Matrix Schrödinger equations and Darboux transformations

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Matrix Schrödinger

equations

and

Darboux transformations

by

V.M. Gontcharenko

A Doctoral Thesis

Submitted in partial fulfilment of the requirements

for the award of

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Abstract

This thesis contains the matrix generalisations of some important results known in the theory of the scalar Schrödinger operators. In the first part we discuss the one-dimensional matrix Schrödinger equations in complex domain. The main results here are the local criteria for the Schrödinger operators to have trivial monodromy and a matrix generalisation of the well-known Duistermaat-Grünbaum theorem giving the description of such operators in terms of Darboux transformations.

In the second part we consider D-integrable matrix Schrödinger operators in many dimensions. The local criteria on singularities of such operators are found and new examples are constructed.

In the last chapter we discuss the soliton solutions of the matrix KdV equations and study the interaction of two solitons.

Keywords: Matrix Schrödinger Operator, Matrix Darboux Transformation, Monodromy in Complex Domain, Calogero-Moser System, Matrix Locus Equations, Matrix Korteweg-de Vries Equation, Solitons.
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Chapter 1

Introduction.

In 1967 Gardner, Green, Kruskal and Miura [32] found a remarkable connection between Korteweg-de Vries (KdV) equation

\[ u_t = 6uu_x - u_{xxx} \]  \hspace{1cm} (1.1)

describing the wave propagation in shallow water and spectral theory of the Schrödinger operator

\[ L = -D^2 + u(x,t), \quad D = \frac{d}{dx} \]  \hspace{1cm} (1.2)

This allowed to solve KdV equation by inverse scattering method (see [33, 53]) in the class of rapidly decreasing potentials. An algebraic explanation of the method of paper [32] was given by Lax [51] in 1968 who showed that KdV equation can be written in the form

\[ \dot{L} = [L, A] \]  \hspace{1cm} (1.3)

where \( A = -4D^3 + 3(uD + Du) \) is a skew-symmetric differential operator of order 3.

Since then the Korteweg-de Vries equation and its higher analogues (see [51], [57])

\[ \dot{L} = [L, A_n] \]

where \( A_n \) is some differential operator of order \( 2n+1 \) were investigated from different points of view. Probably, one of the most remarkable result was the discovery by Novikov [60] and Lax [52] that all the periodic solutions of stationary KdV equation and its higher analogues are the potentials of the Schrödinger operators with finitely many gaps in their spectrum (see also McKean and van Moerbeke [56]). Dubrovin [23] and Flaschka [31] proved that in such a way all the finite-gap periodic Schrödinger operators can be obtained. We should mention that the first examples
of finite-gap potentials have been found by Ince [42] who proved that the Schrödinger operators with Lame potentials

$$u(x) = N(N + 1)\varphi(x)$$

where \(\varphi(x)\) is the classical Weierstrass elliptic functions (see e.g. [69]) have \(N\) gaps in their spectrum. General formulae for finite-gap potentials was obtained by Its and Matveev [40, 41] in terms of theta-functions of hyperelliptic Riemann surfaces branch points of which coincide with gap’s boundaries of the corresponding Schrödinger operator (see also [25, 24]). Krichever developed a general approach to the finite-gap theory based on the notion of Baker-Akhiezer function [45, 46, 47, 48, 49].

Rational solutions of KdV equation were studied by Airault, McKean and Moser [3]. They found that the general form of the rational solution of the KdV equation decaying at infinity is

$$u(x, t) = \sum_{j=1}^{n} \frac{1}{(x - x_j(t))^2}$$

where \(n = \frac{d(d+1)}{2}, d \in \mathbb{Z}_{\geq 0}\) is a "triangular" number and points \(x_j = x_j(t)\) have to satisfy the conditions

$$\sum_{i \neq j} \frac{1}{(x_i - x_j)^3} = 0, \quad i = 1, \ldots, n, \quad (1.4)$$

$$\dot{x}_j = \sum_{i \neq j} \frac{6}{(x_j - x_i)^2} \quad i = 1, \ldots, n. \quad (1.5)$$

They called the set \(x_1, \ldots, x_n\) satisfying (1.4) as locus and equations (1.4) - locus equation. Locus coincides with the set of stationary points of the integrable Calogero-Moser system [13, 59] with the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{n} y_j^2 + \sum_{1 \leq j < k \leq n} (x_j - x_k)^{-2}$$

while (1.5) corresponds to the dynamics given by its integral

$$I = \frac{1}{3} \sum_{j=1}^{n} \left( y_j^3 + \sum_{k \neq j} y_j(x_j - x_k)^{-3} \right).$$

In [1] Adler and Moser have shown that as the functions of \(x\) all the rational solutions of the KdV equation are the potentials for Schrödinger operators \(L = -D^2 + u(x)\) which can be obtained by a chain of Darboux transformations [21] from the operator \(L_0 = -\frac{d^2}{dx^2}\) and all these potentials have the form

$$u = -2\frac{d^2}{dx^2} \log w \quad (1.6)$$
where \( w = w(\psi_1, \psi_2, \ldots, \psi_k) \) is the wronskian of functions \( \psi_1, \psi_2, \ldots, \psi_k \) given by recursive formulae

\[
\psi_1'' = 0, \quad \psi_{j+1}' = \psi_j
\]

for \( j = 1, 2, \ldots \) : \( \psi_1 = x, \psi_2 = \frac{1}{6}x^3 + \tau_1, \ldots \).

It turns out that all such operators \( L \) have a remarkable characterisation discovered by Duistermaat and Grünbaum [27]. Namely, operator \( L = -D^2 + u(x) \) with rational potential \( u(x) \) decaying at infinity can be obtained by finitely many Darboux transformations from \( L_0 = -D^2 \) if and only if \( L \) has trivial monodromy in the complex domain, that is, all the solutions \( \psi \) of the equation

\[
L\psi = \lambda\psi
\]

are meromorphic in \( \mathbb{C} \) for all \( \lambda \).

The generalisation of this result for multidimensional case has been found by Chalykh, Feigin and Veselov [17]. Let us call the Schrödinger operator

\[
L = -\Delta + u(x), \quad x \in \mathbb{C}^n
\]

\( D \)-integrable if there exists a differential operator \( D \) intertwining \( L \) with pure Laplacian \( L_0 = -\Delta \):

\[
LD = DL_0.
\]

Berest and Veselov [7, 8] proved that the singularities of the potential of any \( D \)-integrable operator must be located on a configuration of hyperplanes. If we assume that \( u \) is a rational function decaying at infinity then \( u \) is of the form

\[
u(z) = \sum_{j=1}^{n} \frac{m_j(m_j + 1)(\alpha_j, \alpha_j)}{((\alpha_j, z) + c_j)^2}
\]

for some configuration of hyperplanes \( \Pi_j : (\alpha_j, z) + c_j = 0 \) with multiplicities \( m_j \in \mathbb{Z}_+ \). Chalykh, Feigin and Veselov [17] have found the necessary local conditions on such configurations for (1.7) to be \( D \)-integrable (locus conditions). Chalykh [15] has proved the remarkable fact that these conditions are also sufficient. In dimension 1 this gives another proof of Duistermaat-Grünbaum result.

The main difficulty of the multidimensional case is that there is no effective way to describe all the locus configurations. The Coxeter configurations [10] related to any finite group generated by reflections give the examples of such configurations but do not exhaust all of them: in 1996 Chalykh, Feigin and Veselov [65] found some
non-Coxeter examples satisfying locus equations. Let us notice that for Coxeter configuration $A$ the corresponding Schrödinger operator

$$L = -\Delta + \sum_{\alpha \in A} m_{\alpha}(m_{\alpha} + 1)(\alpha, \alpha)$$

is nothing else but the generalisation of the Calogero-Moser operator suggested by Olshanetsky and Perelomov [63].

The main goal of this thesis is to investigate the generalisation of these results to the case of Schrödinger operators with matrix potentials. Lax was the first to introduce the matrix KdV equation

$$U_t = 3(UU_x + U_z U) - U_{zzz}$$

in relation with the matrix Schrödinger operator

$$L = -\frac{d^2}{dx^2} + U(x, t)$$

where $U(x, t)$ is $d \times d$ matrix (see [51]). Inverse scattering method for the matrix Schrödinger operators with hermitian potential on the half-line $0 < x < \infty$ and, in particular, its uniqueness aspects, were developed by Agranovich and Marchenko [2]. Inverse scattering problem on the real line $-\infty < x < +\infty$ for matrix Schrödinger equation in connection with matrix KdV equation was investigated by Wadati and Kamijo [67]. Much more general class of nonlinear equations integrable by inverse scattering transform was constructed by Calogero and Degasperis [14] (see also [12]) who also discussed matrix solitons. Inverse scattering problem for the general (non-hermitian) matrix Schrödinger operator was studied by Olmedilla and Martinez Alonso [62, 54]. We should also mention the paper [38] by Grinevich who developed the finite-gap theory for the matrix operator

$$\tilde{L} = \Lambda D^2 + U(z)$$

with constant matrix $\Lambda$ having different eigenvalues. The last assumption is essential for the methods of [38] which can not be applied to the case $\Lambda = I$ we consider. The theory of non-commutative versions of integrable nonlinear equations from general point of view has been discussed by Etingof, Gelfand and Retakh [29]. They used the important notion of quasideterminants introduced by Gelfand and Retakh in [34].

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In 1984 Gibbons and Hermsen [36] introduced a matrix generalisation of Calogero-Moser system in connection with matrix time-dependent Schrödinger equation

$$\Psi_t = -\Psi_{xx} + Q\Psi$$  \hfill (1.8)

where

$$Q(x, t) = \sum_{j=1}^{n} \frac{M_j(t)}{(x - x_j(t))^2}.$$  

Recently such spin generalisation of the classical Calogero-Moser system has been discussed in another context by Krichever, Babelon, Billey and Talon [50].

Cherednik [19] seems to be the first to consider matrix generalisations of the quantum Calogero-Moser systems. They have a form

$$L = -\Delta + \sum_{\alpha \in \mathcal{R}_+} \frac{m_\alpha (m_\alpha I - \delta_\alpha)(\alpha, \alpha)}{(x, \alpha)^2}$$

where $\mathcal{R}_+$ is a set of positive roots of any semi-simple Lie group $G$ and $\delta_\alpha$ stands for the reflection with respect to $\alpha$ in any matrix representation of corresponding Weyl group $W$ (for definitions see [39]). He showed that the corresponding quantum system has $n = \text{rank} \ G$ commuting quantum integrals and, therefore, it is integrable in a usual quantum mechanical sense. For the trivial one-dimensional representation we have a scalar Schrödinger operator which is the well-known generalised Calogero-Moser operator introduced by Olshanetsky and Petrelomov (see [63]). Another class of matrix Schrödinger operators related to semi-simple Lie groups was investigated by Etingof and Styrkas [28]. An interesting approach to matrix Calogero-Sutherland Hamiltonians has been recently developed by Bracken and Kamran [11]. Further results in this direction have been found by Chalykh, Veselov and the author in [18].

The structure of the thesis is the following.

In the next chapter we investigate the matrix Darboux transformations for the matrix Schrödinger operators in dimension 1. We discuss in more details the case when potentials of the corresponding Schrödinger operators are rational and the initial operator has zero potential.

In the third chapter we find the local conditions on the potential of the matrix Schrödinger operators with trivial monodromy in the complex domain (matrix locus equations) which generalise corresponding local criteria by Duistermaat and Grünbaum to the matrix case. Then we prove the main result of this chapter which is the matrix generalisation of Duistermaat and Grünbaum theorem [27]. Namely, we prove that all matrix Schrödinger operators with trivial monodromy having the
rational potentials decaying at infinity can be obtained by matrix Darboux transformation applied to $L_0 = -D^2$. We also prove the similar statement for trigonometric case generalising the corresponding result by Chalykh [15].

In the forth chapter we introduce the notion of trivial monodromy for multi-dimensional Schrödinger operators $L$ and derive multidimensional locus conditions. The main result here is the theorem that these conditions are necessary and sufficient for the existence of the intertwining differential operator $D$ such that

$$LD = DL_0$$

in a certain class of matrix Schrödinger operators $L$. We present some examples of such operators and discuss the relation with one-dimensional case.

In the last chapter we discuss the multisoliton solutions of matrix KdV equations and the interaction of two matrix solitons.

Finally, in conclusion some open problems are discussed.
Chapter 2
Matrix Schrödinger operators and Darboux transformations.

2.1 Matrix Darboux transformation and intertwining relations.

Let us consider Schrödinger operator

$$L = -\frac{d^2}{dz^2} + u(z)$$

(2.1)

with the potential $u(z)$. The classical Darboux transformation [21] of $L$ can be defined in the following way. Let us ask for the factorization of $L - \lambda I$

$$L - \lambda I = -(D + f)(D - f), \quad D = \frac{d}{dz}$$

where $\lambda$ is a constant. To do this one can take any solution $\psi$ of the equation

$$L\psi = \lambda \psi$$

and find $f$ from $(D - f)\psi = 0$: $f = \frac{\psi'}{\psi} = (\log \psi)'$. Now we can define a new operator $\tilde{L}$ such that

$$\tilde{L} - \lambda I = -(D - f)(D + f).$$

Iterating this procedure $n$ times applied to the given operator $L = L_0$ we come to the operator $L_n$ satisfying the relation

$$L_n A = A L_0$$

(2.2)

where intertwining operator $A$ is the differential operator of order $n$

$$A = (D - f_n)(D - f_{n-1}) \ldots (D - f_1).$$
It turns out that the converse is also true (see e.g. [66]).

Lemma 2.1. If operators \( L_n = -D^2 + u_n(z) \), \( L_0 = -D^2 + u_0(z) \) and \( A = D^n + a_1(z)D^{n-1} + \ldots + a_n(z) \) satisfy the relation (2.2) then there exists a chain of operators \( L_1, L_2, \ldots, L_n \) such that \( L_kA_k = A_kL_{k-1} \), \( k = 1, \ldots, n \), \( A_k = D - f_k \) and \( A = A_nA_{n-1} \ldots A_1 \). Moreover, operators \( A_k \) define the factorization of \( L_k \) so that

\[
L_{k-1} - \lambda_k I = -(D + f_k)(D - f_k), \quad L_k - \lambda_k I = -(D - f_k)(D + f_k)
\]

for some constants \( \lambda_k \).

Proof. It is easy to check that \( V = \ker A \) is invariant with respect to \( L_0 \). Indeed, if \( \phi \in \ker A \) then from (2.2) we have \( AL_0\phi = 0 \) and \( L_0\phi \in \ker A \). So, we can choose the basis \( \psi_1, \psi_2, \ldots, \psi_n \) and define the subspaces \( V_1 \subset V_2 \subset \ldots \subset V_n = V \) such that \( V_k = < \psi_1, \ldots, \psi_k >, \ k = 1, \ldots n \) are invariant with respect to \( L_0 \). Then \( A \) can be written (see [43])

\[
A = (D - f_n)(D - f_{n-1}) \ldots (D - f_1), \quad f_k = (\log w_k - w_{k-1})', \quad w_0 = 1
\]

where \( w_k = w(\psi_1, \psi_2, \ldots, \psi_k) \) is the wronskian of functions \( \psi_1, \psi_2, \ldots, \psi_k \) and

\[
A(\psi) = \frac{w(\psi_1, \ldots, \psi_n, \psi)}{w(\psi_1, \ldots, \psi_n)}.
\]

Moreover,

\[
V_k = \ker B_k, \quad B_k = A_kA_{k-1} \ldots A_1
\]

and since \( V_k \) is invariant with respect to \( L_0 \) there exists an operator \( L_k \) of the second order such that

\[
L_kB_k = B_kL_0.
\]

Assume that \( k = 1 \). Then \( B_1 = A_1 = D - f_1 \) and the last operator identity has a form

\[
L_1(D - f_1) = (D - f_1)L_0.
\]

It is easy to check that \( L_1 \) is a Schrödinger operator and

\[
L_0 = (D + f_1)(D - f_1) + \lambda, \quad L_1 = (D - f_1)(D + f_1) + \lambda
\]

for some constant \( \lambda \). Analogously, as \( L_2 \) satisfies \( B_2L = L_2B_2 \)

\[
L_2A_2A_1 = A_2A_1L_0 = A_2L_1A_1
\]

we obtain

\[
L_2A_2 = A_2L_1
\]
so that \( L_2 \) can be obtained by Darboux transformation from \( L_1 \). The induction proves the lemma.

Let us consider now the matrix Schrödinger operator

\[
L = -D^2 + U(z), \quad D = \frac{d}{dz}
\]

where potential \( U(z) \) is \( d \times d \) matrix-valued function.

In the matrix case the relation \( L_1(D - f_1) = (D - f_1)L_0 \) does not imply that \( L_0 = (D + f_1)(D - f_1) + \lambda I \) and \( L_1 = (D - f_1)(D + f_1) + \lambda I \). Therefore, the definition of Darboux transformation based on the factorization of the operators seems to be not suitable here. That is why we introduce the following

**Definition.** We say that the operator \( L \) is obtained from another operator \( L_0 = -D^2 + U_0(z) \) by matrix Darboux transformations (MDT) if there exists a differential operator \( A = D^n + a_1(z)D^{n-1} + \ldots + a_n(z) \) such that

\[
LA = AL_0.
\]

In other words, by definition \( L \) is a result of MDT applied to \( L_0 \) if there exists a matrix differential operator \( A \) with a scalar highest coefficient intertwining \( L \) and \( L_0 \). The order \( n \) of \( A \) is called the order of MDT.

For further investigation of MDT we will use the notion of the quasideterminant introduced by Gelfand and Retakh [34].

### 2.2 Quasideterminants and the structure of matrix Darboux transformations.

Let \( R = \text{Mat}_d(\mathbb{C}) \) be an algebra of matrices \( d \times d \) and \( X \) be an \( n \times n \) matrix over \( R \) (in [34] quasideterminants were introduced for any assotiative algebra \( R \)). For any \( 1 \leq i, j \leq n \) let \( r_i(X) \) be the \( i \)-th row and \( c_j(X) \) be the \( j \)-th column of \( X \). Let \( X^{ij} \) be the submatrix of \( X \) obtained by removing the \( i \)-th row and the \( j \)-th column from \( X \). For a row vector \( r \) let \( r^{(j)} \) be \( r \) without the \( j \)-th entry. For the column vector \( c \) let \( c^{(i)} \) be \( c \) without the \( i \)-th entry. Then the quasideterminant is defined as

\[
|X|_{ij} = x_{ij} - r_i(X)^{(j)}(X^{ij})^{-1}c_j(X)^{(i)}
\]

where \( x_{ij} \) is the \( ij \)-th entry of \( X \).
Remark. The term "quasideterminant" may be misleading since it corresponds to a generalisation of the fraction of the determinants of the matrix and its submatrix but not the determinant itself. Namely, if $d = 1$ then

$$|X|_{ij} = (-1)^{(i+j)} \frac{\det X}{\det X^{ij}}.$$ 

In contrast to $\det X$ quasideterminants are not always defined. It depends on the invertibility of matrix $X^{ij}$.

Example. Let us consider the simplest nontrivial case $n = 2$

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$ 

Then

$$|X|_{11} = x_{11} - x_{12}x_{22}^{-1}x_{21},$$

$$|X|_{12} = x_{12} - x_{11}x_{21}^{-1}x_{22},$$

$$|X|_{21} = x_{21} - x_{22}x_{12}^{-1}x_{11},$$

$$|X|_{22} = x_{22} - x_{21}x_{11}^{-1}x_{12}.$$ 

We will use some properties of quasideterminants which are formulated in the following

**Theorem 2.1** ([34, 35]).

1. The quasideterminant $|X|_{ij}$ does not change after the permutation of rows and columns in $X$ provided the element $x_{ij}$ is preserved.

2. If the quasideterminant $|X|_{ij}$ of a matrix $X$ is defined then $|X|_{ij} = 0$ is equivalent to the fact that the $j$-th column of matrix $X$ is a right linear combination of other columns of this matrix (columns are multiplied by the elements of $R$ from the right).

3. Let $Y$ be inverse matrix to $X$: $Y = X^{-1}$. If all the quasideterminants are defined and invertible then the matrix is also invertible and

$$y_{ij} = |X|^{-1}_{ji}.$$ 

4. Suppose we have the following system of linear equations

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = \xi_1, \\ \vdots \quad \ddots \quad \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n = \xi_n. \end{cases}$$
Then
\[ x_i = \sum_{j=1}^{n} |A|^{-1}_{ji} \xi_j. \]

5. (The analogue of Cramer's rule.) Let \( A_1(\xi) \) be obtained by changing the \( l \)-th column of \( A \) on \( \xi = (\xi_1, \ldots, \xi_n)^T \). Then
\[ x_i = |A|^{-1}_{ji} |A_1(\xi)|_{ji}. \]

Let \( \Psi_1(z), \Psi_2(z), \ldots, \Psi_n(z) \) be \( d \times d \)-matrix valued functions. Then we can define the Wronski matrix
\[
W[n] = W(\Psi_1, \Psi_2, \ldots, \Psi_n) = \begin{pmatrix}
\Psi_1 & \cdots & \Psi_n \\
\Psi'_1 & \cdots & \Psi'_n \\
\vdots & \ddots & \vdots \\
\Psi_1^{(n-1)} & \cdots & \Psi_n^{(n-1)}
\end{pmatrix}.
\] (2.5)

**Definition.** A set of elements \( \Psi_1, \ldots, \Psi_n \) is called nondegenerate if \( W(\Psi_1, \ldots, \Psi_n) \) is invertible.

**Theorem 2.2 ([29]).**

1. Let \( \Psi_1, \ldots, \Psi_n \in R \) be a nondegenerate set of \( d \times d \)-matrix valued functions. Then there exists a unique differential operator
\[ A = D^n + a_1 D^{n-1} + \ldots + a_n \]
of order \( n \) with the identity highest coefficient such that \( A\Psi_i = 0 \) for \( i = 1, \ldots, n \) and for any \( \Psi \in R \)
\[ A\Psi = |W(\Psi_1, \ldots, \Psi_n, \Psi)|_{n+1,n+1}. \] (2.6)

2. Let \( A \) be a differential operator of order \( n \) with the identity highest coefficient and \( \Psi_1, \ldots, \Psi_n \) be a set of solutions of the equation \( A\Psi = 0 \) such that for any \( m \leq n \) the set of elements is nondegenerate. Then \( A \) admits a factorization \( A = (D - f_n) \ldots (D - f_1) \) where
\[ f_i = W_i^i W_i^{-1}, \quad W_i = |W(\Psi_1, \ldots, \Psi_i)|_{ii}. \] (2.7)

**Proof.** To prove the first part of the theorem we look for \( A \) in the form \( A = D^n + a_1 D^{n-1} + \ldots + a_n \). We can rewrite the conditions \( A\Psi_i = 0 \) for \( i = 1, \ldots, n \) as
a system of linear differential equations

\[
\begin{aligned}
\psi_1^{(n)} + a_1 \psi_1^{(n-1)} + \ldots + a_n \psi_1 &= 0, \\
\psi_2^{(n)} + a_1 \psi_2^{(n-1)} + \ldots + a_n \psi_2 &= 0, \\
&\vdots \\
\psi_n^{(n)} + a_1 \psi_n^{(n-1)} + \ldots + a_n \psi_n &= 0,
\end{aligned}
\]

or, equivalently,

\[
(\psi_1^{(n)}, \psi_2^{(n)}, \ldots, \psi_n^{(n)}) = -(a_n, a_{n-1}, \ldots, a_1)W[n].
\]

Multiplying both sides of the last equation by the matrix $W^{-1}[n]$ from the right we have

\[
(a_n, \ldots, a_1) = -(\psi_1^{(n)}, \ldots, \psi_n^{(n)})W^{-1}[n].
\]

Then for any $\Psi \in R$ we obtain

\[
[W(\Psi_1, \Psi_2, \ldots, \Psi_n)_{n+1,n+1}] = \\
\psi_1^{(n)} \cdots \psi_n^{(n)} \psi^{(n)}_{n+1,n+1} = \\
\begin{pmatrix}
\psi \\
\psi' \\
\vdots \\
\psi^{(n-1)}
\end{pmatrix} = \\
\begin{pmatrix}
\psi_1^{(n)} \\
\psi_2^{(n)} \\
\vdots \\
\psi_n^{(n)}
\end{pmatrix} W^{-1}[n] = \\
\begin{pmatrix}
\psi^{(n)} \\
(a_n, \ldots, a_1)
\end{pmatrix} = A\Psi.
\]

The second statement of the theorem can be proved by induction. For $n = 1$ it is obvious. Let us assume that it is true for the differential operator $A_{n-1}$ of order $n - 1$ with the identity highest coefficient which annihilates $\Psi_1, \ldots, \Psi_{n-1}$ (by the first part it exists and unique). Set $f_n = W_n W_n^{-1}$ and consider the operator

\[
\tilde{A} = (D - f_n)A_{n-1}.
\]

Obviously,

\[
\tilde{A}\psi_i = 0, \quad i = 1, \ldots, n - 1
\]

and as $A_{n-1}\Psi_n = |W(\Psi_1, \ldots, \Psi_n)|_{n,n}$ we have that

\[
\tilde{A}\Psi_n = (D - f_n)A_{n-1}\Psi_n = (D - f_n)W_n = 0.
\]
So, $\tilde{A} = A$ by uniqueness. This proves the theorem.

Using theorem 2.2 we can describe in more details the structure of matrix Darboux transformation. At the beginning let us notice that in the same way as lemma 2.1 we can prove the following

**Lemma 2.2.** If Schrödinger operators $L$ and $L_0$ satisfy the relation $LA = AL_0$ then $V = \ker A$ is invariant under $L_0$

$$L_0V \subseteq V.$$ Conversely, for any nd-dimensional $L_0$-invariant space $V$ there exists Schrödinger operator $L$ such that $LA = AL_0$ and $\ker A = V$.

Therefore, we can find the flag $V_1 \subset V_2 \subset \ldots \subset V_n$ consisting of the spaces being invariant under $L_0$ such that $\dim V_k = kd$, $V_k = \langle \Psi_1, \Psi_2, \ldots, \Psi_k \rangle$ where $\Psi_k$, $k = 1, \ldots, n$ are $d \times d$ matrices and then choose the corresponding factorization of $A$

$$A = (D - f_n) \ldots (D - f_1)$$

according to the theorem 2.2. Because of the special choice of the flag there is a sequence of the Schrödinger operators $L_1, L_2, \ldots, L_n = L$, $L_k = -D^2 + U_k(z)$ such that

$$L_k(D - f_k) \ldots (D - f_1) = (D - f_k) \ldots (D - f_1)L_0$$

where $k = 1, 2, \ldots, n$ and then

$$L_k(D - f_k) \ldots (D - f_1) = (D - f_k)L_{k-1}(D - f_{k-1}) \ldots (D - f_1)$$

so that

$$L_k(D - f_k) = (D - f_k)L_{k-1}, \quad k = 1, \ldots, n.$$ Thus we obtain

**Proposition 2.1.** Any MDT of order $n$ can be represented as a composition of $n$ MDT of order 1.

Of course, such a representation is not unique.

## 2.3 Relation between the potentials of the Schrödinger operators connected by MDT.

Now we would like to find the formula for the potential of Schrödinger operator $L = -D^2 + U(z)$ obtained by MDT from $L_0 = -D^2 + U_0(z)$. At the beginning let
us mention the following simple

**Lemma 2.3.** Let \( L = -D^2 + U \) and \( L_0 = -D^2 + U_0 \) be Schrödinger operators related by MDT: \( LA = AL_0 \) where \( A = D^n + a_1(z)D^{n-1} + \ldots + a_n(z) \). Then

\[
U = U_0 + 2a_1'(z).
\]

(2.8)

*Proof* is straightforward.

If functions \( \Psi_1, \ldots, \Psi_n \) generate the kernel of the intertwining operator \( A \) then according to the theorem 2.2

\[
A = (D - f_n) \ldots (D - f_1)
\]

with

\[
f_j = W_j'W_j^{-1}, \quad W_j = |W(\Psi_1, \ldots, \Psi_j)|_{jj}, \quad j = 1, \ldots, n.
\]

Since \( a_1(z) = -\sum_{j=1}^{n} f_j \) we have the following relation between the potentials \( U(z) \) and \( U_0(z) \):

\[
U = U_0 - 2 \sum_{j=1}^{n} (W_j'W_j^{-1}).'
\]

It turns out that the last formula can be simplified. Namely, introduce matrix \( Y(\Psi_1, \ldots, \Psi_n) \) which coincides with the Wronski matrix \( W(\Psi_1, \ldots, \Psi_n) \) except at the last row where it has \( \Psi_i^{(n)} \) instead of \( \Psi_i^{(n-1)} \)

\[
Y(\Psi_1, \ldots, \Psi_n) = \begin{pmatrix}
\Psi_1 & \ldots & \Psi_n \\
\vdots & \ddots & \vdots \\
\Psi_1^{(n-2)} & \ldots & \Psi_n^{(n-2)} \\
\Psi_1^{(n)} & \ldots & \Psi_n^{(n)}
\end{pmatrix}
\]

Let \( Y_n = |Y(\Psi_1, \ldots, \Psi_n)|_{nn} \) and \( W_n = |W(\Psi_1, \ldots, \Psi_n)|_{nn} \).

**Lemma 2.4.** For any nondegenerate set of \( d \times d \) matrices \( \Psi_1(z), \Psi_2(z), \ldots, \Psi_n(z) \)

\[
\sum_{k=1}^{n} W_k'W_k^{-1} = Y_nW_n^{-1}
\]

*Proof.* It is enough to prove that

\[
Y_nW_n^{-1} + W'_{n+1}W_{n+1}^{-1} = Y_{n+1}W_{n+1}^{-1}.
\]

(2.9)
By definition,

\[ W_{n+1} = \Psi_{n+1}^{(n)} - (\Psi_1^{(n)}, \ldots, \Psi_n^{(n)})W^{-1}[n] \begin{pmatrix} \Psi_{n+1} \\ \vdots \\ \Psi_{n+1}^{(n-1)} \\ \Psi_{n+1} \end{pmatrix}. \]

Therefore,

\[ W'_{n+1} = \Psi_{n+1}^{(n+1)} - (\Psi_1^{(n+1)}, \ldots, \Psi_n^{(n+1)})W^{-1}[n] \begin{pmatrix} \Psi_{n+1} \\ \vdots \\ \Psi_{n+1}^{(n-1)} \\ \Psi_{n+1} \end{pmatrix} + \]

\[ - (\Psi_1^{(n)}, \ldots, \Psi_n^{(n)})W^{-1}[n] \begin{pmatrix} \Psi_{n+1}^{(1)} \\ \vdots \\ \Psi_{n+1}^{(n)} \\ \Psi_{n+1} \end{pmatrix} - W'[n] W^{-1}[n] \begin{pmatrix} \Psi_{n+1} \\ \vdots \\ \Psi_{n+1}^{(n-1)} \\ \Psi_{n+1} \end{pmatrix} = \]

\[ = Y_{n+1} - (\Psi_1^{(n)}, \ldots, \Psi_n^{(n)})W^{-1}[n] \begin{pmatrix} \Psi_{n+1}^{(1)} \\ \vdots \\ \Psi_{n+1}^{(n)} \\ \Psi_{n+1} \end{pmatrix} - W'[n] W^{-1}[n] \begin{pmatrix} \Psi_{n+1} \\ \vdots \\ \Psi_{n+1}^{(n-1)} \\ \Psi_{n+1} \end{pmatrix} . \]

Consider the vector-column

\[ \xi = \begin{pmatrix} \Psi_{n+1}^{(1)} \\ \vdots \\ \Psi_{n+1}^{(n)} \\ \Psi_{n+1} \end{pmatrix} - W'[n] W^{-1}[n] \begin{pmatrix} \Psi_{n+1} \\ \vdots \\ \Psi_{n+1}^{(n-1)} \\ \Psi_{n+1} \end{pmatrix} . \]

Then, using the theorem 2.1 (part 2) we can conclude

\[ \xi_1 = \Psi'_{n+1} - (\Psi_1', \ldots, \Psi_n') W^{-1}[n] \begin{pmatrix} \Psi_{n+1} \\ \vdots \\ \Psi_{n+1}^{(n-1)} \\ \Psi_{n+1} \end{pmatrix} = \]

\[ = \begin{vmatrix} \Psi_1 & \cdots & \Psi_{n+1} \\ \vdots & \ddots & \vdots \\ \Psi_{n-1} & \cdots & \Psi_{n+1} \\ \Psi_1' & \cdots & \Psi_n' \end{vmatrix}_{n+1, n+1} = 0, \]

\[ \xi_2 = \Psi_{n+1}^{(2)} - (\Psi_1^{(2)}, \ldots, \Psi_n^{(2)}) W^{-1}[n] \begin{pmatrix} \Psi_{n+1} \\ \vdots \\ \Psi_{n+1}^{(n-1)} \\ \Psi_{n+1} \end{pmatrix} = \]
\[ W(n+1) - W(n-1) = 0 \]

and so on. At last,

\[ \xi_{n-1} = \Psi_{n+1}^{(n-1)} - (\Psi_{1}^{(n-1)}, \ldots, \Psi_{n}^{(n-1)}) W^{-1}[n] \]

\[ \begin{vmatrix} \Psi_{1} & \cdots & \Psi_{n+1} \\ \vdots & \ddots & \vdots \\ \Psi_{1}^{(n-1)} & \cdots & \Psi_{n+1}^{(n-1)} \end{vmatrix}_{n+1,n+1} = 0, \]

\[ \xi_{n} = \Psi_{n+1}^{(n)} - (\Psi_{1}^{(n)}, \ldots, \Psi_{n}^{(n)}) W^{-1}[n] \]

\[ \begin{vmatrix} \Psi_{1} & \cdots & \Psi_{n+1} \\ \vdots & \ddots & \vdots \\ \Psi_{1}^{(n)} & \cdots & \Psi_{n+1}^{(n)} \end{vmatrix}_{n+1,n+1} = W_{n+1}. \]

So, we have that \( \xi = (0, 0, \ldots, 0, W_{n+1})^T \) and

\[ W'_{n+1} = Y_{n+1} - (\Psi_{1}^{(n)}, \ldots, \Psi_{n}^{(n)}) W^{-1}[n] \]

\[ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ W_{n+1} \end{pmatrix}. \]

From the properties of the inverse matrix (see theorem 2.1, part 3) it follows that the last column of matrix \( W^{-1}[n] \) is \( (W_{n1}^{-1}, \ldots, W_{nn}^{-1})^T \) and it is possible to rewrite the formula for \( W'_{n+1} \) as follow

\[ W'_{n+1} = Y_{n+1} - (\Psi_{1}^{(n)}, \ldots, \Psi_{n}^{(n)}) \begin{pmatrix} W_{n1}^{-1}W_{n+1} \\ \vdots \\ W_{nn}^{-1}W_{n+1} \end{pmatrix}. \]
Let us insert this expression to (2.9). We obtain

\[ Y_n W_n^{-1} = (\Psi_1^{(n)}, \ldots, \Psi_n^{(n)}) \begin{pmatrix} W_{n1}^{-1} \\ \vdots \\ W_{nn}^{-1} \end{pmatrix}. \]

Substituting the formula for \( Y_n \) into the last expression we have

\[
\Psi_n^{(n)} W_n^{-1} - \left( \Psi_1^{(n)}, \ldots, \Psi_{n-1}^{(n)} \right) W^{-1}[n-1] \begin{pmatrix} \Psi_n \\ \vdots \\ \Psi_{n-2}^{(n-2)} \end{pmatrix} W_n^{-1} = \\
= \left( \Psi_1^{(n)}, \ldots, \Psi_{n-1}^{(n)} \right) \begin{pmatrix} W_{n1}^{-1} \\ \vdots \\ W_{n,n-1}^{-1} \end{pmatrix} + \Psi_n^{(n)} W_n^{-1}.
\]

Here we used that

\[
\left( \Psi_1^{(n)}, \ldots, \Psi_n^{(n)} \right) \begin{pmatrix} W_{n1}^{-1} \\ \vdots \\ W_{nn}^{-1} \end{pmatrix} = \left( \Psi_1^{(n)}, \ldots, \Psi_{n-1}^{(n)} \right) \begin{pmatrix} W_{n1}^{-1} \\ \vdots \\ W_{n,n-1}^{-1} \end{pmatrix} + \Psi_n^{(n)} W_n^{-1}.
\]

So, we should prove that

\[
-(\Psi_1^{(n)}, \ldots, \Psi_{n-1}^{(n)}) W^{-1}[n-1] \begin{pmatrix} \Psi_n \\ \vdots \\ \Psi_{n-2}^{(n-2)} \end{pmatrix} W_n^{-1} = \left( \Psi_1^{(n)}, \ldots, \Psi_{n-1}^{(n)} \right) \begin{pmatrix} W_{n1}^{-1} \\ \vdots \\ W_{n,n-1}^{-1} \end{pmatrix},
\]

for which it is enough to show that

\[
-W^{-1}[n-1] \begin{pmatrix} \Psi_n \\ \vdots \\ \Psi_{n-2}^{(n-2)} \end{pmatrix} W_n^{-1} = \begin{pmatrix} W_{n1}^{-1} \\ \vdots \\ W_{n,n-1}^{-1} \end{pmatrix},
\]

or, equivalently,

\[
-W^{-1}[n-1] = W[n-1] \begin{pmatrix} W_{n1}^{-1} \\ \vdots \\ W_{n,n-1}^{-1} \end{pmatrix}.
\]

As it was mentioned above the last column of matrix \( W^{-1}[n] \) is \( (W_{n1}^{-1}, \ldots, W_{nn}^{-1}) \) so that

\[
\sum_{k=1}^{n} \Psi_k^{(j)} W_{n,k}^{-1} = 0, \quad j = 0, \ldots, n-2,
\]

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or, equivalently,
\[ \sum_{k=1}^{n-1} \Psi_k^{(j)} W_n^{-1} = -\Psi_n^{(j)} W_n^{-1}, \quad j = 0, \ldots, n - 2. \]

Rewriting the last identities in matrix form we obtain
\[ - \begin{pmatrix} \Psi_n & \cdots & \Psi_{n-1} \\ \vdots & \ddots & \vdots \\ \Psi_n^{(n-2)} & \cdots & \Psi_{n-1}^{(n-2)} \end{pmatrix} W_n^{-1} = \begin{pmatrix} W_{n1}^{-1} \\ \vdots \\ W_{nn-1}^{-1} \end{pmatrix} \]
which proves (2.10) and the lemma.

We obtain the following

**Theorem 2.3** (cf. [29]). Let Schrödinger operator $L$ be obtained from another Schrödinger operator $L_0$ by MDT with intertwining operator $A$ of order $n$. If the $L_0$-invariant space $\ker A$ is generated by $d \times d$-matrix valued functions $\Psi_1, \ldots, \Psi_n$ then the potential $U$ of the operator $L$ is
\[ U = U_0 - 2(Y_n W_n^{-1})' \]
where $Y_n = |Y(\Psi_1, \ldots, \Psi_n)|_{nn}$ and $W_n = |W(\Psi_1, \ldots, \Psi_n)|_{nn}$.

### 2.4 Special case $L_0 = -\frac{d^2}{dz^2}$. Schrödinger operators with rational potentials.

Further we confine ourselves to the special case when $L_0 = -\frac{d^2}{dz^2}$. Then the result of MDT is the Schrödinger operator $L$ with the potential
\[ U = -2(Y_n W_n^{-1})' \quad (2.11) \]
and lemma 2.3 means that there exists a constant $nd \times nd$ matrix $C$ such that if $d \times d$ matrices $\Psi_1, \Psi_2, \ldots, \Psi_n$ generate $V = \ker A$ then as $d \times nd$ matrix $\Phi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ satisfies the matrix differential equation
\[ \Phi'' = \Phi C. \quad (2.12) \]

Any transformation
\[ C \rightarrow BCB^{-1}, \quad \det B \neq 0 \quad (2.13) \]
of matrix $C$ corresponds to the linear transformation of the basis of $V$ which does not change the form of $A$. Therefore, according to the formula (2.8) the transformation
(2.13) does not change the result of MDT and $C$ can be assumed to be in Jordan form

\[
C = \begin{pmatrix}
J_1(\lambda_1) & 0 & \ldots & 0 \\
0 & J_2(\lambda_2) & 0 & \ldots \\
0 & \ldots & \ddots & 0 \\
0 & \ldots & 0 & J_m(\lambda_m)
\end{pmatrix}, \quad J(\lambda) = \begin{pmatrix}
\lambda & 1 & 0 \\
0 & \ddots & \ddots & 0 \\
0 & \ldots & \ddots & 1 \\
0 & \ldots & 0 & \lambda
\end{pmatrix}.
\]

In this case $V$ consists of vectors $v_i$ which are quasipolynomials in $z$, that is, have the form $v_i = \sum_{j=1}^{m_1} p_{ij}(z)e^{\lambda_jz}$, $p_{ij}$ are vector polynomials in $z$.

Let us assume now that potential of $L$ rational. Obviously, to get such an operator one has to consider the matrix $C$ with all $\lambda_i = 0$, $i = 1, \ldots, m$. As it follows from (2.11) the singularities of the potential are zeroes of the (usual) determinant of the matrix $W_n$.

Lemma 2.5.

\[
\det W_n = \frac{\det W[n]}{\det W[n-1]}
\]

where $W[n]$ is defined by (2.5).

Proof. Introduce vector-column $\xi$ which consists of the $n-1$ elements of the last column of matrix $W[n]$

\[
\xi = \begin{pmatrix}
\Psi_n \\
\Psi'_n \\
\vdots \\
\Psi^{(n-2)}_n
\end{pmatrix}
\]

and vector-row $\eta$ which consists of the $n-1$ elements of the last row of $W[n]$

\[
\eta = \begin{pmatrix}
\Psi^{(n-1)}_1 \\
\Psi^{(n-1)}_2 \\
\vdots \\
\Psi^{(n-1)}_{n-1}
\end{pmatrix}.
\]

Then

\[
W[n] = \begin{pmatrix}
W[n-1] & \xi \\
\eta & \Psi^{(n-1)}_n
\end{pmatrix}.
\]

Introduce also the matrix $S_n$

\[
S_n = \begin{pmatrix}
W^{-1}[n-1] & -W^{-1}[n] \xi \\
0 & I_d
\end{pmatrix}
\]

where $I_d$ is identity $d \times d$ matrix. Obviously, $\det W[n-1] \det S_n = 1$ and

\[
\det W[n] = \det W[n-1] \det S_n \det W[n] = \det W[n-1] \det(S_n W[n]).
\]
Whence

\[
\frac{\det W[n]}{\det W[n-1]} = \det \left( \begin{array}{ccc}
W[n-1] & \xi \\
\eta & \psi^{(n-1)}_n
\end{array} \right) \left( \begin{array}{cc}
W[n-1]^{-1} & -W^{-1}[n]\xi \\
0 & I_d
\end{array} \right) =
\]

\[
= \det \left( \begin{array}{cc}
I_{(n-1)d} & 0 \\
* & W_n
\end{array} \right) = \det W_n
\]

which proves the lemma.

So, in generic case, when polynomials \(\det W[n]\) and \(\det W[n-1]\) do not have common roots, the set of singularities of \(U(z)\) is defined by the equation

\[
\det W[n] = 0.
\]  

(2.14)

**Proposition 2.2.** The degree \(N_d(n)\) of polynomial \(\det W[n]\) satisfies

\[N_d(n) \leq \frac{nd(nd+1)}{2}.
\]

If \(C\) consists of only one Jordan block then generically

\[N_d(n) = \frac{nd(nd+1)}{2}.
\]

**Proof.** If matrix \(C\) consists of only one Jordan block

\[
C = \begin{pmatrix}
0 & 1 & 0 \\
0 & \ddots & \ddots & 0 \\
0 & \ldots & \ddots & 1 \\
0 & \ldots & 0 & 0
\end{pmatrix}
\]

then \(nd \times nd\) matrix \(W[n]\) has a form

\[
W[n] = \begin{pmatrix}
\zeta^{(2nd-2)} & \ldots & \zeta^{(2)} & \zeta
\end{pmatrix}
\]

where \(\zeta = \zeta(z)\) is the first column of matrix \(W[n]\). The last column \(\zeta^{(2nd-2)}\) can be a vector-polynomial of the first degree or a constant as it is annihilated by \(L_0\).

Assume that it is polynomial of the first degree. Then \(\zeta\) has degree \(2nd - 1\) and \(\zeta^{(2nd)} = 0\). We want to prove that \((\det W[n])^{(N_d(n)+1)} = 0\). To do this we use that for any \(k \times k\) matrix \(A\) the elements of which are smooth functions of \(z\)

\[
(\det A)' = \sum_{r=1}^{k} \det A^r
\]

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where $A^r$ is obtained from $A$ by differentiation of the $r$-th column. We can easily get that

$$(\det W[n])^{nd(nd+1)/2} =$$

$$c \cdot \det \begin{pmatrix} \zeta^{2nd-1} & \zeta^{2nd-2} & \cdots & \zeta^{nd+1} & \zeta^{nd} \end{pmatrix}$$

where $c$ is a positive constant and $(\det W[n])^{nd(nd+1)/2+1} = 0$ because differentiation of any columns of the right-hand side of the last formula gives a degenerate matrix. From the proof it is easy to see that for other $C$ the degree of $\det W[n]$ is less then $N_d(n)$. Proposition follows.

Remark. Let us consider the generic case, when deg $\det W = N_d(n)$ and all the roots of the equation (2.14) are simple. Then from the formulae (2.11) one can derive that the potential $U(z)$ has the form

$$U(z) = \sum_{i=1}^{N_d(n)} \frac{A_i}{(z - z_i)^2}$$

(2.15)

where all the matrices $A_i$ have rank 1.
Chapter 3

Matrix Schrödinger operators with trivial monodromy in the complex domain.

Let us consider Schrödinger operator

\[ L = -D^2 + u(z), \quad D = \frac{d}{dz} \]

with a meromorphic potential \( u(z) \) and the corresponding Schrödinger equation

\[ L\psi(z, \lambda) = \lambda \psi(z, \lambda) \quad (3.1) \]

in the complex plane \( z \in \mathbb{C} \).

Assume that all the singularities of \( u(z) \) are regular, i.e. \( u(z) \) has the poles of order at most 2 (see [68]). In general, if \( z = z_0 \) is a pole of \( u(z) \) it is a branch point of solutions of the equation (3.1), or, in other words, (2.1) has nontrivial monodromy around \( z_0 \) depending on the spectral parameter \( \lambda \). If this is not the case, i.e. all the solutions of the equation (3.1) are single-valued in the complex domain for all \( \lambda \), we say that the corresponding Schrödinger operator has trivial monodromy.

In 1986 Duistermaat and Grünbaum [27] proved the following local criteria for the potential of the Schrödinger operator with trivial monodromy which looks very classical but seems to have been unknown before.

**Theorem 3.1** ([27]). Let \( u(z) \) be an arbitrary meromorphic function in a neighborhood of \( p \in \mathbb{C} \), with Laurent expansion

\[ u(z) = \sum_{r \geq -2} c_r(z - p)^r, \quad c_{-2} \neq 0. \]
Then all eigenfunctions of $L = -D^2 + u(z)$ are single-valued around $p$ if and only if

$$c_{-2} = m(m - 1), \ m \in \mathbb{N}, \ m \geq 2$$

and

$$c_{-1} = c_1 = \ldots = c_{2m-3} = 0.$$ 

Every eigenfunction $\psi$ has a Laurent expansion of the form

$$\psi(z) = (z - p)^{-m+1} \sum_{r=0}^{\infty} d_r (z - p)^r,$$

with

$$d_1 = d_3 = \ldots = d_{2m-3} = 0.$$ 

In this chapter we present the matrix generalisation of this result (theorem 3.3). We start from the standard definition of regular and irregular singular point for the system of ordinary differential equations in the complex domain following essentially Wasow’s book [68].

3.1 Regular and irregular singular points of ordinary differential equations.

Let us consider linear ordinary matrix differential equation of the first order

$$Y' = B(z)Y$$

(3.2)

where $Y(z)$ and $B(z)$ are $d \times d$ matrices. The columns of $Y(z)$ are solutions of the system

$$y' = B(z)y, \ \ y \in \mathbb{C}^d.$$ 

Let us assume that $B(z)$ has a singular point at $z = 0$. If this singularity is a pole of the first order then the equation (3.2) can be, evidently, written in the form

$$zY' = A(z)Y$$

(3.3)

where $A(z)$ is holomorphic at $z = 0$.

Definition ([68]). If a linear differential equation has a singular point at $z = 0$ and permits the representation (3.3) this point is called a regular singular point of the equation (3.2). Otherwise, this point is an irregular singular point of this equation.
Remark. In some books (see e.g. [20, 4]) regular singularities are defined in a different way. Namely, a singular point $z = a$ is called regular if there are no solutions growing faster than some negative degree of $(z - a)$ as $z \to a$. What we call here following Wasow's book [68] as a "regular singularity" is called sometimes a "singularities of the first kind" (or "fuchsian singularity").

The notion of regular singularity can be generalised for a ordinary matrix differential equation of any order. Indeed, let us consider the equation

$$X^{(n)} + A_1(z)X^{(n-1)} + A_2(z)X^{(n-2)} + \ldots + A_n(z)X = 0$$

(3.4)

where $X(z), A_j(z)$ are $d \times d$ matrices. Introduce

$$Y_j = z^{j-1}X^{(j-1)}, \quad j = 1, 2, \ldots, n.$$ 

Then

$$Y'_j = (j - 1)z^{j-2}X^{(j-1)} + z^{j-1}X^{(j)}$$

and

$$zY'_j = (j - 1)z^{j-1}X^{(j-1)} + z^jX^{(j)}.$$ 

Therefore,

$$zY'_j = (j - 1)Y_{j-1} + Y_{j+1}$$

(3.5)

for $j = 1, 2, \ldots, n - 1$ and

$$zY'_n = (n - 1)Y_n + z^nX^{(n)} =$$

(3.6)

$$= (n - 1)Y_n - zA_1(z)Y_n - z^2A_2(z)Y_{n-1} - \ldots - z^nA_n(z)Y_1$$

So, equations (3.5), (3.6) constitutes the system of the form (3.3) with matrix $A(z)$

$$\begin{pmatrix}
0 & I & 0 & \ldots & 0 \\
\vdots & I & I & \ldots & 0 \\
\vdots & \ddots & 2I & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 0 & (n-2)I \\
-z^nA_n(z) & \ldots & \ldots & -z^2A_2(z) & (n-1)I - zA_1(z)
\end{pmatrix}$$

where $I$ is the identity operator and $A(z)$ is $nd \times nd$ matrix. Equation (3.4) has a regular singular point at $z = 0$ if and only if matrix functions $z^jA_j(z), j = 1, \ldots, n$ are holomorphic at $z = 0$, that is, if $A_j(z)$ has a pole of order at most $j$ at $z = 0$. In
particular, Schrödinger equation (3.1) has a regular singularity at \( z = 0 \) if and only if the potential \( U(z) \) has a pole of at most the second order at the origin.

Finally, we would like to mention the following fact which justifies the use of formal series (see below)

**Theorem 3.2 ([68]).** Let \( F(z) = \sum_{r=0}^{\infty} F_r z^r \) be a matrix function which is holomorphic if \( |z| < z_0 \) and

\[
\sum_{r=0}^{\infty} a_r z^r
\]

be a formal vector series satisfying formally to the differential equation

\[
zy' = \left( \sum_{r=0}^{\infty} F_r z^r \right) y.
\]

Then,

\[
y(z) = \sum_{r=0}^{\infty} a_r z^r
\]

converges when \( |z| < z_0 \) and \( y(z) \) is a solution of (3.7).

### 3.2 Local criteria of trivial monodromy.

Let us consider matrix Schrödinger equation

\[
-\frac{d^2 \psi}{dz^2} + U(z) \psi = E \psi
\]

where \( U(z) \) is a meromorphic \( d \times d \) matrix-valued function of \( Z \in \mathbb{C} \).

**Definition.** We say that Schrödinger operator \( L \) has local trivial monodromy around \( z = z_0 \) if any solution of (3.8) is single-valued near \( z = z_0 \) for any \( E \). If any solution of (3.8) is single-valued around any singular point of \( U(z) \) for any \( E \) we say that \( L \) has trivial monodromy in \( \mathbb{C} \) (or simply trivial monodromy).

Let \( z = z_0 \) be a regular singularity of (3.8). Without loss of generality we can assume that \( z_0 = 0 \). Then

\[
U(z) = \frac{C_{-2}}{z^2} + \frac{C_{-1}}{z} + C_0 + C_1 z + \ldots = \sum_{r \geq -2} C_r z^r
\]

where \( C_i \) are constant \( d \times d \) matrices. Let us assume that there is a meromorphic solution of (3.8)

\[
\psi = z^\alpha (v_0 + zv_1 + \ldots + z^k v_k + \ldots)
\]
where \( v_i = v_i(E) \in \mathbb{C}^d \) and \( \alpha \in \mathbb{Z} \). So, we can change the eigenfunction and the potential in (3.8) by their series expansions. We have

\[
-(z^\alpha v_0 + z^{\alpha+1}v_1 + \ldots + z^{\alpha+k}v_k + \ldots)'' + \\
+(C_{-2}z^{-2} + C_{-1}z^{-1} + \ldots)(z^\alpha v_0 + z^{\alpha+1}v_1 + \ldots + z^{\alpha+k}v_k + \ldots) =
\]

\[
=Ez^\alpha(v_0 + zv_1 + \ldots + z^kv_k + \ldots),
\]

or,

\[
-z^{\alpha-2}(\alpha-1)v_0 - z^{\alpha-1}\alpha(\alpha+1)v_1 - z^\alpha(\alpha+1)(\alpha+2)v_2 - \ldots
\]

\[
+z^{\alpha-2}C_{-2}v_0 + z^{\alpha-1}(C_{-1}v_0 + C_{-2}v_1) + \\
z^\alpha(C_{-2}v_2 + C_{-1}v_1 + C_0v_0) + \ldots = E z^\alpha v_0 + \ldots
\]

Comparing the coefficients with the same degrees of \( z \) we obtain

\[
C_{-2}v_0 = (\alpha - 1)\alpha v_0,
\]

\[
C_{-2}v_1 + C_{-1}v_0 = \alpha(\alpha + 1)v_1,
\]

\[
C_{-2}v_2 + C_{-1}v_1 + C_0v_0 = (\alpha + 1)(\alpha + 2)v_2 + Ev_0,
\]

\[
\ldots
\]

Therefore, \( v_0 \) is an eigenvector of matrix \( C_{-2} \) with eigenvalue \( \alpha(\alpha - 1) \).

**Remark.** It is easy to see that for a regular singularity \( z = z_0 \) condition of local trivial monodromy is equivalent to the fact that (3.8) has meromorphic basis of solutions at the vicinity of \( z_0 \) for any \( E \).

**Proposition 3.1.** If the matrix Schrödinger equation (3.8) has a complete basis of solutions which are meromorphic near \( z = 0 \) for some \( E \in \mathbb{C} \) then \( C_{-2} \) is diagonalizable with the eigenvalues \( \lambda_i = m_i(m_i - 1) \), \( m_i \in \mathbb{Z}_+ \).

Let us postpone the proof to the next section.

Let us now assume that \( L \) has local trivial monodromy near \( z = 0 \). Then, according to proposition 3.1, \( V = \mathbb{C}^d \) can be represented as a direct sum of the eigenspaces of \( C_{-2} \):

\[
V = \bigoplus_{m=1}^{M} V_m, \quad \dim V_m = d_m, \quad \sum_{m=1}^{M} d_m = d
\]

where the eigenspace \( V_m \) corresponds to the eigenvalue \( \lambda = m(m-1) \) (some of these spaces can be of dimension \( d_m = 0 \)). Any operator in \( V \)

\[
A : \bigoplus_{i=1}^{M} V_i \to \bigoplus_{i=1}^{M} V_i
\]
can be represented in a block form

\[ A = (A^{ij}), \quad i, j = 1, \ldots, M \]

where \( A^{ij} \) are some operators

\[ A^{ij} : V_i \rightarrow V_j. \]

This corresponds to the representation of the matrix \( A \) of such an operator in a suitable basis in the block form

\[ A = (A^{ij}), \quad 1 \leq i, j \leq M \] (3.10)

where \( A^{ij} \) are \( d_i \times d_j \) matrices and any vector \( \psi \) in \( V \) can be uniquely represented as a sum

\[ \psi = \sum_{i=1}^{M} \psi^i, \quad \psi^i \in V_i. \] (3.11)

Now we want to formulate the main result of this section

**Theorem 3.3.** *(local criteria of trivial monodromy).* A matrix Schrödinger operator \( L \) with a meromorphic potential \( U(z) \) given by (3.9) has trivial monodromy if and only if the following entries (with respect to representation (3.10)) of the corresponding matrix coefficients in the expansions of \( U(z) \) near any of its singular points vanish:

1. \( C^{ij}_l = 0 \) if \( |i - j| \geq l + 1 \)
2. \( C^{ij}_l = 0 \) if \( i + j = l + 3, l + 5, \ldots, l + 2k + 1, \ldots \)

where \( l = -1, 0, \ldots, 2M - 3 \). In particular, the matrix residue \( C_{-1} = 0 \).

The coefficients \( \psi_0, \psi_1, \ldots, \psi_{2M-3} \) of the corresponding expansions of the vector-eigenfunctions

\[ \psi = z^{-M+1}(\psi_0 + z\psi_1 + \ldots + z^k\psi_k + \ldots) \]

satisfy the conditions

1. \( \psi^i_l = 0 \) if \( i + l < M \)
2. \( \psi^i_l = 0 \) if \( i + l = M + 1, M + 3, \ldots M + 2k + 1, \ldots \) (3.13)

and \( l - i \leq M - 3 \).

Before the proof of the theorem we would like to give more explicit description of the conditions on the block structure (3.10) of the coefficients of \( U \) and solutions \( \psi \) of (3.8).
There are no conditions on matrices $C_k$, $k > 2M - 3$. Matrix $C_{2M-3}$ has only one zero on the intersection of the last column and of the last row and matrix $C_{2M-4}$ has two zeroes on the diagonal which is parallel to the secondary one as it is shown below.

$$C_{2M-3} = \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & 0 \end{pmatrix}, \quad C_{2M-4} = \begin{pmatrix} * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & 0 & 0 \end{pmatrix}.$$

Matrix $C_{2M-5}$ has the following structure

$$C_{2M-5} = \begin{pmatrix} * & \cdots & * \\ \vdots & \vdots & \vdots \\ \vdots & \cdots & * \\ \vdots & \cdots & 0 \\ * & 0 & * \\ * & \cdots & 0 & 0 \end{pmatrix}.$$

Further zero and nonzero (about which we have no conditions) diagonals which are parallel to secondary one are alternated. Matrix $C_{M-2}$ has zero secondary diagonal:

$$C_{M-2} = \begin{pmatrix} * & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 & 0 & \vdots \\ 0 & \cdots & \ddots \\ * & \cdots & \cdots \end{pmatrix}.$$

The alternation of the zero and nonzero diagonals is preserved for $C_k$ with $k < M - 2$ but we will have some additional zeroes. Namely, consider the zero diagonal $C^i_j : i + j = c$ which is parallel to secondary one with the smallest $c$. There are two diagonals having common ends with it (at the first row and the first column) which are parallel to the main diagonal. It turns out that their elements are equal to 0 as well as all elements of the matrix $C_k$ in the upper right and lower left parts.
intercepted by them. So, the generic form of the matrix $C_k$, $k < M - 2$ is

$$
\begin{pmatrix}
* & \ldots & * & 0 & \ldots & 0 & 0 \\
* & \ldots & * & \ldots & \ldots & \ldots & \ldots \\
* & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & * & \ldots & * & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & * \\
0 & 0 & \ldots & 0 & 0 & 0 & * 
\end{pmatrix}
$$

In particular,

$$
C_1 = \begin{pmatrix}
* & * & 0 & \ldots & 0 & 0 \\
* & 0 & * & \ldots & \ldots & 0 \\
0 & * & 0 & \ldots & \ldots & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & * & 0 \\
0 & 0 & \ldots & 0 & 0 & * 
\end{pmatrix},
$$

$C_0$ is diagonal and $C_{-1} = 0$.

There are also some conditions on coefficients $\psi_0, \psi_1, \ldots, \psi_{2M-3}$. Namely, all entries of $\psi_0$ except the last one and of $\psi_1$ except the $(M - 1)$-th one are zeros and

$$
\psi_0 = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
* 
\end{pmatrix}, \quad \psi_1 = \begin{pmatrix}
0 \\
\vdots \\
0 \\
* \\
0 
\end{pmatrix}, \quad \psi_2 = \begin{pmatrix}
0 \\
\vdots \\
0 \\
* \\
0
\end{pmatrix}.
$$

In the next coefficients zero and nonzero entries are alternated while the highest nonzero one moves upward. It will be so till we obtain nonzero at the first entry for
Further, 

\[
\psi_{M} = \begin{pmatrix}
* \\
0 \\
* \\
0 \\
0 \\
\vdots
\end{pmatrix}
\]

and \(\psi_{M+k} \ (k = 1, 2, \ldots, M - 3)\) has zeros at the \(k + 3, k + 5, \ldots, (M - 1)\)-th or \(M\)-th entries which depends on if \(M + k\) is even or odd.

### 3.3 Proof of the local criteria.

#### 3.3.1 Resonance equations.

Let \(\lambda = n(n-1)\) be an eigenvalue of \(C_{-2}\) and \(e_n \in \mathbb{C}^d\) be a corresponding eigenvector of \(C_{-2}\)

\[
C_{-2}e_n = n(n-1)e_n.
\]

Then we can construct two formal series \(\varphi^{-n+1}\) and \(\varphi^n\) of the forms

\[
\varphi^{-n+1} = z^{-n+1}(\varphi_0^{-n+1} + \varphi_1^{-n+1}z + \ldots) = z^{-n+1}\sum_{k=0}^{\infty} \varphi_k^{-n+1}z^k
\]

and

\[
\varphi^n = z^n(\varphi_0^n + \varphi_1^n z + \varphi_2^n z^2 + \ldots) = z^n\sum_{k=0}^{\infty} \varphi_k^n z^k
\]

where \(\varphi_0^{-n+1} = \varphi_0^n = e_n\). Substituting expansions (3.9) and (3.14) to the equation (3.8) and comparing the coefficients for the same degrees of \(z\) we have the following infinite number of equations

\[
C_{-2}\varphi_0^{-n+1} = (-n + 1)(-n)\varphi_0^{-n+1},
\]

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\[
\begin{align*}
C_{-2}\varphi_{1}^{-n+1} + C_{-1}\varphi_{0}^{-n+1} &= (-n+2)(-n+1)\varphi_{1}^{-n+1}, \\
C_{-2}\varphi_{2}^{-n+1} + C_{-1}\varphi_{1}^{-n+1} + C_{0}\varphi_{0}^{-n+1} &= \\
&= (-n+3)(-n+2)\varphi_{2}^{-n+1} + E\varphi_{0}^{-n+1}, \\
& \vdots \\
C_{-2}\varphi_{M+n-1}^{-n+1} + \sum_{l=1}^{M+n-1} C_{l-2}\varphi_{M+n-1-l}^{-n+1} &= \\
&= M(M-1)\varphi_{M+n-1}^{-n+1} + E\varphi_{M+n-3}^{-n+1} \\
& \vdots
\end{align*}
\]

It is easy to see that the coefficients \(\varphi_{k}^{-n+1}\) of the expansion (3.14) can be reconstructed uniquely when \(k > M+n-1\) because matrices \(C_{-2} - k(k-1)\) are invertible if \(k > M\). So, we have the system of \(M+n\) equations on coefficients \(\varphi_{k}^{-n+1}\) of series expansion for functions \(\varphi^{-n+1}\)

\[
C_{-2}e_{n} = n(n-1)e_{n},
\]

\[
(C_{-2} - (-n+k)(-n+k+1)I)\varphi_{k}^{-n+1} + \sum_{l=1}^{k} C_{l-2}\varphi_{k-l}^{-n+1} = E\varphi_{k}^{-n+1}, \quad k = 1, \ldots, M+n-1.
\]

where \(\varphi_{k}^{-n+1} = 0\). Their solvability imposes some conditions on coefficients \(C_{k}\).

Analogously, we obtain \((M-n+1)\) equations using expansion (3.15) for \(\varphi^{n}\):

\[
(C_{-2} - (n+k-1)(n+k)I)\varphi_{k}^{n} + \sum_{l=1}^{k} C_{l-2}\varphi_{k-l}^{n} = E\varphi_{k}^{n}, \quad 1 \leq k \leq M-n.
\]

Equations for which \(C_{-2} - (-n+k)(-n+k+1)I\) or \(C_{-2} - (n+k-1)(n+k)I\) are not invertible and the corresponding coefficients \(\varphi_{k}^{-n+1}, \varphi_{k}^{n}\) cannot be reconstructed uniquely are called resonance equations.

**Lemma 3.1.** The number of the linear independent meromorphic solutions of the equation (3.8) is less or equal to the double number of linear independent eigenvectors of \(C_{-2}\).

**Proof.** Let us assume that spectrum of \(C_{-2}\) consists of eigenvalues \(m_{1}(m_{1} - 1), \ldots, m_{n}(m_{n} - 1)\) where \(0 \leq m_{1} < m_{2} < \ldots < m_{n} = M\), \(\dim V_{m_{k}} \geq 1\) and rearrange formal series for \(\varphi^{m_{k}}, \varphi^{-m_{k}+1}, k = 1, \ldots, n\) according to the number of the equations entering the corresponding systems (3.16), (3.17): \(\psi^{1} = \varphi^{m_{n}} = \varphi^{M}, \psi^{2} = \varphi^{m_{n-1}}, \ldots, \psi^{n} = \varphi^{m_{1}}, \psi^{n+1} = \varphi^{-m_{1}+1}, \ldots, \psi^{2n} = \varphi^{-M+1}\). For convenience...
let us introduce also linear spaces \( W_1 = V_{m_1}, W_2 = V_{m_1-1}, \ldots, W_n = V_{m_1}, W_{n+1} = W_n, \ldots, W_{2n-1} = W_2, W_{2n} = W_1 \). To prove the lemma let us consider conditions on \( \psi^1, \psi^2, \ldots, \psi^k \) together as single system of equations \( M_k \). It is enough to show that the rank of \( M_k \) is less or equal to \( \sum_{r=1}^{k} \dim W_r \). If so, for \( k = 2n \) we will have that the number of linear independent solution of the equation (3.8) equals

\[
\sum_{r=1}^{2n} \dim W_r = 2 \sum_{r=1}^{n} \dim V_{m_r},
\]

that is, the double number of linear independent eigenvectors of \( C_{-2} \). For \( k = 1 \) we have only one equation for \( \psi^1 \):

\[
(C_{-2} - M(M - 1))\psi^1_0 = 0
\]

the rank of which is, obviously, equal to \( \dim W_1 \). Let us assume that our statement is true for system \( M_k \) and prove it for \( M_{k+1} \). Without loss of generality we can suppose that \( k + 1 \leq n \), that is, \( \psi^{k+1} = \varphi^{m_n-k} \). \( M_{k+1} \) differs from \( M_k \) by the new series of equations for \( \psi^{k+1} \) which is

\[
(C_{-2} - m_{n-k}(m_{n-k} - 1)I)\psi^{k+1}_0 = 0 \\
(C_{-2} - (m_{n-k} + 1)m_{n-k}I)\psi^{k+1}_1 + C_{-1}\psi^{k+1}_0 = 0 \\
(C_{-2} - (m_{n-k} + 2)(m_{n-k} + 1)I)\psi^{k+1}_2 + C_{-1}\psi^{k+1}_1 + C_0\psi^{k+1}_0 = E\psi^{k+1}_0 \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
\]

If matrix \( C_{-2} - (m_{n-k}+r)(m_{n-k}+r-1)I, \) \( r = 1, 2, \ldots \) is invertible, the corresponding coefficient \( \psi^{k+1}_r \) can be reconstructed uniquely. Let us suppose that it is not invertible. That means that \( m_{n-k} + r = m_s \) for some \( s > n - k \) and if the corresponding equation is solvable we can not reconstruct the projection of \( \psi^{k+1}_r \) onto the eigenspace \( V_{m_s} \). Evidently, we can decompose \( \psi^{k+1}_r \) into the sum \( \psi^{k+1}_r = \xi + \eta \) where \( \xi \in V_{m_s} \) and \( \eta \in \hat{V}_{m_s} \) where \( \hat{V}_{m_s} \) is a complement to \( V_{m_s} : \hat{V}_{m_s} \oplus V_{m_s} = V \). Then \( \eta \) can be reconstructed uniquely. On the other hand, there is the solution \( \varphi^{m_s} \) of (3.8) such that \( \varphi^{m_s}_0 \in V_{m_s} \) and, considering \( \psi^{k+1} - c\varphi^{m_s} \) where \( c \) is a constant we can always assume that \( \xi = 0 \). So, the difference between rank of system \( M_{k+1} \) and that of \( M_k \) is equal to number of linear independent solutions of the equation of (3.18) which is evidently \( \dim V_{m_{n-k}} \). Lemma follows.

**Corollary 3.1.** If \( L \) has trivial monodromy matrix then \( C_{-2} \) is diagonalisable.

In what follows for convenience we assume that the spectrum of \( C_{-2} \) is simple and \( C_{-2} = \text{diag}(0, 2, \ldots, M(M - 1)) \). The general case can be considered similarly.
We will prove that the conditions of the theorem 3.3 are necessary and sufficient for solvability of equations (3.16), (3.17), that is, for finding series expansions of solutions $\varphi^n, n = -M + 1, \ldots, M$ of (3.8). It is easy to check that we can obtain some conditions on matrices $C_k$ if $k = -1, 0, \ldots, 2M - 3$ because only these matrices enter the resonances.

To write them in more convenient and homogenous form we introduce the following notation. Let $s(m, n) = m(m - 1) - n(n - 1), m, n \in \mathbb{Z}$ and assume that eigenvectors $e_n$ can have negative indices. Namely, by definition $e_n = e_{-n+1}$ for $n = 0, -1, -2, \ldots, -M + 1$. Introduce covectors $f_m \in V^*: \langle f_m | e_n \rangle = 1$ if $m = n$ or $m = -n + 1$ and 0 otherwise, $m, n \in \mathbb{Z}$. So, operator $C_{-2}$ can be written in the form

$$C_{-2} = \sum_{n=1}^{M} n(n - 1) |e_n \rangle \langle f_n|$$

and

$$C_{l}^{mn} = \langle f_m | C_l | e_n \rangle$$

is the $mn$-th element of $C_l$. We obtain the following form of the resonance equations

$$\sum_{p} s(p, n + k) |e_p \rangle \langle f_p | \varphi^n_k \rangle + \sum_{l=1}^{k} C_{l-2} \varphi^n_{k-l} = E \varphi^n_{l-2}$$

(3.19)

where $n = -M+1, -M+2, \ldots, M$ is the number of the resonance. $k = 0, 1, \ldots, M - n$ is the number of the equation in a resonance and $p$ is supposed to change from $-M + 1$ to 0 if $n + k \leq 0$ and from 1 to $M$ otherwise, $\varphi^n_{-1} = \varphi^n_{-2} = 0$. For example, if $n = -M + 1$ we get the resonance consisting of $2M$ equations. If $n = M$ the corresponding resonance consists only of the definition of $e_M$.

The conditions on matrices $C_l$ and coefficients $\varphi^n_s$ of the theorem can now be written as

$$\langle f_{n+l+2k} | C_l | e_n \rangle = 0, \ -M + 1 \leq n \leq M - l - 2.$$  

(3.20)

$$\langle f_{r+l+2k} | \varphi^n_s \rangle = 0, k = 1, 2, \ldots$$  

(3.21)

or, equivalently to the (3.21),

$$\varphi^n_s \in \langle e_{r+s}, e_{r+s-2}, \ldots \rangle$$

Notice that in the expansion

$$\varphi^n = z^n (\varphi^n_0 + \varphi^n_1 z + \varphi^n_2 z^2 + \ldots) = z^n \sum_{k=0}^{\infty} \varphi^n_k z^k$$

$\varphi^n_k$ is the coefficient for $z^{n+k}$ for any integer $n$: $-M + 1 \leq n \leq M$.  

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3.3.2 Numbers $\Omega_k(m,n,l)$.

Numbers $\Omega_k(m,n,l)$ are determined by the recursive formula

$$
\Omega_k(m,n,l) = \frac{1}{s(n,n+2)\ldots s(n,n+2k)} + \frac{1}{s(m,n+l+2k)}\Omega_{k-1}(m,n,l)
$$

with $\Omega_0(m,n,l) \equiv 1$. In particular,

$$
\Omega_1(m,n,l) = \frac{1}{s(n,n+2) + \frac{1}{s(m,n+l+2)}},
$$

$$
\Omega_2(m,n,l) = \frac{1}{s(n,n+2)s(n,n+4)} + \frac{1}{s(m,n+l+4)} \left( \frac{1}{s(n,n+2)} + \frac{1}{s(m,n+l+2)} \right).
$$

We can write the general formula for $\Omega_k$ as

$$
\Omega_k(m,n,l) = \sum_{r=0}^{k} \frac{1}{s(n,n+2)\ldots s(n,n+2r)} \frac{1}{s(m,n+l+2(r+1))\ldots s(m,n+l+2k)}.
$$

We will get the conditions on matrices $C_l$, $l = -1, 0, 1, \ldots, 2M - 3$ in the form $\Omega_k(n+l+2(k+1), n, l)\langle f_{n+l+2(k+1)} | C_l | e_n \rangle = 0$, $k = 1, 2, \ldots$. So, to obtain that $\langle f_{n+l+2(k+1)} | C_l | e_n \rangle = 0$ we have to prove that $\Omega_k(n+l+2(k+1), n, l) \neq 0$.

To do this let us notice that if we reduce $\Omega_k(m,n,l)$ to a common denominator then we have in the numerator

$$
\omega_k(m,n,l) = \sum_{r=0}^{k} \frac{s(n,n+2(r+1))\ldots s(n,n+2k)}{s(m,n+l+2(r+1))\ldots s(m,n+l+2k)} \cdot \frac{1}{s(m,n+l+2r)\ldots s(m,n+l+2)}.
$$

Expressions for $\omega_k(n+l+2(k+1), n, l)$ can be simplified as it follows from Proposition 3.2.

$$
\omega_k(n+l+2(k+1), n, l) = 2^{2k}k!(l+2)(l+3)\ldots(l+k+1)
$$

so that $\omega_k(n+l+2(k+1), n, l) \neq 0$ for $l \geq -1$. 

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Proof. Indeed, by definition,

$$\omega_k(n + l + 2(k + 1), n, l) = \sum_{r=0}^{k} \frac{s(n, n + 2(r + 1)) \ldots s(n, n + 2k)}{k-r} \cdot \underbrace{s(n + l + 2(k + 1), n + l + 2r) \ldots s(n + l + 2(k + 1), n + l + 2)}_{r}$$

(3.22)

Using \(s(m, n) = (m - n)(m + n - 1)\) we can calculate

\[
s(n, n + 2(r + 1)) = -2^2(r + 1)(n + r + 1 - 1/2),
\]

\[
s(n, n + 2(r + 2)) = -2^2(r + 2)(n + r + 2 - 1/2),
\]

\[
\ldots \quad \ldots \quad \ldots ,
\]

\[
s(n, n + 2k) = -2^2k(n + k - 1/2),
\]

\[
s(n + l + 2(k + 1), n + l + 2r) = 2^2(k - r + 1)(n + l + (k + r + 1) - 1/2),
\]

\[
s(n + r + 2(k + 1), n + l + 2(r - 1)) = 2^2(k - r + 2)(n + l + (k + r) - 1/2),
\]

\[
\ldots \quad \ldots \quad \ldots ,
\]

\[
s(n + l + 2(k + 1), n + l + 2) = 2^2k(n + l + (k + 2) - 1/2).
\]

So, the \(r\)-th summand of (3.22) can be written as \((x = n - 1/2)\)

\[(-1)^{k-r}2^2k!C_k^r(x + r + 1)(x + r + 2) \ldots (x + k).\]

\[\cdot (x + l + k + 2) \ldots (x + l + k + r)(x + l + k + r + 1)\]

and

\[\omega_k(n + l + 2(k + 1), n, l) = 2^2k!(-1)^k[(x + 1)(x + 2) \ldots (x + k) -
\]

\[-C_k^1(x + 2) \ldots (x + k)(x + k + l + 2) +
\]

\[+C_k^2(x + 3) \ldots (x + k)(x + k + l + 2)(x + k + l + 3) + \ldots
\]

\[+(-1)^{k-1}C_k^{k-1}(x + k)(x + k + l + 2)(x + k + l + 3) \ldots (x + k + l + k) +
\]

\[+(-1)^k(x + k + l + 2) \ldots (x + l + k + 1)].\]

Let \(G(k, l)\) denote the expression in square brackets. We want to prove that \(G(k, l) = (-1)^{k-1}(l + 2)(l + 3) \ldots (l + k + 1)\). As \(G(k, l)\) is a polynomial of the degree \(k\) in \(l\) it is enough to show that \(G(k, -1) = (-1)^{k}k!\) and \(G(k, -2) = G(k, -3) = \ldots = G(k, -k - 1) = 0.\)

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Lemma 3.2. $G(k, -1) = (-1)^kk!$.

Proof. Indeed,

$$G(k, -1) = (x + 1)(x + 2)\ldots(x + k) - C_k^1(x + 2)\ldots(x + k)(x + k + 1) +$$

$$+ C_k^2(x + 3)\ldots(x + k)(x + k + 1)(x + k + 2) + \ldots$$

$$+ (-1)^{k-1}C_k^{k-1}(x + k)(x + k + 1)(x + k + 2)\ldots(x + 2k - 1) +$$

$$+ (-1)^{k}(x + k + 1)\ldots(x + 2k).$$

Let us introduce the shift-operator $T$ such that for any polynomial $f(x)$

$$(Tf)(x) = f(x + 1).$$

Then $1 - T$ reduce a degree of $f(x)$ and

$$(1 - T)^m = \sum_{l=0}^{m}(-1)^lC_m^lT^l. \quad \text{(3.23)}$$

If

$$P_k(x) = x^k + \ldots$$

is any polynomial of the degree $k$ then, evidently,

$$\quad (1 - T)^k(P_k(x)) = (-1)^kk!,$$

$$\quad (1 - T)^l(P_k(x)) = 0, \quad l > k.$$

Using (3.23) we obtain

$$G(k, -1) = (1 - T)^k[(x + 1)(x + 2)\ldots(x + k)] = (-1)^kk!.$$

Lemma proved.

For $l = -2$

$$G(k, -2) = (x + 1)(x + 2)\ldots(x + k) - C_k^1(x + 2)\ldots(x + k)(x + k) +$$

$$+ C_k^2(x + 3)\ldots(x + k)(x + k)(x + k + 1) + \ldots$$

$$+ (-1)^{k-1}C_k^{k-1}(x + k)(x + k + 1)\ldots(x + 2k - 2) +$$

$$+ (-1)^{k}(x + k)\ldots(x + 2k - 1) =$$

$$= (x + k)[(x + 1)(x + 2)\ldots(x + k - 1) - C_k^1(x + 2)(x + 3)\ldots(x + k) + \ldots$$

\footnote{The idea to use shift operators $T$ has been suggested by Prof. A.P.Veselov.}
$$+(-1)^{k-1}C_k^{k-1}(x+k)(x+k+1)\ldots(x+2k-2)+$$
$$+(-1)^k(x+k+1)\ldots(x+2k-1) = (x+k)(1-T)^k[(x+1)(x+2)\ldots(x+k-1)] = 0$$
as operator \((1-T)^k\) reduce the degree of any polynomial at least by \(k\) but the degree of \((x+1)(x+2)\ldots(x+k-1)\) equals \(k-1\). Analogously, for any \(r\), 
\(-2 \geq r \geq -k-1\)

$$G(k,-r) = (x+1)(x+2)\ldots(x+k)-$$

$$-C_k^1(x+2)\ldots(x+k-1)(x+k)(x+k-r+2)+$$
$$+C_k^2(x+3)\ldots(x+k)(x+k-r+2)(x+k-r+3)+\ldots$$
$$+(-1)^{k-1}C_k^{k-1}(x+k)(x+k-r+2)(x+k-r+3)\ldots(x+2k-r)+$$
$$+(-1)^k(x+k-r+2)\ldots(x+2k-r+1)) =$$
$$= (x+k-r+2)\ldots(x+k)[(x+1)(x+2)\ldots(x+k-r+1)-$$
$$-C_k^1(x+2)(x+3)\ldots(x+k-r+2)+\ldots$$
$$+(-1)^{k-1}C_k^{k-1}(x+k)(x+k-r+2)\ldots(x+2k-r)+$$
$$+(-1)^k(x+k-r+2)\ldots(x+2k-r+1)] =$$
$$= (x+k-r+2)\ldots(x+k)(1-T)^k[(x+1)(x+2)\ldots(x+k-r+1)] = 0.$$

We see that \(G(k,l)\) is a polynomial in \(l\) of degree \(k\) the roots of which are \(-2, -3, \ldots, k+1\). So, \(G(k,l) = (-1)^k(l+2)(l+3)\ldots(l+k+1)\). Proposition follows and we obtain that \(\Omega_k(n+l+2(k+1), n,l) \neq 0\) for \(k = 1, 2, \ldots\).

### 3.3.3 Truncated polynomials and the proof that matrix residue

\(C_{-1} = 0\).

Let us consider the \(n\)-th \((n = -M+1, \ldots, M)\) resonance:

$$\sum_p s(p,n)|e_p\rangle\langle f_p|\varphi_0^n\rangle = 0,$$  \((n.0)\)

$$\sum_p s(p,n+1)|e_p\rangle\langle f_p|\varphi_1^n\rangle + C_{-1}\varphi_0^n = 0,$$  \((n.1)\)

$$\sum_p s(p,n+2)|e_p\rangle\langle f_p|\varphi_2^n\rangle + C_{-1}\varphi_1^n + C_0\varphi_0^n = E\varphi_0^n,$$  \((n.2)\)

\[\ldots \ldots \ldots \ldots\]
\[
\sum_p s(p, M - 1)|e_p\langle f_p|\varphi^n_{M-n-1}\rangle + \\
+ \sum_{l=1}^{M-n-1} C_{l-2}\varphi^n_{M-n-1-l} = E\varphi^n_{M-n-3}, \\
(n.M - n - 1)
\]
\[
\sum_p s(p, M)|e_p\langle f_p|\varphi^n_{M-n}\rangle + \sum_{l=1}^{M-n} C_{l-2}\varphi^n_{M-n-l} = E\varphi^n_{M-n-2}. \\
(n.M - n)
\]

It is easy to see that \(\varphi^n_{2k}\) and \(\varphi^n_{2k+1}\) are vector polynomials in \(E\) of degree \(k\). It is more convenient for us to consider truncated polynomials \(\varphi^n_{2k}[k]\) and \(\varphi^n_{2k+1}[k]\), that is, parts of \(\varphi^n_{2k}\) and \(\varphi^n_{2k+1}\) without terms of degree \((k - 1)\) and less.

**Remark.** From the equation
\[
\sum_p s(p, n + k)|e_p\langle f_p|\varphi^n_k\rangle + \sum_{l=1}^k C_{l-2}\varphi^n_{k-l} = E\varphi^n_{k-2}
\]

it follows that, generally speaking, \(\langle f_{n+k}|\varphi^n_k\rangle\) can be of any value. On the other hand, there exists the eigenfunction
\[
\varphi^{n+k} = z^{n+k}(e_{n+k} + \ldots)
\]

and we will always specify \(\varphi^n_k\) subtracting \(\langle f_{n+k}|\varphi^n_k\rangle e_{n+k}\) so that \(\langle f_{n+k}|\varphi^n_k\rangle = 0\).

**Proposition 3.3.**
\[
\varphi^n_{2k}[k] = E^k \frac{|e_n\rangle}{s(n, n + 2) \ldots s(n, n + 2k)},
\]
\[
\varphi^n_{2k+1}[k] = -E^k \sum_p \frac{(f_p|C_{-1}|e_n\rangle}{s(p, n + 2k + 1)} \Omega_k(p, n, -1|e_p\rangle
\]

where \(k = 0, 1, \ldots\) and the sum is supposed to be taken over all \(p \neq n + 1, n + 3, \ldots, n + 2k + 1\) from \(-M + 1\) to \(0\) or from \(1\) to \(M\), \(p\) is identified with \((-p + 1)\) so that \(p \neq n + 2k + 1\) means that \(p \neq n + 2k + 1\) or \(p \neq -n - 2k\) which depend on the range of summation.

**Proof.** If \(k = 0\), obviously, \(\varphi^n_0 = |e_n\rangle\) and from (n.1) we obtain using the remark above
\[
\varphi^n_1 = -\sum_{p, p \neq n+1} \frac{(f_p|C_{-1}|e_n\rangle}{s(p, n + 1)} |e_p\rangle,
\]

Let us assume that proposition is true for \(\varphi^n_{2l}, \varphi^n_{2l+1}\) where \(l = 1, \ldots, k-1\) and prove it for \(l = k\). In the equation (n.2k)
\[
\sum_p s(p, 2k)|e_p\langle f_p|\varphi^n_{2k}\rangle + \sum_{l=1}^{2k} C_{l-2}\varphi^n_{2k-l} = E\varphi^n_{2k-2}
\]
the sum
\[ \sum_{l=1}^{2k} C_{l-2} \varphi_{2k-l}^n \]
is a polynomial of degree \( k - 1 \) in \( E \) according to induction assumption. So,
\[ \sum_p s(p, 2k) |e_p\rangle \langle f_p | \varphi_{2k}^n \rangle = E^k \frac{|e_n\rangle}{s(n, n+2) \ldots s(n, n+2k-2)} \]
and
\[ \varphi_{2k}^n \rangle = E^k \frac{|e_n\rangle}{s(n, n+2) \ldots s(n, n+2k)} . \]

For \( \varphi_{2k+1}^n \) we have the equation
\[ \sum_p s(p, 2k+1) |e_p\rangle \langle f_p | \varphi_{2k+1}^n \rangle + \sum_{l=1}^{2k+1} C_{l-2} \varphi_{2k+1-l}^n = E \varphi_{2k-1}^n, \]
or, removing all terms of degree \( k - 1 \) and lower in \( E \)
\[ \sum_p s(p, 2k+1) |e_p\rangle \langle f_p | \varphi_{2k+1}^n \rangle + C_{-1} \varphi_{2k}^n \rangle = E \varphi_{2k-1}^n \] .

Substituting the expressions for \( \varphi_{2k-1}^n \) and \( \varphi_{2k}^n \) we get
\[ \sum_p s(p, n+2k+1) |e_p\rangle \langle f_p | \varphi_{2k+1}^n \rangle + \]
\[ + E^k \frac{C_{-1}|e_n\rangle}{s(n, n+2) \ldots s(n, n+2k)} = \]
\[ = - E^k \sum_p \frac{\langle f_p | C_{-1} |e_n\rangle}{s(p, n+2k-1)} \Omega_{k-1}(p, n, -1) |e_p\rangle \]
and
\[ \varphi_{2k+1}^n \rangle = - E^k \sum_p \frac{\langle f_p | C_{-1} |e_n\rangle}{s(p, n+2k+1)} . \]

Proposition 3.3 follows.

**Lemma 3.3.**
\[ \langle f_{n+2k+1} | C_{-1} |e_n\rangle = 0, \text{ } k = 0, 1, \ldots ; n = -M + 1, \ldots , M. \]
so that matrix residue \( C_{-1} = 0. \)
Proof. If \( k = 0 \) lemma follows immediately from (n.1). To prove it for \( k \geq 1 \) let us multiply (3.24) by \( \langle f_{n+2k+1} \rangle \). We obtain
\[
\Omega_k(n + 2k + 1, n, -1)\langle f_{n+2k+1} | C_{-1} | e_n \rangle = 0,
\]
and as \( \Omega_k(n + 2k + 1, n, -1) \neq 0 \) we get the proof of the lemma.

3.3.4 End of the proof.

Let us now assume that the conditions of the theorem are fulfilled for \( C_{-1}, C_0, \ldots, C_{l-1} \) and prove it for \( C_l \). We have to show that
\[
\langle f_{n+l+2k} | C_l | e_n \rangle = 0, \quad -M + 1 \leq n \leq M - l - 2, \quad k = 1, 2, \ldots
\]

It is easy to check that \( C_l \) is included to the \( n \)-th resonance where \( n = -M + 1, \ldots, M - l - 2 \). If we fix \( n \) then \( C_l \) appears in the equation (n.1+2)
\[
\sum_p s(p, n + l + 2)e_p \langle f_p | \varphi_{l+2}^n \rangle + \sum_{q=0}^{l-1} C_q \varphi_{l-q}^n + C_l \varphi_0^n = E \varphi_l^n
\]
and the \((l + 2 + m)\)-th equation is of the form \((m \in \mathbb{N})\)
\[
\sum_p s(p, n + l + 2 + m)e_p \langle f_p | \varphi_{l+2+m}^n \rangle + \sum_{q=0}^{l-1} C_q \varphi_{l+m-q}^n + C_l \varphi_m^n = E \varphi_{l+m}^n.
\]

Let \( m = 2k, k = 1, 2, \ldots \). Then, according to the proposition 3.3 \( \sum_{q=1}^{m} C_{l+q} \varphi_{m-q}^n \) is a polynomial of degree \( k - 1 \) in \( E \) and we can write the last equations omitting all the terms of the degree less than \( k \)
\[
\sum_p s(p, n + l + 2 + 2k)e_p \langle f_p | \varphi_{l+2+2k}^n(k) \rangle + \sum_{q=0}^{l-1} C_q \varphi_{l+2k-q}^n(k) +
\]
\[
+ C_l \varphi_{2k}^n[k] = E \varphi_{l+2k}^n[k - 1], \quad k = 0, 1, \ldots
\]

where we obtain the equation (n.1+2) for \( k = 0 \) and \( \varphi_l^0[-1] = 0 \) by definition.

Proposition 3.4. If the theorem 3.3 is true for \( C_r, -1 \leq r \leq l - 1 \) then coefficients \( \varphi_m^n, m = 0, 1, \ldots, l + 1 \) satisfy conditions (3.21), that is,
\[
\varphi_m^n \in \langle e_{n+m}, e_{n+m-2}, \ldots \rangle.
\]
Proof. For \( m = 0 \) the proposition is evident because \( \varphi^n_0 = e_n \). Let us assume that it is true for \( \varphi^n_r, \ r = 0, 1, \ldots, m - 1 \) and prove it for \( \varphi^n_m \). Multiplying the equation \((n, m)\) by \( \langle f_{n+m+2k}, k = 1, 2 \ldots \rangle \) we have

\[
\begin{align*}
\text{s}(n + m + 2k, n + m) & \langle f_{n+m+2k} | \varphi^n_n \rangle + \sum_{q=0}^{m-2} \langle f_{n+m+2k} | C_q | \varphi^n_{m-2-q} \rangle = \\
& = E \langle f_{n+m+2k} | \varphi^n_{m-2} \rangle.
\end{align*}
\]

Let us prove that

\[
\langle f_{n+m+2k} | \varphi^n_{m-2} \rangle = 0
\]

and

\[
\sum_{q=0}^{m-2} \langle f_{n+m+2k} | C_q | \varphi^n_{m-2-q} \rangle = 0.
\]

These identities follow from the induction assumption. Indeed, as \( \varphi^n_{m-2} \in < e_{m-2}, e_{m-4}, \ldots > \) we obtain \( \langle f_{n+m+2k} \varphi^n_{m-2} \rangle = 0, \ k = 1, 2, \ldots \). Because \( \varphi^n_{m-2-q} \in < e_{n+m-q-2}, e_{n+m-q-4}, \ldots > \) we just have to prove that

\[
\langle f_{n+m+2k} | C_q | e_{n+m-q-2p} \rangle = 0, \ p \in \mathbb{N}. \]

Introduce \( n + m - q - 2p = r \). Then the last condition can be rewritten as \( \langle f_{q+r+2(p+k)} | C_q | e_r \rangle = 0 \) which coincide with conditions (3.20) on matrix \( C_r, r \leq l - 1 \). So, (3.26) reduces to

\[
s(n + m + 2k, n + m) \langle f_{n+m+2k} | \varphi^n_n \rangle = 0, \ k = 1, 2, \ldots, \ m = 0, 1, \ldots, l + 1
\]

and as \( s(n + m + 2k, n + m) \neq 0 \) proposition follows.

**Corollary 3.2.** If coefficients \( \varphi^n_m, \ m = 0, 1, \ldots, l + 1 \) satisfy the conditions \( \varphi^n_m \in < e_{n+m}, e_{n+m-2}, \ldots > \) then

\[
\langle f_{n+m+2k} | C_q | \varphi^n_{m-q} \rangle = 0, \ q \leq l - 1, \ k = 1, 2, \ldots
\]

**Proof** follows easily from the conditions (3.20) by substitution \( e_{n+m}, e_{n+m-2}, \ldots \) instead of \( \varphi^n_m \) at the same way as in proposition 3.4.

Let us multiply the equation (3.25) by \( \langle f_{n+l+2+2k} | \). We obtain

\[
\sum_{q=0}^{l-1} \langle f_{n+l+2+2k} | C_q | \varphi^n_{l+2k-q} \rangle + \langle f_{n+l+2+2k} | C_l | \varphi^n_{2k} \rangle = \\
= E \langle f_{n+l+2+2k} | \varphi^n_{l+2k} \rangle, \ k = 1, 2, \ldots
\]

Let us prove now that

\[
\sum_{q=0}^{l-1} \langle f_{n+l+2+2k} | C_q | \varphi^n_{l+2k-q} \rangle = 0.
\]
At the next lemma we prove that functions $\varphi_n^{n+2k}[k]$, $k = 0, 1, \ldots$ satisfy conditions (3.21) and corollary 3.2 is applicable to them.

**Lemma 3.4.** Functions $\varphi_n^{n+2k}[k]$ satisfy

$$\langle f_{n+l+2k+2m} | \varphi_i^{n+2k}[k] \rangle = 0$$

and

$$\langle f_{n+l+2k+2m} | C_q | \varphi_i^{n+2k-q}[k] \rangle = 0, \quad q = 0, \ldots, l - 1$$

and $k = 0, 1, \ldots, m \in \mathbb{N}$

**Proof.** The case $k = 0$ was considered in proposition 3.4. Let us assume that the lemma is proved for $r = 0, 1, k - 1$. Let us prove it for $r = k$. We have from the equation (n.l+2k)

$$\sum_p s(p, n + l + 2k) | e_p \rangle \langle f_p | \varphi_i^{n+2k}[k - 1] \rangle + \sum_{q=0}^{l-1} C_q \varphi_i^{n+2k-2-q}[k - 1] + C_i \varphi_i^{n+2k-2}[k - 1] = E \varphi_i^{n+2k-2}[k - 2].$$

Omitting all the terms of the degree $k - 1$ we obtain

$$\sum_p s(p, n + l + 2k) | e_p \rangle \langle f_p | \varphi_i^{n+2k}[k] \rangle + \sum_{q=0}^{l-1} C_q \varphi_i^{n+2k-2-q}[k] = E \varphi_i^{n+2k-2}[k - 1].$$

Multiplying both sides of this identity by $\langle f_{n+2+2k+2} | \langle f_{n+2+2k+4} | \ldots$ and so on, using induction assumption we obtain the first part of the lemma. Its second part then can be proved in the same way as corollary 3.2.

So, applying the lemma 3.4 for $m = 1$ we can rewrite (3.27) as

$$\langle f_{n+l+2(k+1)} | C_i | \varphi_i^{n+2k}[k] \rangle = E \langle f_{n+l+2(k+1)} | \varphi_i^{n+2k}[k - 1] \rangle, \quad k = 0, 1, \ldots$$

Let us notice that for $k = 0$ we obtain that

$$\langle f_{n+l+2} | C_i | e_n \rangle = 0,$$

that is, the first of the conditions (3.20) on matrix $C_i$ and for $k = 1, 2, \ldots$ proposition 3.3 gives

$$\frac{E^k \langle f_{n+l+2(k+1)} | C_i | e_n \rangle}{s(n, n + 2) \ldots s(n, n + 2k)} = E \langle f_{n+l+2(k+1)} | \varphi_i^{n+2k}[k - 1] \rangle$$

where $k = 1, 2, \ldots$
Lemma 3.5.

\[ \langle f_{n+l+2(k+1)} | \varphi^n_{i+2k} | [k-1] \rangle = \]

\[ = -E^{k-1} \Omega_{k-1} (n + l + 2(k + 1), n, l) \frac{s(n + l + 2(k + l), n + l + 2k)}{s(n + l + 2(k + 1), n + l + 2k)} \langle f_{n+l+2(k+1)} | C_l | e_n \rangle. \]  

(3.30)

\textbf{Proof.} Let us prove more general statement that

\[ \langle f_{n+l+2(k+r)} | \varphi^n_{i+2k} | [k-1] \rangle = \]

\[ = -E^{k-1} \Omega_{k-1} (n + l + 2(k + r), n, l) \frac{s(n + l + 2(k + r), n + l + 2k)}{s(n + l + 2(k + r), n + l + 2k)} \langle f_{n+l+2(k+r)} | C_l | e_n \rangle, \quad r = 1, 2, \ldots \]

If \( k = 1 \) we have to prove that

\[ \langle f_{n+l+2+2r} | \varphi^n_{i+2} \rangle = - \frac{\Omega_0 (n + l + 2 + 2r, n, l)}{s(n + l + 2 + 2r, n + l + 2)} \langle f_{n+l+2+2r} | C_l | e_n \rangle, \]

or, using \( \Omega_0 = 1 \)

\[ \langle f_{n+l+2+2r} | \varphi^n_{i+2} \rangle = - \frac{1}{s(n + l + 2 + 2r, n + l + 2)} \langle f_{n+l+2+2r} | C_l | e_n \rangle. \]

(3.31)

Let us consider the equation \((n.l+2)\)

\[ \sum_p s(p, n + l + 2) | e_p \rangle \langle f_p | \varphi^n_{i+2} \rangle + \sum_{q=0}^{l-1} C_q \varphi^n_{i-q} + C_0 \varphi^n = E \varphi^n. \]

Multiplying by \( \langle f_{n+l+2+2r} \rangle \) and using lemma 3.4 we get

\[ s(n + l + 2 + 2r, n + l + 2) \langle f_{n+l+2+2+2r} | \varphi^n_{i+2} \rangle + \langle f_{n+l+2+2+2r} | C_l | e_n \rangle = 0 \]

which coincides with (3.31).

Let us assume the lemma is proved for \( s = 1, 2, \ldots, k - 1 \). To prove it for \( s = k \) let us write down the equation \((n,n+l+2k)\) omitting all the terms of degree less than \( k - 1 \)

\[ \sum_p s(p, n + l + 2k) | e_p \rangle \langle f_p | \varphi^n_{i+2k} | [k-1] \rangle + \sum_{q=0}^{l-1} C_q \varphi^n_{i+2k-2-q} | [k-1] + \]

\[ + C_0 \varphi^n_{2k-2} | [k-1] = E \varphi^n_{i+2k-2} | [k-2] \]

and multiply it by \( \langle f_{n+l+2+2k} |, r = 1, 2, \ldots \) Using lemma 3.4 we obtain

\[ \sum_{q=0}^{l-1} \langle f_{n+l+2+2+2r} | C_q \varphi^n_{i+2k-2-q} | [k-1] \rangle = 0 \]
\[
\begin{align*}
&\text{and, therefore,} \\
&s(n + l + 2k + 2r, n + l + 2k)|C|\psi^n_{l+2k}[k - 1]| + \\
&+ (f_{n + l + 2k + 2r} |C|\psi^n_{l+2k}[k - 1]) = E(f_{n + l + 2k + 2r} |C|\psi^n_{l+2k}[k - 2]). \tag{3.32}
\end{align*}
\]

From proposition 3.3 we know that
\[
\phi^n_{2k-2}[k - 1] = E^{k-1} \frac{|e_n|}{s(n, n + 2) \cdots s(n, n + 2k - 2)}
\]

and by induction assumption
\[
(f_{n + l + 2k + 2r} |C|\phi^n_{l+2k}[k - 2]) = (f_{n + l + 2(k-1) + 2(r+1)} |C|\phi^n_{l+2(k-1)}[(k - 1) - 1]) = \\
= -E^{k-2} \frac{\Omega_{k-3}(n + l + 2k + 2r, n, l)}{s(n + l + 2k + 2r, n + l + 2k - 2)} \cdot (f_{n + l + 2k + 2r} |C| e_n), \quad r = 1, 2, 3, \ldots.
\]

Finally, substituting the last two formulae into (3.32) we obtain
\[
\begin{align*}
&\frac{1}{s(n, n + 2) \cdots s(n, n + 2k - 2)} \Omega_{k-2}(n + l + 2k + 2r, n, l) \\
&= -E^{k-1} \frac{\Omega_{k-2}(n + l + 2k + 2r, n, l)}{s(n + l + 2k + 2r, n + l + 2k - 2)} (f_{n + l + 2k + 2r} |C| e_n), \\
&\text{or,} \\
&\frac{1}{s(n, n + 2) \cdots s(n, n + 2k - 2)} \Omega_{k-2}(n + l + 2k + 2r, n, l) \\
&= -E^{k-1} \frac{\Omega_{k-2}(n + l + 2k + 2r, n, l)}{s(n + l + 2k + 2r, n + l + 2k - 2)} (f_{n + l + 2k + 2r} |C| e_n).
\end{align*}
\]

In the partial case \( r = 1 \) we obtain the proof of the lemma.

Let us return now to the equation (3.29). Using lemma 3.5 we can rewrite it
\[
E^k \frac{\langle f_{n + l + 2(k+1)} |C| e_n \rangle}{s(n, n + 2) \cdots s(n, n + 2k)} = \\
= -E^k \frac{\Omega_{k-1}(n + l + 2(k + 1), n, l)}{s(n + l + 2(k + 1), n + l + 2k)} \langle f_{n + l + 2(k+1)} |C| e_n \rangle, \\
\text{or,} \\
\left[ \frac{1}{s(n, n + 2) \cdots s(n, n + 2k)} + \frac{\Omega_{k-1}(n + l + 2(k + 1), n, l)}{s(n + l + 2(k + 1), n + l + 2k)} \right] \\
\cdot (f_{n + l + 2(k+1)} |C| e_n) = 0,
\]

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which is equivalent to

$$\Omega_k(n + l + 2(k + 1), n, l)(f_{n+l+2(k+1)}|C_l|e_n) = 0.$$  

As $\Omega_k(n + l + 2(k + 1), n, l) \neq 0$ we have together with (3.28) that

$$\langle f_{n+r+2k}|C_r|e_n \rangle = 0, \quad k = 1, 2, \ldots$$

That finishes the induction step and proves the part of the theorem concerned matrix coefficients $C_l, l = -1, 0, \ldots, 2M - 3$. From proposition 3.4 we get the conditions (3.13) on eigenfunctions. Theorem 3.3 is proved.

**Remark.** It is easy to generalise theorem 3.3 for the case when there are indices $n_1 < \ldots < n_p$ such that

$$\dim V_{n_j} = 0, \quad j = 1, \ldots, p.$$  

We should just assume that the columns and rows of matrices $C_k$ have indices $1, \ldots, n_1, \ldots, \hat{n}_2, \ldots, \hat{n}_p, \ldots d$ where $\hat{n}$ means that the corresponding indices are omitted. In particular, if $\dim V_1 = \dim V_2 = \ldots = \dim V_{n-1} = 0$ (scalar case) we get theorem 3.1.

### 3.4 Formulation of the local criteria in invariant form.

To formulate the theorem 3.3 in more invariant way let us present the matrix $C_{-2}$ in the form

$$C_{-2} = \sum_{j=1}^{k} m_j(m_j - 1)P_j$$  \hspace{1cm} (3.33)

where $P_j$ are commuting projectors to the corresponding eigenspaces $V_j$:

$$P_i P_j = \delta_{ij} P_i, \quad \sum_{j=1}^{k} P_j = I,$$

where

$$\delta_{ij} = \begin{cases} 
1, & i = j, \\
0, & i \neq j 
\end{cases}$$

and $1 \leq m_1 < m_2, \ldots < m_k = M$.

**Theorem 3.4 (local criteria of trivial monodromy in invariant form).** A matrix Schrödinger operator $L$ with a meromorphic potential $U(z)$ given by (3.9) has trivial
monodromy around $z = 0$ if and only if $C_{-2}$ has a form (3.33) and the coefficients $C_l, l = -1, 0, \ldots, 2M - 3$ satisfy the relation

$$P_i C_l P_j = 0$$

when $|m_i - m_j| \geq l + 1$ or $m_i + m_j = l + 3, l + 5, \ldots, l + 2k + 1, \ldots$.

The coefficients $\psi_0, \psi_1, \ldots, \psi_{2M-1}$ of the corresponding expansions of the vector-eigenfunctions

$$\psi = z^{-M+1}(\psi_0 + z\psi_1 + \ldots + z^k\psi_k + \ldots)$$

satisfy the conditions

1. $P_i \psi_l = 0$ if $m_i + l < M$
2. $P_i \psi_l = 0$ if $m_i + l = M + 1, M + 3, \ldots, M + 2k + 1, \ldots$ and $m_i \geq l - M + 3$.

3.5 Matrix Darboux transformations and Schrödinger operators with trivial monodromy.

In section 2.2 we introduced the definition of matrix Darboux transformation and derived the explicit formula for the potentials of the Schrödinger operators obtained by MDT. Using this it is easy to prove the following

**Theorem 3.5.** All matrix Schrödinger operators $L$ obtained by matrix Darboux transformation from $L_0 = -\frac{d^2}{dz^2}$ have trivial monodromy.

**Proof.** Indeed, the kernel of the intertwining operator $A$ is invariant under $L_0$ and, therefore, is generated by quasipolynomials. According to the construction of the intertwining operator given in theorem 2.2 this implies that all the coefficients of $A$ are meromorphic in $\mathbb{C}$ and, therefore, any eigenfunction $\psi$ of $L$ can be written on the form

$$\psi = A(e^{kz}R_1 + e^{-kz}R_2)$$

where $R_1$ and $R_2$ are constant matrices. Obviously, $\psi$ is meromorphic in $\mathbb{C}$ and the theorem follows.

Let us assume now that matrix Schrödinger operator

$$L = -D^2 + U(z)$$

has trivial monodromy, the potential $U(z)$ is rational and decays at infinity. Since all the residues must be equal to zero $U(z)$ can be represented as

$$U(z) = \sum_{k=1}^{N} \frac{A_k}{(z - z_k)^2}.$$  \hspace{1cm} (3.34)
The conditions of the theorem 3.4 give the algebraic system for the matrix coefficients $A_k$ and poles $z_k$ which we will call locus equations. This terminology goes back to the paper [3] by Airault, McKean and Moser who considered this system in the scalar case.

Example. Let us consider the simplest nontrivial case when $d = 2$ and the potential $U(z)$ has only 3 poles of the second order

$$U(z) = \frac{P_u}{(z-u)^2} + \frac{P_v}{(z-v)^2} + \frac{P_w}{(z-w)^2}, \quad (3.35)$$

where $P_u, P_v, P_w$ are matrices with eigenvalues 0 and 2. Notice, that for the case $\text{Spec} C_{-2} = (0, 2)$ conditions of trivial monodromy can be written in the form

$$C_{-1} = 0, \quad [C_{-2}, C_0] = 0, \quad C_{-2}C_1C_{-2} = 0. \quad (3.36)$$

Expanding $U(z)$ near $u, v$ and $w$ and solving the corresponding locus equation we comes to the following

Proposition 3.5. The poles of $U(z)$ $u, v, w$ can be arbitrary complex numbers and they are the only essential parameters in this case. After a suitable transformation $U \to CUC^{-1}$ $U(z)$ has a form (3.35) with

$$P_u = \frac{2}{(u-w)(u-v)} \begin{pmatrix} uv - u^2 & u(u^2 - vw) \\ w - 2u + v & -u(w - 2u + v) \end{pmatrix},$$

$$P_v = \frac{2}{(v-w)(v-u)} \begin{pmatrix} uw - v^2 & v(v^2 - uw) \\ u - 2v + w & -v(u - 2v + w) \end{pmatrix},$$

and

$$P_w = \frac{2}{(w-u)(w-v)} \begin{pmatrix} uv - w^2 & w(w^2 - uw) \\ u - 2w + v & -w(u - 2w + v) \end{pmatrix}.$$
operators with rational potentials obtained by Darboux transformation are regular as it is easy to see from formula (1.6). It turns out that this is not the case for matrix Schrödinger operators (2.3) as one can see from the following example.

Let us consider the case $d = 2$ and equation (2.12)

$$\Phi'' = \Phi C$$

with $n = 1$. If we choose $C = 0$ and

$$\Phi = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$$

then

$$U = -2(\Phi'\Phi^{-1})' = \begin{pmatrix} \frac{2}{z^2} & \frac{4}{z^3} \\ 0 & \frac{2}{z^2} \end{pmatrix}.$$ 

So, the potential has the pole of the third order at $z = 0$, that is, irregular singular point. We will not investigate conditions of trivial monodromy for the corresponding Schrödinger operators in this case.

**Proof of the theorem 3.6.** We borrow the main idea from the recent paper by Chalykh [15]. The potential $U(z)$ must be of the form

$$U(z) = \sum_{j=1}^{N} \frac{A_j}{(z - z_j)^2}$$

where $A_j$ have the eigenvalues $m(m - 1)$ with integer $m$. Let $\lambda_j = M_j(M_j - 1)$ be the maximum of eigenvalues of the matrix $A_j$. Introduce a linear space $V$ consisting of matrix-valued functions $\Psi(z)$, $z \in \mathbb{C}$ satisfying the following conditions:

1. $\Psi(z) \prod_{j=1}^{N} (z - z_j)^{M_j - 1}$ is holomorphic in $\mathbb{C}$.

2. The columns of the coefficients of the Laurent expansion of $\Psi(z)$ at the vicinity of $z_j$, $j = 1, \ldots, N$ satisfy the conditions (3.13) with $M = M_j$.

**Lemma 3.6.** Space $V$ is invariant with respect to $L$.

**Proof.** Let $\Psi(z) \in V$. It is enough to prove that if any column $\psi$ of $\Psi$ satisfy conditions (3.13) at the vicinity of any singular point $z_j$ of $U(z)$ (without loss of generality we assume $z_j = 0$) then any column of $L\Psi$ satisfy (3.13) as well. So, we can suppose that potential of the Schrödinger operator $L$ has a form (3.9) at the vicinity of $z = 0$ with (for notations see section 3.3)

$$C_{-2} = \sum_{n=1}^{M} n(n - 1) |e_n\rangle \langle f_n|$$

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Let us assume that function $\psi$:

$$\psi = \sum_{k \geq -M+1} z^k \psi_k$$

satisfies conditions (3.13), or, equivalently, (3.21):

$$\langle f_{s+2k} | \psi_s \rangle = 0, \ s = -M + 1, -M + 2, \ldots, M - 2, \ k \in \mathbb{N} \tag{3.37}$$

It is enough to prove that function $\varphi = L\psi$ has a form

$$\varphi = \sum_{s \geq -M+1} z^s \varphi_s$$

and satisfies conditions (3.37) as well:

$$\langle f_{s+2k} | \varphi_s \rangle = 0, \ s = -M + 1, -M + 2, \ldots, M - 2 \tag{3.38}$$

Indeed, if

$$\psi = \sum_{s \geq -M+1} z^s \psi_s$$

then

$$\psi' = \sum_{s \geq -M+1} sz^{s-1} \psi_s$$

and

$$\psi'' = \sum_{s \geq -M+1} s(s-1)z^{s-2} \psi_s = M(M-1)\psi_{-M+1}z^{-M-1} + (M-1)(M-2)\psi_{-M+2}z^{-M+1} + \sum_{s \geq -M+1} (s+1)(s+2)z^s \psi_{s+2}.$$ 

We have also using $C_{-1} = 0$ that

$$U\psi = \left( \sum_{l \geq -2} C_l \psi_l \right) \left( \sum_{n \geq -M+1} z^n \psi_n \right) = C_{-2} \psi_{-M+1}z^{-M-1} + C_{-2} \psi_{-M+2}z^{-M} + \sum_{s \geq -M+1} z^s \left( \sum_{l=-2}^{s+M-1} C_l \psi_{s-l} \right).$$

Therefore,

$$\varphi = -\psi'' + U\psi = (C_{-2} - M(M-1))\psi_{-M+1}z^{-M-1} + (C_{-2} - (M-1)(M-2))\psi_{-M+2}z^{-M} + \sum_{s \geq -M+1} z^s \left( -(s+1)(s+2)\psi_{s+2} + \sum_{l=-2}^{s+M-1} C_l \psi_{s-l} \right).$$

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According to (3.37), $\psi_{-M+1}$ is proportional to $|e_M|$ and $\psi_{-M+2}$ is proportional to $|e_{M-1}|$ and

$$(C_{-2} - M(M-1))\psi_{-M+1} = 0, \quad (C_{-2} - (M-1)(M-2))\psi_{-M+2} = 0.$$  

Therefore,

$$\varphi = \sum_{s \geq -M+1} z^s \varphi_s$$

where

$$\varphi_s = (C_{-2} - (s+1)(s+2))\psi_{s+2} - \sum_{l=0}^{s+M-1} C_l \psi_{s-l}, \quad s = -M+1, -M+2, \ldots$$

Now we want to prove that

$$\langle f_{s+2k}\varphi_s \rangle = 0, \quad s = -M+1, -M+2, \ldots, M-2,$$

for $k \in \mathbb{N}$ or,

$$\langle f_{s+2k}\varphi_s \rangle = 0, \quad s = -M+1, -M+2, \ldots, M-2.$$  

Indeed, $\psi_{s+2} < e_{s+2}, e_s, \ldots >$ and as $C_{-2}e_{s+2} = (s+1)(s+2)e_{s+2}$

$$(C_{-2} - (s+1)(s+2))\psi_{s+2} < e_s, e_{s-2}, \ldots >$$

which is equivalent to

$$\langle f_{s+2k}\varphi_s \rangle = 0, \quad k = 1, 2, \ldots$$

To prove that

$$\langle f_{s+2k}\varphi_s \rangle = 0, \quad k = 1, 2, \ldots$$ (3.39)

where $s = -M+1, -M+2, \ldots, M-2, 0 \leq l \leq s + M - 1$ it is enough to notice that

$$\psi_{s-l} < e_{s-l}, e_{s-l-2}, \ldots >$$

and condition $\langle f_{s+2k}\varphi_{s-l} \rangle = 0$ is the same as $\langle f_{s+2k}\varphi_s \rangle = 0, \quad m = 0, 1, \ldots$

which after the substitution $n = s - l - 2m$ is equivalent to

$$\langle f_{n+4+2(k+m)}\varphi_s \rangle = 0, \quad -M + 1 \leq n \leq M - l - 2$$

It is easy to see that according to the theorem 3.3 these are the conditions on $L$ to have trivial monodromy. Then (3.39) and lemma follows.
Let us consider the matrix-function $\Psi_0 = \prod_{j=1}^{N} (z - z_j)^{M_j-1} e^{kz} I$. Evidently, $\Psi_0 \in V$ and, therefore, all matrices

$$\Psi_i = (L + k^2)^i \Psi_0$$

belong to $V$ as well. These functions have the form

$$\Psi_i = P_i(k, z)e^{kz}$$

where $P_i(k, z)$ is a polynomial in $k$ and a rational function in $z$. Since

$$P_{i+1} = \left( -\frac{d^2}{dz^2} - 2k \frac{d}{dz} + U(z) \right) P_i$$

degree of $P_i$ in $z$ at infinity is decreasing with $i$: $\deg_z P_i \leq M - i$, where $M = \sum_{j=1}^{N} (M_j - 1)$. On the other hand, since the space $V$ is invariant under $L$ the degree of the denominator of $P_i$ can not be more than that of $\prod_{j=1}^{N} (z - z_j)^{M_j-1}$. So, there exists such $K$ that $(L + k^2) \Psi_K = 0$. It is easy to see that for $M$ defined above

$$\Psi_M = \left[ (-2)^M M! k^M + \ldots \right] e^{kz} I \neq 0.$$

We claim that $\Psi_{M+1} = (L + k^2) \Psi_M = 0$. Indeed, assume that this is not true. Then for some $K > M \ \Psi_K \neq 0,$

$$\Psi_{K+1} = (L + k^2) \Psi_K = 0.$$

Since

$$P_{K+1} = \left( -\frac{d^2}{dz^2} - 2k \frac{d}{dz} + U(z) \right) P_K = 0$$

and $P_K$ is polynomial in $k$ its highest coefficient has to be constant (cf. lemma 2.5 in [9]). At the same time $P_K$ has to decay at infinity at least as $z^{M-K}$. Thus $K = M$ and

$$L \Psi_M = -k^2 \Psi_M.$$

Replacing $k$ by $\frac{d}{dz}$ we can define a differential operator $A$ such that $\Psi = Ae^{kz} I$. We have

$$L \psi = L A e^{kz} I = -k^2 A e^{kz} I = -A k^2 e^{kz} I = A L_0 e^{kz} I$$

and, therefore, $LA = AL_0$. Thus, $L$ is related to $L_0 = -\frac{d^2}{dz^2}$ by matrix Darboux transformation. The theorem is proved.
3.6 Locus equations and matrix Calogero-Moser system.

Let us consider the generic case (2.15) when the rank of all matrices $A_i$ is 1 and the only nonzero eigenvalue of $A_i$ is 2. Such matrices can be represented as

$$A_i = 2|e_i\rangle\langle f_i|$$

where $|e_i\rangle \in V = \mathbb{C}^d$ is a vector, $\langle f_i| \in V^*$ is a covector such that $\langle f_i|e_i\rangle = 1$ and $|e_i\rangle$ is the eigenvector of $A_i$ with the eigenvalue 2:

$$A_i|e_i\rangle = 2|e_i\rangle\langle f_i|e_i\rangle = 2|e_i\rangle.$$

Covector $\langle f_i|$ defines the kernel of $A_i$ by the relation $\langle f_i|x\rangle = 0$.

If the potential of the Schrödinger operator is given by

$$U(z) = \sum_{i=1}^{N} \frac{2|e_i\rangle\langle f_i|}{(z - z_i)^2}$$

we can expand $U(z)$ near $z = z_i$ as follows

$$U(z) = \frac{2|e_i\rangle\langle f_i|}{(z - z_i)^2} + \sum_{j \neq i} \frac{2|e_j\rangle\langle f_j|}{(z_j - z_i)^2} - \sum_{j \neq i} \frac{|e_j\rangle\langle f_j|}{(z_j - z_i)^3}(z - z_i) + \ldots$$

and the locus equations (3.36) can be written in the following form

$$\left[\left| e_i \right\rangle \left\langle f_i \right|, \sum_{j \neq i} \frac{|e_j\rangle\langle f_j|}{(z_j - z_i)^2} \right] = 0, \quad i = 1, \ldots, N,$$

$$\sum_{j \neq i} \frac{\left| f_i \right\rangle \left\langle e_j \right| \left| f_j \right\rangle \left\langle e_i \right|}{(z_j - z_i)^3} = 0, \quad i = 1, \ldots, N.$$

This is a stationary equations for a spin version of classical Calogero-Moser system when particles possess internal degrees of freedom. Such a system was first introduced by Gibbons and Hermsen in 1984 (see [36]) and has the Hamiltonian

$$H = \sum_{j=1}^{N} \frac{1}{2} \sum_{i=1}^{N} \sum_{i \neq j}^{N} \frac{\langle f_j|e_j\rangle \langle f_j|e_i\rangle}{(z_i - z_j)^2}$$

where $\langle f_j(t)|$ and $|e_j(t)\rangle$ are as above. The equation of the motion are

$$\frac{dz_i}{dt} = p_i,$$

$$\frac{dp_i}{dt} = \sum_{j \neq i} \frac{2\langle f_i|e_j\rangle \langle f_j|e_i\rangle}{(z_i - z_j)^3}.$$
\[
\frac{d|e_i\rangle}{dt} = -\sum_{j \neq i} \frac{|e_j\rangle\langle f_j|e_i\rangle}{(z_i - z_j)^2},
\]
\[
\frac{d|f_i\rangle}{dt} = \sum_{j \neq i} (|e_i\rangle\langle f_j|e_i\rangle|f_j\rangle)}{(z_i - z_j)^2}.
\]
Normalization condition \(|\langle f_i|e_i\rangle = 1\) is invariant under the transformation
\[
|e_j\rangle \rightarrow c_j(t)|e_j\rangle, \quad |f_i\rangle \rightarrow |f_i|c^{-1}(t),
\]
so we can represent the equation of the motion in the form
\[
\frac{d^2 z_i}{dt^2} = 2\sum_{j \neq i} \frac{|\langle f_i|e_j\rangle\langle f_j|e_i\rangle|}{(z_i - z_j)^3},
\]
\[
\frac{d}{dt} (|e_i\rangle\langle f_i|) = -\left[|e_i\rangle\langle f_i|, \sum_{j \neq i} \frac{|e_j\rangle\langle f_j|}{(z_j - z_i)^2}\right].
\]
So, we come to

**Theorem 3.7.** Locus equations for the matrix Schrödinger operator with the potential
\[
U(z) = \sum_{i=1}^{N} \frac{2|e_i\rangle\langle f_i|}{(z - z_i)^2}
\]
coincide with the stationary equations for the generalised Calogero-Moser system.

Theorem 3.6 allows us to construct explicitly some solutions to this complicated algebraic system.

### 3.7 Schrödinger operators with matrix trigonometric potentials.

At the next chapter we study in details two-dimensional matrix Schrödinger operators with trivial monodromy and we will need the following result concerning the operators
\[
\mathcal{L} = -\frac{d^2}{dd^2} + V(\phi)
\]
where
\[
V(\phi) = \sum_{i=1}^{N} \frac{A_i}{\sin^2(\phi - \phi_i)}.
\]
with some matrices \(A_i\), \(\phi_i\) are singularities of \(V(\phi)\).

**Theorem 3.8.** Any Schrödinger operator (3.40) with trigonometric potential (3.41) which satisfies local trivial monodromy conditions at all the singularities is related by MDT to \(L_0 = -D^2\).
Remark. This result is a matrix generalisation of the corresponding theorem for the scalar Schrödinger operator proved by Chalykh in [15] (see also [17]).

Proof. As operator $\mathcal{L}$ has trivial monodromy matrices $A_s$ from the expression (3.41) have to be of the form

$$A_s = \sum_{i=1}^{k_s} m_i^{(s)} (m_i^{(s)} + 1) \begin{pmatrix} \cdots \\ \vdots \end{pmatrix}, \quad 0 \leq m_1^{(s)} < m_2^{(s)} < \ldots < m_{k_s}^{(s)} = M_s$$

with some projectors $P_i^{(s)}$: $P_i^{(s)} P_j^{(s)} = \delta_{ij} P_i^{(s)}$, $\sum_{i=1}^{k_s} P_i^{(s)} = I$.

Let us introduce the linear space $W$ which consists of the $d \times d$ matrix-valued functions $\Psi(\phi)$ such that:

1) $\Psi(\phi) \prod_{\varsigma=1}^{N} \sin^{M_s}(\phi - \phi_s)$ is holomorphic in $C_i$;

2) Columns of the series expansion coefficients of $\Psi(\phi)$ at the vicinity of $\phi = \phi_s$, $s = 1, \ldots, N$ satisfy the locus conditions with $M = M_s$.

Then the matrix locus equations imply that the space $W$ is invariant under $\mathcal{L}$ (see lemma 3.6).

Let us consider the matrix function $\Psi_0(\phi) = Q e^{k \phi} I$ where

$$Q = \prod_{\varsigma=1}^{N} \sin^{M_s}(\phi - \phi_s).$$

Evidently, $\Psi_0 \in W$ and, therefore, functions

$$\Psi_n = (\mathcal{L} + (k + s_n)^2) \ldots (\mathcal{L} + (k + s_1)^2) \Psi_0$$

($s_i \in C$) belong to $W$ as well and they have a form

$$\Psi_n(\phi) = \frac{P_n(\phi)}{Q(\phi)} e^{k \phi}$$

(3.42)

where

$$P_n(\phi) = \sum_{r=G_n}^{H_n} C_r^n e^{r \phi}, \ G_n \leq H_n$$

is a sum of exponents with matrix coefficients $C_r^n$ which is also a quasi-polynomial in $k$. Let us investigate in more details function

$$\Psi(\phi) = \frac{P(\phi)}{Q(\phi)} e^{k \phi} \in W$$

where

$$P(\phi) = \sum_{r=G}^{H} C_r e^{r \phi}, \ G \leq H$$

The interval $[G, H]$ is called the support of $\Psi$: $\text{supp}(\Psi) = [G, H]$. 

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Lemma 3.7.

\[ \text{supp} (\mathcal{L} + k^2) \Psi \subseteq \text{supp} \Psi. \]

Proof. If \( \Psi(\phi) = \frac{P(\phi)}{Q(\phi)} e^{k\phi} \), then

\[ \Psi'(\phi) = \left[ \frac{P'}{Q} - \frac{P'Q'}{Q^2} \right] e^{k\phi} + k \Psi \]

and

\[ \Psi''(\phi) = \left[ \frac{P'' + kP'}{Q} - 2Q'P' + kP \frac{Q''}{Q^2} - kP \frac{Q^2}{Q^2} \right] e^{k\phi} + k^2 \Psi \]

As \( Q = \prod_{i=1}^{N} \sin M_i (\phi - \phi_0) \)

\[ Q' = Q \sum_{j=1}^{N} M_j \frac{\cos(\phi - \phi_j)}{\sin(\phi - \phi_j)} \]

and

\[ Q'' = \left[ \sum_{i \neq j} M_i M_j \frac{\cos(\phi - \phi_i) \cos(\phi - \phi_j)}{\sin(\phi - \phi_i) \sin(\phi - \phi_j)} + \sum_{j=1}^{N} \left( M_j (M_j - 1) \frac{\cos^2(\phi - \phi_j)}{\sin^2(\phi - \phi_j)} - M_j \right) \right]. \]

Using \( \cos^2(\phi - \phi_j) = 1 - \sin^2(\phi - \phi_j) \) and \( M_j (M_j - 1) + M_j = M_j^2 \) we can rewrite the last formula as

\[ Q'' = Q \left[ \sum_{i \neq j} M_i M_j \frac{\cos(\phi - \phi_i) \cos(\phi - \phi_j)}{\sin(\phi - \phi_i) \sin(\phi - \phi_j)} + \sum_{j=1}^{N} \left( \frac{M_j (M_j - 1)}{\sin^2(\phi - \phi_j)} - M_j \right) \right]. \]

Analogously,

\[ Q^2 = \left[ \sum_{i \neq j} M_i M_j \frac{\cos(\phi - \phi_i) \cos(\phi - \phi_j)}{\sin(\phi - \phi_i) \sin(\phi - \phi_j)} - \sum_{j=1}^{N} M_j^2 \left( 1 - \frac{1}{\sin^2(\phi - \phi_j)} \right) \right]. \]

Therefore,

\[ \Psi'' = \left[ P'' + 2kP' - 2(P' + kP) \sum_{j=1}^{N} \frac{M_j \cos(\phi - \phi_j)}{\sin(\phi - \phi_j)} + P \sum_{i \neq j} M_i M_j \frac{\cos(\phi - \phi_i) \cos(\phi - \phi_j)}{\sin(\phi - \phi_i) \sin(\phi - \phi_j)} - \right. \]

\[ - \left. P \sum_{j=1}^{N} \left( \frac{M_j (M_j - 1)}{\sin^2(\phi - \phi_j)} - M_j^2 \right) - 2P \sum_{j=1}^{N} M_j^2 \left( 1 - \frac{1}{\sin^2(\phi - \phi_j)} \right) \right] \frac{e^{k\phi}}{Q} + k^2 \Psi = \]

\[ = \frac{e^{k\phi}}{Q} \left[ P'' + 2kP - P \sum_{j=1}^{n} M_j^2 - 2(P' + kP) \sum_{j=1}^{N} \frac{M_j \cos(\phi - \phi_j)}{\sin(\phi - \phi_j)} + \right. \]

\[ + \left. P \sum_{i \neq j} M_i M_j \frac{\cos(\phi - \phi_i) \cos(\phi - \phi_j)}{\sin(\phi - \phi_i) \sin(\phi - \phi_j)} + P \sum_{j=1}^{N} \frac{M_j (M_j + 1)}{\sin^2(\phi - \phi_j)} \right] + k^2 \Psi. \]
So,

\[
(L + k^2)\Psi = \frac{e^{ik\phi}}{Q} \left[ -\left( P'' + 2kP - P \sum_{j=1}^{N} M_j^2 \right) + 2(P' + kP) \sum_{j=1}^{N} \frac{M_j \cos(\phi - \phi_j)}{\sin(\phi - \phi_j)} - P \sum_{i \neq j} M_i M_j \frac{\cos(\phi - \phi_i) \cos(\phi - \phi_j)}{\sin(\phi - \phi_i) \sin(\phi - \phi_j)} + \sum_{j=1}^{N} \frac{(A_j - M_j(M_j + 1))P}{\sin^2(\phi - \phi_j)} \right].
\]

Expression in the square bracket can be written in the form

\[
\frac{1}{Q[2]} \left[ (-P'' - 2kP' + P \sum_{j=1}^{N} M_j^2)Q[2] - P \sum_{i \neq j} M_i M_j \cos(\phi - \phi_i) \cos(\phi - \phi_j)Q_{ij} + 
+ 2 \sum_{j=1}^{N} (M_j P' + kM_j P) \cos(\phi - \phi_j)Q_j[2] + 
+ \sum_{j=1}^{N} (A_j - M_j(M_j + 1)I)PQ_j[0] \right]
\]

where

\[
Q[2] = \prod_{s=1}^{N} \sin^2(\phi - \phi_j), \quad Q_j[1] = \frac{Q[2]}{\sin(\phi - \phi_j)},
\]

\[
Q_j[0] = \frac{Q[2]}{\sin^2(\phi - \phi_j)}, \quad Q_{ij} = \frac{Q[2]}{\sin(\phi - \phi_i) \sin(\phi - \phi_j)}.
\]

Because the space of function satisfying conditions 1), 2) have to be invariant under the action of the operator \( L \) and it is possible to represent \( L\Psi \) in the form (3.42) there exists trigonometric polynomial \( R \) with matrix coefficients such that

\[
(-P'' - 2kP' + P \sum_{j=1}^{N} M_j^2)Q[2] - P \sum_{i \neq j} M_i M_j \cos(\phi - \phi_i) \cdot 
\cos(\phi - \phi_j)Q_{ij} + 2 \sum_{j=1}^{N} (M_j P' + kM_j P) \cos(\phi - \phi_j)Q_j[2] + 
+ \sum_{j=1}^{N} (A_j - M_j(M_j + 1)I)PQ_j[0] = RQ[2]
\]

Comparing the supports of the terms in left and right hand sides of the last formula using \( \text{supp} P' = \text{supp} P \) and

\[
\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}
\]

we get that \( \text{supp} R \subseteq \text{supp} P \).

We claim that there exists such number \( \lambda \) that \( \text{supp} R \subseteq \text{supp} P \). To prove this let us calculate the coefficient for the highest term \( e^{iH\phi} \) of \( R \). We will denote it \( h(R) \). Let us notice that the support \( [\tilde{G}, \tilde{H}] \) of

\[
\sum_{j=1}^{N} (A_j - M_j(M_j + 1)I)Q_j[0]P
\]
is less than the support \([G, H]\) of other terms in (3.43) so that \(\hat{G} > G\) and \(\hat{H} < H\). That means that the last sum does not affect \(h(R)\) and further reasoning is essentially scalar. Without loss of generality we can assume that \(h(P) = 1\). We have: \(h(P') = iH\), \(h(P'') = -H^2\). Therefore, using (3.44) we obtain

\[
\begin{aligned}
  h(R) \left( \frac{1}{2i} \right)^{2N} & = \left( \frac{1}{2i} \right)^{2N} \left( H^2 - 2kiH + \sum_{j=1}^{N} M_j^2 \right) - \frac{1}{2} \cdot \frac{1}{2} \left( \frac{1}{2i} \right)^{2N-2} \sum_{i \neq j} M_i M_j + \\
& + 2 \cdot \frac{1}{2} \left( \frac{1}{2i} \right)^{2N-1} \sum_{j=1}^{N} (iH M_j + k M_j),
\end{aligned}
\]

so that

\[
h(R) = \sum_{j=1}^{N} M_j^2 + H^2 - 2kiH + \sum_{i \neq j} M_i M_j + 2 \sum_{j=1}^{N} M_j (ik - H).
\]

Then, obviously, \(\lambda = -h(R)\). Let us notice that

\[
k^2 + \lambda = k^2 - \sum_{j=1}^{N} M_j^2 - H^2 + 2kiH - \sum_{i \neq j} M_i M_j - 2 \sum_{j=1}^{N} M_j (ik - H) = (k + i(H - \sum_{j=1}^{N} M_j))^2
\]

and operator

\[
\mathcal{L} + (k + i(H - \sum_{j=1}^{N} M_j))^2
\]

reduces the support of \(\Psi(\phi, k)\). That means that there exists a function \(\Phi = \Psi_M\) such that for some \(s\)

\[
(\mathcal{L} + (k + s)^2)\Phi = 0
\]

Presenting \(\Phi\) in the form \(\Phi = Ae^{(k+s)\phi}I\) for the proper matrix differential operator \(A\) we have

\[
\mathcal{L} \Phi = \mathcal{L} Ae^{(k+s)\phi}I = -(k + s)^2 Ae^{(k+s)\phi}I = -A(k + s)^2 e^{(k+s)\phi}I = A \mathcal{L}_0 e^{(k+s)\phi}I
\]

for any \(k \in \mathbb{R}\) and, therefore,

\[
\mathcal{L} A = A \mathcal{L}_0
\]

and theorem follows.
Chapter 4

Multidimensional integrable Schrödinger operator with matrix potential.

In this chapter we consider the operators

\[ L = -\Delta + U(z) \]

where \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \Delta = \frac{\partial^2}{\partial z_1^2} + \ldots + \frac{\partial^2}{\partial z_n^2} \) and \( U(z) \) is a matrix-valued meromorphic function. Let us assume that there exists a differential operator \( D \) with meromorphic matrix coefficients and constant scalar highest term such that

\[ LD = DL_0 \tag{4.1} \]

where \( L_0 = -\Delta \) is the pure Laplacian acting on the space of vector-valued functions.

We call such Schrödinger operators \( D \)-integrable following to Berest and Veselov [7], [8]. It has been shown in [7], [8] that the singularities of the potential \( U(z) \) of any scalar \( D \)-integrable Schrödinger operator have to be located on a union of the hyperplanes. The proof works also in the matrix case under an additional assumption of the regularity (see below).

4.1 Scalar case.

Let us now remind some properties of scalar \( D \)-integrable Schrödinger operators following essentially [17]. Let \( A \) be a configuration of the hyperplanes \( \Pi_j \) in a complex Euclidean space \( \mathbb{C}^n \) given by the equations \((\alpha_j, z) + c_j = 0, j = 1, \ldots, N,\)
taken with some multiplicities \( m_j \in \mathbb{Z}_+ \). Here \( \alpha_j \in \mathcal{A} \), \( \mathcal{A} \) is a finite set of noncollinear vectors. Corresponding Schrödinger operators have the form

\[
L = -\Delta + \sum_{j=1}^{N} \frac{m_j(m_j + 1)(\alpha_j, \alpha_j)}{((\alpha_j, z) + c_j)^2}.
\]  

(4.2)

In contrast to the one-dimensional situation there is no analogue of Darboux transformation and, in particular, there is no effective procedure to construct examples of D-integrable Schrödinger operators if \( n > 1 \). Nevertheless, as it was discovered recently by Chalykh [15] there exists an effective way the check for a given configuration \( \mathcal{A} \) whether the corresponding Schrödinger operator D-integrable or not. It turned out to be related with the notion of local trivial monodromy for Schrödinger operators \( L = -\Delta + u(z) \) which can be introduced (see [17]) as follows. Let \( u(z) \) have a pole along the hyperplane \( \Pi_\alpha \) defined by the equation \( (\alpha, z) = 0 \) which is assumed to be non-isotropic: \( (\alpha, \alpha) \neq 0 \). Let us assume that there exists a formal solution \( \psi \) of the Schrödinger equation

\[
L\psi = \lambda\psi
\]  

(4.3)

in the form

\[
\psi(z) = \sum_{\mu \geq 0} \psi^\mu_s (\alpha, z)^{\mu + s}
\]  

(4.4)

for some \( \mu < 0 \) where the coefficients \( \psi^\mu_s = \psi^\mu_s (z^\perp) \) are some analytic functions on the hyperplane \( \Pi_\alpha \), \( z^\perp \) is the orthogonal projection of \( z \) onto \( \Pi_\alpha \) and \( \psi^0_0 \neq 0 \). Then the potential \( u(z) \) must have a pole the second order along \( \Pi_\alpha \): the Laurent expansion in the normal direction \( \alpha \) has a form

\[
u(z) = \sum_{k \geq -2} c_k^\mu (\alpha, z)^k
\]  

(4.5)

with \( c_2^\mu = \mu(\mu + 1)(\alpha, \alpha) \).

Moreover, substituting series expansions (4.4) and (4.5) into the Schrödinger equation (4.3) we obtain the following recurrent relations for the coefficients \( \psi^\mu_s \):

\[
(\alpha, \alpha)(\mu(\mu - 1) - (\mu + s)(\mu + s - 1))\psi_s = (\tilde{\Delta} + \lambda)\psi_{s-2} - \sum_{i=-1}^{s-2} c_i \psi_{s-i-2}
\]  

(4.6)

where \( s = 1, 2, \ldots, \psi_{-1} = 0 \) and \( \tilde{\Delta} \) is the Laplacian \( \Delta \) restricted to the hyperplane \( \Pi_\alpha \). We also omitted all the indices \( \alpha \) in the coefficients.

**Definition** ([17]). *Schrödinger operator* \( L = -\Delta + u(z) \) *with meromorphic potential* \( u(z) \) *having a second order pole along the hyperplane* \( \Pi_\alpha : (\alpha, z) = 0 \) *has local trivial monodromy around this hyperplane if*
1) the Laurent coefficient $c_{-2}$ in the expansion (4.5) has the form $c_{-2} = m_\alpha (m_\alpha + 1)(\alpha, \alpha)$ for some $m_\alpha \in \mathbb{Z}_+$.

2) the system (4.6) with $\mu = -m_\alpha$ is compatible for any function $\psi_0$ and for all $\lambda \in \mathbb{C}$.

Conditions of local trivial monodromy in terms of the Laurent expansion (4.5) of the potential $u(z)$ are given in the following

**Theorem 4.1 ([17]).** $L$ has local trivial monodromy around $\Pi_\alpha$ if and only if the coefficients of the normal Laurent expansion of the potential $u(z)$ near $\Pi_\alpha$

$$u(z) = \sum_{k \geq -2} c_k^\alpha (\alpha, z)^k$$

satisfy the following conditions: $c_{-2} = m_\alpha (m_\alpha + 1)(\alpha, \alpha)$ for some $m_\alpha \in \mathbb{Z}_+$, and

$$c_{-1} = c_1^\alpha = c_3^\alpha = \ldots = c_{2m_\alpha - 1}^\alpha \equiv 0$$
on $\Pi_\alpha$. \hspace{1cm} (4.7)

This result can be considered as a multidimensional generalisation of Duistermaat and Grünbaum's result (see [27] and theorem 3.1). Applying the theorem 4.1 to the Schrödinger operator (4.2) we obtain the following

**Corollary 4.1 (see [15]).** Schrödinger operator (4.2) has local trivial monodromy around hyperplane $\Pi_i$ if and only if

$$\sum_{j \neq i} m_j (m_j + 1)(\alpha_i, \alpha_j)(\alpha_i, \alpha_j)^{2s-1} \equiv 0$$
on $((\alpha_j, z) + c_j)^{2s+1}$

on the hyperplane $(\alpha_i, z) + c_i = 0$ for all $s = 1, 2, \ldots, m_i$.

Using the terminology of the paper by Airault, McKean and Moser [3] we call (4.8) locus equation.

**Definition.** If conditions of local trivial monodromy are fulfilled around any singularity hyperplane of $u(z)$ we say that Schrödinger operator (4.2) has trivial monodromy in $\mathbb{C}^n$.

It turns out that conditions of trivial monodromy are necessary and sufficient for operator (4.2) to be D-integrable:

**Theorem 4.2 ([15])** Schrödinger operator $L$ given by formula (4.2) is D-integrable if and only if $L$ has trivial monodromy. Then operator $D$ is given by the Berest's formula [5]

$$D e^{(k, z)} = (L + k^2)^M \left( \prod_{j=1}^N ((\alpha_j, z) + c_j)^{m_j} e^{(k, z)} \right)$$

where $M = \sum_{j=1}^N m_j$. 

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The main aim of this chapter is to generalise theorem 4.1 and 4.2 to the matrix case and construct new examples of matrix D-integrable Schrödinger operators.

4.2 Trivial monodromy in many dimensions.

Let us consider the matrix Schrödinger operator

\[ L = -\Delta + U(z) \]  \hspace{1cm} (4.9)

where \( U(z) \) is a meromorphic \( d \times d \) matrix-valued function having a pole of the second order along the hyperplane \( \Pi_\alpha \) given by \( (\alpha, z) = 0 \) which is assumed to be non-isotropic: \( (\alpha, \alpha) \neq 0 \) (cf. [7, 8, 17]). We will suppose for simplicity that \( (\alpha, \alpha) = 1 \). The Laurent expansion of \( u(z) \) in the normal direction \( \alpha \) at the vicinity of \( \Pi_\alpha \) can be written in the form

\[ U(z) = \sum_{r \geq -2} C_r(\alpha, z)^r \]  \hspace{1cm} (4.10)

where \( C_r = C_r(z^\perp) \) are some analytic \( d \times d \) matrix-valued functions on the hyperplane \( \Pi_\alpha \). Let us suppose that there exists a formal solution of the Schrödinger equation

\[ L\psi(z, \lambda) = \lambda \psi(z, \lambda) \]  \hspace{1cm} (4.11)

of the form

\[ \psi(z) = (\alpha, z)^m \sum_{s \geq 0} \psi_s(\alpha, z)^s \]  \hspace{1cm} (4.12)

for some \( m \), where the coefficients \( \psi_s = \psi_s(\lambda, z^\perp) \), \( r \geq 0 \) are analytic vector-functions on \( \Pi_\alpha \). Substituting series (4.10) and (4.12) into the equation (4.11) one can see that

\[ C_{-2}\psi_0 = m(m - 1)\psi_0, \]

i.e. \( \psi_0 \) is an eigenvector of \( C_{-2} \) with the eigenvalue \( m(m - 1) \).

Definition. We say that Schrödinger operator (4.9) with the potential (4.10) has local trivial monodromy around the hyperplane \( \Pi_\alpha \) if

1) at any point of \( \Pi_\alpha \) matrix \( C_{-2} \) is diagonalizable with eigenvalues having the form \( m(m - 1), m \in \mathbb{Z}_+ \),

2) for any \( m \in \mathbb{Z} \) such that \( m(m - 1) \) is an eigenvalue of \( C_{-2} \) and for any choice of the corresponding eigenvector \( \psi_0(z^\perp) \) there exists a formal solution (4.12) of the equation (4.11) for any \( \lambda \).
So, we can choose the basis $e_1, e_2, \ldots, e_d$ depending in general on the point of the hyperplane $\Pi_\alpha$ in which matrix $C_{-2}$ has a form

$$
\begin{pmatrix}
  m_1(m_1 - 1) & & \\
  & m_2(m_2 - 1) & \\
  & & \ddots \\
  & & & m_d(m_d - 1)
\end{pmatrix}.
$$

Without loss of generality, we can assume that $1 \leq m_1 \leq m_2 \leq \ldots \leq m_d = M$ and similar to the one-dimensional case for any $m_r$, $r = 1, \ldots, d$ we get two systems of resonance equations corresponding to eigenfunctions

$$
\psi^{-r+1}(z) = (\alpha, z)^{-m_r+1} \sum_{s \geq 0} \psi^{-r+1}_s(\alpha, z)^s
$$

and

$$
\psi^r(z) = (\alpha, z)^{m_r} \sum_{s \geq 0} \psi^s(\alpha, z)^s
$$

of $C_{-2}$ with eigenvalue $m_r(m_r - 1)$. The first of them is

$$
C_{-2} e_r = m_r(m_r - 1) e_r,
$$

$$
(C_{-2} - (-m_r + k)(-m_r + k + 1))\psi^{-r+1}_k + \sum_{i=1}^{k} C_{i-2} \psi^{-r+1}_{k-i} = (E + \tilde{\Delta})\psi^{-r+1}_{k-2}, \quad k = 1, \ldots, M + m_r - 1
$$

where $\psi^{-r+1}_k = 0$ and the second one is

$$
(C_{-2} - (m_r + k - 1)(m_r + k))\psi^k + \sum_{i=1}^{k} C_{i-2} \psi^k_{k-i} = (E + \tilde{\Delta})\psi^k_{k-2}, \quad k = 1, \ldots, M - m_r.
$$

where $\tilde{\Delta}$ stands for the Laplacian restricted on the hyperplane $\Pi_\alpha$. At the same way as proposition 3.3 one can prove the following

**Lemma 4.1.** If Schrödinger operator $L$ (4.9) with the potential (4.10) has local trivial monodromy around $\Pi_\alpha$ then matrix residue $C_{-1} \equiv 0$ and

$$
\psi^{r+1}_{2k}[k] = \psi^k_{2k}[k] = E^k \frac{|e_r|}{s(m, m + 2) \ldots s(m, m + 2k)}.
$$
For the further considerations we need the following observation. In general, \( C_{-2} \) might depend on the point at the hyperplane \( \Pi_\alpha \), but this is not the case.

**Theorem 4.3.** If the operator (4.9) has a local trivial monodromy around the hyperplane \( \Pi_\alpha \) then \( C_{-2} \) is a constant matrix on \( \Pi_\alpha \).

**Proof.** We will prove that it is possible to choose eigenvectors \( e_1, e_2, \ldots, e_d \) of \( C_{-2} \) in such a way that they are constants on \( \Pi_\alpha \). We can assume without loss of generality that \( \alpha = (1, 0, \ldots, 0) \). Then the Schrödinger operator can be written in the form

\[
L = -\frac{\partial^2}{\partial z_1^2} - \tilde{\Delta} + U(z)
\]

where \( \tilde{\Delta} = \frac{\partial^2}{\partial z_2^2} + \ldots + \frac{\partial^p}{\partial z_p^2} \). Using truncated polynomials (see section 3.3) we can rewrite systems (4.13), (4.14) as follows omitting equations containing terms \( C_0 \psi_{2k+1} \):

\[
(C_{-2} - (-m_r)(-m_r + 1))\psi_0^{-r+1} = 0,
\]

\[
(C_{-2} - (-m_r + 2)(-m_r + 3))\psi_2^{-r+1} + C_0\psi_0^{-r+1} = (E + \tilde{\Delta})\psi_0^{-r+1},
\]

\[
(C_{-2} - (-m_r + 4)(-m_r + 5))\psi_4^{-r+1}[1] + C_0\psi_2^{-r+1}[1] =
\]

\[
= E\psi_2^{-r+1} + \tilde{\Delta}\psi_2^{-r+1}[1],
\]

\[
(C_{-2} - (-m_r + 6)(-m_r + 7))\psi_6^{-r+1}[2] + C_0\psi_4^{-r+1}[2] =
\]

\[
= E\psi_4^{-r+1}[1] + \tilde{\Delta}\psi_4^{-r+1}[2]
\]

and

\[
(C_{-2} - m_r(m_r - 1))\psi_5^0 = 0,
\]

\[
(C_{-2} - (m_r + 2)(m_r + 1))\psi_2^0 + C_0\psi_0^0 = (E + \tilde{\Delta})\psi_0^0,
\]

\[
(C_{-2} - (m_r + 4)(m_r + 3))\psi_4^1[1] + C_0\psi_2^1[1] = E\psi_2^1 + \tilde{\Delta}\psi_2^1[1],
\]

\[
= E\psi_4^1[1] + \tilde{\Delta}\psi_4^1[2]
\]

Let us introduce covectors \( \{f_s\} = \{f_s(z_2, \ldots, z_n)\} \) satisfying

\[
\langle f_s | e_r \rangle = \delta_{rs}.
\]

Then using (4.15) we obtain that there are the following conditions on matrix \( C_0 \)

\[
\langle f_j | C_0 | e_r \rangle \equiv \langle f_j | \tilde{\Delta} e_r \rangle, \quad j \neq r.
\]
Let us notice now that these conditions should be fulfilled for any choice of eigenvector $|e_r\rangle$ so that for any function $g(z^\perp)$, $z^\perp = (z_2, \ldots, z_n)$

$$\langle f_j | C_0 | g(z^\perp) e_r \rangle \equiv \langle f_j | \tilde{\Lambda}(g(z^\perp) e_r) \rangle, \quad j \neq r,$$

or,

$$g(z^\perp) \langle f_j | C_0 | e_r \rangle \equiv g(z^\perp) \langle f_j | \tilde{\Lambda} e_r \rangle + \langle f_j | \sum_{s=2}^{n} \partial_s g(z^\perp) \partial_s e_k \rangle, \quad j \neq r.$$

Using the obvious identity $g(z^\perp) \langle f_j | C_0 | e_r \rangle \equiv g(z^\perp) \langle f_j | \tilde{\Lambda} e_r \rangle$ we obtain

$$\sum_{s=2}^{n} \langle f_j | \partial_s g(z^\perp) \partial_s e_r \rangle \equiv 0$$

for any function $g(z^\perp)$. Choosing $g(z^\perp) = g(z_s)$ depending only on $z_s$ such that $\partial_s g(z_s) \neq 0$ on $\Pi_\alpha$ we get

$$\langle f_j | \partial_s e_r \rangle \equiv 0 \quad (4.16)$$

for all $s = 2, \ldots, n$, $j \neq r$. Let us prove now that all entries of $e_r$ can be chosen as constants. Indeed, as (4.16) holds for any choice of eigenvector $e_r = (e^1_r, e^2_r, \ldots, e^d_r)$ we can fix it so that $e^r_s$ is a constant. Then we can rewrite (4.16) as a system

$$\begin{cases}
    f^1_r \partial_s e^1_r + f^2_r \partial_s e^2_r + \ldots + f^d_r \partial_s e^d_r \equiv 0, \\
    f^1_{r-1} \partial_s e^1_r + f^2_{r-1} \partial_s e^2_r + \ldots + f^d_{r-1} \partial_s e^d_r \equiv 0, \\
    f^1_{r+1} \partial_s e^1_r + f^2_{r+1} \partial_s e^2_r + \ldots + f^d_{r+1} \partial_s e^d_r \equiv 0, \\
    \vdots \\
    f^1_d \partial_s e^1_r + f^2_d \partial_s e^2_r + \ldots + f^d_d \partial_s e^d_r \equiv 0.
\end{cases}$$

with unknown $\partial_s e^1_r, \ldots, \partial_s e^{r-1}_r, \partial_s e^{r+1}_r, \ldots, \partial_s e^d_r$. Obviously, covectors $f_1, f_2, \ldots, f_d$ are linear independent so the determinant of the system is not zero and the only solution is

$$\partial_s e^k_r = 0,$$

where $k = 1, \ldots, r - 1, r + 1, \ldots, d$. This holds for any $s = 2, \ldots, n$. As $e^r_r$ was also chosen as a constant we can conclude that $e_r, r = 1, \ldots, d$ are constant vectors on $\Pi_\alpha$ and eigenvectors of $C_{-2}$ do not depend on the point of $\Pi_\alpha$ and, therefore, $C_{-2}$ is a constant matrix. Theorem 4.3 follows.

Now similarly to the one-dimensional case one can prove the following

**Theorem 4.4.** A matrix Schrödinger operator (4.9) with a meromorphic potential (4.10) has local trivial monodromy around the hyperplane $\Pi_\alpha$ if and only if
1. $C_{-2}$ is a constant diagonalisable matrix

$$C_{-2} = \sum_{i=1}^{k} m_i (m_i + 1) P_i,$$

where $0 \leq m_1 < m_2 < \ldots < m_k = M$ are some integers, $P_i$ are commuting projectors

$$P_i P_j = \delta_{ij} P_i, \quad \sum_{i=1}^{k} P_i = I.$$

2. The coefficients $C_l$ with $l = -1, 0, \ldots, 2M - 1$ satisfy the following relations

$$P_l C_l P_j = 0 \quad (4.17)$$

if $|m_i - m_j| \geq l + 1$ or $m_i + m_j = l + 1, l + 3, \ldots, l + 2k + 1, \ldots$. In particular, $C_{-1} \equiv 0$ and $[C_0, C_{-2}] \equiv 0$.

The coefficients $\psi_0, \psi_1, \ldots, \psi_{2M-1}$ of the corresponding expansions of the vector-eigenfunctions

$$\psi = (\alpha, z)^{-M}(\psi_0 + (\alpha, z)\psi_1 + \ldots + (\alpha, z)^k\psi_k + \ldots)$$

satisfy the conditions

$$P_l \psi_l \equiv 0 \quad (4.18)$$

if $m_i + l < M$ or $m_i + l = M + 1, M + 3, \ldots, M + 2k + 1, \ldots$ and $m_i \geq l - M + 3$.

4.3 Structure of the singularities of D-integrable matrix Schrödinger operators.

Let us consider D-integrable matrix Schrödinger operator (4.9) with a meromorphic potential $U(z)$.

**Lemma 4.2.** $L$ is D-integrable if and only if it possesses the matrix eigenfunction

$$\Psi(k, z) = P(k, z)e^{(k,z)}$$

such that $P(k, z), k, z \in \mathbb{C}^n$ is polynomial in $k$ with matrix-valued coefficients which are meromorphic in $z$ and constant scalar highest term. $\Psi(k, z)$ satisfies

$$L\Psi = -k^2 \Psi.$$

**Proof.** Indeed, any term of $P(k, z)$ has a form

$$a_\sigma(z)k^\sigma$$

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where \( \sigma = (\sigma_1, \ldots, \sigma_n) \), \( \sigma_j \in \mathbb{Z}_{\geq 0} \) and \( k^\sigma = k_1^{\sigma_1}k_2^{\sigma_2} \ldots k_n^{\sigma_n} \) so that

\[
P(k, z) = \sum_{\sigma} a_\sigma(z) k^\sigma
\]

and we can define operator

\[
D = \sum_{\sigma} a_\sigma(z) \frac{\partial^{\sigma}}{\partial z^\sigma}
\]

where

\[
\frac{\partial^{\sigma}}{\partial z^\sigma} = \frac{\partial^{|\sigma|}}{\partial z_1^{\sigma_1} \ldots \partial z_n^{\sigma_n}}
\]

and \( |\sigma| = \sum_{j=1}^n \sigma_j \). Then, obviously,

\[
\Psi(k, z) = D e^{(k, z)}
\]

and if \( L \Psi(k, z) = -k^2 \Psi(k, z) \) then

\[
L D e^{(k, z)} = L \Psi(k, z) = -k^2 \Psi(k, z) = -D k^2 e^{(k, z)} = DL_0 e^{(k, z)}.
\]

Inversely, if there exists operator \( D \) intertwining \( L \) and \( L_0 = -\Delta \) then function \( \Psi(k, z) = D e^{(k, z)} \) satisfy

\[
L \Psi = L D e^{(k, z)} = DL_0 e^{(k, z)} = -k^2 D e^{(k, z)} = -k^2 \Psi.
\]

Lemma follows.

Let us now write \( P(k, z) \) as a sum of homogenous components \( P_r(k, z) \) in \( k \):

\[
P(k, z) = \sum_{r=0}^m P_r(k, z) \tag{4.19}
\]

where

\[
P_r(k, z) = \sum_{\sigma: |\sigma| = r} a_\sigma k^\sigma.
\]

Then \( L \Psi = -k^2 \Psi \) can be rewritten as the equation for \( P(k, z) \)

\[
[-\Delta - 2(k, \frac{\partial}{\partial z}) + U(z)] P(k, z) = 0.
\]

Substituting (4.19) into the last formula which have to be fulfilled for any \( k \in \mathbb{C}^n \) we get the following system of equations on homogenous components of \( P(k, z) \)

\[
\begin{cases}
-2(k, \frac{\partial}{\partial z}) P_m(k, z) = 0, \\
-2(k, \frac{\partial}{\partial z}) P_{m-1}(k, z) + (-\Delta + U(z)) P_m(k, z) = 0,
\end{cases} \tag{4.20}
\]

\[
\ldots
\]
Lemma 4.3 (Berezin’s lemma [9]). If for differential operator $D$ operator $[D, \Delta]$ has the order no larger than that of $D$ then the highest symbol of $D$ is polynomial in $z$.

Remark. Berezin’s lemma was initially proved for the differential operators with scalar coefficients but it is easy to check that it works also in the matrix case.

If operator $D$ intertwines $L$ and $L_0$ then

$$[D, \Delta] = D\Delta - \Delta D = -LD - \Delta D = -U(z)D$$

whence $D$ satisfies the conditions of Berezin’s lemma and $P_m(k, z)$ is a polynomial in $z$. Moreover, according to the first equation of the system $P(k, z)$ is a constant on each line $z + kt$:

$$\frac{d}{dt} P_m(k, z + kt) = (k, \frac{\partial}{\partial z})P_m(k, z + kt) = 0. \tag{4.21}$$

Therefore, $P_m(k, z + kt) = C(k, z)$ does not depend on $t$.

Next proposition is the first step to prove the basic result of this section that set of singularities of any $D$-integrable Schrödinger operator is a union of hyperplanes.

Proposition 4.1 ("zero-residue lemma" (cf. [7])). The restriction of $U(z)$ on any line $z + kt, t \in \mathbb{C}$ has no residues:

$$\text{Res}_t U(z + kt) = 0.$$

Proof. From the second equation of the system (4.20) changing $z$ by $z + kt$ we obtain

$$-2 \frac{d}{dt} P_{m-1}(k, z + kt) - \Delta P_m(k, z + kt) + U(z + kt)C(k, z) = 0.$$

As it was mentioned above $P_m(k, z)$ and, therefore, $\Delta P_m(k, z)$ are polynomials in $z$ and in $t$. Moreover, $P_{m-1}$ is meromorphic in $t$ so its derivative in $t$ has no residues. Therefore,

$$\text{Res}_t U(z + kt) C(k, z) = 0.$$

From the condition on the highest term of $D$ it is obvious that $\det C(k, z) \neq 0$ and we have

$$\text{Res}_t U(z + kt) = 0$$

for almost all $k \in \mathbb{C}^n$ and, therefore, for all $k \in \mathbb{C}^n$. Lemma is proved.

Let us now assume that the potential of $L$ has a form

$$U(z) = \frac{Q(z)}{r^2(z)}.$$
such that

\[ \int_0^t U(z + k\xi) d\xi \quad (4.22) \]

is a rational function of \( t \) for all \( z \) and \( k \) \((z, k \in \mathbb{C}^n)\). Let \( S = \{ z \in \mathbb{C}^n : \tau(z) = 0 \} \) be the singularity of the potential \( U(z) \) and \( R \subset S \) be any irreducible component of \( S \) which is nondegenerate in the sense that \( d\tau \neq 0 \) on \( R \). Assume that \( Q(z) \neq 0 \) on \( R \).

**Theorem 4.5** (cf. [7, 8]). *Any irreducible nondegenerate component of the singular set of the potential \( U(z) \) with the property (4.22) is a hyperplane.*

**Proof.** The arguments are very close to those in scalar case [7], [8]. Let \( z \) be any point of \( R \): \( \tau(z) = 0 \). Expanding the potential \( U(z + kt) \) as a series in \( t \) near \( z \) we get

\[ U(z + kt) = \sum_{s \geq -M} C_s(k, z)t^s \]

and condition (4.22) means that

\[ C_{-1} = 0. \quad (4.23) \]

On the other hand, we can calculate \( C_{-1}(k, z) \) using the series expansions for \( \tau(z + kt) \) and \( Q(z + kt) \). Indeed, as \( \tau(z) = 0 \)

\[ \tau(z + kt) = \tau(z) + d\tau(k)t + d^2\tau(k)t^2/2 + \ldots = d\tau(k)t + d^2\tau(k)t^2/2 + \ldots \]

where

\[ d\tau(k) = \sum_{i=1}^n \frac{\partial \tau}{\partial z_i} k_i, \quad d^2\tau(k) = \sum_{i,j=1}^n \frac{\partial^2 \tau}{\partial z_i \partial z_j} k_i k_j. \]

Also

\[ Q(z + kt) = Q(z) + dQ(k)t + \ldots \]

Therefore,

\[ \frac{Q(z)}{\tau^2(z)} = \frac{Q(z) + dQ(k)t + \ldots}{(d\tau(k)t + d^2\tau(k)t^2/2 + \ldots)(d\tau(k)t + d^2\tau(k)t^2/2 + \ldots)} = \]

\[ \frac{Q(z) + dQ(k)}{t^2} + \frac{\tau^2(k)}{(d\tau(k))^2(1 + \frac{d^2\tau(k)}{dr(k)} + \ldots)} \]

\[ \frac{Q(z) + dQ(k)}{t} + \frac{dQ(k)}{(d\tau(k))^2} \left(1 - \frac{d^2\tau(k)}{d\tau(k)} t + \ldots \right) = \]

\[ = \frac{Q(z)}{(d\tau(k))^2 t^2} \left[ \frac{dQ(k)}{(d\tau(k))^2} - Q(z) \frac{d^2\tau(k)}{(d\tau(k))^3} \right] \frac{1}{t} + \ldots \]

as \( t \to 0 \). So,

\[ C_{-1}(k, z) = \frac{dQ(k)dr(k) - Q(z)d^2\tau(k)}{(d\tau(k))^3} \]

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and condition (4.23) is equivalent to the following relation

$$dQ(k)dr(k) - Q(z)d^2r(k) = 0$$

for all $z \in R$ and $k \in \mathbb{C}^n$, or, comparing the coefficients for the corresponding $k$

$$\partial_i Q(z)\partial_j r(z) + \partial_j Q(z)\partial_i r(z) - 2Q(z)\partial_i \partial_j r(z) = 0, \quad i, j = 1, \ldots, n \quad (4.24)$$

for all $z \in R$. Obviously, the last identity is fulfilled also for any element of $Q(z)$ or their linear combination.

Let us consider now any curve $\gamma$: $z = z(s)$ in $R$ with a tangent vector $\xi = (\xi^1, \ldots, \xi^n) = \dot{z}(s) \in T_R(z(s))$ where $T_R(z(s))$ denote the tangent bundle over $R$ at the point $z(s)$ (for definitions see [26]). As $\tau(z(s)) \equiv 0$ we have

$$dr(\xi) = 0.$$ 

Let us choose now any nonzero element $q(z) = q_j(z)$ of matrix $Q(z)$ and define the functions

$$\phi_i(s) = \frac{\partial_i \tau(z(s))}{\sqrt{q(z(s))}}$$

Then using (4.24) we obtain

$$\frac{d}{ds} \phi_i(z(s)) = \frac{1}{2\sqrt{q^3}} \sum_{j=1}^{n} \xi^j(s)(2q_i \partial_j - \partial_j q_i \partial_j) =$$

$$= \frac{1}{2\sqrt{q^3}} \sum_{j=1}^{n} \xi^j(s) \partial_i q_j \partial_j \tau = \frac{\partial_i q dr(\xi)}{2\sqrt{q^3}} \equiv 0$$

and

$$\phi_i = c_i, \quad i = 1, \ldots, n$$

for some constants $c_i$ on $\gamma$. Let us consider function $F(z) = \sum_{i=1}^{n} c_i z_i$ on the curve $\gamma$

$$\frac{d}{ds} F(z) \bigg|_{\gamma} = \sum_{j=1}^{n} \partial_j F \frac{dz_j}{ds} = \sum_{j=1}^{n} c_j \frac{dz_j}{ds} = \sum_{j=1}^{n} \phi_j \frac{dz_j}{ds} = \sum_{j=1}^{n} \frac{\partial_i \tau(z(s))}{\sqrt{q(z(s))}} \frac{dz_j}{ds} =$$

$$= \frac{1}{\sqrt{q(z(s))}} \sum_{j=1}^{n} \partial_i \tau(z(s)) \frac{dz_j}{ds} = \frac{dr(\xi)}{\sqrt{q(z(s))}} \equiv 0$$

and $F(z)$ is a constant on $R$ since it is so for any curve $\gamma \in R$. Thus $R$ belongs to a hyperplane $\Pi$ defined by $F(z) = const$ and it coincides with $\Pi$ as $R$ is irreducible.

The theorem follows.

Let us assume now that the potential $U(z)$ of matrix Schrödinger operator is rational and decays at infinity. Let $L$ be D-integrable and all the singularities are
regular, i.e. \( U(z) \) has poles of the second order at most. Then the potential must have the form

\[
U(z) = \sum_{i=1}^{N} \frac{(\alpha_i, \alpha_i) A_i}{((\alpha_i, z) + c_i)^2}
\]  \( \tag{4.25} \)

due to the theorem 4.5:

**Corollary 4.2.** The regular singularities of the matrix potential of any D-integrable Schrödinger operator \( L \) are located on a union of non-isotropic hyperplanes. If such a potential is rational and decaying at infinity it should have a form (4.25).

The coefficient \((\alpha_i, \alpha_i)\) is written at the numerator of the expression (4.25) for the convenience, as this makes the matrices \( A_i \) independent on the choice of the equation of the corresponding hyperplane.

### 4.4 Matrix locus equations and D-integrability

Let us assume now that the operator \( L \) with the potential (4.25) has local trivial monodromy around all the hyperplanes \( \Pi_i : (\alpha_i, z) + c_i = 0 \). We will say in this case that \( L \) has **trivial monodromy**. The local trivial monodromy conditions (4.17) around all the hyperplanes form a highly-overdetermined algebraic system on the configuration of the hyperplanes with prescribed matrices \( A_i \). We will call this system as a **matrix locus equations**.

**Theorem 4.6.** Let \( L \) be a matrix Schrödinger operator (4.9) with a rational potential (4.25) satisfying the matrix locus equations. Then \( L \) is D-integrable.

**Proof.** From the theorem 4.4 it follows that

\[
A_i = \sum_{i=1}^{k_s} m_i^{(s)} (m_i^{(s)} + 1) P_i^{(s)}, \quad 0 \leq m_1^{(s)} < m_2^{(s)} < \ldots < m_{k_s}^{(s)} = M_s
\]

with some projectors \( P_i^{(s)} \cdot P_j^{(s)} P_i^{(s)} = \delta_{ij} P_i^{(s)} \), \( \sum_{i=1}^{k_s} P_i^{(s)} = I \).

Following the main idea of [15] let us introduce a linear space \( V \) consisting of the \( d \times d \) matrix-valued functions \( \Psi(z) \), \( z \in \mathbb{C}^n \) which satisfy the conditions:

1) \( \Psi(z) \left( \prod_{s=1}^{N} ((\alpha_s, z) + c_s)^{M_s} \right) \) is holomorphic in \( \mathbb{C}^n \);

2) columns of the coefficients of the series expansion of \( \Psi(z) \) at the vicinity of hyperplanes \((\alpha_s, z) + c_s = 0, s = 1, \ldots N \) satisfy the conditions (4.18) with \( M = M_s \).

Like it was in one-dimensional case (see lemma 3.6) matrix locus equations (4.17) imply that the space \( V \) is invariant under \( L \) (cf. [15], [37]).

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Let us consider the matrix function $\Psi_0 = \prod_{s=1}^{N} ((\alpha_s, z) + c_s)^{M_s} e^{(k,z)}I$ where $I$ is the identity matrix. Evidently, $\Psi_0 \in V$ and, therefore, all the functions

$$\Psi_i = (L+k^2)^i \Psi_0, \ i = 1, 2, \ldots$$

belong to $V$ as well. These functions have the form

$$\Psi_i = \frac{P_i(k, z) e^{(k,z)}}{\prod_{s=1}^{N} ((\alpha_s, z) + c_s)^{M_s}},$$

where $P_i(k, z)$ are some matrix polynomials in $k, z$. Since

$$P_{i+1} = \phi(-\Delta - 2(k, \frac{\partial}{\partial z}) + U(z))\phi^{-1} P_i, \ \phi = \prod_{s=1}^{N} ((\alpha_s, z) + c_s)^{M_s},$$

the degrees of $P_i$ in $z$ are decreasing with $i$. So, there exists such $j$ that $(L+k^2)\Psi_j = 0$. It is easy to see that for $M = \sum_{s=1}^{N} M_s$

$$\Psi_M = \left[(-2)^{M} M! \prod_{s=1}^{N} (\alpha_s, k)^{M_s} I + \ldots\right] e^{(k,z)} \neq 0,$$

(4.26)

where the dots mean the terms decaying while $z \to \infty$. We claim that $\Psi_{M+1} = (L+k^2)\Psi_M = 0$. Indeed, assume that this is not true. Then for some $j > M$ we have

$$\Psi_{j+1} = (L+k^2)\Psi_j = 0$$

with $\Psi_j \neq 0$. Since

$$P_{j+1} = \phi(-\Delta - 2(k, \frac{\partial}{\partial z}) + U(z))\phi^{-1} P_j = 0$$

and $P_j$ is polynomial in $k$ its highest coefficient $P_j^{(0)}$ has to satisfy the condition

$$(k, \frac{\partial}{\partial z})P_j^{(0)} = 0.$$  

One can show that this implies that $P_j^{(0)}$ must be polynomial in $z$ (see the proof of lemma 4.3 in [9]). On the other hand one can see from (4.26) that $\Psi_j$ for $j > M$ decays as $z \to \infty$. This contradiction means that $L\Psi_M = -k^2\Psi_M$. Presenting $\Psi_M$ in the form $\Psi_M = D e^{(k,z)}I$ for a proper matrix differential operator $D(z, \frac{\partial}{\partial z})$ (see proof of the lemma 4.2) we have

$$L\Psi = LD e^{(k,z)}I = -k^2 D e^{(k,z)}I = -Dk^2 e^{(k,z)}I = DL_0 e^{(k,z)}I$$

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and, therefore,

\[ LD = DL_0. \]

The theorem is proved.

**Remark.** Notice that our proof gives an explicit formula for the intertwining operator

\[ D \left( e^{(k,z) I} \right) = (L + k^2)^M \left( \prod_{s=1}^{N} (\alpha_s, z) + c_s \right)^M e^{(k,z) I}. \]

Such a formula has been discovered in the scalar case by Berest [5].

### 4.5 Two-dimensional case.

Let us consider the matrix locus configurations on the plane in the case when all the lines pass through the origin. In the scalar case essentially all such configurations have been described by Berest and Lutsenko [6] (see also [17]).

In the matrix case in the polar coordinates \((r, \phi)\) the corresponding potential \(U\) has a form

\[ U(r, \phi) = \frac{1}{r^2} V(\phi), \quad (4.27) \]

where \(V(\phi)\) is given by (3.41).

**Proposition 4.1.** Two-dimensional matrix Schrödinger operator

\[ L = -\Delta + U \quad (4.28) \]

with the potential (4.27) has trivial monodromy if and only if the same is true for the one-dimensional Schrödinger operator

\[ \mathcal{L} = -\frac{d^2}{d\phi^2} + V(\phi) \quad (4.29) \]

with trigonometric potential (3.41).

**Proof.** If we expand

\[ V(\phi) = \frac{A_j}{\sin^2(\phi - \phi_j)} + \sum_{k \neq j} \frac{A_k}{\sin^2(\phi - \phi_k)} \]

at the vicinity of \(\phi = \phi_j\) the expansion for the trigonometric case differs from that for rational one because there are some additional terms in holomorphic part of the corresponding series for \(V(\phi)\) due to Laurent expansion of \(1/\sin^2(\phi - \phi_j)\):

\[ \frac{A_j}{\sin^2(\phi - \phi_j)} = \frac{A_j^{-2}}{(\phi - \phi_j)^2} + A_j^0 + A_j^2(\phi - \phi_j)^2 + \ldots \]
It is easy calculate that
\[ A_j^{-2} = A_j, \quad A_j^0 = -\frac{1}{2} A_j, \ldots \]
and, as \( \sin^2 x \) is even function these additional coefficients appears only in expansion coefficients with even indices. Choosing the basis in which \( A_{-2} \) is diagonal we can easily check that there are no locus conditions on the diagonal elements of the coefficients with even indices in this basis, and, therefore, these additional terms do not effect conditions of trivial monodromy. Proposition follows.

**Proposition 4.2.** If the operator (4.29) is obtained by matrix Darboux transformation from the operator \( \mathcal{L}_0 = -\frac{d^2}{d\phi^2} \) then the two-dimensional Schrödinger operator (4.28) is \( D \)-integrable.

**Proof.** Indeed, from the theorem 3.8 it follows that if \( \mathcal{L} \) is obtained by MDT from \( \mathcal{L}_0 \) it has trivial monodromy. From the proposition 4.1 we can conclude that the two-dimensional operator (4.28) has trivial monodromy as well and, therefore, it is \( D \)-integrable according to theorem 4.6.

From proposition 4.1 and 4.2 we obtain the following

**Theorem 4.7.** Two-dimensional matrix Schrödinger operator \( L = -\Delta + U(r, \phi) \) with \( U(r, \phi) = \frac{1}{r^2} V(\phi) \) where \( V(\phi) \) is the potential of the one-dimensional Schrödinger operator \( \mathcal{L} \) related by matrix Darboux transformation to \( \mathcal{L}_0 = -\frac{d^2}{d\phi^2} \) has trivial monodromy and therefore \( D \)-integrable. Conversely, for any \( D \)-integrable operator \( L = -\Delta + U(r, \phi) \) the corresponding one-dimensional operator can be obtained by a matrix Darboux transformation from the operator \( \mathcal{L}_0 \).

Let us derive the explicit formula for \( D \)-integrable operators in the simplest case of three lines with prescribed \( 2 \times 2 \) matrices \( P, Q, \) and \( R \) with the eigenvalues 0 and 2. To do this we will use matrix locus equations rather than Darboux transformation. Let \( V(\phi) \) be of the form
\[
V(\phi) = \frac{P}{\sin^2(\phi - \alpha)} + \frac{Q}{\sin^2(\phi - \beta)} + \frac{R}{\sin^2(\phi - \gamma)}, \tag{4.30}
\]
where \( \phi, \alpha, \beta, \gamma \in \mathbb{C} \); According to the theorem 3.8 the Schrödinger operator \( \mathcal{L} \) has trivial monodromy if its Laurent expansion at each pole \( \phi = \phi_j \)
\[
\frac{C_{-2}}{(\phi - \phi_j)^2} + \frac{C_{-1}}{(\phi - \phi_j)} + C_0 + C_1(\phi - \phi_j) + \ldots
\]
satisfies the conditions
\[
C_{-1} = 0,
\]
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\[ [C_{-2}, C_0] = 0, \quad (4.31) \]
\[ C_{-2} C_1 C_{-2} = 0. \quad (4.32) \]

Applying this to the operator with the potential (4.30) we see that conditions \( C_{-1} = 0 \) are, obviously, fulfilled. Expanding \( V(\phi) \) near \( \phi = \alpha \)

\[ V(\phi) = \frac{P}{(\phi - \alpha)^2} + \left( \frac{Q}{\sin^2(\alpha - \beta)} + \frac{R}{\sin^2(\alpha - \gamma)} + \frac{2P}{3} \right) + \]
\[ + \left( \frac{-4Q \cos(\alpha - \beta)}{\sin^2(\alpha - \beta)} + \frac{-4R \cos(\alpha - \gamma)}{\sin^2(\alpha - \gamma)} \right) (\phi - \alpha) + \ldots, \]

and then near \( \phi = \beta \) and \( \phi = \gamma \) we get the following system of the equations (from (4.31))

\[ \begin{bmatrix} P, \frac{Q}{\sin^2(\alpha - \beta)} + \frac{R}{\sin^2(\alpha - \gamma)} \end{bmatrix} = 0. \quad (4.33) \]
\[ \begin{bmatrix} Q, \frac{P}{\sin^2(\beta - \alpha)} + \frac{R}{\sin^2(\beta - \gamma)} \end{bmatrix} = 0. \quad (4.34) \]
\[ \begin{bmatrix} R, \frac{P}{\sin^2(\gamma - \alpha)} + \frac{Q}{\sin^2(\gamma - \beta)} \end{bmatrix} = 0. \quad (4.35) \]

It is easy to see that (4.35) follows from (4.33) and (4.34). Conditions (4.32) give

\[ P \left( \frac{Q \cos(\alpha - \beta)}{\sin^2(\alpha - \beta)} + \frac{R \cos(\alpha - \gamma)}{\sin^2(\alpha - \gamma)} \right) P = 0. \quad (4.36) \]
\[ Q \left( \frac{P \cos(\beta - \alpha)}{\sin^2(\beta - \alpha)} + \frac{R \cos(\beta - \gamma)}{\sin^2(\beta - \gamma)} \right) Q = 0. \quad (4.37) \]
\[ R \left( \frac{P \cos(\gamma - \alpha)}{\sin^2(\gamma - \alpha)} + \frac{Q \cos(\gamma - \beta)}{\sin^2(\gamma - \beta)} \right) R = 0. \quad (4.38) \]

Let us assume now that

\[ P = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \]
\[ Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \]
\[ R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \]

where

\[ q_{11} + q_{22} = r_{11} + r_{22} = 2 \quad (4.39) \]

and

\[ \det P = \det Q = \det R = 0. \quad (4.40) \]
Then (4.33) gives

\[
\begin{align*}
\frac{q_{12}}{\sin^2(\alpha - \beta)} + \frac{r_{12}}{\sin^2(\alpha - \gamma)} &= 0, \\
\frac{q_{21}}{\sin^2(\alpha - \beta)} + \frac{r_{21}}{\sin^2(\alpha - \gamma)} &= 0
\end{align*}
\]  

(4.41)

and, therefore,

\[
\frac{r_{12}}{q_{12}} = \frac{r_{21}}{q_{21}} = -\frac{\sin^2(\alpha - \gamma)}{\sin^2(\alpha - \beta)}.
\]  

(4.42)

Let us introduce the new notations: \( A = -\frac{\sin^2(\alpha - \gamma)}{\sin^2(\alpha - \beta)} \), \( q = q_{12} \) and \( r = q_{21} \). Then \( Q \) and \( R \) can be written in the form

\[
Q = \begin{pmatrix} q_{11} & q \\ r & q_{22} \end{pmatrix}, \quad R = \begin{pmatrix} r_{11} & Aq \\ Ar & r_{22} \end{pmatrix}
\]

Using (4.34), (4.36) and (4.39) we can obtain the following system of equation for \( q_{22} \) and \( r_{22} \)

\[
\begin{align*}
q_{22} \sin^2(\alpha - \gamma) + r_{22} \sin^2(\alpha - \beta) &= \sin^2(\alpha - \beta) + \sin^2(\alpha - \gamma) - \sin^2(\beta - \gamma), \\
q_{22} \sin^3(\alpha - \gamma) \cos(\alpha - \beta) + r_{22} \sin^3(\alpha - \beta) \cos(\alpha - \gamma) &= 0,
\end{align*}
\]

the solution of which is

\[
q_{22} = -2 \sin^2(\alpha - \beta) \frac{\cos(\alpha - \gamma) \cos(\beta - \gamma)}{\sin(\alpha - \gamma) \sin(\beta - \gamma)}
\]

and

\[
r_{22} = 2 \sin^2(\alpha - \gamma) \frac{\cos(\alpha - \beta) \cos(\beta - \gamma)}{\sin(\alpha - \beta) \sin(\beta - \gamma)}.
\]

Then according to (4.39)

\[
q_{11} = \frac{\cos(\alpha - \beta)}{\sin(\alpha - \gamma) \sin(\beta - \gamma)} T(\alpha, \beta, \gamma)
\]

\[
r_{11} = -\frac{\cos(\alpha - \gamma)}{\sin(\alpha - \beta) \sin(\beta - \gamma)} T(\alpha, \beta, \gamma)
\]

where

\[
T(\alpha, \beta, \gamma) = \sin^2(\alpha - \beta) + \sin^2(\alpha - \gamma) + \sin^2(\beta - \gamma).
\]

It is easy to check that condition (4.37) can be reduced to the identity

\[
2 q_{22} \cos(\beta - \alpha) \frac{\cos(\beta - \alpha)}{\sin^3(\beta - \alpha)} + \frac{(q_{11} r_{11} + 2 q_{11} q_{22} A + q_{22} r_{22}) \cos(\beta - \alpha)}{\sin^3(\beta - \gamma)} = 0
\]
which can be proved using

\[
\cos^2(\alpha - \beta) + \cos^2(\alpha - \gamma) + \cos^2(\beta - \gamma) - 2 \cos(\alpha - \beta) \cos(\beta - \gamma) \cos(\beta - \gamma) = 1.
\]

Condition (4.38) can be treated analogously.

At last, as (4.40) is fulfilled

\[
q'r = q_{11}q_{22} = -2 \frac{\sin^2(\alpha - \beta)}{\sin^2(\alpha - \gamma) \sin^2(\beta - \gamma)} S(\alpha, \beta, \gamma) T(\alpha, \beta, \gamma)
\]

where

\[
S(\alpha, \beta, \gamma) = \cos(\alpha - \beta) \cos(\alpha - \gamma) \cos(\beta - \gamma)
\]

(then \(r_1r_2 = A^2q'\) is satisfied automatically) and we can define

\[
q = -r \frac{\sin(\alpha - \beta) T(\alpha, \beta, \gamma)}{\sin(\alpha - \gamma) \sin(\beta - \gamma)},
\]

\[
\tau = \frac{2}{\tau} \frac{\sin(\alpha - \beta) S(\alpha, \beta, \gamma)}{\sin(\alpha - \gamma) \sin(\beta - \gamma)}
\]

where \(\tau\) is a constant and

\[
Aq = \frac{\tau}{\sin(\alpha - \gamma) T(\alpha, \beta, \gamma)},
\]

\[
Ar = -\frac{2}{\tau} \frac{\sin(\alpha - \gamma) S(\alpha, \beta, \gamma)}{\sin(\alpha - \beta) \sin(\beta - \gamma)}
\]

As \(\tau\) can be eliminated by a suitable transformation

\[
U(z) \rightarrow C U(z) C^{-1}
\]

we can assume without any loss of generality that \(\tau = 1\) and, finally, we obtain that if

\[
P = \begin{pmatrix}
0 & 0 \\
0 & 2
\end{pmatrix}
\]

then

\[
Q = \frac{1}{\sin(\alpha - \gamma) \sin(\beta - \gamma)} \begin{pmatrix}
\cos(\alpha - \beta) T(\alpha, \beta, \gamma) & -\sin(\alpha - \beta) T(\alpha, \beta, \gamma) \\
2 \sin(\alpha - \beta) S(\alpha, \beta, \gamma) & -2 \sin^2(\alpha - \beta) \cos(\alpha - \gamma) \cos(\beta - \gamma)
\end{pmatrix}
\]

and

\[
R = \frac{1}{\sin(\alpha - \beta) \sin(\beta - \gamma)} \begin{pmatrix}
-\cos(\alpha - \gamma) T(\alpha, \beta, \gamma) & \sin(\alpha - \gamma) T(\alpha, \beta, \gamma) \\
-2 \sin(\alpha - \gamma) S(\alpha, \beta, \gamma) & 2 \sin^2(\alpha - \gamma) \cos(\alpha - \beta) \cos(\beta - \gamma)
\end{pmatrix}
\]
Now we want to find such a transformation (4.43) of (4.30) that the corresponding projectors \( P_\alpha, P_\beta \) and \( P_\gamma \) of the new operator \( \tilde{V}(\phi) \) would be in the symmetric form. It is easy to see that \( \ker P = (1,0)^T \), \( \ker Q = (\sin(\alpha - \beta), \cos(\alpha - \beta))^T \) and \( \ker R = (\sin(\alpha - \gamma), \cos(\alpha - \gamma))^T \). Let us assume that

\[
\ker P_\alpha = \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix},
\]

\[
\ker P_\beta = \begin{pmatrix} \sin \beta \\ \cos \beta \end{pmatrix}
\]

and

\[
\ker P_\gamma = \begin{pmatrix} \sin \gamma \\ \cos \gamma \end{pmatrix}.
\]

So, we are looking for the transformation \( C \):

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow p \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix}
\]

\[
C : \begin{pmatrix} \sin(\alpha - \beta) \\ \cos(\alpha - \beta) \end{pmatrix} \rightarrow q \begin{pmatrix} \sin \beta \\ \cos \beta \end{pmatrix} (4.44)
\]

\[
\begin{pmatrix} \sin(\alpha - \gamma) \\ \cos(\alpha - \gamma) \end{pmatrix} \rightarrow r \begin{pmatrix} \sin \gamma \\ \cos \gamma \end{pmatrix}
\]

where \( p, q, \) and \( r \) are constants. From the first of condition (4.44) we get that \( C \) is of the form

\[
C = \begin{pmatrix} p \sin \alpha & b \\ p \cos \alpha & d \end{pmatrix}
\]

where \( b \) and \( d \) are some unknown constants.

The second and the third conditions from (4.44) give the following system of equations

\[
\begin{cases}
 p \sin \alpha \sin(\alpha - \beta) + b \cos(\alpha - \beta) = q \sin \beta \\
 p \cos \alpha \sin(\alpha - \beta) + d \cos(\alpha - \beta) = q \cos \beta \\
 p \sin \alpha \sin(\alpha - \gamma) + b \cos(\alpha - \gamma) = r \sin \gamma \\
 p \cos \alpha \sin(\alpha - \gamma) + d \cos(\alpha - \gamma) = r \cos \gamma
\end{cases} (4.45)
\]

Expressing \( b \) from the first and the third equations of system (4.45) and \( d \) from the second and the forth ones we have that

\[
b = \frac{-p \sin \alpha \sin(\alpha - \beta) + q \sin \beta}{\cos(\alpha - \beta)} = \frac{-p \sin \alpha \sin(\alpha - \gamma) + r \sin \beta}{\cos(\alpha - \gamma)}
\]
\[ d = - \frac{p \cos \alpha \sin(\alpha - \beta)}{\cos(\alpha - \beta)} + \frac{q \cos \beta}{\cos(\alpha - \gamma)} = - \frac{p \cos \alpha \sin(\alpha - \gamma)}{\cos(\alpha - \gamma)} \]

Omitting now \( b \) and \( d \) we obtain the system

\[
\begin{aligned}
    r \sin \gamma \cos(\alpha - \beta) - q \sin \beta \cos(\alpha - \gamma) &= p \sin \alpha \sin(\beta - \gamma) \\
    r \cos \gamma \cos(\alpha - \beta) - q \cos \beta \cos(\alpha - \gamma) &= p \cos \alpha \sin(\beta - \gamma)
\end{aligned}
\]

to express \( q \) and \( r \) through \( p \). Solving it we get

\[ q = -p \tan(\alpha - \gamma) \]

\[ r = -p \tan(\alpha - \beta) \]  \hspace{1cm} (4.46)

So,

\[
\begin{aligned}
    b &= -p \frac{\sin \alpha \sin(\alpha - \beta) \cos(\alpha - \gamma) + \sin \beta \sin(\alpha - \gamma)}{\cos(\alpha - \beta) \cos(\alpha - \gamma)} = \\
    &= -p \frac{\sin \alpha \sin(\alpha - \gamma) \cos(\alpha - \beta) + \sin \gamma \sin(\alpha - \beta)}{\cos(\alpha - \beta) \cos(\alpha - \gamma)} \\
    d &= -p \frac{\cos \alpha \sin(\alpha - \beta) \cos(\alpha - \gamma) + \cos \beta \sin(\alpha - \gamma)}{\cos(\alpha - \beta) \cos(\alpha - \gamma)} = \\
    &= -p \frac{\cos \alpha \sin(\alpha - \gamma) \cos(\alpha - \beta) + \cos \gamma \sin(\alpha - \beta)}{\cos(\alpha - \beta) \cos(\alpha - \gamma)}
\end{aligned}
\]

Here we used the identities

\[ \sin \alpha \sin(\alpha - \beta) \cos(\alpha - \gamma) + \sin \beta \sin(\alpha - \gamma) = \]

\[ = \sin \alpha \sin(\alpha - \gamma) \cos(\alpha - \beta) + \sin \gamma \sin(\alpha - \beta) \]  \hspace{1cm} (4.47)

and

\[ \cos \alpha \sin(\alpha - \beta) \cos(\alpha - \gamma) + \cos \beta \sin(\alpha - \gamma) = \]

\[ = \cos \alpha \sin(\alpha - \gamma) \cos(\alpha - \beta) + \cos \gamma \sin(\alpha - \beta) \]  \hspace{1cm} (4.48)

Taking the arithmetic mean of right and left sides of identities (4.47) and (4.48) we get the expressions for \( b \) and \( d \) which are symmetric with respect to \( \beta \) and \( \gamma \)

\[
\begin{aligned}
    b &= -p \frac{\sin^2 \alpha \cos(\beta + \gamma - \alpha) - \cos \alpha \sin \beta \sin \gamma}{\cos(\alpha - \beta) \cos(\alpha - \gamma)} \\
    d &= p \frac{\cos^2 \alpha \sin(\beta + \gamma - \alpha) - \sin \alpha \cos \beta \cos \gamma}{\cos(\alpha - \beta) \cos(\alpha - \gamma)}
\end{aligned}
\]

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and, whence (since all the entries of matrix $C$ are proportional to $p$ we can assume that $p = 1$),

$$C = \begin{pmatrix}
\sin \alpha & -\sin^2 \alpha \cos(\beta + \gamma - \alpha) - \cos \alpha \sin \beta \sin \gamma \\
\cos \alpha & \cos^2 \alpha \sin(\beta + \gamma - \alpha) - \sin \alpha \cos \beta \cos \gamma \\
\cos(\alpha - \beta) & \cos(\alpha - \gamma)
\end{pmatrix}$$

As $\det C = -\cot(\alpha - \beta) \cot(\alpha - \gamma)$ we finally obtain

$$P_\alpha = \frac{2}{\sin(\alpha - \beta) \sin(\alpha - \gamma)} \xi_\alpha^T \eta_\alpha,$$

(4.49)

where

$$\xi_\alpha = (-\cos \alpha, \sin \alpha), \eta_\alpha = (s(\alpha; \beta, \gamma), c(\alpha; \beta, \gamma)),$$

$$s(\alpha; \beta, \gamma) = \sin^2 \alpha \cos(\beta + \gamma - \alpha) - \cos \alpha \sin \beta \sin \gamma,$$

$$c(\alpha; \beta, \gamma) = \cos^2 \alpha \sin(\beta + \gamma - \alpha) - \sin \alpha \cos \beta \cos \gamma.$$

Projectors $P_\beta$ and $P_\gamma$ can be obtained by corresponding permutations of $\alpha$, $\beta$ and $\gamma$.

**Theorem 4.8** (cf. proposition 3.5). *Any three lines on the plane with prescribed matrices $P_\alpha$, $P_\beta$ and $P_\gamma$ (4.49) where $\alpha$, $\beta$, $\gamma$ are the corresponding angles form a matrix locus configuration. Modulo (4.43) this describes all matrix locus configurations with prescribed $2 \times 2$ matrices having the eigenvalues $0$ and $2$ and three singular lines.*

**Remark.** It is interesting to note that the potential (4.30), (4.49) is symmetric if and only if $\alpha = \vartheta$, $\beta = \vartheta + \frac{\pi}{3}$ and $\gamma = \vartheta + \frac{2\pi}{3}$ for some $\vartheta$ which corresponds to the matrix Calogero-Moser system (see below) related to $A_2$ root system.

### 4.6 Generalised matrix Calogero-Moser operators.

In this section we want to mention some examples of D-integrable Schrödinger operators related to quantum Calogero-Moser system (for details see [18]). Let us consider the following matrix Schrödinger operators

$$L = -\Delta + \sum_{\alpha \in \mathcal{R}_+} \frac{m(\alpha I - \delta_\alpha)(\alpha, \alpha)}{(\alpha, z)^2}.$$  

(4.50)

Here $\mathcal{R}$ is any Coxeter root system [10] in $\mathbb{R}^n$, $\mathcal{R}_+$ is its positive part consisting of the normals to the reflection hyperplanes of the corresponding Coxeter group $G$, $m(\alpha) = m_\alpha$ is a $G$-invariant function on $\mathcal{A}$, $\delta_\alpha$ stands for the reflection with respect...
to } \alpha \text{ in an arbitrary matrix representation } \pi \text{ of the group } G: \hat{s}_\alpha = \pi(s_\alpha). \text{ The operators (4.50) introduced by Cherednik [19] can be considered as matrix version of quantum Calogero-Moser systems [63].}

**Theorem 4.9 ([18]).** The generalised matrix Calogero-Moser operators (4.50) with integer } G \text{-invariant } m_\alpha \text{ are } D \text{-integrable.}

The proof follows from the simple check of the matrix locus conditions and theorem 4.6.

In the scalar case the Calogero-Moser operator admits integrable deformations related to the non-Coxeter configurations of the hyperplanes [65, 16, 17]. These deformations admit a matrix generalisation as well. Let } \mathcal{A} \text{ be a finite set of the hyperplanes } \Pi_\alpha \text{ in a complex Euclidean space } C^n \text{ given by the equations } (\alpha, z) = 0, \text{ taken with some multiplicities } m_\alpha \in \mathbb{Z}_+. \text{ Here } \alpha \in \mathcal{A}, \mathcal{A} \text{ is a finite set of noncollinear vectors. Consider the following matrix Schrödinger operator [18]}

\[ L = -\Delta + U(z) \]

with

\[ U(z) = \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha(m_\alpha - s_\alpha)(\alpha, \alpha)}{(\alpha, z)^2} \]

(4.51)

where } s_\alpha \text{ is the } n \times n \text{ matrix of the reflection}

\[ s_\alpha(z) = z - 2 \frac{(\alpha, z)}{(\alpha, \alpha)} \alpha \]

**Theorem 4.10 ([18]).** Operator } L \text{ has trivial monodromy if and only if the following conditions for the configuration } \mathcal{A} \text{ hold for each } \alpha \in \mathcal{A}

\[ \sum_{\beta \neq \alpha} \frac{m_\beta(m_\beta + 1)(\beta, \beta)(\alpha, \beta)\beta^{2j-1}}{(\beta, z)^{2j+1}} \bigg|_{(\alpha, z) = 0} = 0, \quad j = 1, 2, \ldots, m_\alpha, \quad (4.52) \]

\[ \sum_{\beta \neq \alpha} \frac{m_\beta(\alpha, \beta)\beta^{2j-1}}{(\beta, z)^{2j-1}} \bigg|_{(\alpha, z) = 0} = 0, \quad j = 1, 2, \ldots, m_\alpha. \quad (4.53) \]

In particular, the conditions (4.52), (4.53) are satisfied for the following non-Coxeter configurations } A_n(m) \text{ and } C_{n+1}(m, t) \text{ discovered in [65, 16, 17].}

Configuration } A_n(m) \text{ consists of the following vectors in } \mathbb{R}^{n+1}: e_i - e_j \text{ with multiplicity } m (1 \leq i < j \leq n) \text{ and } e_i - \sqrt{m}e_{n+1} \text{ with multiplicity } 1 (i = 1, \ldots, n) \text{ (for } m = 1 \text{ we have the root system } A_n). \text{ Parameter } m \text{ is also allowed to be negative.
Then one should consider vectors $e_i - e_j$ with the multiplicity $-1 - m$. In the last case we have a complex configuration in $\mathbb{C}^{n+1}$.

Configuration $C_{n+1}(m, l)$ consists of the following set of vectors in $\mathbb{R}^{n+1}$:

$$C_{n+1}(m, l) = \begin{cases} 
  e_i \pm e_j & \text{with multiplicity } k \\
  2e_i & \text{with multiplicity } m \\
  2\sqrt{k}e_{n+1} & \text{with multiplicity } l \\
  e_i \pm \sqrt{k}e_{n+1} & \text{with multiplicity } 1 
\end{cases}$$

where $l$ and $m$ are integer parameters such that $k = \frac{2m + 1}{2l + 1} \in \mathbb{Z}$, $1 \leq i < j \leq n$. If $l = m = k = 1$ the system $C_{n+1}(m, l)$ coincides with the classical root system $C_{n+1}$. As before, the parameters $k, m, l$ may be negative, in that case the corresponding multiplicities should be $-1 - k, -1 - m$ or $-1 - l$ respectively.

**Corollary ([18]).** The matrix Schrödinger operators with potentials (4.51) corresponding to the configurations $A_n(m)$ and $C_{n+1}(m, l)$ are $D$-integrable.
Chapter 5

Multisoliton solutions of the matrix Korteweg-de Vries equation. Interaction of solitons.

It is well-known that the soliton solutions of the classical Korteweg-de Vries (KdV) equation

\[ u_t = 6uu_x - u_{xxx} \]

correspond to the potentials of the Schrödinger operators which have zero reflection coefficient (reflectionless potentials) (see e.g. [32], [61]). In this chapter we will investigate the matrix version of these potentials and their KdV dynamics.

Let us start with some facts from the spectral theory of scalar Schrödinger operator following essentially [61, 30]. Consider the Schrödinger operator

\[ L = -D^2 + u(x), \quad D = \frac{d}{dx}, \quad -\infty < x < +\infty \]

on the line with a real potential \( u(x) \in C[-\infty, +\infty] \) satisfying

\[ \int_{-\infty}^{+\infty} (1 + x^2) |u(x)| \, dx < \infty \]

and the corresponding Schrödinger equation

\[ L\psi = -\frac{d^2\psi}{dx^2} + u(x)\psi = k^2\psi. \]
Let us introduce Jost solutions \((k \in \mathbb{R})\)

\[
f(x, k) = e^{ikx}(1 + o(1)), \quad x \to +\infty,
\]

\[
f'(x, k) = e^{ikx}(ik + o(1)), \quad x \to +\infty,
\]

\[
g(x, k) = e^{-ikx}(1 + o(1)), \quad x \to -\infty,
\]

\[
g'(x, k) = e^{-ikx}(-ik + o(1)), \quad x \to -\infty,
\]

Then there exists the following relation between them:

\[
g(x, k) = f(x, -k)a(k) + f(x, k)b(k)
\]

\[
f(x, k) = g(x, -k)c(k) + g(x, k)d(k)
\]

It is possible to prove that function \(a(k)\) can be analytically continued to the upper half plane \(\mathbb{C}_+\): \(\{k : \text{Im} \ k > 0\}\) so that it is continuous in \(\{k : \text{Im} \ k \geq 0\}\) and \(a(k) \neq 0\) for \(k \in \mathbb{R}\). The number of zeros \(k_j\) of \(a(k)\) in upper half plane is finite and they are simple: \(a'(k_j) \neq 0, j = 1, \ldots, N\). Since \(u(x)\) is real numbers \(k_j\) are located at the imaginary axis and we can assume that \(k_j = i\lambda_j, 0 < \lambda_1 < \ldots < \lambda_N\). So, there is a finite number of negative simple eigenvalues \(-\lambda_j^2\) with eigenfunctions

\[
\psi_0(x) = e^{\lambda_jx}(1 + o(1)), \quad x \to -\infty,
\]

\[
\psi_0(x) = e^{-\lambda_jx}(b_j + o(1)), \quad x \to +\infty.
\]

such that \(\psi_j(x) \in L_2[-\infty, +\infty]\) (see, e.g. [44]). Reflection coefficient is defined for real \(k\) as

\[
r(k) = \frac{b(k)}{a(k)}.
\]

The set \(s = \{k_j = i\lambda_j, b_j, r(k)\}\) are scattering data of Schrödinger operator \(L\). If

\[
H(x) = \sum_{j=1}^{N} \frac{b_j e^{-\lambda_j x}}{ia'(i\lambda_j)} + \frac{1}{2\pi} \int_{\mathbb{R}} r(k) e^{ikx}dk
\]

then the Gelfand-Levitan-Marchenko equation has the form

\[
K(x, y) + H(x + y) + \int_{x}^{+\infty} K(x, z)H(y + z)dz = 0
\]

and

\[
u(x) = -2K'(x), \quad K(x) = K(x, x).
\]
Let us now assume that potential of \( L \) depends on the additional parameter \( t \):
\[
u = u(x, t)
\]
according to KdV equation
\[
u_t = 6\nu \nu_x - \nu_{xxx}.
\]
A remarkable property of this equation is that it can be written in the Lax form
\[
L_t = [L, A], \quad [L, A] = LA - AL
\]
where
\[
L = -D^2 + u(x, t), \quad A = -4D^3 + 3(\nu D + Du).
\]
Using this representation one can show that the evolution of the spectral data is [32]
\[
r(k, t) = r(k, 0)e^{8i\lambda^2 t}, \quad \lambda_j(t) = \lambda_j(0), \quad b_j(t) = b_j(0)e^{8\lambda_j^2 t}.
\]
If reflection coefficient is identically equal to 0: \( r(k, t) = r(k, 0) \equiv 0 \), and Schrödinger operator has just one discrete eigenvalue \(-\lambda^2\) then
\[
u(x, t) = -\frac{2\lambda^2}{\cosh^2 \lambda(x - 4\lambda^2 t - \phi)}, \quad \phi = \frac{1}{2\lambda} \ln b.
\]
If the number of discrete levels is \( N \) then
\[
H(x, t) = \sum_{j=1}^{N} \beta_j e^{-\lambda_j x}, \quad \beta_j = \frac{b_j e^{8\lambda_j^2 t}}{i\alpha'(i\lambda_j)}
\]
and solution of the GLM equations is
\[
u(x, t) = -2 \frac{d^2}{dx^2} \ln \det A(x, t)
\]
where
\[
A_{mn} = \delta_{mn} + \frac{\beta_m}{\lambda_m + \lambda_n} e^{-(\lambda_m + \lambda_n)x}.
\]
It is remarkable that \( \nu(x, t) \) can be interpreted as KdV solution describing the interaction of \( N \) one-solitons and solitons interact in such a way that the only change after the interaction is a change of their phases. Discrete eigenvalues of the Schrödinger operator define the speed \( 4\lambda^2_j \) of the \( j \)-th soliton and if \( \phi_j^- \) and \( \phi_j^+ \) are its phases at \( t \to \pm \infty \) then the phase-shift \( \Delta_j \) of the \( j \)-th soliton is [22, 61]
\[
\Delta_k = \phi_k^+-\phi_k^- = \frac{1}{2\lambda_k} \sum_{n=1}^{k-1} \ln \left(\frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n}\right)^2 - \frac{1}{2\lambda_k} \sum_{n=k+1}^{N} \ln \left(\frac{\lambda_k + \lambda_n}{\lambda_n - \lambda_k}\right)^2.
\]
If \( N = 2 \)
\[
\Delta_1 = \phi_1^+-\phi_1^- = -\frac{1}{\lambda_1} \ln \left(\frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1}\right) < 0
\]
\[
\Delta_2 = \phi_2^+-\phi_2^- = \frac{1}{\lambda_2} \ln \left(\frac{\lambda_1 + \lambda_2}{\lambda_2 - \lambda_1}\right) > 0
\]
(5.1)
Remark. As we can conclude from the last formulae the slow soliton shifts back on $\Delta_1$ after the interaction and the fast one shifts forward on $\Delta_2$. The most intensive interaction is between solitons with similar velocities. If $\Delta \lambda = \lambda_2 - \lambda_1 \ll \lambda$ then [61]

$$\Delta_1 = \frac{1}{\lambda} \ln \frac{2\lambda}{\Delta \lambda} \to \infty$$

as $\Delta \lambda \to 0$. We will see below that two matrix solitons can interact in a different way.

Formulae for multisoliton solutions of KdV equation can be also obtained using the theory of Darboux transformation (see e.g. [55, 58]). Consider the equations

$$\psi_j'' = \lambda_j^2 \psi_j, \quad \psi_j = -4\psi_j'', \quad j = 1, \ldots, N.$$ 

Obviously,

$$\psi_j = a_j e^{\lambda_j x - 4\lambda_j t} + b_j e^{-\lambda_j x + 4\lambda_j t}.$$ 

Then the solution of the KdV equation can be written in the form

$$u(x,t) = -2 \frac{d^2}{dx^2} \ln W[\psi_1, \ldots, \psi_N]$$

where $W[\psi_1, \ldots, \psi_N]$ is the Wronskian of the functions $\psi_1, \ldots, \psi_N$. The KdV solution $u(x,t)$ is nonsingular if signs of the products $a_j b_j$, $j = 1, \ldots, n$ alternate (see [58])

$$a_1 b_1 > 0, \quad a_2 b_2 < 0, \quad a_3 b_3 > 0, \ldots.$$ 

In this chapter we investigate the problem of the interaction of solitons for the matrix KdV equation

$$U_t = 3UU_x + 3U_xU - U_{xxx} \quad (5.2)$$

where $U(x,t)$ is a $d \times d$ matrix.

In the first section we remind some facts from the spectral theory of matrix Schrödinger operator following essentially [54, 62] and explain how to construct the spectral data for $L$ and derive the Gelfand-Levitan-Marchenko (GLM) equations.

In section 2 we show how to solve the GLM equation under the assumption that reflection coefficient is equal to zero. Then we obtain the general formula for the multisoliton solution of the matrix KdV equation (in terms of quasideterminants) and find the evolution of spectral data for the Schrödinger operator $L = -D^2 + U(x,t)$ if $U(x,t)$ is a solution of matrix KdV. Using the results of chapter 2 and [55, 29] we also get the formula for the multisoliton solution via matrix Darboux transformations.
In section 3 we derive the formulae of the phase-shift for the 2-solitons solutions of the matrix KdV equation. It turns out that in matrix case the situation is reacher than in the scalar one. First of all, the matrix amplitude may change. Second, the sign of the phase-shift can be different from the scalar case, i.e. slow soliton can move forward and fast one can move back after the interaction.

In section 4 we study the relation between two approaches (Darboux transformation and Gelfand-Levitan-Marchenko equation) to construct multisoliton solutions of the matrix KdV equation.

5.1 Gelfand-Levitan-Marchenko equations for the matrix Schrödinger operator.

Let us consider the Schrödinger operator

\[ L = -D^2 + U(x), \quad D = \frac{d}{dx} \]

with a \( d \times d \) matrix-valued potential \( U(x) \) (complex in general) satisfying

\[ \int_{-\infty}^{+\infty} (1 + x^2) |U(x)| \, dx < \infty \]

where \( |X| = \max_j \sum_k |X_{jk}| \) for a given matrix \( X \).

Following [54, 62] introduce two eigenvalue problems:

\[ -\Psi'' + U(x) \Psi = k^2 \Psi, \quad (5.4) \]

\[ -\Phi'' + \Phi U(x) = k^2 \Phi \quad (5.5) \]

where \( \Phi = \Phi(x, k) \) and \( \Psi = \Psi(x, k) \) are \( d \times d \) matrix-valued functions. We can define Jost solutions \( F_\pm \) of (5.4) and \( G_\pm \) of (5.5) such that

\[ F_\pm(x, k) = e^{\pm ikx} (I + o(1)), \quad x \to \pm \infty, \]

\[ G_\pm(x, k) = e^{\pm ikx} (I + o(1)), \quad x \to \pm \infty. \]

**Lemma 5.1** ([54]). The Jost solutions \( F_\pm(x, k) \) and \( G_\pm(x, k) \) exist for all \( k \): \( \text{Im} k \geq 0 \) and as functions of \( k \) are analytic when \( \text{Im} k > 0 \) and continious when \( \text{Im} k \geq 0 \). Moreover, they satisfy the conditions

\[ |e^{\mp ikx} F_\pm(x, k)| = 1 + O \left( \frac{1}{|k|} \right), \quad |k| \to \infty, \]

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\[ |e^{\mp ikz}G_\pm(x, k)| = 1 + O\left(\frac{1}{|k|}\right), \quad |k| \to \infty. \]

It is easy to check that the Wronskian of any solutions \( \Phi \) and \( \Psi \) of (5.4) and (5.5)

\[ W[\Psi, \Phi] = \Psi \Phi' - \Psi' \Phi \]

is independent of \( x \) and, therefore, there are the following relations between \( F_\pm(x, k) \) and \( G_\pm(x, k) \)

\[ F_-(x, k) = F_+(x, -k)A(k) + F_+(x, k)B(k), \]
\[ F_+(x, k) = F_+(x, -k)C(k) + F_-(x, k)D(k), \]
\[ G_-(x, k) = A(k)G_+(x, -k) - B(-k)G_+(x, k), \]
\[ G_+(x, k) = C(k)G_-(x, -k) - D(-k)G_-(x, k) \]

and we can find using \((W[\Psi, \Phi])' = 0\) that

\[ A(k) = -\frac{1}{2ik}W[G_+(k), F_-(k)], \]
\[ C(k) = \frac{1}{2ik}W[G_-(k), F_+(k)], \]
\[ B(k) = \frac{1}{2ik}W[G_+(k), F_-(k)], \]
\[ D(k) = -\frac{1}{2ik}W[G_-(k), F_+(k)]. \]

Function

\[ R(k) = B(k)A^{-1}(k). \]

is the matrix analogue of the reflection coefficient. From lemma 5.1 and formulae for \( A(k) \) and \( C(k) \) it is easy to get the following

**Lemma 5.2 ([54]).** The matrix functions \( A(k) \) and \( C(k) \) are analytic in \( \text{Im} \ k > 0 \) and continuous in \( \text{Im} \ k \geq 0 \). Moreover, they satisfy

\[ |A(k) - I| = O\left(\frac{1}{|k|}\right) \]
\[ |C(k) - I| = O\left(\frac{1}{|k|}\right) \]

as \( |k| \to \infty. \)

Let us consider the equation

\[ \det A(k) = 0 \]
which define the poles of $A^{-1}(k)$ (as it was proved in [54] the equation $\det A(k) = 0$ coincide with $\det C(k) = 0$). We assume that

1) they are simple;
2) $A(k)$ is nondegenerate on the real axis.

**Remark.** If $U(x)$ is a hermitian matrix it is possible to prove that conditions 1), 2) are fulfilled automatically (see [67]). If $U(x)$ is nonhermitian these assumptions are essential as $A^{-1}(k)$ can have poles of any order (see [62] where the case of a double pole was investigated).

**Lemma 5.3** ([62]). *If conditions 1), 2) are satisfied for Schrödinger operator $L$ then the number of poles of $A^{-1}(k)$ is finite.*

Let $k_j, j = 1, \ldots, n$ denote the solutions of the equation (5.7) in the upper half plane. The following lemma shows that $k_j, j = 1, \ldots, n$ correspond to the discrete spectrum of the operator (5.3) with vector-eigenfunctions belonging to $L_2(-\infty, \infty)$, that is, they are square integrable on the real line.

**Lemma 5.4** (cf. [62]). *Let $N_j = \text{Res}_{k=k_j} A^{-1}(k), j = 1, \ldots, n$. Then there exist matrices $R_j$ such that*

$$F_-(x, k) N_j = i F_+(x, k) R_j$$

(5.8)

**Proof.** At the vicinity of $k_j$

$$A(k) = A(k_j) + A'(k_j)(k - k_j) + \ldots,$$

$$A^{-1}(k) = \frac{N_j}{k - k_j} + \ldots.$$  

Since $A(k) A^{-1}(k) = A^{-1}(k) A(k) = I$ we have

$$A(k_j) N_j = N_j A(k_j) = 0.$$  

From (5.6)

$$A(k_j) = -\frac{1}{2ik} W[G_+(k_j), F_-(k_j)]$$  

and we obtain

$$W[G_+(k_j), F_-(k_j) N_j] \equiv 0.$$  

On the other hand it is easy to check that

$$W[G_+(k_j), F_+(k_j)] \equiv 0$$  

and the last two identities give the existence of matrices $R_j$ such that

$$F_-(k_j) N_j = i F_+(k_j) R_j.$$  

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Lemma follows.

Notice that in the scalar case $d = 1$

$$R_j = \frac{b_j}{2 \alpha'(i \lambda_j)}$$

So far we discussed the direct problem for matrix Schrödinger operator (5.3). We introduced reflection coefficient $R(k)$, found the discrete spectrum of $L$ and constructed the corresponding spectral matrices $R_j$. It turns out that the potential of (5.3) can be reconstructed from the set $\{k_j, R_j, R(k)\}$ which is called the scattering data for operator $L$. If one introduces the matrix function

$$H(x) = \sum_{j=1}^{n} R_j e^{ik_j x} + \frac{1}{2\pi} \int_{x}^{+\infty} dk R(k) e^{ikx}.$$

Then the GLM equation has the form (cf. [62])

$$K(x, y) + H(x + y) + \int_{x}^{+\infty} K(x, z) H(y + z) dz = 0 \quad (5.9)$$

and

$$U(x) = -2K'(x)$$

where $K(x) = K(x, x)$.

Further we will study the conditions of symmetry of the Schrödinger operator and we shall need the following

Proposition 5.1 ([2]). If $H(x)$ is symmetric then $K(x)$ (and $U(x)$) are symmetric matrices as well.

Proof. If $H(x)$ is symmetric from (5.9) we obtain

$$K(x, x) = -H(2x) - \int_{x}^{+\infty} K(x, z) H(y + z) dz$$

and

$$K'(x, x) = -H'(2x) - \int_{x}^{+\infty} H(y + z) K'(x, z) dz.$$ 

Let us prove that the integrals in the right-hand sides of both formulae coincide. Indeed, according (5.9)

$$H(x + y) = -K(x, y) - \int_{x}^{+\infty} K(x, z) H(y + z) dz =$$

$$=-K'(x, y) - \int_{x}^{+\infty} H(y + z) K'(x, z) dz. \quad (5.10)$$
since $H$ is symmetric. Multiplying (5.10) by $K(x, z)$ from the left and integrating we get

$$\int_{-\infty}^{+\infty} K(x, z)H(x + z)dz = - \int_{-\infty}^{+\infty} K(x, z)K^t(x, z)dz - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K(x, z)H(y + z)K^t(x, y)dydz.$$  

On the other hand, multiplying (5.10) by $K^t(x, z)$ from the right and integrating we have

$$\int_{-\infty}^{+\infty} H(x + z)K^t(x, z)dz = - \int_{-\infty}^{+\infty} K(x, z)K^t(x, z)dz - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K(x, z)H(y + z)K^t(x, y)dydz.$$  

and as the right-hand sides of both formulae coincide proposition follows.

### 5.2 Matrix KdV equation and reflectionless potentials.

Let us assume now that $U = U(x, t)$ is a solution of matrix KdV equation (5.2). It is easy to check that it admits the Lax representation (see [67])

$$\dot{L} = [L, A]$$  

(5.11)

where

$$L = -D^2 + U(x, t), \quad A = 4D^3 - 3(U(x, t)D + DU(x, t)).$$

Now we want to find the evolution of spectral data $R(k, t), k_j(t), R_j(t)$ of the Schrödinger operator with the potential $U(x, t)$ (cf. [32, 61]). Let us consider Jost solution $G(x, k, t) = F_-(x, k, t)$ which now also depends on $t$.

**Notation.** For convenience further we will use the notation $G(x, k, t)$ for $F_-(x, k, t)$ and $F(x, k, t)$ for $F_+(x, k, t)$ so that

$$F(x, k, t) = e^{ikx}(I + o(1)), \quad x \to +\infty$$

$$G(x, k, t) = e^{-ikx}(I + o(1)), \quad x \to -\infty$$
As \( LG = k^2 G \)

\[ \dot{L}G + LG = k^2 \dot{G} \]

and

\[ LAG - ALG + L\dot{G} = k^2 \dot{G}, \quad L(\dot{G} + AG) = k^2(\dot{G} + AG), \]

so that \( \dot{G} = \dot{G} + AG \) is an eigenfunction of \( L \) with the same eigenvalue \( k^2 \). As the asymptotics of \( G \) does not depend on time and \( U(x, t) \to 0 \) as \( x \to \pm\infty \) we have that

\[ \dot{G} = 4i k^3 e^{-ikx}(I + o(1)), x \to -\infty \]

and, therefore, \( \dot{G} = 4i k^3 G \), or

\[ \dot{G} = -AG + 4ik^3 G. \] (5.12)

We know that \( G(x, k, t) = F(x, -k, t)A(k, t) + F(x, k, t)B(k, t) \). If \( x \to +\infty \) this identity becomes

\[ G(x, k, t) = e^{-ikx}A(k, t) + e^{ikx}B(k, t), x \to +\infty. \]

Substituting this into (5.12) and taking the limit \( x \to +\infty \) we obtain

\[ \dot{A}(k, t)e^{-ikx} + \dot{B}(k, t)e^{ikx} = (-4D^3 + 4ik^3 I)(A(k, t)e^{-ikx} + B(k, t)e^{ikx} \]

or,

\[ \dot{A}(k, t)e^{-ikx} + \dot{B}(k, t)e^{ikx} = 8ik^3 B(k, t)e^{ikx} \]

which means that

\[ \dot{A}(k, t) \equiv 0, \quad \dot{B}(k, t) = 8ik^3 B(k, t). \]

As bound states of Schrödinger operator are defined by the equation (5.7) we get

\[ \text{Theorem 5.1 (cf. [14])}. \quad \text{Matrix KdV flow preserves the discrete spectrum of the corresponding matrix Schrödinger operator and the evolution of the spectral data is} \]

\[ A(k, t) = A(k, 0), \quad R_j(t) = R_j(0)e^{8ik^3 t}, j = 1, \ldots, n, \] (5.13)

\[ R(k, t) = R(k, 0)e^{8ik^3 t}. \]

From the proposition 5.1 we obtain

\[ \text{Corollary 5.1}. \quad \text{If initial condition} \ U(x, 0) \ \text{for the matrix KdV equation is symmetric matrix then} \ U(x, t) = UT(x, t) \ \text{for any} \ t. \]
Let us consider now the case $R(k) = 0$ (reflectionless potentials) and suppose that all $k_j : A(k_j) = 0$ are located on the imaginary axis so that $k_j = i\lambda_j$, $0 < \lambda_1 < \ldots < \lambda_n$. In this case

$$H(x) = \sum_{j=1}^{n} R_j e^{-\lambda_j x}$$

and (5.9) becomes

$$K(x, y) + \sum_{j=1}^{n} R_j e^{-\lambda_j (x+y)} + \sum_{j=1}^{n} \int_{x}^{\infty} K(x, z) R_j e^{-\lambda_j (y+z)} dz = 0.$$ 

Assuming that $K(x, y)$ has a form

$$K(x, y) = \sum_{j=1}^{n} K_j(x) e^{-\lambda_j y}$$ (5.14)

integrating and separating the terms with the same $e^{-\lambda_j y}$, $j = 1, \ldots, n$ we get the following linear system of equations

$$\sum_{i=1}^{n} K_i A_{ij} = -R_j e^{-\lambda_i x}, \quad j = 1, \ldots, n$$ (5.15)

where

$$A_{ij} = \delta_{ij} I + \frac{R_j}{\lambda_i + \lambda_j} e^{-(\lambda_i + \lambda_j) x}$$ (5.16)

are $d \times d$ matrices. Let $A$ denote the $n \times n$ matrix the $ij$-th element of which is $d \times d$ matrix $A_{ij}$ (so, $A$ is $nd \times nd$ matrix). Using quasideterminants to write down the solution of (5.15) (see theorem 2.1) and then substituting it to the formula (5.14) we have

$$K(x) = K(x, x) = \sum_{j=1}^{n} |A^{(j)}|_{jj} |A|^{-1}_{jj}$$

where $A^{(j)}$ are obtained from $A$ by differentiating $j$-th row and $|A|_{jj}$ is the $jj$-quasideterminant of $A$. Finally,

$$U(x) = -2K'(x).$$

Summarizing the discussion we obtain the following

**Theorem 5.2** (cf. [29]). *Matrix KdV equation possesses multisoliton solutions*

$$U(x, t) = -2 \left( \sum_{j=1}^{n} |A^{(j)}|_{jj} |A|^{-1}_{jj} \right)'$$ (5.17)

where $A$ is $n \times n$ matrix the $ij$-th entry of which is given by (5.16) and $R_j(t) = R_j(0) e^{8\lambda_j t}$.

Further we consider examples of real 1-soliton and 2-soliton solutions of matrix KdV equation in the case $d = 2$. 

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5.3 1-soliton solution of matrix KdV.

1-soliton solution of matrix KdV equation corresponds to the Schrödinger operator with one discrete eigenvalue \( \lambda \). Let us assume that the corresponding spectral matrix \( R = p \otimes q \) has rank 1 where \( p = (p^1, p^2) \) and \( q = (q^1, q^2) \) are vectors with the scalar product \( (p, q) \neq 0 \). Using (5.13) equation (5.15) (for \( n = 1 \)) can be written in the form

\[
K(x, x) \left( I e^{2\lambda \theta} + \frac{R}{2\lambda} \right) = -R,
\]

where \( \theta = x - 4\lambda^2 t \).

Lemma 5.5. For any 2-vectors \( a = (a_1, a_2) \), \( b = (b_1, b_2) \), \( t = (t_1, t_2) \) and parameters \( \lambda \), \( \theta \)

\[
a \otimes b \cdot (e^{2\lambda \theta} I + t \otimes b)^{-1} = \frac{a \otimes b}{e^{2\lambda \theta} + (t, b)}
\]

Proof. Indeed,

\[
\det \begin{pmatrix}
e^{2\lambda \theta} + t_1 b_1 & t_1 b_2 \\
t_2 b_1 & e^{2\lambda \theta} + t_2 b_2
\end{pmatrix} = e^{2\lambda \theta}(e^{3\lambda \theta} + (p, q))
\]

and

\[
(e^{2\lambda \theta} I + t \otimes b)^{-1} = \frac{1}{e^{2\lambda \theta} + (t, b)}(I + b^\perp \otimes t^\perp e^{-2\lambda \theta})
\]

where \( b^\perp = (b_2, -b_1) \) and \( t^\perp = (t_2, -t_1) \). Whence

\[
a \otimes b \cdot (e^{2\lambda \theta} I + t \otimes b)^{-1} = \frac{a \otimes b}{e^{2\lambda \theta} + (t, b)}(I + b^\perp \otimes t^\perp e^{-2\lambda \theta}) = \frac{a \otimes b}{e^{2\lambda \theta} + (t, b)}.
\]

Lemma is proved.

Then it follows that

\[
K(x, x) = -\frac{R}{e^{2\lambda \theta} + \frac{(p, q)}{2\lambda}}.
\]

For any numbers \( \beta_1, \beta_2 \) and \( \xi \)

\[
e^{2\xi} \left( \beta_1 e^{2\xi} + \beta_2 \right)^2 = \left( \frac{1}{\beta_1 e^{\xi} + \beta_2 e^{-\xi}} \right)^2 = \frac{1}{\sqrt{\beta_1} e^{\xi} + \sqrt{\beta_2} e^{-\xi}}^2 = \frac{1}{\beta_1 \beta_2} \left( e^{\xi} - \frac{1}{2} \ln \frac{\beta_2}{\beta_1} + e^{-(\xi - \frac{1}{2} \ln \frac{\beta_2}{\beta_1})} \right)^2 = \frac{1}{4\beta_1 \beta_2} \text{sech}^2 \left( \xi - \frac{1}{2} \ln \frac{\beta_2}{\beta_1} \right).
\]

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Therefore, 1-soliton solution of matrix KdV equation can be written in the form

\[ U(x, t) = -2K'(x, x) = -2\lambda^2 p \otimes q \frac{\text{sech}^2 \lambda(\theta - \phi)}{(p, q)} \], \quad \phi = \frac{1}{2\lambda} \ln \left( \frac{p_1 q_1}{2\lambda} \right).

We will call \( \frac{2\lambda^2 p \otimes q}{(p, q)} = 2\lambda^2 \frac{R}{\text{tr} R} \) matrix amplitude of the soliton and \( \phi \) - its phase.

### 5.4 2-soliton solution of matrix KdV. Interaction of two solitons.

In this section we investigate 2-soliton solution of matrix KdV equation (5.2). The corresponding Schrödinger operator has two discrete eigenvalues \( -\lambda_1^2 \) and \( -\lambda_2^2 \). We assume that the corresponding spectral matrices are \( R_j = p_j \otimes q_j, \ j = 1, 2 \) where \( p_j = (p_j^1, p_j^2) \) and \( q_j = (q_j^1, q_j^2) \) are vectors, and the scalar products \( (p_j, q_j) \neq 0, \ j = 1, 2 \).

Using (5.13) equations (5.15) can be written in the form

\[
\begin{align*}
K_1 + \frac{1}{2\lambda_1^2} K_1 R_1 e^{-2\lambda_1 x + 8\lambda_1^2 t} + \frac{1}{\lambda_1 + \lambda_2} K_2 R_1 e^{-(\lambda_1 + \lambda_2) x + 8\lambda_1^2 t} &= -R_1 e^{-\lambda_1 x + 8\lambda_1^2 t}, \\
K_2 + \frac{1}{\lambda_1 + \lambda_2} K_1 R_2 e^{-(\lambda_1 + \lambda_2) x + 8\lambda_2^2 t} + \frac{1}{2\lambda_2^2} K_2 R_2 e^{-2\lambda_2 x + 8\lambda_2^2 t} &= -R_2 e^{-\lambda_2 x + 8\lambda_2^2 t}.
\end{align*}
\]

Let us multiply the first equation by \( e^{-\lambda_1 x} \) and the second one - by \( e^{-\lambda_2 x} \). Using the definition of \( \theta_j \) we get

\[
\begin{align*}
K_1 e^{-\lambda_1 x} \left( I + \frac{R_1 e^{-2\lambda_1 \theta_1}}{2\lambda_1} \right) + K_2 e^{-\lambda_2 x} \frac{R_1 e^{-2\lambda_1 \theta_1}}{\lambda_1 + \lambda_2} &= -R_1 e^{-2\lambda_1 \theta_1}, \\
K_1 e^{-\lambda_1 x} \frac{R_2 e^{-2\lambda_2 \theta_2}}{\lambda_1 + \lambda_2} + K_2 e^{-\lambda_2 x} \left( I + \frac{R_2 e^{-2\lambda_2 \theta_2}}{2\lambda_2} \right) &= -R_2 e^{-2\lambda_2 \theta_2}.
\end{align*}
\]

Introducing \( L_j = K_j e^{-\lambda_j x} \) such that according to (5.14)

\[ K(x, x) = L_1(x) + L_2(x) \]

we obtain that \( L_j \) satisfy the following system

\[
\begin{align*}
L_1 e^{2\lambda_1 \theta_1} + L_1 \frac{R_1}{2\lambda_1} + L_2 \frac{R_1}{\lambda_1 + \lambda_2} &= -R_1, \\
L_2 e^{2\lambda_2 \theta_2} + L_1 \frac{R_2}{\lambda_1 + \lambda_2} + L_2 \frac{R_2}{2\lambda_2} &= -R_2.
\end{align*}
\]

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Using theorem 2.1 we can write the solution of this system as follows:

\[
L_1 = - \left( I - \frac{R_2}{\lambda_1 + \lambda_2} \left( e^{2\lambda_1 \theta_1} I + \frac{R_2}{2\lambda_2} \right)^{-1} \right) R_1.
\]

\[
\cdot \left( e^{2\lambda_1 \theta_1} I + \frac{R_1}{2\lambda_1} - \frac{R_2}{\lambda_1 + \lambda_2} \left( e^{2\lambda_2 \theta_2} I + \frac{R_2}{2\lambda_2} \right)^{-1} \frac{R_1}{\lambda_1 + \lambda_2} \right)^{-1}
\]

and

\[
L_2 = - \left( I - \frac{R_1}{\lambda_1 + \lambda_2} \left( e^{2\lambda_2 \theta_1} I + \frac{R_1}{2\lambda_1} \right)^{-1} \right) R_2.
\]

\[
\cdot \left( e^{2\lambda_2 \theta_2} I + \frac{R_2}{2\lambda_2} - \frac{R_1}{\lambda_1 + \lambda_2} \left( e^{2\lambda_1 \theta_1} I + \frac{R_1}{2\lambda_1} \right)^{-1} \frac{R_2}{\lambda_1 + \lambda_2} \right)^{-1}
\]

From lemma 5.5 it follows that

\[
R_j \left( e^{2\lambda_j \theta_j} I + \frac{R_j}{2\lambda_j} \right)^{-1} = \frac{R_j}{e^{2\lambda_j \theta_j} + \frac{(p_j, q_j)}{2\lambda_j}} \quad (5.18)
\]

and

\[
L_1 = - \left( I - \frac{R_2}{(\lambda_1 + \lambda_2)(e^{2\lambda_2 \theta_2} + \frac{(p_2, q_2)}{2\lambda_2})} \right) R_1.
\]

\[
\cdot \left( e^{2\lambda_1 \theta_1} I + \frac{R_1}{2\lambda_1} - \frac{R_2 R_1}{(\lambda_1 + \lambda_2)^2(e^{2\lambda_2 \theta_2} + \frac{(p_2, q_2)}{2\lambda_2})} \right)^{-1}
\]

\[= \left( \frac{R_2}{g_2} - I \right) R_1 \left( e^{2\lambda_1 \theta_1} I + \left( \frac{I}{2\lambda_1} - \frac{R_2}{(\lambda_1 + \lambda_2)g_2} \right) R_1 \right)^{-1}
\]

where

\[g_j = (\lambda_1 + \lambda_2) \left( e^{2\lambda_j \theta_j} + \frac{(p_j, q_j)}{2\lambda_j} \right), \quad j = 1, 2.
\]

Let us notice that

\[R_1 \left( e^{2\lambda_1 \theta_1} I + \left( \frac{I}{2\lambda_1} - \frac{R_2}{(\lambda_1 + \lambda_2)g_2} \right) R_1 \right)^{-1} =
\]

\[= p_1 \otimes q_1 \left( e^{2\lambda_1 \theta_1} I + \left( \frac{I}{2\lambda_1} - \frac{R_2 \otimes q_2}{(\lambda_1 + \lambda_2)g_2} p_1 \otimes q_1 \right) \right)^{-1} =
\]

\[= p_1 \otimes q_1 \left( e^{2\lambda_1 \theta_1} I + \left( \frac{p_1 I}{2\lambda_1} - \frac{(p_1, q_2)}{(\lambda_1 + \lambda_2)g_2} \right) \otimes q_1 \right)^{-1}.
\]
Applying lemma 5.5 again with 
\[ t = \left( \frac{p_1 I}{2 \lambda_1} - \frac{(p_1, q_2)}{(\lambda_1 + \lambda_2)g_2} p_2 \right) \]
we obtain that

\[ R_1 \left( e^{2 \lambda_1 \theta_1} I + \left( \frac{I}{2 \lambda_1} - \frac{R_2}{(\lambda_1 + \lambda_2)g_2} \right) R_1 \right)^{-1} = \]

\[ \left( e^{2 \lambda_1 \theta_1} + \frac{p_1 \otimes q_1}{2 \lambda_1} - \frac{(p_1, q_2)(p_2, q_1)}{(\lambda_1 + \lambda_2)g_2} \right) = \frac{(\lambda_1 + \lambda_2)g_2 R_3}{g_1 g_2 - (p_1, q_2)(p_2, q_1)} \]

and

\[ L_1 = \frac{\lambda_1 + \lambda_2}{g} (R_2 - g_2 I) R_1 \]

where

\[ g = g_1 g_2 - (p_1, q_2)(p_2, q_1). \]  \hspace{1cm} (5.19)

Analogously,

\[ L_2 = \frac{\lambda_1 + \lambda_2}{g} (R_1 - g_1 I) R_2 \]

and

\[ K(x, x) = L_1 + L_2 = \frac{(\lambda_1 + \lambda_2)}{g} (\{R_1, R_2\} - g_2 R_1 - g_1 R_2) \]

where \( \{R_1, R_2\} = R_1 R_2 + R_2 R_1 \). To write down the formula for the potential \( U \) let us introduce function

\[ f = (\lambda_1 + \lambda_2) \left( 2(\lambda_1 + \lambda_2)e^{2 \lambda_1 \theta_1 + 2 \lambda_2 \theta_2} + \frac{\lambda_1}{\lambda_2}(p_2, q_2)e^{2 \lambda_1 \theta_1} + \frac{\lambda_2}{\lambda_1}(p_1, q_1)e^{2 \lambda_2 \theta_2} \right) \]

and matrix

\[ F = 2(\lambda_1 e^{2 \lambda_1 \theta_1} R_2 + \lambda_2 e^{2 \lambda_2 \theta_2} R_1) \]

Then (cf. [14])

\[ U(x, t) = -2K'(x, x) = 2(\lambda_1 + \lambda_2) g F + f (\{R_1, R_2\} - g_2 R_1 - g_1 R_2) \]  \hspace{1cm} (5.20)

It is important to find the conditions when a solution of the matrix KdV equation is non-singular. Analyzing the expression for \( g \) (see (5.19)) it is easy to obtain the following sufficient condition for that.

**Proposition 5.2.** The potential (5.20) is non-singular if \( (p_1, q_1) > 0; (p_2, q_2) > 0 \) and \( \alpha > 0 \) where

\[ \alpha = (p_1, q_1)(p_2, q_2) - \frac{4 \lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2} (p_1, q_2)(q_1, p_2). \]
Let us investigate the behaviour of $U(x, t)$ as $|t| \to \infty$. At the beginning let us consider $U(x, t)$ at the vicinity of the line $\theta_1 = x - 4\lambda_1^2 t = 0$. If $t \to -\infty$ then $\theta_2 \to +\infty$. Subtracting the terms in (5.20) containing $e^{4\lambda_2 \theta_2}$ we obtain that

$$
U(x, t) \sim -4 \frac{\lambda_1 p_1 \otimes q_1 e^{2\lambda_1 \theta_1}}{\left(e^{2\lambda_1 \theta_1} + \frac{(p_1, q_1)}{2\lambda_1}\right)^2}
$$

or, equivalently,

$$
U(x, t) \sim -\frac{2\lambda_1^2 R_1}{\text{tr} R_1} \text{sech}^2 \lambda_1 (\theta_1 - \phi_1^-) = -\frac{2\lambda_1^2 p_1 \otimes q_1}{(p_1, q_1)} \text{sech}^2 \lambda_1 (\theta_1 - \phi_1^-)
$$

where

$$
\phi_1^- = \frac{1}{2\lambda_1} \ln \frac{(p_1, q_1)}{2\lambda_1}.
$$

Analogously, if $t \to +\infty$ then $\theta_2 \to -\infty$ and

$$
U(x, t) \sim -\frac{2\lambda_1^2 p_1 \otimes q_1}{(\hat{p}_1, \hat{q}_1)} \text{sech}^2 \lambda_1 (\theta_1 - \phi_1^+)
$$

where

$$
\hat{p}_1 = (p_2, q_2)p_1 - \frac{2\lambda_2}{(\lambda_1 + \lambda_2)}(p_1, q_2)p_2,
$$

$$
\hat{q}_1 = (p_2, q_2)q_1 - \frac{2\lambda_2}{(\lambda_1 + \lambda_2)}(p_2, q_1)q_2
$$

and

$$
\phi_1^+ = \frac{1}{2\lambda_1} \ln \frac{(p_1, \hat{q}_1)}{2\lambda_1}.
$$

Let us consider now the potential $U(x, t)$ at the vicinity of the line $\theta_2 = x - 4\lambda_2^2 t = 0$. If $t \to -\infty$ $\theta_1 \to -\infty$ and

$$
U(x, t) \sim -\frac{2\lambda_2^2 \hat{p}_2 \otimes \hat{q}_2}{(\hat{p}_2, \hat{q}_2)} \text{sech}^2 \lambda_2 (\theta_2 - \phi_2^-)
$$

where

$$
\hat{p}_2 = (p_1, q_1)p_2 - \frac{2\lambda_1}{(\lambda_1 + \lambda_2)}(p_2, q_1)p_1,
$$

$$
\hat{q}_2 = (p_1, q_1)q_2 - \frac{2\lambda_1}{(\lambda_1 + \lambda_2)}(p_1, q_2)q_1
$$

and

$$
\phi_2^- = \frac{1}{2\lambda_2} \ln \frac{(\hat{p}_2, \hat{q}_2)}{2\lambda_2(p_1, q_1)}.
$$

If $t \to +\infty$ then $\theta_1 \to -\infty$ and

$$
U(x, t) \sim -\frac{2\lambda_2^2 R_2}{\text{tr} R_2} \text{sech}^2 \lambda_2 (\theta_2 - \phi_2^+)
$$

where

$$
\phi_2^+ = \frac{1}{2\lambda_2} \ln \frac{(p_2, q_2)}{2\lambda_2(p_1, q_1)}.
$$

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where
\[ \phi^+_2 = \frac{1}{2\lambda_2} \ln \left( \frac{p_2, q_2}{2\lambda_2} \right). \]

Using (5.20) it is easy to check that along any other line \( \theta = x - 4\lambda^2 t \), \( \lambda > 0, \lambda \neq \lambda_1, \lambda \neq \lambda_2 \) on the plane \( (x, t) \) \( U(x, t) \to 0 \) when \( |t| \to \infty \).

So, we can conclude that \( U(x, t) \) breaks up into two 1-solitons as \( |t| \to \infty \) where the \( j \)-th soliton moves with the speed \( 4\lambda_j^2 \) and centres near \( \theta_j = x - 4\lambda_j^2 t \). The phase-shift \( \Delta_1 \) of the slow soliton is
\[ \Delta_1 = \phi^+_1 - \phi^-_1 = \frac{1}{2\lambda_1} \ln \left( 1 - \frac{4\lambda_1\lambda_2(p_1, q_2)(q_1, p_2)}{(\lambda_1 + \lambda_2)^2(p_1, q_1)(p_2, q_2)} \right) \]
and phase-shift \( \Delta_2 \) of the fast soliton is
\[ \Delta_2 = \phi^+_2 - \phi^-_2 = -\frac{1}{2\lambda_2} \ln \left( 1 - \frac{4\lambda_1\lambda_2(p_1, q_2)(q_1, p_2)}{(\lambda_1 + \lambda_2)^2(p_1, q_1)(p_2, q_2)} \right). \]

From these formulae one can see that \( \Delta_1 \) (\( \Delta_2 \)) can be both positive (negative) or negative (positive). That means that solitons can interact in two different ways: slow soliton can move back and the second one can move forward as it was in the scalar case (see e.g [61]) and, otherwise, slow soliton can move forward and fast one can move back.

Another remarkable feature of the interaction is the change of matrix amplitude (cf. [14]). For the slow soliton amplitude changes
\[ \frac{2\lambda_1^2 R_1}{\text{tr } R_1} \to \frac{2\lambda_2^2 \hat{R}_1}{\text{tr } \hat{R}_1} \]
and for the fast one
\[ \frac{2\lambda_2^2 \hat{R}_2}{\text{tr } \hat{R}_2} \to \frac{2\lambda_1^2 R_2}{\text{tr } R_2} \]
where \( \hat{R}_1 = \hat{p}_1 \otimes \hat{q}_1 \) and \( \hat{R}_2 = \hat{p}_2 \otimes \hat{q}_2 \) as \( t \) goes from \( -\infty \) to \( \infty \). If \((p_1, q_2) = 0 \) \((R_2R_1 = 0)\) or \((p_2, q_1) = 0 \) \((R_1R_2 = 0)\) the phase-shift is equal to zero and interaction appears only as a change of the matrix amplitude. If both of the last scalar product equal zero solitons pass through each other with neither phase-shift nor change of amplitude, so that they do not interact at all (in this case \( R_1 R_2 = R_2 R_1 = 0 \)). This corresponds to the diagonal \( U(x, t) \) with scalar 1-solitons at the diagonal entries.

Let us assume now that \( U(x, t) \) is symmetric: \( U^T(x, t) = U(x, t) \). It is the case when \( R_1 = R_1^T \) and \( R_2 = R_2^T \), or, \( p_1 = q_1 \) and \( p_2 = q_2 \). In this case the sufficient conditions of proposition 5.2 are always satisfied so all such solutions are nonsingular. Then
\[ \Delta_1 = \frac{1}{2\lambda_1} \ln \left( 1 - \frac{4\lambda_1\lambda_2(p_1, p_2)^2}{(\lambda_1 + \lambda_2)^2 p_1^2 p_2^2} \right) = \frac{1}{2\lambda_1} \ln \left( 1 - \frac{4\lambda_1\lambda_2 \cos^2 \beta}{(\lambda_1 + \lambda_2)^2} \right) = \]
\[
= \frac{1}{2\lambda_1} \ln \left( \frac{(\lambda_1 + \lambda_2)^2 (\cos^2 \beta + \sin^2 \beta) - 4\lambda_1 \lambda_2 \cos^2 \beta}{(\lambda_1 + \lambda_2)^2} \right) = \frac{1}{2\lambda_1} \ln \left( \sin^2 \beta + \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2 \cos^2 \beta \right).
\]

and
\[
\Delta_2 = -\frac{1}{2\lambda_2} \ln \left( \sin^2 \beta + \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2 \cos^2 \beta \right)
\]

where \( \beta = \angle(p_1, p_2) \). The change of matrix amplitude is
\[
-2\lambda_1^2 \frac{p_1 \otimes p_1}{|p_1|^2} \rightarrow -2\lambda_1^2 \frac{\xi_1 \otimes \xi_1}{|\xi_1|^2}
\]

and
\[
-2\lambda_2^2 \frac{\xi_2 \otimes \xi_2}{|\xi_2|^2} \rightarrow -2\lambda_2^2 \frac{p_2 \otimes p_2}{|p_2|^2}
\]

where \( \xi_1 = p_1 = p_2 \hat{p}_1 - \frac{2\lambda_1 (p_1, p_2) p_2}{\lambda_1 + \lambda_2} \) and \( \xi_2 = p_2 \hat{p}_2 - \frac{2\lambda_2 (p_1, p_2) p_1}{\lambda_1 + \lambda_2} \). The phase-shifts in this case are
\[
\Delta_1 = \frac{1}{2\lambda_1} \ln \left( \sin^2 \beta + \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2 \cos^2 \beta \right) < 0
\]

and
\[
\Delta_2 = -\frac{1}{2\lambda_2} \ln \left( \sin^2 \beta + \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2 \cos^2 \beta \right) > 0.
\]

Since
\[
\sin^2 \beta + \left( \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \right)^2 \cos^2 \beta < 1
\]

the slow soliton shifts back and the fast one shifts forward.

In contrast to the scalar case [61] we see that solitons with similar velocities do not interact intensively. As \( \lambda_2 \rightarrow \lambda_1, \lambda_1 \neq \lambda_2 \) the phase-shifts
\[
\Delta_1 \rightarrow \frac{1}{2\lambda_1} \ln \sin^2 \beta
\]

\[
\Delta_2 \rightarrow -\frac{1}{2\lambda_2} \ln \sin^2 \beta
\]

but not go to infinity unless \( \beta = 0 \) when matrix case reduces to the scalar one.

5.5 Matrix Darboux transformations and solitons.

It is well known that multisoliton solutions of the scalar KdV equation can be also obtained by Darboux transformations (see e.g. [55]). In the matrix case we can
use the general scheme of section 2.4 and the book [55] to produce solutions of the matrix KdV equation in the form

\[ U = -2(Y_nW_n^{-1})'. \]  

(5.21)

Here

\[ W = W(\Psi_1, \ldots, \Psi_n) = \begin{pmatrix} \Psi_1 & \cdots & \Psi_n \\ \vdots & \ddots & \vdots \\ \Psi_1^{(n-2)} & \cdots & \Psi_n^{(n-2)} \\ \Psi_1^{(n-1)} & \cdots & \Psi_n^{(n-1)} \end{pmatrix}, \]

\[ Y = Y(\Psi_1, \ldots, \Psi_n) = \begin{pmatrix} \Psi_1 & \cdots & \Psi_n \\ \vdots & \ddots & \vdots \\ \Psi_1^{(n-2)} & \cdots & \Psi_n^{(n-2)} \\ \Psi_1^{(n)} & \cdots & \Psi_n^{(n)} \end{pmatrix}, \]

\[ Y_n = |Y|_{n,n}, \ W_n = |W|_{n,n} \text{ and } \Psi_1, \ldots, \Psi_n \text{ are defined as columns of the } d \times nd \text{ matrix } \Phi \text{ satisfying the equation} \]

\[ \Phi'' = \Phi C \]  

(5.22)

where constant \( nd \times nd \) matrix \( C \) can be assumed to be in the form

\[ C = \begin{pmatrix} \lambda_1^2 I_1 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 I_2 & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_m^2 I_m \end{pmatrix} \]

and \( \sum_{k=1}^{m} \dim I_k = nd, n \) is the order of the operator \( A \) intertwining \( L = -D^2 + U \) and \( L_0 = -D^2 \). Time dependence is defined by the equation (see [55, 29])

\[ \Phi_t = -4D^3 \Phi \]

so that the general form of \( \Psi_j \) is

\[ \Psi_j = P_je^{A_j x - 4A_j^2 t} + Q_je^{-A_j x + 4A_j^2 t} \]

where \( P_j, Q_j \) are constant matrices and \( A_j, j = 1, \ldots, n \) are positively defined \( d \times d \) matrices which come from the presentation of \( C \) in the form

\[ C = \begin{pmatrix} \Lambda_1^2 & 0 & \cdots & 0 \\ 0 & \Lambda_2^2 & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_n^2 \end{pmatrix}. \]
As a disadvantage of the MDT method we should mention the difficulty to supply the symmetry condition for the solution.

Let us consider now the simplest nontrivial case when the intertwining operator $A$ is of the first order. Then $C = \Lambda^2$, $\Lambda = \text{diag}(\lambda_1, I_1, \ldots, \lambda_n I_n)$ where $\sum_{k=1}^n \dim I_k = d$. The equation (5.22) can be written in the form

$$\Psi'' = \Psi \Lambda^2.$$  

Then

$$\Psi = Pe^{\Lambda x} + Qe^{-\Lambda x}$$  \hspace{1cm} (5.23)

where $P$ and $Q$ are arbitrary matrices (we assume that they are nondegenerate) and

$$U = -2 \left( \Psi' \Psi^{-1} \right)'$$  \hspace{1cm} (5.24)

**Remark.** Let us notice that in contrast to the scalar case we can obtain multisoliton solutions of matrix KdV equation already at the first step of Darboux transformation.

To compare the solutions (5.24) with the multisoliton solutions discussed above let us find the spectral data $(\lambda_j, R_j)$ in terms of the parameters of MDT $(\Lambda, P, Q)$. Let us introduce $H = \Psi' \Psi^{-1}$. Then

$$H_1 = \lim_{x \to +\infty} H = \lim_{x \to +\infty} (\Psi' e^{-\Lambda x}) (\Psi e^{-\Lambda x})^{-1} = P \Lambda P^{-1}$$

and

$$H_2 = \lim_{x \to -\infty} H = -Q \Lambda Q^{-1}.$$

Functions

$$\Phi_1(x, k) = \begin{vmatrix} \Psi & Ie^{ikx} \\ \Psi' & ikI e^{ikx} \end{vmatrix} = \begin{vmatrix} (ikI - \Psi' \Psi^{-1}) e^{ikx} \end{vmatrix}$$

and

$$\Phi_2(x, k) = \begin{vmatrix} \Psi & Ie^{-ikx} \\ \Psi' & -ikI e^{-ikx} \end{vmatrix} = \begin{vmatrix} (ikI + \Psi' \Psi^{-1}) e^{-ikx} \end{vmatrix}$$

are the solutions of the equation

$$L \Phi = k^2 \Phi, \quad L = -D^2 + U(x)$$  \hspace{1cm} (5.25)

with $U$ defined by (5.24) as it follows from theorem 2.2. As

$$\Phi_1 \to (ikI - H_1)e^{ikx}, x \to +\infty$$

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and
\[ \Phi_2 \rightarrow (-ikI + H_2)e^{-ikx}, x \rightarrow -\infty \]
the Jost solutions \( F(x, k) \) and \( G(x, k) \) of (5.25) are
\[
\begin{align*}
F(x, k) &= \Phi_1(x, k)P(ikI - \Lambda)^{-1}P^{-1}e^{ikx} \\
G(x, k) &= \Phi_2(x, k)Q(ikI - \Lambda)^{-1}Q^{-1}e^{-ikx}.
\end{align*}
\]
Since \( \Phi_1(x, k) = \Phi_2(x, -k) \) (potential (5.24) is reflectionless)
\[
A(k) = F^{-1}(x, -k)G(x, k) = P(ikI + \Lambda)P^{-1}Q(ikI - \Lambda)^{-1}Q^{-1}
\]
Since
\[
\det A(k) = \frac{\det(ikI + \Lambda)}{\det(ikI - \Lambda)}
\]
we see that the discrete spectrum of the Schrödinger operator \( L \) with the potential (5.24) consists of the points \(-\lambda_j^2\) where \( \lambda_j \) are the eigenvalues of \( \Lambda \).

Spectral data \( R_j \) are defined by the conditions (5.8)
\[
(-\lambda_jI + \Psi'\Psi^{-1})e^{\lambda_jx}P\left(\text{Res}_{k=i\lambda_j}(ikI + \Lambda)^{-1}\right)P^{-1} = F(x, k_j)R_j
\]
Since \( F(x, i\lambda_j) = (I + o(1))e^{-\lambda_jx} \) as \( x \rightarrow +\infty \) we obtain taking the limit of the both sides of the last formula that
\[
R_j = \lim_{x \rightarrow +\infty} \left[ (-\lambda_jI + \Psi'\Psi^{-1})PP_jP^{-1}e^{2\lambda_jx} \right]
\]
where
\[
P_j = \text{Res}_{k=i\lambda_j}(ikI + \Lambda)^{-1} = \text{diag}(0, \ldots, 0, I_j, 0, \ldots, 0)
\]
are the projectors onto the eigenspace corresponding to the eigenvalue \( \lambda_j \) of \( \Lambda \).

**Proposition 5.3.** The spectral data \( R_j \) of the Schrödinger operator \( L \) with the potential (5.24) can be written in terms of the parameters of MDT as
\[
R_j = P f(\lambda_j) P^{-1} Q P_j P^{-1}
\]
where \( f(\lambda) = \lambda I + \Lambda \).

**Proof.** We can rewrite (5.26) as
\[
R_j = \lim_{x \rightarrow +\infty} \left[ (-\lambda_j\Psi - \Psi')(P^{-1}\Psi)^{-1}P_jP^{-1}e^{2\lambda_jx} \right].
\]
Then
\[
(P^{-1}\Psi)^{-1} = e^{-\Lambda x}(I + P^{-1}Qe^{-2\Lambda x})^{-1} = e^{-\Lambda x}(I - P^{-1}Qe^{-2\Lambda x} + \ldots)
\]
Further calculation is straightforward using commutativity of $P_j$, $j = 1, \ldots, d$.

We can use this result to write down the conditions on the data of MDT which guarantee the symmetry of the potential. Indeed, the potential (5.24) is symmetric if and only if spectral matrices $R_j$, $j = 1, \ldots, d$ are symmetric $R_j^T = R_j$, or

$$P f(\lambda_j) P^{-1} Q P_j P^{-1} = (P^T)^{-1} P_j Q^T (P^{-1})^T f(\lambda_j) P^T, \quad j = 1, \ldots, d. \quad (5.27)$$

It is easy to check that

$$Q = \sum_{j=1}^d P f^{-1}(\lambda_j) (P^T P)^{-1} P_j. \quad (5.28)$$

is a solution of (5.27) and using proposition 5.3 we obtain the following

**Proposition 5.4.** Potential (5.24) is symmetric if $Q$ is defined by (5.28). In that case

$$R_j = (P^T)^{-1} P_j P^{-1}. \quad (5.29)$$

**Example.** Let us consider the simplest case $d = 2$, $\Lambda = \text{diag}(\lambda_1, \lambda_2), \lambda_1 \neq \lambda_2$. Then the potential (5.24) satisfies matrix KdV equation if

$$\Psi = P(t)e^{\Lambda x} + Q(t)e^{-\Lambda x}$$

where

$$P(t) = Pe^{-4\lambda_1 t}, \quad Q(t) = Qe^{4\lambda_3 t}$$

and

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}.$$

Let us assume that $\det P \neq 0$, $\det Q \neq 0$ and introduce

$$\hat{P} = \begin{pmatrix} p_{11} & q_{12} \\ p_{21} & q_{22} \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} q_{11} & p_{12} \\ q_{21} & p_{22} \end{pmatrix}.$$ 

Then it is easy to check using proposition 5.3 and the results of the previous section that the phase-shifts of solitons can be expressed in terms of the parameters of MDT as

$$\Delta_1 = \frac{1}{2\lambda_1} \ln \frac{\det \hat{P} \det \hat{Q}}{\det P \det Q}, \quad \Delta_2 = \frac{1}{2\lambda_2} \ln \frac{\det P \det Q}{\det \hat{P} \det \hat{Q}}. \quad (5.29)$$

Now

$$\det \Psi = e^{(\lambda_1+\lambda_2)x-4(\lambda_1^3+\lambda_2^3)t} \det P + e^{(\lambda_1-\lambda_2)x-4(\lambda_1^3-\lambda_2^3)t} \det \hat{P} +$$

$$+ e^{(\lambda_2-\lambda_1)x+4(\lambda_1^3-\lambda_2^3)t} \det \hat{Q} + e^{-(\lambda_1+\lambda_2)x+4(\lambda_1^3+\lambda_2^3)t} \det Q$$

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so that $U(x,t)$ is nonsingular $\det \Psi \neq 0$ for any $x$ which is the case if
\[
\det P > 0, \det \hat{P} > 0, \det Q > 0, \det \hat{Q} > 0. \tag{5.30}
\]

Let us now introduce the vectors
\[
e_1 = (p_{11}, p_{21}),
\]
\[
e_2 = (p_{12}, p_{22}),
\]
\[
e_3 = (q_{11}, q_{21}),
\]
\[
e_4 = (q_{12}, q_{22}).
\]

and the angles between them
\[
\alpha = \angle < e_1, e_3 >
\]
\[
\beta = \angle < e_1, e_2 >
\]
\[
\gamma = \angle < e_1, e_4 >
\]

Then conditions (5.30) of nonsingularity are equivalent to $0 < \alpha < \beta < \gamma < \pi$ or $0 < \alpha < \gamma < \beta < \pi$. We have
\[
\frac{\det \hat{P} \det \hat{Q}}{\det P \det Q} = \frac{\sin(\beta - \alpha) \sin \gamma}{\sin \beta \sin(\gamma - \alpha)} = \frac{\cos \alpha - \sin \alpha \cot \beta}{\cos \alpha - \sin \alpha \cot \gamma}.
\]

Since $\cot x$ is decreasing on $(0, \pi)$ we see that if $\beta < \gamma$ then the phase-shifts $\Delta_1 < 0$ ($\Delta_2 > 0$) and if $\beta > \gamma$ then $\Delta_1 > 0$ ($\Delta_2 < 0$).
Chapter 6

Conclusion

The theory of integrable Schrödinger operators with matrix potentials is a very large area which is essentially open. In this thesis we made some progress in the case when the potential is a rational (or trigonometric) matrix-valued function which, of course, is not the most general one. Let us formulate here the main results of this thesis.

- Derivation of the local trivial monodromy criteria for the one-dimensional matrix Schrödinger operator $L = -D^2 + U(z)$.
- Description of all Schrödinger operators $L$ with rational and trigonometric matrix potentials which have trivial monodromy in $\mathbb{C}$ in terms of matrix Darboux transformations.
- Derivation of the trivial monodromy conditions for the matrix Schrödinger operators in many dimensions.
- Proof of the equivalence of the trivial monodromy property and D-integrability for the matrix Schrödinger operators.
- Investigation of the interaction of two matrix KdV solitons.

Let us identify now some problems which seem to be important and should be considered next.

- Investigate the general finite-gap periodic matrix Schrödinger operators. In particular, describe the elliptic matrix finite-gap potentials.
- Investigate the interaction of the matrix KdV solitons in the general case.
- Describe the configurations of hyperplanes which may be the singularities of the D-integrable Schrödinger operators in many dimensions. For a given configuration how many such operators can be constructed?
We should mention that in many dimensions already the scalar case is very difficult and still essentially open (see [17] for the last results in this direction). The matrix case is much more difficult but may clarify some structures which can be important also for the scalar case. A good example is the appearance in the matrix case of the "old axiomatics" for the scalar Baker-Akhiezer function (see [18]) the meaning of which was not clear before.

We believe that the further investigation of the matrix Schrödinger operators and related nonlinear equations will bring more new interesting phenomena which may have important applications in physics.
Bibliography


