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On the Provenance of Hinged-Hinged Frequencies in Timoshenko Beam Theory

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ABSTRACT

An exact differential equation governing the motion of an axially loaded Timoshenko beam supported on a two parameter, distributed foundation is presented. Attention is initially focused on establishing the provenance of those Timoshenko frequencies generated from the hinged-hinged case, both with and without the foundation being present. The latter option then enables an exact, neo-classical assessment of the ‘so called’ two frequency spectra, together with their corresponding modal vectors, to be undertaken when zero, tensile or compressive static axial loads are present in the member. An alternative, ‘precise’ approach, that models Timoshenko theory efficiently, but eliminates the possibility of a second spectrum, is then described and used to confirm the original eigenvalues. This leads to a definitive conclusion regarding the structure of the Timoshenko spectrum. The ‘precise’ technique is subsequently extended to allow, either the full foundation to be incorporated, or either of its component parts individually. An illustrative example from the literature is solved to confirm the accuracy of the approach, the nature of the Timoshenko spectrum and a wider indication of the effects that a distributed foundation can have.

1. Introduction

A knowledge of the vibration of beams and beam systems is a modern requirement across a diverse range of engineering and scientific disciplines, no better typified than in the aerospace industry. Of the vibration theories available, Bernoulli-Euler theory is the simplest and most widely understood. It underpins the majority of practical applications, which involve members with moderate to high slenderness ratios, but requires modification when the members carry a static axial force of significant value. In similar fashion, when members have a low slenderness ratio or vibrate at high frequency, the second order effects of rotary inertia of the cross-section and shear deformation must be taken into account; the so-called Timoshenko theory.

Since its inception in 1921 [1], Timoshenko beam theory has generated an almost relentless flow of research on virtually every aspect of its sphere of influence. Despite this, there are still a surprising number of topics that can benefit from further attention and subsequent explanation. One of these concerns the provenance of frequencies determined through Timoshenko beam theory. Unlike Bernoulli-Euler theory, which is well understood for single members, there has long been conjecture regarding a particular aspect of Timoshenko theory, namely the point discontinuity in the governing differential equation and the subsequent division of frequencies into two spectra. Background to this area of work can be found, for example, in the following papers [2-9], which themselves contain a wide variety of references.

In the paper that follows, a unified, exact member theory that incorporates all the second order effects of static axial load, rotary inertia and shear deformation is presented in a way that enables a critical assessment to be made of the point discontinuity in the Timoshenko equation. This leads to a clarification of the status of the corresponding cut-off frequency, which in turn enables a definitive conclusion to be drawn about the Timoshenko spectrum. The evidence thus gathered is further used to cast light on the corresponding modal vectors and other related topics of interest.

Initially, the exact differential equation governing the motion of a Timoshenko beam supported on a two parameter, distributed foundation is presented in a convenient, non-dimensional form that allows for the possibility of a zero, tensile or compressive static axial load in the member. The basic hinged-hinged relationships required in the remainder of the paper are then developed concisely, in such a way that shows how the continuous spectrum of Bernoulli-Euler (B-E) frequencies are related to their Timoshenko counterparts. The paper then focuses in more detail on the case of an axially loaded, hinged-hinged Timoshenko beam in the absence of an elastic foundation. It is shown how the frequency equation factorizes, the nature of the second spectrum is discussed, together with its influence on mode shape recovery, and a simple, physical comparison is made between any combination of hinged and guided boundary conditions.

A ‘precise’ approach for converging upon any required Timoshenko frequency in an extremely efficient way, is then described. In the absence of a distributed foundation, this involves establishing a stiffness
matrix using Bernoulli-Euler theory that allows exactly for static axial load and shear deformation and then augmenting it with a close approximation to the distributed rotational inertia. Such an approach eliminates the possibility of a second spectrum and when used in conjunction with the Wittrick-Williams algorithm [10], ensures convergence to any required natural frequency with the certain knowledge that none have been missed. This theory is subsequently used to confirm the exact results from the corresponding equations presented earlier, by mapping the step-by-step transformation of the B-E frequencies to their equivalent Timoshenko values. This process underlines the need to reassess the discontinuity in the Timoshenko equation and hence the status of the cut-off frequency.

Attention is then focused on establishing the corresponding eigenvalues when the member is supported on a two-parameter, distributed foundation. Exact solution of the Timoshenko problem in such cases has been considered previously by Capron and Williams [11] and has been shown to be intractable. In contrast, it is shown herein that equivalent solutions to the hinged-hinged case, can be generated through the 'precise' approach, to good accuracy, by implementing a minor modification, and furthermore, that extension to any possible combination of elastic end conditions is equally straightforward. An additional point of note that emerges from the above is that, for the hinged-hinged case, two possible discontinuities in the governing differential equation can now be identified.

Finally, the theory of this paper is applied to a well-known, illustrative problem, from which a number of conclusions are drawn and recommendations made.

2. Theory

An exact, fourth order differential equation governing the motion of an axially loaded Timoshenko beam of length, \( L \), that is supported on a two parameter, distributed foundation, whose lateral and rotational restraining stiffnesses per unit length are \( k_y \) and \( k_\theta \), respectively, has been given by Capron and Williams [11] and can be written in the following non-dimensional form

\[
[D^4 + 2(\Delta - k_y^*)D^2 - (qb^2 - k_y^*)/t]\Theta = 0
\]

(1)

where \( D = d/d\xi \), \( \xi = x/L \) is the non-dimensional length parameter and \( \Theta = V \) or \( \Psi \), where \( V \) and \( \Psi \) are the amplitudes of the lateral displacement and bending slope, respectively.

\[
\Delta = [qp^2 + b^2(r^2 + s^2)]/2t \quad q = 1 - b^2r^2s^2 \quad b^2 = \rho AL^4 \omega^2 / EI \quad t = 1 - s^2 p^2
\]

(2a)

\[
k_1^* = (s^2 k_y^* + tk_\theta^*) / 2t \quad k_2^* = q k_y^* - s^2 k_\theta^* (b^2 - k_y^*)
\]

(2b)

\[
p^2 = PL^2 / EI \quad r^2 = I / AL^2 \quad s^2 = EI / \kappa AGL^2 \quad k_y^* = k_y L^4 / EI \quad k_\theta^* = k_\theta L^2 / EI
\]

(2c)

where \( \rho \), \( E \) and \( G \) are the density, Young’s modulus and shear modulus of the member material respectively, \( A \) and \( I \) are the area and second moment of area of the cross-section, \( \kappa \) is the section shape factor, \( \omega \) is the radian frequency of vibration and \( P \) is the static axial load in the member, which is positive for compression, zero, or negative for tension. The non-dimensional parameters \( b^2 \), \( p^2 \), \( r^2 \) and \( s^2 \) uniquely define the effects of frequency, axial load, rotary inertia and shear deformation, respectively [12,13]. For generality, the work that follows is developed in terms of \( b \) and \( b^2 \) which, for conciseness, are merely referred to as frequencies, while any combination of the remaining effects can be neglected by setting the relevant parameter to zero. Finally, it can be demonstrated that, \( t \), defined by the last of Eqs.(2a), is always positive so long as \( P \) is less than the elastic critical buckling load (when in compression and does not cause inelastic behaviour when in tension).

2.1. Frequency relationships for the hinged-hinged case
For this case, it is convenient to consider Eq.(1) with \( \Theta = V \) and to assume a general solution of the form \( V = C \sin i \pi \xi \), where \( C \) is an arbitrary constant and \( V \) satisfies the boundary conditions. Substituting for \( V \) in Eq.(1) then yields

\[
(i \pi)^4 - 2(\Delta_i - k_i^*) (i \pi)^2 - (q_i b_i^2 - k_i^*) / t = 0 \quad i = 1,2,\ldots,\infty
\]

Eq.(3) can now be used to find any frequency, \( b_i \), for any combination of the non-dimensional parameters defined in Eqs.(2). However, for the context of this paper, it is first necessary to consider the simple B-E beam, for which \( p^2 = r^2 = s^2 = k_y^* = k_\theta^* = 0 \). Substituting these parameters in Eq.(3) yields the \( i^{th} \) B-E frequency, \( b_{0,i} \), as

\[
b_{0,i} = (i \pi)^2 \quad i = 1,2,\ldots,\infty
\]

Attention is now focused on relating the \( i^{th} \) B-E frequency to its counterparts emanating from any desired combination of non-dimensional parameters. This is most easily achieved by embedding Eq.(4) into Eq.(3), to yield

\[
(1 - s^2 p^2) b_{0,i}^2 - \{p^2 + (b_i^2 r^2 - k_\theta^*) (1 - s^2 p^2) + s^2 (b_i^2 - k_y^*)\} b_{0,i}
\]

\[
- (b_i^2 - k_y^*) (1 - s^2 (b_i^2 r^2 - k_\theta^*)) = 0 \quad i = 1,2,\ldots,\infty
\]

which, upon expansion, yields the quadratic frequency equation in the non-dimensional frequency parameter, \( b_i^2 \), as

\[
b_i^4 r^2 s^2 - \{1 + [r^2 (1 - s^2 p^2) + s^2] b_{0,i} + s^2 (r^2 k_y^* + k_\theta^*)\} b_i^2
\]

\[
+ (1 - s^2 p^2) b_{0,i}^2 - \{p^2 - s^2 k_y^* - (1 - s^2 p^2) k_\theta^*\} b_{0,i} + (1 + s^2 k_\theta^*) k_y^* = 0 \quad i = 1,2,\ldots,\infty
\]

A further closed form relationship of interest can be established by ignoring the effects of rotary inertia, i.e. setting \( r^2 = 0 \) in Eq.(6), to yield

\[
b_{psy\theta,i}^2 = \{(1 - s^2 p^2) b_{0,i}^2 - \{p^2 - (1 - s^2 p^2) k_\theta^* - s^2 k_y^*\} b_{0,i} + (1 + s^2 k_\theta^*) k_y^*\}
\]

\[
/[1 + s^2 (b_{0,i} + k_\theta^*)] \quad i = 1,2,\ldots,\infty
\]

where the lowercase subscripts on the left hand side of the equation, other than \( i \), denote the non-dimensional parameter(s) retained.

2.2. Equivalent equations in the absence of a distributed foundation

The effects of a two parameter foundation can be eliminated simply from Eqs.(1), (5), (6) and (7) by setting \( k_y^* = k_\theta^* = 0 \) to yield

\[
[D^4 + 2 AD^2 - q b^2 / t] \Theta = 0 \quad (8)
\]

\[
(1 - s^2 p^2) b_{0,i}^2 - \{p^2 + b_i^2 r^2 (1 - s^2 p^2) + b_i^2 s^2\} b_{0,i} - b_i^2 (1 - b_i^2 r^2 s^2) = 0 \quad i = 1,2,\ldots,\infty
\]

\[
b_i^4 r^2 s^2 - \{1 + [r^2 (1 - s^2 p^2) + s^2] b_{0,i} + (1 - s^2 p^2) b_{0,i}^2 - p^2 b_{0,i}\} = 0 \quad i = 1,2,\ldots,\infty
\]

\[
b_{psy,i}^2 = \{(1 - s^2 p^2) b_{0,i}^2 - p^2 b_{0,i}\}/(1 + s^2 b_{0,i}) \quad i = 1,2,\ldots,\infty
\]
respectively.

Solutions to Eq.(8) can now be found easily, for any required combination of boundary conditions, using the dynamic stiffness technique in conjunction with the Wittrick-Williams algorithm, as described in detail in references [12,13]. Since such solutions are founded in exact member theory and can be converged upon to the accuracy of the host computer, the accuracy of such results is commensurate with that of closed form solutions.

Eq.(8) yields Eq.(9) exactly for the case of hinged-hinged boundary conditions which, on expansion, yields the quadratic equation in $b_i^2$ that is given in Eq.(10). Thus either equation can be used to generate the required frequency parameters, the former by trial values of $b_i$ and the latter from a classical route.

Finally, it is clear that Eqs.(7) and (11) are simple, exact closed form solutions and are typical of a family of such equations formed from various combinations of the non-dimensional parameters, in which either $r^2$, $s^2$ or both are zero. In both equations presented, the effects of axial load and shear deformation are accounted for exactly, while the effects of rotary inertia are ignored. Once more it is straightforward to show that the right hand side of Eq.(11) is always positive, with the result that each frequency generated is related uniquely to a single B-E frequency, which thus forms part of a single continuous spectrum. Such an argument must also hold for Eq.(7), since adding stiffness to a structure cannot reduce the values of its natural frequencies. Furthermore, it is a requirement of any theory that seeks to improve the accuracy of B-E theory, that each frequency is modified without loss or addition of any frequencies. This is clearly satisfied by Eqs.(7), (11) and all similar equations that constitute the family.

In contrast, Eq.(10) shows that although the complete spectrum of Timoshenko frequencies can be determined through an entirely consistent approach, each pair of frequencies now stems from a single B-E frequency and hence the one-to-one relationship between a developed frequency and a unique B-E frequency appears to have been lost. Consider therefore the lower and upper solutions to Eq.(10), which can be written as, respectively,

\[
\begin{align*}
    \beta_{L,i} &= (\alpha_i - \sqrt{\beta_i^2})/2r^2s^2 & \text{and} & \quad \beta_{U,i} &= (\alpha_i + \sqrt{\beta_i^2})/2r^2s^2 \\
    \alpha_i &= 1 + b_{0,i}[r^2(1-s^2p^2) + s^2] & \text{and} & \quad \beta_i &= \alpha_i^2 - 4r^2s^2b_{0,i}[b_{0,i}(1-s^2p^2) - p^2]
\end{align*}
\]

It is then straightforward to show that $\beta_i$, the discriminant of Eq.(10), is always positive and that both roots are also positive, subject to the constraints imposed earlier on $P$.

2.3. The two frequency spectra

Timoshenko beam theory, as defined by Eq.(8), has a discontinuity when

\[
q = 1 - b^2r^2s^2 = 0 \quad \text{i.e. when} \quad b_{co}^2 = 1/r^2s^2
\]

where $b_{co}^2$ defines the cut-off frequency. Beyond this point, the natural frequencies stem from two sources, the first (original) spectrum and the ‘so-called’ second spectrum, as discussed later in this section.

Before that, it is important to deal with the lack of clarity surrounding the cut-off frequency, which lies at the heart of the current debate on this aspect of Timoshenko theory. The problem stems from the fact that the cut-off frequency corresponds to the point discontinuity in the Timoshenko equation and as such has never been ascribed any intrinsic value, over and above defining the lower limit of the second spectrum. However, there are a number of indications that this should not be so. For example, the cut-off frequency can be captured using Timoshenko theory, since Eq.(8) is valid in the range $0 < q < b^2r^2s^2 < 0$ and therefore offers the possibility of infinitely close upper and lower bounds on the cut-off frequency, subject only to the accuracy of the host computer. i.e. the accuracy to which any other frequency can be converged upon. Furthermore, it can be shown that the cut-off frequency and its mode shape correspond
exactly to the lowest frequency and mode shape of an infinite family of pure shear modes that are
governed by a second order differential equation [2]. Since the pure shear mode corresponding to the cut-
off frequency is clearly a possible mode of a hinged-hinged member, it implies that the determinant in a
transcendental dynamic stiffness approach must change sign through zero rather than infinity. Thus any
theory, exact or approximate, that seeks to converge on Timoshenko frequencies can include the cut-off
frequency in its spectrum by ensuring that its root count algorithm increases by one as the trial frequency
passes through the cut-off frequency. Finally, it is shown in Section 3 that there is always a Bernoulli-
Euler frequency that maps to the cut-off frequency. The implications stemming from the above points are
discussed later in Sections 4 and 5.

The factorisation of the hinged-hinged frequency equation, which leads to the two frequency spectra of
Timoshenko beam theory, has been dealt with by a number of authors, as alluded to in the Introduction.
Stephen [8] has also described the extensions necessary to include the related cases of guided-guided and
hinged-guided boundary conditions. However, there are also some simple physical relationships that exist
between these boundary conditions that are of interest and these are presented in the Appendix.

In order to facilitate further comparison, it is necessary to establish the manner in which Eq.(8)
factorises in the case of a hinged-hinged member to yield the two spectra of Timoshenko frequencies.
This is described in detail in [12] and in outline below. Thus from [12], and with the current notation, the
required natural frequencies correspond to

\[
\sin \Phi_i = 0 \quad \text{when} \quad q_i > 0 \quad i = 1, 2, \ldots, \infty \quad (15a)
\]

and

\[
\sin \Phi_i \sin \Lambda_i = 0 \quad \text{when} \quad q_i < 0 \quad i = 1, 2, \ldots, \infty \quad (15b)
\]

where

\[
\Phi_i^2 = \Delta_i + \left(\Delta_i^2 + q_i b_i^2 / t \right)^{1/2} \quad \text{and} \quad \Lambda_i^2 = \Delta_i - \left(\Delta_i^2 + q_i b_i^2 / t \right)^{1/2} \quad (16a, b)
\]

and the right hand sides of Eqs.(16) are defined by Eqs.(2). The frequencies, \(b_{\Phi,i}\), corresponding to
\(\sin \Phi_i = 0\) constitute the first spectrum, while those frequencies, \(b_{\Lambda,i}\), corresponding to \(\sin \Lambda_i = 0\)
constitute the second. The frequencies in the first and second spectrum can therefore be found from,Eqs.(16) by substituting, respectively,

\[
\Phi_i = i\pi \quad \text{and} \quad \Lambda_i = i\pi \quad i = 1, 2, \ldots, \infty \quad (17a, b)
\]

The frequencies \(b_{\Phi,i}\) and \(b_{\Lambda,i}\) are evidently the solutions of Eq.(9) and hence of Eq.(10). They must
therefore be identical to the solutions of Eq.(10) that stem from Eqs.(12), with the result that

\[
b_{\Phi,i} = b_{L,i} \quad \text{and} \quad b_{\Lambda,i} = b_{U,i} \quad (18a, b)
\]

2.3.1. Mode shape retrieval and interpretation

The mode shapes corresponding to those frequencies obtained through Timoshenko theory are
straightforward to develop by any appropriate method. In the current context of converging on the
required frequencies using a stiffness formulation and the Wittrick-Williams algorithm [10], the method
of mode extraction used is described in detail by Hopper and Williams [14] and implemented in the
computer program described in [13]. The mode shape corresponding to the cut-off frequency is the only
possible pure shear mode and is easy to recognise, since the lateral displacement is zero and the bending
slope is a constant value along the length of the member. Hence the modal number is either known by
inspection or can be calculated \textit{a priori} through Eq.(21). All other modes are flexural with those below
the cut-off frequency stemming from the first spectrum and those above the cut-off frequency stemming
from either. For those modes occurring above the cut-off frequency, it is helpful to identify their origin
either from Eqs.(12) or (16) and these will be designated $\Phi_m$ and $\Lambda_n$ modes, corresponding to the first and second spectrum, respectively. Since the modes are functions of sine terms and the member has hinged-hinged supports, all modes will have $m$ or $n$ half sine waves along the full length of the member and the lateral displacement will be either symmetric or antisymmetric about its mid-length depending on whether $m$ or $n$ is odd or even, respectively, with the corresponding rotational displacement being out of phase by $3\pi/2$ for $\Phi_m$ modes and $\pi/2$ for $\Lambda_n$ modes. This is due to the fact that the sign of the shear slope and the sign of the bending slope can be shown to be the same for the $\Phi_m$ modes and opposite for the $\Lambda_n$ modes [15]. It also confirms that the corresponding $\Phi_m$ and $\Lambda_n$ modes, in which $m = n$, satisfy the necessary condition of independence between themselves and every other mode. These points are further clarified in Tables 4 and 5 and Figs. 1 and 2, which highlight a selection of mode shapes stemming from the results of the Numerical Example of Section 3.

2.4. A ‘precise’ approach

In the ‘precise’ approach used herein, the original uniform member of length, $L$, is notionally divided into $NS$ segments of equal length, $LS$, such that

$$NS = 2^M \quad \text{and} \quad LS = L / NS$$

(19a,b)

where $M$ is an integer that ultimately defines the numerical accuracy of the solution. An ‘appropriate’ dynamic stiffness matrix is then formulated for a single datum segment of length $LS$, using a theory that reflects, as accurately as possible, the behaviour of the original member, but which lends itself to simpler solution. The first step is then to determine the number of clamped ended frequencies of the datum segment that have been passed by the trial frequency. An approximation to the original member is then reconstituted from the datum segment in $M$ doubling procedures. In the first of these, two datum segments are joined together end to end to form a new segment of length $2 \times LS$. This implies the addition of two datum stiffness matrices followed by Gauss elimination to eliminate the central node. This is also a necessary step in the Wittrick-Williams algorithm [10] for accumulating the number of natural frequencies passed by the trial frequency. This doubling procedure is then used recursively [16-18] a further $M – 1$ times, until an approximation to the original member has been formed in terms of its dynamic stiffness matrix. Once this has been achieved, the boundary conditions are imposed on the resulting stiffness matrix and the Wittrick-Williams root counting algorithm [10] is completed by establishing the ‘sign count’ of the final matrix i.e. the number of negative leading diagonal elements of the matrix following Gauss elimination in its standard form. Iterative use of this process enables convergence upon the currently required natural frequency, and subsequently on all required natural frequencies to any desired accuracy with the certain knowledge that none have been missed.

2.4.1. The hinged-hinged case in the absence of a distributed foundation

In this case, the ‘appropriate’ datum stiffness matrix is developed using exact B-E theory with exact allowance for axial load and shear deformation. i.e. the dynamic stiffness equivalent of Eq.(11) that clearly has a single, continuous frequency spectrum. The distributed rotary inertia of the datum segment is then calculated and half is allocated as a lumped inertia to each of the direct rotational stiffness locations of the datum stiffness matrix. The member is then reconstituted from the datum matrix in the way described above, with the result that an equivalent Timoshenko member has been formed through a route that has eliminated the possibility of a second spectrum. It can be seen that the ‘precise’ results presented for the numerical example of Section 3 show exceptionally good accuracy when compared to the equivalent exact results.

2.4.2. The hinged-hinged case in the presence of a distributed foundation
The relatively simple boundary conditions of this case enable Eq.(1) to be written in the form of Eq.(5) and subsequently Eq.(6). However, the difficulties in solving Eq.(1) exactly, are also apparent in Eq.(6), where the discriminant of the quadratic equation can be awkwardly volatile and difficult to deal with. Thus it is useful to adopt once more the approach described in Section 2.4.1 above, but this time based on Eq.(7). This presents a more difficult problem than previously, until it is noted that Eq.(11) is Eq.(7) with the foundation stiffnesses removed. It is then a simple matter to use the original datum segment matrix, but now augmented with lumped stiffnesses, equivalent to the original distributed foundation stiffnesses, which are divided equally between the appropriate direct stiffness locations at each end of the datum segment. The process of reconstitution is then undertaken as before.

Finally, it is interesting to note from either Eq.(1) or Eq.(5), that there now appears to be two possible discontinuities, namely at

\[ b_i^2 = \frac{1 + s^2 k_0^*}{r^2 s^2} \text{ and } b_i^2 = k_y^* \quad (20a,b) \]

The first is a variation of the original cut-off frequency, modified solely by the rotational component of foundation stiffness, which therefore maintains its identity as a pure shear mode, while the second is a function of the lateral foundation stiffness only. In the ‘precise’ approach proposed herein, the substitution of approximate lumped foundation stiffnesses to replace the original distributed stiffnesses, relegates the root counting procedures necessary to account for Eqs.(20), to the purely mechanical task of establishing the ‘sign count’ of the reconstituted, dynamic member stiffness matrix, as discussed in Section 2.4.2. In contrast, the complexities imposed by these discontinuities in an exact environment have been discussed in detail by Capron and Williams [11].

2.4.3. The ‘precise’ solution of Eq.(1) with any combination of elastic end conditions

It should be noted that the problem described in Section 2.4.2 requires no specific knowledge of Eq.(1), but only the datum segment matrix of the equivalent Timoshenko beam developed in Section 2.4.1 and its augmentation with equivalent foundation stiffnesses in Section 2.4.2. Thus the only difference between the hinged-hinged case and this one, is the imposition of any combination of nodal elastic boundary conditions at the end of each iteration of the Wittrick-Williams process and their subsequent effect on the ‘sign count’, as discussed in Section 2.4.

3. Numerical example

The problem of a hinged-hinged Timoshenko beam, originally solved by Levinson and Cooke [5] and extended by Stephen [8] is now considered in a variety of ways. The basic member data are as follows. Young’s modulus \( E = 210 \text{ GN/m}^2 \), density \( \rho = 7850 \text{ kg/m}^3 \), Poisson’s ratio \( \nu = 0.3 \), the shear coefficient \( k = 5(1 + \nu)/(6 + 5\nu) \), length \( L = 0.5 \text{m} \), depth \( d = 0.125 \text{m} \) and breadth equal to unity. The axial load in the member when not zero is \( 675.0 \pm \text{GN} \), which is approximately half its Euler load when in compression.

4. Results and discussion

The results emanating from the analysis of the above data are presented in Tables 1 to 7 and Figs. 1 and 2. In each of Tables 1 to 3 and 6 to 7, the columns are numbered to facilitate description and the core results forming the body of the table are the required non-dimensional frequency values, \( b_i \). Each table is divided into three sections, depending on whether the axial load is zero, tensile or compressive. The results in the tensile and compressive sections follow the results for the unloaded member in a predictable way and create no anomalies. The descriptions given thus relate to all three sections of the tables. Tables 4 and 5 are structured differently and together with Figs. 1 and 2 describe the attributes of a selection of mode shapes that typify the motion over the range of frequencies examined.

Table 1 lists the non-dimensional frequency parameters, \( b_i \), stemming from Eq.(11) with \( s^2 = 0 \) (columns 2, 4 and 6) and \( s^2 \neq 0 \) (columns 3, 5 and 7). Consider first the odd numbered columns. It is
clear that each of these results, which allow exactly for the effects of axial load and shear deformation in the member, but make no allowance for rotary inertia, bare a unique relationship to their corresponding B-E frequencies. These results therefore constitute part of an infinite, unbroken single spectrum of frequencies. In contrast, the equivalent Timoshenko frequencies listed in Table 2 are generated in pairs, being the lower and upper solutions of Eq.(10), the governing quadratic equation, which is seeded by a single value. Thus the notion that a single B-E frequency is uniquely related to a single Timoshenko frequency appears to be unfounded. In turn, this throws doubt on a theory that purports to improve the accuracy of all frequencies stemming from the simpler B-E model.

Table 3 compares results from a number of equations in the body of the paper, together with additional supporting information. The exact results in column 2 are generated from Eq.(11) with $s^2 \neq 0$ and provide an upper bound datum for the remainder of the frequencies shown in the table. The results from the ‘precise’ approach, which is described in Section 2.4, are presented in columns 3-6 and correspond to the number of doubling procedures undertaken in each case. $M = 3$ indicates that the reconstituted member would be modelled by eight segments. In contrast, $M = 12$ indicates that it would be modelled by 4096 segments in only twelve doubling procedures. In the latter case, the segment length for the current problem would be less than 0.125mm long. Columns 3-6 therefore chart the convergence of the ‘precise’ frequencies to their exact counterparts in column 7, with increasing $M$. It should be noted that the ‘precise’ results of column 6 are identical to the exact Timoshenko results of column 7, which were determined from a general computer program based on Eq.(8) and using exact member theory [12,13]. Columns 8 and 9 give the lower (first spectrum) and upper (second spectrum) frequencies satisfying Eqs.(12), but now written in ascending order and retained in their respective columns for comparison. The table is rounded off by columns 10 and 11, which note the values of $\Phi$ and $\Lambda$ for each of the first and second spectrum frequencies, respectively. This confirms that all first and second spectrum frequencies correspond to integer multiples of $\pi$ and that the values in columns 8 and 9 also satisfy Eqs.(16). Table 3 is important because it shows beyond all doubt that the ‘precise’ model, which cannot have a second spectrum, yields results for sufficiently large $M$, that match identically with the exact Timoshenko results from any of the sources described in the paper, when written in ascending order. Furthermore, if the results of the precise approach given in column 6, which comprise a single continuous spectrum, are now taken as the datum, there is no indication, to the accuracy of the results presented, that the inherent accuracy of second spectrum frequencies is any different to that of the first spectrum, as has been suggested elsewhere [8].

Table 4 identifies the modal type associated with each of the fifteen frequencies identified in Table 3 according to the procedure described in Section 2.3.1., which clearly parallels the data of Table 3. Thus it can be gleaned from either table, that the frequency/mode sequencing does not adhere to a fixed pattern and hence that, in any analysis, it will always be beneficial to be in a position to define frequency/modal provenance.

Table 5 gives numerical values of the mode shape corresponding to the cut-off frequency, together with a selection of typical first and second spectrum flexural modes, taken across the load categories and which occur predominantly around the cut-of frequency or at higher frequencies. In each case the mode has been scaled by the absolute value of its largest element. It is interesting to note that on close comparison between mode $(p^2 = 0, \Phi_5)$ and mode $(p^2 < 0, \Phi_5)$ there is very little difference between them, thus confirming that an axial load has little effect on such squat members at high frequencies. Although not shown, this pattern is typical across all modes in Table 4 with the same $m$ value and likewise with all modes with the same $n$ value.

The flexural mode shapes given in Table 5 are also reproduced in diagrammatic form in Fig. 1. In this case, each mode shape was constructed by calculating a scale factor for the lateral displacement by dividing the reciprocal of the logarithm of peak rotational displacement by peak lateral displacement. This has the effect of amplifying smaller displacements to make them visually identifiable while imposing negligible effects on the larger values. This is particularly helpful in the case of second spectrum $(\Lambda_n)$ modes, where it can be shown that the lateral displacements are particularly small over a wide range of frequencies [15]. Fig. 2 highlights the important difference that exists between corresponding first and second spectrum modes in which $m = n$. In the first pair, $m = n = 1, p^2 < 0$ and
the lateral displacement is symmetric about the mid-length of the member, while in the second pair \( m = n = 2 \), \( p^2 < 0 \) and the lateral displacement is anti-symmetric. The difference between the modes in each pair is clear, thus confirming their mutual independence, as discussed in Section 2.3.1.

Table 6 presents the results determined from the ‘precise’ approach for \( M = 12 \) when the distributed rotary inertia of the member is scaled by the factors shown. Factors of zero and one therefore yield the results for Eqs. (11) and (8), in columns 2 and 12, respectively. The intermediate values then indicate a smooth and monotonic transition between the two. This shows that there is a unique, one-to-one correspondence between each B-E frequency and its Timoshenko counterpart, which most importantly includes the cut-off frequency. Furthermore, the modal number of the B-E frequency that maps to the cut-off frequency can be determined \textit{a priori} from the problem data, as follows. Eqs. (14) define the cut-off frequency. Thus, substituting these in Eq. (16a) and noting that the cut-off frequency corresponds to the first value of \( \Phi \) that is not an integer multiple of \( \pi \), the required modal number is given by

\[
\text{the smallest integer } > \left( \frac{r^2 + s^2}{(1 - s^2 p^2)} \right)^{1/2} / r s \pi
\]

which can be tested easily using the current data and results.

These facts, together with the fact that Timoshenko theory provides infinitely close bounds on the cut-off frequency, provide a compelling argument that the cut-off frequency should be thought of as a true Timoshenko frequency.

Table 7 gives the non-dimensional frequency parameters, \( b_i \), from the ‘precise’ approach with \( M = 12 \), for a variety of foundation stiffness combinations. The distributed foundation stiffnesses are additionally incorporated into the original ‘precise’ model by adding equivalent lumped stiffnesses into the appropriate locations of the datum segment stiffness matrix. The table shows that; the cut-off frequency remains unchanged when \( k_\theta = 0 \) and rises otherwise; there are no additional frequencies as a consequence of the term \( b_i^2 - k_\theta^2 \) and that the effect of a distributed rotational foundation stiffness is greater than that of its lateral counterpart.

5. Conclusions

The main conclusions that can be deduced from the body of the paper are set out below. Initially, however, it is useful to consider two generic areas of note.

Firstly, the general equations presented allow for any combination of second order effects, including static axial load. Traditionally, this latter option has rarely been considered in Timoshenko theory, despite the fact that it often influences the lower frequencies, creates no anomalies and offers a more complete description of the problem.

Secondly, the addition of a distributed elastic foundation generates an intractable problem from which exact solutions are difficult to obtain. One exception to this is a simple formula that shows how the rotational component of foundation stiffness uniquely raises the cut-off frequency, while retaining its identity as a pure shear mode. On the other hand, the ability to provide general solutions to such problems has been achieved through the use of a simple and extremely efficient ‘precise’ approach that has been shown to offer excellent accuracy.

The remaining conclusions assume the absence of a distributed foundation, but are unaffected by whether or not a static axial load is present in the member. In the first of these, it has been shown that both spectra of Timoshenko frequencies can be determined from the single quadratic equation governing their exact solution. This classical problem is easily solved, but fails to determine the cut-off frequency, since it corresponds to a singularity in the equation at that point. However, this is easily remedied by use of a simple, exact formula or, conversely, the value can be converged upon to any desired accuracy, since the theory is valid at infinitely close lower and upper bounds. The modal vectors corresponding to all such frequencies are easy to obtain by conventional methods, such as forward elimination and backward substitution of the stiffness equations in the present case. A typical selection are presented, both in numerical and graphic form, with particular attention given to confirming the necessary independence of the second spectrum modes.
The complete range of exact Timoshenko frequencies, described in the paragraph above, can also be generated using the ‘precise’ technique, when programmed to emulate Timoshenko theory. In this form, it generates a single spectrum of frequencies that, to the accuracy of presentation, match identically the exact Timoshenko frequencies, including the cut-off frequency. When this is linked to the following facts, namely: that one frequency from the continuous B-E spectrum, whose modal value can be identified \textit{a priori} from the problem data, maps exactly to the cut-off frequency; that all remaining B-E frequencies each map uniquely to a single Timoshenko frequency in either the first or second spectrum; that any theory which seeks to improve accuracy must do so without loss or gain of any frequencies; it is our conjecture that all frequencies in the first and second spectra, together with the cut-off frequency, comprise the single spectrum of Timoshenko frequencies.

References

Appendix  Some simple physical relationships

The uniform beam shown in Fig. A1 (a) and (b) is symmetric about its mid-length. Therefore the modes of vibration in each case may be divided into a set that is symmetric about the mid-length, which may be found, respectively, from the half members of Fig. A2 (aS) and (bS), and an anti-symmetric set given by the half members of Fig. A2 (aA) and (bA).

Fig. A1. Two alternative sets of boundary conditions of a uniform beam of length $L$. The hinged-hinged supports of (a) allow rotation, but prevent displacement, while those of (b) allow precisely the opposite.

Fig. A2. Half members used to find the symmetric (S) and anti-symmetric (A) modes of the beam of Fig. A1.

Fig. A2 (aS) and (bA) clearly both define the same physical problem and therefore give the same natural frequencies. Equally clearly, Fig. A2 (aA) and (bS) are related to each other in exactly the same way as the two cases of Fig. A1 (a) and (b). Hence the argument used to obtain Fig. A2 from Fig. A1, and the associated deductions given above, can be applied recursively until the members of Fig. A2 are of infinitesimal length. Thus it follows that all natural frequencies for the case of Fig. A1 (a) are identical to those of Fig. A1 (b), except that the former has any natural frequencies possessed by an infinitesimally short beam of the type shown in Fig. A2 (aA), whereas the latter has any natural frequencies possessed by an infinitesimally short beam of the type shown in Fig. A2 (bS). This accounts for the fact that the beam of Fig. A1 (a) does not have a rigid body mode, while that of Fig. A1 (b) does.

In the physical argument used above, it is only assumed that the member is uniform. The arguments must therefore apply to any member with any uniform combination of member properties, second order effects and other externally imposed conditions, such as distributed foundations.

The corollary to the above, is that a uniform member of length, $L$, with hinged-guided boundary conditions will have all the natural frequencies corresponding to either the symmetric modes of a hinged-hinged beam or the anti-symmetric modes of a guided-guided beam, of length $2L$ in each case.

The conclusions that can be drawn from the above are:

1. that the non-zero frequencies of a guided-guided beam are identical to those of a hinged-hinged beam; and
2. the frequencies stemming from a beam with any combination of the above boundary conditions, can always be found from a beam in which the frequency equation factorises.
Fig. 1. Diagrammatic representation of the flexural mode shapes given in Table 5. The dashed and solid lines represent lateral displacement and bending slope respectively.
Fig. 2. Comparison of modes \((p^2 < 0, \Phi_1, \Lambda_1)\) and modes \((p^2 < 0, \Phi_2, \Lambda_2)\) in order to illustrate the independence of second spectrum modes with respect to their first spectrum counterparts.
Table 1
Exact non-dimensional frequency parameters, $b_i$, stemming from Eq. (11) with $s^2 = 0$ (columns 2, 4 and 6) and $s^2 \neq 0$ (columns 3, 5 and 7). These results are for the data given in Section 3 and it should be noted that $b_{p,i} = b_{0,i}$ when $p^2 = 0$ etc.

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Table 2
Exact, non-dimensional, Timoshenko frequency parameters, $b_i$, corresponding to the lower, $b_{L,i}$, and upper, $b_{U,i}$, solutions of Eq.(10), which are calculated from Eqs.(12) using the data given in Section 3.

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Table 3
The non-dimensional frequency parameters, $b_i$, given by the ‘precise’ approach for various values of $M$, are compared with those from relevant equations in the body of the paper, together with illustrative supporting results. $b_{co}$ is the cut-off frequency and the data are those given in Section 3.

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$p^2 > 0$

$\Lambda$

$\Phi$

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Table 4

Modal identification for the fifteen exact frequencies of the hinged-hinged Timoshenko beam for each load category, as set out in Table 3. $S$ signifies the Shear mode, while $m$ and $n$ denote the flexural mode numbers corresponding to those frequencies stemming from the first and second spectrum, respectively.

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<td>3</td>
<td>4</td>
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<td>6</td>
<td>7</td>
<td>8</td>
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<tr>
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Table 5

Numerical examples for a selection of typical, normalised mode shapes that can be identified by their $i$ or $b$ values (Table 3) or their $m$ or $n$ values (Table 4). $V_\xi$, $Y_\xi$ and $\Psi_\xi$ are the amplitudes of the lateral displacement, shear slope and bending slope, respectively, at a distance $\xi$ from the left hand end of the member.

$$V_\xi, Y_\xi, \Psi_\xi$$ are the amplitudes of the lateral displacement, shear slope and bending slope, respectively, at a distance $\xi$ from the left hand end of the member.

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<td>$b=108.361$</td>
<td>$b=121.381$</td>
<td>$b=85.8052$</td>
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<tr>
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<td>$i=7, n=1$</td>
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</tr>
<tr>
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<td>$(\Phi_5)$</td>
<td>$(A_1)$</td>
<td>$(\Phi_4)$</td>
</tr>
<tr>
<td>$\xi=x/L$</td>
<td>$V_\xi$</td>
<td>$Y_\xi$</td>
<td>$\Psi_\xi$</td>
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<td>0.000</td>
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<td>-1.000</td>
<td>-0.124</td>
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Table 6
Non-dimensional frequency parameters, $b_i$, from the ‘precise’ approach with $M = 12$ when the distributed rotary inertia is shown. The data are those given in Section 3.

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and distributed rotary inertia factor $=$

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<tr>
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<table>
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<tr>
<td>14</td>
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<td>15</td>
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</table>
### Table 7

Non dimensional frequency parameters, $b_i$, from the ‘precise’ approach with $M = 12$ and the data given in Section 3, for a selection of two parameter foundations. The values in bold type correspond to the cut-off frequencies and the units of $k_\theta$ and $k_y$ are N and Nm$^{-2}$ respectively.

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$p^2 = 0$

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## Footnotes:

*Note: The values in bold type correspond to the cut-off frequencies and the units of $k_\theta$ and $k_y$ are N and Nm$^{-2}$ respectively.*