly suited to roads fitting this chosen model. Furthermore, such an approach is conceptually likely to lead to a control strategy which is unable to adequately handle an unexpected sudden event disturbance such as a brick or pothole. A simpler and more robust approach is to make a minimum number of assumptions about the forthcoming disturbances and to apply the control which is optimal for a state regulator which assumes no forthcoming disturbances, i.e. assume \( d(t) = 0 \) for \( t > t_0 \).

The control decision made at time, \( t_0 \), by such a regulator is based upon the analysis of the ensuing behaviour of the system, and ideally, this behaviour would be evaluated over infinite time. However this would present a numerical optimisation problem that is not practically possible. Hence, the control decision must be based on the analysis of the ensuing behaviour for a finite period of time, \( t_f - t_0 \). This concept is known as 'receding horizon control' (Banks, 1986) and if \( t_f - t_0 \) is sufficiently high then the control decision is sufficiently close to that made with an infinite horizon.

To summarise, the control strategy relates the control input to the current state of the system. At any state the controller should apply the control which minimises \( J \), evaluated over a finite time horizon, \( t_f - t_0 \), assuming no forthcoming disturbances. This is known as a finite time optimal state regulator.

### 3.2 General Optimal Regulator Design.

In this section the solution of the optimal regulator problem from a given initial condition is studied and the open-loop series of controls applied by an optimal state regulator evaluated. The concept of optimal regulator design is described in detail by Bryson and Ho (1969), and a summary of the analysis follows.

There are now no disturbing inputs giving state equations:

\[
\dot{x}(t) = f [x(t), u(t)] \tag{3.2.1}
\]

and for simplicity, \( u(t) \) is a single control input. Also, \( t_0 \) may be set to zero without loss of generality. These form the constraint equations for the optimisation of the
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dynamic cost, \( J \), where

\[
J = \int_0^T L[x(t), u(t)] \, dt .
\] (3.2.2)

Adding the constraint equations to this with a vector of Lagrange multiplier functions, \( p(t) \), gives,

\[
J = \int_0^T \left( L(x(t), u(t)) + p^T(t)[f(x(t), u(t)) - \dot{x}(t)] \right) \, dt .
\] (3.2.3)

The Lagrange multiplier functions \( p(t) \) are usually called costate functions. Defining the Hamiltonian function:

\[
H[x(t), u(t), p(t)] = L[x(t), u(t)] + p^T(t)[f(x(t), u(t)) - \dot{x}(t)] ,
\] (3.2.4)

(3.2.3) may be rewritten as

\[
J = \int_0^T \left( H(x(t), u(t), p(t)) - p^T(t)\dot{x}(t) \right) \, dt .
\] (3.2.5)

Integrating the second term by parts gives

\[
J = p^T(o)x(o) - p^T(t)\dot{x}(t) + \int_0^T \left( H(x(t), u(t), p(t)) + p^T(t)\dot{x}(t) \right) \, dt .
\] (3.2.6)

Consider the small changes in dynamic cost \( \delta J \), caused by small changes in the controls \( \delta u(t) \), and in the state trajectories \( \delta x(t) \):

\[
\delta J = p^T(0)\delta x(0) - p^T(t)\delta x(t) + \int_0^T \left( \frac{\partial H}{\partial x} \delta x(t) + \frac{\partial H}{\partial u} \delta u(t) + \dot{p}^T(t)\dot{x}(t) \right) \, dt .
\] (3.2.7)

\[
= p^T(0)\delta x(0) - p^T(t)\delta x(t) + \int_0^T \left( \frac{\partial H}{\partial x} + \dot{p}^T(t) \right) \delta x(t) + \frac{\partial H}{\partial u} \delta u(t) \right) \, dt
\]

The costates can be chosen such that \( \delta J \) depends only on changes in the controls without having to consider the changes in the states by imposing the following conditions on \( p(t) \):
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\[ p^T(t) = - \frac{\partial H}{\partial x}, \quad p^T(u) = 0, \quad (3.2.8) \]

hence,

\[ \delta J = p^T(0) \delta x(0) + \int_0^t \frac{\partial H}{\partial u} \delta u(t) \ dt. \quad (3.2.9) \]

The open loop series of controls \( u(t) \) which minimises the dynamic cost \( J \), for given initial state values is sought, hence, \( \delta x(0) = 0 \), leaving simply

\[ \delta J = \int_0^t \frac{\partial H}{\partial u} \delta u(t) \ dt. \quad (3.2.10) \]

To minimise \( J \), this expression must be zero for arbitrary change \( \delta u(t) \), and this is true if and only if

\[ \frac{\partial H}{\partial u} = 0, \quad \forall \ t \quad (3.2.11) \]

which is a sufficient and necessary condition for an extremum. To verify that this extremum is a local minimum then the second order variations of \( J \) must be considered see Bryson and Ho (1969).

It is also noted from (3.2.9) that the change in dynamic cost, due to a small change in initial state, \( \delta x(0) \), is given by

\[ \delta J = p^T(0) \delta x(0) \quad (3.2.12) \]

provided the controls \( u(t) \) are either held constant, or are optimised to satisfy (3.2.11). This allows each initial costate value to be interpreted as the local rate of change of dynamic cost due to a change of the corresponding initial state value subject to the stated conditions.

3.3 LQR Optimal Regulator Design.

The most mathematically convenient method of implementation of this control strategy is via the LQR technique; for Linear system models and Quadratic cost functions, this technique designs optimal state Regulators. The system model for the quarter vehicle is a linear one. Hence, if the cost function is chosen to be quadratic this technique can be
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implemented.

To achieve the performance goals, terms costing tyre deformations, suspension deflections and body accelerations must be included. Hence consider a cost function of the form

\[ L[x(t), F_c(t)] = \alpha x_1(t)^2 + \beta x_2(t)^2 + \dot{x}_d(t)^2. \]

Substituting for body acceleration, \( \dot{x}_d(t) \), from the state equations (3.1) gives

\[ L[x(t), F_c(t)] = \alpha x_1(t)^2 + \beta x_2(t)^2 + \left( \frac{F_d(t)}{m_b} \right)^2 \]

which is quadratic in states and controls and includes the appropriate terms.

The solution of the optimal regulator problem with such a cost function gives the Hamiltonian (3.2.4) as

\[ H(t) = \alpha x_1(t)^2 + \beta x_2(t)^2 + \left( \frac{F_d(t)}{m_b} \right)^2 - p_1(t)x_3(t) + p_2(t)x_3(t) - p_2(t)x_4(t) + \frac{p_3(t)}{m_w} (F_d(t) - F_c(t)) + \frac{p_4(t)}{m_0} F_c(t) \]

and hence costate equations from (3.2.8),

\[
\begin{align*}
\dot{p}_1(t) &= -2\alpha x_1(t) - \frac{p_3(t)}{m_w} F_c'(x_1(t)) = -2\alpha x_1(t) - \frac{p_3(t)K_f}{m_w} \\
p_2(t) &= -2\beta x_2(t) \\
p_3(t) &= p_1(t) - p_2(t) \\
p_4(t) &= p_2(t)
\end{align*}
\]

with boundary values,

\[ p^T(\omega) = 0 . \]

For optimal response the control forces, \( F_d(t) \), must satisfy (3.2.11), hence

\[ F_c^*(t) = \frac{m_b^2}{2} \left( \frac{p_3(t)}{m_w} - \frac{p_4(t)}{m_b} \right) \]

where, \( F_c^*(t) \), denotes an optimal control force.
To compute the optimal regulator responses and hence implement the control strategy for this system from given initial conditions with respect to such a cost function, the interdependent system of state and costate differential equations must be solved. The initial conditions of the states and the final values of the costates are given and this type of problem is known as a two point boundary value problem. The quadratic cost function yields linear costate equations which combined with the linear state equations and ideal nature of the control force generator give a two point boundary value problem which is linear. The mathematical convenience of the linearity of this problem permits relatively easy solution, for example, by the linear perturbation techniques discussed in §4.1 and §4.2. However, the optimal controls are more usually derived via the solution of the matrix Ricatti equation (Kwakernaak and Sivan, 1972), which utilises the linear relationship, \( p = Px \), between the states and costates.

Under the LQR conditions, the solution of the infinite time regulator problem yields a control strategy in which the controls are related to the states by a constant linear feedback law. This can be expressed in the form:

\[
F_c(x) = -Kx
\]

(3.3.7)

where \( K \) is known as the optimal gain matrix and can be found from the solution of the matrix Ricatti equation. Numerical algorithms for this problem are commonly available in modern controller design software packages such as P-Matlab or MatrixX. Hence for a given cost function, \( K \) can easily be found and the optimal response of the system to a given initial disturbance easily simulated.

### 3.4 Tuning of LQR Controller and Performance Analysis.

The controller design is now only dependent upon the cost function chosen and the only free parameters in this are the relative weightings \( \alpha \) and \( \beta \). The controller is tuned by making appropriate choices for these. Initially, a controller is required that optimises the ride performance whilst meeting the performance guidelines on handling and workspace usage. This tuning procedure resulted in weightings of \( \alpha = 100000 \) and \( \beta = 1000 \) and the resulting controller is labelled Linear Controller 1.

The eigenvalues of the closed loop system incorporating this controller are \(-3.75 \pm 3.90i\) and \(-14.9 \pm 75.9i\) for the body bounce and wheel hop modes with corresponding eigenvectors \( v_1 \) and \( v_2 \) where:
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\[ v_1 = \begin{pmatrix} -0.005 + 0.005i \\ 0.133 + 0.128i \\ 0.001 + 0.038i \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 0.002 + 0.013i \\ -0.002 - 0.013i \\ 1 \end{pmatrix} \]

The actual wheel hop mode of this system is very similar to that of the quarter vehicle model employing the passive suspension system, however, the body bounce mode is significantly changed. Although the eigenvector defining the overall mode shape is fairly similar, the eigenvalue governing the dynamic behaviour is quite different. The damping ratio is now 0.69 compared with just 0.21 for the passive and the 98% settling time is reduced from 2.42 seconds to 1.07 seconds and the undamped natural frequency from 1.27Hz to 0.86Hz. Such changes to the precise modes of the quarter vehicle model are expected for different suspension systems, however, as mentioned in §2, the behavioural conditions to which they relate remain broadly unchanged.

The performance of the quarter vehicle model employing this linear active controller in response to the extreme initial disturbances has been simulated and is compared with the results when employing the passive system. Fig 3.4.1 shows the wheel hop condition and Fig 3.4.2 the body bounce and the results have also been collated in Table 3.4.

It can be seen that the performance guideline relating to the wheel hop condition has been met with a lower peak tyre deformation observed. The rms results show that the behaviour of this system is very similar to that of the passive for this disturbance with a small reduction in tyre deformations traded for a small increase in body accelerations. This similarity is expected since the governing eigenvalues and eigenvectors are similar.

For the body bounce condition disturbance, the greater damping ratio and shorter settling time of the dominant mode are apparent in the plots. Initially, much greater suspension forces are applied to retard the motion of the body, and this results in increased levels of tyre deflection and body acceleration during the initial phase. However, this permits a much quicker attenuation of the dynamics and results in significant reductions in the rms levels calculated over the first second. Again the performance guideline has been met with a slightly smaller peak suspension deflection.

These results relating to these extreme amplitudes of disturbance are satisfactory but responses to disturbances of a more reasonable amplitude also need to be considered. The responses to initial disturbances a third the magnitude of the extreme wheel hop disturbance and half the magnitude of the body bounce disturbance have also been simulated for both passive and active systems and the results are included in the Table. Since the systems are both linear the results are simply scaled with respect to the extreme amplitude results. However the wisdom of the linear controller is brought into
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question here since the ability to provide good ride quality is unnecessarily impeded in both conditions. In the wheel hop condition, excessive costing of tyre deformations places unnecessary restrictions on the system in a situation where they are too low to critically affect the handling ability. Similarly, in the body bounce condition unnecessary restrictions are placed on workspace usage when bump stop contact is not likely.

To illustrate the improvements in ride that are possible by the removal of such unnecessary restrictions, two further linear controllers have been designed. Linear Controller 2 has a lower weighting on tyre deformations in its cost function with \( \alpha \) set at half its original value and this should improve the ride quality in the wheel hop condition. Similarly, Linear Controller 3 seeks to improve ride quality in the body bounce condition by using a lower weighting on the term costing suspension deflections. For this controller, \( \beta \) is halved and \( \alpha \) reset to its original value. The responses to the initial disturbances of these controllers have also been simulated and the results included in Table 3.4. Figs 3.4.3 and 3.4.4 compare these responses with those of Linear Controller 1 and the passive system following the low amplitude disturbances in wheel hop condition and body bounce condition respectively.

It is apparent that Linear Controller 2 is able to make a significant improvement in ride quality by permitting greater levels of tyre deformation following the wheel hop condition disturbances. This increase is not believed to represent a significant degradation in handling performance at this low amplitude of disturbance, however, it is unable to meet the performance guidelines at the extreme amplitude. Similarly, Linear Controller 3 provides improved ride by permitting increased workspace usage. Again this is acceptable at low amplitudes but the performance guidelines are exceeded at the extreme amplitudes.

These two controllers illustrate the possibility of significantly improving the ride quality at non-extreme amplitudes of disturbance by changing the cost function. It also shows that the LQR technique is unable to make such improvements whilst adhering to the performance guidelines. This is due to the inflexibility of the imposed quadratic structure of the cost function.
Wheel Hop Condition

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>R.M.S values for first 0.5 seconds</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tyre Deformation (mm)</td>
<td>Suspension Deflection (mm)</td>
<td>Body Acceleration (m/s²)</td>
<td>Peak Tyre Deformation (mm)</td>
<td></td>
</tr>
<tr>
<td>Extreme Amplitude</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>6.16</td>
<td>6.35</td>
<td>1.92</td>
<td>25.3</td>
<td></td>
</tr>
<tr>
<td>Active : Linear Controller 1</td>
<td>5.96</td>
<td>6.43</td>
<td>1.98</td>
<td>24.9</td>
<td></td>
</tr>
<tr>
<td>Linear Controller 2</td>
<td>7.15</td>
<td>7.28</td>
<td>1.66</td>
<td>27.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Third Amplitude</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>2.05</td>
<td>2.12</td>
<td>0.640</td>
<td>8.43</td>
<td></td>
</tr>
<tr>
<td>Active : Linear Controller 1</td>
<td>1.99</td>
<td>2.14</td>
<td>0.660</td>
<td>8.30</td>
<td></td>
</tr>
<tr>
<td>Linear Controller 2</td>
<td>2.38</td>
<td>2.43</td>
<td>0.553</td>
<td>9.00</td>
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Body Bounce Condition

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>R.M.S values for first second</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tyre Deformation (mm)</td>
<td>Suspension Deflection (mm)</td>
<td>Body Acceleration (m/s²)</td>
<td>Peak Suspension Deflection (mm)</td>
<td></td>
</tr>
<tr>
<td>Extreme Amplitude</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>5.47</td>
<td>55.6</td>
<td>4.18</td>
<td>110</td>
<td></td>
</tr>
<tr>
<td>Active : Linear Controller 1</td>
<td>4.14</td>
<td>59.5</td>
<td>3.00</td>
<td>106</td>
<td></td>
</tr>
<tr>
<td>Linear Controller 3</td>
<td>3.75</td>
<td>77.3</td>
<td>2.74</td>
<td>127</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Half Amplitude</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>2.74</td>
<td>27.8</td>
<td>2.09</td>
<td>55.0</td>
<td></td>
</tr>
<tr>
<td>Active : Linear Controller 1</td>
<td>2.07</td>
<td>29.8</td>
<td>1.50</td>
<td>53.0</td>
<td></td>
</tr>
<tr>
<td>Linear Controller 3</td>
<td>1.88</td>
<td>38.7</td>
<td>1.37</td>
<td>63.5</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.4 : Response Summary for Initial Disturbance Simulations
Fig 3.4.1: Extreme Amplitude Wheel Hop Condition

- Passive
- Linear Controller 1
Fig 3.4.2: Extreme Amplitude Body Bounce Condition

---

**Passive**

---

**Linear Controller 1**
Fig 3.4.3: Third Amplitude Wheel Hop Condition

- Passive
- Linear Controller 1
- Linear Controller 2
Fig 3.4.4: Half Amplitude Body Bounce Condition

- Passive
- Linear Controller 1
- Linear Controller 3
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One possible way of alleviating this problem is by using an adaptive controller design technique. This would enable different feedbacks resulting from different cost functions to be employed at different disturbance amplitudes. For example, Linear Controller 2 could be employed at low amplitudes of wheel hop condition disturbance switching to Linear Controller 1 at high amplitudes. Possible limitations of techniques of this nature were studied in Gordon, Marsh & Milsted (1991), the approach was not favoured for several reasons including:

- It was found that the choice of robust switching criteria between the various feedbacks required lengthy observations of the vehicle motion.

- Poor or even dangerous results occurred if an unexpected large disturbance was encountered whilst operating at a feedback suited to low levels of disturbance.

- Additional algorithms would be required to circumvent such problems which would result in ad-hoc behaviour and would be hard to justify within any systematic framework.

A preferred approach is to adopt a more flexible structure for the cost function and aim to tune the system to automatically alter the characteristics of the quarter vehicle in line with changes in the road amplitude.

3.5 Use of Non-Quadratic Cost Function.

Consider the addition of some high order terms costing tyre deformation and suspension deflection giving a cost function of the form:

\[ L\{x(t), F_c(t)\} = L_1(x_1(t)) + L_2(x_2(t)) + \left( \frac{F_c(t)}{m_0} \right)^2, \]  

(3.5.1)

where

\[ L_1(x_1(t)) = \alpha x_1(t)^2 + \alpha_1 x_1(t)^{n_1} \quad \text{and} \quad L_2(x_2(t)) = \beta x_2(t)^2 + \beta_1 x_2(t)^{n_2} . \]

The indices \(n_1\) and \(n_2\) can be set to values which are greater than 2 giving \(L_1\) and \(L_2\) the form depicted in Fig 3.5.1. The aim is to promote good ride quality at low amplitudes of disturbance where the additional high order terms will have relatively little effect, by the choice of low values for \(\alpha\) and \(\beta\). The terms will however have a significant effect at high amplitudes and a suitable choice of parameters will enable the 'worst case'...
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performance guidelines to be met.

\[ \text{Cost} \]

\[ \text{Quadratic Term} \quad \text{High Order Term} \quad \text{Total Cost} \]

Fig 3.5.1: Shape of Non Quadratic Cost Function.

It is not possible to generate the optimal controls with respect to cost functions of this structure by the simple methods employed in LQR conditions. The boundary value problem is no longer a linear one and its solution is not trivial nor is there such a simple linear relationship between the states and the optimal controls. The design of nonlinear controllers to implement strategies relating to cost functions of this nature is now to be addressed.

3.6 Definition of Design Problem for Nonlinear Controllers.

To implement the optimal control strategy for such a given cost function, it is essential to relate desired control inputs to state values. For a given state this relationship can be found by the solution of the optimal regulator problem. This problem is now nonlinear and its solution is fully addressed in §4. Its solution will be a basic tool used in the controller design methodology.

It is found that this problem can only be solved efficiently if the control inputs are discretised by applying a zero order hold to each element in the open loop series. This makes a slight modification of the control strategy necessary and the control strategy used will be a discrete time one rather than a continuous time one. However, the length of the zero order hold periods has been chosen to be sufficiently short to have negligible effect on the overall behaviour of the system, see §4.6.1.
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The full controller design methodology is completed in §5 where the development of a nonlinear feedback controller to implement the strategy is considered. The performance aims of this particular nonlinear controller are motivated in this chapter; to meet the performance guidelines and to provide improved ride at low amplitudes. However, the wider aim of the design methodology is to be able to produce a controller to meet whatever performance requirements are given. As for the linear system, the performance of the nonlinear system is to be adjusted via a tuning procedure driven by modification of the parameters in the cost function.
Chapter 4

Solution of Nonlinear Optimal Regulator Problem.

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Nomenclature.

<table>
<thead>
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<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0, t_1, t_2, ..., t_n$</td>
<td>Discrete times for control sequence</td>
</tr>
<tr>
<td>$T$</td>
<td>Zero order hold period</td>
</tr>
<tr>
<td>$u_0, u_1, u_2, ..., u_{n-1}$</td>
<td>Zero order hold control sequence</td>
</tr>
</tbody>
</table>
4. Solution of Nonlinear Optimal Regulator Problem.

In §3 the design of the controller for an active suspension system was considered and a control strategy was proposed based on fundamental principles. This strategy requires solutions of the optimal regulator problem. For linear systems with quadratic cost functions these solutions were calculated with relative ease but for general non-quadratic cost functions the mathematics of the solution of the optimal regulator problem is less trivial since the problem becomes nonlinear. This chapter is devoted to the solution of such problems which form an integral part of the overall controller design methodology.

Here, a non quadratic cost function whose structure is given in (3.5.1) is to be considered; the Hamiltonian is given by:

\[ H(t) = L_1(x_1(t)) + L_2(x_2(t)) + \left( \frac{F_c(t)}{m_b} \right)^2 - p_1(t)x_3(t) + p_2(t)x_3(t) - p_2(t)x_4(t) + \frac{p_3(t)(F(t) - F_c(t)) + p_4(t)F_c(t)}{m_w} \]

from (3.2.4), and similarly the costate equations (3.2.8) take the form

\[ \begin{align*}
    p_1(t) &= -L_1'(x_1(t)) - \frac{p_3(t)K_i}{m_w} \\
    p_2(t) &= -L_2'(x_2(t)) \\
    p_3(t) &= p_1(t) - p_2(t) \\
    p_4(t) &= p_2(t)
\end{align*} \]  

with boundary values

\[ p^T(t) = 0 \]  

Also, from (3.2.11), the optimal controls are given by

\[ F_c^*(t) = \frac{m_b^2}{2} \left( \frac{p_3(t)}{m_w} - \frac{p_4(t)}{m_b} \right) \]  

It is seen that the non quadratic nature of the cost function causes the costate equations to be nonlinear and hence yields a nonlinear two point boundary value problem.

The solution of this problem is sought in §4.1-5 and the resulting performance of the optimal nonlinear regulators examined in §4.6. Initially, some standard solution
Solution of Nonlinear Optimal Regulator Problem.

Techniques are analysed. Both the 'Shooting Method' (§4.1) and the 'Backwards Sweep Method' (§4.2) are linear perturbation methods and are ultimately unable to cope efficiently with the nonlinearity of the present problem. They are also hampered by the highly unstable nature of the costate equations, and this is also true for the 'Direct Optimisation' method (§4.3). Using the method described in §4.4, approximate solutions to the regulator problem can be found; however, this method is also ultimately unsatisfactory since it is not possible to specify the initial states from which solutions are obtained. The method in §4.5 which involves discretising the control inputs gives satisfactory solutions and this method is ultimately used to solve the problem. The 'unsuccessful' methods are now summarised for completeness; further details are available elsewhere (Marsh, Gordon & Milsted, 1989 and Bryson & Ho, 1969).

4.1 Shooting Method.

Here guesses are made for the initial costate values; these added to the known initial state values form a complete initial condition. The differential equations can be solved forward from this point and the final conditions noted. Generally, the final costates reached will not be zero and hence, the state/costate trajectory is not optimal. The shooting method perturbs the guess made for each initial costate value independently and repeats the solution noting the final costate values. This enables the transition matrix (which approximates the mapping from initial costates to final costates) to be evaluated. This matrix is then used to extrapolate for the initial costate values that will produce zero final costate values and hence an optimal trajectory. This process is then iterated until a solution is achieved. This method is guaranteed to work without iteration for systems of linear differential equations, but for our problem it was found that the level of nonlinearity combined with the instability of the equations prevented the optimal trajectory from being found.

4.2 Backwards Sweep Method.

This method is conceptually similar to the 'Shooting Method' described above; it also involves an initial guess for a point on the optimal trajectory, solves the equations and from the solution obtained extrapolates for the optimal solution. The method is more efficient and more robust to nonlinearities and instabilities than the shooting method. However, solution was dependent upon the proximity of the initial guess to the optimal trajectory and generally solutions could only be achieved with some prior knowledge. This was gained by initially solving for the underlying linear boundary value problem with a quadratic cost function. The high order term in the cost function was then slowly added until the required cost function was reached. These successive solutions provided good initial guesses for the following case. For each solution an acceleration parameter was used, the value of which must also be optimised adding a further
complication. Overall, as for the shooting method, the level of nonlinearity and instability in our problem make this method inappropriate for use here.

4.3 Direct Optimisation.

Here the aim is to find the optimal trajectory by optimising the initial costate values to achieve the minimum dynamic cost. This method has the advantage that no assumptions are made about the linearity of the equations and a general nonlinear parameter optimisation problem is posed. However, difficulties arise due to the unstable nature of the equations causing the cost surface to be highly irregular. This makes cost gradient information evaluated at any particular point unreliable and also causes many local minima. Some success was had in achieving an optimal solution using downhill simplex optimisation (Press et al, 1986, Nelder & Mead, 1965). This is a particularly robust optimisation technique that does not rely upon cost gradient information but it is not particularly efficient in terms of its asymptotic rate of convergence.

The irregularity of the cost surface is a function of the time period over which an optimal solution is sought, solutions for short time periods can be obtained relatively easily. These can then be used as intelligent initial guesses when considering longer time periods. This approach is not wholly reliable as under certain circumstances, the nature of the problem can change dramatically as the time period is increased. An approach which enabled some solutions to be achieved was one of restarting the solution, (Marsh et al, 1989). Generally, solutions could not efficiently be found using this method due to the unstable nature of the costate equations. This leads to a highly sensitive relationship between the initial costate values and the resulting trajectory. Such a high sensitivity is very difficult to handle numerically with sufficient accuracy to enable solutions to be obtained.

4.4 Approximate Solution in Backwards Time.

The previous methods all incurred problems caused by both the nonlinear and unstable nature of the equations. The problem can be approximated to a linear one for small values of the states since the quadratic terms in the cost function are dominant here. Hence, if a very small initial state vector is chosen, approximate initial costates can be evaluated from linear theory. This point can now be considered as an intermediate point on an approximately optimal trajectory from a high amplitude initial state. The nonlinear equations can be integrated backwards in time from this intermediate point to evaluate a complete approximation to a solution of the nonlinear problem.
Solution of Nonlinear Optimal Regulator Problem.

A solution is required from a stated initial condition and each chosen intermediate point yields a sample of approximate solutions to the nonlinear problem but the exact location of these solutions can not easily be determined apriori. In principle there is a complete mapping to the whole of the state space from the chosen intermediate point; however, this mapping is ill conditioned due to the dominant 'mode' shapes, (n.b. the system does not strictly have well defined modes due to its nonlinearity but it still has behavioural patterns). This causes a natural tendency for solutions to lie in certain regions of the state space. Numerical inaccuracies in the solution of the differential equations in backwards time combined with the ill conditioning of the mapping make it practically impossible to generate solutions in significant areas of the state space. Since the aim is to solve the nonlinear optimal regulator problem from given initial state values, this restriction is unacceptable.

4.5 Parameter Optimisation for Discrete Optimal Controls.

Previous methods have sought to solve the optimal regulator problem via the solution of the two point boundary value problem. The optimal controls which solve this problem and minimise the dynamic cost are continuous time functions. Here, an approximation to this problem will be considered where the controls are discretised. They are now to be zero order held for periods of T seconds between times $t_0, t_1, t_2, ..., t_n$ (Fig 4.5.1) and if $T$ is sufficiently small then the solution of this discretised problem is a good approximation to the solution of the continuous time problem.

Fig 4.5.1: Discretisation of Control Inputs.
Solution of Nonlinear Optimal Regulator Problem.

The optimal controls can now be specified as a finite set of parameters $u_0, u_1, u_2, \ldots, u_{n-1}$ which can be chosen to minimise $J$ by a standard parameter optimisation technique. Let us consider the detail of this optimisation, for given initial states, an extremum is reached when $\delta J = 0 \quad \forall \delta u(t)$. From (3.2.10)

$$\delta J = \int_{t_0}^{t_f} \frac{\partial H}{\partial u} \delta u(t) \, dt , \quad (4.5.1)$$

and for continuous $u(t)$ this requires:

$$\frac{\partial H}{\partial u} = 0, \quad \forall \quad t \quad (4.5.2)$$

from 3.2.11, however, now that the controls are discretised, this can be re-expressed as:

$$\delta J = \int_{t_0}^{t_1} \frac{\partial H}{\partial u_0} \delta u_0(t) \, dt + \int_{t_1}^{t_2} \frac{\partial H}{\partial u_1} \delta u_1(t) \, dt + \ldots + \int_{t_{n-1}}^{t_n} \frac{\partial H}{\partial u_{n-1}} \delta u_{n-1}(t) \, dt , \quad (4.5.3)$$

and an optimal solution now requires that

$$\int_{t_i}^{t_{i+1}} \frac{\partial H}{\partial u_i} \, dt = 0 , \quad (i=0,1,2,\ldots,n-1) . \quad (4.5.4)$$

The cost gradients with respect to the parameters can be found from (4.5.3) by bringing the changes in the controls, $\delta u_0, \delta u_1, \delta u_2, \ldots, \delta u_{n-1}$, outside the integrals since they are constant over the periods concerned, thus:

$$\delta J = \delta u_0 \int_{t_0}^{t_1} \frac{\partial H}{\partial u_0} \, dt + \delta u_1 \int_{t_1}^{t_2} \frac{\partial H}{\partial u_1} \, dt + \ldots + \delta u_{n-1} \int_{t_{n-1}}^{t_n} \frac{\partial H}{\partial u_{n-1}} \, dt . \quad (4.5.5)$$

Hence, the cost gradients are given by:

$$\frac{\partial J}{\partial u_i} = \int_{t_i}^{t_{i+1}} \frac{\partial H}{\partial u_i} \, dt , \quad (i=0,1,2,\ldots,n-1) . \quad (4.5.6)$$

This allows a gradient based optimisation algorithm to be employed.
The solution of optimisation problems of this nature is extensively discussed by Press et al (1986). A basic algorithm for parameter optimisation when cost gradient information is available is **steepest descent**. In this an initial guess is made for the parameter values and the cost and cost gradients evaluated for that point. The direction of most rapid decrease of cost in parameter space called the steepest descent direction is defined by the direction of the cost gradient vector. A line minimisation is then carried out in this direction and the solution of this gives a new point where the cost and cost gradients are re-evaluated. A new line minimisation is then performed along the new steepest descent direction. This process continues until the convergence criteria are met.

A recognised problem with first order gradient methods of this type is the slow rate of asymptotic convergence to the minimum. In Bryson and Ho (1969), the use of a second order method is suggested to overcome this problem. Another well recognised problem with the steepest descent algorithm is that it tends to perform many successive line searches along orthogonal directions. These problems can be overcome by the use of a **conjugate gradient** algorithm which is designed to eliminate the problem of repeated searches in orthogonal directions and gives convergence rates similar to those of second order methods. The implementation of this algorithm was found to greatly reduce the number of iterations required and hence speed up the solution.

To start the algorithm, initial guesses for \( u_0, u_1, u_2, \ldots, u_{n-1} \) must be made. Consider the linear feedback controller which optimises the LQR problem relating to the quadratic part of the cost function. Sampling the control force inputs employed by this controller gives a reasonable choice for these initial guesses. Using these guesses, the state equations are now integrated forwards from the given initial conditions and the total dynamic cost and final state values are recorded. These final state values along with the known final costate values from the boundary condition form a starting point for the backwards time integration of the costate equations. This gives the cost gradients with respect to the parameters from

\[
\frac{\partial J}{\partial u_i} = \int_t^{t_{n+1}} \frac{\partial H}{\partial u_i} \, dt = \int_t^{t_{n+1}} \left[ \frac{2u_i}{m_b} \frac{p_3(t)}{m_B} + \frac{p_4(t)}{m_b} \right] \, dt = \text{, i=0,1,2,...,n-1 } (4.5.7)
\]

The conjugate gradient optimisation algorithm can now be started from this point with this vector of cost gradients. The costs for the line minimisation are simply evaluated by integrating the state equations for the relevant set of controls and when new cost gradients are required, they can be found by subsequent backwards time integration of the costate equations.
Solution of Nonlinear Optimal Regulator Problem.

The convergence criterion considers the sum of the changes in discrete controls over successive solutions of the line minimisations. Once this quantity remains below the convergence threshold for 10 consecutive iterations then convergence is assumed.

The method is much more efficient than the previous ones but still takes a significant amount of computation. A typical solution requires 50-150 iterations and takes 4.5 minutes of cpu time on a Sun Sparc 2 workstation.

4.6 Performance of Optimal Nonlinear Regulator.

In this section solutions of the nonlinear optimal regulator problem are to be considered, they illustrate the deployment of the control strategy with respect to the non-quadratic cost function from §3.5. Initially, the following reference cost function parameter values are used:

$$\alpha = 16000, \alpha_1 = 2 \times 10^{12}, \beta = 500, \beta_1 = 5 \times 10^{11}, n_1 = 6 \text{ and } n_2 = 10$$

and the set of optimal control sequences produced with respect to this are denoted Nonlinear Regulators 1.

Responses have been evaluated for the extreme wheel-hop and body bounce condition initial disturbances from §2.4 and also for the non-extreme amplitude initial disturbances considered in §3.4. However, to evaluate the solutions, suitable choices must be made for $T$, the zero order hold period and $t_f$, the horizon time; these choices are analysed in §4.6.1. The performance of the optimal regulators for this cost function are compared with the performance of the passive and linear active systems for similar disturbances in §4.6.2.

These reference cost function parameter values have been carefully chosen since they give regulator responses which broadly meet the system performance targets, i.e. they meet the performance guidelines on handling ability and workspace usage and offer improved ride quality at non-extreme amplitudes of disturbance. However, a full performance analysis including simulations of forced responses can only be performed once a closed loop controller is designed, see §5, and such an analysis is presented in §6.

As previously stated, the methodology must be capable of tuning the controller to modify system performance via the adjustment of the cost function. To illustrate this ability, a preliminary investigation of the effects of cost function parameters on the behaviour of the optimal regulators is performed in §4.6.3. The knowledge gained
Solution of Nonlinear Optimal Regulator Problem.

from this can be used to guide the adjustment of the cost function parameters when tuning the controller.

4.6.1. Choice of the values of T and tf.

The choice of these values can be made independently of each other providing reasonable values are used. Initially, the horizon time tf was considered with T set at 1ms. Solutions to the problem were evaluated from both the wheel hop and body bounce initial disturbances for various values of tf. The resulting initial optimal control values are given in Table 4.6.1.

It can be seen that the horizon time chosen does affect the choice of initial optimal control and that the effect is greater in the body bounce condition which is slower settling. The greater the horizon time, the closer the value for the initial optimal control to that for the infinite time regulator. However, the amount of computation required to generate a solution also increases. From these results, it seems reasonable to use a horizon time of 1.5 seconds, though further investigation may show that a smaller value may cause little degradation in the control strategy deployed throughout the state space with significant savings in computational time.

<table>
<thead>
<tr>
<th>Horizon Time, tf (s)</th>
<th>Initial Optimal Control, u0, (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wheel Hop Condition</td>
<td>Body Bounce Condition</td>
</tr>
<tr>
<td>0.5</td>
<td>5153</td>
</tr>
<tr>
<td>0.75</td>
<td>5150</td>
</tr>
<tr>
<td>1.0</td>
<td>5149</td>
</tr>
<tr>
<td>1.5</td>
<td>5149</td>
</tr>
</tbody>
</table>

Table 4.6.1: Effect of Horizon Time on Initial Optimal Control.

Now consider the choice of the zero order hold period, T. It is required to be short enough to give solutions which are approximately equal to those for the continuous time optimal regulator. For this cost function and these initial disturbances, solutions with T = 10, 1 and 0.1 ms have been evaluated, and are presented in Figs 4.6.1 and
Solution of Nonlinear Optimal Regulator Problem.

4.6.2. The total dynamic costs for these simulations have also been calculated and are presented in Table 4.6.2. It can be seen that the solutions converge as $T$ is reduced and it is reasonable to conclude that a value of $T=1\text{ms}$ is sufficiently short.

<table>
<thead>
<tr>
<th>Zero Order Hold Period (ms)</th>
<th>Dynamic Cost, $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Wheel Hop Condition</td>
</tr>
<tr>
<td>10</td>
<td>5.090</td>
</tr>
<tr>
<td>1</td>
<td>4.868</td>
</tr>
<tr>
<td>0.1</td>
<td>4.866</td>
</tr>
</tbody>
</table>

Table 4.6.2: Effect of Zero Order Hold Period on Dynamic Cost.
Fig 4.6.1: Effect of Z.O.H. Period on Extreme Amplitude Wheel Hop Condition

- \( T = 10 \text{ms} \)
- \( T = 1 \text{ms} \)
- \( T = 0.1 \text{ms} \)
Fig 4.6.2: Effect of Z.O.H. Period on Extreme Amplitude Body Bounce Condition
4.6.2 Comparison with Passive and LQR.

Here, the responses produced by the nonlinear optimal regulators are compared with those produced by the passive system and by Linear Controller 1. The time histories are plotted in Figs 4.6.3-6 and the results data presented in Table 4.6.3.

For the extreme wheel hop condition it can be seen that the performance guideline has comfortably been met with a peak tyre deformation of just 19.3mm. The overall rms tyre deformations are slightly less and the body accelerations slightly greater than those of the other systems. A closer analysis of the time histories reveals how the rate of attenuation of wheel hop is amplitude dependent; at high amplitudes the attenuation is relatively quick and at low amplitudes the attenuation relatively slow. The rapid attenuation that occurs initially, up to say 0.025 seconds, restricts the peak tyre deformation to its low value but causes the large body accelerations. Thereafter, the rate of attenuation is slower and the body accelerations are lower. A further obvious difference between the responses is the low frequency drift of the suspension deflection of the nonlinear system. This causes a significant increase in the rms value. However, since the amplitude of these suspension deflections is small, this is not considered to degrade the performance.

The performance guideline has also comfortably been met for the extreme body bounce condition disturbance with a peak suspension deflection of just 86.6mm. In §3.4 it was found that the linear active system provided much greater damping and quicker settling in this condition than the passive system. Here, it can be seen that this is also true of the nonlinear active system. A more rapid initial attenuation of the motion which restricts the peak suspension deflection to a low level causes greater rms body accelerations than Linear Controller 1. However, once the amplitude becomes low the attenuation slows and the body accelerations become lower.

Following the third amplitude wheel hop condition it can be seen that the nonlinear system is capable of making a significant improvement in the ride quality. This is indicated by the significant reduction in body accelerations and as for Linear Controller 2, this is achieved at the acceptable expense of greater tyre deformations. Ride improvements are also apparent following the half amplitude body bounce disturbance and again extra workspace usage has been traded for these in a similar manner to Linear Controller 3.

To summarise the results of the nonlinear optimal regulators for this cost function; the performance guidelines have been comfortably met whilst significant improvements have been made in the ride quality at non-extreme amplitudes of disturbance.
Wheel Hop Condition.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Tyre Deformation (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Extreme Amplitude</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>6.16</td>
<td>6.35</td>
<td>1.92</td>
<td>25.3</td>
</tr>
<tr>
<td>Active: Linear Controller 1</td>
<td>5.96</td>
<td>6.43</td>
<td>1.98</td>
<td>24.9</td>
</tr>
<tr>
<td>Nonlinear Regulator 1</td>
<td>5.16</td>
<td>13.5</td>
<td>2.56</td>
<td>19.3</td>
</tr>
<tr>
<td><strong>Third Amplitude</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>2.05</td>
<td>2.12</td>
<td>0.640</td>
<td>8.43</td>
</tr>
<tr>
<td>Active: Linear Controller 1</td>
<td>1.99</td>
<td>2.14</td>
<td>0.660</td>
<td>8.30</td>
</tr>
<tr>
<td>Nonlinear Regulator 1</td>
<td>2.94</td>
<td>3.01</td>
<td>0.447</td>
<td>9.10</td>
</tr>
</tbody>
</table>

Body Bounce Condition.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Suspension Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Extreme Amplitude</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>5.47</td>
<td>55.6</td>
<td>4.18</td>
<td>110</td>
</tr>
<tr>
<td>Active: Linear Controller 1</td>
<td>4.14</td>
<td>59.5</td>
<td>3.00</td>
<td>106</td>
</tr>
<tr>
<td>Nonlinear Regulator 1</td>
<td>4.47</td>
<td>44.2</td>
<td>3.39</td>
<td>86.6</td>
</tr>
<tr>
<td><strong>Half Amplitude</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>2.74</td>
<td>27.8</td>
<td>2.09</td>
<td>55.0</td>
</tr>
<tr>
<td>Active: Linear Controller 1</td>
<td>2.07</td>
<td>29.8</td>
<td>1.50</td>
<td>53.0</td>
</tr>
<tr>
<td>Nonlinear Regulator 1</td>
<td>2.06</td>
<td>34.3</td>
<td>1.42</td>
<td>58.6</td>
</tr>
</tbody>
</table>

Table 4.6.3: Response Summary for Initial Disturbance Simulations.
Fig 4.6.3: Extreme Amplitude Wheel Hop Condition

- Passive
- Linear Controller 1
- Nonlinear Regulator 1
Fig 4.6.4: Extreme Amplitude Body Bounce Condition

- Passive
- Linear Controller 1
- Nonlinear Regulator 1

Tyre Deformation (m)

Suspension Deflection (m)

Body Acceleration (m/s²)
Fig 4.6.5: Third Amplitude Wheel Hop Condition

- Passive
- Linear Controller 1
- Nonlinear Regulator 1
Fig 4.6.6: Half Amplitude Body Bounce Condition

- Passive
- Linear Controller 1
- Nonlinear Regulator 1
4.6.3 Effects of Cost Function Parameters on Regulator Responses.

In this section, the effects on the regulator responses due to changes in the cost function parameter values are investigated. Firstly, the value of each weighting constant is to be doubled independently and the resulting optimal control sequences are denoted Nonlinear Regulators 2-5 respectively, see Table 4.6.4. Then the effects of changing the indices of the high order terms with corresponding changes in weighting constants are examined, these optimal sequences are denoted Nonlinear Regulators 6&7.

<table>
<thead>
<tr>
<th>Nonlinear Regulators</th>
<th>Parameter Values</th>
<th>n₁</th>
<th>n₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2 \times 10^{12}$</td>
<td>16000</td>
<td>5x10^{11}</td>
</tr>
<tr>
<td>2</td>
<td>$4 \times 10^{12}$</td>
<td>16000</td>
<td>5x10^{11}</td>
</tr>
<tr>
<td>3</td>
<td>$2 \times 10^{12}$</td>
<td>32000</td>
<td>5x10^{11}</td>
</tr>
<tr>
<td>4</td>
<td>$2 \times 10^{12}$</td>
<td>16000</td>
<td>1x10^{12}</td>
</tr>
<tr>
<td>5</td>
<td>$2 \times 10^{12}$</td>
<td>16000</td>
<td>5x10^{11}</td>
</tr>
<tr>
<td>6</td>
<td>$6 \times 10^{8}$</td>
<td>16000</td>
<td>5x10^{11}</td>
</tr>
<tr>
<td>7</td>
<td>$2 \times 10^{12}$</td>
<td>16000</td>
<td>2.5x10^{7}</td>
</tr>
</tbody>
</table>

Table 4.6.4: Cost Function Parameter Values for Various Nonlinear Regulators.

First consider the effects of changing the weighting constants; it is expected that the effect of $\alpha_1$ will be greatest at high amplitudes of tyre deformation which occur at high levels of disturbance exciting the wheel hop condition and that those of $\alpha$ will be most prominent at lower amplitudes. Similarly, $\beta_1$ is expected to dictate the high amplitude body bounce behaviour and $\beta$ the low amplitude.

The regulator responses may be compared by examination of the time histories, Nonlinear Regulators 1,2 and 3 operating from full and third amplitude wheel hop condition and full and half amplitude body bounce condition initial disturbances are compared in Figs 4.6.7-10. As expected, $\alpha_1$ has a significant effect on the high amplitude wheel hop condition behaviour causing a substantial reduction in the peak tyre deformation. It also has a slight effect at medium to low amplitude behaviour in this condition. The high amplitude body bounce condition response is also affected by $\alpha_1$; the increase made has caused a reduction in the high amplitude tyre deformations which has affected the whole behaviour. The increase in $\alpha$ has principally affected the
Solution of Nonlinear Optimal Regulator Problem.

low and medium amplitude wheel hop condition behaviour, and a slight change is detected in the low amplitude body bounce condition but this is not so significant.

Nonlinear Regulators 4 and 5 are compared with the reference system in Figs 4.6.11-14, the expected effect of \( \beta_1 \) on the high amplitude body bounce condition is apparent as is that of \( \beta \) on the low amplitude behaviour. It can also be seen that changes in these parameters have virtually no effect on the wheel hop condition behaviour.

The high order terms in the cost function were included to dictate the behaviour of the system at high amplitudes of disturbance and to allow the weighting constants on the quadratic terms to be chosen to give desirable behaviour at low amplitudes. The amplitude at which the control of the behaviour is transferred from the quadratic terms to the high order terms is dependent upon the indices used for the high order terms and the relative values for the gains. The effect of changes in these indices combined with appropriate changes in the gains is considered here. The index of the tyre deformation term is reduced from 6 to 4; this will change the overall shape of \( L_1(x_1) \) since a fourth order term will have relatively greater effect at low \( x_1 \) and relatively less effect at high \( x_1 \) than a sixth order term. Appropriate adjustment has also been made to \( \alpha_1 \) so that \( L_1(x_1) \) is generally of a comparable amplitude (the values are equal at \( x_1 = 17.3 \text{mm} \)). Similarly the effect of reducing the index on the high order suspension deflection term from 10 to 6 is to be investigated. Again an appropriate adjustment has been made to \( \beta_1 \). Here \( L_2(x_2) \) remains constant at \( x_2 = 84.1 \text{mm} \). These changes are incorporated in Nonlinear Regulators 6 and 7.

The responses of these regulators are compared with the reference for the four initial disturbances in Figs 4.6.15-18. As expected, it can be seen that Nonlinear Regulator 6 allows greater tyre deformations at high amplitudes of wheel hop motion and less at low amplitudes than the reference. The effect can also be detected in the responses in the body bounce condition but is much less significant. Also the expected differences between Nonlinear Regulator 7 and the reference are apparent in the body bounce behaviour and, as for the other changes made to the suspension deflection terms, there is virtually no change in the wheel hop condition behaviour.
Fig 4.6.7: Extreme Amplitude Wheel Hop Condition

- Nonlinear Regulator 1
- Nonlinear Regulator 2
- Nonlinear Regulator 3
Fig 4.6.8: Third Amplitude Wheel Hop Condition

- Nonlinear Regulator 1
- Nonlinear Regulator 2
- Nonlinear Regulator 3
Fig 4.6.9: Extreme Amplitude Body Bounce Condition

- Nonlinear Regulator 1
- Nonlinear Regulator 2
- Nonlinear Regulator 3
Fig 4.6.10: Half Amplitude Body Bounce Condition

- Nonlinear Regulator 1
- Nonlinear Regulator 2
- Nonlinear Regulator 3
Fig 4.6.11: Extreme Amplitude Wheel Hop Condition

- Nonlinear Regulator 1
- Nonlinear Regulator 4
- Nonlinear Regulator 5
Fig 4.6.12: Third Amplitude Wheel Hop Condition

- Tyre Deformation (m)
- Suspension Deflection (m)
- Body Acceleration (m/s²)

- Nonlinear Regulator 1
- Nonlinear Regulator 4
- Nonlinear Regulator 5

Time (secs)
Fig 4.6.13: Extreme Amplitude Body Bounce Condition

- Nonlinear Regulator 1
- Nonlinear Regulator 4
- Nonlinear Regulator 5
Fig 4.6.14: Half Amplitude Body Bounce Condition

- Tyre Deformation (m)
- Suspension Deflection (m)
- Body Acceleration (m/s²)

Nonlinear Regulator 1
Nonlinear Regulator 4
Nonlinear Regulator 5

Time (secs)
Fig 4.6.15: Extreme Amplitude Wheel Hop Condition

- Nonlinear Regulator 1
- Nonlinear Regulator 6
- Nonlinear Regulator 7

Tyre Deformation (m)

Suspension Deflection (m)

Body Acceleration (m/s²/g)

Time (secs)
Fig 4.6.16: Third Amplitude Wheel Hop Condition

- Nonlinear Regulator 1
- Nonlinear Regulator 6
- Nonlinear Regulator 7
Fig 4.6.17: Extreme Amplitude Body Bounce Condition

- Nonlinear Regulator 1
- Nonlinear Regulator 6
- Nonlinear Regulator 7
Fig 4.6.18: Half Amplitude Body Bounce Condition

Tyre Deformation (m)

Suspension Deflection (m)

Body Acceleration (m/s²)

- Nonlinear Regulator 1
- Nonlinear Regulator 6
- Nonlinear Regulator 7
Chapter 5

Implementation of Control Strategy via Nonlinear Feedback.

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<th>Title</th>
<th>Page</th>
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Nomenclature.

- A: Design matrix in overconstrained equations
- \( A_w \): Design matrix in weighted overconstrained equations
- a: Vector of free parameters in feedback law.
- b: Data vector in overconstrained equations.
- \( b_w \): Data vector in weighted overconstrained equations.
- E: Error criterion for least squares fit.
- e: Term in error criterion.
- \( \phi \): Basis functions for feedback law.
- \( \psi \): Basis functions for fitting function in example.
- \( J^* \): Cost potential.
- k: Vector of free parameters in example.
- \( k \): Scale of grid for design sample.
- \( \eta \): J-efficiency of feedback controller.
- P: Projection matrix.
- U: Matrix of singular vectors of P.
- v: Vector forming row of A.
- \( v_1 \): Vector forming row 1 of A.
- \( v_2 \): Vector forming row 2 of A.
- \( \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \): Normalised state variables.
- \( \bar{x} \): Normalised state vector.
- \( \mathbf{x}^* \): State vector on an optimal trajectory.
- \( X \): Practically available region of state space.
- \( x_p \): Design Sample.
- \( x \): Independent variable in example.
- \( y \): Independent variable in example.
- \( \gamma \): Fit parameter used in example.
- z: Dependent variable in example.
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To summarise from the previous two chapters, the desired control strategy is the following; at any given state apply the control which would be applied by an optimal regulator operating from that point. For LQR, the implementation of this strategy is simple since the optimal controls can be exactly described by a simple linear function of the states. However, for nonlinear optimal regulators there is no such simple relationship.

A method has been devised for solving the nonlinear optimal regulator problem and hence, the control to be applied at any given state can be evaluated. A "Utopian" way of implementing this control strategy would be to solve the optimal regulator problem on-line. At each point in time the optimal control input would be calculated for the prevailing state. However this is not practicable due to the large amount of computation required for each solution. A more realisable approach is to use knowledge gained from off-line solutions of the nonlinear optimal regulator problem to design a feedback law to implement the control strategy on-line.

There are many possible formats in which to express this feedback law; a look-up table could be used, or a nonlinear analytical function, and it is potentially a good application for a neural network. A look-up table containing the values of the optimal controls for chosen points in the state space calculated off-line could be stored in ROM on the vehicle and this could then be used to implement the feedback strategy on-line. However, many off-line solutions of the optimal regulator problem would be required to build a look-up table with a sufficiently fine grid to avoid significant interpolation errors. This approach would also make the tuning procedure very time consuming as a new table is required for each cost function considered. The advent of efficient learning and implementation algorithms for neural networks makes them a possible candidate for this task, but their use has not been investigated in detail to date. However, the learning procedure is likely to be slow and computationally expensive if many solutions of the nonlinear optimal regulator problem are necessary.

Concern for simplicity leads us to consider the implementation of the control strategy via a continuous analytical feedback function to be known as the feedback law. This chapter is devoted to the fitting of this law to the optimal mapping and validating that a good approximation of the desired controls throughout the state space is given.
5.1 Fitting a Feedback law.

For any given state, the control applied is evaluated by the feedback law, denoted $F_c(x, a)$, which is designed to represent the optimal control, $F_c^*(x)$, throughout the state space. This feedback function is parameterised by a vector, $a$, of adjustable parameters and is of the form

$$F_c(x, a) = \sum_i a_i \phi_i(x) \quad (5.1.1)$$

where $\phi_i(x)$ are basis functions. These parameters, $a$, are chosen such that

$$F_c(x_p, a) \approx F_c^*(x_p), \quad p = 1, 2, \ldots N \quad (5.1.2)$$

where $x_p$, $p = 1, 2, \ldots N$, is a sample of states known as the design sample. Each $x_p$ forms the initial condition for a solution of the nonlinear optimal regulator problem thus providing $F_c^*(x_p)$. The choice of the basis functions, $\phi_i(x)$, determines the potential ability of the feedback law to represent the optimal controls, and this choice is to be discussed in §5.2.

The choice of the design sample determines how accurately the $a_i$ are fixed and a dense, well distributed sample should provide a good choice for $a$, however, it is also desirable to keep the size of the sample to a minimum due to the computational expense of calculating the $F_c^*(x_p)$. So, the choice of the design sample must be compromised and given careful consideration, this is discussed in §5.3.

Once the basis functions and design sample have been selected, the parameters $a$ must be chosen to satisfy (5.1.2) and this can be formalised by the use of an error criterion, $E$, such that the parameters are chosen to minimise $E$. The criterion used seeks to limit the sub-optimality of the system performance caused by the application of incorrect control at each state, and the choice of this criterion is discussed in detail in §5.4. The choice of $a$ can be simplified mathematically by the use of a weighted linear least squares optimisation technique which is computationally efficient. This technique is applicable if $F_c(x, a)$ is linear in $a$ and the error criterion is of the least squares type:

$$E = \sum_{p=1}^{N} \varepsilon_p$$

where

$$\varepsilon_p = w_p(x)(F_c(x_p, a) - F_c^*(x_p))^2 \quad (5.1.3)$$
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and \( w_p(x) \) are the weightings.

For each point in the design sample a fitting equation can be formed:

\[
F_c(x_p, a) = F_c^*(x_p), \quad p = 1, 2, \ldots, N, \tag{5.1.4}
\]

giving a set of over-constrained equations which can be expressed in the form

\[
A_a = b, \tag{5.1.5}
\]

where \( A_a \) is known as the design matrix and \( b \) the data vector. After each equation is multiplied by the square root of the weighting, the equations can be re-expressed:

\[
A_wa = b_w. \tag{5.1.6}
\]

The parameters, \( a \), which minimise the error criterion, \( E \), can then be evaluated by the standard least squares solution (Strang, 1980):

\[
a = \left( A_w^T A_w \right)^{-1} A_w^T b_w. \tag{5.1.7}
\]

5.2 Choice of Basis Functions.

In this section the choice of the basis functions, \( \phi_i(x) \), which define the structure of \( F_c(x, a) \) from (5.1.1) is considered. The potential ability of the feedback law to represent the optimal controls is highly dependent upon the choice of these functions, and once they are chosen, the free parameters \( a \) will be optimised subject to the error criterion evaluated across the design sample.

It is known that under LQR conditions a linear feedback law gives a perfect representation of the optimal controls. However, such a structure is unlikely to be able to adequately represent the nonlinear relationship between \( x \) and \( F_c^*(x) \) prevailing here. The structure must be given more freedom and there are many choices for the terms which can give this extra freedom; sinusoidal terms could be used to give a Fourier type structure, or exponential terms could be used. However, for simplicity, the addition of general higher order polynomial terms is to be investigated.

Further examination of the two point boundary value problem which is solved for the optimal controls, \( F_c^*(x) \), reveals that; for linear state equations and cost functions which are even functions of \( x \), the state-costate system of §4 will be odd. This means that if a given state, \( x \), has associated costates, \( p \), then the associated costates for \(-x\) are
-p etc., and hence,

$$F_c^*(x) = -F_c^*(-x)$$

(5.2.1)

So only odd-order basis functions need be considered to represent $F_c^*(x)$. For simplicity, only third order polynomial terms have been added to the linear structure giving a feedback law described by the 24 basis functions below:

$$
\begin{align*}
\Phi_1 & \quad \Phi_2 & \quad \Phi_3 & \quad \Phi_4 & \quad \Phi_5 & \quad \Phi_6 & \quad \Phi_7 & \quad \Phi_8 \\
x_1 & \quad x_2 & \quad x_3 & \quad x_4 & \quad x_1^3 & \quad x_2^3 & \quad x_3^3 & \quad x_4^3 \\
\Phi_9 & \quad \Phi_{10} & \quad \Phi_{11} & \quad \Phi_{12} & \quad \Phi_{13} & \quad \Phi_{14} & \quad \Phi_{15} & \quad \Phi_{16} \\
x_1^2x_2 & \quad x_1^2x_3 & \quad x_1^2x_4 & \quad x_2^2x_1 & \quad x_2^2x_3 & \quad x_2^2x_4 & \quad x_3^2x_1 & \quad x_3^2x_2 \\
\Phi_{17} & \quad \Phi_{18} & \quad \Phi_{19} & \quad \Phi_{20} & \quad \Phi_{21} & \quad \Phi_{22} & \quad \Phi_{23} & \quad \Phi_{24} \\
x_3^2x_4 & \quad x_4^2x_1 & \quad x_4^2x_2 & \quad x_4^2x_3 & \quad x_1x_2x_3 & \quad x_1x_2x_4 & \quad x_1x_3x_4 & \quad x_2x_3x_4
\end{align*}
$$

5.3 Choice of Design Sample.

The free parameters $a$ are to be optimised to minimise an error criterion thus ideally providing the best fit for the given basis functions. Since only a certain domain of the space is practically available to the quarter vehicle model, the fitting exercise is restricted to this domain thus enabling a better representation to be achieved within it. The domain of practically available states is of course somewhat arbitrary, but on the basis of simulations and measurements on a test vehicle, the following limits have been chosen:

$$
\begin{align*}
x_1 & : \pm 0.025 \text{ m}, \\
x_2 & : \pm 0.1 \text{ m}, \\
x_3 & : \pm 2.5 \text{ m/s}, \\
x_4 & : \pm 1.25 \text{ m/s},
\end{align*}
$$

and this region is denoted $X$.

In practice only a limited number of points may be investigated. These form the design sample, $\{x_p, p = 1, 2, \ldots N\}$, and a simple but effective strategy for choosing these from within $X$ must be developed. To enable the feedback law to give a good representation across $X$, it is reasonable to expect that the design sample needs to provide information
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for the optimisation about all areas of X. The density of the sample must be compromised however, due to the computational cost of the evaluation of each $F_c \circ \left(x_p^*\right)$. Furthermore, the *conditioning* of the fit is an important consideration in the choice of the design sample.

If the chosen data do not clearly distinguish between two or more of the basis functions then different combinations of basis functions and parameter values can describe the data equally well. This phenomenon is known as ill conditioning and can cause spurious values to be assigned to the parameters and ill-defined parameters can cause great inaccuracies in the fit of data outside the design sample where there is a clear distinction between the particular basis functions.

The ability to fit across the design sample itself is representative of the potential ability of the basis functions to fit over the whole domain. Therefore, an effective test of the quality of the fit and the conditioning is to compare the fit achieved over an independent sample of points, known as the *test sample*, with that achieved over the design sample. If the fit is ill-conditioned then the quality of the fit is likely to be significantly worse over the independent sample.

To illustrate some of the problems associated with describing a complex mapping by a relatively simple analytic function, a simple example is considered.

**5.3.1 Simple Example to Illustrate Effects of Choice of Basis Functions and Design Sample.**

The problem is to fit the surface defined by:

$$z = e^{3x} + e^{4y}$$

by the fitting function

$$z = f(x, y, k)$$

where $k$ is a vector of free parameters, over the domain: $0 \leq x \leq 1$, $0 \leq y \leq 1$.

The fitting function is described by

$$f(x, y, k) = \sum_i k_i \psi_i(x, y),$$

where $\psi_i(x, y)$ are basis functions.
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The surface is depicted in Fig 5.3.1 and the free parameters, \( k \), are to be chosen by least squares optimisation with no weightings. Initially, it will be illustrated how the potential ability to fit is dependent upon the choice of the basis functions; three sets of basis functions are considered:

1) a constant term plus all linear functions of \( x \) and \( y \): \( \psi_1 = 1, \psi_2 = x, \psi_3 = y \),
2) these terms plus \( x^2 \) and \( y^2 \): \( \psi_1 - \psi_3 \) plus \( \psi_4 = x^2, \psi_5 = y^2 \),
3) these terms plus \( x^3 \) and \( y^3 \): \( \psi_1 - \psi_5 \) plus \( \psi_6 = x^3, \psi_7 = y^3 \).

Since, in this example, the true mapping can be easily evaluated, a design sample which is very dense can be used to give the 'best' set of parameters for each set of basis functions and hence, maximise their potential to fit. The design sample used in each case consists of all of the nodes on a square mesh of side 0.01 placed on the xy plane, hence, there are 101 nodes on each axis and 10201 nodes in total.

The fitting equation

\[
\sum_i k_i \psi_i(x_p,y_p) = e^{3x_p} + e^{4y_p} , \quad p = 1, 2, 3, ..., 10201
\]

is formed for each point in this design sample and collated into the overconstrained equations described by

\[ A k = b \]

which are solved for the parameters, \( k \), that minimise the error criterion

\[
E = \sum_{p=1}^{10201} \left( \sum_i (k_i \psi_i(x_p,y_p)) - \left( e^{3x_p} + e^{4y_p} \right) \right)^2
\]

by the standard least squares solution.

The vectors of best parameters for each set of basis functions are found to be:

1) \( k^T = [-10.2 43.5 16.8] \),
2) \( k^T = [6.66 -6.90 -34.9 23.7 78.4] \),
3) \( k^T = [0.759 6.76 23.9 -10.62 -69.21 22.9 98.4] \).
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The ability to fit is measured by the fitting parameter

\[ \gamma = 1 - \frac{E}{\sum p b_p^2} \]

which is a standard measure of fit. The values of this parameter in each case are found to be:

1) \( \gamma = 93.15 \% \),
2) \( \gamma = 99.34 \% \),
3) \( \gamma = 99.97 \% \).

These results clearly show the dependence of the potential ability to fit upon the choice of basis functions and that the greater the freedom in the basis functions, the greater the potential ability to fit. This may lead one to expect that even better fits could be obtained by including more higher order basis functions and this is almost certainly true in this simple, smooth example. However, for less well behaved surfaces and where there are constraints on the size of the design sample, the increase in number or complexity of the basis functions may make the problem of choosing a well conditioned set of parameters more demanding. In such cases, within practical limits, the best fit may be achieved with relatively simple basis functions.

The above study was carried out using an ideal design sample, now the situation where this is not practically possible is considered. For set 2 of basis functions the parameters are chosen using a design sample whose size must be compromised and is restricted to 9 points. Several choices of design sample are considered to illustrate how the choice of the parameters is affected by the conditioning of the design sample.

The ultimate test of conditioning and the quality of the fit is to test on the whole domain, or at least on the very dense ideal sample and, since in this example the computational effort is small, the dense ideal sample can be used as a test sample.

A simple measure of the conditioning is also given by the condition number of the design matrix, A. This is the ratio of the largest singular value to the smallest singular value of A and a well conditioned optimisation is characterised by a low condition number. However, as will be seen, there are certain constraints on the use of the condition number as a measure of conditioning.
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Design Sample 1: \[ \begin{bmatrix} 0 & 0.125 & 0.25 & 0.375 & 0.5 & 0.625 & 0.75 & 0.875 & 1.0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

This sample is, as might be expected, totally ill-conditioned since none of the basis functions involving \( y \) is tested, hence, the parameters relating to these basis functions are undefined. The design matrix, \( A \), does not have full rank; the nullity is 2 and hence, 2 of the singular values are zero giving \( A \) an infinite condition number. The 'optimal' parameters can not strictly be evaluated since \( A^T A \) is singular, see (5.1.7).

Design Sample 2: \[ \begin{bmatrix} 0 & 0.125 & 0.25 & 0.375 & 0.5 & 0.625 & 0.75 & 0.875 & 1.0 \\ 0 & 0.125 & 0.25 & 0.375 & 0.5 & 0.625 & 0.75 & 0.875 & 1.0 \end{bmatrix} \]

This sample is again totally ill-conditioned since, throughout the sample, the data can be equally described by the basis function \( x \) and the basis function \( y \) and similarly by \( x^2 \) and \( y^2 \). Again, \( A \) has nullity. In fact the rank is 3. It has an infinite condition number and the 'optimal' parameters can not be evaluated.
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Design Sample 3: $$\begin{bmatrix} 0 & 0.5 & 1.0 & 0 & 0.5 & 1.0 & 0 & 0.5 & 1.0 \\ 0 & 0 & 0 & 0.125 & 0.125 & 0.125 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

This sample is poorly spread and badly conditioned since it only contains values of y up to 0.25 and hence, none of the basis functions involving y is adequately tested and some of the parameters are ill-defined. A has a condition number of 175 and the optimal parameters are


The fit parameter evaluated over the ideal design sample is: $\gamma = 79.78\%$, which is significantly worse than that achieved by the 'best' parameters for these basis functions.

Design Sample 4: $$\begin{bmatrix} 0 & 0.5 & 1.0 & 0.5 & 1.0 & 0.475 & 0.95 & 0.45 & 0.9 \\ 0.45 & 0.9 & 0.475 & 0.95 & 0.5 & 1.0 & 0.5 & 1.0 \end{bmatrix}$$

This sample is also poorly spread and badly conditioned since all the data points are too close to $x=y$ and hence, do not clearly distinguish between the basis functions $x$ and $y$ or between $x^2$ and $y^2$. Again, this results in the ill-definition of some of the parameters and A has a condition number of 206. The optimal parameters produced are

$$k^T = [2.0336.9-65.5-11.9110]$$.
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The fit parameter, \( \gamma = 96.22\% \), is still significantly worse than that for the 'best' parameters.

Design Sample 5: 

\[ \begin{bmatrix} 0 & 0.5 & 1.0 & 0 & 0.5 & 1.0 & 0 & 0.5 & 1.0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1.0 & 1.0 & 1.0 \end{bmatrix} \]

This sample is well distributed throughout the fitting domain and is relatively well conditioned; \( A \) has a condition number of 17.3 and the optimal parameters are

\[ k^T = \begin{bmatrix} 2.00 & -5.16 & -28.0 & 24.2 & 81.6 \end{bmatrix} \]

The fit parameter is, \( \gamma = 97.74\% \) which is much closer to that of the ideal design sample.

In this simple example it seems that all that is required for good conditioning is a good spread of points. However, for more complex problems the requirement may not be so trivial but it is likely to be a good guide to the initial choice of a design sample.

From the results obtained for design samples 3, 4 and 5, it can be seen how the condition number gives a measure of the conditioning; for design samples 3 and 4, the condition numbers are relatively high and the fits to the whole surface relatively poor. However, for design sample 5, the condition number is relatively low and the fit relatively good. However, it can also be seen that the measure of the conditioning given by the condition number is not absolute; design sample 3 has a lower condition number than design sample 4 yet gives a worse fit. The ability of the condition number to describe the conditioning is now further examined.
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Design Sample 6:

\[
\begin{bmatrix}
0 & 5 & 10 & 0 & 5 & 10 & 0 & 5 & 10 \\
0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1.0 & 1.0 & 1.0 \\
\end{bmatrix}
\]

For this case the problem is re-scaled such that \(0 \leq x \leq 10\), and design sample 5 is effectively repeated at this new scaling. The least squares fit is effectively the same with the chosen parameters scaled accordingly and the fit parameter exactly the same. However, the condition number for this sample is 728. This illustrates that to achieve a meaningful condition number, the numerical values of the basis functions at the design points must be of the same order of magnitude. This can be broadly achieved by scaling the independent variables in the fitting function, to achieve a similar nominal amplitude.

5.3.2 Design Sample for Fitting of Nonlinear Feedback Law.

The example considered illustrates the need to consider conditioning when choosing a design sample and how the condition number of the design matrix can be used as an indication of this. Another advantage of the condition number for an unweighted least squares solution is that it can be computed directly from the design matrix which does not require the evaluation of the true mapping. However, the choice of the parameters for the feedback law is made by weighted least squares and the design matrix, \(A_w\), requires the computation of the optimal mapping to evaluate the weightings. Hence, the evaluation of the condition number is no longer computationally cheap. However, the condition number of the unweighted matrix can still be cheaply evaluated and will give a warning of any likely conditioning problems. I.e. if \(A\) is poorly conditioned, then the weightings are not likely to drastically improve matters and \(A_w\) is likely to also be poorly conditioned. However, the good conditioning of \(A\) does not ensure the good conditioning of \(A_w\) since the weightings may conceivably spoil the conditioning. Nevertheless, it is thought well worthwhile to examine the conditioning of \(A\).

It was noted in the example above that to produce numerically meaningful condition numbers, the independent variables must be scaled so that each basis function takes nominally similar amplitudes. For the fitting of the feedback law, the independent
variables are the state variables and the fit is to be performed in terms of normalised state variables. These are related to the state variables by dividing each by its maximum practically available value, thus:

\[
\begin{align*}
\bar{x}_1 &= \frac{x_1}{0.025}, \quad \bar{x}_2 = \frac{x_2}{0.1}, \quad \bar{x}_3 = \frac{x_3}{2.5}, \quad \bar{x}_4 = \frac{x_4}{1.25} .
\end{align*}
\] (5.3.1)

A strategy for choosing a design sample for a given set of basis functions is required; a reasonable starting point is to consider all points on a four dimensional grid of side length equal to 1 in normalised state space, this gives the following 81 points:

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

which are depicted across a typical 2D slice of the state space in Fig 5.3.2.

![Fig 5.3.2: First Design Sample Depicted Across Typical 2D Slice of State Space](image)

It is found that, due to the oddness of (5.1.4), equations formulated from any state, \( \bar{x} \), are identical to those formulated from a state, \( -\bar{x} \). Hence 40 of the points are redundant and also, the origin provides a trivial equation. This leaves an effective design sample of 40, which is a reasonable size, with points well distributed throughout \( X \). Examinining the conditioning of \( A \), it is found that the columns are not all independent, and in fact the rank of \( A \) is only 20 and hence, the parameters can not be optimised for
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this design sample.

The choice of sample must be reconsidered; the four dimensional grid is to be preserved since this provides a good distribution of points but the side length reduced to $\frac{2}{3}$. Noting the redundancy of points that are the negative of points already chosen, only the positive half of the grid in one of the states, say $\bar{x}_1$, need be considered. This yields the following grid of 128 points

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \frac{1}{2} \\ \frac{1}{2} \\ -1 \frac{1}{2} \end{pmatrix} \ldots \text{etc.,}$$

again depicted across a typical 2D slice of state space in Fig 5.3.3.

A now has full rank with a condition number of 9.12, but does this mean that this sample is likely to be well conditioned? For comparison, several randomly chosen design samples consisting of 128 points each were considered and the mean condition number of the resulting 'A' matrices was approximately 11. It is believed that a random sample of this size is likely to produce data which clearly distinguishes between the basis functions and hence give acceptable conditioning. Therefore, the condition number of the chosen sample suggests that it will provide reasonable conditioning.

Fig 5.3.3: Second Design Sample Depicted Across Typical 2D Slice of State Space
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This grid may form a reasonable design sample. However, its size is larger than is desirable and a way of reducing this whilst maintaining the conditioning is sought. An approach that may be taken is to choose the sample of 24 states from the grid of 128 which yields the best conditioning. The condition number of a matrix is, by definition, related to the orthogonality of the basis vectors which define the column space; if these vectors are orthogonal, the condition number is 1, whereas if any two are parallel then the condition number will be infinite. A sample of 24 states produces an A matrix that is square and hence, the orthogonality of its column space is equal to the orthogonality of its row space. Furthermore, since each row of A is derived from a single state vector, each basis vector of the row space relates directly to a single point in the design sample. Thus a 24 point design sample with optimal column space orthogonality, and hence condition number, can be selected from the grid of 128. This orthogonality is informally optimised as follows:

Make an arbitrary choice from the grid for the state which defines the first row vector \( v_1 \), then search the remaining 127 states in the grid for the state that produces the row vector, \( v_2 \), which is most orthogonal to the first. These two rows now form the basis of a two dimensional row space, \( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \), and the remaining 126 states are now searched for the state that produces the row vector that is most orthogonal to \( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \). The orthogonality of a given vector \( v \) to the row space \( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) is measured by the direction cosine between \( v \) and its projection into this space. This projection is found by pre-multiplying \( v \) by the projection matrix, \( P \), of the space where

\[
P = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \left( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)^T \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \left( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)^T ,
\]

(5.3.2)

see Strang (1980), hence, \( v \) should be chosen to minimise \( \frac{|Pv|}{|v|} \). This procedure is repeated until the 24 dimensional row space with optimal orthogonality has been built.

It should be noted that the computation of this algorithm can be significantly accelerated by considering the singular value decomposition of the matrix which defines the row space at each stage. The resulting matrix of singular vectors, \( U \), gives an orthonormal basis for the row space which significantly simplifies \( P \):

\[
P = U \left[U^T \ U \right]^{-1} U^T = UIU^T = UU^T
\]

(5.3.3)

and hence,
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\[ |Pv| = |UU^Tv| = |U^Tv| \]  \hspace{1cm} (5.3.4)

since U is orthonormal.

The overall solution reached depends upon the arbitrary choice made for \( v_1 \), hence, all possibilities should be considered if the true optimum is sought. However, it was found that for three different \( v_1 \) considered, the orthogonality of the resulting row space was similar, and hence the orthogonality produced is assumed to be reasonably optimal.

Starting with normalised state vector \([1 1 1]\)\(^T\), the 'best' sample of 24 states, shown in Fig 5.3.4, produces an A matrix with a condition number of 33.8. Though this is higher than that for the original grid of 128, it does not indicate any ill-conditioning.

So the size of the sample has been reduced from 128 to 24 states whilst maintaining a reasonable condition number, and this could be used as the design sample. However, these 24 states do not provide a very dense sample and the quality of the fit is likely to improve if more points are added. Again this should be done in a careful and controlled way to maintain good conditioning. Choosing 24 from 128 was a particularly tractable problem since it allowed the orthogonality of the row space to be related to the condition number, and the basis vectors of row space, unlike those of column space, are one to one mapped from the states sampled. To select a sample of any other size from this grid would involve a much greater amount of computation.

One simple method of adding further points is to repeat this proven sampling policy from this grid on grids with different amplitudes, i.e.

\[ \bar{x}_p, \quad p = 25, 26, \ldots, 48 = k \bar{x}_p, \quad p = 1, 2, \ldots, 24. \]

This will produce a design sample where good spread is guaranteed and reasonable conditioning is expected. The design sample used in this work is of this basic pattern repeated at amplitudes \( k = 0.75, 0.5 \) and 0.25. This sample of 96 points produces an A matrix whose condition number is 22.7 which gives no reason to suspect ill-conditioning.

It should be further noted that randomly chosen design samples of this size will, on average, produce a condition number of approximately 15 and hence, could be expected to offer similar conditioning. The use of such samples is not discredited by this work and the problem of how to choose the most effective design sample within the given constraints remains only partially solved.
Implementation of Control Strategy via Nonlinear Feedback.

\[
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\vdots \\
\tilde{x}_{24}
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 \\
\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 \\
\frac{1}{3} & \frac{1}{3} & -1 & -\frac{1}{3} \\
1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\
\frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \\
1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & 1 & -\frac{1}{3} \\
\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -1 \\
\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\
1 & \frac{1}{3} & \frac{1}{3} & -1 \\
1 & -\frac{1}{3} & \frac{1}{3} & 1 \\
\frac{1}{3} & -1 & -1 & -\frac{1}{3} \\
\frac{1}{3} & 1 & -1 & \frac{1}{3}
\end{bmatrix}
\]

Fig 5.3.4: 'Optimised' Design Sample.
5.4 J-efficiency of the Fitted Controller and an Error Criterion for the Choice of a.

A formal measure of the ability of \( F_c(x, a) \) to represent \( F_c^*(x) \) at any given state, \( x \), is sought in this section and this is to be defined as the 'J-efficiency' of the fit for that state. The definition is related to that of efficiency in the classical sense, being the ratio of useful work done to total work done. This analysis shall also be used to formalise the error criterion which is to be based upon the sub-optimality caused by imperfect representation. Firstly some further fundamental concepts shall be introduced.

During any period of time, the dynamic cost, \( J \), (3.1.1) can be thought of as the cost incurred by the quarter vehicle system model, and for any given situation the amount of cost incurred is dependent upon the control applied. Define the cost potential, \( J^*(x) \), as the cost that will be incurred in infinite time from a given state, \( x \), when optimal controls, \( u^*(t) \), are applied and no further disturbances are received. The optimal trajectory is to be denoted, \( x^*(t) \), \( 0 \leq t \leq \infty \). Hence for such an optimal trajectory, the total cost incurred is equivalent to the cost potential of the initial state:

\[
\int_0^\infty L[x^*(t), u^*(t)] \, dt = J^*(x(0))
\]

and this identity is true for any state. Consider two points on the optimal trajectory, \( x^*(t) \) and \( x^*(t + T) \); the difference in cost potential at these two points is equivalent to the cost incurred along the trajectory over this period; i.e.

\[
[J^*(x^*(t)) - J^*(x^*(t + T))] = \int_t^{t+T} L[x^*(t), u^*(t)] \, dt - \int_t^{t+T} L[x^*(t), u^*(t)] \, dt = \int_0^\infty L[x^*(t), u^*(t)] \, dt
\]

(5.4.2)

However, if for a similar period sub-optimal controls, \( u(t) \), are applied then not all cost incurred results in an equivalent decrease of cost potential, hence,

\[
\int_0^\infty L[x(t), u(t)] \, dt > [J^*(x(t)) - J^*(x(t + T))]
\]

(5.4.3)

The 'J-efficiency', \( \eta \), of the feedback control \( F_c(x, a) \) is defined as follows; at any given state, \( x \), and for a small (infinitesimal) period of time, \( \delta t \),
Implementation of Control Strategy via Nonlinear Feedback.

\[ \eta = \frac{\text{Reduction in Cost Potential}}{\text{Cost Incurred}} = \frac{J^*(x(t)) - J^*(x(t+\delta t))}{\int_{t}^{t+\delta t} L[x(t), u(t)] dt} \]  

(5.4.4)

which is analogous to the ratio of useful work out to the total work input, the classical definition of efficiency in a mechanical system.

Consider the increase in total cost, \( \delta J \), resulting from the application of a sub-optimal control, \( u(t) \), at a given state, \( x \), for a small (infinitesimal) period of time, \( \delta t \); this is equal to the difference between cost incurred and the reduction in cost potential:

\[ \delta J = \int_{t}^{t+\delta t} L[x(t), u(t)] dt - [J^*(x) - J^*(x(t+\delta t))] \].  

(5.4.5)

The initial cost potential, \( J^*(x) \), can be substituted by terms for the cost incurred and the final cost potential on an optimal trajectory; from the identity (5.4.2) this gives:

\[ \delta J = \int_{t}^{t+\delta t} L[x(t), u(t)] dt - \int_{t}^{t+\delta t} L[x^*(t), u^*(t)] dt - [J^*(x(t+\delta t)) - J^*(x(t+\delta t))] \]

\[ = L[x(t), u(t)] \delta t - L[x^*(t), u^*(t)] \delta t + J^*(x(t+\delta t)) - J^*(x^*(t+\delta t)) \]  

(5.4.6)

n.b. the integrals have been approximated by the integrand multiplied by the short integration period, \( \delta t \). From (3.1.1), the states on the sub-optimal and optimal trajectories at time \( t+\delta t \) are approximately given by

\[ x(t+\delta t) = x(t) + f(x(t), u(t)) \delta t \text{ and } x^*(t+\delta t) = x(t) + f(x(t), u(t)) \delta t. \]  

(5.4.7)

The change in cost potential due to a change in initial state is given in (3.2.12) i.e.

\[ J^*(x(t+\delta t)) - J^*(x^*(t+\delta t)) = pT(t)[x(t+\delta t) - x^*(t+\delta t)] \]  

(5.4.8)

Now substituting from (5.4.7) gives

\[ J^*(x(t+\delta t)) - J^*(x^*(t+\delta t)) = pT(t)[f(x(t), u(t)) - f(x(t), u^*(t))] \delta t. \]  

(5.4.9)

Hence, (5.4.6) can now be re-written as
Implementation of Control Strategy via Nonlinear Feedback.

\[
\delta J = L[x(t), u(t)]\delta t - L[x(t), u^*(t)]\delta t \\
+ p^T(t)f(x(t), u(t))\delta t - p^T(t)f(x(t), u^*(t))\delta t \\
= (H[x(t), u(t), p(t)] - H[x(t), u^*(t), p(t)])\delta t \\
\text{(5.4.10)}
\]

This increase in cost is therefore related to an increase of the value of the Hamiltonian (3.2.4) which behaves rather like entropy in a thermodynamic system in so much as it is constant for a totally efficient optimal system but will increase if any inefficiency or sub-optimality is introduced.

To quantify this increase in cost due to the application of a sub-optimal control and hence find the J-efficiency, \( H[x(t), u(t), p(t)] \) is expressed as a Taylor's series expanded about \( H[x(t), u^*(t), p(t)] \); (5.4.10) then gives

\[
\delta J = \left[ H[x(t), u^*(t), p(t)] + \frac{\partial H}{\partial u} \delta u(t) + \frac{1}{2!} \frac{\partial^2 H}{\partial u^2} \delta u(t)^2 + \frac{1}{3!} \frac{\partial^3 H}{\partial u^3} \delta u(t)^3 + \ldots \right] \delta t \text{(5.4.11)}
\]

where

\[
\delta u(t) = u(t) - u^*(t) \\
\text{.}
\]

Since the Taylor's expansion is about \( u = u^*(t) \), \( \frac{\partial H}{\partial u} = 0 \) from (3.2.11). Also from the definition of \( H[x(t), u(t), p(t)] \) in (3.2.4) and from the fact that \( L[x(t), u(t)] \) is quadratic in \( u(t) \) and \( f[x(t), u(t)] \) is linear in \( u(t) \), it follows that \( \frac{\partial^2 H}{\partial u^2} = \frac{\partial^2 L}{\partial u^2} = \frac{2}{m_b} \), \( \frac{\partial^3 H}{\partial u^3} = 0 \) and all higher order partial derivatives are zero.

This leaves simply

\[
\delta J = \frac{\delta u(t)^2}{m_b^2} \delta t \\
\text{(5.4.12)}
\]

So for any state, the increase in cost due to application of sub-optimal control for a short (infinitesimal) period of time is proportional to the square of the error in the control multiplied by the length of the period. Referring to (5.4.4) and (5.4.5) it can be deduced that the J-efficiency of \( F_a(x, a) \) at \( x \) is given by

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Implementation of Control Strategy via Nonlinear Feedback.

\[ \eta = 1 - \frac{\delta J}{\int_{t_0}^{t_{*}} L[x(t), u(t)] \, dt} = 1 - \frac{(F_c(x, a) - F_c^*(x))^2}{m_b^2 L[x, F_c(x, a)]} \]  

It should be noted that \( \eta = 1 \) for the optimal regulator, \( \eta = 0 \), if the applied control makes no reduction in the cost potential and \( \eta \) can, in principle, be negative if the control action actually increases the cost potential.

A reasonable choice for the error criterion is to aim to maximise the \( J \)-efficiency of the feedback controller, hence for each state in the design sample, let the term in \( E \) be

\[ \varepsilon = 1 - \eta = \frac{(F_c(x, a) - F_c^*(x))^2}{m_b^2 L[x, F_c(x, a)]} . \]  

However, this is not of the required format (5.1.3) for the application of linear least squares optimisation for \( a \) since the weighting function is an explicit function of \( a \). If the weighting is approximated by replacing \( F_c(x, a) \) by \( F_c^*(x) \) which has already been calculated then the correct format is achieved and this slight change of weighting is only likely to have a subtle effect on the choice of \( a \), thus let:

\[ \varepsilon = \frac{(F_c(x, a) - F_c^*(x))^2}{m_b^2 L[x, F_c^*(x)]} \]  

where the weightings, \( w(x) \), are now given by

\[ w(x) = \frac{1}{L[x, F_c^*(x)]} \]  

and the constant factor of \( \frac{1}{m_b^2} \) can be ignored.
Implementation of Control Strategy via Nonlinear Feedback.

5.5 Results and Testing of the Fitting Procedure.

The ability of \( F_c(x, a) \) to represent \( F_c^*(x) \) must be tested and validated and this can be done 'statically' by evaluating the J-efficiency of the feedback at given states. Firstly, the representation is to be examined throughout the design sample, as mentioned. This gives a measure of the ability of the basis functions to represent the optimal behaviour. Then a test sample of 20 points is to be considered and this gives a more representative test of the fit and examines the conditioning of the choice of \( a \) by testing the fit at non-design points.

The fit can also be tested 'dynamically' by comparing the dynamic behaviour of the closed loop system with feedback defined by \( F_c(x, a) \) with that of the optimal open loop system. The only situation in which the optimal open loop response can easily be calculated is for an initial condition disturbance, and 'dynamic' testing is hence restricted to such conditions. Some of the points in the test sample are to be used as initial conditions and the comparisons made will be principally visual, though the sub-optimality can be quantified to some extent by the increase in dynamic cost.

The results of the fitting procedure for the cost function with reference parameters from §4.6 are presented here. \( F_c^*(x) \) was evaluated at the 96 points in the design sample via the solution of the nonlinear optimal regulator taking approximately 7 hours of cpu time on the Sun Sparc 2 workstation. Equation (5.1.6) is then formed and solved for the parameters, \( a \), this least squares optimisation is very computationally efficient and takes negligible cpu time.

The potential ability of the basis functions to fit the optimal data is measured by the J-efficiency of \( F_c(x, a) \) evaluated at the points in the design grid. The J-efficiencies are above 99% at 71 of the 96 design points. However, the fit is quite poor at 2 of the points with efficiencies of -116% and 9.5% recorded. This reduced the mean efficiency to 94.8%, but the quality of the representation over the majority of the design grid is good suggesting that these basis functions are capable of giving an acceptable fit over the majority of \( X \).

As a preliminary guide to the conditioning of the fit offered by the chosen design sample, the condition number of \( A \) was considered. This was assumed to be of a similar order of magnitude to the condition number of \( A_w \) and hence able to provide an indication of any ill-conditioning. Now that \( A_w \) itself has been evaluated its condition number can be calculated and this was found to be 83.6. This is of a similar order of magnitude to the condition number of \( A \) and again does not indicate any ill-conditioning.

A more effective test of the conditioning and of the quality of the chosen \( a \) for these
basis functions is to examine the J-efficiencies of the fit throughout the test sample. This requires the calculation of $F_c^*(x)$ at the 20 test sample points which takes a further 1.5 hours of computation time. They were found to have a mean value of 96.3% with just one point having an efficiency less than 85% and with 14 of the 20 above 97%. Since the 'static' performance of the feedback law on the random test sample is similar to that observed on the design sample, it is reasonable to conclude that the fit is well conditioned and that the chosen a enable the feedback law to provide a reasonable approximation of the optimal controls throughout $X$.

In an attempt to relate the J-efficiency to dynamic performance, dynamic tests of the fit have been performed. These compare the responses of the nonlinear feedback closed loop system with the optimal responses for three initial condition disturbances. These disturbances have been chosen from the test sample; they are the point at which the J-efficiency was best, the point at which it was worst and the median point. The results for best, worst and median are plotted in Figs 5.5.1, 5.5.2 and 5.5.3 and their comparative dynamic costs are given in Table 5.5.1.

<table>
<thead>
<tr>
<th>Initial Condition Disturbance</th>
<th>J-efficiency of Initial Point</th>
<th>Dynamic Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Optimal</td>
</tr>
<tr>
<td>Best</td>
<td>100 %</td>
<td>4.300</td>
</tr>
<tr>
<td>Worst</td>
<td>75.7 %</td>
<td>5.944</td>
</tr>
<tr>
<td>Median</td>
<td>98.2 %</td>
<td>4.091</td>
</tr>
</tbody>
</table>

Table 5.5.1: Comparative Dynamic Costs for Initial Condition Disturbances.

It should be noted that this test is dissimilar to the static tests since the optimal and feedback systems occupy dissimilar states at all times other than the initial condition and, hence, a direct comparison of the controls or calculation of the J-efficiencies is not meaningful.

The results show that the dynamic behaviour of the feedback controller is a reasonable approximation to the optimal in all these cases and that the approximation is slightly better in the cases where the initial approximation is known to be superior. The increases in dynamic cost which quantify the sub-optimality are fairly small.
Fig 5.5.1: Dynamic Test of Fit at Best Test Point

Tyre Deformation (m)

Suspension Deflection (m)

Body Acceleration (m/s²)

Time (secs)
Fig 5.5.2: Dynamic Test of Fit at Median Test Point

Optimal

Suspension Deflection (m)

Body Acceleration (m/s/s)

Notes:
- Optimal
- Feedback

Time (secs)
Fig 5.5.3: Dynamic Test of Fit at Worst Test Point

- Optimal
- Feedback
Chapter 6.

Performance Comparisons.

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6. Performance Comparisons.

The performance of the quarter vehicle model employing some nonlinear feedback controllers and the passive suspension system will be compared in this section. The results for the passive system have already been discussed in §2 and these will initially be compared with results obtained using a nonlinear controller designed with respect to a cost function with the reference parameter values from §4.6. This controller is denoted Nonlinear Feedback 1.

In §4.6.2 the initial condition responses of the optimal Nonlinear Regulators 1, referring to this cost function, were compared with those of the passive suspension system. It was found that they were able to meet the performance guidelines on handling and workspace usage at the extreme amplitudes of disturbance and also give improved ride at non-extreme amplitudes of disturbance. In view of the dynamic accuracy of the feedback representation of the optimal controls observed in §5, it is expected that the feedback controller will be able to achieve similar results. The results of performance simulations following such initial condition disturbances for Nonlinear Feedback 1 are presented and compared in §6.1.

The simulation of the response to disturbing inputs provides a more comprehensive and realistic test of the performance of the system and simulations of the responses to the disturbing inputs considered in §2 will be studied in §6.2. Again, it is hoped that Nonlinear Feedback 1 will enable the active system to match the passive system for handling ability and ability to operate within the workspace usage constraints, whilst offering a superior ride quality.

Since no precise method of evaluation of performance of the quarter vehicle model is known, nor are any precise performance requirements universally accepted, the performance assessments made in this section are not definitive. In reality, the whole controller design procedure would require further testing beyond that considered in this work. This is likely to lead to the desire to make further changes to the behaviour of the system which would be brought about by tuning. In §6.3 a tuning exercise will be demonstrated by analysing the changes in behaviour of the closed loop system due to changes in the parameter values of the underlying cost function.
6.1 Initial Condition Performance.

For completeness and as a simple test of performance, the responses to the initial conditions considered in §2 will be examined here. As stated in the introduction to this chapter, the performance of Nonlinear Feedback 1 is expected to be very similar to that of Nonlinear Regulators 1 for these conditions. The results are summarised in Table 6.1 and the time histories plotted in Figs 6.1.1-4.

It can be seen that the behaviour of the feedback controller is broadly similar to that of the optimal regulators further validating the dynamic ability of the feedback to implement the control strategy. For the extreme amplitude wheel hop condition, the slight discrepancy between the optimal controls and the feedback causes a slightly greater peak tyre deformation of 20.3 mm against 19.3mm, but this is still well within the performance guideline of 25.3mm. This slight difference in response can also be observed in the rms body acceleration figures which are slightly lower for the feedback system, this corresponds to the slight increase in tyre deformations. The responses to the third amplitude disturbance in this condition are virtually identical.

The extreme amplitude body bounce condition response again shows a slight dissimilarity between the behaviour of the feedback controller and the optimal regulator, during the initial phase, the feedback controller applies slightly greater suspension forces causing greater body accelerations and tyre deformations. These greater forces also reduce the peak suspension deflection from 86.6mm to 83.6mm which is again comfortably within the performance guideline of 110mm. For the half amplitude disturbance in this condition, the responses are very similar.

From these responses, it can be seen that the nonlinear feedback controller produces a quarter vehicle system model whose characteristics are amplitude dependent - in particular, as observed for the nonlinear optimal regulator responses in §4.6.2, the rate of attenuation of the disturbance reduces at lower amplitudes. The similarity between the feedback and optimal regulator responses suggests that the cost function parameters for given performance guidelines can initially be chosen on the basis of the performance of the optimal regulators and that when a feedback is designed for these parameters, the resulting closed loop system will behave in a similar way. The parameters have been chosen such that the simple performance guidelines are comfortably met and now the feedback controller can be subjected to the more comprehensive forced disturbance tests.

To summarise for these simple tests, this nonlinear active system is able to meet the performance guidelines for handling ability and workspace usage set by the passive system at the extreme amplitudes of disturbance and also produce improved ride quality for non extreme amplitude disturbances.
Wheel Hop Condition.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>R.M.S values for first 0.5 seconds</th>
<th>Peak Tyre Deformation (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tyre Deformation (mm)</td>
<td>Suspension Deflection (mm)</td>
</tr>
<tr>
<td>Extreme Amplitude</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>6.16</td>
<td>6.35</td>
</tr>
<tr>
<td>Active : Nonlinear Regulator 1</td>
<td>5.16</td>
<td>13.5</td>
</tr>
<tr>
<td>Nonlinear Feedback 1</td>
<td>5.54</td>
<td>16.3</td>
</tr>
<tr>
<td>Third Amplitude</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>2.05</td>
<td>2.12</td>
</tr>
<tr>
<td>Active : Nonlinear Regulator 1</td>
<td>2.94</td>
<td>3.01</td>
</tr>
<tr>
<td>Nonlinear Feedback 1</td>
<td>3.00</td>
<td>3.05</td>
</tr>
</tbody>
</table>

Body Bounce Condition.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>R.M.S values for first second</th>
<th>Peak Suspension Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tyre Deformation (mm)</td>
<td>Suspension Deflection (mm)</td>
</tr>
<tr>
<td>Extreme Amplitude</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>5.47</td>
<td>55.6</td>
</tr>
<tr>
<td>Active : Nonlinear Regulator 1</td>
<td>4.47</td>
<td>44.2</td>
</tr>
<tr>
<td>Nonlinear Feedback 1</td>
<td>4.50</td>
<td>43.7</td>
</tr>
<tr>
<td>Half Amplitude</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
<td>2.74</td>
<td>27.8</td>
</tr>
<tr>
<td>Active : Nonlinear Regulator 1</td>
<td>2.06</td>
<td>34.3</td>
</tr>
<tr>
<td>Nonlinear Feedback 1</td>
<td>2.07</td>
<td>32.4</td>
</tr>
</tbody>
</table>

Table 6.1: Response Summary for Initial Disturbance Simulations.
Fig 6.1.1: Extreme Amplitude Wheel Hop Condition

- Passive
- Nonlinear Regulator 1
- Nonlinear Feedback 1
Fig 6.1.2: Extreme Amplitude Body Bounce Condition

- Passive
- Nonlinear Regulator 1
- Nonlinear Feedback 1
Fig 6.1.3: Third Amplitude Wheel Hop Condition

- Passive
- Nonlinear Regulator 1
- Nonlinear Feedback 1

Tyre Deformation (m)

Suspension Deflection (m)

Body Acceleration (m/s²)
Fig 6.1.4: Half Amplitude Body Bounce Condition

- Passive
- Nonlinear Regulator 1
- Nonlinear Feedback 1
6.2 Comparison of Forced Responses.

The forced responses of the nonlinear feedback controller, Nonlinear Feedback 1, will now be examined and compared with those of the passive system. Disturbing inputs similar to those considered in §2 are used to test the performance and enable direct comparisons to be made. These include the responses to the two stochastic disturbance models (Gaussian white noise and the low-pass filtered noise) and simulations of the vehicle traversing the measured roads.

The Gaussian white noise disturbance model principally causes excitation in the wheel hop condition and this is used at both extreme and non-extreme amplitudes to examine the balance between ride quality and handling ability. Conversely, the low-pass filtered disturbance model principally excites the body bounce condition and is again applied at both extreme and non-extreme amplitudes to examine the balance between ride quality and workspace usage. In addition to these, simulations of the traversing of roads provide a test of model behaviour which is more easily related to experienced vehicle behaviour.

The simple initial condition test of the behaviour in the wheel hop condition suggested that the handling ability of the nonlinear active system is at least comparable with that of the passive system and that the ride quality is superior at non-extreme amplitudes of disturbance. Firstly, the responses to the Gaussian white noise disturbance model are to be considered, and it is hoped that for these more comprehensive tests of behaviour, the nonlinear active system will also be able to match the handling ability at extreme disturbance amplitudes and produce superior ride quality at non-extreme amplitudes.

A summary of the results is shown in Table 6.2.1 and it can be seen that at the extreme amplitude of disturbance the rms body accelerations for the nonlinear active system are significantly lower, thus illustrating a significantly superior ride quality. The peak tyre deformation is similar to that of the passive but the rms is significantly greater, suggesting that the shape of the distribution is dissimilar to the Gaussian shape for the linear passive system. To compare the handling ability offered by the systems, these distributions must be considered in greater detail.

To compare the shapes of the two distributions a probability plot (Box and Draper, 1987) has been drawn, see Fig 6.2.1. The time histories of the simulated tyre deformations have been sampled at a constant sampling interval and plotted in ascending order against the normal score. A sample taken from a normal distribution would approximately produce a straight line on such a probability plot; the quality of the approximation being dependent upon the size of the sample, and in the limit as the sample size tends to infinity, the plot will produce a perfect straight line.
Performance Comparisons.

Gaussian White Noise Disturbance.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Tyre Deformation (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive</td>
<td>6.36</td>
<td>19.3</td>
<td>2.30</td>
<td>26.6</td>
</tr>
<tr>
<td>Active: Nonlinear Feedback 1</td>
<td>7.62</td>
<td>23.5</td>
<td>1.68</td>
<td>25.9</td>
</tr>
<tr>
<td>(1/2 amp)</td>
<td>4.51</td>
<td>11.8</td>
<td>0.712</td>
<td>15.8</td>
</tr>
</tbody>
</table>

Filtered Noise Disturbance.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Suspension Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive</td>
<td>3.52</td>
<td>35.2</td>
<td>2.75</td>
<td>99.9</td>
</tr>
<tr>
<td>Active: Nonlinear Feedback 1</td>
<td>1.31</td>
<td>40.2</td>
<td>0.920</td>
<td>102</td>
</tr>
<tr>
<td>(1/2 amp)</td>
<td>0.510</td>
<td>24.5</td>
<td>0.324</td>
<td>71.6</td>
</tr>
</tbody>
</table>

Table 6.2.1: Response Summary for Stochastic Disturbance Models.

The samples produced have been divided throughout by their respective rms values to give a similar gradient at the origin to aid visual comparison. The plot clearly shows that the line for the sample taken from the response of the passive system is approximately linear whereas that from the nonlinear active system is not. This implies that, as expected, the tyre deformation time history of the passive system is normally distributed and that of the nonlinear active system is not. The fall off of the curve at high amplitudes suggests that the shape of the distribution for the nonlinear active system has a lower peak to rms ratio than a normal distribution. This feature of the shape of the distribution is also apparent in the results quoted in Table 6.2.1 and enables the nonlinear active system to allow a greater rms tyre deformation whilst matching the peak tyre deformation of the passive system.

An absolute measure of handling ability can not be drawn directly from these results since a definitive relationship between handling ability and tyre deformations is not known. However, in §2 it was suggested that the peak tyre deformations were a good indication of handling ability and hence, it is reasonable to conclude that the handling
Fig 6.2.1: Probability Plot of Sampled Tyre Deformation Time Histories

- Passive
- Nonlinear Active
Fig 6.2.2: Power Spectral Densities of Body Accelerations

- Passive
- Nonlinear Active
- Extreme Disturbance
- Nonlinear Active
- Non-Extreme Disturbance

Frequency (Hz)
ability is comparable to that of the passive system. Furthermore, this feature of the shape of the distribution of tyre deformations is advantageous since the greater rms value allows improvements in ride to be made.

The response of the nonlinear active system has also been simulated for a disturbance half the amplitude of this extreme level and the results included in the Table. Since the passive system is linear, its response would simply be a linear scaling of its response for the extreme amplitude disturbance. It can be seen that at this non-extreme amplitude of disturbance, further relative improvement has been made in the ride quality. This is quantified by a disproportionate reduction in rms body accelerations by a factor of 2.36 compared to those for the extreme amplitude disturbance. Conversely, both peak and rms tyre deformations are disproportionately increased.

The power spectral densities of the body accelerations for both passive and nonlinear active systems are compared in Fig 6.2.2. The spectrum produced by the nonlinear active system operating at the half amplitude of disturbance has been multiplied by 4 to normalise it for comparison with the spectra produced at the extreme amplitude of disturbance. It can be seen that the reduction in body accelerations over the passive system occurs principally at low frequencies. Hedrick and Butsuen (1988) discovered that there is an invariant point in the transfer function from road input to body acceleration for a quarter vehicle model, i.e. the transfer gain at a certain point is independent of the suspension system used in the model. This invariant point occurs at approximately 12Hz for this model and is also apparent in this plot.

The comparison of the spectra produced by the nonlinear active system operating at different amplitudes shows how it is able to naturally adapt its characteristics to the disturbance level. At the half amplitude it has an effective transfer function which transmits significantly less power into body accelerations than the effective transfer function for the extreme amplitude. The disproportionate reduction in the transmission to body accelerations occurs at all frequencies other than the invariant point.

Now the filtered noise disturbance model is to be considered to test the performance in the body bounce condition. The simple initial disturbance tests of the behaviour in this condition showed that at the extreme amplitude disturbance, the ride performance of the nonlinear active system was significantly superior to the passive system and the workspace usage performance guideline was comfortably met. It was also noted that further relative improvements in ride were apparent at non-extreme amplitudes of disturbance. Again it is hoped that this level of performance is maintained for these more comprehensive tests of behaviour in this condition.

From the summary of the results, also shown in Table 6.2.1, the rms body accelerations illustrate that the ride offered by the nonlinear active system is again
Performance Comparisons.

significantly superior to that of the passive system. The peak suspension deflections are similar but the nonlinear active system has a significantly greater rms, again suggesting that the shapes of the distributions are dissimilar with the nonlinear active having a smaller peak to rms ratio.

Neither distribution will be normal for this non-Gaussian input; however, the difference between the shapes of the distributions is similar to the difference between the shapes of the tyre deformation distributions following the wheel hop condition test. Again the nonlinear active system is able to utilise a greater rms whilst matching the peak level. The results of this test suggest that the workspace usage performance of the two systems is similar since they are equally capable of avoiding the bump stops.

The response of the nonlinear active system to the half amplitude disturbance has also been simulated and again it can be seen that further relative improvements in ride quality have been made. The rms body accelerations are disproportionately reduced at this level being a factor of 2.84 lower than those at the extreme disturbance level and as expected, both peak and rms suspension deflections are disproportionately increased. Again, the nonlinear active system displays its ability to change its characteristics in line with the disturbance inputs.

To summarise the results of the simulations for the stochastic disturbance models; it was found that this nonlinear active system was able to approximately match the peak tyre deformations and suspension deflections of the passive system at the extreme amplitudes of disturbance. This suggests that the handling ability and workspace usage attributes of the performance are comparable with the passive system. The shapes of the distributions of the simulated time histories of these states are quite different to those of the passive system and significantly greater rms levels are utilised for similar peaks. This aids the system to produce a significantly superior ride quality. Further relative improvements in ride are also achieved at non-extreme amplitudes of disturbance by the natural adaptation of the controller to the amplitude of the input, thus changing the characteristics of the quarter vehicle as appropriate.

The results of the comparative simulations of the systems traversing the measured roads are now presented and a summary of the results given in Table 6.2.2. In §2 it was noted that the A127 simulation was principally a test of the balance between handling ability and ride performance and that the disturbance was not too extreme. Conversely, the Lower Dunton Road was seen to be quite a severe disturbance and tested all aspects of the performance.

The comparison of rms body acceleration results for the A127 simulation shows that the nonlinear active system offers a significantly superior ride quality whilst the tyre deformations suggest that the handling ability is slightly worse. The rms tyre
deformation is significantly greater but, as observed for the stochastic disturbances, the ratio from rms to peak is less and the actual peak value is only slightly greater than that of the passive. The level of workspace usage is significantly increased compared to the passive system but it still remains well within the permissible limits and hence, is not a problem here.

The results for the Lower Dunton Road show that again, the nonlinear active system offers a significantly superior ride quality to the passive system and its adherence to the limits on workspace usage is good with a peak suspension deflection significantly lower than that of the passive. The shape of the distribution of suspension deflections was again found to be different; a significantly greater rms was utilised aiding ride performance. However, a greater peak tyre deformation is observed which suggests that here the handling performance may be slightly inferior to that of the passive.

A127 @ 30 m/s.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Tyre Deformation (mm)</th>
<th>Peak Suspension Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive</td>
<td>2.30</td>
<td>13.1</td>
<td>1.21</td>
<td>12.3</td>
<td>35.0</td>
</tr>
<tr>
<td>Active: Nonlinear Feedback 1</td>
<td>2.82</td>
<td>18.6</td>
<td>0.521</td>
<td>12.9</td>
<td>47.1</td>
</tr>
</tbody>
</table>

Lower Dunton Road @ 20 m/s.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Tyre Deformation (mm)</th>
<th>Peak Suspension Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive</td>
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<td>26.4</td>
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<td>80.9</td>
</tr>
<tr>
<td>Active: Nonlinear Feedback 1</td>
<td>3.75</td>
<td>32.8</td>
<td>0.906</td>
<td>17.0</td>
<td>69.7</td>
</tr>
</tbody>
</table>

Table 6.2.2 Response Summary for Simulations of Traversing Measured Roads.
6.3 Tuning of the Feedback Controller.

The controller, Nonlinear Feedback 1, demonstrated in the previous section was derived from the reference cost function of §4.6 and gave broadly favourable results. However, it was pointed out in the introduction to this chapter that no absolute performance criteria are given for the system and in this section, it will be demonstrated how the system performance can be modified by tuning the controller.

The assumed aim is to reduce high amplitude tyre deformations and thus improve the handling performance of the quarter vehicle model. Such high amplitude tyre deformations are most prevalent following the extreme amplitude Gaussian white noise disturbance test and in the simulation of traversing the Lower Dunton Road and reductions shall principally be sought for these disturbances.

In §4.6.3, the effects on the behaviour of the optimal regulators of changes in the cost function parameters were investigated. The trends discovered showed that, as expected, increasing $\alpha_1$ caused a reduction in the peak tyre deformation recorded in response to the extreme amplitude wheel hop condition initial disturbance. In that study the increase made in $\alpha_1$ was from the reference value of $2 \times 10^{12}$ to $4 \times 10^{12}$ and the corresponding decrease in high amplitude tyre deformations was fairly small. Here greater reductions are sought and the trend as $\alpha_1$ is increased beyond this from $2 \times 10^{12}$ to $6 \times 10^{12}$ and then to $2 \times 10^{13}$ shall be investigated.

One natural consequence of the reduction in tyre deformation of this regulator response was the corresponding increase in body acceleration and this trend can also be expected for this tuning exercise. The regulator responses also showed that increasing $\alpha_1$ affected the optimal behaviour of the system following the other initial disturbances; a small reduction in low level tyre deformations and a corresponding increase in body accelerations were observed following the third amplitude wheel hop condition initial disturbance. For the body bounce condition tests, slight reductions in tyre deformations were again observed and these were relatively greater at high amplitudes. These reductions forced slight increases in suspension deflections and body accelerations. These side effects were not so significant as the principal effect at the extreme wheel hop disturbance but they suggest that these changes to the cost function parameters are likely to change the performance of the system following all types and amplitudes of disturbance.

To perform this tuning exercise two new nonlinear feedback controllers have been designed. They relate to the reference cost function with the two stages of increase to $\alpha_1$ as mentioned above, these shall be called Nonlinear Feedback 2 and Nonlinear Feedback 3. It is hoped that the implementation procedure of §5 is capable of creating
feedback controllers which show trends in performance which are consistent with the changes made to the cost function parameters.

As a preliminary investigation of the behaviour of the active suspension system employing these new controllers, simulations of responses to the simple initial condition disturbances are considered. These also provide a check on the implementation procedure by validating that the changes in the closed loop responses are in line with the changes in the cost function parameters.

The results are summarised in Table 6.3.1 and the time history plots shown in Figs 6.3.1-4. For the extreme amplitude wheel hop condition, it can be seen that as expected, the increases made to $\alpha_1$ cause significant reductions in the high amplitude tyre deformations and corresponding increases in the body accelerations. Greater levels of workspace usage also result but the amplitude of these is still small and they are not of concern.

Quite significant side effects are apparent at the third amplitude wheel hop condition disturbance where reductions in low amplitude tyre deformations and corresponding increases in body accelerations are observed. These side effects are similar to those observed in the regulator trends of §4.6.3. The body bounce condition tests also show the expected reductions in the high amplitude tyre deformations and corresponding slight increases in suspension deflection. There are slight effects on the body acceleration at the extreme amplitude but these are not really strong enough to establish a trend and the effects at the half amplitude are uncorrelated.
Wheel Hop Condition.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>R.M.S values for first 0.5 seconds</th>
<th>Peak Tyre Deformation (mm)</th>
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<tr>
<td></td>
<td>Tyre Deformation (mm)</td>
<td>Suspension Deflection (mm)</td>
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<td>Extreme Amplitude</td>
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<td></td>
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<td>Passive</td>
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<tr>
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<tr>
<td>Nonlinear Feedback 2</td>
<td>4.49</td>
<td>21.5</td>
</tr>
<tr>
<td>Nonlinear Feedback 3</td>
<td>3.55</td>
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Third Amplitude

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<th>Peak Suspension Deflection (mm)</th>
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<tr>
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</tr>
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<td>2.22</td>
<td>3.44</td>
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</table>

Body Bounce Condition.

<table>
<thead>
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<th>R.M.S values for first second</th>
<th>Peak Suspension Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tyre Deformation (mm)</td>
<td>Suspension Deflection (mm)</td>
</tr>
<tr>
<td>Extreme Amplitude</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passive</td>
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<td>55.6</td>
</tr>
<tr>
<td>Active: Nonlinear Feedback 1</td>
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<td>43.7</td>
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Half Amplitude

<table>
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<tr>
<th>Suspension System</th>
<th>R.M.S values for first second</th>
<th>Peak Suspension Deflection (mm)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Tyre Deformation (mm)</td>
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<tr>
<td>Nonlinear Feedback 3</td>
<td>1.99</td>
<td>34.0</td>
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Table 6.3.1 : Response Summary for Initial Disturbance Simulations.
Fig 6.3.1: Extreme Amplitude Wheel Hop Condition

- Nonlinear Feedback 1
- Nonlinear Feedback 2
- Nonlinear Feedback 3

Tyre Deformation (m)

Suspension Deflection (m)

Body Acceleration (m/s²)

Time (secs)

0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5

0 0.01 0.015 0.02 0.025

0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5

0 40 35 30 25 20 15 10 5 0

0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5
Fig 6.3.2: Extreme Amplitude Body Bounce Condition

- Nonlinear Feedback 1
- Nonlinear Feedback 2
- Nonlinear Feedback 3
Fig 6.3.3: Third Amplitude Wheel Hop Condition

- Nonlinear Feedback 1
- Nonlinear Feedback 2
- Nonlinear Feedback 3
Fig 6.3.4: Half Amplitude Body Bounce Condition

- Nonlinear Feedback 1
- Nonlinear Feedback 2
- Nonlinear Feedback 3
Performance Comparisons.

The results of the simulations of the behaviour in response to the stochastic disturbance models are presented in Table 6.3.2. It can be seen that the tyre deformations observed in response to the extreme amplitude Gaussian white noise model are reduced as sought. Both peak and rms levels are reduced with the distribution shape apparently remaining approximately the same. These results illustrate an improvement in the handling ability of the quarter vehicle model following this tuning procedure; also apparent is a corresponding degradation in the ride quality illustrated by the increased levels of body acceleration. However, even Nonlinear Feedback 3 is able to offer a better ride quality than the passive system whilst producing a slightly lower rms tyre deformation and a significantly lower peak.

The levels of tyre deformation in response to the non-extreme amplitude of Gaussian white noise are also reduced with a corresponding increase in body accelerations. This degradation in ride performance at this non-extreme amplitude is an undesirable side effect of the tuning but this is expected for these changes in cost function parameters from the knowledge gained from both the optimal regulator and closed loop initial condition responses. Nevertheless, Nonlinear Feedbacks 2&3 still exhibit a change in characteristics with disturbance amplitude in a similar manner to those illustrated for Nonlinear Feedback 1 in Fig 6.2.2. In fact, the reduction in body accelerations between the extreme amplitude and the half amplitude disturbances for Nonlinear Feedback 3 is even more disproportionate with a reduction factor of 2.53.

The tuning procedure has also had a slight effect on the results of the filtered noise model tests. The tests are principally an examination of the behaviour of the quarter vehicle model in the body bounce condition and are hence less sensitive to changes in $\alpha_1$. However, at the extreme amplitude of disturbance, there is a clear trend towards a reduction in workspace usage and an increase in body accelerations. A further trend that may be observed in these tests is the increase in rms tyre deformations, though it is a weak effect and the amplitudes concerned are very small. These increases may appear to be strange in light of the changes made to the cost function. However, they are feasible since the optimality of the controller does not strictly hold for these disturbance conditions.

The trend in the increase of body accelerations is also apparent in the simulated responses to the half amplitude filtered noise disturbance though it is not as strong as at the extreme amplitude. A similar trend is observed in the tyre deformations, though these are now at a very small amplitude and there is no significant effect upon the workspace usage.
Gaussian White Noise Disturbance.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>R.M.S Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Tyre Deformation (mm)</th>
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<tbody>
<tr>
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<td>Extreme Amplitude</td>
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<td></td>
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Filtered Noise Disturbance.

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<th>Body Acceleration (m/s²)</th>
<th>Peak Suspension Deflection (mm)</th>
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<tr>
<td></td>
<td>Extreme Amplitude</td>
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<td></td>
<td></td>
</tr>
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<td>3.52</td>
<td>35.2</td>
<td>2.75</td>
<td>99.9</td>
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<td>Nonlinear Feedback 2</td>
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<td>1.00</td>
<td>98.9</td>
</tr>
<tr>
<td>Nonlinear Feedback 3</td>
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<td></td>
<td>Half Amplitude</td>
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<td>50.0</td>
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<td>71.6</td>
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<td>0.375</td>
<td>68.2</td>
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Table 6.3.2: Response Summary for Stochastic Disturbance Models.
Performance Comparisons.

These results for the stochastic disturbance models show that the aim of the tuning procedure to improve the handling ability of the quarter vehicle model has been achieved. As predicted, these improvements are most apparent in the response to the extreme amplitude Gaussian white noise model where an inevitable degradation in the ride quality is also observed. These results were anticipated from the trends seen in the behaviour of the optimal regulators and the results for the simple initial condition disturbance tests. They illustrate how the tuning procedure is able to re-balance the various attributes of the performance.

Some undesirable effects of the procedure are also observed, these include the degradation of ride quality at the non extreme amplitude in the wheel hop condition and at both amplitudes in the body bounce condition. Again, these side effects were anticipated from the preliminary investigations. A further tuning procedure involving further adjustments to the parameters in the cost function could enable some of this degradation to be reclaimed.

The effect of the tuning on the performance of the vehicle traversing the measured roads has also been investigated and the response matrices are presented in Table 6.3.3. The results for the A127 again show the trend of improvements in handling against degradation in ride. This is clearly illustrated by the rms levels of the appropriate responses. The peak tyre deformation observed for Nonlinear Feedback 3 is now less than that for the passive though the rms is still slightly greater. The rms body acceleration, though substantially worse than Nonlinear Feedback 1, is still significantly lower than that for the passive. The tuning has caused no obvious change in the workspace usage which still remains well within the acceptable limits for this simulation.

For the Lower Dunton Road the improvement in handling ability is also apparent with the active system now able to beat the peak tyre deformation of the passive. The tuning has again caused a degradation in the ride quality but it is still significantly superior to that of the passive. A reduction in both peak and rms workspace usage has also resulted, though the rms remains greater than that of the passive.

Overall, the tuning exercise has demonstrated how the performance of the system can be adjusted via modification of the parameters in the cost function and significant improvements in handling ability have been made but these are at the cost of degradations in ride quality.
Performance Comparisons.

A127 @ 30 m/s.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>R.M.S. Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Tyre Deformation (mm)</th>
<th>Peak Suspension Deflection (mm)</th>
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<td>1.21</td>
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<td>0.521</td>
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<td>47.1</td>
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<td>0.587</td>
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<td>18.3</td>
<td>0.665</td>
<td>11.4</td>
<td>46.1</td>
</tr>
</tbody>
</table>

Lower Dunton Road @ 20m/s.

<table>
<thead>
<tr>
<th>Suspension System</th>
<th>Tyre Deformation (mm)</th>
<th>R.M.S. Suspension Deflection (mm)</th>
<th>Body Acceleration (m/s²)</th>
<th>Peak Tyre Deformation (mm)</th>
<th>Peak Suspension Deflection (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Passive</td>
<td>3.73</td>
<td>26.4</td>
<td>2.18</td>
<td>15.4</td>
<td>80.9</td>
</tr>
<tr>
<td>Active Nonlinear Feedback 1</td>
<td>3.75</td>
<td>32.8</td>
<td>0.906</td>
<td>17.0</td>
<td>69.7</td>
</tr>
<tr>
<td>Nonlinear Feedback 2</td>
<td>3.48</td>
<td>32.0</td>
<td>0.984</td>
<td>16.2</td>
<td>67.1</td>
</tr>
<tr>
<td>Nonlinear Feedback 3</td>
<td>3.26</td>
<td>29.7</td>
<td>1.09</td>
<td>15.2</td>
<td>65.6</td>
</tr>
</tbody>
</table>

Table 6.3.3: Response Summary for Simulations of Traversing Measured Roads.
Chapter 7.

Discussion and Conclusions.

7.1 Initial Study of Quarter Vehicle Model Incorporating Nominal Passive Suspension System. 133

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7. Discussion and Conclusions.

In this chapter the results and knowledge gained during the development and demonstration of the controller design methodology are discussed and summarised. Ideas for future study which may improve the methodology have become apparent and these are also discussed and will be listed at the end of the chapter. Finally, conclusions are drawn from the study to evaluate the methodology.

7.1 Initial Study of Quarter Vehicle Model Incorporating Nominal Passive Suspension System.

It has been assumed that the simple quarter vehicle model enables meaningful comparative performance assessments of various suspension systems to be made and hence, is an adequate tool for the demonstration of the methodology. However, its ability to provide absolute performance assessments is not proved here and is beyond the scope of this work.

In this study the behaviour of a model employing a nominal linear passive suspension system has been characterised by its modal properties. Although the modes of a quarter vehicle model are dependent upon the suspension system used and, strictly do not even exist when a nonlinear suspension system is used, analysis of the behavioural conditions to which they broadly relate still gives a useful way of assessing the performance of the model. The two behavioural conditions are known as the wheel hop condition and the body bounce condition and the ride to handling and ride to workspace usage trade-offs can be defined by analysing the behaviour in each of these conditions respectively.

The controller design methodology is demonstrated for an ideal active system used in this quarter vehicle model and some performance guidelines are created from the performance of the model incorporating the passive suspension system. These guidelines aim to ensure that the handling ability and the ability to adhere to the workspace constraints of the active system are comparable to those of the passive. Having met these guidelines, the design seeks to produce an active system that improves the ride performance of the quarter vehicle. The guidelines are; the peak tyre deformation observed on an initial condition test that principally excites the wheel hop condition and the peak suspension deflection observed on another initial condition test that principally excites the body bounce condition. The amplitudes of these initial disturbances are chosen to be extreme to test the system to its operating limits.

Some further, more comprehensive tests of performance are also carried out on the model incorporating the passive system. These consider the forced responses of the
system to disturbance inputs and are also used for comparison with the active systems. Both disturbance inputs generated from stochastic models and from measured road profiles are considered.

7.2 Development of Control Strategy for Active Systems and Implementation Under LQR Conditions.

The principles of time domain optimal control give a suitable basis to the controller design methodology for active suspension systems. A control strategy has been developed which is as follows: at a given state, apply the control that minimises the dynamic cost for the given cost function over a finite time horizon assuming zero forthcoming disturbances.

The credibility of the assumption of zero forthcoming disturbances is questionable; it has yielded a relatively simple and implementable control strategy though it may produce poor results when disturbances which have a significant level of auto-correlation are encountered. Under such circumstances, a stochastic model of the road profile in conjunction with observed knowledge of the recent profile history could enable an estimate to be made for the 'most likely' sequence of forthcoming disturbances over the time horizon. The control strategy could then be modified to produce the controls which minimise the dynamic cost subject to these specified 'most likely' forthcoming disturbances.

This method equates the problem of minimising the expected cost for unknown but stochastically defined disturbances to that of minimising the actual cost for a deterministically defined disturbing input and relies on the certainty equivalence principle, where, the deterministic disturbance is said to be the certainty equivalent of the stochastic model. It should be noted that, for a linear system model and a quadratic cost function, a zero disturbance is the certainty equivalent of Gaussian white noise and hence the similar controls produced by the LQG and LQR techniques, although this is not strictly true for a nonlinear system model or a non-quadratic cost function, assumption of zero disturbance approximates to assumption of Gaussian white noise distributed disturbances. N.B. both processes have zero expected levels of auto-correlation.

Further investigation into the implementation of a control strategy based on this principle for stochastic models of the road profile may lead to performance improvements under circumstances where there is a significant level of auto-correlation in the disturbance. However, as discussed in chapter 3, such an approach is likely to be less robust to unexpected sudden event disturbances. There are other approaches that could be taken such as 'minimax' which would take the action that minimises the
maximum value of the cost assuming a 'worst case' input from within a certain range of possible disturbances, however, an investigation of these other approaches is beyond the scope of this work.

The use of preview, either wheelbase or look-ahead, to improve the knowledge of forthcoming disturbances would almost certainly enhance the strategy (Hac, 1992). It would enable the system to make a more intelligent control decision and hence, prepare itself for future events. However, this area of study is not included in this work.

In this work the system model considered is linear and the control actuation assumed to be unconstrained and, for comparison, the control strategy is initially implemented for quadratic cost functions which enables the standard LQR technique to be used. This technique is mathematically convenient and is offered by many standard controller design software packages. A linear feedback law is produced which gives a perfect implementation of the control strategy for these conditions.

It yields linear closed loop active systems whose overall performance is superior to that of the reference passive system. The cost function parameters are chosen to give a system that meets the performance guidelines. A clear improvement in ride performance is apparent in the body bounce condition behaviour and this is visible in the response to the initial disturbance test that principally excites this condition. The improvement is also apparent from comparison of the eigenvalues which indicate a dramatic reduction in the settling time and an improvement in the damping ratio for this condition. However, the response for the initial disturbance test that principally excites the wheel hop condition is virtually identical to that of the passive, as are the wheel hop mode eigenvalue and eigenvector.

Using LQR, further significant improvements in the ride quality at non-extreme amplitudes of disturbance can be made by making adjustments to the weighting parameters in the cost function; however, it is found that there is insufficient flexibility in this design technique to produce a single controller which is able to achieve such improvements whilst meeting the performance guidelines at extreme disturbance conditions. This motivates the use of more flexible non-quadratic cost functions which is beyond the scope of the LQR technique but is permissible in this general nonlinear design methodology.

7.3 Evaluation of Control Strategy for Non-Quadratic Cost functions.

To evaluate the optimal control for a given state for such non-quadratic cost functions requires the solution of a nonlinear two point boundary value problem. This is not trivial, and after much investigation a reliable and efficient solution method was developed. This requires the discretisation of the controls which can then be optimised
Discussion and Conclusions

using a gradient based parameter optimisation procedure. It was found by successive
reduction of the zero order hold period that 1ms is sufficiently short to give responses
which approximate to those given by an optimal continuous time controller. The effect
of the horizon time was also investigated and it was found that as it was increased, the
initial optimal behaviour of the system converged and that 1.5 seconds was a sufficient
time horizon.

The solution of this two point boundary value problem allows the implementation of the
control strategy for chosen points and hence, the optimal responses for a reference non-
quadratic cost function can be compared with initial condition responses for passive and
LQR active systems. By appropriate choice of the parameters, the extra freedom in the
cost function allows regulators to be designed which give responses which match the
performance guidelines set by the passive system at extreme amplitudes of disturbance
and also provide significant improvements in ride at non-extreme amplitudes. The
ability of the designer to tailor the behaviour of the system via the choice of the cost
function parameters, i.e. tune the system, is also illustrated.

The computational effort of the solution of this two point boundary value problem is
significant and any way of reducing this would be very beneficial. Ways of choosing a
more intelligent initial guess for the solution to start the optimisation algorithm may
produce a slight acceleration and the algorithm itself or the convergence criterion may
be further tuned but these have been fairly well developed and such effort is not likely
to lead to significant reductions. Numerical accuracy may be sacrificed for speed but
this is likely to lead to unreliable, unrobust solutions and is not recommended. Also, if
accurate solutions are sought, the solution time is not greatly reduced by the extension
of the zero order hold period even though this reduces the number of parameters to be
optimised. However, the computation time is sensitive to the horizon time and the
results suggest that, in many cases, this may be reduced to 1 second or maybe even
below, without a drastic effect on the initial optimal behaviour. Hence, it is suggested
that the effects of the reduction of the time horizon could be investigated to accelerate
the solution of this problem.


A direct real time implementation of the control strategy is not practicable as this would
require the on-line solution of the nonlinear two point boundary value problem for
which the computational effort is too great. Hence, a feedback law is used to
implement the strategy in real time and the design of this is included in the
methodology. It was chosen to describe the feedback law by an analytical function
relating the states to the control.
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Overall, the implementation is fairly successful as validated by both the dynamic comparisons of open loop and closed loop responses and the static evaluations of the fit quantified by the J-efficiencies. However, the fit is not perfect and the precise effect of the fitting errors on the overall performance of the closed loop system is difficult to evaluate since comparisons with truly optimal responses can only be made for initial condition disturbances.

A major limiting factor in the design of the feedback law is the computation time; currently the design takes about seven hours per set of cost function parameters and hence, a tuning exercise requiring iteration of cost function parameters is quite time consuming. The fit could almost certainly be improved but this is likely to result in increased design times, conversely, the design time could be reduced but this would again almost certainly be at the expense of the quality of the fit. Sacrificing the quality of the fit is discouraged since it provides the crucial link between closed loop system performance and the cost function which is central to the whole design methodology.

The quality of the fit is dependent upon two factors; the choice of the basis functions and the subsequent choice of the free parameters. Generally, for any given set of basis functions the quality of fit is dependent upon the design grid used to evaluate the optimal parameters; a larger, more dense design grid will, up to a certain point, provide a better fit though require greater computational effort. However, care must be taken in the choice of the grid to avoid ill-conditioning problems. A detailed conclusive study into how to make a good choice for the design grid for a given set of basis functions has not been thoroughly carried out; however, the work has resulted in a methodology which gives reasonable results. Furthermore, a study of the effect on the fit of the choice of the design grid may give an insight into the trade-off between size of design grid and quality of fit.

An increase in the number and order of the basis functions would increase the flexibility of the feedback law and hence, increase its potential to fit. However, this would also increase the scale of the conditioning and sampling density requirements for a well defined set of coefficient parameters to be chosen. The inflexibility of the feedback law with the basis functions used is highlighted by its inability to give a good representation at a few of the points in the design sample as characterised by some relatively poor J-efficiencies.

A detailed study of the shape of the mapping from the state space to the optimal control space has not been carried out due to the expense of solving the two point boundary value problem which is required to define the map at any point. However, from the solutions considered, it is believed that this mapping has quite a few irregular undulations and hence, it is perhaps unreasonable to expect a global third order function to fit it accurately. Furthermore, the shape of the map is dependent upon the cost
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function parameter values and it is reasonable to assume that the more significant the high order terms become, the less smooth the mapping.

Further work may consider the addition of some higher order basis functions to try to handle this problem and give a better fit. This must be done in a systematic way else it will become computationally expensive and inefficient. To add a few select higher order basis functions and possibly remove some that are ineffective whilst subtly changing the design grid to ensure the selection of a well defined set of parameters is not a trivial task but it may give substantial rewards in terms of the fitting ability. Perhaps a better approach would be to consider one of the alternative implementation methods mentioned; however, all the methods mentioned suffer from the requirement, and hence computational expense, of many solutions of the nonlinear two point boundary value problem.

7.5 Performance of Nonlinear Closed Loop Active System.

The performance of the quarter vehicle model employing the nonlinear active system that results from the execution of the design methodology for the reference cost function parameter values has been comprehensively tested. Firstly, initial condition responses are considered and the behaviour of the model is generally favourable; the nonlinear active system enables it to comfortably meet the performance guidelines and also produce improved ride performance at non-extreme amplitudes of disturbance.

Forced responses are also considered, providing more comprehensive tests of performance. In response to disturbances generated by an extreme amplitude Gaussian white noise model, the nonlinear active system is able to give a significant improvement in ride quality compared to the passive system. The peak tyre deformations were similar, suggesting that the handling abilities were comparable. However, the shape of the distribution of the tyre deformation time history for the nonlinear active system is clearly non Gaussian with a lower peak to rms ratio. Hence, the nonlinear system produces a greater rms tyre deformation for a similar peak and this assists it in reducing the body accelerations.

The results for a non-extreme Gaussian white noise disturbance show a further relative improvement in ride quality and the comparison of the power spectral densities shows the change in the effective transfer function, from road disturbance to body accelerations, for the nonlinear system operating at different levels of disturbance. This illustrates how the nonlinear active suspension system is able to change the characteristics of the quarter vehicle model in line with the disturbance amplitude. Hence, the nonlinear controller naturally exhibits the properties of an adaptive algorithm without all the drawbacks of implementing such an algorithm.
Discussion and Conclusions

As for the passive system, the Gaussian white noise disturbance model principally excites the wheel hop condition and examines the trade-off between ride and handling. Conversely, the filtered noise disturbance model principally excites the body bounce condition and examines the trade-off between ride and workspace usage. For the extreme amplitude disturbance, the peak suspension deflections are similar, illustrating that the systems have similar ability to adhere to the constraints placed on the workspace. Again, the shape of the distribution of the time history of the suspension deflections for the nonlinear active system is clearly different to that of the passive with a lower peak to rms ratio and hence, it uses substantially more rms workspace for the similar peak which again aids it to reduce the rms body accelerations.

From the comparison of body acceleration power spectral densities for the Gaussian white noise disturbance, it is noted that the reductions principally occur between 0.5Hz and 5Hz. For this filtered noise disturbance model the input power is all below 3Hz and hence, as expected, the improvement in ride quality compared to the passive system is greater than for the Gaussian white noise disturbance. The natural adaptation to the disturbance amplitude is also apparent for this disturbance model with further relative improvements in ride at a non-extreme disturbance amplitude.

The results of simulations of traversing the measured roads confirm the ability of the nonlinear active system to improve the quarter vehicle performance under more realistic circumstances. For both the A127 and the Lower Dunton Road, the active system produces significantly better ride and the improvement is relatively greater for the Lower Dunton Road. This is expected from the stochastic disturbance results since the input spectra relating to the Lower Dunton road contains relatively more power in the 0.5Hz to 5Hz range. The A127 principally excites the wheel hop condition and hence provides a test of the handling ability of the quarter vehicle. The peak tyre deformations are similar for the passive and nonlinear active suspension systems though again the shapes of the distributions are dissimilar with the nonlinear active having a greater rms. The levels of suspension deflection are fairly low and hence not of concern, with both systems remaining well within the workspace constraints.

The Lower Dunton Road excites both conditions and hence tests both handling ability and adherence to workspace constraints. The peak suspension deflections are large though both systems manage to remain within the constraints. The value for the passive suspension system is clearly greater than that for the nonlinear active system and hence it is reasonable to conclude from this test that the ability of the nonlinear active system to adhere to the workspace constraints is at least comparable to that of the passive. Again, the shapes of the distributions of the suspension deflections are significantly different; the nonlinear active system has a significantly greater rms. However, the peak tyre deformation observed for the nonlinear active system is clearly greater than that for the passive system suggesting that the improvements in ride and
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change in shape of suspension deflection distribution are made at some expense to the handling ability of the model.

Overall, the simulated results obtained by the implementation of the controller design methodology for this reference cost function are promising, although, the performance criteria on which these assessments are based are somewhat artificial. To fully design an active suspension system, further testing on more complex and realistic models and on real vehicles must be performed which may subsequently illustrate the need for modifications to these criteria. Hence, the design methodology must enable the controller to be redesigned to tune the performance of the system in line with such modifications. This ability is demonstrated by the tuning exercise considered, where the system performance is changed in line with changes made to the underlying cost function.

7.6 Conclusions.

The results of this study suggest that the nonlinear controller design methodology is capable of producing effective controllers for computer-controlled suspension systems. It has been demonstrated for the specific case of an ideal active system employed in a quarter vehicle model. The methodology allows the use of non-quadratic cost functions which in turn give greater freedom to the designer to tackle the complex design problems of a computer-controlled suspension system.

Having set some artificial performance criteria for the quarter vehicle model, the performance employing an ideal active system employing a nonlinear controller produced by the methodology was found to be superior to that employing a nominal passive system. Some of the improvement could be attributed to the greater capabilities of ideal active hardware but a significant proportion is directly attributable to the effectiveness of the controller. The optimal nonlinear controller improves ride performance by both the improved utilisation of the available suspension workspace and the improved use of non-critical levels of tyre deformation through non-Gaussian behaviour, which can only be achieved by a nonlinear algorithm. It also displays the inherent nonlinear ability to adapt the characteristics of the quarter vehicle model to the amplitude of the prevailing motions.

The methodology can, in principle, be used to produce controllers which seek to give system performance as defined by a broad range of performance goals. It also enables the designer to adjust the performance of the quarter vehicle by modifying parameters in the cost function and this ability is illustrated by the tuning exercise considered.

Though the application is restricted here to ideal active systems, the methodology is applicable to computer-controlled suspension systems with other hardware and will
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potentially produce effective controllers.

To use a controller designed by this methodology in practice on a vehicle will require a state estimator. This requirement is similar to that of an optimal linear controller produced by a standard technique or indeed, any full state feedback controller. The sensitivity of the control to the ability to estimate the states may be greater for such nonlinear controllers and further investigation of this may prove necessary. However, knowledge of the expected accuracy of the state information could be incorporated into the fitting of the feedback law and states whose observability is poor made less important or ignored. The nonlinear feedback laws produced by the methodology will require greater on-line computation than a standard linear feedback law. However, the requirement should not present a great problem to a modern processor.

Overall, it can be concluded that this nonlinear controller design methodology potentially gives the designer greater ability to design effective controllers for computer-controlled suspension systems than standard linear design techniques. However, it requires significantly greater computational effort and the methodology is in an early stage of development and some further refinement of the numerical methods is necessary to reduce the design time.

7.7 Recommendations for Further Work.

1) The numerical solution of the nonlinear two point boundary value problem merits further investigation since it is the backbone of the whole controller design methodology. It is suggested that the use of a shorter horizon time may significantly reduce computational effort without great detriment to the overall methodology.

2) A more accurate implementation of the control strategy should be sought, probably by the inclusion of other higher order basis functions in the feedback law. However, to permit the efficient computation of a well conditioned choice of coefficient parameters, the method of choosing the design sample should also be reconsidered.

3) The credibility of the assumption of zero forthcoming disturbances, made in the underlying control strategy, should be investigated. The detrimental effect of this should be quantified and the option of optimising the control with respect to a stochastic disturbance model studied.

4) The potential of the methodology to design effective controllers for systems which require description by a nonlinear system model or have constraints on the control actuation should be investigated. This is another area in which the design constraints of the standard linear design techniques are potentially restrictive.
5) The development of the methodology to permit application to more complex and realistic vehicle models should be investigated. Half or even full vehicle models would enable handling considerations to be included in the design and performance testing along with the problem of attitude control. Such models would also enable the potential enhancement given by wheelbase preview to be considered.
References.


References.


References.


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References.


