Robust Waveform Design for Multistatic Cognitive Radars

GAIA ROSSETTI AND SANGARAPILLAI LAMBOOTHARAN, (Senior Member, IEEE)
Wolfson School of Mechanical, Electrical and Manufacturing Engineering, Loughborough University, Loughborough LE11 3TU, U.K.

Corresponding author: Gaia Rossetti (g.rossetti@lboro.ac.uk)

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ABSTRACT
In this paper, we propose robust waveform techniques for multistatic cognitive radars in a signal-dependent clutter environment. In cognitive radar design, certain second order statistics such as the covariance matrix of the clutter, are assumed to be known. However, exact knowledge of the clutter parameters is difficult to obtain in practical scenarios. Hence, we consider the case of waveform design in the presence of uncertainty on the knowledge of the clutter environment and propose both worst-case and probabilistic robust waveform design techniques. Initially, we tested our multistatic, signal-dependent model against existing worst-case and probabilistic methods. These methods appeared to be over conservative and generic for the considered scenario. We therefore derived a new approach where we assume uncertainty directly on the radar cross-section and Doppler parameters of the clutters. Accordingly, we propose a clutter-specific stochastic optimization that, by using Taylor series approximations, is able to determine robust waveforms with specific signal to interference and noise ratio outage constraints.

INDEX TERMS
Waveform design, robust optimization techniques, stochastic optimization techniques, cognitive radars.

I. INTRODUCTION
Cognitive radars are based on three main concepts: intelligent signal processing; feedback from the receiver to the transmitter and preservation of the information content of radar returns [1]. This innovative idea, supported by new flexible computing architectures and solid state transmitters, has been subjected to extended research in the areas of target detection [2], [3]; target tracking [4], [5]; waveform design [6], [7] and many others [8]. In a Cognitive Radar Network (CRN), several radars could work together in a cooperative manner [9] combining the benefits of both cognition and diversity offered by Multiple Input Multiple Output (MIMO) radars. These radar systems employ multiple transmit waveforms and have the ability to jointly process signals received at multiple receiver antennae [10]. Distributed radars provide potential advantages such as enlarging coverage areas, improving detection performance and many others [11]. The disadvantages of such radar networks are the interference among radar transceivers, the requirement of precise location information of sensing nodes, synchronization and the need for data fusion [11].

In recent years many works considered both cognitive radars and MIMO systems, but not much work has been carried out about the added value of merging these two concepts. Optimization of waveform lies at the heart of radar design. Various criteria have been used for waveform design. For example, one of the earlier works [12] considered mutual information (MI) for the waveform design. Further developments of this concept can be found in [13] and [14], where the mutual information between subsequent radar returns is used as a figure of merit for ultra-wideband waveform optimization. In [15] maximization of MI between the target impulse response and the received echoes is used to improve the target detection and feature extraction performance. In [16], MI is used for robust information extraction in distributed multiple-radar systems (DMRSs). The works in [7] and [17] are aimed at sequentially designing a desired waveform by minimizing the Bayesian Cramér-Rao bound under some constraints such as constant-modulus and peak-to-average-power ratio for the transmit beamforming. Finally, [11] provides a methodology for the design of a family of frequency-selective orthonormal wavelets. Within the context of cognition, matched illumination has been proposed in [18] and [19]. A comprehensive outline of SNR-based and matched illumination-based waveform design techniques has been proposed in [18] for both known and stochastic targets, in a monostatic radar scenario.
The work in [19] discusses matched illumination waveform design for a multistatic through-the-wall radar system where the target is assumed to be stationary and with a known impulse-response. Furthermore, the work in [20] proposes an optimal waveform design method and a fast hierarchical optimization method to optimize a wideband cognitive radar (WCR) waveform for joint target radar signature (TRS) estimation and target detection.

Normally, the clutter environment is determined by prior information such as previous radar measurements, land cover databases or by using estimates based on training signals. However, it is difficult to obtain an exact description of the clutter. Therefore, radar waveform design should take into account uncertainties associated with clutter parameters. Traditionally, this can be done in two ways: with worst-case optimization techniques [21] and with stochastic optimization techniques [22]. An interesting example of robust radar waveform in the presence of signal-dependent interference can be found in [23]. This work deals with the robust iterative optimization of the transmit signal and receive filter bank for the case of a monostatic radar and assumes unknown Doppler shift of the target.

In this article we propose three different robust optimization techniques. The first two techniques employ traditional worst-case optimization and probabilistic (stochastic) optimization, respectively. Both methods are used for robust radar waveform design in the presence of uncertainty on the clutter-plus-noise covariance matrix. The third technique considers a novel approach where the uncertainty is assumed directly on the radar cross-section and Doppler parameters of the clutter rather than on the estimated clutter-plus-noise covariance matrix. The latter is solved using Taylor approximations and stochastic optimization.

This paper is organized as follows. In Section II the general description of the model of the system is provided. This includes the depiction of environment and application, an outline of the relevant nomenclature and the formulation of the optimization problem. The Signal to Interference and Noise Ratio (SINR) equation that will be used for the Semidefinite Programming (SDP) is presented in Subsection II-A along with the description of the method implemented for the receiver filter optimization. The orthogonal codes optimization is presented in Subsection II-B. Section III describes the robust optimization techniques that are the objective of this paper. Subsection III-A introduces worst-case optimization techniques, whereas Subsection III-B proposes stochastic optimization techniques. Finally, Subsection III-C introduces a novel approach for clutter-specific stochastic optimization for the case of signal-dependent clutter. The performance of the proposed algorithms is analyzed and compared in Section IV followed by conclusions in Section V.

II. SYSTEM MODEL

We consider a multistatic cognitive radar system with centralized cognition. In this model the $M$ radars transmit $M$ mutually orthogonal signals in order to detect a target within a surveillance area. It is assumed that all radars can process the received signal, extract useful information and forward decisions to a centralized processor. The processor performs the joint optimization, determines the most appropriate $M$ waveforms for the subsequent transmissions of the radars and conveys them through a local backbone communication network.

Let us denote the signal of length $N$ transmitted by Radar-$i$ as $s_{i}(j) = [s_{i}(1) s_{i}(2) \ldots s_{i}(N)]^{T} \in \mathbb{C}^{N}$ with $i = 1, 2, \ldots, M$. In the considered model, we assume scattering elements to be located in an area of $N_{c}$ range rings, each of them subdivided into $L$ azimuth bins. The index $b_{r}$ is used to denote the range delay. We select Radar-1 as a reference radar for the other radars towards the relative position of the target and clutter. To account for the range position of the bins with respect to Radar-1, we considered a time-shift matrix $J_{i,b_{r}}$ described in [8], where Radar-$j$ is the transmitting radar, Radar-$i$ is the receiving radar and $b_{r}$ is the relative range delay. Radar-1 has therefore a zero-shift with respect to the position of the target and the corresponding time-shift matrix is $J_{1,1,0}$. On the other hand, the matrix $J_{j,b_{r}}$ with $i \neq 1$ accounts for the delays of the signal originating from the other radars. The signal received by the generic Radar-$i$ can be represented by the column-vector $x_{i}$. This encloses the signals sent by every radar and subsequently scattered by the target as follows:

$$x_{i} = \sum_{j=1}^{M} \left( \alpha_{T,(j,i)} J_{j,b_{r}} (s_{j} \odot p(v_{T,(j,i)})) \right) + c_{i} + n_{i}, \quad (1)$$

where $\alpha_{T,(j,i)}$ is the complex parameter that accounts for the propagation and backscattering effects of the channel experienced by the waveform sent by Radar-$j$ and received by Radar-$i$, $p(v_{T,(j,i)} = [1 \quad e^{j2\pi \nu_{T,(j,i,0)}} \ldots \quad e^{j2\pi (N-1)\nu_{T,(j,i,0)}}]^{T}$ is the temporal Doppler steering vector as defined in [8] and $v_{T,(j,i)}$ is the normalized target Doppler frequency for the channel. The target parameter, as seen by each radar, will be characterized by the variance $\sigma^{2}_{T,(j,i)} = \mathbb{E}[(\alpha_{T,(j,i)})^{2}]$ and mean $\mathbb{E}[\alpha_{T,(j,i)}] = 0$. This corresponds to the Radar Cross-Section (RCS) of the target. Similarly, for each illuminated clutter scatterer $\sigma^{2}_{c,(j,i,b)} = \mathbb{E}[(\alpha_{c,(j,i,b)})^{2}]$ and $\mathbb{E}[\alpha_{c,(j,i,b)}] = 0$. We also denote the normalized Doppler frequency of the clutter as uniformly distributed between $\tilde{\nu}_{c,(i,b)} - \epsilon$ and $\tilde{\nu}_{c,(i,b)} + \epsilon$. The noise $n_{i}$ is considered to be a zero-mean white Gaussian noise characterized by $\mathbb{E}[n_{i}] = 0$ and $\mathbb{E}[n_{i}n_{j}^{H}] = \sigma^{2}_{n} I$.

In this model the clutter is considered to be signal-dependent. We denote its instantaneous received component as seen by Radar-$i$ as $c_{i}$ and its covariance matrix (as defined in [24]) as:

$$\Psi_{c,(i)} = \mathbb{E}[c_{i}c_{i}^{H}] = \sum_{j=1}^{M} \sum_{b=0}^{B} \sigma^{2}_{c,(j,i,b)} J_{j,b} \text{diag}(s_{j}) \times \Phi_{j,b} \text{diag}(s_{j}^{H}) J_{j,b}^{H}, \quad (2)$$
where $\Phi_{ij,ib}(l,m)\equiv e^{2\pi i_{j,ib}(l-m)}\sin[\pi e_{ij,ib}(l-m)]/(\pi e_{ij,ib}(l-m))$, where $(l,m)$ indicates the position within the matrix. We assume the radars have an estimate of the clutter statistics, however, possibly with some estimation errors. The aim is therefore to determine optimal waveforms and receiver filters that will achieve a certain SINR performance in the presence of uncertainty on the clutter parameters. The proposed optimization is of an iterative nature where the receiver filter and the waveforms are designed alternatively by optimizing the SINR. Specifically, starting from a given receiver filter $w_{ij}(t-1)$ at iteration $(t-1)$, we search for the admissible radar codes $s_{ij}(t)$ that maximize the SINR subject to various constraints. When the waveforms are determined, we estimate the new adaptive receiver filter $w_{ij}(t)$ which maximizes the SINR corresponding to the waveforms $s_{ij}(t)$. A set of known waveforms with desired auto-correlation and cross-correlation properties will be utilized for initialization purposes.

A. RECEIVE FILTER OPTIMIZATION

The first step consists of determining the receiver filter for a given set of radar waveforms. The SINR at Radar-$i$ can be written as:

$$\text{SINR}_{ij}(t) = \frac{\mathbf{w}_i^H \sum_{j=1}^{M} \left( \sigma_{T,ij}(t) \mathbf{J}_{ij,ib} \right) \left( \mathbf{s}_{ij} \odot p(v_{T,ij}(t)) \right)^2}{\mathbf{w}_i^H (\sum_{j=1}^{M} \mathbf{\Psi}_{c,ij} + \sigma_n^2 \mathbf{I}) \mathbf{w}_i},$$

(4)

where $\sigma_{T,ij}(t)$ is the standard deviation of the target parameter. The optimum receiver filter vectors $\mathbf{w}_{ij}$ are obtained as the generalized eigenvector of the largest generalized eigenvalue of the matrix pair $(\mathbf{A}_{ij}, \mathbf{B}_{ij})$.

B. ORTHOGONAL WAVEFORMS OPTIMIZATION

The second step consists of optimizing the radar waveforms. The proposed algorithm requires the following constraints on the codes:

- All waveforms should be aimed at being mutually orthogonal or nearly orthogonal: $-\varrho \leq \mathbf{J}_{ij,ib}^H \mathbf{s}_{ij} \leq \varrho$, where $\varrho$ is a positive value very close to zero and $i \neq j$;
- All radars need to transmit finite energy (here assumed to be one): $\|s_{ij}\|^2 = 1$;
- In order to maintain good auto-correlation and cross-correlation properties, the estimated waveform $\mathbf{s}_{ij}$ cannot diverge more than a specific amount from an initial waveform with desired features $\mathbf{s}_{ij}(t)$, i.e. $\|\mathbf{s}_{ij} - \mathbf{s}_{ij}(t)\|^2 \leq \delta$.

We now reformulate the equations in order to develop convex optimization techniques. The power of the desired signal component of the received signal at the $i$-th radar is written in terms of the transmitted waveforms as [24]:

$$\sum_{j=1}^{M} \left| \mathbf{s}_{ij}^H \left( \sigma_{T,ij}(t) \mathbf{J}_{ij,ib} \right) (\mathbf{w}_{ij} \odot p(v_{T,ij}(t))) \right|^2 = \text{tr}(\mathbf{s}_i^H \mathbf{R}_{ij} \mathbf{s}_i),$$

(5)

where:

$$\mathbf{R}_{ij} = \text{blkdiag}(\mathbf{R}_{ij,1}, \mathbf{R}_{ij,2}, ..., \mathbf{R}_{ij,M}),$$

$$\mathbf{R}_{ij,1} = \mathbf{E}[(\mathbf{w}_{ij} \odot p(v_{T,ij}(t)))],$$

$$\mathbf{r}_{ij,1} = \sigma_{T,ij}(t) \mathbf{J}_{ij,ib} \mathbf{w}_{ij} \odot p(v_{T,ij}(t)),$$

$$\mathbf{s} = \begin{bmatrix} \mathbf{s}_{ij}^T(1) & \cdots & \mathbf{s}_{ij}^T(M) \end{bmatrix}^T,$$

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_{ij}^T(1) & \cdots & \mathbf{w}_{ij}^T(M) \end{bmatrix}^T,$$

(6)

(7)

(8)

(9)

(10)

where $\text{blkdiag}$ is defined as the operator for block diagonalization. Similarly, the power of the clutter returns at the $i$-th radar can be calculated as:

$$\mathbf{w}_i^H \left( \sum_{j=1}^{M} \mathbf{\Psi}_{c,ij} \right) \mathbf{w}_i,$$

(11)

where, as previously mentioned, $\mathbf{\Psi}_{c,ij}$ is the covariance matrix of the clutter. Extending the work in [8] to the case of multiple radars, the received interference power can also be written as:

$$\mathbf{w}_i^H \left( \sum_{j=1}^{M} \mathbf{\Psi}_{c,ij} \right) \mathbf{w}_i = \mathbf{w}_i^H (\mathbf{\Psi}_{c,ij}) \mathbf{w}_i = \mathbf{s}_i^H (\mathbf{\Theta}_{c,ij}) \mathbf{s}_i,$$

(12)

where:

$$\mathbf{\Theta}_{c,ij} = \text{blkdiag}(\mathbf{\Theta}_{c,ij,1}, \mathbf{\Theta}_{c,ij,2}, ..., \mathbf{\Theta}_{c,ij,M}).$$

(13)

The relationship between $\mathbf{s}_{ij}$, $\mathbf{\Theta}_{c,ij}$, $\mathbf{s}$ and $\mathbf{w}$ consists in the fact that $\mathbf{\Psi}_{c,ij}$ is the clutter covariance matrix as a function of the waveforms transmitted by all radars $\mathbf{s}$, and $\mathbf{\Theta}_{c,ij}$ is the clutter covariance matrix as a function of the receiver filter $\mathbf{w}_{ij}$.

The denominator of the SINR$_{ij}$ can therefore be rewritten as:

$$\text{tr}(\mathbf{s}_i^H (\mathbf{\Theta}_{c,ij}^* + \frac{\sigma_n^2}{M} \|\mathbf{w}_{ij}\|^2 \mathbf{I}) \mathbf{s}_i) = \text{tr}(\mathbf{s}_i^H \mathbf{Z}_{ij} \mathbf{s}_i).$$

(14)

Let us now define the optimization function as well as the constraints needed for the convex optimization problem. By following the guidelines provided for the single radar scenario in [8], the numerator and denominator of SINR$_{ij}$ can be reorganized in the form $\text{tr}(\mathbf{Q}_{nr,ij} \mathbf{X})$ and $\text{tr}(\mathbf{Q}_{dr,ij} \mathbf{X})$, where:

$$\mathbf{Q}_{sr,ij} = \begin{bmatrix} \mathbf{R}_{ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{Q}_{dr,ij} = \begin{bmatrix} \mathbf{Z}_{ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{s}_i^H & \mathbf{s}_i^* \\ \mathbf{s}_i & u \end{bmatrix},$$

(15)

where $u$ is the Charnes-Cooper variable transformation needed for the optimization. The orthogonality constraint can
be written as \( tr(Q_{\text{orth}(j,i)}X) \leq \varrho \) and \( tr(Q_{\text{orth}(j,i)}X) \geq -\varrho \), where:

\[
Q_{\text{orth}(j,i)} = \begin{bmatrix}
m_{(j,i)}^T j_{(i,b_i)}^T j_{(i,b_i)} m_{(j)} & 0 \\
0 & 0
\end{bmatrix},
\]

\[
m_{(j)} = \begin{bmatrix}
0_N(1) & \ldots & 0_N(i) & \ldots & 0_N(k) & \ldots & 0_N(M)
\end{bmatrix},
\]

where \( m_{(j)} \) is a vector matrix of size \( NN \times M \) that contains all zeros for the exception of an \( N \times N \) identity matrix at matrix position \( i \), with \( i = 1 \ldots M \).

The unit norm constraint at the \( i \)-th radar is written as:

\[
tr(Q_{\text{pw}(i)}X) = 1, \quad \text{where} \quad Q_{\text{pw}(i)} = \begin{bmatrix}
m_{(i)}^H m_{(i)} & 0 \\
0 & 0
\end{bmatrix}.
\]

The constraint on the deviation of the waveform from an initial waveform can be written as \( tr(Q_{\text{init}(i)}X) \leq 0 \), where:

\[
Q_{\text{init}(i)} = \begin{bmatrix}
m_{(i)}^H m_{(i)} & -m_{(i)}^H m_{(i)} s_0 \\
-s_0 m_{(i)}^H m_{(i)} & s_0 m_{(i)}^H m_{(i)} s_0 - \delta
\end{bmatrix}.
\]

Furthermore, we will be using the Charnes-Cooper variable transformation similarly to the work in [8]: \( Q_{\text{CC}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

From this starting point, different optimization problems can be formulated. For the cases of non-robust optimization, the reader is referred to the works in [24], [25], and [26]. For multistatic radars that wish to maximize the SINR at a specific radar while keeping the SINR of the other radars at a satisfactory level, the work in [24] adopted convex optimization approaches. This proposed a waveform optimization using an iterative approach for the design of receiver filter and radar waveforms. The waveforms at each iteration are obtained using the following optimization problem:

\[
\max_X tr(Q_{\text{dir}(i)}X)
\]

\[
s.t. \quad tr(Q_{\text{dir}(i)}X) = 1
\]

\[
tr(Q_{\text{dir}(i)}X) - \text{SINR}_{\text{min}} tr(Q_{\text{dir}(i)}X) \geq 0 \quad \forall j, j \neq i
\]

\[
tr(Q_{\text{orth}(j,i)}X) \geq -\varrho u \quad \forall i, j, j \neq i
\]

\[
tr(Q_{\text{orth}(j,i)}X) \leq \varrho u \quad \forall i, j, j \neq i
\]

\[
tr(Q_{\text{pw}(i)}X) = u \quad \forall i
\]

\[
tr(Q_{\text{init}(i)}X) \leq 0 \quad \forall i
\]

\[
tr(Q_{\text{CC}}X) = u
\]

\[
X \succeq 0, u \geq 0
\]

(20)

Suppose that the desired SINR for all radars except Radar- \( i \) is SINR\(_{\text{goal}}\). It is unlikely to achieve this value at the first iteration, as the initial waveforms and the receiver filters are not optimized enough to meet the goal. As the SINRs of the radars are expected to improve at every iteration, we start with a small goal SINR, namely SINR\(_{\text{min}}\), for the first iteration. As the iterations of the inner loop progress, we increase this minimum goal by a small constant amount \( \Delta_{\text{step}} \) until the problem is infeasible or SINR\(_{\text{min}}\) reaches SINR\(_{\text{goal}}\). Once the waveforms are obtained in this inner loop iteration process, the code goes back to the outer loop and the receiver filter is optimized as described in Subsection II-A. The SINR\(_{\text{min}}\) is increased by \( \Delta_{\text{step}} \) and the optimization in (20) is repeated until the solution turns out to be infeasible. It needs to be noted that a problem that becomes infeasible in the inner loop for a specific SINR\(_{\text{min}}\) might provide again feasible results after the receiver filter optimization is performed in the outer loop. The specific steps involved in the optimization are summarized from a simulation-oriented perspective in Table 1. These include both the inner loop steps that have just been described as well as the outer loop steps (that is, the sequential optimization of the receiver filter and waveform design).

### Table 1. Outline of the optimization method from a simulation-oriented perspective

**feasible** is a parameter that is set to one as long as the SDP provides defined numerical results and \( t \) is the iteration number.

**Initialization:**
- Parameter Initialization:
  - SINR\(_{\text{iteration}}\) = 0;
  - step is set;
  - SINR\(_{\text{goal}}\) is set;
- Known waveform \( s(0) \) with desired features.

**while** \( \| \text{SINR}_t (t-1) - \text{SINR}_t (t) \| > \zeta \)

**Filter Optimization:** as described in (4);

**Waveform Optimization:**

\[
\text{SINR}_{\text{min}} = \text{SINR}_{\text{iteration}};
\]

**while** \( \text{feasible} \ & \ \text{SINR}_{\text{min}} \leq \text{SINR}_{\text{goal}} \)

\[
\text{perform } \text{cvx} \text{ with } \text{SINR}_{\text{min}} = \text{SINR}_{\text{iteration}} \quad \text{as described in (20)};
\]

\[
\text{if } \text{feasible} = 0
\]

\[
s = s(t-1)
\]

\[
\text{else}
\]

\[
\text{SINR}_{\text{iteration}} = \text{SINR}_{\text{iteration}} + \text{step}.
\]

\[
\text{end}
\]

\[
\text{end}
\]

### III. ROBUST AND STOCHASTIC OPTIMIZATION TECHNIQUES

The convex optimization in (20) assumes perfect knowledge of the second order statistics of both the signal-dependent clutter and the additive noise. This scenario is not always practically feasible or realistic. It is therefore important to take into account the mismatch between actual and presumed values of the clutter plus noise covariance matrix.

In order to tackle the problem of optimization affected by parameter uncertainty, two main approaches can be undertaken. The first one is robust or worst-case optimization and
the second one is stochastic optimization. In the first case the uncertainty model is deterministic and set-based whereas in the latter case the uncertainty has a probabilistic description. Robust Optimization techniques are computationally tractable and the development of fast interior point methods for convex optimization sparked an interest in this field [27].

A. WORST-CASE OPTIMIZATION TECHNIQUES

We hereby outline a worst-case robust optimization technique that assumes uncertainty in the clutter plus noise covariance matrix. In a worst-case robust optimization model the decision maker constructs a solution that is feasible for any realization of the uncertainty in a given set. This formulation is inherently that of a max-min problem and is the most rigorous approach to account for the mismatch [27].

As described in Section II, we chose to optimize the SINR at Radar-\(i\) whilst satisfying a specific \(\text{SINR}_{\text{goal}}\) for all Radar-\(j\) with \(j \neq i\). However, in the presence of an error on the estimate of the clutter plus noise covariance matrix, we will not always be able to achieve the desired \(\text{SINR}_{\text{goal}}\) due to the mismatch between the real covariance matrix and the assumed covariance matrix for the clutter plus noise. In order to describe the robust approach, we assume the estimate of the covariance matrix \(\mathbf{Q}_{\text{dr},i}\) in (20) with an error as follows:

\[
\mathbf{Z}_{i\beta} = \mathbf{Z}_i + \Delta \beta, \quad (21)
\]

where, with reference to (14), \(\mathbf{Z}_i\) is the presumed interference plus noise covariance matrix and \(\mathbf{Z}_{i\beta}\) denotes its actual value. The subscript \(\beta\) in the error matrix \(\Delta \beta\) indicates that the mismatch between the expected and received covariance matrix is bounded through the constant value \(\beta\) in the Frobenius norm:

\[
||\Delta \beta||_F \leq \beta. \quad (22)
\]

The denominator of \(\text{SINR}_i\) can be rewritten as:

\[
\text{tr}(\mathbf{s}^H(\mathbf{Z}_i + \Delta \beta)\mathbf{s}). \quad (23)
\]

The robust worst-case optimization problem is consequently formulated as:

\[
\max_s \quad \max_{||\Delta \beta||_F \leq \beta} \text{tr}(\mathbf{s}^H(\mathbf{Z}_i + \Delta \beta)\mathbf{s}). \quad (24)
\]

This can be modified in the following well-known equivalent formulation thanks to the Lagrangian multipliers method [21], [28]:

\[
\max_{\mathbf{s}} \text{tr}(\mathbf{s}^H(\mathbf{Z}_i + \beta \mathbf{I})\mathbf{s}). \quad (25)
\]

The final robust waveform optimization problem can be expressed as the one in (20) but modifying the constraint on the Radar-\(j\) as:

\[
\text{tr}(\mathbf{Q}_{\text{dr},j}X) - \text{SINR}_{\text{min}}\text{tr}(\mathbf{Q}_{\text{dr},i}X) \geq 0 \quad \forall j, j \neq i, \quad (26)
\]

where \(\mathbf{Q}_{\text{dr},i} = \begin{bmatrix} \mathbf{Z}_i + \beta \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\).

B. STOCHASTIC OPTIMIZATION TECHNIQUES

The problem associated with worst-case optimization techniques is that they result in overly-conservative methods as they aim at satisfying the SINR for worst-case errors. For this reason, most of the time the achieved SINR is much greater than the required SINR. By utilizing statistical knowledge on the error of the covariance matrix, it is possible to achieve robustness against the uncertainty with a certain outage probability [22]. As the RCS and Doppler values change randomly, it is more efficient to exploit the statistical nature of these errors.

The SINR constraints in the optimization be reformulated as:

\[
P_j = \Pr\left(\frac{\text{tr}(\mathbf{s}^H\mathbf{R}_j\mathbf{s})}{\text{tr}(\mathbf{s}^H(\mathbf{Z}_i + \mathbf{E}_i)\mathbf{s})} \geq \text{SINR}_{\text{goal}}\right), \quad (27)
\]

with \(P_j \geq p_{ij}, j \neq i\). Similarly to Section III-A, we need to apply robustness only to those radars with specific SINR requirements. \(P_j\) defines the probability that the \(j\)th user achieves the required \(\text{SINR}_{\text{goal}}\) and \(p_{ij}\) is a preselected threshold value. In (27), \(\Pr(\cdot)\) identifies the probability operator and \(\mathbf{E}_i\) is the error matrix. \(\mathbf{E}_i\) is a block diagonal matrix (as is \(\mathbf{Z}_i\)) where each of the inner matrices has been modeled as a Hermitian matrix whose elements are taken from the distribution \(\mathcal{CN}(0, \sigma^2_{ij})\). Naming \(S = ss^H\), the variance of \(\text{tr}(\mathbf{E}_i\mathbf{S})\) can be therefore calculated as:

\[
\mathbb{E}[\text{tr}(\mathbf{E}_i\mathbf{S})\text{tr}(\mathbf{E}_i\mathbf{S})^*] = \sum_{q=0}^{M-1} \sum_{l=1}^{N} \sum_{m=1}^{N} \sigma^2_{e_{iq}} \left[S_l + qM, l + qM + M\right] \left[S_l + qM, l + qM + M\right]^* = \sigma^2_{e_{iq}} \left[S_{iq} + qM, iq\right] \sum_{l=1}^{N} \sum_{m=1}^{N} \sigma^2_{e_{iq}}, \quad (28)
\]

where \((l, m)\) identifies the matrix cell. (27) can be reformulated as:

\[
P_j = \Pr\left(\text{tr}(\mathbf{Z}_i + \mathbf{E}_i) \leq \gamma_j\right), \quad (29)
\]

where:

\[
\gamma_j = \frac{\text{tr}(\mathbf{R}_j\mathbf{S})}{\text{SINR}_{\text{goal}}}, \quad \text{and } S = ss^H. \quad (30)
\]

We define the random variable \(\gamma_j = \text{tr}(\mathbf{Z}_i + \mathbf{E}_i)\). This is a real variable because both \(\mathbf{Z}_i + \mathbf{E}_i\) and \(S\) are Hermitian and, as by [22, Lemma 1], is assumed to have the probability distribution \(\gamma_j \sim \mathcal{N}(\text{tr}(\mathbf{Z}_i\mathbf{S}), M\sigma^2_{e_{ij}})\). The probability of achieving the required \(\text{SINR}_{\text{goal}}\) is therefore calculated as:

\[
P_j = \int_{-\infty}^{\gamma_j} \frac{1}{\sqrt{2\pi}\sigma_{e_{ij}}\sqrt{M}} \exp\left(-\frac{(\gamma_j - \mu_j)^2}{2\sigma_{e_{ij}}^2 M}\right) dy, \quad (31)
\]

where \(\mu_j = \text{tr}(\mathbf{Z}_i\mathbf{S})\). Using the error function \(\text{erf}(\cdot)\) solution of the Gaussian integral, (31) can be rewritten as:

\[
P_j = \frac{1}{2} + \frac{1}{2} \text{erf}\left(\frac{\gamma_j - \mu_j}{\sqrt{2\sigma_{e_{ij}}\sqrt{M}}}\right) \geq p_{ij}, \quad (32)
\]

hence,

\[
\gamma_j - \mu_j \geq \text{erf}^{-1}(2p_{ij} - 1)\sqrt{2\sigma_{e_{ij}}\sqrt{M}}. \quad (33)
\]
Eq 34:  
\[
\frac{\text{tr}(\mathbf{R}_{ij}S)}{\text{SINR}_{\text{min}}} - \frac{\text{tr}(\mathbf{Z}_{ij}S)}{\text{SINR}_{\text{min}}} \geq \delta_{e_{ij}},
\]

where  
\[
\delta_{e_{ij}} = \text{erf}^{-1}(2p_{ij} - 1)\sqrt{2\sigma_{e_{ij}}\sqrt{M}},
\]
and  
\[
||S|| = ||s^{H}|| = \text{tr}(ss^{H}) = M \text{ since } s \text{ is a vector containing the } M \text{ radar waveforms. Writing the condition for stochastic robustness so that it is more convenient in light of the SDP formulation we obtain:}
\]
\[
\text{tr}(\mathbf{R}_{ij}S) - \text{SINR}_{\text{min}}\text{tr}\left(\left(\mathbf{Z}_{ij} + \frac{\delta_{e_{ij}}}{\text{tr}(S)}\right)S\right) \geq 0,
\]

which leads to the convex optimization problem constraint:  
\[
\text{tr}(\mathbf{Q}_{\text{ar},ij}X) - \text{SINR}_{\text{min}}\text{tr}(\mathbf{Q}_{\text{ar},ij}X) \geq 0 \quad \forall j, j \neq i,
\]

where  
\[
\mathbf{Q}_{\text{ar},ij} = \begin{bmatrix}
\mathbf{Z}_{ij} + \frac{\delta_{e_{ij}}}{\text{tr}(S)} & 0 \\
0 & 0
\end{bmatrix}.
\]

\[\text{C. CLUTTER-SPECIFIC STOCHASTIC OPTIMIZATION}\]

The methods described so far are applicable to uncertainties introduced directly to the clutter-plus-noise covariance matrix, hence they are very generic and over-conservative. In most cases, the covariance matrix will be constructed using the estimates of the underlying parameters of the clutter such as radar cross-section and Doppler. Hence, in order to prove the validity of the previous models as well as to investigate new optimization techniques aimed at enhanced accuracy, a clutter parameter-specific stochastic optimization is proposed. The clutter covariance matrix is a function of the RCS and Doppler of the clutter. Hence in the presence of uncertainty, this can be expressed for radar  \(i\) as:

\[
\Theta_{c,\text{rob},(i)}(\epsilon) = \sum_{j=1}^{M} \sum_{b=0}^{B} \left(\sigma_{c,ij,b}^{2} + \epsilon_{\text{RCS},ij,b}^{2}\right) \times \text{diag}(\mathbf{J}_{ij,b}^{T}w_{ij,b}^{*}) \Phi_{\text{e,rob}(i,j,b)} \text{diag}(\mathbf{J}_{ij,b}^{T}w_{ij,b}),
\]

with  \(i, j = 1, \ldots, M\) and where the matrix accounting for the Doppler shift is:

\[
\Phi_{\text{e,rob}(i,j,b)}(l,m) = e^{i2\pi \nu_{ij,b}(l-m)} \sin[\pi(\epsilon_{ij,b} + \nu_{ij,b})(l-m)] + \frac{1}{\pi(\epsilon_{ij,b} + \nu_{ij,b})},
\]

and:

-  \(\epsilon_{\text{RCS},ij,b} \sim \mathcal{CN}(0, \sigma_{\text{RCS},ij,b}^{2})\) defines the statistics of the uncertainty on the radar cross-section;
-  \(\nu_{ij,b} \sim \mathcal{CN}(0, \sigma_{\nu_{ij,b}}^{2})\) provides the statistics associated to the uncertainty on the Doppler interval.

For notational convenience the subscript \((i)\) will be hereafter omitted. The reader will therefore need to keep in mind that all of the following equations refer to a receiving/transmitting radar \((i)\) even if not directly specified.

In order to develop robust optimization techniques, we expand the elements of the matrix  \(\Phi_{c,\text{rob}}\) using Taylor series as a function of the error term  \(\epsilon_{\nu}\). This results in the following expression:

\[
\Phi_{c,\text{rob}}(l, m) = e^{i2\pi \nu(l-m)} \left(\frac{\sin[\pi \epsilon(l-m)]}{\pi \epsilon(l-m)}\right)\epsilon_{\nu} + \frac{1}{\pi \epsilon(l-m)} \left(\cos[\pi \epsilon(l-m)] - \frac{\sin[\pi \epsilon(l-m)]}{\pi \epsilon(l-m)}\right)\epsilon_{\nu} + \frac{1}{2} \left(2 - (\pi \epsilon(l-m))^{2}\right)\left(\frac{\sin[\pi \epsilon(l-m)]}{\pi \epsilon(l-m)}\right)^{2} - \cos[\pi \epsilon(l-m)]\epsilon_{\nu}^{2}.
\]

where  \((l, m)\) identifies the position of the element within the matrix. Substituting (40) into (38) leads to (41). This can also be written in the form  \(\Theta_{c,\text{rob}} = \Theta_{c} + \Theta_{\text{RCS}} + \Theta_{e} + \Theta_{\text{RCS}e} + \Theta_{e2} + \Theta_{\text{RCS}e2}\). Some necessary remarks on (41):

- The notation "~" marks the difference between the original matrix  \(\Theta_{c,\text{rob}}\) and the Taylor series-approximated matrix  \(\hat{\Theta}_{c,\text{rob}}\):
-  \(\Theta_{c}\) is the error-free clutter covariance matrix;
-  \(\Theta_{\text{RCS}}\) is the covariance matrix carrying the uncertainty on the radar cross-section of the clutter;
-  \(\Theta_{e}\) is the covariance matrix carrying the uncertainty on the Doppler of the clutter;
-  the expected value of  \(\Theta_{\text{RCS}e}\) can be assumed to be zero since it contains a multiplication between the two errors which are very small and uncorrelated;
-  the terms  \(\Theta_{c}\) and  \(\Theta_{e2}\) will contribute to the mean of  \(\hat{\Theta}_{c,\text{rob}}\);
-  the terms  \(\Theta_{\text{RCS}}\) and  \(\Theta_{e}\) will contribute to the variance of  \(\hat{\Theta}_{c,\text{rob}}\);
-  the term  \(\Theta_{\text{RCS}e2}\) will contribute with a mean value to the variance of  \(\Theta_{e2}\).

In other words, the new clutter covariance matrix can be re-written as the error-free clutter covariance matrix plus a series of signal-dependant error matrices. The denominator of the SINR can be therefore written as:

\[
y = \text{tr}((\mathbf{Z} + \Theta_{\text{RCS}}^{*} + \Theta_{e}^{*} + \Theta_{\text{RCS}e2}^{*} + \Theta_{e2}^{*})S),
\]

where  \(\mathbf{Z} = \Theta_{c}^{*} + \sigma_{e}^{2}I\) (please refer to (14)). The statistics of  \(y\) are derived hereafter.

The expected value of  \(y\) is:

\[
\mu = \mathbb{E}(y) = \mathbb{E}(\text{tr}((\mathbf{Z} + \Theta_{\text{RCS}}^{*} + \Theta_{e}^{*} + \Theta_{\text{RCS}e2}^{*} + \Theta_{e2}^{*})S))
\]

\[
= \mathbb{E}(\text{tr}(\mathbf{Z}S)) + \text{tr}(\mathbf{ZS}) + \text{tr}(\hat{\Theta}_{e2}^{*}S),
\]

where:

\[
\hat{\Theta}_{e2} = \sum_{j=1}^{M} \sum_{b=1}^{B} \sigma_{c,ij,b}^{2} \text{diag}(\mathbf{J}_{ij,b}^{T}w_{ij,b}) \Phi_{e2,ij,b} \text{diag}(\mathbf{J}_{ij,b}^{T}w_{ij,b}),
\]
The statistics of the matrix elements of $\mathbf{2}$ can be calculated as:

$$\tilde{\Theta}_{c,\text{rob},(i)} = \sum_{j=1}^{M} \sum_{b=1}^{B} \sigma_{c,j,b}^2 \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* \Phi_{c,j,b} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}$$

$$+ \sum_{j=1}^{M} \sum_{b=1}^{B} \varepsilon_{\text{RCS},j,b} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* \Phi_{c,j,b} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}$$

$$+ \sum_{j=1}^{M} \sum_{b=1}^{B} \sigma_{c,j,b}^2 \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* \Phi_{c,j,b} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}$$

$$+ \sum_{j=1}^{M} \sum_{b=1}^{B} \varepsilon_{\text{RCS},j,b} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* \Phi_{c,j,b} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}$$

$$+ \sum_{j=1}^{M} \sum_{b=1}^{B} \sigma_{c,j,b}^2 \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* \Phi_{c,j,b} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}$$

$$+ \sum_{j=1}^{M} \sum_{b=1}^{B} \varepsilon_{\text{RCS},j,b} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* \Phi_{c,j,b} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}$$

$$= \Theta_{c,(i)} + \Theta_{\text{RCS},(i)} + \Theta_{e_v,(i)} + \Theta_{\text{RCS},e_v,(i)} + \Theta_{\text{RCS},e_o,(i)} + \Theta_{\text{RCS},e_o2,(i)} \text{ for } i = \{1, 2, \ldots M\}$$

(41)

where:

$$\Phi_{o2}(l, m) = \mathbb{E} [\Phi_{o2}(l, m)]$$

$$\mathbb{E} [\varepsilon_{v}^2] = \sigma_{v}^2$$

In the above, the mean of $\Theta_{\text{RCS}}, \Theta_{e_v}$ and $\Theta_{\text{RCS},e_o2}$ goes to zero and $\mathbb{E} [\varepsilon_{v}^2] = \sigma_{v}^2$ as consequence of the Gaussian variables being distributed with $\varepsilon_{\text{RCS},j,b} \sim \mathcal{CN}(0, \sigma_{\text{RCS},j,b}^2)$ and $\varepsilon_{v,j,b} \sim \mathcal{CN}(0, \sigma_{v,j,b}^2)$. The second order statistics of $\varepsilon$ can be calculated as:

$$\mathbb{E} [\varepsilon^2] = \mathbb{E} \left[ \frac{1}{e^2} \left( \frac{2 - (\varepsilon_{\text{RCS},j,b}^2) \sin[\pi \varepsilon_{\text{RCS},j,b}^2]}{[\pi \varepsilon_{\text{RCS},j,b}^2]} \right) + \frac{1}{e^2} \left( \frac{2 - (\varepsilon_{v,j,b}^2) \sin[\pi \varepsilon_{v,j,b}^2]}{[\pi \varepsilon_{v,j,b}^2]} \right) \right]$$

(45)

The statistics of the matrix elements of $\Theta_{\text{RCS}}$ and $\Theta_{e_v}$ can be derived as follows:

$$\Theta_{\text{RCS}}(l, m) \sim \mathcal{CN} \left( 0, \mathbf{A}_{\text{RCS}}^2(l, m) \sigma_{\text{RCS}}^2 \right),$$

$$\Theta_{e_v}(l, m) \sim \mathcal{CN} \left( 0, \mathbf{A}_{e_v}^2(l, m) \sigma_{e_v}^2 \right),$$

(47)

(48)

where:

$$\mathbf{A}_{\text{RCS}} = \sum_{j=1}^{M} \sum_{b=1}^{B} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* \Phi_{e_v,(j,b)} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)},$$

$$\mathbf{A}_{e_v} = \sum_{j=1}^{M} \sum_{b=1}^{B} \sigma_{c,j,b}^2 \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* \mathbf{K} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)},$$

(49)

and:

$$\mathbf{K}(l, m) = e^{(2\pi \nu \tilde{v} - l)} \cdot \frac{1}{e} \left( \cos[\pi \varepsilon_{\text{RCS},j,b}^2] - \frac{\sin[\pi \varepsilon_{\text{RCS},j,b}^2]}{[\pi \varepsilon_{\text{RCS},j,b}^2]} \right).$$

(50)

Also,

$$\mathbb{E} [\text{tr}((\Theta_{\text{RCS}}^* + \Theta_{\text{RCS},e_o2}) S \text{tr}(\Theta_{\text{RCS}}^* + \Theta_{\text{RCS},e_o2}) S)]$$

$$= \mathbb{E} [\text{tr}(\mathbf{A}_{\text{RCS}}^2 S \text{tr}(\mathbf{A}_{\text{RCS},e_o2}^2 S) + \text{tr}(\sigma_{\text{RCS}}^2 S \text{tr}(\mathbf{A}_{\text{RCS},e_o2}^2 S))]$$

$$= \sigma_{\text{RCS}}^2 \sum_{j=1}^{M} \sum_{b=1}^{B} \text{tr} \left( \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* (\Phi_{e_v,(j,b)} + \tilde{\Phi}_{o2}) \right) \right)^2$$

$$= \sigma_{\text{RCS}}^2 \| \mathbf{v}_{\text{RCS}} \|^2,$$

(51)

(52)

where $\mathbf{v}_{\text{RCS}}$ is a vector of dimension $MB \times 1$ containing in each element the value $\text{tr} \left( \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* (\Phi_{e_v,(j,b)} + \tilde{\Phi}_{o2}) \right)$ for a specific radar $j$ and range-azimuth bin $b$ and

$$\mathbf{A}_{e_o2} = \sum_{j=1}^{M} \sum_{b=1}^{B} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}^* \tilde{\Phi}_{o2} \text{diag}(\mathbf{J}^T_{j,b}) \mathbf{w}_{(i)}.$$
where the expected value would also be zero. The variance of the uncertainty related to the Doppler can be calculated in the same way:

\[
\mathbb{E}\{\text{tr}(\mathbf{r}_e^* \mathbf{S}) \text{tr}(\mathbf{r}_e \mathbf{S}^*)\} = \mathbb{E}\{\text{tr}(\mathbf{r}_e \mathbf{A}_e^* \mathbf{S}) \text{tr}(\mathbf{r}_e \mathbf{A}_e^* \mathbf{S})\}
\]

\[
= \sigma_{v_e}^2 \sum_{j=1}^{M} \sum_{b=1}^{B} \text{tr}(\text{diag}(J^T_{(j,b)} \mathbf{w}^* \mathbf{w}) \mathbf{K}_{(j,b)} \text{diag}(J^T_{(j,b)} \mathbf{w} \mathbf{w}) \mathbf{S})^2
\]

\[
= \sigma_{v_e}^2 \|v_v\|^2,
\]

(54)

where \(v_v\) is a vector of dimension \(MB \times 1\) containing in each element the value \(\text{tr}(\text{diag}(J^T_{(j,b)} \mathbf{w}^* \mathbf{w}) \mathbf{K}_{(j,b)} \text{diag}(J^T_{(j,b)} \mathbf{w} \mathbf{w}) \mathbf{S})^2\) for a specific radar \(j\) and range-azimuth bin \(b\). In this model the variance depends on the signal as well as other parameters specifically related to the scenario under investigation. The variance of \(y\) is written as:

\[
\mathbb{E}\{y^2\} = \sigma_{\text{RCS}}^2 \|\text{RCS}\|^2 + \sigma_{v_e}^2 \|v_v\|^2
\]

\[
= \|\{\sigma_{\text{RCS}} \text{RCS}; \sigma_{v_e} v_v\}\|^2.
\]

(55)

Similarly to the case described in Subsection III-B:

\[
P = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \|v_v\|} \exp\left(-\frac{(y - \mu)^2}{2\|v_v\|^2}\right) \text{dy},
\]

(56)

where \(v_v = \{\sigma_{\text{RCS}} \text{RCS}; \sigma_{v_e} v_v\}\). This leads to the second Order Cone Programming (SOCP) convex constraint:

\[
\frac{\text{tr}(\mathbf{R}(\theta^*) \mathbf{S})}{\text{SINR}_{\text{min}}} - \text{tr}(\mathbf{Z}(\theta^*) + \mathbf{\bar{\Theta}}_{\text{init}}^*\mathbf{S}) \geq \delta_{p_{(j)}} \|v_v\|,
\]

(57)

where \(\delta_{p_{(j)}} = \text{erf}^{-1}(2p_{(j)} - 1)\sqrt{2}\).

IV. PERFORMANCE ANALYSIS

In order to evaluate the performance of the proposed algorithms, we perform Monte Carlo simulations for the case of \(M = 2\) radars. The SINR of the first radar is maximized while requiring the second radar to achieve a desired SINR_{goal}. The SINR achieved by the optimization for the second radar is investigated for both the robust and non-robust cases. The initial waveforms \(s_{0,1}\) and \(s_{0,2}\) are Fractional Fourier Waveforms of length \(N = 64\) as developed in [29]. These waveforms provide very good auto-correlation and cross-correlation properties (refer to Figure 1) granting therefore good range resolution while maintaining orthogonality between the two radar waveforms. We assume the scatterers to be located in \(N_c = 4\) range rings. The number of azimuth cells in each ring is \(L = 8\). As for the parameters of the target, the various radar cross-sections are set randomly to \(\sigma_{11,T}^2 = 0.5823, \sigma_{21,T}^2 = 0.6036, \sigma_{22,T}^2 = 0.5953\) and \(\sigma_{12,T}^2 = 0.6203\). The target Doppler values are set randomly to \(v_{11,T} = 0.0141, v_{21,T} = 0.0237, v_{22,T} = 0.0249\) and \(v_{12,T} = 0.0044\). The clutter power as seen by the radars is \(\sigma_{11,c}^2 = \sigma_{21,c}^2 = \sigma_{22,c}^2 = \sigma_{12,c}^2 = 1\). We also set the noise variance to \(\sigma_n^2 = 0.25\) and the Doppler frequency is uniformly distributed around its mean value of \(\bar{v}_c = 0.0267\) with a spread of \(\epsilon = 0.02\). Finally, we set the maximum acceptable deviation to the initial waveform to \(\delta = 0.1\) and the orthogonality threshold to \(\varrho = 0.05\). For solving the SDP problem, we used CVX MATLAB Software for Disciplined Convex Programming [30]. The waveform optimization was solved as described in (20). The full optimization method (that is, including the iterations between waveform and filter optimizations) has been described in Section II. The reader may refer to Table 1 for the outline of the optimization method from a simulation-oriented perspective. For all simulations the parameters used were \(\Delta_{\text{step}} = 0.1, \text{SINR}_{\text{iteration}} = 0\) at the initialization stage and SINR_{goal} = 2dB. SINR_{goal} refers to Radar−2 since SINR_{1} will be maximized in the objective function of the SDP. The number of Monte-Carlo experiments for the simulation results is 10000.

A. PERFORMANCE ANALYSIS OF WORST-CASE OPTIMIZATION TECHNIQUES

In order to test the algorithm for worst-case optimization techniques, the Frobenius norm bound of the error matrix was set to \(\beta = 0.18\). This \(\beta\) value corresponds to the 1% of the Frobenius norm of the error-free covariance matrix of the clutter. As seen in Figure 2, the required SINR_{goal} of 2dB was over satisfied with robust optimization techniques but the non-robust case achieved the required SINR of 2dB only half of the times. As expected, the results are over-conservative.
TABLE 2. Stochastic optimization results. Comparison between the achievable percentage of a desired SINR Goal with stochastic waveform optimization techniques and non-robust waveform optimization techniques.

<table>
<thead>
<tr>
<th>Desired percentage</th>
<th>70%</th>
<th>80%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obtained percentage with stochastic optimization techniques</td>
<td>70.2</td>
<td>79.8</td>
<td>89.5</td>
</tr>
<tr>
<td>Obtained percentage without stochastic optimization techniques</td>
<td>50.5</td>
<td>49.7</td>
<td>49.6</td>
</tr>
</tbody>
</table>

for the worst-case optimization techniques since the achieved SINR is always higher than the required one by a considerable margin.

The SINR achieved by Radar-1, i.e. the radar whose SINR is maximized, was equal to \( \text{SINR}_{1,\text{max}} = 3.73 \text{dB} \) on average.

B. PERFORMANCE ANALYSIS OF STOCHASTIC OPTIMIZATION TECHNIQUES

In order to test the stochastic optimization techniques, the standard deviation of the error was set to \( \sigma_{e_j} = 0.01 \). We selected this value so that \( \sqrt{M} \sigma_{e_j} \) is 4% of the mean of \( \text{tr}(Z_j S) \) (please refer to the probability distribution of \( \gamma_j \) in Subsection III-B). This has been tested for an SINR achievement rate of 70%, 80% and 90%. As seen in Figures 3, 4 and 5 respectively, the robust algorithm provides the desired SINR with the desired percentage. On the other hand, the non-robust algorithm was able to achieve the desired SINR of 2dB only about half of the times. The specific values have been provided in Table 2. The average SINR achieved by Radar-1 was equal to \( \text{SINR}_{1,\text{max}} = 3.76 \text{dB} \).

C. PERFORMANCE ANALYSIS OF CLUTTER-SPECIFIC STOCHASTIC OPTIMIZATION

In order to test the clutter-specific stochastic optimization, we set \( \sigma_{\text{RCS}}^2 = \left( \frac{\sigma_{\text{RCS}}}{\langle \sigma_{\text{RCS}} \rangle} \right)^2 \) or, in other words, we set the standard deviation of the error of the RCS of the clutter to 20% of the RCS of the clutter. Similarly, we set \( \sigma_{\nu}^2 = \left( \frac{\sigma_{\nu}}{\langle \sigma_{\nu} \rangle} \right)^2 \) i.e. 20% of \( \epsilon \). The results obtained through the Monte Carlo simulations for clutter-specific stochastic optimization have been provided in the second row of Table 3 as well as in the green histograms in Figures 6, 7 and 8. As it can be seen, by using the proposed optimization method, there generally is a very good match between the desired and the obtained SINR percentages. However, a 2.5% mismatch occurs for the 90% case. This is a consequence of the Taylor series approximation of the covariance matrix. Nonetheless, during this Monte-Carlo simulation, the value of 1.99dB was achieved 90% of the time, showing how this mismatch is actually negligible.

It needs to be noted that we could have considered uncertainty also on the average of the Doppler \( \bar{\nu} \). The methodology proposed in this work is still applicable to this case. However,
TABLE 3. Results for signal-dependent clutter i.e. for error applied directly to the RCS and Doppler of the clutter. Comparison between the achievable percentage of the desired SINR\(_{\text{goal}}\) by using the proposed optimization that assumes uncertainty on the clutter parameters directly (row 2), the ordinary stochastic optimization (row 3) and non-robust optimization (row 4).

<table>
<thead>
<tr>
<th>Desired percentage</th>
<th>70%</th>
<th>80%</th>
<th>90%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Obtained percentage with clutter-specific stochastic optimization techniques</td>
<td>69.8</td>
<td>79.4</td>
<td>87.5</td>
</tr>
<tr>
<td>Obtained percentage with ordinary stochastic optimization techniques</td>
<td>6.6</td>
<td>7.0</td>
<td>6.5</td>
</tr>
<tr>
<td>Obtained percentage with non-robust optimization techniques</td>
<td>1.3</td>
<td>0.7</td>
<td>0.4</td>
</tr>
</tbody>
</table>

incorporating error to the average Doppler in (39) will lead to additional terms in the covariance matrix in (41). For clarity of the description of the algorithm, we therefore considered uncertainty only on the Doppler spread.

We also compared the above results with non-robust optimization and with the ordinary stochastic method described in Subsection III-B. In order to compare the parameter-specific uncertainty with the stochastic method that considers the uncertainty directly on the covariance matrix of the clutter, the same level of uncertainty needs to be used in both optimizations. To estimate the variance \(\sigma^2\) of the equivalent level of uncertainty, we introduced errors directly to the clutter parameters as in (38) and computed the difference between the true covariance matrix and the error-free covariance matrix:

\[
\tilde{E} = \Theta_{c,\text{rob}} - \Theta_c,
\]

(58)

where \(\Theta_{c,\text{rob}}\) and \(\Theta_c\) are defined in (41). Once \(\tilde{E}\) is obtained, \(\mathbb{E}[\text{tr}(\tilde{E}S_\text{c})]\) is computed using the 10000 Monte-Carlo runs, and the equivalent variance of the error \(\sigma^2\) for the ordinary optimization was obtained using (28) as \(\frac{\mathbb{E}[\text{tr}(\tilde{E}S_\text{c})]}{M}\). It should be noted that \(\tilde{E}\) is a function of the receiver filter \(w_i\) and the percentage of SINR achievement rate. Hence we computed the equivalent error terms for the final value of \(w_i\) as obtained by the proposed stochastic optimizations for each percentage 70%, 80% and 90%. The variance \(\sigma^2\) thus obtained has then been used for the ordinary stochastic optimization.

The results obtained for non-robust optimization, ordinary stochastic optimization and clutter-specific optimization for the case of when the error is applied directly to the radar cross section and the Doppler spread, are depicted in Figures 6, 7 and 8 and summed up in Table 3. As it can be seen, there is a significant difference between the desired and obtained SINR\(_{\text{goal}}\) for the case of ordinary stochastic optimization i.e. obtained by assuming that the error is directly applied to the clutter plus noise covariance matrix.

\[
\text{FIGURE 6. The SINR goal of 2dB was required to be achieved 70% of the time. Comparison between non-robust optimization, ordinary stochastic optimization and the clutter-specific optimization proposed in this work. The required SINR}_\text{goal} \text{ of 2 dB was achieved 69.8% of times with clutter-specific stochastic optimization. The required SINR}_\text{goal} \text{ was achieved 6.6% with the more generic stochastic optimization method and 1.3% with non-robust optimization.}
\]

\[
\text{FIGURE 7. The SINR goal of 2dB was required to be achieved 80% of the time. Comparison between non-robust optimization, ordinary stochastic optimization and the clutter-specific optimization proposed in this work. The required SINR}_\text{goal} \text{ of 2 dB was achieved 79.4% of times with clutter-specific stochastic optimization. The required SINR}_\text{goal} \text{ was achieved 7.0% with the more generic stochastic optimization method and 0.7% with non-robust optimization.}
\]

The reason is that the assumption of errors applied directly to the covariance matrix is not sufficiently accurate to describe the structure of the error. As a matter of fact, there is almost no difference between the results obtained with non-robust optimization and with ordinary stochastic optimization, proving how this model is an over-simplification when considering signal-dependent clutter.

It should be noted that only Radar-2 is required to achieve a specific SINR. But for Radar-1, the aim was to maximize its achievable SINR. Hence robust formulation is applicable to
only Radar-2. However, as we assumed various realizations of the clutter parameters, the SINR achieved by Radar-1 varied slightly but with a mean value of 3.78 dB, as shown in Figure 9. It needs to be noted that, in principle, the proposed techniques are applicable to more than two radars. However, in practice, the performance in the presence of more than two radars will be very limited in the presence of severely cluttered environment. For example, in our simulation model, two radars will be very limited in the presence of severely cluttered environment. Assuming uncertainty on the clutter statistics, we have proposed worst-case robust optimization and stochastic robust optimization methods. While non-robust optimization methods are unable to achieve the required SINR\(_{\text{goal}}\), the worst-case robust optimization is always able to achieve the desired SINR for the worst-case clutter statistics. The stochastic robust optimization is able to achieve the goal SINR with a specified outage probability in the presence of uncertainty on the clutter covariance matrix. Finally, the proposed algorithm that assumes uncertainty directly on the clutter parameter is able to achieve the desired probability of SINR\(_{\text{goal}}\) with a small margin error due to Taylor series approximation. However, this method is able to outperform the ordinary stochastic robust optimization method significantly due to possible preservation of the structure of the error matrix.

V. CONCLUSION

We have considered the problem of robust waveform design for multistatic cognitive radars in a signal-dependent clutter environment. Assuming uncertainty on the clutter statistics, we have proposed worst-case robust optimization and stochastic robust optimization methods. While non-robust optimization methods are unable to achieve the required SINR\(_{\text{goal}}\), the worst-case robust optimization is always able to achieve this goal SINR, however, this method is over-conservative as it aims to achieve the desired SINR for the worst-case clutter statistics. The stochastic robust optimization is able to achieve the goal SINR with a specified outage probability in the presence of uncertainty on the clutter covariance matrix. Finally, the proposed algorithm that assumes uncertainty directly on the clutter parameter is able to achieve the desired probability of SINR\(_{\text{goal}}\) with a small margin error due to Taylor series approximation. However, this method is able to outperform the ordinary stochastic robust optimization method significantly due to possible preservation of the structure of the error matrix.

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REFERENCES


