Stokes’ Phenomenon arising from the confluence of two simple poles

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Additional Information:

- A Doctoral Thesis. Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University.

Metadata Record: https://dspace.lboro.ac.uk/2134/28357

Publisher: © Horrobin, Calum

Rights: This work is made available according to the conditions of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0) licence. Full details of this licence are available at: https://creativecommons.org/licenses/by-nc-nd/4.0/

Please cite the published version.
Stokes’ Phenomenon Arising from the Confluence of Two Simple Poles

by
Calum Horrobin

A Doctoral Thesis
Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy of Loughborough University

31st January 2018
© by Calum Horrobin 2018
Abstract

We study certain confluences of equations with two Fuchsian singularities which produce an irregular singularity of Poincaré rank one. We demonstrate a method to understand how to pass from solutions with power-like behavior which are analytic in neighbourhoods to solutions with exponential behavior which are analytic in sectors and have divergent asymptotic behavior. We explicitly calculate the Stokes’ matrices of the confluent system in terms of the monodromy data, specifically the connection matrices, of the original system around the merging singularities. The confluence of Gauss’ hypergeometric equation gives an excellent opportunity to show our approach with a concrete example. We explicitly show how the Stokes’ data arise in the confluences of the isomonodromic deformation problems for the Painlevé equations $P_{VI} \rightarrow P_V$ and $P_V \rightarrow P_{III}^{D_6}$.

Key word and phrases. Painlevé equations, hypergeometric differential equations, confluence, monodromy, isomonodromic deformations, asymptotic expansions, analytic functions.
Acknowledgements

I am enormously grateful to my supervisor Professor Marta Mazzocco for her continual support and encouragement throughout my studies. I must also thank my former teacher Dr Jort van Mourik for giving me the opportunity to do research with Marta. Thank you to all of the staff at Loughborough University, I have thoroughly enjoyed learning as much as I can. Finally, thank you to all of my family and friends, especially my Dad Simon, my Mum Sue, my sister Kate and my partner Yanwen.
Table of Contents

Chapter 1 Introduction 1

Chapter 2 General Theoretical Background 8
  2.1 Linear Meromorphic ODEs .............................. 8
  2.2 Defining Monodromy Data .............................. 14
  2.3 A Result of Glutsyuk ................................. 18

Chapter 3 Hypergeometric Differential Equations 24
  3.1 Gauss’ Hypergeometric Differential Equation .......... 25
    3.1.1 Solutions ....................................... 26
    3.1.2 Monodromy Data ................................ 31
  3.2 Kummer’s Confluent Hypergeometric Equation .......... 43
    3.2.1 Solutions ....................................... 44
    3.2.2 Monodromy Data ................................ 51
    3.2.3 Gevrey Asymptotics and a Result of Ramis and Martinet . 66
  3.3 A Confluence from Gauss to Kummer ................. 69
    3.3.1 Limits of Solutions ............................... 70
    3.3.2 Limits of Monodromy Data ......................... 90

Chapter 4 The Sixth and Fifth Painlevé Equations 100
  4.1 Background ........................................... 101
    4.1.1 Auxiliary Linear System for $P_{VI}$ ............ 101
    4.1.2 Auxiliary Linear System for $P_{V}$ ............ 106
    4.1.3 A Confluence Procedure from $P_{VI}$ to $P_{V}$ .... 113
  4.2 From Fuchsian Singularities to Fuchsian Singularities .. 120
    4.2.1 Obtaining $Y_{5}^{(0)}(\lambda_5)$ from $Y_{6}^{(0)}(\lambda_6)$ .. 121
    4.2.2 Obtaining $Y_{5}^{(t)}(\lambda_5)$ from $Y_{6}^{(t)}(\lambda_6)$ ... 123
Table of Contents

4.3 From Fuchsian Singularities to an Irregular One .................................. 126
   4.3.1 Taking a Term-By-Term Limit of the Solution $Y_{6}^{(\infty)}(\lambda_6)$ .... 127
   4.3.2 Taking a Term-By-Term Limit of the Solution $Y_{6}^{(1)}(\lambda_6)$ .... 132
   4.3.3 Obtaining the Solutions $Y_{5}^{(\infty,k)}(\lambda_5)$ .......................... 141
4.4 Limits of Monodromy Data ................................................................. 147

Chapter 5 The Fifth and Third Painlevé Equations ................................. 154
   5.1 Background ................................................................. 155
      5.1.1 Auxiliary Linear System for $P_{III}^{\text{D}_6}$ .............. 155
      5.1.2 A Confluence Procedure from $P_V$ to $P_{III}^{\text{D}_6}$ .......... 160
   5.2 From Fuchsian Singularities to an Irregular Singularity ...................... 164
      5.2.1 Taking a Term-By-Term Limit of the Solution $Y_{5}^{(0)}(\lambda_5)$ .... 164
      5.2.2 Taking a Term-By-Term Limit of the Solution $Y_{5}^{(t)}(\lambda_5)$ .... 166
      5.2.3 Obtaining the True Solutions $Y_{3}^{(0,k)}(\lambda_3)$ ............... 167
   5.3 Limits of Monodromy Data ....................................................... 170

Chapter 6 Conclusions and Outlook ......................................................... 173

References .................................................. 174
Chapter 1

Introduction

In this thesis we study the confluence of two simple poles in a system of two first order linear ordinary differential equations which produces a new system with an irregular singularity of Poincaré rank one. We are primarily concerned with understanding the behavior of the monodromy data in the confluence limit. We answer the question of how the monodromy data around the merging singularities of the original system tend to the two Stokes’ matrices attached to the newly created double pole of the confluent system.

The results of this thesis solve this problem by making use of a certain existence theorem by Glutsyuk [Glu]. Essentially, this states that there exist diagonal matrices $K_\varepsilon$ and $K_{-\varepsilon}$ such that the limit,

$$\lim_{\varepsilon \to 0} K_{-\varepsilon}^{-1} C K_\varepsilon,$$

where $C$ is the connection matrix between the merging simple poles, gives one Stokes’ matrix if $\varepsilon$ is taken along a certain ray and the other Stokes’ matrix if taken along the opposite ray. However, this existence theorem is limited in that there is no explanation of how to calculate the diagonal matrices $K_\varepsilon$ and $K_{-\varepsilon}$. Our main results demonstrate a procedure in three cases to calculate such diagonal matrices and thus produce the Stokes’ matrices in terms of limits of the monodromy data of the original equation explicitly. The cases we study are: the confluence of Gauss’ hypergeometric differential equation and of the auxiliary linear systems associated to the Painlevé equations for $P_{VI} \to P_V$ and $P_V \to P_{III}^{D_6}$.

The motivation for our work comes from the desire to better understand the confluences
of the Painlevé equations. At the start of the twentieth century, Paul Painlevé (1853 - 1933) worked on finding new transcendental functions. To this end, he turned his attention to the problem of classifying all second order ODEs of the form,

$$\frac{d^2 y}{dt^2} = R\left(\frac{dy}{dt}, y, t\right), \quad t \in \mathbb{C},$$

where $R$ is a rational function of $\frac{dy}{dt}$, $y$ and $t$, which satisfy the following:

**Painlevé property.** The general solution has no movable critical points, that is, singularities that are determined by the initial conditions can only be poles.

Up to a Möbius transformation,

$$y(t) \mapsto \frac{r_1(t)y(t) + r_2(t)}{r_3(t)y(t) + r_4(t)}, \quad t \mapsto r_5(t),$$

where all $r_j(t)$ are analytic functions of $t$, Painlevé and his students managed to classify all such equations into a canonical list of fifty. Returning to one of Painlevé’s original motives, this list was then reduced to six equations by factoring out those which could be integrated in terms of known functions. These six equations are known as the Painlevé equations, they are listed below with complex parameters $\alpha, \beta, \gamma$ and $\delta$:

$$\begin{align*}
\frac{d^2 y}{dt^2} &= 6y^2 + t, \quad P_I \\
\frac{d^2 y}{dt^2} &= 2y^3 + ty + \alpha, \quad P_{II} \\
\frac{d^2 y}{dt^2} &= \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{1}{t} (\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \quad P_{III} \\
\frac{d^2 y}{dt^2} &= \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2} y^3 + 4ty^2 + 2\left(t^2 - \alpha\right) y + \frac{\beta}{y}, \quad P_{IV} \\
\frac{d^2 y}{dt^2} &= \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{y - 1}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{(y - 1)^2}{t^2} \left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y + 1)}{y - 1}, \quad P_V \\
\frac{d^2 y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y - 1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{1}{t - 1}\left(\frac{1}{y} + \frac{1}{y - 1}\right)\left(\frac{dy}{dt}\right) + \frac{y(y - 1)(y - t)}{t^2(t - 1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma (t - 1)}{(y - 1)^2} + \frac{\delta (t - 1)}{(y - t)^2}\right), \quad P_{VI}
\end{align*}$$
The general solutions of the Painlevé equations, that is, solutions in terms of all parameters of the equation and two arbitrary constants of integration, are transcendental and irreducible. Loosely speaking, this means that, for general parameter values, the general solutions cannot be expressed by simple operations in terms of elementary functions, for a precise definition see [Nis1, Nis2, Ume].

The Painlevé equations $P_I - P_{VI}$ can be found through isomonodromic deformation problems of certain linear differential equations. The groundwork of the theory of isomonodromic deformations and its connections to equations with the Painlevé property was pioneered by Richard Fuchs in his discovery of the $P_{VI}$ equation in 1905 [Fuc] which, in turn, was based on the work of his father, Immanuel Lazarus Fuchs. Schlesinger [Sch] continued this work in his studies of isomonodromic deformation problems of linear systems of equations with an arbitrary number of Fuchsian singularities. In the 1970s and 1980s, the theory was significantly developed further by a number of authors including: Jimbo, Miwa, Ueno, Okamoto, Flaschka, Newell, Its, Kitaev, Fokas and Novokshenov [JMU, Oka1, FN, Its, IKF, IN].

Consider the following system of first order linear equations,

\[
\frac{\partial Y}{\partial \lambda} = A(\lambda, t)Y, \tag{1.1}
\]
\[
\frac{\partial Y}{\partial t} = B(\lambda, t)Y, \tag{1.2}
\]

where $A, B \in \mathfrak{sl}_2(\mathbb{C})$ are functions of $\lambda$ and $t$ which are rational in $\lambda$. The compatibility condition of these two equations is the following,

\[
\frac{\partial A}{\partial t} - \frac{\partial B}{\partial \lambda} = [B, A]. \tag{1.3}
\]

It is known [JM] that for each Painlevé equation, matrices $A$ and $B$ can be found such that the compatibility condition (1.3) is equivalent to the given Painlevé equation. For each Painlevé equation, the matrices $A$ and $B$ depend on the Painlevé transcendent $y$ and its derivative $y_t$. The first order linear equations (1.1) and (1.2) are called the auxiliary linear systems of the Painlevé equations. These equations are constructed to
satisfy the isomonodromic deformation property that, given a solution of these equations, its monodromy data around the poles of \( A(\lambda, t) \) are constant as \( t \) varies locally.

In this thesis we are interested in confluences of the isomonodromic deformation problems associated to the Painlevé equations. By this, we mean making certain substitutions on the parameters and variables and taking a limit which transforms one linear system into another. This process typically merges two singularities or maps a diagonalisable matrix to a non-diagonalisable matrix. In the cases where the leading matrices at each pole are diagonalisable, the \((2 \times 2)\) linear systems for each Painlevé equation have a different partition of four, which corresponds to the orders of the poles in the equation. We illustrate the confluence scheme of the linear systems in terms of their singularity structures below.

![Figure 1: Confluences of the auxiliary linear systems for the Painlevé equations.](image)

There are two distinct cases of the Painlevé equation \( P_V \) and three cases of \( P_{III} \) according to various conditions that their parameters satisfy. We also remark that two \((2 \times 2)\) isomonodromic deformation problems have been found for \( P_{II} \), one by Jimbo and Miwa [JM] and another by Flaschka and Newell [FN], these are not gauge equivalent. We are concerned with the general cases: when \( \delta \neq 0 \) in \( P_V \) and \( \gamma \delta \neq 0 \) in \( P_{III} \) (known as \( P_{III}^{D_6} \) in Sakai’s classification [Sak]). We are primarily interested in the confluences \( P_{VI} \to P_V \) and \( P_V \to P_{III}^{D_6} \) because these are the cases where two simple poles merge to form a double pole.
The monodromy data, including Stokes’ data, of the auxiliary linear systems for each Painlevé equation are constant along solutions of the associated Painlevé equation and, due to irreducibility, cannot be written down in closed form. In the recent work [CMR], Chekhov, Mazzocco and Rubtsov consider the degeneration of the monodromy cubic surfaces and are able to obtain the confluence procedure for the Painlevé equations in geometric terms. The problem of expressing the Stokes’ matrices as direct limits of the monodromy data was left open. Our work provides an analytic answer to understand explicitly how to take limits of the monodromy data around two merging simple poles and produce the Stokes’ matrices at the newly formed double pole.

If we take a formal limit passage from one auxiliary linear system (1.1)-(1.2) to another, then this also produces a formal limit passage between their compatibility conditions (1.3). In this way, we are able to write the leading asymptotic behavior of the Painlevé transcendent in terms of the next Painlevé transcendent and its derivative under our confluence procedures, this is shown in Theorems 4.4 and 5.3. One of our main tasks is to make sense of these formal limit passages by understanding how to pass from the solutions of the original system to the solutions of the confluent system. This is a non-trivial question because the behaviors of solutions around simple poles and double poles are very different. In particular, this involves passing from a solution with power-like behavior which converges in a disk to solutions with exponential behavior which are analytic in a sector and asymptotic to a divergent series.

To give some insight into the solution to this problem, we explain that we should re-write the fundamental solutions at the merging singular points in a specific way. For example, in the confluence of Gauss’ hypergeometric equation, we transform our solution around $x = \infty$, which in canonical form is made of terms of the form $x^{-\alpha - n}$ for some parameter $\alpha$, to include the term,

$$(1 - x)^{\gamma - \alpha - \beta}.$$ 

Now, under the substitution $x = \frac{z}{\alpha}$, observe how this term behaves under the limit
\[ \alpha \to \infty: \lim_{\alpha \to \infty} \left(1 - \frac{z}{\alpha}\right)^{\gamma - \alpha - \beta} = e^z, \]

which shows how to asymptotically pass from power-like behavior to exponential behavior. In addition, to give some more insight into our results, we note that a term-by-term limit of the new series in the transformed solution produces the divergent series which gives the asymptotic behavior of solutions of the confluent equation. In the case of the hypergeometric equations, this phenomenon is shown in Lemmas 3.10 and 3.11; for the confluence \( P_{VI} \to P_V \), we have Lemmas 4.5 and 4.7 and for the confluence \( P_V \to P_{III}^{D_6} \), we have Lemmas 5.1 and 5.2.

In [Kit3] Kitaev proposed an approach in which these problems are bypassed studying how to pass between solutions at singular points which remain simple poles under the confluence procedure. He is then able to use connection formulae and the cyclic relation among the monodromy matrices to retrieve the Stokes’ matrices. This approach does not apply to the confluence of the auxiliary linear systems \( P_V \to P_{III}^{D_6} \) because it does not tackle the problem directly, namely, it relies on the existence of simple poles which survive the confluence limit of which there are none for the \( P_{III}^{D_6} \) system. Our procedure produces the Stokes’ data by using only knowledge of the monodromy data at the merging singular points. This is the novelty of our work.

In this thesis we also deal with Gauss’ hypergeometric equation and its confluence to Kummer’s hypergeometric equation. Our first motivation to study the hypergeometric equations is to enrich our understanding about the intricacies of differential equations, analytic continuation and Stokes’ phenomenon. Secondly, the confluence of the hypergeometric equations give an excellent opportunity to demonstrate our method of producing the Stokes’ matrices from the monodromy data of the merging simple poles by performing calculations with explicit formulae. At the same time, we felt it is beneficial to write a self-contained, comprehensive chapter for the hypergeometric equations from which the inexperienced reader could study and learn from. There have been several works which deal with the confluence of Gauss’ hypergeometric monodromy data to that of Kummer’s. The approach by Lambert and Rousseau [LR] is similar to that.
of Kitaev [Kit3] in that they tackle the problem by staying near a Fuchsian singularity which survives the confluence limit. Watanabe [Wat] uses integral representations of Gauss’ hypergeometric \( _2F_1 \) function to produce the Mellin-Barnes integral representations of solutions of Kummer’s equation under a limiting process, this is possible due to a uniformity condition. Using limits of solutions, Watanabe is then able to deduce limits of monodromy data. However, both of these approaches rely on the fact that the hypergeometric equations have solutions and monodromy data expressible in closed-form. This does not apply to the auxiliary linear systems associated to the Painlevé equations. We provide a new approach which does not require explicit formulae nor the existence of additional simple poles which survive the confluence limit.

**Statement of contributions and new results**

We study three confluence procedures of linear ODEs which involve merging two simple poles to form a double pole. These cases are: the confluence of Gauss’ hypergeometric differential equation and of the auxiliary linear systems associated to the Painlevé equations for \( P_{VI} \rightarrow P_V \) and \( P_V \rightarrow P_{III}^{D_6} \). In each case, we deduce the solutions of the confluent equation as explicit limits of solutions of the original equation, by expanding on a certain existence theorem of Glutsyuk [Glu]. We are then able to deduce the monodromy data, including Stokes’ data, of the confluent system as limits of the monodromy data of the original system. In the confluence of the hypergeometric equations, the monodromy data were already explicitly known in terms of the parameters of the equations; our contribution here is to derive the relation between a connection matrix of Gauss’ equation and the Stokes’ matrices of Kummer’s equation using a method that had not been done before. Our approach is original since it does not require explicit formulae for the solutions and monodromy data nor the knowledge of monodromy data around additional simple poles which survive the confluence procedure.

In the confluence of the auxiliary linear systems \( P_{VI} \rightarrow P_V \), the novelty of our method is that we are able to deduce the Stokes’ matrices attached to the double pole by only using information about the monodromy data around the two merging simple poles. In the third demonstration of our approach, we are able to understand how Stokes’ phenomenon arises in the confluence of the auxiliary linear systems \( P_V \rightarrow P_{III}^{D_6} \), which had not been studied previously. Our new results are stated as Main Theorems 1-6.
Chapter 2

General Theoretical Background

Here, we provide some general theoretical background to support our work. In Section 2.1 we deal with the local analysis of solutions of linear meromorphic differential equations. The global analysis of solutions is then studied in Section 2.2, where we define the monodromy data of linear meromorphic differential equations. These two sections give us the opportunity to introduce our notation and establish the underlying fundamental ideas involved. While our general overview here is quite brief, in each of the cases we consider in this thesis we will cover all results in detail. We restrict our attention to the cases that are pertinent to our work, namely, \((2 \times 2)\) equations with poles of order at most two and whose leading matrices are diagonalisable with eigenvalues that do not differ by an integer. Comprehensive accounts of the more general cases can be found in [BJL, FIKN, Inc, JMU, Sib, Was, WW]. In Section 2.3 we state an important theorem by Glutsyuk which will be used in the proofs of our main theorems.

### 2.1 Linear Meromorphic ODEs

Since Gauss' hypergeometric differential equation is classically presented as a scalar equation, we begin by considering equations of the following form,

\[
\frac{d^2 y}{dx^2} + p_1(x) \frac{dy}{dx} + p_2(x) y = 0, \quad x \in \mathbb{C},
\]  

(2.1)

where \(p_1(x)\) and \(p_2(x)\) are rational functions of \(x\). In the following, the nature of the point \(x = \infty\) should be treated by transforming the equation using \(x = w^{-1}\) and examining the nature of the point \(w = 0\).

**Definition 2.1.** We call the point \(x = a\) a Fuchsian singularity of equation (2.1) if
and only if the coefficients have the following form,

\[ p_1(x) = (x - a)^{-1}P_1(x) \quad \text{and} \quad p_2(x) = (x - a)^{-2}P_2(x), \]

where \( P_1(x) \) and \( P_2(x) \) are analytic in a neighbourhood of \( x = a \).

We recall some facts about Fuchsian singularities and the method of Frobenius. Let \( x = a \) be a Fuchsian singularity of equation (2.1). We make the following ansatz of the form of a solution,

\[ y(x) = (x - a)^r \sum_{n=0}^{\infty} c_n(x - a)^n, \]

where \( r \in \mathbb{C} \) and \( c_n, \ n \geq 1 \) are to be determined and \( c_0 \neq 0 \) is arbitrary. After substituting this expression into the left hand side of equation (2.1), \( r \) is determined by equating the coefficient of \( (x - a)^{r-2} \) to zero, this is called the indicial equation. The terms \( c_n, \ n \geq 1 \), are determined recursively by equating the coefficients of \( (x - a)^{n+r-2} \).

It can be shown [Inc] §15.2 that the indicial equation will be a polynomial in \( r \) of degree 2 and that, if these roots do not differ by an integer, then the two solutions thus found are linearly independent [Inc] §15.31. Furthermore, the series \( \sum c_n(x - a)^n \) will converge in a sufficiently small neighbourhood of \( x = a \) [Inc] §16.2. The solutions of the indicial equation relative to \( x = a \) are called the exponents of the singularity \( x = a \) and they are denoted as \( r_{a,1} \) and \( r_{a,2} \). Equations with only Fuchsian singularities are called Fuchsian equations. Suppose equation (2.1) is Fuchsian with singularities \( x = a_1, \ldots, a_t, \infty \) which have exponents \( r_{1,1}, r_{1,2}, \ldots, r_{t,1}, r_{t,2} \) and \( r_{\infty,1}, r_{\infty,2} \) respectively. The sum of all exponents of each singularity must satisfy Fuchs’ condition [Inc] §15.4, namely,

\[ \sum_{s=1}^{t} (r_{s,1} + r_{s,2}) + (r_{\infty,1} + r_{\infty,2}) = t - 1. \quad (2.2) \]

We proceed with the analysis of equations of the following form,

\[ \frac{dY}{dx} = A(x)Y, \quad x \in \mathbb{C}, \quad (2.3) \]

where \( A(x) \) is a \((2 \times 2)\) meromorphic matrix, that is, the entries of \( A(x) \) are rational functions of \( x \). We remark that it is easy to pass from scalar equations to matrix
equations and vice versa. For example, the matrix given by,

$$Y(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ (x-a)y'_1(x) & (x-a)y'_2(x) \end{pmatrix},$$

is a fundamental solution of the matrix equation,

$$\frac{dY}{dx} = \begin{pmatrix} 0 & \frac{1}{x-a} \\ -(x-a)p_2(x) & -p_1(x) + \frac{1}{x-a} \end{pmatrix} Y,$$

if and only if $y_1(x)$ and $y_2(x)$ are linearly independent solutions of equation (2.1). In general, the column vectors of the $(2 \times 2)$ fundamental solution $Y(x)$ of a matrix equation of the form (2.3) can be constructed as linear combinations of the derivatives of a function which satisfies a certain scalar second order linear ODE.

**Definition 2.2.** We call the point $x = a$ a Fuchsian singularity of equation (2.3) if $A(x)$ has a simple pole at $x = a$.

**Definition 2.3.** Let $x = a$ be a pole of $A(x)$. If every fundamental solution $Y(x)$ of equation (2.3) has at most power-like behavior as $x \to a$ along a ray then the point $x = a$ is called a regular singularity.

The following theorem holds true,

**Theorem 2.1.** Let $x = a$ be a Fuchsian singularity of equation (2.3), in other words,

$$(x-a)A(x) = \sum_{n=0}^{\infty} A_n(x-a)^n,$$

where the series converges for $|x-a| < \rho$ for some $\rho > 0$. If the eigenvalues of $A_0$ do not differ by an integer then there exists an analytic fundamental solution of the form,

$$Y^{(a)}(x) = R_a \sum_{n=0}^{\infty} G_n(x-a)^n(x-a)^{\Theta_a}, \quad x \in \Omega_a$$

with the series involved being convergent in the neighbourhood,

$$\Omega_a = \{ x : |x-a| < \rho, \ \eta \leq \arg(x-a) < \eta + 2\pi \}, \quad \text{for any } \eta \in \mathbb{R},$$
where $R_a^{-1}A_0R_a = \Theta_a$ is a diagonal matrix and,

$$
\sum_{n=1}^{\infty} nG_n(x-a)^{n-1} + \frac{1}{x-a} \sum_{n=0}^{\infty} G_n(x-a)^n \Theta_a
\quad = \quad \frac{1}{x-a} \sum_{n=0}^{\infty} A_n(x-a)R_a \sum_{n=0}^{\infty} G_n(x-a)^n.
$$

**Proof.** See for instance [Was] Chapter II. We note that Wasow's use of the term *regular* corresponds to our use of the term *Fuchsian*.

We note, after setting $G_0 := I$, the coefficients $G_n, n \geq 1$, of the series in Theorem 2.1 are uniquely determined by equating the coefficients of $(x-a)^{n-1}$ in the final equation.

The condition that the eigenvalues of $A_0$ do not differ by an integer is called nonresonance, this is the case we are concerned with throughout our work. The condition on $\arg(x-a)$ in the above fundamental solution is to make the term $(x-a)^{\Theta_a}$ single-valued. Fundamental solutions defined on different sheets of the Riemann surface of the logarithm are related to each other by simple multiplication on the right by the so-called local monodromy matrix $e^{2\pi i \Theta_a}$.

Theorem 2.1 shows that, if a singularity of (2.3) is irregular then it is non-Fuchsian. We make the following definition.

**Definition 2.4.** We call the point $x = a$ an irregular singularity of Poincaré rank one of equation (2.3) if $A(x)$ has a double pole at $x = a$ and its leading matrix there is diagonalisable.

In the following, we place the irregular singularity at $x = \infty$, to fit with Kummer’s hypergeometric differential equation and the auxiliary linear system associated to $P_V$.

Before we state the main theorem concerned with solutions at irregular singular points, we recall the following definition of an asymptotic series, due to Poincaré.

**Definition 2.5.** Let $f(x)$ be a function defined in a set $\sigma \subseteq \mathbb{C}$, if

$$
 x^N \left( f(x) - \sum_{n=0}^{N} a_n x^{-n} \right)
$$
tends to zero as $x \to \infty$, $x \in \sigma$, for all $N \geq 0$, then we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n},$$

as $x \to \infty$, $x \in \sigma$.

**Theorem 2.2.** Let $x = \infty$ be an irregular singularity of Poincaré rank one of equation (2.3), in other words,

$$A(x) = \sum_{n=0}^{\infty} A_n x^{-n},$$

where the series converges for $|x| > R$, for some sufficiently large number $R$. Denote by $\mu_1$ and $\mu_2$ the eigenvalues of $A_0$. If $\mu_1 \neq \mu_2$ then, for all $k \in \mathbb{Z}$, there exist fundamental solutions $Y^{(\infty,k)}(x)$ which are analytic in the sectors,

$$\Sigma_k := \{ x : |x| > R, \pi(k-1) - \delta < \arg(x) < k\pi \},$$

for any number $\delta$ such that $0 < \delta < \pi$, which satisfy,

$$Y^{(\infty,k)}(x) \sim R_{\infty} \sum_{n=0}^{\infty} G_n x^{-n} x^{-\Theta_{\infty}} \exp \left( x \left( \begin{array}{cc} \mu_1 & 0 \\ 0 & \mu_2 \end{array} \right) \right), \quad \text{as } x \to \infty, \ x \in \Sigma_k, \quad (2.4)$$

where $R_{\infty}^{-1} A_0 R_{\infty} = \left( \begin{array}{cc} \mu_1 & 0 \\ 0 & \mu_2 \end{array} \right)$, $\Theta_{\infty}$ is the diagonal matrix formed from the diagonal entries of $-(R_{\infty}^{-1} A_1 R_{\infty})$ and,

$$- \sum_{n=1}^{\infty} n G_n x^{-n-1} + \sum_{n=0}^{\infty} G_n x^{-n} \left( \begin{array}{cc} \mu_1 & 0 \\ 0 & \mu_2 \end{array} \right) \frac{\Theta_{\infty}}{x} \right) = R_{\infty}^{-1} \sum_{n=0}^{\infty} A_n x^{-n} R_{\infty} \sum_{n=0}^{\infty} G_n x^{-n}.$$

Moreover, each solution $Y^{(\infty,k)}(x)$ is uniquely specified by the relation (2.4).

**Proof.** This is an instance of a well-known general result whose proof can be found in several texts such as [BJL,Was,Sib].

We note that, after setting $G_0 := I$, the coefficients of $x^{-1}$ in the above equation uniquely determine the off-diagonal part of $G_1$ and, in general, the coefficients of $x^{-n}$.
uniquely determine the diagonal part of $G_{n-1}$ and the off-diagonal part of $G_n$, for $n \geq 2$. Furthermore, for general parameter values, this uniquely constructed series diverges for all $x \in \mathbb{C}$.

**Definition 2.6.** Let $x = \infty$ be an irregular singularity of Poincaré rank one of equation (2.3) and define the solutions $Y^{(\infty,k)}(x)$ and sectors $\Sigma_k$ as in Theorem 2.2 above. For $k \in \mathbb{Z}$, we call the matrices $S_k \in \text{GL}_2(\mathbb{C})$ defined by the relation,

$$Y^{(\infty,k+1)}(x) = Y^{(\infty,k)}(x)S_k, \quad x \in \Sigma_{k+1} \cap \Sigma_k,$$

the Stokes’ matrices of equation (2.3) at the point $x = \infty$.

It is clear from the definition of the sectors in Theorem 2.2 that, for all $k \in \mathbb{Z}$, the projection of the sectors $\Sigma_k$ and $\Sigma_{k+2}$ onto the plane coincide. Fundamental solutions defined on such sheets of the Riemann surface of the logarithm are related to each other by simple multiplication on the right by the so-called local monodromy exponent $e^{2\pi i \Theta_\infty}$. In this sense, there are only two fundamentally distinct solutions $Y^{(\infty,k)}(x)$ and hence only two fundamentally distinct Stokes’ matrices $S_k$, namely when $k$ is even and when $k$ is odd.

From the asymptotic relation (2.4) and Definition 2.6, we have the following relation,

$$\exp \left( x \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \right) x^{-\Theta_\infty} S_k x^{\Theta_\infty} \exp \left( -x \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \right) \sim I, \quad (2.5)$$

as $x \to \infty$ with $x \in \Sigma_{k+1} \cap \Sigma_k$. This shows the importance of the eigenvalues of the leading matrix of $A(x)$ at $x = \infty$ and their ordering in the asymptotic relation (2.4). Observe that the dominant term in this expression changes according to the sign of the real part of the difference of the eigenvalues of $A_0$, that is, as $\text{Re}(\mu_1 - \mu_2)$ changes sign. This results in one type of Stokes’ matrix being unipotent, upper triangular and the other type being unipotent, lower triangular. This motivates the following definition.

**Definition 2.7.** Let $x = \infty$ be an irregular singularity of Poincaré rank one of equation
Denote the eigenvalues of the leading matrix at infinity as follows,
\[
\text{eigenv} (A_0) = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix},
\]
by assumption we have \(\mu_1 \neq \mu_2\). The rays,
\[
r_{j,k} := \{ x : \text{Re}(x(\mu_j - \mu_k)) = 0, \ \text{Im}(x(\mu_j - \mu_k)) > 0 \}, \quad j, k = 1, 2,
\]
are called Stokes’ rays.

In fact, it follows from similar reasoning as in (2.5) that we may extend the openings of the sectors \(\Sigma_k\) from Theorem 2.2, on which our fundamental solutions \(Y^{(\infty,k)}(x)\) are analytic, up to (but not including) the Stokes’ ray not already contained in that sector.

In the following chapters this is the practice we will follow.

### 2.2 Defining Monodromy Data

We proceed with the global analysis of solutions of linear meromorphic differential equations. Let \(A(x)\) in equation (2.3) have poles at the points \(x = a_1, \ldots, a_m, \infty\). We recall the following facts and definitions, following [Sib].

**Theorem 2.3 [Sib].** If \(U \subseteq \mathbb{C}\{a_1, \ldots, a_m\}\) is a disk, then there exists a fundamental solution \(Y(x)\) of equation (2.3) analytic on \(U\).

**Definition 2.8.** Let \(\gamma : [0, 1] \to \mathbb{C}\{a_1, \ldots, a_m\}\) be a continuous, orientable curve and let \(\{U_1, \ldots, U_N\}\) be an ordered collection of disks, where,
\[
U_n = \{ x : |x - x_n| < r_n \}, \quad n = 1, \ldots, N,
\]
for some \(x_n \in \mathbb{C}\) and \(r_n \in \mathbb{R}^+\). If there exists an ordered collection of intervals
\( \{I_1, \ldots, I_N\} \) such that:

\[
\{\gamma(t) : t \in [0, 1]\} \subset \bigcup_{n=1}^{N} U_n,
\]

\( I_n \cap I_m \neq \emptyset \iff \left| n - m \right| = 1, \)

\( t \in I_n \implies \gamma(t) \in U_n, \quad n = 1, \ldots, N, \)

\( U_n \cap U_{n+1} \neq \emptyset, \quad n = 1, \ldots, N - 1, \)

then we call \( \{U_1, \ldots, U_N\} \) a covering of \( \gamma \). See Figure 2 below.

\[
\begin{align*}
\text{Figure 2: A covering of the curve } &\gamma : [0, 1] \to \mathbb{C}\setminus\{a_1, \ldots, a_m\}. \\
\end{align*}
\]

Due to Theorem 2.3, there exists an ordered sequence of fundamental solutions \( \{Y_1(\lambda), \ldots, Y_N(\lambda)\} \) defined on each respective domain \( U_1, \ldots, U_N \).

**Theorem 2.4 [Sib].** Let \( \gamma : [0, 1] \to \mathbb{C}\setminus\{a_1, \ldots, a_m\} \) be a continuous, orientable curve, let \( \{U_1, \ldots, U_N\} \) be a covering of \( \gamma \) and let \( \{Y_1, \ldots, Y_N\} \) be an ordered sequence of fundamental solutions defined on each respective domain \( U_1, \ldots, U_N \). For \( N \geq 2 \), define constant matrices \( c_n \in GL_2(\mathbb{C}) \) as follows,

\[
c_n = Y_n(\lambda)^{-1}Y_{n-1}(\lambda), \quad \text{for } x \in U_n \cap U_{n+1}, \quad n = 2, \ldots, N.
\]

The matrix \( C(\gamma; U_1, \ldots, U_N) \in GL_2(\mathbb{C}) \) defined as,

\[
C(\gamma; U_1, \ldots, U_N) = \begin{cases} I, & \text{if } N = 1, \\ \prod_{n=2}^{N} c_n, & \text{if } N \geq 2, \end{cases}
\]

is determined by the homotopy class of \( \gamma, U_1 \) and \( U_N \).
Now, let the poles $x = a_1, \ldots, a_m, \infty$ of equation (2.3) be simple poles and let $Y^{(k)}(x)$ be the solutions analytic in the domains $\Omega_k$ as defined in Theorem 2.1, see Figure 3 below. For $j, k \in \{1, \ldots, m, \infty\}$, let $\gamma_{j,k} : [0, 1] \to \mathbb{C}\{a_1, \ldots, a_m\}$ be a continuous, orientable curve with $\gamma_{j,k}(0) \in \Omega_j$ and $\gamma_{j,k}(1) \in \Omega_k$. Due to the final statement of Theorem 2.4, we denote $C(\gamma_{j,k}; U_1, U_N)$ simply as $C^{kj}$, with the understanding that we have fixed the homotopy class of $\gamma_{j,k}$ once and for all according to the branch cuts in the definitions of the domains $\Omega_k$. We use Theorem 2.4, with the domains $U_1 \subset \Omega_j$ and $U_N \subset \Omega_k$ and with the fundamental solutions $Y_1(x) = Y^{(j)}(x)$ and $Y_N(x) = Y^{(k)}(x)$, to define the analytic continuation of the fundamental solution $Y^{(j)}(x)$ along the curve $\gamma_{j,k}$ to be,

$$\gamma_{j,k} \left[ Y^{(j)} \right] (x) = Y^{(k)}(x)C^{kj}.$$ 

This defines the connection matrices $C^{kj} \in \text{GL}_2(\mathbb{C})$ between Fuchsian singular points of equation (2.3).

We choose to normalise the monodromy data of equation (2.3) with respect to the fundamental solution $Y^{(\infty)}(x)$ at infinity. For $k = 1, \ldots, m$, let $\gamma_k$ be a continuous and orientable curve $\gamma_k : [0, 1] \to \mathbb{C}\{a_1, \ldots, a_m\}$ with $\gamma_k(0) = \gamma_k(1) \in \Omega_{\infty}$, which encircles the singularity $x = a_k$ in the positive direction. The curves $\gamma_1, \ldots, \gamma_m$ are illustrated in Figure 3 below, note that $\gamma_{\infty} := \gamma_{m}^{-1} \ldots \gamma_{1}^{-1}$. In a similar manner as above, we use Theorem 2.4 to define the analytic continuation of $Y^{(\infty)}(x)$ along $\gamma_k$ to be,

$$\gamma_k \left[ Y^{(\infty)} \right] (x) = Y^{(\infty)}(x)M_k.$$ 

This defines the monodromy matrices $M_k \in \text{GL}_2(\mathbb{C})$ around Fuchsian singular points of equation (2.3). We note that this defines a monodromy antirepresentation of the fundamental group,

$$p : \pi_1 (\mathbb{C}\{a_1, \ldots, a_m\}, \infty) \to \text{GL}_2(\mathbb{C}) : [\gamma_k] \mapsto M_k,$$

since $p$ is an antihomomorphism. Due to the fact that each curve $\gamma_k$ can be decomposed as a product of three curves: $\gamma_{\infty} \hat{\gamma}_k \gamma_{\infty}$, where $\hat{\gamma}_k : [0, 1] \to \Omega_k$ encircles $a_k$ in the
positive direction, the monodromy matrices $M_k$ have the following form:

$$M_k = (C^{k∞})^{-1} e^{2πiΘ_k} C^{k∞},$$

for $k = 1, \ldots, m$, and $M_∞ = e^{2πiΘ_∞}$.

Furthermore, these matrices satisfy the following cyclic relation:

$$M_∞M_mM_{m-1} \ldots M_1 = I.$$  \hspace{1cm} (2.6)

We now consider the case where $x = ∞$ is an irregular singularity of Poincaré rank one of equation (2.3), as in Kummer’s hypergeometric differential equation and the auxiliary linear system associated to $PV$. Let $Y^{(∞,k)}(x)$ be the fundamental solutions analytic in the sectors $Σ_k$ as defined in Theorem 2.2. Choosing to normalise the monodromy data with respect to the fundamental solution $Y^{(∞,0)}(x)$ in the sector $Σ_0$, the above Definition 2.8 and Theorem 2.4 carry over. However, since the solution is only defined in a sector, its analytic continuation around a closed loop which encircles infinity involves crossing Stokes’ rays. The form of the monodromy matrix around the irregular point becomes,

$$M_∞ = S_0S_1 e^{2πiΘ_∞},$$

where $S_k$ are the Stokes’ matrices defined in Definition 2.6. We note that the cyclic relation (2.6) remains true.

In the case of the auxiliary linear system for $P^D_{III}$, where both infinity and zero are
irregular singularities of Poincaré rank one, we must choose with which fundamental solutions to normalise our monodromy data. If we choose the solution \( Y(\infty,0)(x) \) in the sector \( \Sigma_0^{(\infty)} \) at infinity and the solution \( Y(0,0)(x) \) in the sector \( \Sigma_0^{(0)} \) at zero, then by similar reasoning we find:

\[
M_{\infty} = S_0 S_1 e^{2\pi i \Theta_{\infty}} \quad \text{and} \quad M_0 = \left( C^{0\infty} \right)^{-1} S_0 S_1 e^{2\pi i \Theta_0} C^{0\infty},
\]

where,

\[
M_{\infty} M_0 = I.
\]

### 2.3 A Result of Glutsyuk

We now turn our attention to an important result of Glutsyuk found in [Glu], which deals with limits of solutions at merging simple poles under a generic confluence procedure. We are concerned with the case where two simple poles of a \((2 \times 2)\) linear equation merge to form an irregular singularity of Poincaré rank one. In particular, we deal with the generic case that the eigenvalues of the leading matrices at the merging poles do not differ by an integer and the eigenvalues of the leading matrix at the double pole are distinct. In summary, Glutsyuk shows that it is possible to compute the Stokes’ matrices at the double pole from appropriately normalised monodromy data around the merging simple poles. We remark that Glutsyuk’s work also covers the generalisation when \((k+1)\) simple poles merge to form an irregular singularity of Poincaré rank \(k\), again with generic conditions on the eigenvalues of leading matrices; the result here is that it is possible to compute products of Stokes’ matrices from appropriately normalised monodromy data around the merging simple poles.

Consider the following differential equation,

\[
\frac{\partial Y}{\partial \lambda} = \frac{A(\lambda, \varepsilon)}{(\lambda - \varepsilon)(\lambda + \varepsilon)} Y, \quad A(\lambda, \varepsilon) \in \text{GL}_2(\mathbb{C}), \tag{2.7}
\]

with \(A(\lambda, \varepsilon)\) a holomorphic matrix about \(\lambda = \pm \varepsilon\) such that \(A(\pm \varepsilon, \varepsilon) \neq 0\) for sufficiently
small $\varepsilon \geq 0$ satisfying the following limit,

$$\lim_{\varepsilon \to 0} A(\lambda, \varepsilon) = A(\lambda, 0).$$

Hence, the non-perturbed, or *confluent*, equation,

$$\frac{\partial Y}{\partial \lambda} = \frac{A(\lambda, 0)}{\lambda^2} Y,$$  \hspace{1cm} (2.8)

has an irregular singularity at $\lambda = 0$ of Poincaré rank one. Moreover, it is assumed that the eigenvalues of the leading matrix of $A(\lambda, \varepsilon)$ at $\lambda = \pm \varepsilon$ do not differ by an integer and the eigenvalues of the leading matrix of $A(\lambda, 0)$ at $\lambda = 0$ are distinct.

Remark 2.1. It is without loss of generality that we have placed the Fuchsian singularities at $\lambda = \pm \varepsilon$, which tend to the irregular singularity $\lambda = 0$ of Poincaré rank one as $\varepsilon \to 0$, since there is the freedom of making a conformal transformation. We are using these singularities in order to closely follow Glutsyuk’s work. When we deal with the confluences of Gauss’ hypergeometric equation and of the linear system associated to $P_{VI}$ we will be merging two Fuchsian singularities at infinity, rather than at zero.

We first solve the perturbed equation (2.7). We define neighbourhoods $\Omega_{\pm \varepsilon}$ of the points $\lambda = \pm \varepsilon$ respectively whose radii are less than $2|\varepsilon|$ and with branch cuts made along the straight line passing through the points $\lambda = -\varepsilon, 0, \varepsilon$, as illustrated in Figure 4 below. From Theorem 2.1, equation (2.7) has fundamental solutions $Y^{(\pm \varepsilon)}(\lambda)$ which are analytic in the cut disks $\Omega_{\pm \varepsilon}(\varepsilon)$ of the following form,

$$Y^{(\pm \varepsilon)}(\lambda) = \left( \sum_{n=0}^{\infty} G_{n, \pm \varepsilon}(\lambda \mp \varepsilon)^n \right) (\lambda \mp \varepsilon)^{\lambda \pm \varepsilon}, \quad \lambda \in \Omega_{\pm \varepsilon},$$

where $G_{0, \pm \varepsilon}$ are fixed matrices which diagonalise the leading terms of $A(\lambda, \pm \varepsilon)$ and all other terms of the series are determined by certain recursion formulae.
We now turn our attention to the confluent equation (2.8). Denote by $\mu_1$ and $\mu_2$ the eigenvalues of the leading matrix of $A(\lambda, 0)$ at $\lambda = 0$ (by assumption, $\mu_1 \neq \mu_2$) and let,

$$r_{i,j} = \left\{ \lambda : \text{Re} \left( \frac{\mu_i - \mu_j}{\lambda} \right) = 0, \text{Im} \left( \frac{\mu_i - \mu_j}{\lambda} \right) > 0 \right\}, \quad i, j \in \{1, 2\},$$

be the Stokes’ rays. We denote by $\mathcal{S}_0$ and $\mathcal{S}_1$ open sectors whose union is a punctured neighbourhood of $\lambda = 0$, each of which: has an opening greater than $\pi$; contains only one Stokes’ ray and does not contain the other Stokes’ ray at its boundary. An illustration of such Stokes’ rays and sectors is given below.

We can cover all of the sheets of the Riemann surface of the logarithm at $\lambda = 0$ by
extending the notation as follows,

\[ \lambda \in \mathcal{J}_{k+2} \iff \lambda e^{-2\pi i} \in \mathcal{J}_k. \]

From Theorem 2.2, there exists a number \( R \) sufficiently large such that, for all \( k \in \mathbb{Z} \), there exist fundamental solutions \( Y^{(0,k)}(\lambda) \) of the non-perturbed equation (2.8) analytic in the sectors \( \mathcal{J}_k \) such that,

\[ Y^{(0,k)}(\lambda) \sim \left( \sum_{n=0}^{\infty} H_n \lambda^n \right) \lambda^{\Theta_0} \exp \left( \lambda^{-1} \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \right), \quad \text{as } \lambda \to 0, \lambda \in \mathcal{J}_k, \]

where \( H_0 \) is a fixed matrix which diagonalises the leading term of \( A(\lambda,0) \) at \( \lambda = 0 \), all other terms of the series and the diagonal matrix \( \Theta \) are uniquely determined by certain recursion relations. Each solution \( Y^{(0,k)}(\lambda) \) is uniquely specified by the above asymptotic relation.

We define open sectors \( \sigma_{\pm \varepsilon}(\varepsilon) \subset \Omega_\pm \) with base points at \( \lambda = \pm \varepsilon \) respectively whose openings do not contain the branch cut between \( -\varepsilon \) and \( \varepsilon \) as illustrated in Figure 6 below.

We impose the condition that, as \( \varepsilon \to 0 \) along a ray, the sector \( \sigma_\varepsilon(\varepsilon) \) (resp. \( \sigma_{-\varepsilon}(\varepsilon) \)) is translated along a ray to zero and becomes in agreement with the sector \( \mathcal{J}_{k+1} \) (resp. \( \mathcal{J}_k \)), for some \( k \in \mathbb{Z} \). We write this condition as follows,

\[ \lim_{\varepsilon \to 0} \sigma_\varepsilon(\varepsilon) = \mathcal{J}_{k+1} \quad \text{and} \quad \lim_{\varepsilon \to 0} \sigma_{-\varepsilon}(\varepsilon) = \mathcal{J}_k. \]  

(2.9)

**Theorem 2.5** [Glu]. Let the fundamental solutions \( Y^\varepsilon(\lambda) \), \( Y^{-\varepsilon}(\lambda) \) and \( Y^{(0,k)}(\lambda) \)
and the sectors \( \sigma_\varepsilon(\varepsilon) \), \( \sigma_{-\varepsilon}(\varepsilon) \) and \( \mathcal{S}_k \) be defined as above. There exist diagonal matrices \( K_\varepsilon \) and \( K_{-\varepsilon} \) such that we have the following limits,

\[
\lim_{\varepsilon \to 0} Y^{(\varepsilon)}(\lambda) \bigg|_{\lambda \in \sigma_\varepsilon(\varepsilon)} K_\varepsilon = Y^{(0,k+1)}(\lambda),
\]

\[
\lim_{\varepsilon \to 0} Y^{(-\varepsilon)}(\lambda) \bigg|_{\lambda \in \sigma_{-\varepsilon}(\varepsilon)} K_{-\varepsilon} = Y^{(0,k)}(\lambda),
\]

uniformly for \( \lambda \in \mathcal{S}_{k+1}, \mathcal{S}_k \) respectively, as \( \varepsilon \) belongs to a fixed ray.

**Remark 2.2.** It is well-known that, when solving a linear ordinary differential equation around a Fuchsian singular point, the maximal radius we may take for the neighbourhood on which we can define an analytic solution is the distance to the nearest singularity. For the perturbed equation (2.7), as \( \varepsilon \) becomes arbitrarily small it is clear from the hypotheses on \( A(\lambda, \varepsilon) \) that the closest singularity to \( \lambda = \pm \varepsilon \) will be \( \lambda = \mp \varepsilon \) respectively. We have illustrated the domains \( \Omega_{\pm \varepsilon} \) in Figure 4 with the maximal radii for which it is possible to define analytic solutions. Observe that the neighbourhoods of analyticity of the fundamental solutions diminish as \( \varepsilon \to 0 \). The intelligent part of restricting the fundamental solutions \( Y^{(\pm \varepsilon)}(\lambda) \) to the sectors \( \sigma_{\pm \varepsilon}(\varepsilon) \) as drawn in Figure 6, rather than the neighbourhoods \( \Omega_{\pm \varepsilon} \), is that the radii of these sectors need not be restricted to the distance to the nearest singularity. Indeed, by construction, the singularity \( \lambda = \pm \varepsilon \) will not be inside the sector \( \sigma_{\mp \varepsilon}(\varepsilon) \) respectively. In particular, this means that the radii of these sectors need not vanish.

By the same reasoning as in the previous remark, it is without loss of generality that we may assume \( \sigma_\varepsilon(\varepsilon) \cap \sigma_{-\varepsilon}(\varepsilon) \neq \emptyset \) for \( \varepsilon \) sufficiently close to zero. Accordingly, since we have two fundamental solutions defined on this intersection, they must be related by multiplication by a constant invertible matrix on the right, namely,

\[
Y^{(\varepsilon)}(\lambda) = Y^{(-\varepsilon)}(\lambda)C, \quad \lambda \in \sigma_\varepsilon(\varepsilon) \cap \sigma_{-\varepsilon}(\varepsilon),
\]

for some connection matrix \( C \in \text{GL}_2(\mathbb{C}) \). Similarly, the two fundamental solutions \( Y^{(0,0)}(\lambda) \) and \( Y^{(0,1)}(\lambda) \) of the confluent equation must be related to each other by multiplication by a constant invertible matrix on the right on the intersection \( \mathcal{S}_0 \) and...
\[ Y^{(0,1)}(\lambda) = Y^{(0,0)}(\lambda)S, \quad \lambda \in \mathcal{S}_0, \]  
for some Stokes’ matrix \( S \in \text{GL}_2(\mathbb{C}) \).

**Corollary 2.1 [Glu].** Let the fundamental solutions \( Y^{(\varepsilon)}(\lambda), Y^{(-\varepsilon)}(\lambda) \) and \( Y^{(0,k)}(\lambda) \) and the sectors \( \sigma_{\varepsilon}(\varepsilon), \sigma_{-\varepsilon}(\varepsilon) \) and \( \mathcal{S}_k \) be defined as above; let \( K_{\pm\varepsilon} \) be matrices satisfying Theorem 2.5 and let \( C \) and \( S \) be the matrices defined by (2.10) and (2.11) respectively. We have the following limit,

\[
\lim_{\varepsilon \to 0} K_{-\varepsilon}^{-1}CK_\varepsilon = S, \tag{2.12}
\]

as \( \varepsilon \) belongs to a fixed ray.

In (2.12) it is clear how to obtain one of the Stokes’ matrices at the point \( \lambda = 0 \) of the confluent equation. In order to obtain the second Stokes’ matrix we take \( \varepsilon \to 0 \) along the opposite ray to the one already considered. Rather than having the limits in (2.9), we would now have, for example, that \( \sigma_{\varepsilon}(\varepsilon) \) tends to \( \mathcal{S}_k \) and \( \sigma_{-\varepsilon}(\varepsilon) \) tends to \( \mathcal{S}_{k-1} \). In this way, we use the limit in (2.12) to produce the other Stokes’ matrix. We will explain all of these details and calculate everything explicitly for each of the cases we consider.
Chapter 3

Hypergeometric Differential Equations

In this chapter we analyse Gauss’ hypergeometric differential equation,

\[ x(1-x)\frac{d^2y}{dx^2} + (\gamma - (\alpha + \beta + 1)x) \frac{dy}{dx} - \alpha \beta \, y = 0, \]  

(3.1)

where \( x \in \mathbb{C} \), and Kummer’s confluent hypergeometric differential equation,

\[ z \frac{d^2\tilde{y}}{dz^2} + (\gamma - z) \frac{d\tilde{y}}{dz} - \beta \, \tilde{y} = 0, \]  

(3.2)

where \( z \in \mathbb{C} \). For brevity, we will refer to these equations simply as Gauss’ and Kummer’s equations respectively; despite there being other equations attached with the names of Gauss and Kummer, in this work we only mean (3.1) and (3.2). It is beneficial to review the hypergeometric differential equations because it allows us to define monodromy data, explain Stokes’ phenomenon and demonstrate a confluence procedure with explicit examples. This also gives us the important opportunity to demonstrate our procedure of producing the monodromy data of the confluent equation, including Stokes’ data, of limits of the monodromy data of the original equation using explicit formulae. It is for this purpose that we distinguish between \( y \) and \( \tilde{y} \), with independent variables \( x \) and \( z \), of Gauss’ and Kummer’s equations respectively. This chapter is organised as follows: we first review Gauss’ and Kummer’s equations and their monodromy data in sections 3.1 and 3.2, following the classical analysis of [Inc, WW, BE]; in Section 3.3.1 we then study a confluence procedure from Gauss’ equation to Kummer’s and explain how to relate the solutions of these equations via certain limits; finally, in Section 3.3.2, we prove how to take limits of monodromy data.
3.1 Gauss’ Hypergeometric Differential Equation

In the following, \( \alpha, \beta \) and \( \gamma \) are complex parameters. The following lemma will be used to pass from the scalar form of Gauss’ hypergeometric differential equation (3.1), which is traditionally studied, and a certain \((2 \times 2)\) linear equation.

**Lemma 3.1.** Under the assumptions \( \alpha \neq 0, \gamma \neq \beta \neq 1 \) and \( \alpha \neq \beta - 1 \), the matrix

\[
Y(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ \Psi(y_1, y_1'; x) & \Psi(y_2, y_2'; x) \end{pmatrix},
\]

where,

\[
\Psi(y_k, y_k'; x) = \frac{\alpha (\beta - \gamma + (\alpha + 1 - \beta)x)y_k(x) + x(x-1)(\alpha + 1 - \beta)y_k'(x)}{\alpha(\beta - 1)(\beta - \gamma)},
\]

is a fundamental solution of the equation

\[
\frac{\partial Y}{\partial x} = \left( \frac{A_0}{x} + \frac{A_1}{x - 1} \right) Y,
\]

if and only if \( y_1(x) \) and \( y_2(x) \) are linearly independent solutions of Gauss’ hypergeometric equation (3.1),

\[
x(1 - x) y''(x) + (\gamma - (\alpha + \beta + 1)x) y'(x) - \alpha \beta y(x) = 0.
\]

**Proof.** By direct substitution. \( \square \)

The singularities of (3.1) are \( x = 0, 1, \infty \), which are all Fuchsian since \( \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \) has simple poles at \( x = 0, 1 \) and \( \frac{-\alpha \beta}{x(1-x)} \) has simple poles at \( x = 0, 1 \). We see that \( x = \infty \) is Fuchsian by transforming the equation using \( x = w^{-1} \) and applying similar reasoning.
at $w = 0$. The exponents around $x = 0$ are $1 - \gamma$ and 0, around $x = 1$ are $\gamma - \alpha - \beta$ and 0 and around $x = \infty$ are $\alpha$ and $\beta$. Observe that they indeed satisfy Fuchs’ condition (2.2) with $t = 2$:

$$1 - \gamma + \gamma - \alpha - \beta + \alpha + \beta = 1.$$ 

We make the assumption that these exponents do not differ by an integer, namely $\gamma$, $\gamma - \alpha - \beta$, $\alpha - \beta \notin \mathbb{Z}$, which is a non-resonance condition.

### 3.1.1 Solutions

We define the following disks with chosen branches, as illustrated in Figure 7 below:

$$\Omega_0 = \{x : |x| < 1, -\pi \leq \arg(x) < \pi\},$$

$$\Omega_1 = \{x : |x - 1| < 1, -\pi \leq \arg(1 - x) < \pi\},$$

$$\Omega_\infty = \{x : |x| > 1, -\pi \leq \arg(-x) < \pi\},$$

![Figure 7: Chosen disks with branch cuts. Note that we can visualise $\Omega_\infty$ as $\overline{C \setminus \Omega_0}$, where (...) denotes the closure set.]

It is well-known that the solutions of equation (3.1) are expressible in terms of Gauss’ hypergeometric $\binom{a_1, \ldots, a_p}{b_1, \ldots, b_q} F_1$ series, where the generalised hypergeometric series is defined as,

$$p F_q \left( \binom{a_1, \ldots, a_p}{b_1, \ldots, b_q} ; w \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{w^n}{n!}$$

where the notation $(\cdot)_n$ denotes the Pochhammer symbol, also known as the rising factorial, defined as,

$$(c)_n := \frac{\Gamma(c + n)}{\Gamma(c)} = \begin{cases} \prod_{j=0}^{n-1} (c + j), & n \geq 1, \\ 1, & n = 0. \end{cases}$$
We note that this series becomes a polynomial if one of $a_1, \ldots, a_p$ is a negative integer and is not well-defined if one of $b_1, \ldots, b_q$ is a negative integer, these cases will not be relevant to our work. Using d’Alembert’s ratio test or otherwise, for $a, b, c \in \mathbb{C}\setminus\mathbb{Z}^{\leq 0}$ we find that the series,

$$\binom{2}{F}_1 \left( \begin{array}{c} a, b \\ c \end{array} ; w \right),$$

converges for $|w| < 1$ and diverges for $|w| > 1$. We have the following three pairs of linearly independent local solutions $y_1^{(k)}(x)$ and $y_2^{(k)}(x)$ of (3.1) defined in the neighbourhoods $\Omega_k$ around each singular point:

$$y_1^{(0)}(x) = x^{1-\gamma} \binom{2}{F}_1 \left( \begin{array}{c} \alpha + 1 - \gamma, \beta + 1 - \gamma \\ 2 - \gamma \end{array} ; x \right), \quad x \in \Omega_0, \quad (3.6)$$

$$y_2^{(0)}(x) = \binom{2}{F}_1 \left( \begin{array}{c} \alpha, \beta \\ \gamma \end{array} ; x \right),$$

$$y_1^{(1)}(x) = (1 - x)^{-\alpha-\beta} \binom{2}{F}_1 \left( \begin{array}{c} \gamma - \alpha, \gamma - \beta \\ \gamma + 1 - \alpha - \beta \end{array} ; 1 - x \right), \quad x \in \Omega_1, \quad (3.7)$$

$$y_2^{(1)}(x) = \binom{2}{F}_1 \left( \begin{array}{c} \alpha, \beta \\ \alpha + \beta + 1 - \gamma \end{array} ; 1 - x \right),$$

$$y_1^{(\infty)}(x) = (-x)^{-\alpha} \binom{2}{F}_1 \left( \begin{array}{c} \alpha, \alpha + 1 - \gamma \\ \alpha + 1 - \beta \end{array} ; x^{-1} \right), \quad x \in \Omega_{\infty}, \quad (3.8)$$

$$y_2^{(\infty)}(x) = (-x)^{-\beta} \binom{2}{F}_1 \left( \begin{array}{c} \beta, \beta + 1 - \gamma \\ \beta + 1 - \alpha \end{array} ; x^{-1} \right),$$

Remark 3.1. In order to keep with the classical approach by Whittaker and Watson [WW] and Bateman and Erdélyi [BE], we use the solutions (3.7) of Gauss’ equation around one with the base $(1 - x)$, as opposed to $(x - 1)$, and the solutions (3.8) around infinity with the base $(-x)$, as opposed to simply $x$.

Lemma 3.2. We have the following local fundamental solutions of the matrix hyper-
Chapter 3  Hypergeometric Differential Equations

geometric equation (3.5):
\[
Y^{(0)}(x) = R_0 G_0(x) x^{\Theta_0}, \quad x \in \Omega_0, \quad (3.9)
\]
\[
Y^{(1)}(x) = R_1 G_1(x) (1 - x)^{\Theta_1}, \quad x \in \Omega_1, \quad (3.10)
\]
\[
Y^{(\infty)}(x) = R_\infty G_\infty(x) (-x)^{-\Theta_\infty}, \quad x \in \Omega_\infty, \quad (3.11)
\]

where \( R_k \) and \( \Theta_k \) are the following matrices:
\[
R_0 = \begin{pmatrix} 1 & 1 \\ \frac{\alpha+1-\gamma}{\alpha(\beta-\gamma)} & \frac{1}{\beta-1} \end{pmatrix}, \quad R_1 = \begin{pmatrix} 1 & 1 \\ \frac{1}{\alpha} & \frac{\alpha+1-\gamma}{(\beta-1)(\beta-\gamma)} \end{pmatrix}, \quad R_\infty = \begin{pmatrix} 1 & 0 \\ 0 & \frac{(\beta-\alpha)(\alpha+1-\beta)}{\alpha(\beta-1)(\beta-\gamma)} \end{pmatrix},
\]
\[
\Theta_0 = \begin{pmatrix} 1 - \gamma & 0 \\ 0 & 0 \end{pmatrix}, \quad \Theta_1 = \begin{pmatrix} \gamma - \alpha - \beta & 0 \\ 0 & 0 \end{pmatrix}, \quad \Theta_\infty = \begin{pmatrix} \alpha & 0 \\ 0 & \beta - 1 \end{pmatrix},
\]

which satisfy \( R_k^{-1} A_k R_k = \Theta_k \), and \( G_k(x) \) are the following series:
\[
G_0(x) = 2F_1 \left( \begin{array}{c} \alpha + 1 - \gamma, \beta - \gamma \\ 1 - \gamma \end{array} ; x \right),
\]
\[
x(\alpha+1-\gamma)(1-\beta)2F_1 \left( \begin{array}{c} \alpha + 2 - \gamma, \beta + 1 - \gamma \\ 3 - \gamma \end{array} ; x \right),
\]
\[
x\gamma(1-\gamma)2F_1 \left( \begin{array}{c} \alpha + 1, \beta \\ \gamma + 1 \end{array} ; x \right),
\]
\[
x\gamma(1-\gamma)2F_1 \left( \begin{array}{c} \alpha, \beta - 1 \\ \gamma - 1 \end{array} ; x \right),
\]
\[
G_1(x) = 2F_1 \left( \begin{array}{c} \gamma - \alpha - 1, \gamma - \beta \\ \gamma - \alpha - \beta \end{array} ; 1 - x \right),
\]
\[
(1-x)(\beta-1)(\beta-\gamma)2F_1 \left( \begin{array}{c} \gamma - \alpha, \gamma + 1 - \beta \\ \gamma + 2 - \alpha - \beta \end{array} ; 1 - x \right),
\]

Page 28 of 178
\[
\frac{(1-x)\alpha(\alpha+1-\gamma)}{(\alpha+\beta-\gamma)(\alpha+\beta+1-\gamma)} \quad _2F_1\left(\begin{array}{c}
\alpha+1, \beta \\
\alpha+\beta+2-\gamma
\end{array} ; 1-x \right) \quad _2F_1\left(\begin{array}{c}
\alpha, \beta-1 \\
\alpha+\beta-\gamma
\end{array} ; 1-x \right),
\]

\[
G_{\infty}(x) = \left( _2F_1\left(\begin{array}{c}
\alpha, \alpha+1-\gamma \\
\alpha+1-\beta
\end{array} ; x^{-1} \right),
\right)
\] 

\[
\frac{\alpha(\beta-1)(\beta-\gamma)(\gamma-\alpha-1)}{[\alpha-\beta)(\alpha+1-\beta)(\alpha+2-\beta)]x} \quad _2F_1\left(\begin{array}{c}
\alpha+1, \alpha+2-\gamma \\
\alpha+3-\beta
\end{array} ; x^{-1} \right),
\]

\[-\frac{1}{x} _2F_1\left(\begin{array}{c}
\beta, \beta+1-\gamma \\
\beta+1-\alpha
\end{array} ; x^{-1} \right),
\]

\[-\frac{1}{x} _2F_1\left(\begin{array}{c}
\beta-1, \beta-\gamma \\
\beta-\alpha-1
\end{array} ; x^{-1} \right).
\]

**Proof.** The solutions (3.9), (3.10) and (3.11) can be found by reducing equation (3.5) to Birkhoff normal form. Let \( \Theta_k \) be the diagonal matrices as given above. Around each singular point we make a gauge transformation:

\[
Y^{(0)}(x) = R_0 \sum_{n=0}^{\infty} g_{n,0} x^n \hat{Y}^{(0)}(x), \quad Y^{(1)}(x) = R_1 \sum_{n=0}^{\infty} g_{n,1} (1-x)^n \hat{Y}^{(1)}(x)
\]

\[
Y^{(\infty)}(x) = R_{\infty} \sum_{n=0}^{\infty} g_{n,\infty} x^{-n} \hat{Y}^{(\infty)}(x),
\]

where \( R_k \) are invertible matrices to be determined, \( g_{0,0} = g_{0,1} = g_{0,\infty} = I \) and all other terms of the series are to be determined, such that \( \hat{Y}^{(k)}(x) \) satisfies:

\[
\frac{\partial}{\partial x} \hat{Y}^{(0)} = \frac{\Theta_0}{x} \hat{Y}^{(0)}, \quad \frac{\partial}{\partial x} \hat{Y}^{(1)} = \frac{\Theta_1}{x-1} \hat{Y}^{(1)}, \quad \frac{\partial}{\partial x} \hat{Y}^{(\infty)} = -\frac{\Theta_{\infty}}{x} \hat{Y}^{(\infty)}.
\]

The matrices \( R_k \) and the remaining coefficients of the three series are determined by substituting the expressions for \( Y^{(k)}(x) \) into equation (3.5) and equating the coefficients of powers of \( x^n, (1-x)^n \) and \( x^{-n} \) respectively. In the first cases, for \( n = 0 \), we find the conditions \( R_k^{-1} A_k R_k = \Theta_k \), so we are free to use the expressions for \( R_k \) as given in the
statement of the lemma. For \( n \geq 1 \), we find the following recursion formulae:

\[
ng_{n,0} + [g_{n,0}, \Theta_0] = -R_0^{-1}A_1R_0 \sum_{l=0}^{n-1} g_{l,0},
\]

\[
ng_{n,1} + [g_{n,1}, \Theta_1] = -R_1^{-1}A_0R_1 \sum_{l=0}^{n-1} g_{l,1},
\]

\[
ng_{n,\infty} + [g_{n,\infty}, \Theta_\infty] = -R_\infty^{-1}A_1R_\infty \sum_{l=0}^{n-1} g_{l,\infty}.
\]

It can be verified that the general solutions of these recursion equations are:

\[
g_{n,0} = \begin{pmatrix} (\alpha+1-\gamma)_n(\beta-\gamma)_n \\ (1-\gamma)_{n+l} \\ (1-\beta)(\alpha+1-\gamma)_n(\beta+1-\gamma)_{n-1} \\ (1-\gamma)_{n+1}(n-1)! \end{pmatrix},
\]

\[
g_{n,1} = \begin{pmatrix} (\gamma-\alpha-1)_n(\gamma-\beta)_n \\ (\gamma-\alpha)_{n+l} \\ (1-\beta)(\gamma-\alpha)_{n-1}(\gamma-\beta)_n \\ (\gamma-\alpha-\beta)_{n+1}(n-1)! \end{pmatrix},
\]

\[
g_{n,\infty} = \begin{pmatrix} (\alpha)_n(\alpha+1-\gamma)_n \\ (\alpha+1-\beta)_n n! \\ (1-\beta)(\beta-\gamma)(\alpha)(\alpha+1-\gamma)_n \\ (\alpha-\beta)_{n+2}(n-1)! \end{pmatrix},
\]

which are indeed the coefficients of the series \( G_0(x) \), \( G_1(x) \) and \( G_\infty(x) \) as given above.

\[\square\]

**Lemma 3.3.** Let \( Y^{(k)}(x) \) be the local fundamental solutions given in (3.9)-(3.11). Also let \( y_1^{(k)}(x) \) and \( y_2^{(k)}(x) \) be the local bases of solutions (3.6)-(3.8) and denote by \( Y(y_1, y_2; x) \) the matrix function given by (3.3). We have,

\[
Y^{(k)}(x) = Y \left( y_1^{(k)}, y_2^{(k)}; x \right), \quad x \in \Omega_k,
\]

for \( k = 0, 1, \infty \).

**Proof.** We find the result by direct substitution of the local solutions (3.6)-(3.8) into expression (3.3) and, if necessary, through the use of the following identity, which is provable using Gauss’ contiguous relations,

\[
_2F_1 \left( \begin{array}{c} a, b \\ c \end{array} ; w \right) \equiv _2F_1 \left( \begin{array}{c} a, b-1 \\ c-1 \end{array} ; w \right) - wa(b-c) \frac{1}{c(c-1)} _2F_1 \left( \begin{array}{c} a+1, b \\ c+1 \end{array} ; w \right), \quad |w| < 1.
\]

Page 30 of 178
Remark 3.2. The matrices $R_k$, $k = 0, 1$ and $\infty$, in the above solutions (3.9), (3.10) and (3.11) have been chosen to satisfy $R_k^{-1}A_k R_k = \Theta_k$, where $A_\infty := -A_0 - A_1$. There is some arbitrariness here, for example the transformation, 

$$R_k \mapsto R_k D_k, \text{ with } D_k \text{ a diagonal, } \text{GL}_2(\mathbb{C}) \text{ matrix,}$$

does not affect the relation $R_k^{-1}A_k R_k = \Theta_k$. From Lemma 3.2, these transformations are equivalent to the following transformations on the matrix solutions,

$$Y^{(k)}(x) \mapsto Y^{(k)}(x) D_k,$$

and, from Lemma 3.3, we see this is equivalent to the following transformations on the scalar solutions,

$$y_1^{(k)}(x) \mapsto y_1^{(k)}(x) (D_k)_{1,1}, \quad y_2^{(k)}(x) \mapsto y_2^{(k)}(x) (D_k)_{2,2}.$$

We have made fixed choices for these matrices so that the leading behaviors of the solutions are fixed. Our reason for making these choices is to ensure we have limit passages to the solutions of Kummer’s confluent hypergeometric equation with the correct leading behaviors, we pay attention to this when we analyse a confluence procedure in Section 3.3.

3.1.2 Monodromy Data

We now define the monodromy data of Gauss’ hypergeometric equation (3.1) and recall how to express them in explicit form [BE, WW]. We then spend the remainder of this subsection deriving these classical formulae by following the approach of representing solutions using Barnes integrals.

When defining local solutions, we have been specific about identifying which sheet of the Riemann surface of the logarithm we are restricting our local solutions to at each singular point. We may extend the definitions of our local fundamental solutions...
We proceed with the global analysis of solutions. Let \( Y^{(k)}(x) \) to other sheets \( e^{2m \pi i \Omega_k}, k = 0, 1, \infty \), by analytically continuing along a closed loop encircling the singularity \( x = 0, 1, \infty \). This action simply means that our solution becomes multiplied by the corresponding exponent \( e^{2m \pi i \Theta_k} \), for \( k = 0, 1 \) and \( \infty \), \( m \in \mathbb{Z} \).

Note that, for \( k = 0 \) and \( 1 \), the analytic continuation of \( Y^{(k)}(x) \) around its singularity in the positive direction means \( m > 0 \) in the previous sentence; while, for \( k = \infty \), it means \( m < 0 \). The diagonal matrices \( e^{2\pi i \Theta_k} \) are called the local monodromy exponents of the singularities.

We proceed with the global analysis of solutions. Let \( Y^{(0)}(x) \), \( Y^{(1)}(x) \) and \( Y^{(\infty)}(x) \) be the fundamental solutions of the hypergeometric equation as defined in the previous section. Denote by \( \gamma_{j,k} \) \( [Y^{(j)}] \) \( (x) \) the analytic continuation of \( Y^{(j)}(x) \) along an orientable curve \( \gamma_{j,k} : [0, 1] \to \mathbb{C} \) with \( \gamma_{j,k}(0) \in \Omega_j \) and \( \gamma_{j,k}(1) \in \Omega_k \), for \( j, k = 0, 1, \infty \).

We have the following connection formulae:

\[
\gamma_{j,k} \left[ Y^{(j)} \right] (x) = Y^{(k)}(x)C^{kj}, \tag{3.12}
\]

where:

\[
C^{0\infty} = \begin{pmatrix}
    e^{\pi i (\gamma - 1) \Gamma(\alpha + 1 - \beta) \Gamma(\gamma - 1) / \Gamma(\alpha) \Gamma(\gamma - \beta)} & e^{\pi i (\gamma - 1) \Gamma(\beta + 1 - \alpha) \Gamma(\gamma - 1) / \Gamma(\beta) \Gamma(\gamma - \alpha)} \\
    \Gamma(\alpha + 1 - \beta) \Gamma(1 - \gamma) / \Gamma(\alpha + 1 - \gamma) & \Gamma(\beta + 1 - \alpha) \Gamma(1 - \gamma) / \Gamma(\beta + 1 - \gamma)
\end{pmatrix}, \tag{3.13}
\]

\[
C^{1\infty} = \begin{pmatrix}
    e^{\pi i (\gamma - 1) \Gamma(\alpha + 1 - \beta) \Gamma(\alpha + \beta - \gamma) / \Gamma(\alpha) \Gamma(\alpha + 1 - \gamma)} & e^{\pi i (\gamma - 1) \Gamma(\beta + 1 - \alpha) \Gamma(\alpha + \beta - \gamma) / \Gamma(\beta) \Gamma(\alpha + 1 - \gamma)} \\
    \Gamma(\alpha + 1 - \beta) \Gamma(1 - \gamma) / \Gamma(\alpha + 1 - \gamma) & \Gamma(\beta + 1 - \alpha) \Gamma(1 - \gamma) / \Gamma(\beta + 1 - \gamma)
\end{pmatrix}, \tag{3.14}
\]

\[
C^{01} = \begin{pmatrix}
    \Gamma(\gamma - 1) \Gamma(\gamma - \beta) / \Gamma(\gamma - \alpha) & \Gamma(\beta + 1 - \alpha) \Gamma(\gamma - 1) / \Gamma(\beta + 1 - \gamma) \\
    \Gamma(\gamma - 1) \Gamma(\gamma - \beta) / \Gamma(\gamma - \alpha) & \Gamma(\beta + 1 - \alpha) \Gamma(\gamma - 1) / \Gamma(\beta + 1 - \gamma)
\end{pmatrix}. \tag{3.15}
\]

We choose to normalise the monodromy data of Gauss’ hypergeometric equation with the fundamental solution \( Y^{(\infty)}(x) \). Denote by \( \gamma_k \) \( [Y^{(\infty)}] \) \( (x) \) the analytic continuation of \( Y^{(\infty)}(x) \) along an orientable, closed curve \( \gamma_k : [0, 1] \to \mathbb{C} \) with \( \gamma_k(0) = \gamma_k(1) \in \Omega_{\infty} \), \( k = 0, 1 \), which encircles the singularity \( x = 0, 1 \) respectively in the positive (anticlockwise) direction. The curves \( \gamma_0 \) and \( \gamma_1 \) are illustrated in Figure 8 below, note that
\[ \gamma_\infty := \gamma_1^{-1} \gamma_0^{-1}. \]

We have:

\[ \gamma_k \left[ Y^{(\infty)} \right] (x) = Y^{(k)}(x) M_k, \quad k = 0, 1, \infty, \]

where,

\[ M_0 = \left( C^{0,\infty} \right)^{-1} e^{2 \pi i \Theta_0} C^{0,\infty}, \quad M_1 = \left( C^{1,\infty} \right)^{-1} e^{2 \pi i \Theta_1} C^{1,\infty}, \quad M_\infty = e^{2 \pi i \Theta_\infty}. \quad (3.16) \]

These matrices satisfy the cyclic relation,

\[ M_\infty M_1 M_0 = I. \quad (3.17) \]

Figure 8: Curves defining the monodromy matrices \( M_k \) of Gauss’ hypergeometric differential equation.

**Definition 3.1.** We define the monodromy data of Gauss’ hypergeometric equation (3.1) as the set,

\[ \mathcal{M} := \left\{ (M_0, M_1, M_\infty) \in (\text{GL}_2(\mathbb{C}))^3 \left| \begin{array}{c}
M_\infty M_1 M_0 = I, \\
M_\infty = e^{2 \pi i \Theta_\infty} \\
\text{eigenv}(M_k) = e^{2 \pi i \Theta_k}, \ k=0,1
\end{array} \right. \right\} / \text{GL}_2(\mathbb{C}) \quad (3.18) \]

where \( \text{eigenv}(M_k) = e^{2 \pi i \Theta_k} \) means that the eigenvalues of \( M_k \) are given as the elements of the diagonal matrix \( e^{2 \pi i \Theta_k} \) and the quotient is by global conjugation by a diagonal matrix.
Counting the dimension of the set of monodromy data, we have,

\[ \dim_{\mathbb{C}} (\mathcal{M}) = 12 - 4 - 2 - 5 - 1 = 0. \]

We note that the quotient by global conjugation comes from the freedom of multiplying our fundamental solution \( Y^{(\infty)}(x) \), with which we chose to normalise our monodromy data, on the right by a diagonal, invertible matrix \( D \).

**Remark 3.3.** The connection matrices can be retrieved from the monodromy matrices, up to a freedom of multiplication on the left by diagonal, invertible matrices. In other words, the monodromy matrix,

\[ M_k = (C^{k\infty})^{-1} e^{2\pi i \Theta_k} C^{k\infty}, \]

is invariant under the transformation \( C^{k\infty} \mapsto D_k C^{k\infty} \), with \( D_k \) a diagonal, invertible matrix. The connection matrices (3.13)-(3.15) of Gauss’ equation will play an important role in Section 3.3.2, where we will produce the Stokes’ matrices of Kummer’s equation from certain limits of these connection matrices.

**Deriving the Monodromy Data Formulae**

The remainder of this subsection is spent deriving the classical formulae (3.13)-(3.15). This is a worthwhile exercise as it gives a greater understanding of how to analytically continue solutions and compute their monodromy data; this also adds a good measure of completeness as we will use the explicit formulae for the monodromy data of the hypergeometric equations at the end of this chapter in section 3.3.2. Our approach is based on the readings of [BE,WW,AAR]

We will work with the following Barnes integral,

\[ \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} I(s, x) \, ds \quad \text{where} \quad I(s, x) = \frac{\Gamma(\alpha + s)\Gamma(\beta + s)\Gamma(-s)}{\Gamma(c + s)}(-x)^s, \quad (3.19) \]

with \( |\arg(-x)| < \pi \) and whose path of integration is along the imaginary axis with indentations as necessary so that the poles of \( \Gamma(\alpha + s)\Gamma(\beta + s) \) lie on its left and the poles of \( \Gamma(-s) \) lie on its right, as shown in Figure 9 below. It is always possible to
construct such a path as long as $\alpha$ and $\beta \notin \mathbb{Z}^{\leq 0}$, which is a general assumption since the case in which $\alpha$ or $\beta \in \mathbb{Z}^{\leq 0}$ corresponds to some of the solutions in (3.6)-(3.8) being polynomials.

We will prove the following proposition, which is sufficient to derive the connection formulae (3.13)-(3.15).

**Proposition 3.1.** The integral as given by (3.19) satisfies the following properties:

1. for $|\arg(-x)| < \pi$, 
   \[
   \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} I(s, x) \, ds,
   \]
   defines an analytic function of $x$;

2. for $|\arg(-x)| < \pi$ and $|x| < 1$, 
   \[
   \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} I(s, x) \, ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} y_2^{(0)}(x),
   \]
   where $y_2^{(0)}(x)$ is the solution of Gauss’ equation as given by (3.6);

3. for $|\arg(-x)| < \pi$ and $|x| > 1$, 
   \[
   \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} I(s, x) \, ds = \frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\gamma - \alpha)} y_1^{(\infty)}(x) + \frac{\Gamma(\beta)\Gamma(\alpha - \beta)}{\Gamma(\gamma - \beta)} y_2^{(\infty)}(x),
   \]
   where $y_1^{(\infty)}(x)$ and $y_2^{(\infty)}(x)$ are the solutions of Gauss’ equation as given by (3.8).
Proof. This proof is organised into three parts to prove each statement consecutively.

1. Analyticity of the integral
We use Euler’s reflection formula \(\Gamma(-s)\Gamma(s+1) = -\pi \csc(\pi s)\) to re-write the integrand,

\[
I(s, x) = -\frac{\Gamma(\alpha + s)\Gamma(\beta + s)}{\Gamma(c + s)\Gamma(s + 1)} \frac{\pi}{\sin(\pi s)} (-x)^s.
\] (3.20)

Using the following asymptotic expansion of the Gamma function [WW] §13.6,

\[
\Gamma(s + a) = s^{s+a-\frac{1}{2}} e^{-s} \sqrt{2\pi}(1 + o(1)), \text{ with } |s| \text{ large,}
\] (3.21)

which is valid for \(|\arg(s + a)| < \pi\), we deduce,

\[
\frac{\Gamma(\alpha + s)\Gamma(\beta + s)}{\Gamma(c + s)\Gamma(s + 1)} = O\left(|s|^{|\alpha + \beta - \gamma - 1|}\right), \text{ as } |s| \to \infty.
\] (3.22)

Writing \(\sin(\pi s) = \frac{1}{2i}(e^{i\pi s} - e^{-i\pi s})\) we also deduce,

\[
\sin(\pi s) = O\left(e^{|s|\pi}\right), \text{ as } |s| \to \infty,
\] (3.23)

along the contour of integration (the imaginary axis). Combining (3.22) and (3.23), the integrand has the following asymptotic behavior,

\[
I(s, x) = O\left(|s|^{|\alpha + \beta - \gamma - 1|}e^{-|s|\pi}(-x)^s\right), \text{ as } |s| \to \infty,
\]

along the contour of integration, we therefore need only consider the analyticity of the following integral,

\[
\int_{-i\infty}^{+i\infty} e^{-|s|\pi}(-x)^s \, ds = i \int_{0}^{\infty} e^{-\sigma e^{i\arg(x)}} d\sigma - i \int_{0}^{\infty} e^{-\sigma e^{-i\arg(x)}} d\sigma.
\] (3.24)

We recall the following lemma, see for instance [WW] §5.32,

Lemma 3.4. If \(f : \mathbb{R} \to \mathbb{R}\) is a continuous function such that \(|f(t)| \leq Ke^{rt}\) for constants \(K\) and \(r\), then the integral \(\int_{0}^{\infty} f(t)e^{-\lambda t} \, dt\) defines an analytic function of \(\lambda\)
for $r < \text{Re}(\lambda)$.

Applying this lemma to the first integral in (3.24), with $r = -\pi$, $K = 1$ and $\lambda = \arg(-x)$, we find an analytic function for $-\pi < \arg(-x)$. Applying this lemma to the second integral in (3.24), with $r = -\pi$, $K = 1$ and $\lambda = -\arg(-x)$, we find an analytic function for $\arg(-x) < \pi$. This concludes the proof that the integral (3.19) defines an analytic function for $-\pi < \arg(-x) < \pi$.

2. Representing $y_2^{(0)}(x)$ using a Barnes integral

We write $I(s, x)$ as in (3.20) and consider the following integral,

$$\frac{1}{2\pi i} \int_{C_N} I(s, x) \, ds,$$

for $N \in \mathbb{N}_{\geq 0}$, where $C_N$ is the following semicircle,

$$C_N = \left\{ s = \left(N + \frac{1}{2}\right) e^{i\theta} : \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\}.$$

Let $s \in C_N$, using formula (3.21) from above, we deduce the following asymptotic behavior,

$$\frac{\Gamma(\alpha + s)\Gamma(\beta + s)}{\Gamma(\gamma + s)\Gamma(s + 1)} = O \left(N^{\alpha+\beta-\gamma-1}\right), \quad \text{as } N \to \infty,$$

and, using $\sin(\pi s) = \frac{1}{2i}(e^{i\pi s} - e^{-i\pi s})$,

$$\frac{(-x)^s}{\sin(\pi s)} = O \left(e^{(N+\frac{1}{2})(\cos(\theta) \log |x|-\sin(\theta)\arg(-x) - \pi| \sin(\theta)|)}\right), \quad \text{as } N \to \infty.$$

Since $|\arg(-x)| < \pi$, we write $|\arg(-x)| \leq \pi - \delta$ for some $\delta > 0$, so that,

$$\pm \arg(-x) + \pi \geq \delta \quad \Leftrightarrow \quad \sin(\theta)\arg(-x) + |\sin(\theta)|\pi \geq |\sin(\theta)|\delta,$$

$$\quad \Leftrightarrow \quad e^{-\sin(\theta)\arg(-x)-\pi|\sin(\theta)|} \leq e^{-|\sin(\theta)|\delta}. \quad (3.27)$$

Combining (3.25)-(3.27), the integrand has the following asymptotic behavior for $s \in C_N$,

$$I(s, x) = O \left(N^{\alpha+\beta-\gamma-1}e^{(N+\frac{1}{2})(\cos(\theta) \log |x|-|\sin(\theta)|\delta)}\right), \quad \text{as } N \to \infty.$$
Since \( \cos(\theta) \) and \( |\sin(\theta)| \) are even functions, we need only consider \( \theta \in [0, \frac{\pi}{2}] \). For \( \theta \in [0, \frac{\pi}{4}] \), \( \cos(\theta) \geq \frac{1}{\sqrt{2}} \) and for \( \theta \in [\frac{\pi}{4}, \frac{\pi}{2}] \), \( \sin(\theta) \geq \frac{1}{\sqrt{2}} \). Henceforth, we impose the condition that \( |x| < 1 \), or equivalently \( \log |x| < 0 \). For \( s \in \mathbb{C} \) we deduce:

\[
I(s, x) = \begin{cases} 
\mathcal{O}
\left(N^{\alpha+\beta-\gamma-1} \left(e^{\left(N+\frac{1}{2}\right)i \log |x|}\right)ight), & \theta \in [0, \frac{\pi}{4}], \\
\mathcal{O}
\left(N^{\alpha+\beta-\gamma-1} \left(e^{\left(N+\frac{1}{2}\right)i \log |x| - \delta}\right)\right), & \theta = \frac{\pi}{4}, \\
\mathcal{O}
\left(N^{\alpha+\beta-\gamma-1} e^{-\left(N+\frac{1}{2}\right)i \log |x|}\right), & \theta \in (\frac{\pi}{4}, \frac{\pi}{2}],
\end{cases}
\]

as \( N \to \infty \). This shows that the integral of \( I(s, x) \) along the semicircle \( C_N \) tends to zero as \( N \) tends to infinity, for \( |x| < 1 \) and \( |\arg(-x)| < \pi \). Due to Cauchy’s theorem, we have,

\[
\frac{1}{2\pi i} \left(\int_{-\infty}^{+\infty} - \int_{+\infty}^{-\infty} - \int_{C_N} + \int_{-\infty}^{-\infty} \right) I(s, x) \, ds = -\sum_{n=0}^{N} \text{Res } I(s, x). \tag{3.28}
\]

We note that there is a minus sign since the path of integration is a contour oriented clockwise, see Figure 10 below.

![Figure 10: Paths of integration along the imaginary axis and the semicircle \( C_N \) as in (3.28).](image-url)
Using $\text{Res}_{\lambda=-n} \Gamma(\lambda) = \frac{(-1)^n}{n!}$, for $n \geq 0$, we compute the residues to find,

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} I(s, x) \, ds = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{\Gamma(\alpha + n)\Gamma(\beta + n)}{\Gamma(\gamma + n)\Gamma(n + 1)} x^n,$$

for $|x| < 1$ and $|\arg(-x)| < \pi$ and the desired result is proved after noting $\frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \equiv (\alpha)_n$.

3. The analytic continuation of $y_2^{(0)}(x)$ for $|x| > 1$

The technique to derive the connection formulae is similar to that already used in the second part of this proof, the main difference being that we will now consider taking an integral on the left hand side of the imaginary axis. For $N \in \mathbb{N}$ consider the integral,

$$\frac{1}{2\pi i} \int_{C'_N} I(s, x) \, ds,$$

where $C'_N$ is the semicircle,

$$C'_N = \left\{ s = Ne^{i\theta} : \theta \in \left[ -\frac{3\pi}{2}, -\frac{\pi}{2} \right] \right\}.$$

We summarise the results, following a similar procedure as before. Using (3.21) we deduce,

$$\frac{\Gamma(\alpha + s)\Gamma(\beta + s)\Gamma(-s)}{\Gamma(\gamma + s)} = O\left( N^{\alpha + \beta - \gamma - 1} e^{-N\pi |\sin(\theta)|} \right),$$

for $s \in C'_N$ as $N \to \infty$, and hence,

$$I(s, x) = O\left( N^{\alpha + \beta - \gamma - 1} e^{N\cos(\theta)|\log |x| - \sin(\theta)\arg(-x) - \pi|\sin(\theta)|)} \right),$$

$$= O\left( N^{\alpha + \beta - \gamma - 1} e^{N\cos(\theta)|\log |x| - |\sin(\theta)|\delta)} \right),$$

where $\delta$ is a small positive number such that $|\arg(-x)| \leq \pi - \delta$. Clearly $\cos(\theta)$ and $-|\sin(\theta)|$ are both non-positive for $\theta \in \left[ -\frac{3\pi}{2}, -\frac{\pi}{2} \right]$ and they are never both simultaneously zero. Furthermore, for $|x| > 1$ we have $\log |x| > 0$, so that the integral of $I(s, x)$ along the semicircle $C'_N$ tends to zero as $N$ tends to infinity, for $|x| > 1$ and
Due to Cauchy’s theorem, we have,

\[
\frac{1}{2\pi i} \left( \int_{-\infty}^{+i\infty} - \int_{C_N'} - \int_{N_i}^{+i\infty} \right) I(s, x) \, ds = \sum_{n=0}^{M_1(N)} \text{Res}_{s=\alpha-n} I(s, x) + \sum_{n=0}^{M_2(N)} \text{Res}_{s=\beta-n} I(s, x),
\]

where \( M_1(N) \) and \( M_2(N) \) are the number of poles \(-\alpha, -\alpha-1, \ldots\) and \(-\beta, -\beta-1, \ldots\) which lie to the right of the semicircle respectively. Clearly \( M_1(N) \) and \( M_2(N) \) become infinite as \( N \) tends to infinity, see Figure 11 below.

![Figure 11: Paths of integration along the imaginary axis and the semicircle \( C'_N \) as in (3.29).](image)

We compute the residues to find,

\[
\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} I(s, x) \, ds = \frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\gamma - \alpha)} (-x)^{-\alpha} \lim_{N \to \infty} \sum_{n=0}^{M_1(N)} \frac{(\alpha)_n(\alpha + 1 - \gamma)_n}{(\alpha + 1 - \beta)_n x^n} \\
+ \frac{\Gamma(\beta)\Gamma(\alpha - \beta)}{\Gamma(\gamma - \beta)} (-x)^{-\beta} \lim_{N \to \infty} \sum_{n=0}^{M_2(N)} \frac{(\beta)_n(\beta + 1 - \gamma)_n}{(\beta + 1 - \alpha)_n x^n},
\]

for \(|x| > 1\) and \(|\arg(-x)| < \pi\) and the desired result is proved.

We conclude these computations by explaining how Proposition 3.1 leads to the formu-
lae (3.13)-(3.15). Let $\gamma_{j,k}$ be a curve as described at the beginning of this subsection. The second statement in proposition 3.1 shows how to represent Gauss’ $\, _2F_1$ series using a Barnes integral. Due to the analyticity of this integral, as shown in the first statement, the third statement provides the formula for the analytic continuation of Gauss’ hypergeometric series beyond its radius of convergence. That is to say,

$$\gamma_{0,\infty} \left[ y_2^{(0)} \right] (x) = \frac{\Gamma(\alpha - \beta)\Gamma(\gamma)}{\Gamma(\alpha - \gamma)\Gamma(\beta)} y_1^{(\infty)} (x) + \frac{\Gamma(\beta - \alpha)\Gamma(\gamma)}{\Gamma(\beta - \gamma)\Gamma(\alpha)} y_2^{(\infty)} (x).$$

By manipulating the parameters as follows: $\alpha \mapsto \alpha + 1 - \gamma$, $\beta \mapsto \beta + 1 - \gamma$, $\gamma \mapsto 2 - \gamma$ and multiplying through by $x^{1-\gamma}$ we also deduce,

$$\gamma_{0,\infty} \left[ y_1^{(0)} \right] (x) = -e^{-i\pi\gamma} \frac{\Gamma(\beta - \alpha)\Gamma(2 - \gamma)}{\Gamma(1 - \alpha)\Gamma(\beta + 1 - \gamma)} y_1^{(\infty)} (x)$$

$$-e^{-i\pi\gamma} \frac{\Gamma(\alpha - \beta)\Gamma(2 - \gamma)}{\Gamma(1 - \beta)\Gamma(\alpha + 1 - \gamma)} y_2^{(\infty)} (x),$$

recall that we have selected a branch of $\log(x)$ in the definition of our solutions (3.6) around zero so $x^{1-\gamma}$ is well-defined. These factors constitute the entries of the connection matrix,

$$\left( \gamma_{0,\infty} \left[ y_1^{(0)} \right] (x), \gamma_{0,\infty} \left[ y_2^{(0)} \right] (x) \right) = \left( y_1^{(\infty)} (x), y_2^{(\infty)} (x) \right) C^{\infty 0},$$

where,

$$C^{\infty 0} = \begin{pmatrix} -e^{-i\pi\gamma} \frac{\Gamma(\beta - \alpha)\Gamma(2 - \gamma)}{\Gamma(1 - \alpha)\Gamma(\beta + 1 - \gamma)} & \frac{\Gamma(\alpha - \beta)\Gamma(\gamma)}{\Gamma(\beta - \gamma)\Gamma(\alpha)} \\ -e^{-i\pi\gamma} \frac{\Gamma(\alpha - \beta)\Gamma(2 - \gamma)}{\Gamma(1 - \beta)\Gamma(\alpha + 1 - \gamma)} & \frac{\Gamma(\beta - \alpha)\Gamma(\gamma)}{\Gamma(\beta - \gamma)\Gamma(\alpha)} \end{pmatrix},$$

which is indeed the inverse of the connection matrix $C^{0\infty}$ as given by (3.13). To find the analytic continuation of the solutions around $x = 1$ we manipulate the variable $x$ as well as the parameters. From the transformations $\alpha \mapsto \alpha$, $\beta \mapsto \beta$, $\gamma \mapsto \alpha + \beta + 1 - \gamma$
and \( x \mapsto 1 - x \), we have,

\[
\gamma_{1,\infty} \left[ g_2^{(1)} \right](x) = 
\begin{align*}
e^{-i\pi \alpha} & \frac{\Gamma(\beta - \alpha) \Gamma(\alpha + \beta + 1 - \gamma)}{\Gamma(\beta) \Gamma(\beta + 1 - \gamma)} (1 - x)^{-\alpha} \, _2F_1 \left( \frac{\alpha, \gamma - \beta}{\alpha + 1 - \beta}; (1 - x)^{-1} \right) \\
& + e^{-i\pi \beta} \frac{\Gamma(\alpha - \beta) \Gamma(\alpha + \beta + 1 - \gamma)}{\Gamma(\alpha) \Gamma(\alpha + 1 - \gamma)} (1 - x)^{-\beta} \, _2F_1 \left( \frac{\beta, \gamma - \alpha}{\beta + 1 - \alpha}; (1 - x)^{-1} \right),
\end{align*}
\]

and from the transformations \( \alpha \mapsto \gamma - \alpha \), \( \beta \mapsto \gamma - \beta \), \( \gamma \mapsto \gamma + 1 - \alpha - \beta \) and \( x \mapsto 1 - x \),

\[
\gamma_{1,\infty} \left[ g_1^{(1)} \right](x) = 
\begin{align*}
e^{i\pi (\beta - \gamma)} & \frac{\Gamma(\beta - \alpha) \Gamma(\gamma + 1 - \alpha - \beta)}{\Gamma(1 - \alpha) \Gamma(\gamma - \alpha)} (1 - x)^{-\alpha} \, _2F_1 \left( \frac{\alpha, \gamma - \beta}{\alpha + 1 - \beta}; (1 - x)^{-1} \right) \\
& + e^{i\pi (\alpha - \gamma)} \frac{\Gamma(\alpha - \beta) \Gamma(\gamma + 1 - \alpha - \beta)}{\Gamma(1 - \beta) \Gamma(\gamma - \beta)} (1 - x)^{-\beta} \, _2F_1 \left( \frac{\beta, \gamma - \alpha}{\beta + 1 - \alpha}; (1 - x)^{-1} \right),
\end{align*}
\]

both for \(|\arg(x - 1)| < \pi \) and \(|x - 1| > 1\). After applying Kummer's transformation,

\[
(1 - x)^{-a} \, _2F_1 \left( \frac{a, c - b}{a + 1 - b}; (1 - x)^{-1} \right) = (-x)^{-a} \, _2F_1 \left( \frac{a, a + 1 - c}{a + 1 - b}; x^{-1} \right),
\]

which is valid for \(|\arg(x - 1)| < \pi \), \(|\arg(-x)| < \pi \), \(|x - 1| > 1\) and \(|x| > 1\), we deduce the connection matrix,

\[
\left( \gamma_{0,\infty} \left[ g_1^{(1)} \right](x), \, \gamma_{0,\infty} \left[ g_2^{(1)} \right](x) \right) = \left( y_1^{(\infty)}(x), \, y_2^{(\infty)}(x) \right) \, C^{\infty 1},
\]

where,

\[
C^{\infty 1} = 
\begin{pmatrix}
e^{i\pi (\beta - \gamma)} \frac{\Gamma(\beta - \alpha) \Gamma(\gamma + 1 - \alpha - \beta)}{\Gamma(1 - \alpha) \Gamma(\gamma - \alpha)} & e^{-i\pi \alpha} \frac{\Gamma(\beta - \alpha) \Gamma(\alpha + \beta + 1 - \gamma)}{\Gamma(\beta) \Gamma(\beta + 1 - \gamma)} \\
e^{i\pi (\alpha - \gamma)} \frac{\Gamma(\alpha - \beta) \Gamma(\gamma + 1 - \alpha - \beta)}{\Gamma(1 - \beta) \Gamma(\gamma - \beta)} & e^{-i\pi \beta} \frac{\Gamma(\alpha - \beta) \Gamma(\alpha + \beta + 1 - \gamma)}{\Gamma(\alpha) \Gamma(\alpha + 1 - \gamma)}
\end{pmatrix},
\]

which is indeed the inverse of the connection matrix \( C^{1\infty} \) as given by (3.14). The
connection matrix $C^{01}$ as in (3.15) can be deduced from the relation,

$$C^{01} = C^\infty C^{0\infty}.$$ 

### 3.2 Kummer’s Confluent Hypergeometric Equation

We use $z$ as the variable of Kummer’s confluent hypergeometric equation, we also write tilde above some of the functions and parameters to distinguish from the Gauss hypergeometric equation. We recall the following,

**Lemma 3.5.** Under the assumption $(\beta - 1)(\beta - \gamma) \neq 0$, the matrix

$$\tilde{Y}(z) = \begin{pmatrix} \tilde{y}_1(z) & \tilde{y}_2(z) \\ \tilde{\Psi}(\tilde{y}_1, \tilde{y}_1'; z) & \tilde{\Psi}(\tilde{y}_2, \tilde{y}_2'; z) \end{pmatrix},$$

where,

$$\tilde{\Psi}(\tilde{y}_k, \tilde{y}_k'; z) = \frac{(z + \beta - \gamma) \tilde{y}_k(z) - z \tilde{y}_k'(z)}{(\beta - 1)(\beta - \gamma)},$$

is a fundamental solution of the equation

$$\frac{\partial \tilde{Y}}{\partial z} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{\tilde{A}_0}{z} \right) \tilde{Y}, \text{ where } \tilde{A}_0 = \begin{pmatrix} \beta - \gamma & (1 - \beta)(\beta - \gamma) \\ 1 & 1 - \beta \end{pmatrix},$$

if and only if $\tilde{y}_1(z)$ and $\tilde{y}_2(z)$ are linearly independent solutions of Kummer’s confluent hypergeometric equation (3.2),

$$z \tilde{y}'' + (\gamma - z) \tilde{y}' - \beta \tilde{y} = 0.$$ 

Kummer’s confluent hypergeometric equation (3.2) has one Fuchsian singularity at $z = 0$, since $\frac{\gamma - z}{z}$ and $\frac{-\beta}{z}$ have simple poles at $z = 0$, and an irregular singularity at $z = \infty$ of Poincaré rank one. The exponents of the singularity $z = 0$ are $1 - \gamma$ and $0$ and at $z = \infty$ are $\gamma - \beta$ and $\beta - 1$. We make the non-resonance assumption $\gamma \notin \mathbb{Z}$.
3.2.1 Solutions

Kummer’s confluent hypergeometric equation has an irregular singularity at $z = \infty$ of Poincaré rank one and, as such, solutions around this point exhibit Stokes’ phenomenon. In this section, we will state some definitions and theorems which precisely describe fundamental solutions of Kummer’s equation at the irregular point. We then follow the classical approach to show that these solutions can be expressed in closed form by certain Mellin-Barnes integrals. This analysis allows us to explicitly compute the monodromy data, including Stokes’ matrices, of Kummer’s equation in the following section and thus obtain a richer understanding of Stokes’ phenomenon. We first define our fundamental solution at the Fuchsian singularity $z = 0$.

We define the following disks with chosen branches,

$$
\tilde{\Omega}_0^\pm = \left\{ z : -\pi \pm \frac{\pi}{2} \leq \arg(z) < \pi \pm \frac{\pi}{2} \right\},
$$

where the choice of sign $\pm$ corresponds to a choice of branch cut along the positive or negative imaginary axis respectively.

We deliberately leave the ambiguity in the choice of sign here, this will be explained in Section 3.3.1 when analysing a confluence procedure. Essentially, we will produce two limit passages from the fundamental solution of Gauss’ equation around $x = 0$ to the fundamental solution of Kummer’s equation around $z = 0$ by taking the confluence parameter along two different directions. We will find that the disk with a chosen branch $\Omega_0$, where $Y^{(0)}(x)$ is defined, will depend on the direction in which we take our confluence parameter, in one case this disk will become the disk $\tilde{\Omega}_0^+$ and in the other case $\tilde{\Omega}_0^-$. We have the following standard pair of linearly independent local solutions of (3.2):

$$
\tilde{y}_1^{(0)}(z) = z^{1-\gamma} {}_1F_1\left( \beta + 1 - \gamma, \frac{\beta + 1 - \gamma}{2 - \gamma}; z \right),
$$

$$
\tilde{y}_2^{(0)}(z) = {}_1F_1\left( \frac{\beta}{\gamma}; z \right),
$$

$z \in \tilde{\Omega}_0^\pm. \quad (3.32)$
Using d’Alembert’s ratio test, we see that \( _1F_1 \left( \begin{array}{c} a \\ b \end{array} ; z \right) \) converges for all \( z \in \mathbb{C} \), so these solutions are indeed analytic in the punctured disk \( \tilde{\Omega}_0^+ \).

**Lemma 3.6.** We have the following local fundamental solution of the matrix hypergeometric equation (3.31):

\[
\tilde{Y}^{(0)}(z) = \tilde{R}_0 H_0(z) z \tilde{\Theta}_0, \quad z \in \tilde{\Omega}_0^+,
\]

(3.33)

where \( \tilde{R}_0 \) and \( \tilde{\Theta}_0 \) are the following matrices:

\[
\tilde{R}_0 = \begin{pmatrix}
1 & 1 \\
\frac{1}{\beta - \gamma} & \frac{1}{\gamma - 1}
\end{pmatrix} \quad \text{and} \quad \tilde{\Theta}_0 = \begin{pmatrix}
1 - \gamma & 0 \\
0 & 0
\end{pmatrix},
\]

which satisfy \( \tilde{R}_0^{-1} \tilde{A}_0 \tilde{R}_0 = \tilde{\Theta}_0 \), and \( H_0(z) \) is the following series:

\[
H_0(z) = \left( \begin{array}{ccc}
_1F_1 \left( \begin{array}{c} \beta - \gamma \\ 1 - \gamma \end{array} ; z \right) & \frac{z(\gamma - \beta)}{\gamma(\gamma - 1)} _1F_1 \left( \begin{array}{c} \beta \\ \gamma + 1 \end{array} ; z \right) \\
\frac{z(1 - \beta)}{(1 - \gamma)(2 - \gamma)} _1F_1 \left( \begin{array}{c} \beta + 1 - \gamma \\ 3 - \gamma \end{array} ; z \right) & _1F_1 \left( \begin{array}{c} \beta - 1 \\ \gamma - 1 \end{array} ; z \right)
\end{array} \right).
\]

**Proof.** The solution can be found by reducing equation (3.31) to Birkhoff normal form. Let \( \tilde{\Theta}_0 \) be the diagonal matrix given above. Around zero we make a gauge transformation,

\[
\tilde{Y}^{(0)}(z) = \tilde{R}_0 \sum_{n=0}^{\infty} h_{n,0} z^n \tilde{Y}^{(0)}(z),
\]

where \( \tilde{R}_0 \) is an invertible matrix to be determined, \( h_{0,0} = I \) and all other terms of the series are to be determined, such that \( \tilde{Y}^{(0)}(z) \) satisfies,

\[
\frac{\partial}{\partial z} \tilde{Y}^{(0)}(z) = \frac{\tilde{\Theta}_0}{z} \tilde{Y}^{(0)}(z).
\]

The matrices \( \tilde{R}_0 \) and the remaining coefficients of the series are determined by substituting the expression for \( \tilde{Y}^{(0)}(z) \) into equation (3.31) and equating coefficients of powers of \( z^n \). For \( n = 0 \), we find the condition \( \tilde{R}_0^{-1} \tilde{A}_0 \tilde{R}_0 = \tilde{\Theta}_0 \), so we are free to use
the expression for \( \tilde{R}_0 \) as given in the statement of the lemma. For \( n \geq 1 \), we find the following recursion formula,

\[
 nh_{n,0} + \left[ h_{n,0}, \tilde{\Theta}_0 \right] = \tilde{R}_0^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{R}_0 h_{n-1,0}.
\]

It can be verified that the general solution of this recursion equation is,

\[
h_{n,0} = \begin{pmatrix} \frac{(\beta - \gamma)_n}{(1 - \gamma)_n!} & \frac{\gamma - \beta}{\gamma(\gamma - 1)(\gamma - 1)_n(n - 1)!} \\ 1 - \beta & \frac{(\beta + 1 - \gamma)_n}{(1 - \gamma)(2 - \gamma)(3 - \gamma)_n(n - 1)!} \end{pmatrix},
\]

which are indeed the coefficients of the series \( H_0(z) \) as given above.

**Lemma 3.7.** Let \( \tilde{Y}^{(0)}(z) \) be the local fundamental solution given in (3.33). Also let \( \tilde{y}_1^{(0)}(z) \) and \( \tilde{y}_2^{(0)}(z) \) be the local basis of solutions (3.32) and denote by \( \tilde{Y}(\tilde{y}_1, \tilde{y}_2; z) \) the matrix function given by (3.30). We have,

\[
 \tilde{Y}^{(0)}(z) = \tilde{Y} \left( \tilde{y}_1^{(0)}(z), \tilde{y}_2^{(0)}(z); z \right), \quad z \in \tilde{\Omega}_0^\pm.
\]

**Proof.** We find the result by direct substitution of the local solutions (3.32) into expression (3.30) and through the use of the following identity, which can be derived from the same relation used in the proof of Lemma 3.3,

\[
 _1F_1 \left( \begin{array}{c} b \\ c \end{array} ; z \right) = _1F_1 \left( \begin{array}{c} b - 1 \\ c - 1 \end{array} ; z \right) - \frac{z(b - c)}{c(c - 1)} _1F_1 \left( \begin{array}{c} b \\ c + 1 \end{array} ; z \right), \quad z \in \mathbb{C}.
\]

We now turn our attention to the irregular singularity \( z = \infty \).

**Definition 3.2.** The rays \( \{ z : \text{Re}(z) = 0, \text{Im}(z) > 0 \} \) and \( \{ z : \text{Re}(z) = 0, \text{Im}(z) < 0 \} \) are called the Stokes’ rays of Kummer’s hypergeometric differential equation (3.2).

We note that these rays constitute the borderline where the behavior of \( e^z \) changes, as \( z \to \infty \); that is to say, on one side of each of these rays we have \( e^z \to 0 \), whereas on the other side of each ray we have \( e^z \to \infty \). This is a key aspect of Stokes’ phenomenon and plays a role in understanding the following theorem.
Theorem 3.1. Let,

\[ \tilde{\Sigma}_k = \left\{ z : -\frac{\pi}{2} < \arg(z) - k\pi < \frac{3\pi}{2} \right\}. \]

For all \( k \in \mathbb{Z} \), there exists a solution \( \tilde{Y}^{(\infty,k)}(z) \) of equation (3.31) analytic in the sector \( \tilde{\Sigma}_k \) such that,

\[ \tilde{Y}^{(\infty,k)}(z) \sim \tilde{R}_\infty \left( \sum_{n=0}^{\infty} h_{n,\infty} z^{-n} \right) \begin{pmatrix} e^z z^{\beta-\gamma} & 0 \\ 0 & z^{1-\beta} \end{pmatrix}, \quad \text{as } z \to \infty, \ z \in \tilde{\Sigma}_k, \quad (3.34) \]

where \( \tilde{R}_\infty \) is the following matrix,

\[ \tilde{R}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{(\beta-1)(\beta-\gamma)} \end{pmatrix}, \]

and \( H_\infty(z) \) is the following series,

\[ H_\infty(z) = \begin{pmatrix} \frac{2 F_0 (1-\beta, \gamma-\beta; z^{-1})}{(1-\beta)(\beta-\gamma)} & \frac{-1}{z} 2 F_0 (\beta, \beta+1-\gamma; -z^{-1}) \\ \frac{2 F_0 (2-\beta, \gamma+1-\beta; z^{-1})}{z} & 2 F_0 (\beta-1, \beta-\gamma; -z^{-1}) \end{pmatrix}. \]

Moreover, each solution \( \tilde{Y}^{(\infty,k)}(z) \) is uniquely specified by the relation (3.34).

Proof. A proof of the existence of fundamental solutions \( \tilde{Y}^{(\infty,k)}(z) \) which are analytic on sectors \( \tilde{\Sigma}_k \) may be found in [BJL]. We make an ansatz that the asymptotic behavior of these solutions has the following form,

\[ \tilde{Y}^{(\infty,k)}(z) \sim \tilde{R}_\infty H_\infty(z) \exp \left( \int_{-\infty}^{z} \left( \Lambda_0 + \frac{\Lambda_1}{z'} \right) dz' \right), \quad \text{as } z \to \infty, \ z \in \tilde{\Sigma}_k, \]

where,

\[ \tilde{R}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & \frac{-1}{(\beta-1)(\beta-\gamma)} \end{pmatrix}, \]

\( \Lambda_0 \) and \( \Lambda_1 \) are constant, diagonal matrices to be determined and \( H_\infty(z) \) is the series,

\[ H_\infty(z) = \sum_{n=0}^{\infty} h_{n,\infty} z^{-n}, \]
where the coefficients $h_{n,\infty}$ are to be determined. Since this function is to represent the asymptotic behavior of true solutions of equation (3.31), the following equation must be satisfied,

$$
- \sum_{n=1}^{\infty} nh_{n,\infty} z^{-n-1} + \left( \sum_{n=0}^{\infty} h_{n,\infty} z^{-n} \right) \left( \Lambda_0 + \frac{\Lambda_1}{z} \right)
= \tilde{R}^{-1}_{\infty} \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{\tilde{A}_0}{z} \right) \tilde{R}_{\infty} \left( \sum_{n=0}^{\infty} h_{n,\infty} z^{-n} \right).
$$

By setting $h_{0,\infty} = I$ and equating powers of $z^{-n}$ in this equation, for $n = 0$ and 1, we find:

$$
\Lambda_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \Lambda_1 = \begin{pmatrix} \beta - \gamma & 0 \\ 0 & 1 - \beta \end{pmatrix},
$$

and, for $n \geq 1$, we find the recursion equation,

$$
\begin{bmatrix} h_{n,\infty} \\ 1 0 \\ 0 0 \end{bmatrix} = (n - 1)h_{n-1,\infty} + h_{n-1,\infty} \begin{pmatrix} \gamma - \beta & 0 \\ 0 & \beta - 1 \end{pmatrix} + \tilde{R}_{\infty}^{-1} \tilde{A}_0 \tilde{R}_{\infty} h_{n-1,\infty}.
$$

It can be verified that the general solution of this equation is,

$$
h_{n,\infty} = \begin{pmatrix} \frac{(1-\beta)_n (\gamma-\beta)_n}{n!} & \frac{(\beta)_{n-1} (\beta+1-\gamma)_{n-1}}{(-1)^n (n-1)!} \\ \frac{(1-\beta)(\beta-\gamma)(2-\beta)_{n-1} (\gamma+1-\beta)_{n-1}}{(n-1)!} & \frac{(\beta-1)_{n-1} (\beta-\gamma)_n}{(-1)^n n!} \end{pmatrix},
$$

which are indeed the coefficients in the asymptotic series given in Theorem 3.1.

To prove the final statement, concerned with the uniqueness of solutions, let $\tilde{Y}^{(\infty,k)}(z)$ denote another fundamental solution of equation (3.31) which is analytic on the sector $\tilde{\Sigma}_k$ and has the correct asymptotic behavior, namely,

$$
\tilde{Y}^{(\infty,k)}(z) \sim \tilde{R}_{\infty} \left( \sum_{n=0}^{\infty} h_{n,\infty} z^{-n} \right) \begin{pmatrix} e^{z \beta - \gamma} & 0 \\ 0 & z^{1-\beta} \end{pmatrix}, \quad \text{as } z \to \infty, \ z \in \tilde{\Sigma}_k.
$$

Since $\tilde{Y}^{(\infty,k)}(z)$ and $\tilde{Y}^{(\infty,k)}(z)$ are fundamental solutions defined on the same sector,
there exists a constant matrix \( C \in \text{GL}_2(\mathbb{C}) \) such that,

\[
\tilde{Y}^{(\infty,k)}(z) = \hat{Y}^{(\infty,k)}(z)C, \quad z \in \bar{\Sigma}_k.
\]

Using the asymptotic relations (3.34) and (3.36), we deduce the following,

\[
\begin{pmatrix}
  e^z z^{\beta-\gamma} & 0 \\
  0 & z^{1-\beta}
\end{pmatrix} C
\begin{pmatrix}
  e^{-z} z^{\gamma-\beta} & 0 \\
  0 & z^{\beta-1}
\end{pmatrix}
\sim I, \quad \text{as } z \to \infty, \quad z \in \bar{\Sigma}_k.
\]

From this relation, we immediately see that \((C)_{1,1} = (C)_{2,2} = 1\). Moreover, since there exists rays belonging to \(\bar{\Sigma}_k\) along which each exponential, \(e^z\) and \(e^{-z}\), explodes as \(z \to \infty\), we conclude that \((C)_{1,2} = (C)_{2,1} = 0\).

Remark 3.4. The matrices \(\tilde{R}_0\) and \(\tilde{R}_\infty\) in the above solutions (3.33) and (3.34) have been chosen to satisfy \(\tilde{R}_0^{-1} \tilde{A}_0 \tilde{R}_0 = \tilde{\Theta}_0\) and,

\[
\begin{bmatrix}
  \tilde{R}_\infty, \\
  1 & 0 \\
  0 & 0
\end{bmatrix} = 0.
\]

As in Remark 3.2, which deals with Gauss’ equation, there is some arbitrariness here, for example transformations of the form,

\(\tilde{R}_k \mapsto \tilde{R}_k D_k\), with \(D_k\) a diagonal, \(\text{GL}_2(\mathbb{C})\) matrix.

However, we have made fixed choices for the matrices \(\tilde{R}_0\) and \(\tilde{R}_\infty\); that is to say, the leading behaviors of the solutions \(\tilde{Y}^{(0)}(z)\) and \(\tilde{Y}^{(\infty,k)}(z)\) are fixed. In any case, using different matrices would not cause a problem because when we apply Glutsyuk’s Theorem 2.5 in Section 3.3.1 the arbitrariness will be absorbed in the diagonal matrices \(K_{\infty}^\pm(\alpha)\) and \(K_1^\pm(\alpha)\), to be defined in Section 3.3.1.

Remark 3.5. The asymptotic relation (3.34) means, by definition, for all \(m \in \mathbb{Z}\) and for all closed subsectors \(\sigma \subset \bar{\Sigma}_k\),

\[
| z^m \left( \tilde{Y}^{(\infty,k)}(z) \begin{pmatrix} e^{-z} z^{\gamma-\beta} & 0 \\ 0 & z^{\beta-1} \end{pmatrix} - \sum_{n=0}^{m} h_{n,\infty} z^{-n} \right) | \to 0, \quad \text{as } z \to \infty, \quad z \in \sigma.
\]
We denote the asymptotic behavior of true solutions at infinity as in (3.34) by,

\[
\tilde{Y}_f(\infty)(z) = \left( \sum_{n=0}^{\infty} h_{n,\infty} z^{-n} \right) \begin{pmatrix} e^{z} z^{\beta-\gamma} & 0 \\ 0 & z^{1-\beta} \end{pmatrix}, \quad z \in \tilde{\Sigma}_k.
\]

The series \( H_\infty(z) = \sum_{n=0}^{\infty} h_{n,\infty} z^{-n} \) defines a formal gauge transformation which maps equation (3.31) to,

\[
\frac{\partial}{\partial z} \tilde{Y}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} \beta - \gamma & 0 \\ 0 & 1 - \beta \end{pmatrix} \tilde{Y},
\]

via the transformation \( \tilde{Y}(z) = \tilde{R}_\infty H_\infty(z) \tilde{Y}(z) \). We define the coefficient of \( \frac{1}{z} \) in the new equation to be \( -\tilde{\Theta}_\infty \), namely,

\[
\tilde{\Theta}_\infty := \begin{pmatrix} \gamma - \beta & 0 \\ 0 & \beta - 1 \end{pmatrix} \equiv -\text{diag}(\tilde{A}_0).
\]

In the generic case \( a, b \notin \mathbb{Z}^{\leq 0} \), d’Alembert’s ratio test shows that the series \( _2F_0(a, b; z^{-1}) \) diverges for all \( z \in \mathbb{C} \). In this sense, the asymptotic behavior \( \tilde{Y}_f(\infty)(z) \) is a formal fundamental solution.

From the asymptotic relation (3.34), it is clear that the solutions,

\[
\tilde{Y}(\infty,k+2)(z) \quad \text{and} \quad \tilde{Y}(\infty,k)(ze^{-2\pi i}) e^{2\pi i \tilde{\Theta}_\infty},
\]

have the same asymptotic behavior as \( z \to \infty \) in the sector \( z \in \tilde{\Sigma}_{k+2} \). By the last statement of Theorem 3.1, we therefore conclude that,

\[
\tilde{Y}(\infty,k+2)(z) = \tilde{Y}(\infty,k)(ze^{-2\pi i}) e^{-2\pi i \tilde{\Theta}_\infty}, \quad z \in \tilde{\Sigma}_{k+2}.
\]

In this sense, all solutions \( \tilde{Y}(\infty,k)(z) \) are categorised into two fundamentally distinct cases, namely, when \( k \) is even and when \( k \) is odd.

**Remark 3.6.** Using expression (3.30) in Lemma 3.5, the formal fundamental solution
\( \tilde{Y}_f^{(\infty)} \) of (3.31) corresponds to the following standard formal basis of solutions of (3.2),

\[
\begin{align*}
\tilde{y}_{1,f}^{(\infty)}(z) &= e^z z^{\beta-\gamma} \, _2F_0(\gamma-\beta, 1-\beta; z^{-1}), \\
\tilde{y}_{2,f}^{(\infty)}(z) &= -z^{-\beta} \, _2F_0(\beta, \beta+1-\gamma; -z^{-1}).
\end{align*}
\]  

(3.39)

### 3.2.2 Monodromy Data

We now define the monodromy data, including Stokes’ data, of Kummer’s equation (3.2) and recall how to express them in explicit form [BE, WW]. We then spend the rest of this subsection deriving these classical formulae by representing solutions using Mellin-Barnes integrals.

**Definition 3.3.** Let \( \tilde{Y}^{(\infty,k)}(z) \) be the fundamental solutions given in Theorem 3.1 and define sectors,

\[
\tilde{\Pi}_k := \tilde{\Sigma}_k \cap \tilde{\Sigma}_{k+1} \equiv \left\{ z : |z| > 0, \frac{\pi}{2} < \arg(z) - k\pi < \frac{3\pi}{2} \right\},
\]

as illustrated in Figure 12 below. We define Stokes’ matrices \( \tilde{S}_k \in \text{SL}_2(\mathbb{C}) \) as follows,

\[
\tilde{Y}^{(\infty,k+1)}(z) = \tilde{Y}^{(\infty,k)}(z)\tilde{S}_k, \quad z \in \tilde{\Pi}_k.
\]  

(3.40)

**Figure 12:** Sectors \( \tilde{\Pi}_0, \tilde{\Pi}_{-1}, \tilde{\Sigma}_0 \) and \( \tilde{\Sigma}_{-1} \) projected onto the plane \( \mathbb{C} \setminus \{0\} \).

The positive and negative imaginary axes are Stokes’ rays.
Remark 3.7. Combining Definition 3.3 with the relation (3.38), we have:

\[
\widetilde{Y}^{(\infty,k+2)}(z) = \widetilde{Y}^{(\infty,k)}(ze^{-2\pi i}) e^{-2\pi i \tilde{\Theta}_\infty}, \quad z \in \tilde{\Sigma}_{k+2}, \quad \text{by (3.38)},
\]

\[
= \widetilde{Y}^{(\infty,k+1)}(z) \tilde{S}_{k+1}, \quad z \in \tilde{\Sigma}_{k+2} \cap \tilde{\Sigma}_{k+1}, \quad \text{by (3.40)},
\]

\[
= \widetilde{Y}^{(\infty,k-1)}(ze^{-2\pi i}) \tilde{S}_{k-1} e^{-2\pi i \tilde{\Theta}_\infty}, \quad z \in \tilde{\Sigma}_{k+2} \cap \tilde{\Sigma}_{k+1}, \quad \text{by (3.38)},
\]

\[
= \widetilde{Y}^{(\infty,k-1)}(ze^{-2\pi i}) e^{-2\pi i \tilde{\Theta}_\infty} \tilde{S}_{k+1}, \quad z \in \tilde{\Sigma}_{k+2} \cap \tilde{\Sigma}_{k+1}, \quad \text{by (3.40)}.
\]

Finally, we deduce the following relation,

\[
e^{-2\pi i \tilde{\Theta}_\infty} \tilde{S}_{k+1} = \tilde{S}_{k-1} e^{-2\pi i \tilde{\Theta}_\infty},
\]

which shows that Kummer’s equation has only two types of Stokes’ matrices \( \tilde{S}_k \) which are fundamentally different: one with \( k \) odd and the other with \( k \) even.

Lemma 3.8. We have the following classical formulae:

\[
\tilde{S}_0 = \begin{pmatrix} 1 & \frac{2\pi i}{\Gamma(\beta) \Gamma(1-\beta-\gamma)} e^{2\pi i (\gamma-2\beta)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{S}_{-1} = \begin{pmatrix} 1 & 0 \\ \frac{2\pi i}{\Gamma(1-\beta) \Gamma(\gamma-\beta)} & 1 \end{pmatrix}. \quad (3.41)
\]

Lemma 3.8 is proved at the end of this subsection.

We choose to normalise our monodromy data with respect to the fundamental solution \( \widetilde{Y}^{(\infty,0)}(z) \). Denote by \( \gamma_{\infty,0} \left[ \widetilde{Y}^{(\infty,0)} \right](z) \) the analytic continuation of \( \widetilde{Y}^{(\infty,0)}(z) \) along an orientable curve \( \gamma_{\infty,0} : [0,1] \to \mathbb{C} \) with \( \gamma_{\infty,0}(0) \in \tilde{\Sigma}_0 \) and \( \gamma_{\infty,0}(1) \in \tilde{\Omega}_0^\pm \). We have,

\[
\gamma_{\infty,0} \left[ \widetilde{Y}^{(\infty,0)} \right](z) = \widetilde{Y}^{(0)}(z) \tilde{C}_{0\infty},
\]

where,

\[
\tilde{C}_{0\infty} = \begin{pmatrix} e^{2\pi i (\beta-1)} \frac{\Gamma(\gamma-1)}{\Gamma(\gamma-\beta)} & -\frac{\Gamma(\gamma-1)}{\Gamma(\beta)} \\ e^{2\pi i (\beta-\gamma)} \frac{\Gamma(1-\gamma)}{\Gamma(1-\beta)} & -\frac{\Gamma(1-\gamma)}{\Gamma(\beta+1-\gamma)} \end{pmatrix}. \quad (3.42)
\]

Denote by \( \gamma_0 \left[ \widetilde{Y}^{(\infty,0)} \right](z) \) the analytic continuation of \( \widetilde{Y}^{(\infty,0)}(z) \) along an orientable, closed curve \( \gamma_0 : [0,1] \to \mathbb{C} \) with \( \gamma_0(0) = \gamma_0(1) \in \tilde{\Sigma}_0 \) which encircles the singularity.
$z = 0$ in the positive (anti-clockwise) direction. The curve $\gamma_0$ is illustrated below, note that $\gamma_\infty := \gamma_0^{-1}$.

![Figure 13: Curves defining the monodromy matrices $\tilde{M}_k$ of Kummer’s hypergeometric differential equation.](image)

We have,

$$
\gamma_k \left[ \tilde{Y}^{(\infty,0)} \right] (z) = Y^{(\infty,k)}(z) \tilde{M}_k, \quad k = 0, \infty,
$$

where,

$$
\tilde{M}_0 = \left( \tilde{C}^{0\infty} \right)^{-1} e^{2\pi i \tilde{\Theta}_0 \tilde{C}^{0\infty}} \quad \text{and} \quad \tilde{M}_\infty = \tilde{S}_0 e^{2\pi i \tilde{\Theta}_\infty} \tilde{S}_{-1}.
$$

These matrices satisfy the cyclic relation,

$$
\tilde{M}_\infty \tilde{M}_0 = I.
$$

**Definition 3.4.** We define the monodromy data of Kummer’s hypergeometric differential equation (3.2) as the set,

$$
\mathcal{M} := \left\{ \begin{array}{c}
\left( \tilde{M}_0, \tilde{S}_0, \tilde{S}_{-1} \right) \\
\in (\text{GL}_2(\mathbb{C}))^3
\end{array} \right| \begin{array}{l}
\tilde{S}_0 \text{ is unipotent, upper triangular,} \\
\tilde{S}_{-1} \text{ is unipotent, lower triangular,} \\
\tilde{S}_0 e^{2\pi i \tilde{\Theta}_0} \tilde{S}_{-1} \tilde{M}_0 = I, \\
\text{eigenv} \left( \tilde{M}_0 \right) = e^{2\pi i \tilde{\Theta}_0}
\end{array} \right\} / \text{GL}_2(\mathbb{C})
$$

where *unit triangular* means triangular with 1’s along the diagonal, eigenv($\tilde{M}_0$) = $e^{2\pi i \tilde{\Theta}_0}$.
means that the eigenvalues of $\widetilde{M}_0$ are given as the elements of the diagonal matrix $e^{2\pi i \tilde{\Theta}_0}$ and the quotient is by global conjugation by a diagonal matrix.

Counting the dimension of the set of monodromy data, we have,

$$\dim_{\mathbb{C}} \left( \mathcal{M} \right) = 12 - 3 - 3 - 4 - 1 - 1 = 0.$$ 

The quotient by global conjugation comes from the freedom of multiplying our fundamental solution $\mathcal{Y}^{(\infty,0)}(z)$, with which we chose to normalise our monodromy data, on the right by a diagonal, invertible matrix $D$.

**Deriving the Monodromy Data Formulae**

The remainder of this subsection is dedicated to deriving the classical formulae (3.41)-(3.42). This is a valuable exercise in its own right as it gives us a richer understanding of Stokes’ phenomenon using a concrete example. Our approach is to use Mellin-Barnes integrals to represent the fundamental solutions $\mathcal{Y}^{(\infty,k)}(z)$, as defined in Theorem 3.1, for which we are able to compute their analytic continuations. Our analysis of Mellin-Barnes integrals is based on Whittaker and Watson’s [WW] §16, who study a different form of the confluent hypergeometric differential equation but is equivalent to ours using analytic transformations.

Define the following functions,

\[
\begin{align*}
\tilde{y}_1^{(\infty,-1)}(z) &= e^{-i\pi(\beta-\gamma)}e^{z}\varphi(\gamma - \beta, \gamma; e^{i\pi}z), \\
\tilde{y}_2^{(\infty,-1)}(z) &= -\varphi(\beta, \gamma; z), \\
\tilde{y}_1^{(\infty,0)}(z) &= e^{i\pi(\beta-\gamma)}e^{z}\varphi(\gamma - \beta, \gamma; e^{-i\pi}z), \\
\tilde{y}_2^{(\infty,0)}(z) &= -\varphi(\beta, \gamma; z),
\end{align*}
\]

where $\varphi$ is the Mellin-Barnes integral,

\[
\varphi(\beta, \gamma; z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(s) \Gamma(\beta - s) \Gamma(\beta + 1 - \gamma - s)}{\Gamma(\beta) \Gamma(\beta + 1 - \gamma)} z^{s-\beta} ds,
\]

whose path of integration is along the imaginary axis with indentations as necessary so that the poles of $\Gamma(s)$ lie on its left and the poles of $\Gamma(\beta - s) \Gamma(\beta + 1 - \gamma - s)$ lie on its
right, as shown in Figure 14 below. When dealing with $\varphi(\beta, \gamma; z)$ it is to be understood that $\arg(z)$ belongs to an interval of length at most $2\pi$, as in (3.46) and (3.47), so that we have a well-defined function.

![Figure 14: Path of integration in the Mellin-Barnes integral $\varphi(\beta, \gamma; z)$, the dots represent the poles of the integrand.]

**Proposition 3.2.** Let $\widetilde{Y}^{(\infty,k)}(z)$ be the fundamental solutions defined in Theorem 3.1. Also let $\tilde{y}_1^{(\infty,k)}(z)$ and $\tilde{y}_2^{(\infty,k)}(z)$, $k = -1, 0$, be the functions defined in (3.46) and (3.47) and denote by $\widetilde{Y} (\tilde{y}_1, \tilde{y}_2; z)$ the matrix function given by (3.30). We have,

$$
\widetilde{Y} \left( \tilde{y}_1^{(\infty,k)}, \tilde{y}_2^{(\infty,k)}; z \right) = \widetilde{Y}^{(\infty,k)}(z), \quad z \in \tilde{\Sigma}_k,
$$

(3.49)

for $k = -1, 0$.

**Proof.** We prove this proposition in three steps: we first show that the functions $\tilde{y}_1^{(\infty,k)}(z)$ and $\tilde{y}_2^{(\infty,k)}(z)$ are analytic on their respective sectors; using this fact, we secondly show that these functions satisfy Kummer’s equation (3.2); finally, we show that these functions have the correct asymptotic behavior (3.34). By the uniqueness statement of Theorem 3.1, these conditions are sufficient to conclude (3.49).

1. **Analyticity of $\tilde{y}_1^{(\infty,k)}(z)$ and $\tilde{y}_2^{(\infty,k)}(z)$**

We require formula (3.21) and Lemma 3.4, as used in the derivation of Gauss’ mon-
ody data formulae at the end of Section 3.1.2. Using (3.21), we have the following behavior in the integrand of \( \varphi(a, c; z) \),

\[
\Gamma(s) \Gamma(\beta - s) \Gamma(\beta + 1 - \gamma - s) = O\left(e^{-\frac{3\pi}{2}|s|} s^{2\beta - \gamma - \frac{1}{2}}\right), \quad \text{as } |s| \to \infty \quad (3.50)
\]

along the contour of integration. We therefore need only consider the analyticity of the following integral,

\[
\int_{-i\infty}^{+i\infty} e^{-\frac{3\pi}{2}|s|} z^{s-\beta} \, ds \\
\equiv i \int_{0}^{\infty} e^{-\frac{3\pi}{2}|\sigma|} z^{-\beta} e^{i\sigma (\log|z|+i\arg(z))} \, d\sigma - i \int_{0}^{\infty} e^{-\frac{3\pi}{2}|\sigma|} z^{-\beta} e^{-i\sigma (\log|z|+i\arg(z))} \, d\sigma.
\]

Applying Lemma 3.4 to the first integral, with \( r = -\frac{3\pi}{2}, K = 1 \) and \( \lambda = \arg(z) \), we find an analytic function for \(-\frac{3\pi}{2} < \arg(z)\). Applying Lemma 3.4 to the second integral, with \( r = -\frac{3\pi}{2}, K = 1 \) and \( \lambda = -\arg(z) \), we find an analytic function for \( \arg(z) < \frac{3\pi}{2} \).

We conclude that \( \varphi(\beta, \gamma; z) \) defines analytic functions \( \tilde{y}_{1}^{(\infty, -1)}(z) \) and \( \tilde{y}_{2}^{(\infty, 0)}(z) \) on their respective sectors \( \tilde{\Sigma}_{-1} \) and \( \tilde{\Sigma}_{0} \). It therefore follows that \( \tilde{y}_{1}^{(\infty, -1)}(z) \) and \( \tilde{y}_{1}^{(\infty, 0)}(z) \) are also analytic functions, since \( \varphi(\gamma - \beta - 1, \gamma; e^{i\pi z}) \) must be analytic on \( z \in \tilde{\Sigma}_{-1} \) and \( \varphi(\gamma - \beta - 1, \gamma; e^{-i\pi z}) \) must be analytic on \( z \in \tilde{\Sigma}_{0} \).

2. **Showing \( \tilde{y}_{1}^{(\infty,k)}(z) \) and \( \tilde{y}_{2}^{(\infty,k)}(z) \) satisfy the Kummer’s equation (3.2)**

We will now substitute \( \varphi(\beta, \gamma; z) \) for \( \tilde{y}(z) \) into the left hand side of Kummer’s equation (3.2) and show that the result is zero. Having established the analyticity of \( \varphi(\beta, \gamma; z) \) on the sectors \( \tilde{\Sigma}_{-1} \) and \( \tilde{\Sigma}_{0} \), we can compute the derivatives of this integral by taking the derivatives inside the integral. After multiplying through by \( 2\pi i \Gamma(\beta) \Gamma(\beta + 1 - \gamma) \)
to cancel all multiplicative constant terms, we find,

\[
(z \varphi''(\beta, \gamma; z) + (\gamma - z) \varphi'(\beta, \gamma; z) - \beta \varphi(\beta, \gamma; z)) 2\pi i \Gamma(\beta) \Gamma(\beta + 1 - \gamma) = \int_{-i\infty}^{+i\infty} \Gamma(s) \Gamma(\beta + 2 - s) \Gamma(\beta + 1 - \gamma - s) z^{s-\beta-1} \, ds \\
- \int_{-i\infty}^{+i\infty} \gamma \Gamma(s) \Gamma(\beta + 1 - s) \Gamma(\beta + 1 - \gamma - s) z^{s-\beta-1} \, ds \\
+ \int_{-\infty}^{+\infty} \Gamma(s) \Gamma(\beta + 1 - s) \Gamma(\beta + 1 - \gamma - s) z^{s-\beta} \, ds \\
- \int_{-\infty}^{+\infty} (\beta) \Gamma(s) \Gamma(\beta - s) \Gamma(\beta + 1 - \gamma - s) z^{s-\beta} \, ds
\]

\[
= \int_{-1-i\infty}^{-1+i\infty} \Gamma(s + 1) \Gamma(\beta - \gamma - s) z^{s-\beta} (\Gamma(\beta + 1 - s) - \gamma \Gamma(\beta - s)) \, ds \\
- \int_{-1-i\infty}^{+1+i\infty} \Gamma(s) \Gamma(\beta + 1 - \gamma - s) z^{s-\beta} ((\beta) \Gamma(\beta - s) - \Gamma(\beta + 1 - s)) \, ds
\]

\[
= \left( \int_{-1-i\infty}^{+1+i\infty} \Gamma(s + 1) \Gamma(\beta - s) \Gamma(\beta + 1 - \gamma - s) z^{s-\beta} \, ds \right) - \int_{-1-i\infty}^{+1+i\infty} \Gamma(s) \Gamma(\beta - s) \Gamma(\beta + 1 - \gamma - s) z^{s-\beta} \, ds. \tag{3.51}
\]

Due to the choice of the path of integration, the final integrand has no poles between the contours of integration, see Figure 15 below. Therefore, due to Cauchy’s theorem, the expression equals zero and we have shown that \( \varphi(\beta, \gamma; z) \) satisfies Kummer’s confluent hypergeometric equation (3.2) on \( z \in \bar{\Sigma}_{-1} \) and \( \bar{\Sigma}_0 \).

Figure 15: Paths of integration in (3.51), the dots represent poles of the integrand. Note the crucial detail that \( s = 0 \) is not a pole of the integrand, so there are no singularities between the two paths.
Observe the following differential identity,
\[ z \frac{d^2}{dz^2} (e^z f(-z)) + (\gamma - z) \frac{d}{dz} (e^z f(-z)) - \beta e^z f(-z) \equiv e^z \left( z \frac{d^2}{dz^2} f(z) - (\gamma - (-z)) \frac{d}{dz} f(z) - (\gamma - \beta) f(z) \right). \]

Given that \( \varphi(\beta, \gamma; z) \) satisfies Kummer’s equation (3.2), it follows that the right hand side of this identity equals zero for \( f(-z) = \varphi(\gamma - \beta, \gamma; -z) \). Looking at the left hand side of the identity, we deduce that \( e^z \varphi(\gamma - \beta, \gamma; -z) \) also satisfies equation (3.2).

3. Asymptotic behavior of \( \tilde{y}_1^{(\infty,k)}(z) \) and \( \tilde{y}_2^{(\infty,k)}(z) \) for large \(|z|\)
Recalling the formal solutions given in Remark 3.6, we will deduce the following asymptotics, for \( j \in \{0, -1\} \):
\[
\tilde{y}_1^{(\infty,j)}(z) \sim e^z z^{\beta - \gamma} \, _2F_0\left( \gamma - \beta, 1 - \beta; z^{-1} \right), \quad \text{as } z \to \infty, \ z \in \tilde{\Sigma}_j, \quad (3.52)
\]
\[
\tilde{y}_2^{(\infty,j)}(z) \sim -z^{-\beta} \, _2F_0\left( \beta, \beta + 1 - \gamma; -z^{-1} \right), \quad \text{as } z \to \infty, \ z \in \tilde{\Sigma}_j. \quad (3.53)
\]

Denote the integrand of \( \varphi(\beta, \gamma; z) \) by,
\[
I(s, z) = \frac{\Gamma(s) \Gamma(\beta - s) \Gamma(\beta + 1 - \gamma - s)}{\Gamma(\beta) \Gamma(\beta + 1 - \gamma)} z^{s - \beta}, \quad (3.54)
\]
and let \( \tau \) be a large, positive real number. For \( N \geq 0 \), consider the path of integration along the rectangle \( R \) with vertices at \( \pm i\tau \) and \( -N - \frac{1}{2} \pm i\tau \), with indentations so that the poles of the integrand are separated as usual and with a positive orientation as shown in Figure 16 below.
Figure 16: Path of integration around the rectangle $R$, the dots represent the poles of the integrand of $\varphi(\beta, \gamma; z)$.

By Cauchy’s theorem, we have,

$$
\frac{1}{2\pi i} \int_R I(s, z) \, ds = \frac{1}{2\pi i} \left( \int_{-N - \frac{1}{2} - i\tau}^{-i\tau} + \int_{-i\tau}^{+i\tau} + \int_{+i\tau}^{-N - \frac{1}{2} + i\tau} + \int_{-N - \frac{1}{2} + i\tau}^{-i\tau} \right) I(s, z) \, ds.
$$

$$
= \sum_{n=0}^{N} \text{Res} I(s, z),
$$

We examine these integrals in the limit $\tau \to \infty^+$ one-by-one, using the asymptotics (3.21) of the Gamma function:

1. By writing $s = x - i\tau$ in the first integral we obtain,

$$
e^{\tau \left( \arg(z) - \frac{3\pi}{2} \right)} \int_{-N - \frac{1}{2}}^{0} O \left( |z|^{x} \tau^{\Re(2\beta - \gamma) - x - \frac{1}{2}} \right) \, dx,
$$

which tends to zero as $\tau \to \infty^+$, thanks to $\arg(z) < \frac{3\pi}{2}$.

2. In the limit $\tau \to \infty^+$, the second integral becomes $\varphi(\beta, \gamma; z)$, by definition.

3. Similarly to the first integral, by writing $s = x + i\tau$ in the third integral, we
obtain,

\[ e^{-\tau (\arg(z) + \frac{3\pi}{2})} \int_{0}^{-N-\frac{1}{2}} \mathcal{O} \left( |z|^x \tau \Re(2\beta-\gamma) - x - \frac{1}{2} \right) \, dx, \]

which also tends to zero as \( \tau \to \infty^+ \), thanks to \( \arg(z) > -\frac{3\pi}{2} \).

4. We write \( s = -N - \frac{1}{2} + iy \) in the fourth integral to obtain,

\[
\int_{-N-\frac{1}{2} + iy}^{-N-\frac{1}{2} - \tau} \mathcal{I}(s, z) \, ds = i \left( \int_{0}^{-\tau} + \int_{\tau}^{0} \right) \mathcal{I} \left( -N - \frac{1}{2} + iy, z \right) \, dy \\
= i z^{-N-\frac{1}{2} - \beta} \left( \int_{0}^{\tau} \mathcal{O} \left( |y|^{N+\Re(2\beta-\gamma)-1} e^{-y\left( \frac{3\pi}{2} - \arg(z) \right)} \right) \, dy \\
- \int_{\tau}^{0} \mathcal{O} \left( |y|^{N+\Re(2\beta-\gamma)-1} e^{-y(\arg(z) + \frac{3\pi}{2})} \right) \, dy \right). \tag{3.55}
\]

Using the fact that \( \lim_{\tau \to \infty^+} \int_{0}^{\tau} e^{-ky} \, dy \) for \( k > 0 \) exists, the limit as \( \tau \to \infty^+ \) of the fourth integral exists and is of order \( \mathcal{O} \left( |z|^{-N-\frac{1}{2} - \beta} \right) \) as \( \tau \to \infty^+ \), thanks to \( |\arg(z)| < \frac{3\pi}{2} \).

Summarising the above analysis, we have shown that for large \( \tau \),

\[
\varphi(\beta, \gamma; z) = \sum_{n=0}^{N} \text{Res} I(s, z) + \mathcal{O} \left( |z|^{-N-\frac{1}{2} - \beta} \right), \tag{3.56}
\]

\[
= z^{-\beta} \sum_{n=0}^{N} \frac{(\beta)_n (\beta + 1 - \gamma)_n}{(-z)^n n!} + \mathcal{O} \left( |z|^{-N-\frac{1}{2} - \beta} \right),
\]

where we have used the formula \( \text{Res} \Gamma(\lambda) = \frac{(-1)^n}{n!} \), for \( n \geq 0 \), to calculate the residues. This proves (3.53). Moreover, for \( N \geq 0 \), we can immediately deduce,

\[
e^{\mp \pi i(\beta - \gamma/2)} e^{\mp \pi i \gamma} \varphi(\gamma, \beta; e^{\pm \pi i} z) = e^{\mp \frac{\pi i}{2} (\beta - \gamma)} z^{\beta - \gamma} \sum_{n=0}^{N} \frac{(\gamma - \beta)_n (1 - \beta)_n}{z^n n!} + \mathcal{O} \left( e^{\mp \pi i} |z|^{-N-\frac{1}{2} + \beta - \gamma} \right),
\]

which proves (3.52). \( \square \)

**Remark 3.8.** The expression (3.56) is valid for all finite \( N \). In order to take the limit as \( N \to \infty \) it is important to understand that (3.56) becomes an asymptotic result. This is because the integrals in (3.55) depend on \( N \) and, in particular, they diverge as
$N \to \infty^+$, hence the interchange between limits $\lim_{N \to \infty} +$ and $\lim_{\tau \to \infty} +$ is not justified here.

Having established how to represent the fundamental solutions $\tilde{Y}^{(\infty,k)}(z)$ using Mellin-Barnes integrals, we now show how to analytically continue them to $z = 0$. We will prove the following proposition, which is sufficient to deduce the monodromy data formulae (3.41)-(3.42).

**Proposition 3.3.** Let $\tilde{y}_1^{(0)}(z)$ and $\tilde{y}_2^{(0)}(z)$ be the solutions as given in (3.32). For $-\pi \pm \frac{\pi}{2} < \arg(z) < \pi \pm \frac{\pi}{2}$, the integral as given by (3.48) satisfies,

$$\varphi(\beta, \gamma; z) = \frac{\Gamma(\gamma - 1)}{\Gamma(\beta)} \tilde{y}_1^{(0)}(z) + \frac{\Gamma(1 - \gamma)}{\Gamma(\beta + 1 - \gamma)} \tilde{y}_2^{(0)}(z).$$

**Proof.** Let $I(s, z)$ be the integrand of $\varphi(\beta, \gamma; z)$ as given by (3.54). For large $\tau > 0$ and an integer $N > 0$, we now consider the integral around the rectangle $R'$ with vertices $\pm i\tau$ and $N + \frac{1}{2} \pm i\tau$, with indentations along the imaginary axis as usual and with a negative orientation as shown in Figure 17 below. Our analysis of this integral is analogous to that of the integral around the rectangle $R$, which lies to the left of the imaginary axis.

![Figure 17: Path of integration around the rectangle $R'$, the dots represent the poles of the integrand of $\varphi(\beta, \gamma; z)$.](image)
By Cauchy’s theorem, we have,
$$\frac{1}{2\pi i} \int_{R'} I(s, z) \, ds \equiv \frac{1}{2\pi i} \left( \int_{-\infty}^{-i\tau} + \int_{-i\tau}^{i\tau} + \int_{i\tau}^{N+\frac{1}{2}+i\tau} + \int_{N+\frac{1}{2}+i\tau}^{-\infty} \right) I(s, z) \, ds$$
$$= - \sum_{n=0}^{M_1(N)} \text{Res}_{s=\beta+1-n} I(s, z) - \sum_{n=0}^{M_2(N)} \text{Res}_{s=\beta+n} I(s, z)$$
where $M_1(N)$ and $M_2(N)$ are the number of poles $\beta + 1 - \gamma$, $\beta + 2 - \gamma$, … and $\beta$, $\beta + 1$, … which lie inside the rectangle respectively. We examine these integrals under the limit $\tau \to \infty^+$ one-by-one, using the asymptotics (3.21) of the Gamma function:

1. By writing $s = x - i\tau$ in the first integral we obtain,
$$e^{\tau (\arg(z) - \frac{3\pi}{2})} \int_{N+\frac{1}{2}}^{0} O \left(|z|^x \tau^\Re(2\beta - \gamma) - x - \frac{1}{2}\right) \, dx,$$
which tends to zero as $\tau \to \infty^+$, thanks to $\arg(z) < \frac{3\pi}{2}$.

2. In the limit $\tau \to \infty^+$, the second integral becomes $\varphi(\beta, \gamma; z)$, by definition.

3. Similarly to the first integral, by writing $s = x + i\tau$ in the third integral, we obtain,
$$e^{-\tau (\arg(z) + \frac{3\pi}{2})} \int_{i\tau}^{N+\frac{1}{2}+i\tau} O \left(|z|^x \tau^\Re(2\beta - \gamma) - x - \frac{1}{2}\right) \, dx,$$
which also tends to zero as $\tau \to \infty^+$, thanks to $\arg(z) > -\frac{3\pi}{2}$.

4. We write $s = N + \frac{1}{2} + iy$ in the fourth integral, to obtain,
$$\int_{N+\frac{1}{2}+i\tau}^{N+\frac{1}{2}+i\tau} I(s, z) \, ds = i \int_{0}^{\tau} \int_{0}^{N+\frac{1}{2}+i\tau} \left( N + \frac{1}{2} + iy, z \right) \, dy$$
$$= i z^{N+\frac{1}{2}-\beta} \left( \int_{0}^{\tau} O \left(|y|^{-N+\Re(2\beta - \gamma)} - 2 e^{-y \left(\frac{3\pi}{2} - \arg(z)\right)}\right) \, dy \right) - \int_{0}^{\tau} O \left(|y|^{-N+\Re(2\beta - \gamma)} - 2 e^{-y \left(\frac{3\pi}{2} + \arg(z)\right)}\right) \, dy \right). \quad (3.57)$$
Using the fact that $\lim_{\tau \to \infty^+} \int_{0}^{\tau} e^{-ky} \, dy$ for $k > 0$ exists, the limit as $\tau \to \infty^+$
of fourth integral exists, thanks to $|\arg(z)| < \frac{3\pi}{2}$. Moreover, for $|z|$ sufficiently small, this limit exists uniformly with respect to large $N$, due to the minus sign in the exponent of $|y|$. In particular, for $|z|$ sufficiently small,

$$\lim_{N\to\infty^+}\int_{N+i\frac{1}{2}}^{N+i\frac{3}{2}} I(s, z) \, ds = 0.$$ 

Summarising the above analysis, we have shown the following,

$$\varphi(\beta, \gamma; z) = - \sum_{n=0}^{M_1(N)} \text{Res}_{s=\beta+1-\gamma+n} I(s, z) - \sum_{n=0}^{M_2(N)} \text{Res}_{s=\beta+n} I(s, z) + \lim_{\tau \to \infty^+} \int_{N+i\frac{1}{2}}^{N+i\frac{3}{2}} I(s, z) \, ds,$$

where the convergence of the limit of this integral is uniform with respect to $N \to \infty^+$. As such, we may interchange the limits $\lim_{\tau \to \infty^+}$ and $\lim_{N \to \infty^+}$ as follows,

$$\varphi(\beta, \gamma; z) = - \lim_{N \to \infty^+} \sum_{n=0}^{M_1(N)} \text{Res}_{s=\beta+1-\gamma+n} I(s, z) - \lim_{N \to \infty^+} \sum_{n=0}^{M_2(N)} \text{Res}_{s=\beta+n} I(s, z)$$

$$+ \lim_{\tau \to \infty^+} \lim_{N \to \infty^+} \int_{N+i\frac{1}{2}}^{N+i\frac{3}{2}} I(s, z) \, ds,$$

$$= - \sum_{n=0}^{\infty} \text{Res}_{s=\beta+1-\gamma+n} I(s, z) - \sum_{n=0}^{\infty} \text{Res}_{s=\beta+n} I(s, z)$$

$$+ \lim_{\tau \to \infty^+} \lim_{N \to \infty^+} \int_{N+i\frac{1}{2}}^{N+i\frac{3}{2}} I(s, z) \, ds,$$

$$= - \sum_{n=0}^{\infty} \text{Res}_{s=\beta+1-\gamma+n} I(s, z) - \sum_{n=0}^{\infty} \text{Res}_{s=\beta+n} I(s, z) + \lim_{\tau \to \infty^+} 0.$$

We compute the residues to find,

$$\varphi(\beta, \gamma; z) = \frac{\Gamma(\gamma - 1)}{\Gamma(\beta)} z^{1-\gamma} \sum_{n=0}^{\infty} \frac{(\beta + 1 - \gamma)_{n} z^{n}}{(2 - \gamma)_{n} n!} + \frac{\Gamma(1 - \gamma)}{\Gamma(\beta + 1 - \gamma)} \sum_{n=0}^{\infty} \frac{(\beta)_{n} z^{n}}{(\gamma)_{n} n!},$$

for $z \in \tilde{\Sigma}_1$ and $\tilde{\Sigma}_0$ and the desired result is proved. \qed

**Remark 3.9.** Continuing with the issue raised in Remark 3.8, the fact is that integrating along the rectangle $R$ to the left of the imaginary axis is only able to produce an asymptotic result because we do not have uniform convergence with respect to $N$ in the integrals (3.55). This is to be expected, since we know $\varphi(\beta, \gamma; z)$ is analytic on
sectors $\tilde{\Sigma}_{-1}$ and $\tilde{\Sigma}_0$, it certainly cannot be equal to a divergent $\,_{2}F_{0}$ series. However, when integrating along the rectangle $R'$ to the right of the imaginary axis we produce an equality with a linear combination of convergent series, namely this is the analytic continuation of the solutions at $z = \infty$ to $z = 0$. This is shown in (3.57), because the integrals here converge as $\tau \to \infty^+$ uniformly with respect to large $N$.

We conclude these computations by using Proposition 3.3 to prove the formulae (3.41)-(3.42) of Lemma 3.8.

**Proof of Lemma 3.8.** Recall from the definitions (3.46) and (3.47) of solutions,

\[ \tilde{y}_2^{(\infty,0)}(z) = -\varphi(\beta, \gamma; z) \quad \text{and} \quad \tilde{y}_1^{(\infty,0)}(z) = e^{i\pi(\beta-\gamma)}e^z\varphi(\gamma-\beta, \gamma; e^{-i\pi}z), \quad z \in \tilde{\Sigma}_0. \]

Let $\gamma_{\infty,0}$ be a curve as described at the beginning of this subsection. Proposition 3.2 shows how to represent the solutions of Kummer’s equation (3.2) around $z = \infty$ using a Mellin-Barnes integral. Due to the analyticity of this integral, as shown in the first part of the proof of Proposition 3.2, Proposition 3.3 provides the formula for the analytic continuation of these solutions to $z = 0$. That is to say,

\[ \gamma_{\infty,0} \left[ \tilde{y}_2^{(\infty,0)} \right](z) = -\frac{\Gamma(\gamma - 1)}{\Gamma(\beta)} \tilde{y}_1^{(0)}(z) - \frac{\Gamma(1 - \gamma)}{\Gamma(\beta + 1 - \gamma)} \tilde{y}_2^{(0)}(z). \]

By manipulating the parameters and variable as follows: $\beta \mapsto \gamma - \beta$, $\gamma \mapsto \gamma$, $z \mapsto e^{i\pi}z$, we also deduce,

\[ \gamma_{\infty,0} \left[ \tilde{y}_1^{(\infty,0)} \right](z) = e^{i\pi(\beta-\gamma)}\frac{\Gamma(\gamma - 1)}{\Gamma(\beta)} e^{-i\pi(1-\gamma)}z^{1-\gamma}e^z \,_{1}F_{1}\left(1 - \beta \atop 2 - \gamma ; -z \right) + e^{i\pi(\beta-\gamma)}\frac{\Gamma(1 - \gamma)}{\Gamma(1 - \beta)} e^{z} \,_{1}F_{1}\left(\gamma - \beta \atop \gamma ; -z \right). \]

After applying Kummer’s transformation,

\[ e^z \,_{1}F_{1}\left(\begin{array}{c}a \\ c \end{array}; -z \right) \equiv \,_{1}F_{1}\left(\begin{array}{c}c - a \\ c \end{array}; z \right), \]
we deduce the connection matrix as given in (3.42), namely,

\[
\begin{pmatrix}
\gamma_{\infty,0} \left[ \tilde{y}_1^{(\infty,0)} \right](z), \quad \gamma_{\infty,0} \left[ \tilde{y}_2^{(\infty,0)} \right](z)
\end{pmatrix} = \begin{pmatrix}
\tilde{y}_1^{(0)}(z), \quad \tilde{y}_2^{(0)}(z)
\end{pmatrix} \tilde{C}^{0\infty},
\]

where,

\[
\tilde{C}^{0\infty} = \begin{pmatrix}
e^{i\pi(\beta-1)} \frac{\Gamma(\gamma-1)}{\Gamma(\gamma-\beta)} & -\frac{\Gamma(\gamma-1)}{\Gamma(\beta)} \\
e^{i\pi(\beta-\gamma)} \frac{\Gamma(1-\gamma)}{\Gamma(1-\beta)} & -\frac{\Gamma(1-\gamma)}{\Gamma(\beta+1-\gamma)}
\end{pmatrix}.
\]

We now turn our attention to proving the formulae (3.41) for Stokes’ matrices. By Definition 3.3 of the Stokes’ matrices \( \tilde{S}_k \) and by the asymptotic behavior (3.34) of the fundamental solutions \( \tilde{Y}^{(\infty,k)}(z) \), we have,

\[
\begin{pmatrix}
z^{\beta-\gamma}e^z & 0 \\
0 & z^{1-\beta}
\end{pmatrix} \tilde{S}_k \begin{pmatrix}
z^{\gamma-\beta}e^{-z} & 0 \\
0 & z^{\beta-1}
\end{pmatrix} \sim I, \quad \text{as } z \to \infty, \quad \text{arg}(z) - k\pi \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right).
\]

From this relation we easily deduce that \( \tilde{S}_{-1} \) is lower triangular and \( \tilde{S}_0 \) is upper triangular, both with unit diagonals. Denote by \( \tilde{s}_{-1} \) and \( \tilde{s}_0 \) the (2,1) and (1,2) elements of the matrices \( \tilde{S}_{-1} \) and \( \tilde{S}_0 \) respectively. With the knowledge of the connection matrix \( \tilde{C}^{0\infty} \), we use the cyclic relation (3.44) as follows,

\[
\tilde{C}^{\infty0}e^{2\pi i \tilde{s}_0} \tilde{C}^{0\infty} = \left( \tilde{S}_{-1} \right)^{-1} e^{-2\pi i \tilde{s}_0} \left( \tilde{S}_0 \right)^{-1},
\]

\[
\Leftrightarrow \begin{pmatrix}
\frac{e^{2\pi i(\beta-\gamma)}}{\Gamma(1-\beta)\Gamma(\gamma-\beta)} & \frac{-2\pi i e^{-i\pi \gamma}}{\Gamma(\beta+1-\gamma)} \\
\frac{-2\pi i e^{2\pi i(\beta-\gamma)}}{\Gamma(1-\beta)\Gamma(\gamma-\beta)} & 1 - e^{2\pi i(\beta-\gamma)} + e^{2\pi i(1-\gamma)}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 0 \\
-\tilde{s}_{-1} & 0
\end{pmatrix} \begin{pmatrix}
e^{2\pi i(\beta-\gamma)} & 0 \\
0 & e^{2\pi i(1-\beta)}
\end{pmatrix} \begin{pmatrix}
1 & -\tilde{s}_0 \\
0 & 1
\end{pmatrix},
\]

\[
\Leftrightarrow \begin{cases}
\tilde{s}_{-1} = \frac{2\pi i}{\Gamma(1-\beta)\Gamma(\gamma-\beta)}, \\
\tilde{s}_0 = \frac{2\pi i}{\Gamma(\beta)\Gamma(\beta+1-\gamma)}e^{i\pi(\gamma-2\beta)},
\end{cases}
\]

which are indeed the Stokes’ multipliers found in the formulae (3.41) for the Stokes’ matrices.

**Remark 3.10.** If we had chosen to normalise the monodromy data of Kummer’s equation with respect to the fundamental solution \( \tilde{Y}^{(\infty,-1)}(z) \) then the signs of the exponents in
would be inverted. Furthermore, the monodromy matrix around infinity would change as $\tilde{M}_\infty \mapsto \tilde{S}_0^{-1} \tilde{M}_\infty \tilde{S}_0$.

### 3.2.3 Gevrey Asymptotics and a Result of Ramis and Martinet

We close this subsection about Kummer's confluent hypergeometric differential equation by examining Gevrey asymptotics and stating a result of Ramis and Martinet. This also gives us the opportunity to show a contemporary approach to the theory of Stokes' phenomenon, which we have learned from [Bal,Put]. The contents of this additional subsection will not be necessary for our main theorems in Section 3.3, we include it for the curiosity of the reader.

We recall some definitions and facts regarding asymptotic theory. In the following, keep in mind that the role of the letter $k$ will mirror the concept of a linear differential equation having a pole of Poincaré rank $k$, so that for Kummer’s equation we are specifically concerned with $k = 1$. Denote by $\mathbb{C}[[z^{-1}]]$ the field of formal series in $z^{-1}$.

**Definition 3.5.** Let $f$ be a function analytic in a sector $\tilde{\Sigma}$. We say that $f$ has the series $\hat{f} = \sum_{n=0}^{\infty} f_n z^{-n} \in \mathbb{C}[[z^{-1}]]$ as its Gevrey asymptotic expansion of order $k^{-1}$ as $z \to \infty$, $z \in \tilde{\Sigma}$, denoted $f \simeq_{\frac{1}{k}} \hat{f}$, if for every closed subsector $\sigma$ of $\tilde{\Sigma}$, there exists a constant $K > 0$ such that, for all $N \in \mathbb{N}$ and $z \in \sigma$,

$$
\left| z^N \left( f(z) - \sum_{n=0}^{N-1} f_n z^{-n} \right) \right| \leq K^N \Gamma \left( 1 + \frac{N}{k} \right) .
$$

We denote by $\mathcal{A}_{\frac{1}{k}}(\tilde{\Sigma})$ the set of analytic functions on $\tilde{\Sigma}$ which have a Gevrey asymptotic expansion of order $k^{-1}$.

Gevrey asymptotics is a stronger definition than the usual one of Poincaré because it specifies how the right hand side of the inequality (3.59) depends on $N$. In Poincaré’s definition of an asymptotic series the precise dependence on $N$ is not relevant. If we denote by $\mathcal{A}(\tilde{\Sigma})$ the set of analytic functions on a sector $\tilde{\Sigma}$ which admit an asymptotic expansion then we have,

$$
\mathcal{A}(\tilde{\Sigma}) \supset \mathcal{A}_{\frac{1}{2}}(\tilde{\Sigma}) \supset \mathcal{A}_{\frac{1}{3}}(\tilde{\Sigma}) \supset \mathcal{A}_{\frac{1}{4}}(\tilde{\Sigma}) \supset \ldots ,
$$

Page 66 of 178
since the asymptotic expansion (3.21) of the Gamma function implies:

\[
\frac{\Gamma \left(1 + \frac{N}{k+1}\right)}{\Gamma \left(1 + \frac{N}{k}\right)} \to 0 \text{ as } N \to \infty.
\]

We note that, if \( f \in \mathcal{A}_k^\circlearrowleft(\Sigma) \), with \( f \simeq \sum_{n=0}^\infty f_n z^{-n} \), then these coefficients satisfy \( |f_n| < K^N \Gamma \left(1 + \frac{n}{k}\right) \), for some positive constant \( K \) and \( n \geq 1 \). To see this, we add the following inequalities:

\[
|f(z) - \sum_{n=0}^{N-1} f_n z^{-n}| \leq |z|^{-N} K^N \Gamma \left(1 + \frac{N}{k}\right),
\]

\[
|f(z) - \sum_{n=0}^{N} f_n z^{-n}| \leq |z|^{-N-1} K^{N+1} \Gamma \left(1 + \frac{N+1}{k}\right),
\]

to obtain the following inequality for \( f_N \),

\[
|f_N| \leq K^N \Gamma \left(1 + \frac{N}{k}\right) + |z|^{-1} K^{N+1} \Gamma \left(1 + \frac{N+1}{k}\right),
\]

from which we immediately find the claimed property by taking the limit \( z \to \infty \). This motivates the following definition.

**Definition 3.6.** We call a series \( \hat{f} = \sum_{n=0}^\infty f_n z^{-n} \in \mathbb{C}[[z]] \) a Gevrey series of order \( k^{-1} \) if there exists a positive constant \( K \) such that, \( |f_n| < K^n \Gamma \left(1 + \frac{n}{k}\right) \) for all \( n \geq 1 \). We denote by \( \mathbb{C}[[z]]_{\frac{1}{k}} \) the set of all Gevrey series of order \( k^{-1} \).

Consider the map \( J : \mathcal{A}_k^\circlearrowleft(\Sigma) \to \mathbb{C}[[z]]_{\frac{1}{k}} \) which maps an analytic function \( f \) on the sector \( \Sigma \) to its Gevrey asymptotic expansion of order \( k^{-1} \). We recall the following result, see for instance [Bal,Put].

**Theorem 3.2.** Assume \( k > \frac{1}{2} \). The set \( \mathcal{A}_k^\circlearrowleft(\Sigma) \) is a differential algebra and the map \( J \) is a homomorphism. Moreover, if the sector \( \Sigma \) has an opening less than \( \frac{\pi}{k} \), then \( J \) is surjective, otherwise, if \( \Sigma \) has an opening greater than \( \frac{\pi}{k} \), then \( J \) is injective.

This remarkable theorem draws the connection between Gevrey asymptotics and Stokes’ phenomenon. Given a formal Gevrey series of order \( k^{-1} \), this theorem shows that there is a unique analytic function on a sector of opening greater than \( \frac{\pi}{k} \) which has that series as its Gevrey asymptotic expansion of order \( k^{-1} \). Observe that this is exactly
parallel to the theory of Stokes’ phenomenon: given a differential equation with a pole of Poincaré rank \( k \) and a formal fundamental series solution at that point, there are unique analytic fundamental solutions on a sectors of openings greater than \( \pi / k \) with the prescribed formal series as their asymptotic expansions.

Let \( \varphi(\beta, \gamma; z) \) be defined as in (3.48). Ramis and Martinet prove the following result.

**Theorem [RM].** The function \( z^a \varphi(a, c; z) \) has \( \, _2F_0(a, a + 1 - c; -z^{-1}) \) as its Gevrey asymptotic expansion of order one as \( z \to \infty \) with \( |\arg(z)| < \frac{3\pi}{2} \). Similarly, \( (-z)^{c-a} \varphi(c-a, c; -z) \) has \( \, _2F_0(c-a, 1-a; z^{-1}) \) as its Gevrey asymptotic expansion of order one with \( |\arg(-z)| < \frac{3\pi}{2} \).

We have seen in the first part of the proof of Proposition 3.2 that \( \varphi(a, c; z) \) and \( \varphi(c-a, c; -z) \) are analytic in the sectors \( \Sigma_{-1} \) and \( \Sigma_{0} \). In particular, since these sectors have openings greater than \( \pi \), Theorem 3.2 states that the map \( J : A_1(\Sigma_+) \to \mathbb{C}[[z]] \) is injective. In other words, there are unique analytic functions on these sectors which have the formal series solutions,

\[
\begin{align*}
&z^{-a} \, _2F_0(a, a + 1 - c; -z^{-1}) \quad \text{and} \quad (-z)^{c-a} e^z \, _2F_0(c-a, 1-a; z^{-1}) ,
\end{align*}
\]

as their Gevrey asymptotic expansions of order 1. Since we have seen that Gevrey asymptotics imply asymptotics in the usual sense, recall (3.60), this implies that such analytic functions on these sectors are in fact solutions to Kummer’s equation (3.2), by the uniqueness statement in Theorem 3.1. Since the formal series solutions (3.61) are clearly linearly independent, Ramis and Martinet’s Theorem shows that the functions,

\[
\varphi(a, c; z) \quad \text{and} \quad e^z \varphi(c-a, c; -z),
\]

constitute a fundamental set of solutions of Kummer’s equation. Compared with our proof of this fact, stated as Proposition 3.2, it is satisfying to deduce this from a different perspective.
3.3 A Confluence from Gauss to Kummer

In this Section we analyse a confluence procedure from Gauss’ hypergeometric differential equation (3.1) to Kummer’s confluent equation (3.2). We are primarily concerned with understanding how to produce the monodromy data \( \tilde{\mathcal{M}} \) of Kummer’s equation, as defined in Section 3.2.2, from the monodromy data \( \mathcal{M} \) of Gauss’ equation, as defined in Section 3.1.2, under the confluence procedure. However, we require that the derivation of these limits of monodromy data must be achieved in a manner which is repeatable for the Painlevé equations, where the monodromy data in its full generality is transcendental, in other words the correspondence between solutions and monodromy data is not explicit. Our approach is to understand how to produce the fundamental solutions of Kummer’s equation, as defined in Section 3.2.1, by taking certain limits of the fundamental solutions of Gauss’ equation, as defined in Section 3.1.1. This allows us to solve our problem since the fundamental solutions of Gauss’ and Kummer’s equations are related by connection matrices and Stokes’ matrices, the details of these steps are given in Sections 3.3.1 and 3.3.2. Since the monodromy data of the hypergeometric equations are given in closed-form, we may also prove our result directly by performing the explicit computations, this is given at the end of Section 3.3.2.

We first explain how the confluence procedure works. We make the substitution \( x = \frac{z}{\alpha} \), which maps the singularities of Gauss’ equation from \( x = 0, 1, \infty \) to \( z = 0, \alpha, \infty \) respectively. Observe what happens to Gauss’ equation (3.1),

\[
x(1 - x) \frac{d^2}{dx^2} y(x) + (\gamma - (\alpha + \beta + 1)x) \frac{dy}{dx}(x) - \alpha \beta y(x) = 0,
\]

\[
\Leftrightarrow \frac{z}{\alpha} \left( \frac{\alpha - z}{\alpha} \right) \alpha^2 \frac{d^2}{dz^2} y_{zz} + \left( \gamma - (\alpha + \beta + 1) \frac{z}{\alpha} \right) \alpha y_z - \alpha \beta y = 0,
\]

\[
\Leftrightarrow z \frac{d^2}{dz^2} y_{zz} + (\gamma - z) y_z - \beta y - \frac{1}{\alpha} \left( z^2 y_{zz} + (\beta + 1) y_z \right) = 0.
\]

As a heuristic argument, one can see that the final equation symbolically becomes Kummer’s equation (3.2) as \( \alpha \to \infty \),

\[
z \frac{d^2}{dz^2} \tilde{y}_{zz} + (\gamma - z) \tilde{y}_z - \beta \tilde{y} = 0,
\]
and a double pole is created at \( z = \infty \) as the two simple poles \( z = \alpha \) and \( \infty \) merge. This demonstration lacks mathematical diligence, of course, and merely suggests that there may be a means of obtaining solutions of Kummer’s equation by taking limits as \( \alpha \to \infty \) of certain solutions of Gauss’ equation under the substitution \( x = \frac{z}{\alpha} \). This is indeed the case and we prove it in the following section.

Remark 3.11. This demonstration of the confluence procedure can also be understood from the perspective of the \((2 \times 2)\) linear systems. We make the substitution \( x = \frac{z}{\alpha} \) in (3.5) and differentiate,

\[
\frac{\partial Y}{\partial z} = \frac{1}{\alpha} \left( \frac{\alpha A_0}{z} + \frac{A_1}{\frac{z}{\alpha} - 1} \right) Y,
\]

\[
= \frac{1}{\alpha} \left( \frac{\alpha}{z(\alpha + 1 - \beta)} \begin{pmatrix} \alpha(\beta - \gamma) & \alpha(1 - \beta)(\beta - \gamma) \\ \alpha + 1 - \gamma & (1 - \beta)(\alpha + 1 - \gamma) \end{pmatrix} + \frac{\alpha}{(z - \alpha)(\alpha + 1 - \beta)} \begin{pmatrix} \alpha(\gamma - \alpha - 1) & \alpha(\beta - 1)(\beta - \gamma) \\ \gamma - \alpha - 1 & (\beta - 1)(\beta - \gamma) \end{pmatrix} \right) Y,
\]

\[
= \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} \beta - \gamma & (1 - \beta)(\beta - \gamma) \\ 1 & 1 - \beta \end{pmatrix} + \mathcal{O}(\alpha^{-1}) \right) Y,
\]

\[
= \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{\tilde{A}_0}{z} + \mathcal{O}(\alpha^{-1}) \right) Y.
\]

One can see that the final equation symbolically becomes Kummer’s equation (3.31) as \( \alpha \to \infty \) and an irregular singularity is produced at \( z = \infty \) as the two simple poles \( z = \alpha \) and \( z = \infty \) merge.

3.3.1 Limits of Solutions

As outlined above, our confluence procedure is to introduce the new variable \( z \) by the substitution \( x = \frac{z}{\alpha} \) and take the limit \( \alpha \to \infty \). For the remainder of this chapter we must be careful in which way we are taking \( \alpha \) to infinity, for example it would be inconvenient for us if \( \alpha \) spiralled towards infinity. We will consider two limits along fixed rays: one with \( \arg(\alpha) = \frac{\pi}{2} \) and the other with \( \arg(\alpha) = -\frac{\pi}{2} \). These particular directions along which \( \alpha \) tends to infinity turn out to be the most convenient. These two directions have little effect on our solutions at the Fuchsian points \( x = 0 = z \).
However, we need to take care when considering limits of our solutions at the merging Fuchsian points \( x = 1 \) and \( \infty \) as we will find that the results depend on the direction of \( \alpha \).

**Obtaining \( \tilde{Y}^{(0)}(z) \) from \( Y^{(0)}(z) \)**

Since the substitution \( x = \frac{z}{\alpha} \) and limit \( \alpha \to \infty \) do not interfere with the nature of the Fuchsian singularity \( x = 0 \), corresponding to \( z = 0 \), we will deal with this case first.

We note that this part will not be necessary to prove our main results concerned with producing the Stokes’ matrices of Kummer’s equation. We will use the hypergeometric equations as an opportunity to prove the following lemma, which is a specific case of a more-general result concerned with uniform convergence and the interchange of limits.

**Lemma 3.9.** We have the following limit,

\[
\lim_{\alpha \to \infty} 2F_1 \left( \frac{\alpha, \beta}{\gamma}; \frac{z}{\alpha} \right) = 1F_1 \left( \frac{\beta}{\gamma}; z \right).
\]

In the following proof of this lemma we denote by \( a_\infty \) the point ‘at infinity’ which is adjoined to the complex plane \( \mathbb{C} \) so that the stereographical projection from the sphere \( S^2 \) to the extended complex plane \( \overline{\mathbb{C}} = \mathbb{C} \cup \{ a_\infty \} \) is an isomorphism. For example, in the chart where we project from the ‘north pole’, that is the point \((0, 0, 1)\), the projection \( \pi : S^2 \to \mathbb{C} \cup \{ a_\infty \} \) maps \((0, 0, 1) \mapsto a_\infty \) and \((0, 0, -1) \mapsto 0 \).

**Proof.** This proof is inspired by [Tit]. Let,

\[
f_n(\alpha) := \frac{(\alpha)_n (\beta)_n}{\alpha^n (\gamma)_n n!} z^n, \quad F_m(\alpha) := \sum_{n=0}^{m} f_n(\alpha),
\]

\[
F(\alpha) := \lim_{m \to \infty} F_m(\alpha) \quad \text{and} \quad r_m(\alpha) := F(\alpha) - F_m(\alpha),
\]

so that,

\[
|F(\alpha) - F(a_\infty)| = |F_m(\alpha) - F_m(a_\infty) + r_m(\alpha) - r_m(a_\infty)|
\]

\[
\leq |F_m(\alpha) - F_m(a_\infty)| + |r_m(\alpha)| + |r_m(a_\infty)|.
\]
Observe that,

\[
F(\alpha) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{\alpha^n} \frac{(\beta)_n}{(\gamma)_n n!} z^n = 2F_1 \left( \begin{array}{c} \alpha, \beta \\ \gamma \\ \alpha \end{array} ; \frac{z}{\alpha} \right),
\]

and in particular,

\[
F(a_{\infty}) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\gamma)_n n!} z^n = 1F_1 \left( \begin{array}{c} \beta \\ \gamma \\ z \end{array} \right),
\]

hence, to establish the required limit it is equivalent to show that \(F(\alpha)\) is continuous at \(\alpha = a_{\infty}\). In other words, we need to show: for all \(\varepsilon > 0\), the left hand side of (3.62) is less than \(\varepsilon\) for \(|\alpha|\) sufficiently large.

For each \(n\), the function \(f_n(\alpha)\) is a polynomial in \(\alpha^{-1}\) since,

\[
\frac{(\alpha)_n}{\alpha^n} = 1 + \sum_{k=1}^{n-1} \alpha^{-k} c_k,
\]

where \(c_k\) are some constants (Stirling numbers of the first kind to be precise). The functions \(F_m(\alpha)\), being a finite sum of \(f_n(\alpha)\), are therefore continuous at \(\alpha = a_{\infty}\), in other words:

\[
\forall \varepsilon > 0 \exists \alpha' : |\alpha| > |\alpha'| \Rightarrow |F_m(\alpha) - F_m(a_{\infty})| < \varepsilon, \quad (3.63)
\]

where,

\[
F_m(a_{\infty}) = \sum_{n=0}^{m} f_n(a_{\infty}) = \sum_{n=0}^{m} \frac{(\beta)_n}{(\gamma)_n n!} z^n.
\]

By d’Alembert’s ratio test, the series \(F(a_{\infty})\) converges for all \(z \in \mathbb{C}\) and the series \(F(\alpha)\) converges for \(|z| < |\alpha|\), which is uniform in \(\alpha\) for sufficiently large \(|\alpha|\). In other words,

\[
\forall \varepsilon > 0 \exists N : m > N \Rightarrow |r_m(a_{\infty})| < \varepsilon, \quad (3.64)
\]
and,

\[\forall \varepsilon > 0 \exists N \text{ independent of } \alpha : m > N \Rightarrow |r_m(\alpha)| < \varepsilon, \quad \forall \alpha : |\alpha| > |\alpha'| \text{ for some } \alpha'.\]

(3.65)

Combining (3.63), (3.64) and (3.65) with (3.62), there exists an \(\alpha'\) such that,

\[|\alpha| > |\alpha'| \Rightarrow |F(\alpha) - F(a_\infty)| < 3\varepsilon.\]

Hence, \(F(\alpha)\) is continuous at \(\alpha = a_\infty\) and the desired result is proved. It is important to note that this conclusion would have failed if \(F(\alpha)\) had not been uniformly convergent in \(\alpha\); if, instead of (3.65), there did not exist an \(N\) independent of \(\alpha\) then \(|r_m(\alpha)| < \varepsilon\) would not hold for \(|\alpha|\) sufficiently large.

Remark 3.12. While Lemma 3.9 establishes a rather elementary limit, this is an instance of an important result concerned with uniform convergence and the interchange of limits. The general result is stated as follows:

If a sequence \(F_m(\varepsilon)\) of functions, each continuous for \(\varepsilon\) sufficiently small, converges uniformly with respect to \(\varepsilon\), then \(\lim_{\varepsilon \to 0} \lim_{m \to \infty} F_m(\varepsilon) = \lim_{m \to \infty} \lim_{\varepsilon \to 0} F_m(\varepsilon)\).

This general statement can be easily proved in the same spirit as the proof of Lemma 3.9, see for instance [Tit]. We will use this result in Section 4.2 when dealing with the confluence of the \(P_{VI}\) linear system to the \(P_V\) one around singular points which remain Fuchsian.

We define,

\[\omega^+_0(\alpha) = \left\{ z : |z| < |\alpha|, \quad -\pi \pm \frac{\pi}{2} \leq \arg(z) < \pi \pm \frac{\pi}{2} \right\},\]

so that, for \(\arg(\alpha) = \pm \frac{\pi}{2}\), \(x \in \Omega_0 \iff z \in \omega^+_0(\alpha)\). Since the radius of this neighbourhood clearly becomes infinite as \(\alpha \to \infty\), if \(z \in \omega^+_0(\alpha)\) for all \(|\alpha|\) sufficiently large, then \(z \in \tilde{\Omega}^+_0\), where \(\tilde{\Omega}^+_0\) is the domain in our definition of the fundamental solutions of Kummer’s equation around \(z = 0\) as given in Section 3.2.1.

**Theorem 3.3.** Let \(y_k^{(0)}(x)\) and \(\tilde{y}_k^{(0)}(z)\), \(k = 1, 2\), be defined as in (3.6) and (3.32)
respectively. For \( \arg(\alpha) = \pm \frac{\pi}{2} \), we have the following limits,

\[
\lim_{\alpha \to \infty} \frac{y_1^{(0)}(z\alpha^{-1})}{\alpha^{1-\gamma}} = \frac{\tilde{y}_1^{(0)}(z)}{z},
\quad z \in \tilde{\Omega}_0^\pm.
\]

\[
\lim_{\alpha \to \infty} \frac{y_2^{(0)}(z\alpha^{-1})}{\alpha^{1-\gamma}} = \frac{\tilde{y}_2^{(0)}(z)}{z},
\quad z \in \tilde{\Omega}_0^\pm.
\]

**Proof.** We have already noted in the paragraph before this theorem how the domain \( \omega_0^\pm \) tends to the domain \( \tilde{\Omega}_0^\pm \). Using Lemma 3.9, we compute the limits as follows,

\[
\lim_{\alpha \to \infty} \frac{y_1^{(0)}(z\alpha^{-1})}{\alpha^{1-\gamma}} = \lim_{\alpha \to \infty} z^{1-\gamma} \frac{2F_1\left(\frac{\alpha + 1 - \gamma}{2 - \gamma}, \frac{\beta + 1 - \gamma}{2 - \gamma}, \frac{z}{\alpha}\right)}{\Gamma(1/2)} = \tilde{y}_1^{(0)}(z),
\quad z \in \tilde{\Omega}_0^\pm,
\]

and

\[
\lim_{\alpha \to \infty} \frac{y_2^{(0)}(z\alpha^{-1})}{\alpha^{1-\gamma}} = \lim_{\alpha \to \infty} \frac{2F_1(0; \frac{\alpha}{\alpha}, \frac{\beta}{\alpha}; z)}{\Gamma(1/2)} = \tilde{y}_2^{(0)}(z),
\quad z \in \tilde{\Omega}_0^\pm,
\]

as required.

**Remark 3.13.** The factor \( \alpha^{1-\gamma} \) in the first limit of Theorem 3.3 is necessary because of the term,

\( x^{1-\gamma} \equiv z^{1-\gamma}\alpha^{\gamma-1}, \)

in the solution \( y_1^{(0)}(x) \), as given in (3.6).

**Remark 3.14.** We have stated Theorem 3.3 in terms of the solutions of the *scalar* hypergeometric equations (3.1) and (3.2). The limits (3.66) can be equivalently stated in terms of the solutions of the \((2 \times 2)\) equations (3.5) and (3.31): for \( \arg(\alpha) = \pm \frac{\pi}{2} \),

\[
\lim_{\alpha \to \infty} \frac{Y^{(0)}(z\alpha^{-1})}{\alpha^{\Theta_0}} = \tilde{Y}^{(0)}(z),
\quad z \in \tilde{\Omega}_0^\pm.
\]

(3.67)

To see how this is equivalent to (3.66), recall Lemmas 3.3 and 3.7. From the viewpoint of working with the \((2 \times 2)\) equations, Theorem 3.3 can be proved in an analogous
manner: for the diagonalising matrices we have that,

$$\lim_{\alpha \to \infty} R_0 = \lim_{\alpha \to \infty} \left( \begin{array}{cc} 1 & 1 \\ \frac{\alpha+1-\gamma}{\alpha(\beta-\gamma)} & \frac{1}{\beta-1} \end{array} \right) = \left( \begin{array}{cc} 1 & 1 \\ \frac{1}{\beta-\gamma} & \frac{1}{\beta-1} \end{array} \right) = \tilde{R}_0,$$

and for the series, using Lemma 3.9,

$$\lim_{\alpha \to \infty} G_0 \left( z^{\alpha^{-1}} \right) = \lim_{\alpha \to \infty} \left( \begin{array}{c} _2F_1 \left( \alpha+1-\gamma, \beta-\gamma; \frac{z}{\alpha} \right) \\ 1-\gamma \\
\frac{z(\alpha+1-\gamma)(1-\beta)}{\alpha(1-\gamma)(2-\gamma)} \ _2F_1 \left( \alpha+2-\gamma, \beta+1-\gamma; 3-\gamma; \frac{z}{\alpha} \right) \right),$$

$$= \left( \begin{array}{c} _1F_1 \left( \beta-\gamma; \frac{z}{1-\gamma} \right) \\ 1-\gamma \\
\frac{z(\gamma-\beta)}{\gamma(\gamma-1)} \ _1F_1 \left( \beta; \frac{z}{\gamma+1} \right) \\ \frac{z(1-\beta)}{(1-\gamma)(2-\gamma)} \ _1F_1 \left( \beta+1-\gamma; 3-\gamma; \frac{z}{\gamma-1} \right) \\ 1-\gamma \end{array} \right) = H_0(z).$$

**Obtaining the solutions** $Y^{(\infty,k)}(z)$

We now turn our attention to the main problem of how to obtain fundamental solutions at the double pole of the confluent equation from solutions at the merging simple poles of the original equation. We first examine the behavior of the fundamental solutions at $x = \infty$, as given in (3.8). Observe that these solutions are expressed using the Gauss $\text{2F}_1$ series in the variable $x^{-1} \equiv \frac{\alpha}{z}$, which diverge for $|x^{-1}| > 1 \Leftrightarrow |z| < |\alpha|$. In this case, we clearly do not have uniform convergence with respect to $\alpha$ and so we cannot deal with limits of solutions in the same way as in Lemma 3.9. Instead, we will solve our problem using Lemma 3.10, below, and Glutsyuk’s Theorem 2.5.

The fundamental set of solutions (3.8) are written in canonical form. However, we will
rewrite the solution \( y_1^{(\infty)}(x) \) using one of Kummer’s relations as follows,

\[
y_1^{(\infty)}(x) = (-x)^{-\alpha} _2F_1 \left( \begin{array}{c} \alpha, \alpha + 1 - \gamma \\ \alpha + 1 - \beta \end{array}; x^{-1} \right), \quad x \in \Omega_{\infty},
\]

\[
= (-x)^{\beta - \gamma} (1 - x)^{\gamma - \alpha - \beta} _2F_1 \left( \begin{array}{c} 1 - \beta, \gamma - \beta \\ \alpha + 1 - \beta \end{array}; x^{-1} \right), \quad x \in \hat{\Omega}_{\infty}, \tag{3.68}
\]

where the new domain \( \hat{\Omega}_{\infty} \) is defined as,

\[
\hat{\Omega}_{\infty} = \{ x : |x| > 1, -\pi \leq \arg(-x) < \pi, -\pi \leq \arg(1 - x) < \pi \}.
\]

There is no need to rewrite the solution \( y_2^{(\infty)}(x) \) as given in (3.8) as it is already in a suitable form, this is explained in Lemma 3.10 below. We note that the above two forms of the solution \( y_1^{(\infty)}(x) \) are equivalent on the domain \( \Omega_{\infty} \cap \hat{\Omega}_{\infty} \). The condition imposed on \( \arg(1 - x) \) in \( \hat{\Omega}_{\infty} \) is only necessary to deal with the term \( (1 - x)^{\gamma - \alpha - \beta} \). After making the substitution \( x = \frac{z}{\alpha} \) and taking the limit \( \alpha \to \infty \), the condition \( |\arg(1 - x)| < \pi \) does not play a role because the term \( (1 - x)^{\gamma - \alpha - \beta} \) tends to a single-valued function of \( z \), namely,

\[
\left( 1 - \frac{z}{\alpha} \right)^{\gamma - \alpha - \beta} = \exp \left( (\gamma - \alpha - \beta) \log \left( 1 - \frac{z}{\alpha} \right) \right),
\]

\[
= \exp \left( (\gamma - \alpha - \beta) \left( -\frac{z}{\alpha} + \mathcal{O}\left( \alpha^{-2} \right) \right) \right),
\]

\[
= e^{z} \left( 1 + \mathcal{O}\left( \alpha^{-1} \right) \right). \tag{3.69}
\]

This computation shows how to asymptotically pass from power-like behavior to exponential behavior as \( \alpha \to \infty \). Moreover, with this new form of \( y_1^{(\infty)}(x) \) we are ready to state the following lemma.

**Lemma 3.10.** Let \( y_2^{(\infty)}(x) \) be given by (3.8) and \( y_1^{(\infty)}(x) \) be given in its new form by (3.68). After the substitution \( x = \frac{z}{\alpha} \), the terms of these series tend to the terms in the formal series solutions \( \tilde{y}_{1,f}^{(\infty)}(z) \) and \( \tilde{y}_{2,f}^{(\infty)}(z) \) as given by (3.39), namely we have the
following limits:

\[
\lim_{\alpha \to \infty} \frac{(1 - \beta)_n (\gamma - \beta)_n \alpha^n}{(\alpha + 1 - \beta)_n n! z^n} = \frac{(\gamma - \beta)_n (1 - \beta)_n}{n! z^n},
\]
\[
\lim_{\alpha \to \infty} \frac{(\beta)_n (\beta + 1 - \gamma)_n \alpha^n}{(\beta + 1 - \alpha)_n n! z^n} = (-1)^n \frac{(\beta)_n (\beta + 1 - \gamma)_n}{n! z^n}.
\]

**Proof.** By direct computation, using

\[
\frac{\alpha^n}{(\alpha + 1 - \beta)_n} = 1 + O(\alpha^{-1}) \quad \text{and} \quad \frac{\alpha^n}{(\beta + 1 - \alpha)_n} = (-1)^n + O(\alpha^{-1}).
\]

At first sight, this lemma is concerned with two limits, whose proofs are straightforward after noting the above asymptotics. Looking more closely, Lemma 3.10 shows that *term-by-term* limits of the solutions,

\[
y_1^{(\infty)}(z \alpha^{-1}) (\gamma - \beta) \quad \text{and} \quad -y_2^{(\infty)}(z \alpha^{-1}) (-\alpha)^{\beta - 1},
\]

produce the formal solutions,

\[
\tilde{y}_1^{(\infty)}(z) \quad \text{and} \quad \tilde{y}_2^{(\infty)}(z),
\]

respectively; this provides some re-assurance that the confluence procedure is working as we would want it to, at least on a formal level. The factors \((-\alpha)^{\beta - 1}\) and \((-\alpha)^{-\beta}\) in (3.70) are necessary because of the terms,

\[
(-x)^{\beta - 1} \equiv z^{\beta - 1} (-\alpha)^{\gamma - \beta} \quad \text{and} \quad (-x)^{-\beta} \equiv z^{-\beta} (-\alpha)^{\beta},
\]

in the solutions \(y_1^{(\infty)}(x)\) and \(y_2^{(\infty)}(x)\) respectively. We note that the direction in which \(\alpha \to \infty\) is not yet important for this lemma. The importance of this lemma is shown in the proof of our Main Theorem 1.

**Remark 3.15.** Lemma 3.10 is stated in terms of the solutions of the *scalar* hypergeometric equations (3.1) and (3.2). From the viewpoint of working with the \((2 \times 2)\) equations
(3.5) and (3.31), we rewrite the solution $Y^{(\infty)}(x)$, as given in (3.11), as follows,

$$
Y^{(\infty)}(x) = R_\infty \sum_{n=0}^{\infty} g_{n,\infty} x^{-n} (-x)^{-\Theta_\infty}, \quad x \in \Omega_\infty,
$$

$$
= R_\infty \sum_{n=0}^{\infty} \hat{g}_{n,\infty} x^{-n} (-x)^{-\Theta_\infty - \Theta_1 (1-x)^{\Theta_1}}, \quad x \in \hat{\Omega}_\infty,
$$

(3.71)

where $\hat{g}_{0,\infty} = I$ and we find all other coefficients $\hat{g}_{n,\infty}$, $n \geq 1$, from the recursive relation,

$$
n \hat{g}_{n,\infty} + [\hat{g}_{n,\infty}, \Theta_\infty Y] = -R_\infty^{-1} A_{1Y} R_\infty Y \sum_{l=0}^{n-1} \hat{g}_{l,\infty} + \sum_{l=0}^{n-1} \hat{g}_{l,\infty} \Theta_1.
$$

This recursion equation only differs from that for $g_{n,\infty}$, given in the proof of Lemma 3.2, by the final summation term. We find the solution to this equation is,

$$
\hat{g}_{n,\infty} = \begin{pmatrix}
\frac{(1-\beta)n(\gamma-\beta)}{(\alpha+1-\beta)n!} & -\frac{(\beta)n(\beta+1-\gamma)_{n-1}}{(\beta-1)n(n-1)!} \\
\frac{\alpha(1-\beta)(\beta-\gamma)(\alpha+1-\gamma)}{(\alpha-\beta)(\alpha+1-\beta)^2(\alpha+2-\beta)} & \frac{(\alpha-\beta)(\beta-\gamma)}{(\alpha-1)n(n-1)!}
\end{pmatrix} \cdot (\alpha)^{\beta-n} \gamma^{-\beta} z^{\beta-\gamma} (1-\frac{z}{\alpha})^{\gamma-\alpha-\beta} 0
$$  

(3.72)

The transformation (3.71) is analogous to Kummer’s relation (3.68). We note that,

$$
Y^{(\infty)} \left( \frac{z}{\alpha} \right) = R_\infty \sum_{n=0}^{\infty} \hat{g}_{n,\infty} \alpha^n z^{-n} \begin{pmatrix}
(-\alpha)^{\gamma-\beta} z^{\beta-\gamma} (1-\frac{z}{\alpha})^{\gamma-\alpha-\beta} 0 \\
0
\end{pmatrix},
$$

$$
= R_\infty \begin{pmatrix}
1 & 0 \\
0 & \alpha^{-1}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & \alpha
\end{pmatrix} \sum_{n=0}^{\infty} \hat{g}_{n,\infty} \alpha^n z^{-n} \begin{pmatrix}
1 & 0 \\
0 & \alpha^{-1}
\end{pmatrix} \begin{pmatrix}
z^{\beta-\gamma} (1-\frac{z}{\alpha})^{\gamma-\alpha-\beta} 0 \\
0 & z^{1-\beta}
\end{pmatrix} \begin{pmatrix}
(-\alpha)^{\gamma-\beta} 0 \\
0 & -(-\alpha)^{\beta}
\end{pmatrix}.
$$

The limits analogous to those in Lemma (3.10) are stated as follows: we have the following limit of the leading matrix,

$$
\lim_{\alpha \to \infty} R_\infty \begin{pmatrix}
1 & 0 \\
0 & \alpha^{-1}
\end{pmatrix} = \lim_{\alpha \to \infty} \begin{pmatrix}
1 & 0 \\
0 & \frac{\alpha(\beta-\alpha)(\alpha+1-\beta)}{\alpha(\beta-1)(\beta-\gamma)}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & \alpha^{-1}
\end{pmatrix},
$$

$$
= \begin{pmatrix}
1 & 0 \\
0 & \frac{-1}{(\beta-1)(\beta-\gamma)}
\end{pmatrix} = \tilde{R}_\infty,
$$

Page 78 of 178
and for the terms of the new series,

\[
\lim_{\alpha \to \infty} \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \alpha^\alpha \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = h_{n,\infty},
\]

where \( \hat{g}_{n,\infty} \) and \( h_{n,\infty} \) are given by (3.72) and (3.35) respectively. Hence, we understand that a term-by-term limit of the solution,

\[
\hat{Y}^{(\infty)} \left( \frac{z}{\alpha} \right) \begin{pmatrix} \alpha \beta - \gamma \\ \gamma + 1 - \alpha - \beta \end{pmatrix},
\]

produces the formal solution \( \hat{Y}_f^{(\infty)}(z) \), which is analogous to (3.70).

We now turn our attention to the fundamental solutions at \( x = 1 \), as given in canonical form in (3.7). Observe that these solutions are expressed using Gauss’ hypergeometric \( \, _2F_1 \) series in the variable \((1 - x) \equiv (1 - \frac{z}{\alpha})\), which diverge for \(|1 - x| > 1 \Leftrightarrow |z - \alpha| > |\alpha|\).

As with the fundamental solutions at \( x = \infty \), we do not have uniform convergence with respect to \( \alpha \) here. Rather than keeping these solutions in canonical form, we use two more of Kummer’s relations to rewrite them as follows,

\[
y_1^{(1)}(x) = (1 - x)^{\gamma - \alpha - \beta} \, _2F_1 \left( \begin{array}{c} \gamma - \alpha, \gamma - \beta \\ \gamma + 1 - \alpha - \beta \end{array} ; 1 - x \right), \quad x \in \Omega_1,
\]

\[
y_2^{(1)}(x) = \, _2F_1 \left( \begin{array}{c} \alpha, \beta \\ \alpha + \beta + 1 - \gamma \end{array} ; 1 - x \right), \quad x \in \hat{\Omega}_1,
\]

where the new domain \( \hat{\Omega}_1 \) is defined as,

\[
\hat{\Omega}_1 = \{ x : |1 - x^{-1}| < 1, \ -\pi \leq \arg(x) < \pi, \ -\pi \leq \arg(1 - x) < \pi \}.
\]

We note that the two forms of these solutions are equivalent on the domain \( \Omega_1 \cap \hat{\Omega}_1 \).
There is a very simple philosophical reason why we rewrite the series in these solutions with \((1 - x^{-1})^n\), rather than \((1 - x)^n\): after the change of variable \(x = \frac{z}{\alpha}\), we want to produce a formal series in \(z^{-n}\). Similarly as before, the computations ending in (3.69) show how the solution \(y^{(1)}_1(x)\) asymptotically passes from power-like behavior to exponential behavior as \(\alpha \to \infty\). Moreover, the terms of the series in these new forms of \(y^{(1)}_1(x)\) and \(y^{(1)}_2(x)\) satisfy the lemma below.

**Lemma 3.11.** Let \(y^{(1)}_1(x)\) and \(y^{(1)}_2(x)\) be given in their new forms by (3.73) and (3.74) respectively. After the substitution \(x = \frac{z}{\alpha}\), the terms of these series tend to the terms in the formal series solutions \(\tilde{y}^{(\infty)}_{1,f}(z)\) and \(\tilde{y}^{(\infty)}_{2,f}(z)\) as given by (3.39), namely we have the following limits:

\[
\lim_{\alpha \to \infty} \frac{(\gamma - \beta)_n(1 - \beta)_n(z - \alpha)^n}{(\gamma + 1 - \alpha - \beta)_n n! z^n} = \frac{(\gamma - \beta)_n(1 - \beta)_n}{n! z^n},
\]
\[
\lim_{\alpha \to \infty} \frac{(\beta + 1 - \gamma)_n(\beta)_n(z - \alpha)^n}{(\alpha + \beta + 1 - \gamma)_n n! z^n} = (-1)^n \frac{(\beta)_n(\beta + 1 - \gamma)_n}{n! z^n}.
\]

**Proof.** By direct computation, after expanding the powers of \((z - \alpha)\) and the Pochhammer symbols to find,

\[
\frac{(z - \alpha)^n}{(\gamma + 1 - \alpha - \beta)_n} = 1 + O\left(\alpha^{-1}\right) \quad \text{and} \quad \frac{(z - \alpha)^n}{(\alpha + \beta + 1 - \gamma)_n} = (-1)^n + O\left(\alpha^{-1}\right).
\]

This lemma shows that *term-by-term* limits of the solutions,

\[
y^{(1)}_1(z \alpha^{-1}) \alpha^{\beta - \gamma} \quad \text{and} \quad -y^{(1)}_2(z \alpha^{-1}) \alpha^{-\beta},
\]

produce the formal solutions,

\[
\tilde{y}^{(\infty)}_{1,f}(z) \quad \text{and} \quad \tilde{y}^{(\infty)}_{2,f}(z),
\]

respectively. The factors \(\alpha^{\beta - \gamma}\) and \(\alpha^{-\beta}\) in (3.75) are necessary because of the terms,

\[
x^{\beta - \gamma} \equiv z^{\beta - \gamma} \alpha^{\gamma - \beta} \quad \text{and} \quad x^{-\beta} \equiv z^{-\beta} \alpha^\beta,
\]

in the solutions \(y^{(1)}_1(x)\) and \(y^{(1)}_2(x)\) respectively. We note that the direction in which \(\alpha \to \infty\) is not yet important for this lemma. The importance of this lemma is shown...
in the proof of our Main Theorem 1.

Remark 3.16. Similarly as in Remark 3.15, we may consider the viewpoint of working with the \((2 \times 2)\) equations (3.5) and (3.31) and rewrite the solution \(Y^{(1)}(x)\), as given in (3.10), as follows,

\[
Y^{(1)}(x) = R_1 \sum_{n=0}^{\infty} g_{n,1} (1 - x)^n (1 - x)^{\Theta_1}, \quad x \in \Omega_1,
\]

\[
= R_1 \sum_{n=0}^{\infty} \hat{g}_{n,1} (1 - x^{-1})^n x^{-\Theta_\infty - \Theta_1} (1 - x)^{\Theta_1}, \quad x \in \hat{\Omega}_1, \tag{3.76}
\]

where \(\hat{g}_{0,1} = I\) and we find all other coefficients \(\hat{g}_{n,1}, n \geq 1\), from the recursive equation,

\[
[\hat{g}_{n,1}, \Theta_1] + n \hat{g}_{n,1} = (n - 1) \hat{g}_{n-1,1} + \hat{g}_{n-1,1}(\Theta_1 + \Theta_\infty) + R_1^{-1} A_0 R_1 \hat{g}_{n-1,1}.
\]

This recursion equation differs quite significantly from that for \(g_{n,1}\), given in the proof of Lemma 3.2. We find the solution to this equation is,

\[
\hat{g}_{n,1} = \left( \begin{array}{cc} \frac{(1 - \beta)_{n}(\gamma - \beta)_{n}}{\gamma + \alpha - \beta \cdot n!} & \frac{(\beta)_{n}(\beta + 1 - \gamma)_{n}}{(\alpha + \beta + 1 - \gamma) \cdot n!} - \frac{(\beta)_{n-1}(\beta + 1 - \gamma)_{n-1}}{(\alpha + \beta + 1 - \gamma) \cdot n!} \\
\frac{1}{\alpha} \left( \frac{(2 - \beta)_{n}(\gamma + 1 - \beta)_{n}}{\gamma + 1 - \alpha - \beta \cdot n!} - \frac{(2 - \beta)_{n-1}(\gamma + 1 - \beta)_{n-1}}{\gamma + 1 - \alpha - \beta \cdot (n-1)!} \right) & \frac{\gamma + 1 - \alpha}{\alpha + \beta + 1 - \gamma} \cdot n!
\end{array} \right).
\tag{3.77}
\]

The transformation (3.76) is analogous to Kummer’s relations (3.73) and (3.74). We note that,

\[
Y^{(1)} \left( \frac{z}{\alpha} \right) = R_1 \sum_{n=0}^{\infty} \hat{g}_{n,1} \left( 1 - \frac{\alpha}{z} \right)^n \begin{pmatrix} \alpha^{-\beta} z^{\beta - \gamma} & (1 - \frac{\alpha}{z})^{\gamma - \alpha - \beta} & 0 \\ 0 & \alpha^{\beta - 1} z^{1 - \beta} \end{pmatrix}
\]

\[
\equiv R_1 \begin{pmatrix} 1 & 0 \\ 0 & -\alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix} \sum_{n=0}^{\infty} \hat{g}_{n,1} \left( 1 - \frac{\alpha}{z} \right)^n \begin{pmatrix} 1 & 0 \\ 0 & -\alpha^{-1} \end{pmatrix} \begin{pmatrix} z^{\beta - \gamma} & (1 - \frac{\alpha}{z})^{\gamma - \alpha - \beta} & 0 \\ 0 & \alpha^{\beta - 1} z^{1 - \beta} \end{pmatrix}
\]

The limits analogous to those in Lemma 3.11 are stated as follows: we have the following
limit of the leading matrix,

$$\lim_{\alpha \to \infty} R_1 \left( \begin{array}{cc} 1 & 0 \\ 0 & -\alpha^{-1} \end{array} \right) = \lim_{\alpha \to \infty} \left( \frac{1}{\alpha} \right) \left( \begin{array}{cc} \frac{1}{\alpha+1-\gamma} & 1 \\ 0 & \alpha^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -\alpha^{-1} \end{array} \right),$$

$$= \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{-1}{(\beta-1)(\beta-\gamma)} \end{array} \right) = \tilde{R}_\infty,$$

and for the terms of the new series,

$$\lim_{\alpha \to \infty} \left( \begin{array}{cc} 1 & 0 \\ 0 & -\alpha \end{array} \right) \left( -\alpha \right) \hat{g}_{n,1} \left( \begin{array}{cc} 1 & 0 \\ 0 & -\alpha^{-1} \end{array} \right) = h_{n,\infty},$$

where $\hat{g}_{n,1}$ and $h_{n,\infty}$ are given by (3.77) and (3.35) respectively. Hence, we understand that a term-by-term limit of the solution,

$$Y^{(1)}(z) \left( \begin{array}{c} \frac{\alpha}{\beta} \\ \beta-\gamma \end{array} \right) = \alpha^{\beta-\gamma} \left( \begin{array}{cc} 0 & \alpha^{-\beta} \\ 0 & -\alpha^{-1} \end{array} \right),$$

produces the formal solution $\tilde{Y}^{(\infty)}(z)$, which is analogous to (3.75).

Having understood how to take term-by-term limits of the series solutions of Gauss’ equation around $x = 1$ and $\infty$ to produce the formal solutions of Kummer’s equation around $z = \infty$, we now show how to apply Glutsyuk’s Theorem 2.5 to Gauss’ hypergeometric equation. Let $\eta \in (0, \frac{\pi}{2})$ be some fixed value. We define the following sectors,

$$\tilde{\mathcal{F}}_k := \left\{ z : \arg(z) - k\pi \in \left( \eta - \frac{\pi}{2}, \frac{3\pi}{2} - \eta \right) \right\}, \quad (3.78)$$

we note that if $z \in \tilde{\mathcal{F}}_k$ then $z \in \tilde{\Sigma}_k$. The presence of $\eta$ is to ensure that the boundaries of the sectors $\tilde{\mathcal{F}}_k$ do not contain a Stokes’ ray, as is necessary in the hypothesis of Glutsyuk’s Theorem 2.5. We note that this condition is not satisfied by the sectors $\tilde{\Sigma}_k$ defined in Theorem 3.1, which are the maximal sectors on which we can define single-valued analytic fundamental solutions.
We also define the following sectors,

\[
\sigma_{\alpha}(\alpha) := \left\{ z : |1 - \frac{\alpha}{z}| < |\alpha|^2, \arg\left(\frac{z}{\alpha}\right) \in (\eta - \pi, \pi - \eta), \arg\left(1 - \frac{z}{\alpha}\right) \in (\eta - \pi, \pi - \eta) \right\},
\]

\[
\sigma_{\infty}(\alpha) := \left\{ z : \arg(-z\alpha^{-1}) \in (\eta - \pi, \pi - \eta), \arg\left(1 - \frac{z}{\alpha}\right) \in (\eta - \pi, \pi - \eta) \right\}.
\]

We note that if \( z \) is sufficiently close to \( \alpha \) with \( z \in \sigma_{\alpha}(\alpha) \) then \( x = \frac{z}{\alpha} \in \hat{\Omega}_1 \) and if \( z \) is sufficiently large with \( z \in \sigma_{\infty}(\alpha) \) then \( x = \frac{z}{\alpha} \in \hat{\Omega}_{\infty} \). These sectors will be the new domains of our solutions \( y_{1}^{(1)}(z\alpha^{-1}) \), \( y_{2}^{(1)}(z\alpha^{-1}) \) and \( y_{1}^{(\infty)}(z\alpha^{-1}) \), \( y_{2}^{(\infty)}(z\alpha^{-1}) \) respectively, they are illustrated below.

![Figure 18: Sectors \( \sigma_{\alpha}(\alpha) \) and \( \sigma_{\infty}(\alpha) \).](image)

Compared with the domains \( \hat{\Omega}_1 \) and \( \hat{\Omega}_{\infty} \), which are disks with branch cuts, the sectors \( \sigma_{\alpha}(\alpha) \) and \( \sigma_{\infty}(\alpha) \) have larger radii and do not contain any part of the branch cut between \( \alpha \) and \( \infty \). We can analytically extend our solutions \( y_{k}^{(1)}(z\alpha^{-1}) \) and \( y_{k}^{(\infty)}(z\alpha^{-1}) \), \( k = 1, 2 \), to these larger domains because the singularity \( z = \infty \) (resp. \( z = \alpha \)) can never lie inside the sector \( \sigma_{\alpha}(\alpha) \) (resp. \( \sigma_{\infty}(\alpha) \)) or on its boundary. That is the key reason to restrict our solutions to sectors rather than disks.

We examine the sector \( \sigma_{\alpha}(\alpha) \) more closely. From the first condition,

\[
|1 - \frac{\alpha}{z}| < |\alpha|^2 \iff \left| \frac{1}{\alpha} - \frac{1}{z} \right| < |\alpha|,
\]

observe that as \( \alpha \to \infty \) the radius of this sector becomes infinite, indeed the above inequality becomes simply \( |z| > 0 \). Furthermore, as \( \alpha \to \infty \) along a ray, the base point of the sector \( \sigma_{\alpha}(\alpha) \) is translated along that ray, tending to infinity. We illustrate this phenomenon in Figure 19 below.
Figure 19: As \( \alpha \to \infty \) along a ray, the sector \( \sigma_\alpha(\alpha) \) is translated along the branch cut and becomes in agreement with the sector \( \tilde{\Phi} := \{ z : |\arg(\frac{z}{\alpha})| < \pi - \eta \} \).

In the two limit directions we are concerned with, for \( \arg(\alpha) = \pm \frac{\pi}{2} \), we have,

\[
\arg\left(\frac{z}{\alpha}\right) \in (\eta - \pi, \pi - \eta) \iff \arg(z) \in \left(\eta - \pi \pm \frac{\pi}{2}, \pi \pm \frac{\pi}{2} - \eta \right),
\]

For the sector \( \sigma_\infty(\alpha) \), whose base point is already fixed at infinity, we have,

\[
\arg\left(-\frac{z}{\alpha}\right) \in (\eta - \pi, \pi - \eta) \iff \arg(z) \in \left(\eta \pm \frac{\pi}{2}, 2\pi \pm \frac{\pi}{2} - \eta \right),
\]

recall from (3.69) that the condition on \( \arg(1 - \frac{z}{\alpha}) \) in \( \sigma_\infty(\alpha) \) does not play a role after taking the limit. With these considerations in mind, we write,

\[
\lim_{\alpha \to \infty \atop \arg(\alpha) = -\frac{\pi}{2}} \sigma_\alpha(\alpha) = \tilde{\mathcal{S}}_{-1}, \quad \lim_{\alpha \to \infty \atop \arg(\alpha) = -\frac{\pi}{2}} \sigma_\infty(\alpha) = \tilde{\mathcal{S}}_0,
\]

\[
\lim_{\alpha \to \infty \atop \arg(\alpha) = \frac{\pi}{2}} \sigma_\alpha(\alpha) = \tilde{\mathcal{S}}_0, \quad \lim_{\alpha \to \infty \atop \arg(\alpha) = \frac{\pi}{2}} \sigma_\infty(\alpha) = \tilde{\mathcal{S}}_1.
\]

We now apply Glutsyuk’s Theorem 2.5 with the \((2 \times 2)\) hypergeometric equation (3.5) in place of the perturbed equation and the confluent hypergeometric equation (3.31) in place of the non-perturbed equation. Glutsyuk’s theorem asserts the existence of
invertible diagonal matrices $K^\pm_\infty(\alpha)$ and $K^\pm_1(\alpha)$ such that:

$$\lim_{\alpha \to \infty \atop \arg(\alpha) = \frac{\pi}{2}} Y^{(1)}(z\alpha^{-1}) \bigg|_{z \in \sigma_{\alpha}(\alpha)} K^-_1(\alpha) = \tilde{Y}^{(\infty,-1)}(z),$$  \hspace{1cm} (3.81)

$$\lim_{\alpha \to \infty \atop \arg(\alpha) = -\frac{\pi}{2}} Y^{(\infty)}(z\alpha^{-1}) \bigg|_{z \in \sigma_{\alpha}(\alpha)} K^-_\infty(\alpha) = \tilde{Y}^{(\infty,0)}(z),$$  \hspace{1cm} (3.82)

uniformly for $z \in \tilde{F}_-$ and $z \in \tilde{F}_0$ respectively, and:

$$\lim_{\alpha \to \infty \atop \arg(\alpha) = \frac{\pi}{2}} Y^{(1)}(z\alpha^{-1}) \bigg|_{z \in \sigma_{\alpha}(\alpha)} K^+_1(\alpha) = \tilde{Y}^{(\infty,0)}(z),$$  \hspace{1cm} (3.83)

$$\lim_{\alpha \to \infty \atop \arg(\alpha) = -\frac{\pi}{2}} Y^{(\infty)}(z\alpha^{-1}) \bigg|_{z \in \sigma_{\alpha}(\alpha)} K^+_\infty(\alpha) = \tilde{Y}^{(\infty,1)}(z),$$  \hspace{1cm} (3.84)

uniformly for $z \in \tilde{F}_0$ and $z \in \tilde{F}_1$ respectively. We note that since we are considering two limits, namely one with $\arg(\alpha) = \frac{\pi}{2}$ and another with $\arg(\alpha) = -\frac{\pi}{2}$, we have distinguished the diagonal matrices in each case with a superscript $+$ or $-$ respectively.

Due to the asymptotics of the fundamental solutions of Kummer’s equation as given in Theorem 3.1, each of these four limits is asymptotic to the formal fundamental solution $\tilde{Y}^{(\infty)}_f(z)$ as $z \to \infty$ with $z$ belonging to the corresponding sector.

Equivalently, from the viewpoint of studying the classical scalar hypergeometric equations (3.1) and (3.2), Glutsyuk’s Theorem 2.5 asserts the existence of scalars $k^\pm_{1,\infty}(\alpha)$, $k^\pm_{2,\infty}(\alpha)$, $k^\pm_{1,1}(\alpha)$ and $k^\pm_{2,1}(\alpha)$ such that, for $j \in \{1, 2\}$:

$$\lim_{\alpha \to \infty \atop \arg(\alpha) = -\frac{\pi}{2}} y^{(1)}_j(z\alpha^{-1}) \bigg|_{z \in \sigma_{\alpha}(\alpha)} k^-_{j,1}(\alpha) = \tilde{y}^{(\infty,-1)}_j(z),$$  \hspace{1cm} (3.85)

$$\lim_{\alpha \to \infty \atop \arg(\alpha) = -\frac{\pi}{2}} y^{(\infty)}_j(z\alpha^{-1}) \bigg|_{z \in \sigma_{\alpha}(\alpha)} k^-_{j,\infty}(\alpha) = \tilde{y}^{(\infty,0)}_j(z),$$  \hspace{1cm} (3.86)

uniformly for $z \in \tilde{F}_-$ and $\tilde{F}_0$ respectively, and:

$$\lim_{\alpha \to \infty \atop \arg(\alpha) = \frac{\pi}{2}} y^{(1)}_j(z\alpha^{-1}) \bigg|_{z \in \sigma_{\alpha}(\alpha)} k^+_{j,1}(\alpha) = \tilde{y}^{(\infty,0)}_j(z),$$  \hspace{1cm} (3.87)

$$\lim_{\alpha \to \infty \atop \arg(\alpha) = \frac{\pi}{2}} y^{(\infty)}_j(z\alpha^{-1}) \bigg|_{z \in \sigma_{\alpha}(\alpha)} k^+_{j,\infty}(\alpha) = \tilde{y}^{(\infty,1)}_j(z),$$  \hspace{1cm} (3.88)
uniformly $z \in \tilde{T}_0$ and $\tilde{T}_1$ respectively.

Having applied Glutsyuk’s theorem to our confluence of the hypergeometric equation, we now focus on understanding what we can deduce about these scalars $k_{j,\infty}^\pm(\alpha)$ and $k_{j,1}^\pm(\alpha)$, $j = 1, 2$. We are ready to state our first main theorem.

**Main Theorem 1.** If $k_{j,\infty}^\pm(\alpha)$ and $k_{j,1}^\pm(\alpha)$ are scalars satisfying (3.85)-(3.88), then these numbers satisfy the following limits,

\begin{align*}
\lim_{\alpha \to \infty \atop \arg(\alpha) = \pm \pi/2} k_{1,\infty}^\pm(\alpha) (-\alpha)^{\gamma-\beta} &= 1, \\
\lim_{\alpha \to \infty \atop \arg(\alpha) = \pm \pi/2} -k_{2,\infty}^\pm(\alpha) (-\alpha)^{\beta} &= 1, \\
\lim_{\alpha \to \infty \atop \arg(\alpha) = \pm \pi/2} k_{1,1}^\pm(\alpha) \alpha^{\gamma-\beta} &= 1, \\
\lim_{\alpha \to \infty \atop \arg(\alpha) = \pm \pi/2} -k_{2,1}^\pm(\alpha) \alpha^{\beta} &= 1.
\end{align*}

To prove this theorem we will use two elementary lemmas.

**Lemma 3.12.** Let $s, t : \mathbb{C} \to \mathbb{C}$ such that $\lim_{\alpha \to \infty} s(\alpha) t(\alpha)$ exists. If the limit $\lim_{\alpha \to \infty} s(\alpha)$ exists and is non-zero, then $\lim_{\alpha \to \infty} t(\alpha)$ exists.

**Proof.** Since $\lim_{\alpha \to \infty} s(\alpha)$ exists and is non-zero, the limit $\lim_{\alpha \to \infty} s(\alpha)^{-1}$ exists and is non-zero by the reciprocal limit law. Hence, the limit,

$$
\lim_{\alpha \to \infty} t(\alpha) s(\alpha) \lim_{\alpha \to \infty} s(\alpha)^{-1} = \lim_{\alpha \to \infty} t(\alpha) s(\alpha) s(\alpha)^{-1} \equiv \lim_{\alpha \to \infty} t(\alpha),
$$

exists by the product limit law. \qed

We have learned the following lemma, concerned with asymptotic series, from [Was].

**Lemma 3.13.** Let $f(w)$ be holomorphic in an open sector $\sigma$ at $w = 0$ and let $\sigma^*$ be a closed, proper subsector of $\sigma$. If,

$$
f(w) \sim \sum_{n=0}^{\infty} a_n w^n, \quad \text{as } w \to 0, \ w \in \sigma,
$$

Page 86 of 178
then:

\[ a_n = \frac{1}{n!} \lim_{w \to 0} f^{(n)}(z), \]

where \( f^{(n)}(w) \) denotes the \( n^{th} \) derivative of \( f(w) \).

**Proof.** The hypothesis on \( f(z) \) implies that for all \( n \geq 0 \) we have the following,

\[ f^{(n)}(z) \sim \sum_{l=0}^{\infty} \frac{(l + n)!}{l!} a_{l+n} z^l, \quad \text{as } z \to 0, \quad z \in \sigma^*, \]

see for instance [Was] for this fact. Now, by definition of an asymptotic series, we deduce that the first term is given by,

\[ n! a_n = \lim_{z \to 0} z^n f^{(n)}(z), \]

and the result is proved. \( \Box \)

Lemma 3.13 has a converse statement, however this will not be necessary to prove our main results. We are ready to prove our Main Theorem 1.

**Proof of our Main Theorem 1.** In either case \( \arg(\alpha) = \frac{\pi}{2} \) or \( -\frac{\pi}{2} \), let \( \mathcal{S}^* \) be a closed, proper subsector of \( \tilde{\mathcal{S}}_1 \) or \( \tilde{\mathcal{S}}_0 \) respectively. Combining the statements (3.86) and (3.88), together with the asymptotic behavior (3.34), we have,

\[ \lim_{\alpha \to \infty} \arg(\alpha) = \pm \frac{\pi}{2}, \quad y_1^{(\infty)}(z\alpha^{-1}) \mid_{z \in \sigma_\infty(\alpha)} k^\pm_{1, \infty}(\alpha) \sim \bar{y}_1^{(\infty)}(z), \quad \text{as } z \to \infty, \quad z \in \mathcal{S}^*. \quad (3.93) \]

We now re-write \( y_1^{(\infty)}(z\alpha^{-1}) \) using Kummer’s transformation as in (3.68),

\[ y_1^{(\infty)}(z\alpha^{-1}) \mid_{z \in \sigma_\infty(\alpha)} = z^{\beta-\gamma}(\alpha^{-\gamma})(1 - \frac{z}{\alpha})^{-\alpha-\beta} \sum_{n=0}^{\infty} \frac{(1 - \beta)_n(\gamma - \beta)_n \alpha^n}{(\alpha + 1 - \beta)_n! z^n} \mid_{z \in \sigma_\infty(\alpha)}. \]

We apply Lemma 3.12 with,

\[ s(\alpha) = \left(1 - \frac{z}{\alpha}\right)^{-\alpha-\beta}, \]

\[ t(\alpha) = z^{\beta-\gamma}(\alpha^{-\gamma}) \sum_{n=0}^{\infty} \frac{(1 - \beta)_n(\gamma - \beta)_n \alpha^n}{(\alpha + 1 - \beta)_n! z^n} \mid_{z \in \sigma_\infty(\alpha)} k^\pm_{1, \infty}(\alpha). \]
Observe that the hypotheses of Lemma 3.12 hold: the limits,
\[
\lim_{\alpha \to \infty} \frac{s(\alpha) t(\alpha)}{s(\alpha)} = \pm \frac{\pi}{2}
\]
exist, since \( s(\alpha) t(\alpha) \equiv y_1^{(\infty)}(z) \), and the limits,
\[
\lim_{\alpha \to \infty} \frac{s(\alpha)}{s(\alpha)} = \pm 1
\]
exist and are non-zero by (3.69). We therefore deduce,
\[
\lim_{\alpha \to \infty} \frac{s(\alpha)}{s(\alpha)} = \pm 1
\]
by the product limit law. Combining this with (3.93) and writing \( z^{(\infty)}(z) \) as in (3.39), we have,
\[
\sum_{n=0}^{\infty} \frac{(1 - \beta)(\gamma - \beta)n^{\alpha_n}}{(\alpha + 1 - \beta)n!z^n} \bigg|_{z \in \sigma_{\infty}(\alpha)} \sim \sum_{n=0}^{\infty} \frac{(1 - \beta)(\gamma - \beta)n^{\alpha_n}}{n!z^n},
\]
as \( z \to \infty \) for \( z \in \mathcal{S}^* \).

We now define \( w = z^{-1} \) so that \( w \to 0 \iff z \to \infty \) and we can apply Lemma 3.13 to find,
\[
\frac{(\gamma - \beta)(1 - \beta)n^{\alpha_n}}{n!} = \frac{1}{n!} \lim_{w \to 0} \frac{d^n}{dw^n} \lim_{\alpha \to \infty} \frac{1}{\arg(\alpha) = \pm \frac{\pi}{2}} \sum_{l=0}^{\infty} \frac{(1 - \beta)(\gamma - \beta)l^{\alpha_l}w^l}{(\alpha + 1 - \beta)l!} \bigg|_{w \to 0} \frac{(\gamma - \beta)(1 - \beta)n^{\alpha_n}}{n!z^n},
\]
We proceed to treat the limits on the right hand side with special care. We first note that, due to the uniformity of the limits (3.86) and (3.88), we may interchange the
limit in $\alpha$ with the derivative and the limit in $w$ as follows,

$$
\frac{(\gamma - \beta)_n(1 - \beta)_n}{n!} =
\frac{1}{n!} \lim_{\alpha \to \infty} \lim_{w \to 0} \frac{d^n}{dw^n} \left[ \sum_{l=0}^{\infty} \frac{(1 - \beta)_l(\gamma - \beta)_l \alpha^l w^l}{(\alpha + 1 - \beta)_l l!} \right]_{w^{-1} \in \sigma_\infty(\alpha)} (-\alpha)^{\gamma - \beta} k_{1,\infty}^\pm(\alpha).
$$

The next step is to notice that the series inside the limits on the right hand side represents an analytic function (or at least its analytic extension to the sector $\sigma_\infty(\varepsilon)$ does). We may therefore interchange the derivative and series as follows,

$$
\frac{(\gamma - \beta)_n(1 - \beta)_n}{n!} =
\frac{1}{n!} \lim_{\alpha \to \infty} \lim_{w \to 0} \frac{d^n}{dw^n} \left[ \sum_{l=0}^{\infty} \frac{(1 - \beta)_l(\gamma - \beta)_l \alpha^l w^l}{(\alpha + 1 - \beta)_l l!} \right]_{w^{-1} \in \sigma_\infty(\alpha)} (-\alpha)^{\gamma - \beta} k_{1,\infty}^\pm(\alpha) =
\frac{1}{n!} \lim_{\alpha \to \infty} \lim_{w \to 0} \frac{d^n}{dw^n} \left[ \sum_{l=0}^{\infty} \frac{(l + n)! (1 - \beta)_{l+n}(\gamma - \beta)_{l+n} \alpha^{l+n} w^l}{(\alpha + 1 - \beta)_{l+n}(l + n)!} \right]_{w^{-1} \in \sigma_\infty(\alpha)} (-\alpha)^{\gamma - \beta} k_{1,\infty}^\pm(\alpha).
$$

Furthermore, due to the analyticity of the series on the right hand side, its limit as $w \to 0$ certainly exists and is simply equal to the first term of the series. We finally deduce,

$$
\frac{(\gamma - \beta)_n(1 - \beta)_n}{n!} = \frac{1}{n!} \lim_{\alpha \to \infty} \lim_{\arg(\alpha) = \pm \frac{\pi}{2}} \frac{(1 - \beta)_n(\gamma - \beta)_n \alpha^n}{(\alpha + 1 - \beta)_n n!} (-\alpha)^{\gamma - \beta} k_{1,\infty}^\pm(\alpha). \quad (3.94)
$$

We use Lemma 3.12 once more, this time with,

$$
\begin{align*}
\quad & s(\alpha) = (1 - \beta)_n(\gamma - \beta)_n \alpha^n \\
\quad & t(\alpha) = (-\alpha)^{\gamma - \beta} k_{1,\infty}^\pm(\alpha).
\end{align*}
$$

Observe that the hypotheses of Lemma 3.12 hold: the limits,

$$
\lim_{\alpha \to \infty} \lim_{\arg(\alpha) = \pm \frac{\pi}{2}} s(\alpha)t(\alpha),
$$

exist, since these are (3.94) and, crucially, the limits,

$$
\lim_{\alpha \to \infty} \lim_{\arg(\alpha) = \pm \frac{\pi}{2}} s(\alpha),
$$
exist and are non-zero by Lemma 3.10. Therefore, since the limit of \( t(\alpha) \) must exist,

\[
\lim_{\alpha \to \infty} \frac{(1 - \beta)_n(\gamma - \beta)_n \alpha^n}{(\alpha + 1 - \beta)_n n!} \frac{(-\alpha)^{\gamma - \beta} k_{1, \infty}^\pm(\alpha)}{n!} \frac{\lim_{\alpha \to \infty} \arg(\alpha) = \pm \frac{\pi}{2}}{n!} \frac{(-\alpha)^{\gamma - \beta} k_{1, \infty}^\pm(\alpha)}{n!}.
\]

Comparing with the left hand side of (3.94) we deduce the desired result (3.89). The limit (3.90) can be proved by using \( y_2^{(\infty)}(z \alpha^{-1}) \) as given by (3.8). The limits (3.91) and (3.92) can be proved using \( y_1^{(1)}(z \alpha^{-1}) \) and \( y_2^{(1)}(z \alpha^{-1}) \) as given by (3.73) and (3.74) and using Lemma 3.11 in place of Lemma 3.10.

Remark 3.17. Returning to the point of view of studying the hypergeometric equations as the \((2 \times 2)\) equations (3.5) and (3.31), our Main Theorem 1 may be equivalently stated as follows. If \( K_{1, \infty}^\pm(\alpha) \) and \( K_{\infty}^\pm(\alpha) \) are diagonal matrices satisfying (3.81)-(3.84), then they satisfy the following:

\[
\lim_{\alpha \to \infty} \frac{(-\alpha)^{\gamma - \beta} }{n!} \frac{\lim_{\alpha \to \infty} \arg(\alpha) = \pm \frac{\pi}{2}}{n!} \frac{(-\alpha)^{\gamma - \beta} k_{1, \infty}^\pm(\alpha)}{n!} = I, \quad (3.95)
\]

\[
\lim_{\alpha \to \infty} \frac{(-\alpha)^{\gamma - \beta} }{n!} \frac{\lim_{\alpha \to \infty} \arg(\alpha) = \pm \frac{\pi}{2}}{n!} \frac{(-\alpha)^{\gamma - \beta} k_{1, \infty}^\pm(\alpha)}{n!} = I. \quad (3.96)
\]

These limits can be proved in an analogous way to the limits in our Main Theorem 1 by using Remarks 3.15 and 3.16 in place of Lemmas 3.10 and 3.11 respectively.

### 3.3.2 Limits of Monodromy Data

Summarising the results so far, in section 3.3.1 we showed how term-by-term limits of the solutions of Gauss’ equation around \( x = \infty \) and \( x = 1 \) produce the formal solutions of Kummer’s equation around \( z = \infty \). We then explained how Glutsyuk’s Theorem 2.5 asserts the existence of certain scalars which multiply Gauss’ solutions so that their true limits exist and are equal to the solutions of Kummer’s equation analytic in sectors at \( z = \infty \). We have also proved our Main Theorem 1, which establishes some important limits which these factors must satisfy. We now bring these results together to prove our second main theorem, concerned with explicitly producing the set of monodromy
data $\tilde{M}$ from the set $M$.

**Main Theorem 2.** Define the monodromy data of Gauss’ equation as given in (3.13)-(3.18) and of Kummer’s equation as in (3.41)-(3.45). We have the following limits,

\begin{align}
\lim_{\arg(\alpha) = \frac{\pi}{7}} \left( \begin{array}{cc}
\alpha^\gamma \beta & 0 \\
0 & -\alpha^\beta \\
\end{array} \right) C^{1\infty} \left( \begin{array}{cc}
(-\alpha)^{\beta-\gamma} & 0 \\
0 & -(-\alpha)^{-\beta} \\
\end{array} \right) &= \tilde{S}_0, \\
\lim_{\arg(\alpha) = -\frac{\pi}{7}} \left( \begin{array}{cc}
\alpha^\gamma \beta & 0 \\
0 & -\alpha^\beta \\
\end{array} \right) C^{1\infty} \left( \begin{array}{cc}
(-\alpha)^{\beta-\gamma} & 0 \\
0 & -(-\alpha)^{-\beta} \\
\end{array} \right) &= \tilde{S}_1, \\
\lim_{\arg(\alpha) = -\frac{\pi}{7}} \left( \begin{array}{cc}
\alpha^\gamma \beta & 0 \\
0 & 1 \\
\end{array} \right) C^{0\infty} \left( \begin{array}{cc}
(-\alpha)^{\beta-\gamma} & 0 \\
0 & -(-\alpha)^{-\beta} \\
\end{array} \right) &= \tilde{C}_0, \\
\lim_{\arg(\alpha) = \frac{\pi}{7}} \left( \begin{array}{cc}
\alpha^\gamma \beta & 0 \\
0 & 1 \\
\end{array} \right) C^{0\infty} \left( C^{1\infty} \right)^{-1} \left( \begin{array}{cc}
\alpha^{\beta-\gamma} & 0 \\
0 & -\alpha^{-\beta} \\
\end{array} \right) &= \tilde{C}_\infty.
\end{align}

Furthermore, as immediate consequences of the above limits of connection matrices, we have the following limits of monodromy matrices,

\begin{align}
\lim_{\arg(\alpha) = \frac{\pi}{7}} \left( \begin{array}{cc}
(-\alpha)^{\gamma-\beta} & 0 \\
0 & -(-\alpha)^{-\beta} \\
\end{array} \right) M_0 \left( \begin{array}{cc}
(-\alpha)^{\beta-\gamma} & 0 \\
0 & -(-\alpha)^{-\beta} \\
\end{array} \right) &= \tilde{M}_0, \\
\lim_{\arg(\alpha) = -\frac{\pi}{7}} \left( \begin{array}{cc}
\alpha^\gamma \beta & 0 \\
0 & -\alpha^\beta \\
\end{array} \right) C^{1\infty} M_0 \left( C^{1\infty} \right)^{-1} \left( \begin{array}{cc}
(-\alpha)^{\beta-\gamma} & 0 \\
0 & -(-\alpha)^{-\beta} \\
\end{array} \right) &= \tilde{M}_0, \\
\lim_{\arg(\alpha) = -\frac{\pi}{7}} \left( \begin{array}{cc}
(-\alpha)^{\gamma-\beta} & 0 \\
0 & -(-\alpha)^{-\beta} \\
\end{array} \right) M_\infty M_1 \left( \begin{array}{cc}
(-\alpha)^{\beta-\gamma} & 0 \\
0 & -(-\alpha)^{-\beta} \\
\end{array} \right) &= \tilde{M}_\infty, \\
\lim_{\arg(\alpha) = \frac{\pi}{7}} \left( \begin{array}{cc}
\alpha^\gamma \beta & 0 \\
0 & -\alpha^\beta \\
\end{array} \right) C^{1\infty} M_\infty M_1 \left( C^{1\infty} \right)^{-1} \left( \begin{array}{cc}
\alpha^{\beta-\gamma} & 0 \\
0 & -\alpha^{-\beta} \\
\end{array} \right) &= \tilde{M}_\infty.
\end{align}

As part of the proof of this theorem, we will use the following elementary lemma which is an analogue of Lemma 3.12 for matrices.

**Lemma 3.14.** Let $f(\alpha)$ and $g(\alpha)$ be matrices such that $\lim_{\alpha \to \infty} f(\alpha) g(\alpha)$ exists.

**i)** If $\lim_{\alpha \to \infty} \det(f(\alpha))$ exists and is non-zero and $\det(f(\alpha)) \neq 0$ for all $\alpha$ sufficiently large and if the limit $\lim_{\alpha \to \infty} f(\alpha)$ exists and is invertible, then the limit $\lim_{\alpha \to \infty} g(\alpha)$ exists.
ii) If \( \lim_{\alpha \to \infty} \det(g(\alpha)) \) exists and is non-zero and \( \det(g(\alpha)) \neq 0 \) for all \( \alpha \) sufficiently large and if the limit \( \lim_{\alpha \to \infty} g(\alpha) \) exists, then the limit \( \lim_{\alpha \to \infty} f(\alpha) \) exists.

**Proof.** For i), We begin by noting that if the limits \( \lim_{\alpha \to \infty} a_j(\alpha) \) exist and equal \( A_j \) for \( j = 1, \ldots, n \), then the limit of any polynomial \( P(a_1, \ldots, a_n) \) satisfies:

\[
\lim_{\alpha \to \infty} P(a_1, \ldots, a_n) = P(A_1, \ldots, A_n).
\]

Now, since \( \det(g(\alpha)) \neq 0 \) and since \( \lim_{\alpha \to \infty} g(\alpha) \) exists and is invertible, the limit \( \lim_{\alpha \to \infty} g^{-1}(\alpha) \) exists, since the inverse \( g^{-1}(\alpha) = \frac{1}{\det(g(\alpha))} \text{Adj}(g(\alpha)) \) is polynomial in the entries of \( g(\alpha) \), where \( \text{Adj} \) means the adjugate matrix (namely, the transpose of the matrix of cofactors). Furthermore, since the entries of \( f(\alpha) \equiv f(\alpha)g(\alpha)g^{-1}(\alpha) \) are polynomials in the entries of \( f(\alpha)g(\alpha) \) and \( g^{-1}(\alpha) \), the limit,

\[
\lim_{\alpha \to \infty} f(\alpha) \equiv \lim_{\alpha \to \infty} f(\alpha)g(\alpha)g^{-1}(\alpha) = \lim_{\alpha \to \infty} f(\alpha)g(\alpha) \lim_{\alpha \to \infty} g^{-1}(\alpha),
\]

exists. The case ii) is analogous. \( \square \)

**Proof of our Main Theorem 2.** Let \( \sigma_\alpha(\alpha) \) and \( \sigma_\infty(\alpha) \) be the sectors defined in (3.79) and (3.80) respectively. As mentioned previously, if \( z \in \sigma_\alpha(\alpha) \) then \( x \in \Omega_1 \) and if \( z \in \sigma_\infty(\alpha) \) then \( x \in \Omega_\infty \), so that the connection matrix \( C^{1\infty} \) remains valid for the solutions \( Y^{(1)}(z\alpha^{-1}) \) and \( Y^{(\infty)}(z\alpha^{-1}) \) restricted to the sectors \( \sigma_\alpha(\alpha) \) and \( \sigma_\infty(\alpha) \) respectively. Since the radii of these sectors do not diminish as \( \alpha \to \infty \), for \( |\alpha| \) sufficiently large we must have,

\[
\sigma_\alpha(\alpha) \cap \sigma_\infty(\alpha) \neq \emptyset,
\]

recall Figure 18. Therefore, for \( |\alpha| \) sufficiently large, we have,

\[
Y^{(\infty)}(z\alpha^{-1}) = Y^{(1)}(z\alpha^{-1}) C^{1\infty}, \quad z \in \sigma_\alpha(\alpha) \cap \sigma_\infty(\alpha).
\]

(3.105)

Let \( \tilde{F}_k \) be the sectors defined in (3.78). To prove the first limit (3.97), we first give a proof of Glutsyuk’s Corollary 2.1 in our case. We multiply by the matrices \( K_\infty^+(\alpha) \)
and $K_1^+(\alpha)$ and take the limit $\alpha \to \infty$, with $\text{arg}(\alpha) = \frac{\pi}{2}$, so that (3.105) becomes,

$$
\lim_{\alpha \to \infty} \left. Y^{(\infty)}(z \alpha^{-1}) \right|_{z \in \sigma_\infty(\alpha)} K_1^+(\alpha) = \lim_{\alpha \to \infty} \left. Y^{(1)}(z \alpha^{-1}) \right|_{z \in \sigma_\alpha(\alpha)} K_1^+(\alpha) \left( K_1^+(\alpha) \right)^{-1} C^{1,\infty} K_1^+(\alpha),
$$

(3.106)

for $z \in \tilde{\mathcal{F}}_0 \cap \tilde{\mathcal{F}}_1$. We apply Lemma 3.14 i) with,

$$
f(\alpha) = \left. Y^{(1)}(z \alpha^{-1}) \right|_{z \in \sigma_\alpha(\alpha)} K_1^+(\alpha) \quad \text{and} \quad g(\alpha) = \left( K_1^+(\alpha) \right)^{-1} C^{1,\infty} K_1^+(\alpha).
$$

Observe that the hypotheses of Lemma 3.14 hold: the limit,

$$
\lim_{\alpha \to \infty} \left. Y^{(\infty,1)}(z) \right|_{z \in \tilde{\mathcal{F}}_0 \cap \tilde{\mathcal{F}}_1} = \tilde{Y}^{(\infty,0)}(z), \quad \text{by } (3.84), \quad \text{and the limit,}
$$

$$
\lim_{\alpha \to \infty} \left. Y^{(\infty,0)}(z) \right|_{z \in \tilde{\mathcal{F}}_0 \cap \tilde{\mathcal{F}}_1} = \tilde{Y}^{(\infty,0)}(z), \quad \text{by } (3.83), \quad \text{which is clearly invertible because it is a fundamental solution. For all } \alpha, \text{ } f(\alpha) \text{ is also clearly invertible because it is a fundamental solution. The limit,}
$$

$$
\lim_{\alpha \to \infty} \left. g(\alpha) = \lim_{\alpha \to \infty} \left( K_1^+(\alpha) \right)^{-1} C^{1,\infty} K_1^+(\alpha),
$$

therefore exists and, from (3.106),

$$
\tilde{Y}^{(\infty,1)}(z) = \tilde{Y}^{(\infty,0)}(z) \lim_{\alpha \to \infty} \left( K_1^+(\alpha) \right)^{-1} C^{1,\infty} K_1^+(\alpha), \quad z \in \tilde{\mathcal{F}}_0 \cap \tilde{\mathcal{F}}_1.
$$

Recall that if $z \in \tilde{\mathcal{F}}_k$ then $z \in \tilde{\Sigma}_k$ and recall Definition 3.3 of Stokes' matrices, namely we have,

$$
\tilde{Y}^{(\infty,1)}(z) = \tilde{Y}^{(\infty,0)}(z) \tilde{S}_0, \quad z \in \tilde{\Sigma}_0 \cap \tilde{\Sigma}_1.
$$
We conclude that,
\[
\lim_{\alpha \to \infty} (K_1^+(\alpha))^{-1} C^{1\infty} K^+_\infty(\alpha) = \tilde{S}_0,
\]
which is precisely Glutsyuk’s Corollary 2.1 in our case. Combining this with (3.95) and (3.96), we compute,
\[
\tilde{S}_0 = \lim_{\alpha \to \infty} \left( K_1^+(\alpha) \right)^{-1} C^{1\infty} K^+_\infty(\alpha),
\]
where we have implicitly used Lemma 3.14 again, this proves the first limit (3.97) of the theorem. To prove the second limit (3.98), we multiply by the matrices \(K^-\infty(\alpha)\) and \(K_1^- (\alpha)\) and take the limit \(\alpha \to \infty\), with \(\arg(\alpha) = -\frac{\pi}{2}\), so that (3.105) becomes,
\[
\lim_{\alpha \to \infty} \arg(\alpha)^{-1} \left( Y^{(\infty)}(z\alpha^{-1}) \right)_{z \in \sigma_{\infty}(\alpha)} K^-\infty(\alpha),
\]
for \(z \in \tilde{\mathcal{S}}_{-1} \cap \tilde{\mathcal{F}}_0\). By following a similar procedure as above, using Lemma 3.14 and the relations (3.81) and (3.82), we deduce,
\[
\lim_{\alpha \to \infty} \left( K_1^- (\alpha) \right)^{-1} C^{1\infty} K^-\infty(\alpha) = \tilde{S}_{-1}.
\]
Combining this with \((3.95)\) and \((3.96)\), we compute,

\[
\tilde{S}^{-1} = \lim_{\alpha \to \infty} \frac{K^{-}_{1}(\alpha)}{}^{-1} \left( K^{-}_{1}(\alpha) \right) C^{1\infty} K^{-}_{\infty}(\alpha),
\]

\[
= \lim_{\alpha \to \infty} \left( \frac{\alpha^{-\beta} 0 \alpha^{-\gamma} 0 \alpha^{-\beta}}{0 -\alpha^{-\beta}} \right) C^{1\infty} K^{-}_{\infty}(\alpha) \left( \frac{(-\alpha)^{\gamma-\beta} 0 \alpha^{-\beta} \gamma -\beta -\alpha \beta}{0 -\alpha^{-\beta}} \right),
\]

\[
= \lim_{\alpha \to \infty} \left( \frac{\alpha^{-\beta} 0 \alpha^{-\beta}}{0 -\alpha^{-\beta}} \right) \left( \frac{\alpha^{-\beta} \gamma -\beta -\alpha \beta}{0 -\alpha^{-\beta}} \right),
\]

where we have implicitly used Lemma 3.14, this proves the second limit \((3.98)\) of the theorem.

To prove the third limit \((3.99)\) we first note that the curve \(\gamma_{\infty_{0}}\) which defines the connection matrix \(C^{0\infty}\) survives the confluence limit. In other words, after the substitution \(x = z_{\alpha}\), the curve does not diminish or become broken under the limit \(\alpha \to \infty\). This fact is expressed as follows,

\[
\lim_{\alpha \to \infty} \gamma_{\infty_{0}} \left[ Y^{(\infty)} K^{-}_{\infty}(\alpha) \right] (z_{\alpha}^{-1}) = \gamma_{\infty_{0}} \left[ \tilde{Y}^{(\infty,0)} \right] (z),
\]

or equivalently, using the domains \(\omega_{\infty_{0}}(\alpha)\) and \(\tilde{\Omega}_{0}^{-}\) defined in Sections 3.3.1 and 3.2.1 respectively,

\[
\lim_{\alpha \to \infty} Y^{(0)} (z_{\alpha}^{-1}) \bigg|_{z \in \omega_{\infty_{0}}(\alpha) \bigg( C^{0\infty} (C^{1\infty})^{-1} K^{-}_{\infty}(\alpha) = \tilde{Y}^{(0)}(z) \tilde{C}^{0\infty}, \quad z \in \tilde{\Omega}_{0}^{-}. \bigg)
\]

Combining this with the limits \((3.67)\) and \((3.95)\), we deduce the required result \((3.99)\)
as follows,

\[
\lim_{\alpha \to \infty} Y^{(0)}(z^{\alpha^{-1}}) \bigg|_{z \in \omega_0^{-}(\alpha)} \left( \begin{array}{cc} \alpha^{1-\gamma} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \alpha^{\gamma-1} & 0 \\ 0 & 1 \end{array} \right) C^{0\infty} K^-_\infty(\alpha) \left( \begin{array}{cc} (\gamma\beta-\gamma) & 0 \\ 0 & (\alpha^{-\beta}) \end{array} \right)
\]

\[
= \tilde{Y}^{(0)}(z) \lim_{\alpha \to \infty} \left( \begin{array}{cc} \alpha^{\gamma-1} & 0 \\ 0 & 1 \end{array} \right) C^{0\infty} \left( \begin{array}{cc} (\gamma\beta-\gamma) & 0 \\ 0 & (\alpha^{-\beta}) \end{array} \right), \quad z \in \tilde{\Omega}_0^-, \quad \lim_{\alpha \to \infty} \arg(\alpha) = -\frac{\pi}{2}
\]

\[
\therefore \lim_{\alpha \to \infty} \left( \begin{array}{cc} \alpha^{\gamma-1} & 0 \\ 0 & 1 \end{array} \right) C^{0\infty} \left( \begin{array}{cc} (\gamma\beta-\gamma) & 0 \\ 0 & (\alpha^{-\beta}) \end{array} \right) = \tilde{C}^{0\infty},
\]

where we have implicitly used Lemma 3.14.

The proof of (3.100) is similar: the curve \( \gamma_{10} \) which defines the connection matrix \( C^{01} \equiv C^{0\infty} (C^{1\infty})^{-1} \) survives the confluence limit. The substitution \( x = \frac{z}{\alpha} \) and limit \( \alpha \to \infty \) certainly translates one of the base points of the curve, but not in such a way that the length of the curve vanishes or the homotopy of the curve is affected. This fact is expressed as follows,

\[
\lim_{\alpha \to \infty} \gamma_{10} \left[ Y^{(1)}(\alpha) K_1^+(\alpha) \right] (z^{\alpha^{-1}}) = \gamma_{0\infty} \left[ \tilde{Y}^{(\infty,0)}(\alpha) \right] (z),
\]

or equivalently,

\[
\lim_{\alpha \to \infty} Y^{(0)}(z^{\alpha^{-1}}) \bigg|_{z \in \omega_0^{+}(\alpha)} C^{0\infty} K_1^+(\alpha) = \tilde{Y}^{(0)}(z) \tilde{C}^{0\infty}, \quad z \in \tilde{\Omega}_0^+.
\]

Combining this with the limits (3.67) and (3.96), we deduce the required result (3.100)
as follows,

\[
\lim_{\alpha \to \infty} Y^{(0)}(z\alpha^{-1}) \bigg|_{z \in \omega_0^+} \left( \begin{array}{cc} \alpha^{1-\gamma} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \alpha^{\gamma-1} & 0 \\ 0 & 1 \end{array} \right) C^{0\infty} (C^{1\infty})^{-1}
\]

\[
= \tilde{Y}^{(0)}(z) \lim_{\alpha \to \infty} \left( \begin{array}{cc} \alpha^{\gamma-1} & 0 \\ 0 & 1 \end{array} \right) C^{0\infty} (C^{1\infty})^{-1} \left( \begin{array}{cc} \alpha^{\beta-\gamma} & 0 \\ 0 & \alpha^{-\beta} \end{array} \right), \quad z \in \tilde{\Omega}_0^+,
\]

\[
= \tilde{Y}^{(0)}(z) \tilde{C}^{0\infty}, \quad z \in \tilde{\Omega}_0^+,
\]

\[
\Leftrightarrow \lim_{\alpha \to \infty} \left( \begin{array}{cc} \alpha^{\gamma-1} & 0 \\ 0 & 1 \end{array} \right) C^{0\infty} (C^{1\infty})^{-1} \left( \begin{array}{cc} \alpha^{\beta-\gamma} & 0 \\ 0 & \alpha^{-\beta} \end{array} \right) = \tilde{C}^{0\infty},
\]

where we have implicitly used Lemma 3.14.

Having deduced the limits (3.99) and (3.100) of connection matrices, the limits (3.101) and (3.102) follow directly since \( M_0 = (C^{0\infty})^{-1} e^{2\pi i \Theta_0} C^{0\infty} \) and \( \Theta_0 \equiv \tilde{\Theta}_0 \). For (3.101), we have,

\[
\lim_{\alpha \to \infty} \left( \begin{array}{cc} \gamma & -\beta \\ 0 & 1 \end{array} \right) M_0 \left( \begin{array}{cc} \alpha^{1-\gamma} & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} \alpha^{\beta-\gamma} & 0 \\ 0 & \alpha^{-\beta} \end{array} \right)
\]

\[
= \lim_{\alpha \to \infty} \left( \begin{array}{cc} \alpha^{\gamma-\beta} & 0 \\ 0 & \alpha^{-\beta} \end{array} \right) \left( \begin{array}{cc} \alpha^{\gamma-\beta} & 0 \\ 0 & \alpha^{-\beta} \end{array} \right) M_0 \left( \begin{array}{cc} \alpha^{\beta-\gamma} & 0 \\ 0 & \alpha^{-\beta} \end{array} \right)
\]

\[
= \left( \tilde{C}^{0\infty} \right)^{-1} e^{2\pi i \tilde{\Theta}_0} \tilde{C}^{0\infty} = \tilde{M}_0,
\]

as required. The limit (3.102) is analogous. Finally, the limits (3.103) and (3.104) are immediately found from (3.101) and (3.102) after using the cyclic relations (3.17) and (3.44) to write \( M_\infty M_1 = M_0^{-1} \) and \( \tilde{M}_\infty = \tilde{M}_0^{-1} \).  

\[\square\]
Explicit computations of limits of monodromy data

In this section, we provide another proof of the statements (3.97) and (3.98) of our Main Theorem 2 by direct calculation. This is an instructive computation to explicitly see how the Stokes’ matrices arise. We will use the following classical facts:

\[\lim_{\alpha \to \infty} \frac{a^{c-b} \Gamma(a+b)}{\Gamma(a+c)} = 1, \text{ as } a \to \infty, |\arg(a)| < \pi, \quad (3.108)\]

\[\Gamma(a) \equiv \frac{\pi}{\sin(\pi a) \Gamma(1 - a)}, \quad (3.109)\]

\[\lim_{a \to \infty} e^{i\pi a} \csc(\pi a) = 2i \text{ for } \text{Im}(a) < 0. \quad (3.110)\]

The proof of (3.110) is elementary, the proofs of (3.108) and (3.109) can be found in [WW] and [BE].

Let \(C^{1\infty}\) be given by (3.14). Using \((-\alpha) \equiv \alpha e^{i\pi}\), we calculate,

\[
\begin{pmatrix}
\alpha^{\gamma-\beta} & 0 \\
0 & -\alpha^{-\beta}
\end{pmatrix}
\begin{pmatrix}
(-\alpha)^{\beta-\gamma} & 0 \\
0 & -(\alpha)^{\beta}
\end{pmatrix}
\begin{pmatrix}
\alpha^{\gamma-\beta} & 0 \\
0 & -\alpha^{-\beta}
\end{pmatrix}

= \begin{pmatrix}
e^{i\pi(\gamma-\beta)} \frac{\Gamma(\alpha+1-\beta) \Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha) \Gamma(\alpha+1-\gamma)} & e^{i\pi(\gamma-\alpha)} \frac{\Gamma(\beta+1-\alpha) \Gamma(\alpha+\beta-\gamma)}{\Gamma(\beta) \Gamma(\beta+1-\gamma)} \\
e^{i\pi(\gamma-\alpha)} \frac{\Gamma(\alpha+1-\beta) \Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\beta) \Gamma(\gamma-\beta)} & e^{i\pi(\beta+1-\alpha) \Gamma(\gamma-\alpha-\beta)} \frac{\Gamma(\gamma-\alpha-\gamma-2\beta) \Gamma(\beta+1-\alpha) \Gamma(\alpha+\beta-\gamma)}{\Gamma(1-\beta) \Gamma(\gamma-\beta)}
\end{pmatrix}.
\]

Using (3.108), we find for the (1,1) and (2,2) elements:

\[
\lim_{\alpha \to \infty} \frac{\Gamma(\alpha+1-\beta) \Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha) \Gamma(\alpha+1-\gamma)} = 1,
\]

and \(\lim_{\alpha \to \infty} \frac{\Gamma(\beta+1-\alpha) \Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\beta) \Gamma(\gamma-\beta)} = 1\), respectively, as required. We rewrite the (1,2) and (2,1) elements using (3.109) as
follows:

\[-e^{\pi i (\gamma - \alpha - \beta)} \alpha^{\gamma - 2\beta} \frac{\Gamma(\beta + 1 - \alpha) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\beta) \Gamma(\beta + 1 - \gamma)} = \frac{-e^{\pi i (\gamma - \alpha - \beta)}}{\sin(\pi (\alpha + \beta - \gamma))} \alpha^{\gamma - 2\beta} \frac{\Gamma(\beta + 1 - \alpha)}{\Gamma(\gamma + 1 - \alpha - \beta) \Gamma(\beta) \Gamma(\beta + 1 - \gamma)},\]

and,

\[-e^{\pi i (\alpha + \beta - \gamma)} \alpha^{2\beta - \gamma} \frac{\Gamma(\alpha + 1 - \beta) \Gamma(\gamma - \alpha - \beta)}{\Gamma(1 - \beta) \Gamma(\gamma - \beta)} = \frac{-e^{\pi i (\alpha + \beta - \gamma)}}{\sin(\pi (\gamma - \alpha - \beta))} \alpha^{2\beta - \gamma} \frac{\Gamma(\alpha + 1 - \beta) \pi}{\Gamma(\alpha + \beta + 1 - \gamma) \Gamma(1 - \beta) \Gamma(\gamma - \beta)},\]

respectively. As \(\alpha \to \infty\), the dominant terms in these expressions are \(e^{\mp i \pi \alpha}\) respectively; observe that, if \(\arg(\alpha) = \pm \frac{\pi}{2}\) then \(e^{\pm i \pi \alpha} \to 0\) as \(\alpha \to \infty\), as required. Finally, for the most important computations, we have:

\[
\lim_{\arg(\alpha) \to -\frac{\pi}{2}} \frac{-e^{\pi i (\alpha + \beta - \gamma)}}{\sin(\pi (\gamma - \alpha - \beta))} \alpha^{2\beta - \gamma} \frac{\Gamma(\alpha + 1 - \beta) \Gamma(\alpha + \beta + 1 - \gamma) \pi}{\Gamma(1 - \beta) \Gamma(\gamma - \beta)} \to 2i \quad \text{by (3.110)}
\]

\[
= \frac{2\pi i}{\Gamma(1 - \beta) \Gamma(\gamma - \beta)} \equiv (S_{-1})_{2,1},
\]

and,

\[
\lim_{\arg(\alpha) \to \frac{\pi}{2}} \frac{-e^{\pi i (\gamma - \alpha - \beta)}}{\sin(\pi (\alpha + \beta - \gamma))} \alpha^{\gamma - 2\beta} \frac{\Gamma(\beta + 1 - \alpha) \Gamma(\beta + 1 - \gamma) \pi}{\Gamma(\beta) \Gamma(\beta + 1 - \gamma)} \to 2i \quad \text{by (3.110)} \to e^{i \pi (\gamma - 2\beta)} \text{ by (3.108)}
\]

\[
= \frac{2\pi i e^{i \pi (\gamma - 2\beta)}}{\Gamma(\beta) \Gamma(\beta + 1 - \gamma)} \equiv (S_0)_{1,2},
\]

as required. \(\square\)
Chapter 4

The Sixth and Fifth Painlevé Equations

In this chapter we consider two first order linear ODEs whose monodromy preserving deformations are described by the sixth and fifth Painlevé equations respectively, as found by Jimbo and Miwa [JM]. The linear system for $P_{VI}$ consists of a differential equation with four Fuchsian singularities and the one for $P_V$ with two Fuchsian singularities and one irregular singularity of Poincaré rank one. It is well known that, by imposing certain rescalings on the parameters and variables, the sixth Painlevé equation confluences to the fifth one. Inspired by [Kit3], we give a confluence procedure from the auxiliary linear system of $P_{VI}$ to that of $P_V$, this involves merging two simple poles to create a double pole. The formal limit passage among the auxiliary linear systems allows us to write the leading asymptotic behavior of the $P_{VI}$ transcendent in terms of the $P_V$ transcendent and its derivative, this is stated in Theorem 4.4. The main focus of this chapter is to understand how to relate the monodromy data of the linear systems under the confluence procedure. We begin by recalling from [JM] how the sixth and fifth Painlevé equations are equivalent to certain linear systems through the theory of isomonodromic deformations. We then analyse how the solutions of the $P_{VI}$ linear system behave under the confluence procedure; around the surviving Fuchsian singularities we have straightforward limits while around the merging singularities we use Glutsyuk’s Theorem 2.5. Following this, we prove our Main Theorem 3, which shows a crucial property that the matrices involved in Glutsyuk’s Theorem must satisfy. This enables us to deduce the monodromy data of the $P_V$ system, including Stokes’ matrices, as limits of the monodromy data of the $P_{VI}$ system, this is stated as our Main Theorem 4.
4.1 Background

In this section we recall the auxiliary linear systems for $P_{VI}$ and $P_{V}$, as found by Jimbo and Miwa [JM] in the theory of isomonodromic deformations of $(2 \times 2)$ ODEs. In subsections 4.1.1 and 4.1.2 we define the solutions of these equations and their monodromy data. In subsection 4.1.3 we then demonstrate our confluence procedure from the linear system of $P_{VI}$ to that of $P_{V}$ on the formal level of the equations.

Throughout this chapter, a subscript 6 or 5 shows the relation to the linear system of $P_{VI}$ or $P_{V}$ respectively. Unlike in Chapter 3, we now study traceless equations and so we frequently make use of the notation for the third Pauli matrix $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

4.1.1 Auxiliary Linear System for $P_{VI}$

The sixth Painlevé equation is derived as the compatibility equation (1.3) of the following first order differential equations,

\[
\frac{\partial Y_6}{\partial \lambda_6} = \left( \frac{A_{06}(t_6)}{\lambda_6} + \frac{A_{16}(t_6)}{\lambda_6 - t_6} + \frac{A_{16}(t_6)}{\lambda_6 - 1} \right) Y_6, \tag{4.1}
\]

\[
\frac{\partial Y_6}{\partial t_6} = -\frac{A_{16}(t_6)}{\lambda_6 - t_6} Y_6, \tag{4.2}
\]

where,

\[
A_{k6}(t_6) = \begin{pmatrix} \frac{\theta_{k6}}{2} + z_{k6} & -u_{k6} z_{k6} \\ \frac{u_{k6} + z_{k6}}{u_{k6}} & -\frac{\theta_{k6}}{2} - z_{k6} \end{pmatrix}, \quad k = 0, t, 1, \tag{4.3}
\]

where $u_{k6}$ and $z_{k6}$ are functions of $t_6$ and $\theta_{k6}$ are parameters. Given a solution of equation (4.1), the condition for the solution to satisfy equation (4.2) is precisely the condition for the monodromy data of the solution to be independent of $t_6$. We assume:

1. $A_{06} + A_{16} + A_{16} = -\frac{\theta_{k6}}{2} \sigma_3$;
2. there exist $R_{k6}$ such that $R_{k6}^{-1} A_{k6} R_{k6} = \frac{\theta_{k6}}{2} \sigma_3$, $k = 0, t, 1$;
3. nonresonance, namely $\theta_{k6} \notin \mathbb{Z}$ for $k = 0, t, 1, \infty$. 

Page 101 of 178
Chapter 4  The Sixth and Fifth Painlevé Equations

Remark 4.1. Using expression (4.3), we note that the first assumption is equivalent to the three following equations:

\[
\sum_{k=0,t,1} \frac{\theta_{k6}}{2} + z_{k6} = -\frac{\theta_{\infty 6}}{2}, \quad \sum_{k=0,t,1} u_{k 6} z_{k 6} = 0, \quad \sum_{k=0,t,1} \frac{\theta_{k6} + z_{k6}}{u_{k 6}} = 0. \quad (4.4)
\]

Also, the matrices \( R_{k6} \) in the second assumption are determined up to multiplication on the right by diagonal matrices. In sections (4.2)-(4.3) it will be convenient to use this freedom so that \( R_{k6} \) satisfy certain conditions. Our method is still completely general because, for any choice of \( R_{k6} \), we can bury the freedom of multiplication on the right by a diagonal matrices inside the diagonal matrices asserted by Glutsyuk’s Theorem 2.5.

Equation (4.1) has four Fuchsian singularities at \( \lambda_6 = 0, t_6, 1 \) and \( \infty \). By comparing terms at each of these poles, we find that the compatibility condition (1.3) of (4.1)-(4.2) is equivalent to the following system of first order ODEs:

\[
A_{06}' = t_6^{-1} [A_{16}, A_{06}],
\]

\[
A_{t6}' = t_6^{-1} [A_{06}, A_{16}] + (1 - t_6)^{-1} [A_{t6}, A_{16}],
\]

\[
A_{16}' = (1 - t_6)^{-1} [A_{16}, A_{t6}].
\]

(4.5), (4.6), (4.7)

We recall the following fundamental result due to Jimbo and Miwa.

**Theorem 4.1 [JM].** If we define \( y_6(t_6) \) by the equation,

\[
\left( \frac{A_{06}}{y_6} + \frac{A_{t6}}{y_6 - t_6} + \frac{A_{16}}{y_6 - 1} \right)_{1,2} = 0,
\]

then the system of first order ODEs (4.5)-(4.7) implies that \( y_6(t_6) \) satisfies the equation \( P_{VI} \) (see Introduction) with \( \alpha = \frac{1}{2} (\theta_{\infty 6} - 1)^2, \beta = -\frac{1}{2} \theta_{06}^2, \gamma = \frac{1}{2} \theta_{16}^2 \) and \( \delta = \frac{1}{2} (1 - \theta_{t6})^2 \).

Remark 4.2. From the definition (4.8) of \( y_6(t_6) \) and expression (4.3) for \( A_{k6}, k = 0, t, 1 \), we note that \( y_6 \) is equivalently expressed as,

\[
y_6(t_6) = \frac{t_6 u_{06}(t_6) z_{06}(t_6)}{(1 + t_6) u_{06}(t_6) z_{06}(t_6) + u_{06}(t_6) z_{16}(t_6) + t_6 u_{16}(t_6) z_{16}(t_6)},
\]

(4.9)
assuming the denominator is non-zero. It is immediately clear that this expression is invariant under the simultaneous transformation $u_{k6} \mapsto u_{k6}U$, $k = 0, t, 1$, for some function $U$. In fact, Theorem 4.1 shows that, up to simultaneous conjugation by a diagonal matrix, the compatibility condition (1.3) of (4.1)-(4.2) is equivalent to the equation $P_{VI}$.

We define the following disks with chosen branches around each singularity:

$$
\Omega_{06} = \{ \lambda_6 : |\lambda_6| < \rho_{06}, -\pi \leq \arg(\lambda_6) < \pi \},
\Omega_{t6} = \{ \lambda_6 : |\lambda_6 - t_6| < \rho_{t6}, -\pi \leq \arg(\lambda_6 - t_6) < \pi \},
\Omega_{16} = \{ \lambda_6 : |\lambda_6 - 1| < \rho_{16}, -\pi \leq \arg(1 - \lambda_6) < \pi \},
\Omega_{\infty 6} = \{ \lambda_6 : |\lambda_6| > \rho_{\infty 6}, 0 \leq \arg(\lambda_6) < 2\pi \}.
$$

for some $\rho_{k6} > 0$, see Figure 20 below. Since we are solving linear equations, the maximum value we could take for the radius of convergence $\rho_{k6}$, $k = 0, 1, t, \infty$, is the distance to the nearest singularity. For example, we could take $\rho_{06} = \min(|t_6|, 1)$, $\rho_{t6} = \min(|t_6|, |t_6 - 1|)$, $\rho_{16} = \min(|t_6 - 1|, 1)$ and $\rho_{\infty 6} = \max(|t_6|, 1)$. In Sections 4.2 and 4.3.3 we will be more specific about the radii for the domains of our local fundamental solutions.

We have the following local fundamental solutions of equation (4.1) which are analytic in the neighbourhoods $\Omega_{k6}$ around each singular point:

$$
Y_{6}^{(0)}(\lambda_6) = R_{06} \left( \sum_{n=0}^{\infty} G_{n,0} \lambda_6^n \right) \lambda_6^{\frac{\theta_{06}}{2} \sigma_3}, \quad \lambda_6 \in \Omega_{06}, \quad (4.10)
$$

$$
Y_{6}^{(t)}(\lambda_6) = R_{t6} \left( \sum_{n=0}^{\infty} G_{n,t} (\lambda_6 - t_6)^n \right) (\lambda_6 - t_6)^{\frac{\theta_{t6}}{2} \sigma_3}, \quad \lambda_6 \in \Omega_{t6}, \quad (4.11)
$$

$$
Y_{6}^{(1)}(\lambda_6) = R_{16} \left( \sum_{n=0}^{\infty} G_{n,1} (\lambda_6 - 1)^n \right) (1 - \lambda_6)^{\frac{\theta_{16}}{2} \sigma_3}, \quad \lambda_6 \in \Omega_{16}, \quad (4.12)
$$

$$
Y_{6}^{(\infty)}(\lambda_6) = \left( \sum_{n=0}^{\infty} G_{n,\infty} \lambda_6^{-1} \right) \lambda_6^{-\frac{\theta_{\infty 6}}{2} \sigma_3}, \quad \lambda_6 \in \Omega_{\infty 6}, \quad (4.13)
$$
respectively, where $G_{n,k} = I$ and all other terms of each series are determined by the recursive formulae below:

\[
\begin{align*}
   nG_{n,0} + \left[ G_{n,0}, \frac{\theta_{06}}{2} \sigma_3 \right] &= -\sum_{l=0}^{n-1} R_{06}^{-1} (A_{16} + A_{16}t_6^{l-n}) R_{06} G_{l,0}, \\
   nG_{n,t} + \left[ G_{n,t}, \frac{\theta_{16}}{2} \sigma_3 \right] &= -\sum_{l=0}^{n-1} R_{16}^{-1} (A_{06}(-t_6)^{l-n} + A_{16}(1 - t_6)^{l-n}) R_{16} G_{l,t}, \\
   nG_{n,1} + \left[ G_{n,1}, \frac{\theta_{16}}{2} \sigma_3 \right] &= \sum_{l=0}^{n-1} (-1)^l R_{16}^{-1} (A_{06} + A_{16}(1-t_6)^{-l-1}) R_{16} G_{l,1}, \\
   -nG_{n,\infty} + \left[ G_{n,\infty}, -\frac{\theta_{\infty6}}{2} \sigma_3 \right] &= \sum_{l=0}^{n-1} (A_{16} + t_6^{n-l}A_{16}) G_{l,\infty}.
\end{align*}
\]

(4.14) (4.15) (4.16) (4.17)

We may extend the definitions of our local solutions $Y_6^{(k)}(\lambda_6)$ to other sheets $e^{2\pi i \Omega_k}$, $k = 0, t, 1, \infty$, by analytically continuing along a closed loop encircling the singularity $\lambda_6 = 0, t_6, 1, \infty$. This action simply means that our solution becomes multiplied by the so-called local monodromy matrix $e^{m \pi i \theta_{66} \sigma_3}$, for $k = 0, t, 1$ and $\infty$, $m \in \mathbb{Z}$. Note that, for $k = 0, t$ and $1$, the analytic continuation of $Y_6^{(k)}(\lambda_6)$ around its singularity in the positive direction means $m > 0$ in the previous sentence, while, for $k = \infty$, it means $m < 0$.

We proceed with the global analysis of solutions. Let $Y^{(k)}(\lambda_6)$ be the fundamental solutions of equation (4.1) defined above. We choose to normalise the monodromy data of equation (4.1) with the fundamental solution $Y_6^{(\infty)}(\lambda_6)$. Denote by $\gamma_{j,k}[Y_6^{(j)}](\lambda_6)$ the analytic continuation of $Y_6^{(j)}(\lambda_6)$ along an orientable curve $\gamma_{j,k} : [0, 1] \rightarrow \mathbb{C}$, with $\gamma_{j,k}(0) \in \Omega_j$ and $\gamma_{j,k}(1) \in \Omega_k$, for $j, k = 0, t, 1, \infty$. This defines connection matrices $C_6^{kj} \in \text{GL}_2(\mathbb{C})$ as follows,

\[
\gamma_{j,k} \left[ Y_6^{(j)} \right](\lambda_6) = Y_6^{(k)}(\lambda_6)C_6^{kj}.
\]

(4.18)

Denote by $\gamma_k[Y_6^{(\infty)}](\lambda_6)$ the analytic continuation of $Y_6^{(\infty)}(\lambda_6)$ along an orientable, closed curve $\gamma_k : [0, 1] \rightarrow \mathbb{C}$ with $\gamma_k(0) = \gamma_k(1) \in \Omega_\infty$, $k = 0, t, 1$, which encircles the singularity $\lambda_6 = 0, t_6, 1$ respectively in the positive direction. The curves $\gamma_0, \gamma_t$ and $\gamma_1$ are illustrated in Figure 20 below, note that $\gamma_\infty := \gamma_1^{-1}\gamma_t^{-1}\gamma_0^{-1}$. This defines
monodromy matrices $M_{k6} \in \text{SL}_2(\mathbb{C})$ as follows,

$$
\gamma_k \left[ Y_6^{(\infty)} \right](\lambda_6) = Y_6^{(\infty)} (\lambda_6) M_{k6},
$$

(4.19)

where,

$$
M_{k6} = (C_6^{ik\infty})^{-1} e^{i\pi\theta_{k6}\sigma_3} C_6^{ik\infty}, \text{ for } k = 0, t, 1, \text{ and } M_{\infty6} = e^{i\pi\theta_{\infty6}\sigma_3}.
$$

(4.20)

Figure 20: Curves which define the monodromy matrices $M_{k6}$ of the $P_{VI}$ linear system.

The analytic continuations of $Y_6^{(\infty)}(\lambda_6)$ around these curves defines a monodromy antirepresentation of the fundamental group,

$$
p : \pi_1(\mathbb{C}\setminus\{0, t_6, 1\}, \infty) \to \text{SL}_2(\mathbb{C}), \quad [\gamma_k] \mapsto M_{k6},
$$

since the map $p$ is an antihomomorphism. This results in the following cyclic relation,

$$
M_{\infty6}M_{t6}M_{06} = I.
$$

(4.21)

**Definition 4.1.** We define the monodromy data of equation (4.1) as the set,

$$
\mathcal{M}_6 := \left\{ (M_{06}, M_{t6}, M_{t6}, M_{\infty6}) \in (\text{SL}_2(\mathbb{C}))^4 \left| \begin{array}{c} M_{\infty6}M_{t6}M_{06} = I, \ M_{\infty6} = e^{i\pi\theta_{\infty6}\sigma_3} \\
\text{eigenv}(M_{k6}) = e^{\pm i\pi\theta_{k6}}, \ k=0, t, 1 \end{array} \right. \right\} / \text{SL}_2(\mathbb{C})
$$

(4.22)

where the quotient is by global conjugation by a diagonal matrix.
Remark 4.3. Counting the dimension of the set of monodromy data, we have,

$$\dim_{\mathbb{C}}(\mathcal{M}_6) = 12 - 3 - 2 - 4 - 1 = 2.$$ 

Compared with the set of monodromy data for Gauss’ equation, the extra degree of freedom is caused by the presence of the fourth Fuchsian singularity at $\lambda_6 = t_6$. One can interpret these two dimensions as transcendental functions of the initial conditions of the $P_{VI}$ equation.

Remark 4.4. The connection matrices can be retrieved from the monodromy matrices, up to a freedom of multiplication on the left by diagonal, invertible matrices. The connection matrices defined by (4.18) will play an important role in Section 4.4, where we will produce the Stokes’ matrices of the auxiliary linear system of $P_V$ from certain limits of these connection matrices.

### 4.1.2 Auxiliary Linear System for $P_V$

The system of linear ODEs which we use below for the auxiliary linear problem of $P_V$ is slightly different to the original one found in [JM], the difference being that we have made the substitution $\lambda_5 = t_5 \lambda_5^{JM}$, where $\lambda_5^{JM}$ is the variable used by Jimo and Miwa. This modification is beneficial for us for two reasons. Firstly, this makes our confluence from $P_{VI}$ to $P_V$ more transparent because the singularity $\lambda_5^{JM} = 1$ is transformed to $\lambda_5 = t_5$, which corresponds to $\lambda_6 = t_6$ in the confluence scheme. Secondly, whereas the positions of the Stokes’ rays in the Jimbo-Miwa equation depend on $t_5$, the positions of the Stokes’ rays in our equation are fixed and coincide with those of Kummer’s confluent hypergeometric differential equation (3.2). We consider the following system of linear ODEs,

$$\frac{\partial Y_5}{\partial \lambda_5} = \left(\frac{\sigma_3}{2} + \frac{A_{05}(t_5)}{\lambda_5} + \frac{A_{45}(t_5)}{\lambda_5 - t_5}\right) Y_5, \quad (4.23)$$

$$\frac{\partial Y_5}{\partial t_5} = -\frac{A_{45}(t_5)}{\lambda_5 - t_5} Y_5, \quad (4.24)$$
where,
\[
A_{05}(t_5) = \begin{pmatrix} z_5 + \frac{\theta_05}{2} & -u_5(z_5 + \theta_05) \\ \frac{z_5}{u_5} & -z_5 - \frac{\theta_05}{2} \end{pmatrix}, \tag{4.25}
\]
\[
A_{t5}(t_5) = \begin{pmatrix} -z_5 - \frac{\theta_05 + \theta_{\infty5}}{2} & u_5y_5(z_5 + \frac{\theta_05 + \theta_{\infty5}}{2}) \\ \left(z_5 + \frac{\theta_05 - \theta_{\infty5}}{2}\right)u_5^{-1}y_5^{-1} & z_5 + \frac{\theta_05 + \theta_{\infty5}}{2} \end{pmatrix}, \tag{4.26}
\]
where \(u_5, y_5\) and \(z_5\) are functions of \(t_5\) and \(\theta_05, \theta_{t5}\) and \(\theta_{\infty5}\) are parameters. Given a solution of equation (4.23), the condition for the solution to satisfy equation (4.24) is precisely the condition for the monodromy data of the solution to be independent of \(t_5\). We assume:

1. there exist \(R_{k5}\) such that \(R_{k5}^{-1}A_{k5}R_{k5} = \frac{\theta_05}{2}\sigma_3, k = 0, t;\)
2. nonresonance, namely \(\theta_{k5} \notin \mathbb{Z}\) for \(k = 0, t.\)

Remark 4.5. The matrices \(R_{k5}\) in the first assumption are determined up to multiplication on the right by diagonal matrices. In Chapter 5 Section 5.2 it will be convenient to use this freedom so that \(R_{k5}\) satisfy certain conditions. This does not affect the generality of our method because, for any choice of \(R_{k5}\), the freedom of multiplication on the right by diagonal matrices can be buried inside the diagonal matrices asserted by Glutsyuk’s Theorem 2.5.

Equation (4.23) has two Fuchsian singularities at \(\lambda_5 = 0, t_5\) and an irregular singularity of Poincaré rank one at \(\lambda_5 = \infty\). By comparing terms at each of these poles, we have that the compatibility condition (1.3) of (4.23)-(4.24) is equivalent to the following system of first order ODEs:

\[
A_{05}' = t_5^{-1} [A_{t5}, A_{05}], \tag{4.27}
\]
\[
A_{t5}' = \left[\frac{\sigma_3}{2}, A_{t5}\right] + t_5^{-1} [A_{05}, A_{t5}]. \tag{4.28}
\]
This system of ODEs is equivalent to three first order ODEs:

\[
y_5' = -\frac{1}{2t_5} (3\theta_0 - \theta_{t_5} + \theta_{\infty_5} + 4z_5 - 2y_5(t_5 + 2\theta_0 + \theta_{\infty_5} + 4z_5) + y_5^2(\theta_0 + \theta_{t_5} + \theta_{\infty_5} + 4z_5));
\]

\[
z_5' = \frac{1}{2t_5y_5} (\theta_0(\theta_{t_5} - \theta_0 - \theta_{\infty_5}) + z_5(\theta_{t_5} - 3\theta_0 - \theta_{\infty_5} - 2z_5) + y_5^2(\theta_0 + \theta_{t_5} + \theta_{\infty_5} + 2z_5));
\]

\[
u_5' = \frac{1}{2t_5y_5} (y_5 - 1)(\theta_{t_5} - \theta_0 - \theta_{\infty_5} - 2z_5 + y_5(\theta_0 + \theta_{t_5} + \theta_{\infty_5} + 2z_5)).
\]

In the Hamiltonian formulation of \(P_V\), \(q = y_5(t_5)\) is the coordinate and \(p = z_5(t_5)\) is the conjugate momentum. We note that the third equation for \(u_5(t_5)\) appears from the freedom of global conjugation of the system (4.23)-(4.24) by a diagonal matrix. We have the following fundamental theorem due to Jimbo and Miwa.

**Theorem 4.2 [JM].** Given a solution \(Y_5\) of (4.23)-(4.24), then \(y_5\) satisfies the equation \(P_V\) (see Introduction), with \(\alpha = \frac{1}{2} (\theta_0 + \theta_{t_5} + \theta_{\infty_5})^2\), \(\beta = -\frac{1}{2} (\theta_0 + \theta_{t_5} - \theta_{\infty_5})^2\), \(\gamma = \theta_0 + \theta_{t_5}\) and \(\delta = -\frac{1}{2}\).

We define the following disks with chosen branches, for some \(\rho_k > 0\):

\[
\Omega_{05}^\pm = \left\{ \lambda_5 : |\lambda_5| < \rho_{05}, \ -\pi \pm \frac{\pi}{2} \leq \arg(\lambda_5) < \pi \pm \frac{\pi}{2} \right\},
\]

\[
\Omega_{t5}^\pm = \left\{ \lambda_5 : |\lambda_5 - t_5| < \rho_{t5}, \ -\pi \pm \frac{\pi}{2} \leq \arg(\lambda_5 - t_5) < \pi \pm \frac{\pi}{2} \right\}.
\]

We deliberately leave the ambiguity in the choices of signs here, these will be explained in the Section 4.2 when analysing the limit passage from Fuchsian singularities to Fuchsian singularities under our confluence procedure. Essentially, we will produce two limit passages from the solutions of (4.1) around \(\lambda_6 = 0, t_6\) to the solutions of (4.23) around \(\lambda_5 = 0, t_5\) by taking the confluence parameter along two different directions. This is analogous to the discussion in Section 3.2.1 in the case of the hypergeometric equation.

**Remark 4.6.** As before, the maximum value we could take for the radius \(\rho_k\), \(k = 0, t\), is the distance to the nearest singularity. This also applies to the radius \(\rho_{\infty_5}\) of the
sectors around infinity in Theorem 4.3 below. We remark that these maximal values are: \( \rho_{05} = \rho_{t5} = \rho_{\infty5} = |t_5| \), though it will not be necessary to use these.

We have the following local fundamental solutions of equation (4.23) which are analytic in the neighbourhoods \( \Omega_{k5}^\pm \) around each singular point:

\[
Y_5^{(0)}(\lambda_5) = R_{05}\left(\sum_{n=0}^{\infty} H_{n,0} \lambda_5^n\right) \lambda_5^{\theta_{05}\sigma_3/2}, \quad \lambda_5 \in \Omega_{05}^\pm, \tag{4.32}
\]

\[
Y_5^{(t)}(\lambda_5) = R_{t5}\left(\sum_{n=0}^{\infty} H_{n,t}(\lambda_5 - t_5)^n\right) (\lambda_5 - t_5)^{\theta_{t5}\sigma_3/2}, \quad \lambda_5 \in \Omega_{t5}^\pm, \tag{4.33}
\]

respectively, where \( H_{n,0} = H_{n,t} = I \) and all other terms of each series are uniquely determined by the following recursive relations, for \( n \geq 1 \):

\[
nH_{n,0} + \left[H_{n,0}, \frac{\theta_{05}}{2} \sigma_3\right] = R_{05}^{-1} \frac{\sigma_3}{2} R_{05} H_{n-1,0} - R_{05}^{-1} A_{05} R_{05} \sum_{l=0}^{n-1} t_5^{l-n} H_{l,0}, \tag{4.34}
\]

\[
nH_{n,t} + \left[H_{n,t}, \frac{\theta_{t5}}{2} \sigma_3\right] = R_{t5}^{-1} \frac{\sigma_3}{2} R_{t5} H_{n-1,t} - R_{t5}^{-1} A_{05} R_{t5} \sum_{l=0}^{n-1} (-t_5)^{l-n} H_{l,t}. \tag{4.35}
\]

**Definition 4.2.** The rays \( \{\lambda_5 : \text{Re}(\lambda_5) = 0, \text{Im}(\lambda_5) > 0\} \) and \( \{\lambda_5 : \text{Re}(\lambda_5) = 0, \text{Im}(\lambda_5) < 0\} \) are called Stokes’ rays. We note that they coincide with the Stokes’ rays of Kummer’s equation because the real part of the differences in the eigenvalues of the leading matrix of (4.23) around \( \lambda_5 = \infty \) and of the leading matrix (3.31) around \( z = \infty \) are the same (namely, both are equal to 1).

**Theorem 4.3.** For some \( \rho_{\infty5} > 0 \), let,

\[
\Sigma_k = \left\{\lambda_5 : |\lambda_5| > \rho_{\infty5}, -\frac{\pi}{2} < \arg(\lambda_5) - k\pi < \frac{3\pi}{2}\right\}. \tag{4.36}
\]

For all \( k \in \mathbb{Z} \), there exists a solution \( Y_5^{(\infty,k)}(\lambda_5) \) of equation (4.23) analytic in the sector \( \Sigma_k \) such that,

\[
Y_5^{(\infty,k)}(\lambda_5) \sim \left(\sum_{n=0}^{\infty} H_{n,\infty} \lambda_5^{-n}\right) \lambda_5^{\theta_{\infty5}/2} e^{\frac{\lambda_5}{2} \sigma_3}, \quad \text{as } \lambda_5 \rightarrow \infty, \lambda_5 \in \Sigma_k, \tag{4.37}
\]

where \( H_{0,\infty} = I \) and all other terms of the series are uniquely determined by the
following recursive relation, for \( n \geq 1 \),

\[
\begin{align*}
\left[ H_{n,\infty}, \frac{\sigma_3}{2} \right] &= (n - 1)H_{n-1,\infty} + H_{n-1,\infty} \frac{\theta_{\infty_5}}{2} \sigma_3 + (A_{05} + A_{t5}) H_{n-1,\infty} \\
&\quad + \sum_{l=0}^{n-2} \frac{n-l}{t_5} A_{t5} H_{l,\infty}.
\end{align*}
\] (4.38)

Moreover, each solution \( Y_5^{(\infty,k)}(\lambda_5) \) is uniquely specified by the relation (4.37).

**Proof.** A proof of the existence of solutions \( Y_5^{(\infty,k)}(\lambda_5) \) which are analytic on sectors \( \Sigma_k \) may be found in [BJL]. The series \( \sum_{n=0}^{\infty} H_{n,\infty} \lambda_5^n \) is a formal gauge transformation which maps equation (4.23) into its Birkhoff normal form, the recursive relation is found by equating the coefficients of \( \lambda_5^{-n} \). The final statement of the theorem can be proved in the exact same manner as in the proof of Theorem 3.1. \( \square \)

We denote the asymptotic behavior of true solutions of (4.23) at infinity as in (4.37) by,

\[
Y_5^{(\infty)}(\lambda_5, f) = \left( \sum_{n=0}^{\infty} H_{n,\infty} \lambda_5^{-n} \right) \lambda_5^{\frac{\theta_{\infty_5}}{2}} e^{\frac{i}{2} \sigma_3}.
\] (4.39)

We call this function a *formal* solution in the sense that the series diverges for general parameters \( \theta_{05}, \theta_{t5} \) and \( \theta_{\infty 5} \). The asymptotic relation (4.37) means, by definition, for all \( m \in \mathbb{N} \) and for all closed subsectors \( \sigma \subset \Sigma_k \),

\[
\left| \lambda^m \left( Y_5^{(\infty,k)}(\lambda_5) \lambda_5^{\frac{\theta_{\infty_5}}{2}} e^{\frac{i}{2} \sigma_3} - \sum_{n=0}^{m} H_{n,\infty} \lambda_5^{-n} \right) \right| \to 0, \text{ as } \lambda_5 \to \infty, \lambda_5 \in \sigma.
\]

From the asymptotic relation (4.37), it is clear that the solutions,

\[
Y_5^{(\infty,k+2)}(\lambda_5) \text{ and } Y_5^{(\infty,k)}(\lambda_5 e^{-2\pi i}) e^{-i\pi \theta_{\infty 5} \sigma_3}
\]

have the same asymptotic behavior as \( \lambda_5 \to \infty \) in the sector \( \lambda_5 \in \Sigma_{k+2} \). By the last statement of Theorem 4.3, we therefore conclude that,

\[
Y_5^{(\infty,k+2)}(\lambda_5) \equiv Y_5^{(\infty,k)}(\lambda_5 e^{-2\pi i}) e^{-i\pi \theta_{\infty 5}}, \quad \lambda_5 \in \Sigma_{k+2}.
\] (4.40)
In this sense, all solutions $Y_{5}^{(\infty,k)}(\lambda_5)$ are categorised into two fundamentally distinct cases, namely, when $k$ is even and when $k$ is odd.

**Definition 4.3.** Let $Y_{5}^{(\infty,k)}(\lambda_5)$ be the fundamental solutions given in Theorem 4.3 and define sectors,

$$
\Pi_k := \Sigma_k \cap \Sigma_{k+1} \equiv \left\{ \lambda_5 : |\lambda_5| > N, \frac{\pi}{2} < \arg(\lambda_5) - k\pi < \frac{3\pi}{2} \right\},
$$

as illustrated in Figure 21 below. We define Stokes’ matrices $S_k \in \text{SL}_2(\mathbb{C})$ as follows,

$$
Y_{5}^{(\infty,k+1)}(\lambda_5) = Y_{5}^{(\infty,k)}(\lambda_5)S_k, \quad \lambda_5 \in \Pi_k.
$$

(4.41)

![Figure 21: Sectors $\Pi_0$, $\Pi_{-1}$, $\Sigma_0$ and $\Sigma_{-1}$ projected onto the plane $\mathbb{C}\backslash\{0\}$. The positive and negative imaginary lines are Stokes’ rays.](image)

We can combine Definition 4.3 with the relation (4.40) to deduce,

$$
e^{-\pi i \theta_5 \sigma_3} S_{k+1} = S_{k-1} e^{-\pi i \theta_5 \sigma_3},
$$

which shows that equation (4.23) has only two types of Stokes’ matrices $S_k$ which are fundamentally different: one with $k$ odd and one with $k$ even. Moreover, from the asymptotic relation (4.37), we deduce,

$$
\lambda_5^{-\frac{\sigma_2}{2} \sigma_3} e^{\frac{\lambda_5}{2} \sigma_3} S_k e^{-\frac{\lambda_5}{2} \sigma_3} \lambda_5^{\frac{\sigma_2}{2} \sigma_3} \sim I, \quad \text{as } \lambda_5 \to \infty, \quad \arg(\lambda_5) - k\pi \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right),
$$
from which it is easy to see that the matrices $S_{2k}$ are upper triangular, the matrices $S_{2k+1}$ are lower triangular and all Stokes’ matrices have unit diagonal.

We choose to normalise the monodromy data of equation (4.23) with respect to the fundamental solution $Y_5^{(∞,0)}(λ_5)$. Denote by $γ_{∞k}[Y_5^{(∞,0)}](λ_5)$ the analytic continuation of $Y_5^{(0)}(λ_5)$ along an orientable curve $γ_{∞k} : [0, 1] → C$, with $γ_{∞k}(0) ∈ Σ_0$ and $γ_{∞k}(1) ∈ Ω_k$, $k = 0, t$. This defines monodromy matrices $C_{k∞} ∈ GL_2(C)$ as follows,

$$γ_{∞k} \left[ Y_5^{(∞,0)} \right] (λ_5) = Y_5^{(k)}(λ_5)C_{k∞}^{∞}. \tag{4.42}$$

Denote by $γ_k \left[ Y_5^{(∞,0)} \right] (λ_5)$ the analytic continuation of $Y_5^{(∞,0)}(λ_5)$ along an orientable, closed curve $γ_k : [0, 1] → C$, with $γ_k(0) = γ_k(1) ∈ Σ_0$, $k = 0, t$, which encircles the singularity $λ_5 = 0, t_5$ respectively in the positive direction. The curves $γ_0$ and $γ_t$ are illustrated in Figure 22 below, note that $γ_∞ := γ_t^{-1}γ_0^{-1}$. This defines monodromy matrices $M_{k5} ∈ SL_2(C)$ as follows,

$$γ_k \left[ Y_5^{(∞,0)} \right] (λ_5) = Y_5^{(∞,0)}(λ_5)M_{k5}, \tag{4.43}$$

where,

$$M_{k5} = (C_{k∞}^{∞})^{-1} e^{iπθ_{k5}\sigma_3}C_{k∞}^{k∞} \text{ for } k = 0, t \quad \text{and} \quad M_{∞5} = S_0 e^{iπθ_{∞5}\sigma_3}S_{-1}. \tag{4.44}$$

Similar as before, this defines a monodromy antirepresentation of the fundamental group,

$$p : π_1 \left( C \setminus \{0, t_5\}, ∞ \right) → SL_2(C), \quad [γ_k] → M_{k5},$$

since $p$ is an antihomomorphism, which implies that we have the following cyclic relation,

$$M_{∞5}M_{t5}M_{05} = I. \tag{4.45}$$
Chapter 4  The Sixth and Fifth Painlevé Equations

Definition 4.4. We define the monodromy data of equation (4.23) as the set,

$$\mathcal{M}_5 := \{ (M_{05}, M_{t5}, S_0, S_{-1}) \in (\text{SL}_2(\mathbb{C}))^4 \mid
\begin{align*}
S_0 &\text{ is unipotent, upper triangular,} \\
S_{-1} &\text{ is unipotent, lower triangular,} \\
S_0 e^{\pi i \theta_{\infty_5} \sigma_3} S_{-1} M_{t5} M_{05} &= I, \\
\text{eigenv} (M_{k5}) &= e^{\pi i \theta_{k5} \sigma_3}, \ k=0,t
\end{align*}
\} / \text{SL}_2(\mathbb{C})$$

(4.46)

where the quotient is by global conjugation by a diagonal matrix.

Remark 4.7. Counting the dimension of the set of monodromy data, we have,

$$\dim_{\mathbb{C}} (\mathcal{M}_5) = 12 - 2 - 2 - 3 - 2 - 1 = 2.$$  

Compared with the set of monodromy data for Kummer’s equation, the extra degree of freedom here is caused by the presence of the additional Fuchsian singularity at $\lambda_5 = t_5$. One can interpret the extra two dimensions as transcendental functions of the initial conditions of the $P_V$ equation.

4.1.3  A Confluence Procedure from $P_{VI}$ to $P_V$

In this section, we outline a confluence procedure from the auxiliary linear system (4.1)-(4.2) of $P_{VI}$ to that (4.23)-(4.24) of $P_V$. The procedure we use has been inspired by [Kit3]. We make substitutions on the $P_{VI}$ variables as follows,

$$\lambda_6 = \varepsilon \lambda_5, \ t_6 = \varepsilon t_5,$$

(4.47)
and the following substitutions on parameters,

$$\theta_{06} = \theta_{05}, \quad \theta_{16} = \theta_{15}, \quad \theta_{16} = -\varepsilon^{-1}, \quad \theta_{\infty6} = \theta_{\infty5} + \varepsilon^{-1}. \quad (4.48)$$

The substitutions (4.47) map the Fuchsian singularities $\lambda_6 = 0, t_6, 1$ and $\infty$ of equation (4.1) to $\lambda_5 = 0, t_5, \varepsilon^{-1}$ and $\infty$ respectively, so that two simple poles merge as $\varepsilon \to 0$. In this sense, we say that we are confluencing the simple poles $\lambda_6 = 1$ and $\infty$ of equation (4.1). In Proposition 4.1 below, we show that, under the substitutions (4.47)-(4.48) and the additional assumption below, the coalescence of $\lambda_6 = 1$ and $\infty$ produces an irregular singularity of Poincaré rank one at $\lambda_5 = \infty$.

We make the following assumption on the matrices $A_{06}(t_6)$ and $A_{16}(t_6)$ of the $PVI$ linear system,

**Main Assumption 1.** There exists an open sector $E \subset \mathbb{C}$, with its base point at the origin, an open domain $T \subset \mathbb{C}$ and sequences of matrices $A_{06}^{(n)}(t_5)$ and $A_{16}^{(n)}(t_5)$, $n \geq 1$, such that,

$$A_{06}(\varepsilon t_5) \sim A_{05}(t_5) + \sum_{n=1}^{\infty} \varepsilon^n A_{06}^{(n)}(t_5), \quad \text{as } \varepsilon \to 0, \ \varepsilon \in E, \quad (4.49)$$

$$A_{16}(\varepsilon t_5) \sim A_{15}(t_5) + \sum_{n=1}^{\infty} \varepsilon^n A_{16}^{(n)}(t_5), \quad \text{as } \varepsilon \to 0, \ \varepsilon \in E, \quad (4.50)$$

uniformly for all $t_5 \in T$.

With this assumption on the asymptotic behavior of $A_{06}(t_6)$ and $A_{16}(t_6)$ as $\varepsilon \to 0$, we naturally extend their definitions to the point $\varepsilon = 0$ in the following way:

$$A_{06}(t_6)|_{\varepsilon=0} := \lim_{\varepsilon \to 0, \ \varepsilon \in E} A_{06}(\varepsilon t_5) = A_{05}(t_5),$$

and

$$A_{16}(t_6)|_{\varepsilon=0} := \lim_{\varepsilon \to 0, \ \varepsilon \in E} A_{16}(\varepsilon t_5) = A_{15}(t_5).$$

**Remark 4.8.** The reason for including the uniformity condition with respect to $t_5 \in T$ in the above assumption is so that the derivatives of the left hand sides of (4.49) and (4.50) are asymptotic to the derived series on the right hand sides of (4.49) and (4.50).
respectively, that is to say:

\[
\frac{d}{dt_5} A_{06}(\varepsilon t_5) \sim \frac{d}{dt_5} A_{05}(t_5) + \sum_{n=1}^{\infty} \varepsilon^n \frac{d}{dt_5} A_{06}^{(n)}(t_5), \text{ as } \varepsilon \to 0, \varepsilon \in E,
\]

both uniformly for all \( t_5 \in T \). In particular, these imply the following limits:

\[
\lim_{\varepsilon \to 0, \varepsilon \in E} \frac{d}{dt_5} A_{06}(\varepsilon t_5) = \frac{d}{dt_5} A_{05}(t_5) \quad \text{and} \quad \lim_{\varepsilon \to 0, \varepsilon \in E} \frac{d}{dt_5} A_{16}(\varepsilon t_5) = \frac{d}{dt_5} A_{15}(t_5), \tag{4.51}
\]

both for all \( t_5 \in T \).

By the first assumption in the linear system of \( P_{VI} \), we have,

\[
A_{16} = -\frac{\theta_\infty}{2} \sigma_3 - (A_{06} + A_{16}),
\]

using (4.48) and the asymptotic behaviors (4.49)-(4.50) of \( A_{06}(t_6) \) and \( A_{16}(t_6) \), we therefore have,

\[
A_{16}(\varepsilon t_5) \sim -\frac{\sigma_3}{2\varepsilon} - \left( \frac{\theta_\infty}{2} \sigma_3 + A_{05}(t_5) + A_{15}(t_5) \right) + \sum_{n=1}^{\infty} \varepsilon^n A_{16}^{(n)}(t_5), \text{ as } \varepsilon \to 0, \varepsilon \in E,
\]

(4.52)

uniformly for all \( t_5 \in T \), where \( A_{16}^{(n)} \equiv -(A_{06}^{(n)} + A_{16}^{(n)}) \). In particular, we see that,

\[
\lim_{\varepsilon \to 0 \atop \varepsilon \in E} \varepsilon A_{16}(\varepsilon t_5)|_{t_5 \in T} = -\frac{\sigma_3}{2}.
\]

The following proposition establishes how the above substitutions (4.47)-(4.48) and our Main Assumption 1 produce a limit passage from the auxiliary linear system for \( P_{VI} \) to that of \( P_V \) by confluencing the simple poles \( \lambda_6 = 1, \infty \) and producing an irregular singularity at \( \lambda_5 = \infty \).

**Proposition 4.1.** Under the substitutions (4.47) - (4.48) and our Main Assumption
1, we have,
\[
\lim_{\varepsilon \to 0} \frac{\partial \lambda_6}{\partial \lambda_5} \left( \frac{A_{06}(t_6)}{\lambda_6} + \frac{A_{16}(t_6)}{\lambda_6 - t_6} + \frac{A_{16}(t_6)}{\lambda_6 - 1} \right) = \left( \frac{\sigma_3}{2} + \frac{A_{05}(t_5)}{\lambda_5} + \frac{A_{15}(t_5)}{\lambda_5 - t_5} \right),
\]
and,
\[
\lim_{\varepsilon \to 0} \frac{\partial t_6 A_{16}(t_6)}{\partial t_5 \lambda_6 - t_6} = \frac{A_{15}(t_5)}{\lambda_5 - t_5},
\]
with \( \varepsilon \in E \) and \( t_5 \in T \).

**Proof.** Let \( \varepsilon \in E \) and \( t_5 \in T \), as in Main Assumption 1. For the first limit,
\[
\lim_{\varepsilon \to 0} \frac{\partial \lambda_6}{\partial \lambda_5} \left( \frac{A_{06}(t_6)}{\lambda_6} + \frac{A_{16}(t_6)}{\lambda_6 - t_6} + \frac{A_{16}(t_6)}{\lambda_6 - 1} \right) = \lim_{\varepsilon \to 0} \varepsilon \left( \frac{A_{06}(\varepsilon t_5)}{\varepsilon \lambda_5} + \frac{A_{16}(\varepsilon t_5)}{\varepsilon \lambda_5 - \varepsilon t_5} + \frac{A_{16}(\varepsilon t_5)}{\varepsilon \lambda_5 - 1} \right),
\]
\[
= \left( \frac{A_{05}(t_5)}{\lambda_5} + \frac{A_{15}(t_5)}{\lambda_5 - t_5} + \frac{\sigma_3}{2} \right),
\]
as required. For the second limit,
\[
\lim_{\varepsilon \to 0} \varepsilon A_{06}(t_6) = \lim_{\varepsilon \to 0} \varepsilon \left( \frac{A_{16}(\varepsilon t_5)}{\varepsilon \lambda_5 - \varepsilon t_5} \right) = \frac{A_{15}(t_5)}{\lambda_5 - t_5},
\]
as required.

**Understanding the Confluence as a Limit Passage from \( P_{VI} \) to \( P_V \)**

Having shown that the substitutions (4.47)-(4.48) and our Main Assumption 1 provide a limit passage from the system (4.1)-(4.2) to the system (4.23)-(4.24), we now prove that, fundamentally, they also provide a limit passage from the compatibility condition of the \( P_{VI} \) system to that of \( P_V \). Since we have seen how the compatibility conditions of these systems is equivalent to the Painlevé equations \( P_{VI} \) and \( P_V \), this will show that we have a confluence of the Painlevé equations from \( P_{VI} \) to \( P_V \).

**Theorem 4.4.** Let \( Y_6(\lambda_6) \) be a solution of the linear system (4.1)-(4.2) and define \( y_6(t_6) \) as in (4.9), namely,
\[
y_6(t_6) = \frac{t_6 u_{06} z_{06}}{(1 + t_6) u_{06} z_{06} + u_{16} z_{16} + t_6 u_{16} z_{16}},
\]
which solves \( P_{VI} \) (recall Theorem 4.1), then, under the substitutions (4.47)-(4.48) and
Chapter 4  The Sixth and Fifth Painlevé Equations

the conditions of our Main Assumption 1, \( y_6(t_6) \) has the following asymptotic behavior,

\[
y_6(\varepsilon t_5) \sim \varepsilon t_5 \left[ \frac{\theta_\infty - \theta_{06} - \theta_{t5} - 2(t_5 - 2\theta_{06} + \theta_\infty) y_5(t_5)}{2} \right] + (\theta_\infty - 3\theta_{05} + \theta_{t5}) y_5(t_5)^2 + 2t_5 \frac{dy_5}{dt_5} \left[ (y_5(t_5) - 1) \left( \theta_{05} - \theta_\infty + \theta_{t5} \right) + 2(t_5 - \theta_{05} - \theta_{t5}) y_5(t_5) + (\theta_{05} + \theta_\infty + \theta_{t5}) y_5(t_5)^2 - 2t_5 \frac{dy_5}{dt_5} \right]^{-1}
\]

\[
+ \sum_{n=2}^\infty \varepsilon^n y_6^{(n)}(t_5), \quad \text{as } \varepsilon \to 0, \quad \varepsilon \in E,
\]

(4.53)

uniformly for all \( t_5 \in T \), where \( y_5(t_5) \) satisfies \( P_V \) and \( y_6^{(n)}(t_5) \) are certain functions.

**Proof.** There are two parts to this proof. We first explain how to derive expression (4.53) using the prescribed asymptotic behavior in our Main Assumption 1. Secondly, we show that \( y_5(t_5) \) satisfies the \( P_V \) equation by proving that the compatibility equation of the \( P_{VI} \) linear system tends to that of the \( P_V \) linear system. For the first part, by considering the expression (4.3) for \( A_{k6} \) and expressions (4.25) and (4.26) for \( A_{06} \) and \( A_{t5} \), it is easy to see that the prescribed behaviors in (4.49) and (4.50) are achieved if and only if:

\[
z_{06}(\varepsilon t_5) \sim z_5(t_5) + \sum_{n=1}^\infty \varepsilon^n z_6^{(n)}(t_5),
\]

(4.54)

\[
z_{t6}(\varepsilon t_5) \sim -z_5(t_5) - \frac{\theta_{05} + \theta_{t5} + \theta_\infty}{2} + \sum_{n=1}^\infty \varepsilon^n z_6^{(n)}(t_5),
\]

(4.55)

\[
u_{06}(\varepsilon t_5) \sim u_5(t_5)(z_5(t_5) + \theta_{05}) + \sum_{n=1}^\infty \varepsilon^n u_6^{(n)}(t_5),
\]

(4.56)

\[
u_{t6}(\varepsilon t_5) \sim u_5(t_5) y_5(t_5) + \sum_{n=1}^\infty \varepsilon^n u_6^{(n)}(t_5),
\]

(4.57)

all as \( \varepsilon \to 0, \varepsilon \in E \), uniformly for \( t_5 \in T \), where \( z_6^{(n)}(t_5) \) and \( u_6^{(n)}(t_5) \) are certain functions. Moreover, from the first assumption in the linear problem of \( P_{VI} \), or more specifically from the equations in (4.4), we deduce the following asymptotic behavior
for \( z_{16}(t_6) \) and \( u_{16}(t_6) \):

\[
z_{16}(\varepsilon t_5) \sim -\frac{\varepsilon}{4y_5(t_5)} (\theta_{05} - \theta_{15} + \theta_{\infty5} - 2z_5(t_5)(y_5(t_5) - 1)) \times \ldots
\]

\[
(y_5(t_5)(\theta_{05} + \theta_{15} + \theta_{\infty5} + 2z_5(t_5)) - 2(z_5(t_5) + \theta_{05})) + \sum_{n=2}^{\infty} \varepsilon^n z^{(n)}_{16}(t_5), \quad (4.58)
\]

\[
u_{16}(\varepsilon t_5) \sim -\frac{2u_5(t_5)y_5(t_5)}{\varepsilon(\theta_{05} - \theta_{15} + \theta_{\infty5} - 2z_5(t_5)(y_5(t_5) - 1))} + \sum_{n=0}^{\infty} \varepsilon^n u^{(n)}_{16}(t_5), \quad (4.59)
\]

both as \( \varepsilon \to 0, \varepsilon \in E \), uniformly for \( t_5 \in T \), where \( z^{(n)}_{16}(t_5) \) and \( u^{(n)}_{16}(t_5) \) are certain functions. We find the desired expression (4.53) by substituting these asymptotic behaviors (4.54)-(4.59) into (4.9) and using (4.29) to remove \( z_5(t_5) \).

For the second part, we note that the compatibility conditions (4.5)-(4.7) must be satisfied, since \( Y_6(\lambda_6) \) is a solution of (4.1)-(4.2). Starting with (4.5) and remembering to substitute (4.47)-(4.48), we have,

\[
\frac{d}{dt_6} A_{06}(\varepsilon t_5) = t_6^{-1} [A_{06}(\varepsilon t_5), A_{06}(\varepsilon t_5)] \Leftrightarrow \varepsilon^{-1} \frac{d}{dt_5} A_{06}(\varepsilon t_5) = \varepsilon^{-1} t_5^{-1} [A_{16}(\varepsilon t_5), A_{06}(\varepsilon t_5)].
\]

Multiplying through by \( \varepsilon \) and taking the limit \( \varepsilon \to 0, \varepsilon \in E \), recall (4.51), we produce equation (4.27),

\[
A'_{05}(t_5) = t_5^{-1} [A_{15}(t_5), A_{05}(t_5)].
\]

Now looking at (4.6),

\[
\frac{d}{dt_6} A_{16}(\varepsilon t_5) = t_6^{-1} [A_{06}(\varepsilon t_5), A_{16}(\varepsilon t_5)] + (1 - t_6)^{-1} [A_{16}(\varepsilon t_5), A_{16}(\varepsilon t_5)],
\]

\[
\Leftrightarrow \varepsilon^{-1} \frac{d}{dt_5} A_{16}(\varepsilon t_5) = \varepsilon^{-1} t_5^{-1} [A_{06}(\varepsilon t_5), A_{16}(\varepsilon t_5)] - \left[ A_{16}(\varepsilon t_5), \frac{\sigma_3}{2\varepsilon} + A_{06}(\varepsilon t_5) + A_{16}(\varepsilon t_5) \right].
\]

Multiplying through by \( \varepsilon \) and taking the limit \( \varepsilon \to 0, \varepsilon \in E \), recall (4.51), we produce equation (4.28),

\[
A'_{15}(t_5) = \left[ \frac{\sigma_3}{2}, A_{15}(t_5) \right] + t_5^{-1} [A_{05}(t_5), A_{15}(t_5)].
\]
Finally, looking at (4.7),

\[
\frac{d}{dt_6} A_{16}(\varepsilon t_5) = (1 - t_6)^{-1} [A_{16}(\varepsilon t_5), A_{16}(\varepsilon t_5)],
\]

\[
\Leftrightarrow \varepsilon^{-1} \frac{d}{dt_5} \left( -\frac{\sigma_3}{2\varepsilon} - A_{06}(\varepsilon t_5) - A_{06}(\varepsilon t_5) \right)
\]

\[
= \frac{1}{1 - \varepsilon t_5} \left[ -\frac{\sigma_3}{2\varepsilon} - A_{06}(\varepsilon t_5) - A_{06}(\varepsilon t_5), A_{06}(\varepsilon t_5) \right].
\]

Multiplying through by \(\varepsilon\) and taking the limit \(\varepsilon \to 0, \varepsilon \in E\), recall (4.51), produces,

\[
A'_{05}(t_5) + A'_{15}(t_5) = \left[ \frac{\sigma_3}{2}, A_{15}(t_5) \right],
\]

which is consistent with (4.27) and (4.28), as required. This proves that the matrices \(A_{05}(t_5)\) and \(A_{15}(t_5)\), as introduced in our Main Assumption 1, satisfy (4.27)-(4.28) and thus \(y_5(t_5)\) satisfies the P\(V\) equation, by Theorem 4.2.

In the following sections, we will primarily be working with matrices \(A_k, A_05\) and \(A_15\), rather than the functions \(z_{k6}, u_{k6}, z_5, u_5\) and \(y_5\) which parameterise them. While expression (4.58) shows the leading asymptotic behavior of \(z_{16}\), the following lemma shows how to write this in terms of matrices. This particular result will be used in Sections 4.3.1 and 4.3.2.

**Lemma 4.1.** The diagonal part of the matrix \(A^{(1)}_{16}\), that is the coefficient of \(\varepsilon\) in the asymptotic expansion of \(A_{16}\) as in (4.52), is equal to,

\[
\text{diag} \left( A^{(1)}_{16} \right) = \left( \frac{\theta_{05}}{2} \sigma_3 + A_{05} + A_{15} \right)^2 \sigma_3.
\]

**Proof.** Using expression (4.3) for \(A_{16}\), we see that the diagonal part of its coefficient of \(\varepsilon\) under the substitutions (4.48) and the conditions of our Main Assumption 1 is equal to the coefficient of \(\varepsilon\) in \(z_{16}\). By direct computation using the matrices (4.25) and (4.26), we find the above expression is in agreement with (4.58).

We note that the sector \(E\) in our Main Assumption 1 may not be unique. Proposition 4.1 demonstrates our confluence procedure on the formal level of the equations. Throughout the following Sections 4.2 - 4.4 we need to be careful in which way we are taking \(\varepsilon\) to zero, for example it would be inconvenient for us if \(\varepsilon\) spiralled towards zero.
Similar to the hypergeometric chapter, we will consider two limits along the fixed rays 
\[ \arg(\varepsilon) = \pm \frac{\pi}{2}, \]
in each case we assume that these directions are contained inside some sector \( E \) satisfying (4.49) and (4.50). For brevity, we state once and for all it is to be understood that \( \varepsilon \in E \) and \( t_5 \in T \) for the remainder of this chapter, where \( E \) and \( T \) are chosen to satisfy (4.49) and (4.50).

### 4.2 From Fuchsian Singularities to Fuchsian Singularities

Inspired by the procedure used by Kitaev in [Kit3], we now examine the behavior of solutions at the Fuchsian singular points under the confluence procedure. The contents of this section will only be used to understand how to produce the monodromy matrices around the surviving Fuchsian singularities. This section will not be necessary to prove our main results concerned with producing the Stokes’ matrices of the \( P_V \) linear system.

Under the change of variables (4.47), the Fuchsian singularities \( \lambda_6 = 0 \) and \( t_6 \) of (4.1) are mapped to \( \lambda_5 = 0 \) and \( t_5 \) respectively, which clearly do not depend on \( \varepsilon \). Since the confluence procedure does not interfere with the nature of these Fuchsian singularities, we will deal with solutions around these points first. In this section we will show how to express the fundamental solutions \( Y_5^{(0)}(\lambda_5) \) and \( Y_5^{(t)}(\lambda_5) \) in terms of the fundamental solutions \( Y_6^{(0)}(\lambda_6) \) and \( Y_6^{(t)}(\lambda_6) \) respectively.

Let \( R_{k6} \) and \( R_{k5}, k = 0, t, \) be the diagonalising matrices described in the assumptions of the \( P_{VI} \) and \( P_V \) linear systems. We now make fixed choices once and for all of the matrices \( R_{66} \) and \( R_{65} \) such that, under the substitutions (4.47)-(4.48) and our Main Assumption 1, they have the following asymptotic behavior,

\[
R_{k6}(\varepsilon t_5) \sim R_{k5}(t_5) + \sum_{n=1}^{\infty} \varepsilon^n R_{k6}^{(n)}(t_5), \quad k = 0, t, \quad \text{as } \varepsilon \to 0,
\]

where \( R_{k6}^{(n)}(t_5) \) are some sequences of matrices. We note that it is always possible to
make a choice of such diagonalising matrices because,

\[ R_k^6 \theta_k^6 \sigma_3 R_k^{-1} = A_k^6 \sim A_k^5 + \sum_{n=1}^{\infty} \varepsilon^n A_k^{(n)} \] as \( \varepsilon \to 0 \),

\[ = R_k^5 \theta_k^5 \sigma_3 R_k^{-1} + \sum_{n=1}^{\infty} \varepsilon^n A_k^{(n)}. \]

In other words, since the matrix \( A_k^6 \) has leading term \( A_k^5 \), \( k = 0, t \), there exists a re-scaling of the eigenvectors of \( A_k^6 \) such that their leading terms are the eigenvectors of \( A_k^5 \), this is a simple but cumbersome exercise in linear algebra which we omit. This can also be confirmed by computing explicit formulae for the matrices \( R_k^6 \) and substituting (4.54)-(4.59), the result is that the leading term is a diagonalising matrix for \( A_k^5 \).

### 4.2.1 Obtaining \( Y_5^{(0)}(\lambda_5) \) from \( Y_6^{(0)}(\lambda_6) \)

**Lemma 4.2.** Let \( G_{n,0} \) and \( H_{n,0} \) be defined by (4.14) and (4.34) respectively. Under the substitutions (4.47)-(4.48) and our Main Assumption 1 we have,

\[ \lim_{\varepsilon \to 0} \varepsilon^n G_{n,0} = H_{n,0}, \]

for all \( n \).

**Proof.** We will prove this lemma by proving the broader result that,

\[ G_{n,0} \sim \varepsilon^{-n} \left( H_{n,0} + \varepsilon G_{n,0}^{(n+1)} + \ldots \right), \]

for some sequence of matrices \( G_{n,0}^{(>n)} \), for all \( n \). The case \( n = 0 \) is true by definition, recall \( G_{n,0} := I =: H_{n,0} \). We now assume,

\[ G_{k,0} \sim \varepsilon^{-k} \left( H_{k,0} + \varepsilon G_{k,0}^{(k+1)} + \ldots \right), \quad (4.61) \]

for some sequence of matrices \( G_{k,0}^{(>k)} \), for \( k = 0, \ldots, (n-1) \) and \( n \geq 1 \). We examine the recursion equation (4.14), which determines the matrix \( G_{n,0}, n \geq 1 \). Under the substitutions (4.47)-(4.48), our Main Assumption 1 and the inductive assumption (4.61), we
have,
\[
n G_{n,0} + \left[ G_{n,0}, \frac{\theta_{05}}{2} \sigma_3 \right] = -\sum_{l=0}^{n-1} R_{06}^{-1} \left( A_{16} + A_{46} (\varepsilon t_5)^{l-n} \right) R_{06} G_{l,0},
\]
\[
\sim -\sum_{l=0}^{n-1} (R_{05}^{-1} + \ldots) \left( -\frac{\sigma_3}{2 \varepsilon} + A_{16}^{(0)} + \ldots + (A_{45} + \varepsilon A_{46}^{(1)} + \ldots) (\varepsilon t_5)^{l-n} \right) \]
\[
(R_{05} + \ldots) \varepsilon^{-l} \left( H_{l,0} + \varepsilon G_{l,0}^{(l+1)} + \ldots \right), \quad \text{as } \varepsilon \to 0.
\]

The right hand side of this asymptotic has the following leading term,
\[
\varepsilon^{-n} \left( R_{05}^{-1} \frac{\sigma_3}{2} R_{05} H_{n-1,0} - R_{05}^{-1} A_{45} R_{05} \sum_{l=0}^{n-1} t_5^{l-n} H_{l,0} \right)
\]
\[
= \varepsilon^{-n} \left( n H_{n,0} + \left[ H_{n,0}, \frac{\theta_{05}}{2} \sigma_3 \right] \right) \quad \text{since this is (4.34)}.
\]

Comparing the diagonal elements and off-diagonal elements of the asymptotic,
\[
n G_{n,0} + \left[ G_{n,0}, \frac{\theta_{05}}{2} \sigma_3 \right] \sim \varepsilon^{-n} \left( n H_{n,0} + \left[ H_{n,0}, \frac{\theta_{05}}{2} \sigma_3 \right] \right) + \ldots, \quad \text{as } \varepsilon \to 0,
\]
we deduce that (4.61) holds for \( k = n \) and the desired result is proved. \( \square \)

We define,
\[
\omega_{05}^{\pm}(\varepsilon) = \left\{ \lambda_5 : |\lambda_5| < \frac{\rho_{06}}{|\varepsilon|}, -\pi \pm \frac{\pi}{2} \leq \arg(\lambda_5) < \pi \pm \frac{\pi}{2} \right\},
\]
so that, for \( \arg(\varepsilon) = \pm \frac{\pi}{2} \), \( \lambda_6 \in \Omega_{06} \iff \lambda_5 \in \omega_{05}^{\pm}(\varepsilon) \). Since our confluence procedure (4.47) rescales \( \lambda_6 \) by a factor of \( \varepsilon \) we must ensure that the domains on which our local fundamental solutions are defined do not vanish in the limit \( \varepsilon \to 0 \). We now choose \( \rho_{06} \) such that \( \lim_{\varepsilon \to 0} \frac{\rho_{06}}{|\varepsilon|} = \rho_{05} \). For example, if we had chose the maximal radius \( \rho_{06} = \min(|t_6|, 1) = \min(|\varepsilon t_5|, 1) \) then we would find \( \lim_{\varepsilon \to 0} \frac{\rho_{06}}{|\varepsilon|} = |t_5| \), which is the maximal radius \( \rho_{05} \) we could choose for the domain \( \Omega_{05}^{\pm} \) of our fundamental solution \( Y_5^{(0)}(\lambda_5) \).

In general, note that if \( \lambda_5 \in \omega_{05}^{\pm}(\varepsilon) \) for all \( |\varepsilon| \) sufficiently small, then \( \lambda_5 \in \Omega_{05}^{\pm} \). We have the following theorem.

**Theorem 4.5.** Let \( Y_6^{(0)}(\lambda_6) \) and \( Y_5^{(0)}(\lambda_5) \) be the local fundamental solutions given by
(4.10) and (4.32) respectively. For \( \text{arg}(\varepsilon) = \pm \frac{\pi}{2} \), we have the following limits,

\[
\lim_{\varepsilon \to 0} Y_6^{(0)}(\varepsilon \lambda_5) \varepsilon^{-\frac{\theta_6}{\lambda_5} \sigma_3} = Y_5^{(0)}(\lambda_5), \quad \lambda_5 \in \Omega_{05}^\pm.
\]

**Proof.** We have already noted in the paragraph before this theorem how the domain \( \omega_{05}^\pm(\varepsilon) \) tends to the domain \( \Omega_{05}^\pm(\varepsilon) \). We know that the series,

\[
\sum_{n=0}^{\infty} G_{n,0}(\varepsilon \lambda_5)^n, \quad \lambda_5 \in \omega_{05}^\pm(\varepsilon),
\]

converges. Since the radius of convergence does not diminish as \( \varepsilon \to 0 \), the convergence of this series is uniform in \( \varepsilon \). Moreover, since we also know that the series,

\[
\sum_{n=0}^{\infty} H_{n,0} \lambda_5^n, \quad \lambda_5 \in \Omega_{05}^\pm,
\]

converges, we use Lemma 4.2 to deduce,

\[
\lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} G_{n,0}(\varepsilon \lambda_5)^n = \sum_{n=0}^{\infty} \varepsilon^n G_{n,0} \lambda_5^n = \sum_{n=0}^{\infty} H_{n,0} \lambda_5^n, \quad \lambda_5 \in \Omega_{05}^\pm.
\]

We recall that this reasoning was already explicitly demonstrated in the proof of Lemma 3.9 in the case of Gauss’ \(_2F_1\) hypergeometric series. We therefore have,

\[
\lim_{\varepsilon \to 0} Y_6^{(0)}(\varepsilon \lambda_5) \varepsilon^{-\frac{\theta_6}{\lambda_5} \sigma_3} = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} G_{n,0}(\varepsilon \lambda_5)^n (\varepsilon \lambda_5)^{-\frac{\theta_6}{\lambda_5} \sigma_3} \varepsilon^{-\frac{\theta_6}{\lambda_5} \sigma_3},
\]

\[
= \sum_{n=0}^{\infty} H_{n,0} \lambda_5^n \lambda_5^{-\frac{\theta_6}{\lambda_5} \sigma_3} = Y_5^{(0)}(\lambda_5), \quad \lambda_5 \in \Omega_{05}^\pm,
\]

and the result is proved.

### 4.2.2 Obtaining \( Y_5^{(t)}(\lambda_5) \) from \( Y_6^{(t)}(\lambda_6) \)

**Lemma 4.3.** Let \( G_{n,t} \) and \( H_{n,t} \) be defined by (4.15) and (4.35) respectively. Under the substitutions (4.47)-(4.48) and our Main Assumption 1 we have,

\[
\lim_{\varepsilon \to 0} \varepsilon^n G_{n,t} = H_{n,t},
\]
for all \( n \).

Proof. We will prove this lemma by proving the broader result that,

\[
G_{n,t} \sim \varepsilon^{-n} \left( H_{n,t} + \varepsilon G_{n,t}^{(n+1)} + \ldots \right),
\]

for some sequence of matrices \( G_{n,t}^{(n)} \), for all \( n \). The case \( n = 0 \) is true by definition, recall \( G_{n,t} := I := H_{n,t} \). We now assume,

\[
G_{k,t} \sim \varepsilon^{-k} \left( H_{k,t} + \varepsilon G_{k,t}^{(k+1)} + \ldots \right),
\]

(4.63)

for some sequence of matrices \( G_{k,t}^{(k)} \), for \( k = 0, \ldots, (n-1) \) and \( n \geq 1 \). We examine the recursion relation (4.15), which determines the matrix \( G_{n,t}, n \geq 1 \). Under the substitutions (4.47)-(4.48), our Main Assumption 1 and the inductive assumption (4.63) we have,

\[
nG_{n,t} + \left[ G_{n,t}, \frac{\theta_{t5}}{2\sigma_3} \right] = -\sum_{l=0}^{n-1} R_{t6}^{-1} \left( A_{06}(-t_6)^{l-n} + A_{16}(1-t_6)^{l-n} \right) R_{t6} G_{l,t},
\]

\[
\sim -\sum_{l=0}^{n-1} \left( R_{t5}^{-1} + \ldots \right) \left[ A_{05} + \varepsilon A_{06}^{(1)} + \ldots \right] (-\varepsilon t_5)^{l-n}
\]

\[
+ \left( -\frac{\sigma_3}{2\varepsilon} + A_{16}^{(0)} + \ldots \right) (1 - \varepsilon t_5)^{l-n}
\]

\[
( R_{t6} + \ldots ) \varepsilon^{-l} \left( H_{l,t} + \varepsilon G_{l,t}^{(l+1)} + \ldots \right), \text{ as } \varepsilon \to 0.
\]

The right hand side of this asymptotic relation has the following leading term,

\[
\varepsilon^{-n} \left( R_{t5}^{-1} \frac{\sigma_3}{2} R_{t5} H_{n-1,t} - R_{t5}^{-1} A_{05} R_{t5} \sum_{l=0}^{n-1} (-t_5)^{l-n} H_{l,t} \right)
\]

\[
= \varepsilon^{-n} \left( nH_{n,t} + \left[ H_{n,t}, \frac{\theta_{t5}}{2\sigma_3} \right] \right) \text{ since this is (4.35)}.
\]

Comparing the diagonal and off-diagonal elements of the following asymptotic relation,

\[
nG_{n,t} + \left[ G_{n,t}, \frac{\theta_{t5}}{2\sigma_3} \right] \sim \varepsilon^{-n} \left( nH_{n,t} + \left[ H_{n,t}, \frac{\theta_{t5}}{2\sigma_3} \right] \right) + \ldots, \text{ as } \varepsilon \to 0,
\]

we deduce that (4.63) holds for \( k = n \) and the desired result is proved. \( \square \)
We define, 
\[ \omega_{t_5}^\pm(\varepsilon) = \left\{ \lambda_5 : |\lambda_5 - t_5| < \frac{\rho t_6}{|\varepsilon|}, -\pi \mp \frac{\pi}{2} \leq \arg(\lambda_5 - t_5) < \pi \mp \frac{\pi}{2} \right\}, \]
so that, for \( \arg(\varepsilon) = \pm \frac{\pi}{2} \), \( \lambda_6 \in \Omega t_6 \iff \lambda_5 \in \omega_{t_5}^\pm(\varepsilon) \). We now choose \( \rho t_6 \) such that \( \lim_{\varepsilon \to 0} \rho t_6 |\varepsilon| = \rho_0 t_5 \). For example, if we had chose the maximal radius \( \rho t_6 = \min(|t_6|, |t_6 - 1|) = \min(|\varepsilon t_5|, |\varepsilon t_5 - 1|) \) then we would find \( \lim_{\varepsilon \to 0} \rho t_6 = |t_5| \), which is the maximal radius \( \rho t_5 \) we could choose for the domain \( \Omega_{t_5}^\pm \) of our fundamental solution \( Y_{t_5}^\pm(\lambda_5) \). In general, note that if \( \lambda_5 \in \omega_{t_5}^\pm(\varepsilon) \) for all \( |\varepsilon| \) sufficiently small, then \( \lambda_5 \in \Omega_{t_5}^\pm \). We have the following theorem.

**Theorem 4.6.** Let \( Y_{t_6}^{(i)}(\lambda_6) \) and \( Y_{t_5}^{(i)}(\lambda_5) \) be the local fundamental solutions given by (4.11) and (4.33) respectively. For \( \arg(\varepsilon) = \pm \frac{\pi}{2} \), we have the following limits,

\[
\lim_{\varepsilon \to 0} Y_{t_6}^{(i)}(\varepsilon \lambda_5) e^{-\frac{\varepsilon \lambda_5}{2}} = Y_{t_5}^{(i)}(\lambda_5), \quad \lambda_5 \in \Omega_{t_5}^\pm. \tag{4.64}
\]

**Proof.** We have already noted in the paragraph before this theorem how the domain \( \omega_{t_5}^\pm(\varepsilon) \) tends to the domain \( \Omega_{t_5}^\pm(\varepsilon) \). We know that the series,

\[
\sum_{n=0}^\infty G_{n,t}(\varepsilon \lambda_5 - \varepsilon t_5)^n, \quad \lambda_5 \in \omega_{t_5}^\pm(\varepsilon),
\]
converges. Since the radius of convergence does not diminish as \( \varepsilon \to 0 \), the convergence of this series is uniform in \( \varepsilon \). Moreover, since we also know that the series,

\[
\sum_{n=0}^\infty H_{n,t}(\lambda_5 - t_5)^n, \quad \lambda_5 \in \Omega_{t_5}^\pm,
\]
converges, we use Lemma 4.3 to deduce,

\[
\lim_{\varepsilon \to 0} \sum_{\lambda_5 \in \omega_{t_5}^\pm(\varepsilon)} G_{n,t}(\varepsilon \lambda_5 - \varepsilon t_5)^n = \sum_{n=0}^\infty \left( \lim_{\varepsilon \to 0} \varepsilon^n G_{n,t}(\lambda_5 - t_5)^n \right) = \sum_{n=0}^\infty H_{n,t}(\lambda_5 - t_5)^n,
\]
for $\lambda_5 \in \Omega_{t5}^\pm$. We therefore have,

$$\lim_{\varepsilon \to 0} Y^{(t)}_6(\varepsilon \lambda_5) \varepsilon^{-\frac{\theta_{\infty_5}}{2} \sigma_3} = \lim_{\varepsilon \to 0} R_{t6} \sum_{n=0}^{\infty} G_{n,t}(\varepsilon \lambda_5 - \varepsilon t_5)^n (\varepsilon \lambda_5 - \varepsilon t_5)^{\frac{\theta_{\infty_5}}{2} \sigma_3} \varepsilon^{-\frac{\theta_{\infty_5}}{2} \sigma_3},$$

$$= R_{t6} \sum_{n=0}^{\infty} H_{n,t}(\lambda_5 - t_5)^n (\lambda_5 - t_5)^{\frac{\theta_{\infty_5}}{2} \sigma_3} = Y^{(t)}_5(\lambda_5), \quad \lambda_5 \in \Omega_{t5}^\pm,$$

and the result is proved.

\[\square\]

### 4.3 From Fuchsian Singularities to an Irregular One

Under the changes of variables (4.47), the Fuchsian singularities $\lambda_6 = \infty$ and 1 of (4.1) are mapped to $\lambda_5 = \infty$ and $\varepsilon^{-1}$ respectively. As $\varepsilon \to 0$ the two simple poles merge and, as seen from Proposition 4.1, a double pole at $\lambda_5 = \infty$ is produced in the equation. In this section, we first show how to produce the formal series solution $Y^{(\infty)}_{6,f}(\lambda_5)$ by rewriting the solutions $Y^{(\infty)}_6(\lambda_6)$ and $Y^{(1)}_6(\lambda_6)$ and taking term-by-term limits, this is the subject of Subsections 4.3.1 and 4.3.2. In Subsection 4.3.3, we then use Glutsyuk’s Theorem 2.5 to produce the fundamental solutions $Y^{(\infty,k)}_5(\lambda_5)$ by taking limits of solutions in sectors. At the end of this section we will prove our first main theorem for this chapter, which establishes certain limits which the matrices of Glutsyuk’s theorem must satisfy.

In the following, we use the notation $\text{diag}(\ldots)$ and $\text{off}(\ldots)$ to mean,

$${\text{diag}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad {\text{off}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$ 

An important case to note is that, from the expressions (4.25) and (4.26) for $A_{05}$ and $A_{t5}$,

$$\text{diag}(A_{05} + A_{t5}) = -\frac{\theta_{\infty_5}}{2} \sigma_3. \quad (4.65)$$

We also use the notation $m^{(k)}$ to mean the coefficient of $\varepsilon^k$ in the formal series expansion
of the matrix \( m \) around \( \varepsilon = 0 \), for example, from (4.52),

\[
A_{16}^{(-1)} = \frac{\sigma_3}{2\varepsilon} \quad \text{and} \quad A_{16}^{(0)} = -\left(\frac{\theta_{\infty}}{2} \sigma_3 + A_{05} + A_{t5}\right),
\]

in particular, from (4.65), this means \( \text{diag}(A_{16}^{(0)}) = 0 \).

### 4.3.1 Taking a Term-By-Term Limit of the Solution \( Y_6^{(\infty)}(\lambda_6) \)

We rewrite the solution \( Y_6^{(\infty)}(\lambda_6) \), as given in (4.13), in the following way,

\[
Y_6^{(\infty)}(\lambda_6) = \left(\sum_{n=0}^{\infty} \hat{G}_{n,\infty} \lambda_6^n\right) \lambda_6^{-\frac{\theta_{16} + \theta_{46}}{2} \sigma_3} (1 - \lambda_6)^{\frac{\theta_{16}}{2} \sigma_3}, \quad \lambda_6 \in \hat{\Omega}_{\infty 6},
\]

where the new domain \( \hat{\Omega}_{\infty 6} \) is defined as,

\[
\hat{\Omega}_{\infty 6} = \{ \lambda_6 : |\lambda_6| > \rho_{\infty 6}, \; 0 \leq \arg(\lambda_6) < 2\pi, \; -\pi \leq \arg(1 - \lambda_6) < \pi \},
\]

\( \hat{G}_{0,\infty} := I \) and the coefficients \( \hat{G}_{n,\infty}, \; n \geq 1 \), are determined by the recursion relation given in Lemma 4.4 below. We note that this new form of the solution \( Y_6^{(\infty)}(\lambda_6) \) is equivalent to that given in (4.13) on the domain \( \Omega_{\infty 6} \cap \hat{\Omega}_{\infty 6} \).

#### Lemma 4.4

Setting \( \hat{G}_{0,\infty} := I \), the general formula for \( \hat{G}_{n,\infty} \) with \( n \geq 1 \) is given below,

\[
-n \hat{G}_{n,\infty} + \left[ \hat{G}_{n,\infty}, -\frac{\theta_{\infty}}{\sigma_3} \right] = \sum_{l=0}^{n-1} \left( A_{16} + t_6^{n-l} A_{t6} \right) \hat{G}_{l,\infty} - \sum_{l=0}^{n-1} \hat{G}_{l,\infty} \frac{\theta_{16}}{2} \sigma_3.
\]

#### Proof

Substituting the new form (4.66) of \( Y_6^{(\infty)}(\lambda_6) \) into equation (4.1) gives,

\[
-n \hat{G}_{n,\infty} \lambda_6^{-n-1} + \left( \sum_{n=0}^{\infty} \hat{G}_{n,\infty} \lambda_6^n \right) \left[ -\frac{\theta_{\infty}}{2\lambda_6} \sigma_3 + \frac{\theta_{16}}{2} \sigma_3 \sum_{n=2}^{\infty} \lambda_6^{-n} \right] = \left( -\frac{\theta_{\infty}}{2\lambda_6} \sigma_3 + \sum_{n=2}^{\infty} \lambda_6^{-n} \left( A_{16} t_6^{n-1} + A_{t6} \right) \right) \sum_{n=0}^{\infty} \hat{G}_{n,\infty} \lambda_6^{-n}.
\]

The term inside the \( [ ] \) brackets on the left hand side of the equals sign comes from differentiating the term \( \lambda_6^{-\frac{\theta_{\infty} + \theta_{16}}{2} \sigma_3} (1 - \lambda_6)^{\frac{\theta_{16}}{2} \sigma_3} \) and expanding in powers of \( \lambda_6^{-1} \). The term inside the \( ( ) \) brackets on the right hand side of the equals sign is equivalently
\[
\left( \frac{A_{05}}{\lambda_6} + \frac{A_{16}}{\lambda_6 - \lambda_6} + \frac{A_{16}}{\lambda_6 - 1} \right) \text{ after expanding powers in powers of } \lambda_6^{-1}. \]

For \( n \geq 1 \), equating the coefficients of \( \lambda_6^{-n-1} \) in (4.68) produces the desired formula (4.67).

**Remark 4.9.** We remark that the final term \(-\sum_{l=0}^{n-1} \hat{G}_{n,\infty}^{\hat{G}_{n,\infty}} \theta_{\infty}^{\frac{\theta_{\infty}}{2}} \sigma_3 \) in the recursive formula (4.67) is the only difference with the recursive formula (4.17) for the coefficients \( G_{n,\infty} \).

**Remark 4.10.** We note that the extra condition imposed on \( \arg(1 - \lambda_6) \) in \( \hat{\Omega}_{\infty}^{\hat{\Omega}_{\infty}} \) is only necessary to deal with the term \( (1 - \lambda_6)^{\frac{\theta_{\infty}}{2}} \sigma_3 \). After making the substitutions (4.47)-(4.48) and taking the limit \( \varepsilon \to 0 \), the condition \( |\arg(1 - \lambda_6)| < \pi \) does not play a role because the term \( (1 - \lambda_6)^{\frac{\theta_{\infty}}{2}} \sigma_3 \) tends to a single-valued function of \( \lambda_5 \), namely,

\[
\lim_{\varepsilon \to 0} (1 - \lambda_6)^{\frac{\theta_{\infty}}{2}} \sigma_3 = \lim_{\varepsilon \to 0} (1 - \varepsilon \lambda_5)^{-\frac{\theta_{\infty}}{2} \sigma_3} = e^{\frac{\lambda_5}{2} \sigma_3}. \tag{4.69}
\]

This shows how to produce the exponential behavior found in \( Y_{5,f}^{(\infty)}(\lambda_5) \), this is analogous to (3.69) in the case of the hypergeometric equations. Moreover, under the substitutions (4.47), we have,

\[
\lambda_6^{\frac{\theta_{\infty}}{2} \sigma_3} = e^{-\frac{\theta_{\infty}}{2} \sigma_3} \lambda_5^{\frac{\theta_{\infty}}{2} \sigma_3},
\]

which produces the term \( \lambda_5^{\frac{\theta_{\infty}}{2} \sigma_3} \) as found in \( Y_{5,f}^{(\infty)}(\lambda_5) \). The term \( \varepsilon^{-\frac{\theta_{\infty}}{2} \sigma_3} \) will be dealt with in our Main Theorem 3.

**Lemma 4.5.** Let \( \hat{G}_{n,\infty} \) and \( H_{n,\infty} \) be defined by (4.67) and (4.38) respectively. Under the substitutions (4.47) - (4.48) and our Main Assumption 1,

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \hat{G}_{1,\infty} = H_{1,\infty}, \tag{4.70}
\]

and, for \( n \geq 2 \), there exists a choice of \( \hat{G}_{n-1,\infty}^{(n)} \) such that,

\[
\lim_{\varepsilon \to 0} \varepsilon^{-n} \hat{G}_{n,\infty} = H_{n,\infty}. \tag{4.71}
\]

**Proof.** We first show that (4.70) is correct by explicit computation. From the recursive relation (4.38) with \( n = 1 \), we have,

\[
\left[ H_{1,\infty}, \frac{\sigma_3}{2} \right] = \frac{\theta_{\infty}}{2} \sigma_3 + A_{05} + A_{15} \iff (H_{1,\infty}) = \left( \frac{\theta_{\infty}}{2} \sigma_3 + A_{05} + A_{15} \right) \sigma_3. \tag{4.72}
\]
Now, looking at the diagonal part of (4.38) with $n = 2$,

$$
0 = \text{diag} \left( H_{1,\infty} + H_{1,\infty} \frac{\theta_{\infty 5}}{2} \sigma_3 + (A_{05} + A_{t5}) H_{1,\infty} + t_5 A_{t5} \right). 
$$

(4.73)

Note that,

$$
\text{diag} \left( H_{1,\infty} \frac{\theta_{\infty 5}}{2} \sigma_3 + (A_{05} + A_{t5}) H_{1,\infty} \right) = \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right) \text{off} (H_{1,\infty}) \text{ by (4.65)},
$$

$$
= \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right)^2 \sigma_3 \text{ by (4.72)}.
$$

Hence, (4.73) becomes,

$$
\text{diag} (H_{1,\infty}) = - \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right)^2 \sigma_3 - \text{diag} (t_5 A_{t5}). 
$$

(4.74)

We look at equation (4.67) with $n = 1$ which determines $\hat{G}_{1,\infty}$,

$$
- \hat{G}_{1,\infty} + \left[ \hat{G}_{1,\infty}, - \frac{\theta_{\infty 6}}{2} \sigma_3 \right] = A_{16} + t_6 A_{16} - \frac{\theta_{16}}{2} \sigma_3.
$$

We now multiply through by $\varepsilon^{-1}$ and make the substitutions (4.47)-(4.48) along with our Main Assumption 1 to find,

$$
- \varepsilon^{-1} \hat{G}_{1,\infty} + \left[ \varepsilon^{-1} \hat{G}_{1,\infty}, - \frac{\theta_{\infty 5}}{2} \sigma_3 \right] + \frac{1}{\varepsilon} \left[ \varepsilon^{-1} \hat{G}_{1,\infty}, - \frac{\sigma_3}{2} \right]
$$

$$
\sim \varepsilon^{-1} \left( \varepsilon^{-1} A_{16}^{(-1)} + A_{16}^{(0)} + \varepsilon A_{16}^{(1)} + \ldots \right) + t_6 A_{16} + \frac{\sigma Y}{2 \varepsilon^2}, \quad \text{as } \varepsilon \to 0,
$$

$$
\equiv - \varepsilon^{-1} \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right) + A_{16}^{(1)} + t_5 A_{t5} + \varepsilon \left( A_{16}^{(2)} + A_{16}^{(1)} \right) + \ldots, \quad (4.75)
$$

where the terms indicated cancel with each other due to $A_{16}^{(-1)} = - \frac{\sigma_3}{2}$. Notice that the right hand side of this asymptotic at order $\varepsilon^{-1}$ is off-diagonal only, recall (4.65). From this, and since the left hand side of (4.75) has the following form,

$$
\varepsilon^{-1} \begin{pmatrix}
- (\hat{G}_{1,\infty})_{1,1} & (\hat{G}_{1,\infty})_{1,2} (\varepsilon^{-1} + \theta_{\infty 5} - n) \\
-(\hat{G}_{1,\infty})_{2,1} (\varepsilon^{-1} \theta_{\infty 5} + n) & - (\hat{G}_{1,\infty})_{2,2}
\end{pmatrix},
$$

Page 129 of 178
where \( \hat{G}_{1,\infty} \) and \( \hat{G}_{1,\infty}^{(2)} \) are some matrices. For the diagonal part of \( \hat{G}_{1,\infty}^{(1)} \), we look at the diagonal part of the right hand side of (4.75) at the next order, \( \varepsilon^0 \),

\[
\text{diag} \left( -\hat{G}_{1,\infty}^{(1)} \right) = \text{diag} \left( A_{16}^{(1)} + t_5 A_{t5} \right),
\]

\[
\equiv \text{diag} \left( \left( \frac{\theta_{\infty5}}{2} \sigma_3 + A_{05} + A_{t5} \right)^2 \sigma_3 + t_5 A_{t5} \right) \text{ by (4.60)},
\]

from which we conclude that \( \text{diag}(\hat{G}_{1,\infty}^{(1)}) = \text{diag}(H_{1,\infty}) \), recall (4.74). This proves (4.70).

At the induction step, we now assume that,

\[
\hat{G}_{k,\infty} \sim \varepsilon^k \left( H_{k,\infty} + \varepsilon \hat{G}_{k,\infty}^{(k+1)} + \ldots \right), \quad \text{as } \varepsilon \to 0,
\]

(4.76)

for \( k = 1, \ldots, (n-1) \) and \( n \geq 2 \), for some sequence of matrices \( \hat{G}_{k,\infty}^{(>k)} \). We aim to prove the same holds for \( k = n \). We look at equation (4.67) with \( n \mapsto (n-1) \),

\[
(1-n)\hat{G}_{n-1,\infty} + \left[ \hat{G}_{n-1,\infty}, -\frac{\theta_{\infty6}}{2} \sigma_3 \right] = \sum_{l=0}^{n-2} \left( A_{16} + t_6^{n-l} A_{t6} \right) \hat{G}_{l,\infty} - \sum_{l=0}^{n-2} \hat{G}_{l,\infty} \frac{\theta_{16}}{2} \sigma_3.
\]

(4.77)

We subtract (4.77) from (4.67) to produce the following equation,

\[
-n\hat{G}_{n,\infty} + \left[ \hat{G}_{n,\infty}, -\frac{\theta_{\infty6}}{2} \sigma_3 \right] = -(n-1)\hat{G}_{n-1,\infty} + \left[ \hat{G}_{n-1,\infty}, -\frac{\theta_{\infty6}}{2} \sigma_3 \right] + A_{16} \hat{G}_{n-1,\infty}
\]

\[
+ \sum_{l=0}^{n-1} t_6^{n-l} A_{16} \hat{G}_{l,\infty} - \sum_{l=0}^{n-2} t_6^{n-1-l} A_{t6} \hat{G}_{l,\infty} - \hat{G}_{n-1,\infty} \frac{\theta_{16}}{2} \sigma_3.
\]

(4.78)

We now multiply through by \( \varepsilon^{-n} \) and make the substitutions (4.47)-(4.48) along with
our Main Assumption 1 to find,

\[
-n\varepsilon^{-n}\hat{G}_{n,\infty} + \left[\varepsilon^{-n}\hat{G}_{n,\infty}, -\frac{\theta_{\infty 5}}{2}\sigma_3\right] + \frac{1}{\varepsilon} \left[\varepsilon^{-n}\hat{G}_{n,\infty}, -\frac{\sigma_3}{2}\right]
\]

\[
\sim -(n - 1)\varepsilon^{-1}\varepsilon^{-1-n}\hat{G}_{n-1,\infty} + \varepsilon^{-1} \left[\varepsilon^{-1-n}\hat{G}_{n-1,\infty}, -\frac{\theta_{\infty 5}}{2}\sigma_3\right] + \varepsilon^{-2} \left[\varepsilon^{-1-n}\hat{G}_{n-1,\infty}, -\frac{\sigma_3}{2}\right]
\]

\[
+ \left(\varepsilon^{-1}A_{16}^{(-1)} + A_{16}^{(0)} + \varepsilon A_{16}^{(1)} + \ldots\right)\varepsilon^{-1}\varepsilon^{-1-n}\hat{G}_{n-1,\infty} + \sum_{l=0}^{n-1} t_5^{-1-l} A_{16} \varepsilon^{-l}\hat{G}_{l,\infty}
\]

\[
- \varepsilon^{-1} \sum_{l=0}^{n-2} t_5^{-1-l} A_{16} \varepsilon^{-l}\hat{G}_{l,\infty} + \varepsilon^{-2} \varepsilon^{-1-n}\hat{G}_{n-1,\infty}\frac{\sigma_3}{2}, \quad \text{as } \varepsilon \to 0, \quad (4.78)
\]

where the terms indicated cancel with each other, due to $A_{16}^{(-1)} = -\frac{\sigma_3}{2}$. Looking on the right hand side of this asymptotic at order $\varepsilon^{-1}$, remembering the inductive assumption (4.76), we have,

\[
-(n - 1)H_{n-1,\infty} + \left[H_{n-1,\infty}, -\frac{\theta_{\infty 5}}{2}\sigma_3\right] + A_{16}^{(0)} H_{n-1,\infty} - \sum_{l=0}^{n-2} t_5^{-1-l} A_{16} H_{l,\infty}
\]

\[
= -(n - 1)H_{n-1,\infty} - H_{n-1,\infty}\frac{\theta_{\infty 5}}{2}\sigma_3 - (A_{05} + A_{15}) H_{n-1,\infty} - \sum_{l=0}^{n-2} t_5^{-1-l} A_{15} H_{l,\infty},
\]

\[
= \left[H_{n,\infty}, -\frac{\sigma_3}{2}\right] \quad \text{since this is (4.38)}.
\]

In particular, we notice that the right hand side of (4.78) at order $\varepsilon^{-1}$ is off-diagonal only. From this, and since the left hand side of (4.78) has the following form,

\[
\varepsilon^{-n} \begin{pmatrix}
-n(\hat{G}_{n,\infty})_{1,1} & (\hat{G}_{n,\infty})_{1,2} (\varepsilon^{-1} + \theta_{\infty 5} - n) \\
-(\hat{G}_{n,\infty})_{2,1} (\varepsilon^{-1} + \theta_{\infty 5} + n) & -n(\hat{G}_{n,\infty})_{2,2}
\end{pmatrix}
\]

we conclude that,

\[
\hat{G}_{n,\infty} \sim \varepsilon^n \left(\hat{G}_{n,\infty}^{(n)} + \varepsilon\hat{G}_{n,\infty}^{(n+1)} + \ldots\right), \quad \text{as } \varepsilon \to 0,
\]

where $\text{off}(\hat{G}_{n,\infty}^{(n)}) = \text{off}(H_{n,\infty})$, as required, and $\hat{G}_{n,\infty}^{(n)}$ is some sequence of matrices.
For the diagonal part of \( \hat{G}_{n,\infty}^{(1)} \), we look at (4.78) at the next order, \( \varepsilon^0 \),
\[
\text{diag} \left( \hat{G}_{n,\infty}^{(n)} \right) = \text{diag} \left( - (n - 1) \hat{G}_{n-1,\infty}^{(n)} + A_{16}^{(0)} \hat{G}_{n-1,\infty}^{(n)} + A_{16}^{(1)} \hat{G}_{n-1,\infty}^{(n-1)} \right)
+ \sum_{l=0}^{n-1} t_5^{n-l} A_{15} \hat{G}_{l,\infty}^{(l)} - \sum_{l=0}^{n-2} t_5^{n-l-1} \left( A_{15} \hat{G}_{l+1,\infty}^{(l+1)} + A_{16}^{(1)} \hat{G}_{l,\infty}^{(l)} \right),
\]
and, from the left hand side of (4.78), this is equal to \( \text{diag} \left( \hat{G}_{n,\infty}^{(n)} \right) \). From (4.79), we define \( \text{diag} \left( \hat{G}_{n-1,\infty}^{(n)} \right) \) such that,
\[
\lim_{\varepsilon \to 0} \text{diag} \left( \hat{G}_{n,\infty}^{(n)} \right) = \text{diag} \left( H_{n,\infty} \right).
\]
We are free to do this since \( \text{diag} \left( \hat{G}_{n-1,\infty}^{(n)} \right) \) has not been specified previously in the induction process. With this condition, we have,
\[
\hat{G}_{n,\infty} \sim \varepsilon^n \left( H_{n,\infty} + \varepsilon \hat{G}_{n,\infty}^{(n+1)} + \ldots \right),
\]
and the result (4.71) is proved for all \( n \geq 2 \).

Henceforth, we assume that we have made such a choice of \( \hat{G}_{n-1,\infty}^{(n)} \), for all \( n \geq 2 \), so that the limits in Lemma 4.5 hold.

4.3.2 Taking a Term-By-Term Limit of the Solution \( Y_{6}^{(1)}(\lambda_6) \)

We rewrite the solution \( Y_{6}^{(1)}(\lambda_6) \), as given in (4.12), in the following way,
\[
Y_{6}^{(1)}(\lambda_6) = R_{16} \left( \sum_{n=0}^{\infty} \hat{G}_{n,1} \left( 1 - \lambda_6^{-1} \right)^n \right) \lambda_6^{-\frac{\phi_{16} + \phi_{\infty}}{2} \sigma_3} (1 - \lambda_6)^{\frac{\phi_{16}' \sigma_3}{2}}, \quad \lambda_6 \in \hat{\Omega}_{16},
\]
where, for some \( \hat{\rho}_{16} > 0 \), the new domain \( \hat{\Omega}_{16} \) is defined as,
\[
\hat{\Omega}_{16} = \left\{ \lambda_6 : \left| 1 - \lambda_6^{-1} \right| < \hat{\rho}_{16}, \ -\pi \leq \arg(\lambda_6) < \pi, \ -\pi \leq \arg(1 - \lambda_6) < \pi \right\},
\]
\( \hat{G}_{0,1} := I \) and the coefficients \( \hat{G}_{n,1}, \ n \geq 1 \), are determined by the recursion relation given in Lemma 4.6 below. We note that this new form of the solution is equivalent to that given in (4.12) on the domain \( \Omega_{16} \cap \hat{\Omega}_{16} \). There is a very simple philosophical
reason why we rewrite the series in $Y^{(1)}_6(\lambda_6)$ with $(1 - \lambda_6^{-1})^n$, rather than $(1 - \lambda_6)^n$: after the change of variable $\lambda_6 = \varepsilon \lambda_5$, we want to produce a formal series in $\lambda_6^{-n}$. The maximal radius of convergence we could choose for the domain $\hat{\Omega}_{16}$ is now $\hat{\rho}_{16} = |1 - t_6^{-1}|$.

In (4.80), we have rewritten $Y^{(1)}_6(\lambda_6)$ to include the terms $\lambda_6^{-\theta_{16} + \theta_\infty 6}/2$ and $(1 - \lambda_6)^{-\theta_\infty 6}/2$ which produce the correct terms as in $Y^{(\infty)}_5(\lambda_5)$, recall Remark 4.10.

**Lemma 4.6.** Setting $\hat{G}_{0,1} := I$, the general formula for $\hat{G}_{n,1}$ with $n \geq 1$ is given below,

$$\left[\hat{G}_{n,1}, \frac{\theta_{16}}{2} \sigma_3\right] + n\hat{G}_{n,1} = (n - 1)\hat{G}_{n-1,1} + \hat{G}_{n-1,1} + \frac{\theta_{16} + \theta_\infty 6}{2} \sigma_3$$

$$+ R_{16}^{-1} A_{06} R_{16} \hat{G}_{n-1,1} - R_{16}^{-1} A_{06} R_{16} t_6^{-1} \sum_{l=0}^{n-1} (1 - t_6^{-1})^{l-n} \hat{G}_{l,1}. \quad (4.81)$$

**Proof.** We show this proof in detail, in case the series in $(1 - \lambda_6^{-1})^n$ seems unnatural at first. We note the following identities:

$$\frac{1}{\lambda_6} \equiv 1 - (1 - \lambda_6^{-1}), \quad \frac{1}{\lambda_6 - 1} \equiv (1 - \lambda_6^{-1})^{-1} - 1,$$

$$\frac{1}{\lambda_6 - t_6} \equiv \frac{\lambda_6^{-1} t_6^{-1}}{t_6^{-1} - \lambda_6^{-1}} \equiv \frac{\lambda_6^{-1} t_6^{-1}}{(t_6^{-1} - 1) \left(1 + \frac{1 - \lambda_6^{-1}}{t_6^{-1} - 1}\right)}$$

$$\equiv (1 - (1 - \lambda_6^{-1})) \frac{t_6^{-1}}{t_6^{-1} - 1} \sum_{n=0}^{\infty} \frac{(1 - \lambda_6^{-1})^n}{(1 - t_6^{-1})^n},$$

the last line is not strictly an identity, it is only valid for $\frac{1 - \lambda_6^{-1}}{1 - t_6^{-1}} < 1$, but this is satisfied by definition of the domain $\hat{\Omega}_{16}$ of the fundamental solution $Y^{(1)}_6(\lambda_6)$. 

Page 133 of 178
Substituting the new form \((4.80)\) of \(Y_6^{(1)}(\lambda_6)\) into equation \(4.1\) gives,

\[
\sum_{n=1}^{\infty} n \hat{G}_{n,1} \left[ (1 - \lambda_6^{-1})^{n-1} - 2 (1 - \lambda_6^{-1})^{n} + (1 - \lambda_6^{-1})^{n+1} \right] \equiv (1 - \lambda_6^{-1})^{n-1} \lambda_6^{-2}
\]

\[+ \left( \sum_{n=0}^{\infty} \hat{G}_{n,1} (1 - \lambda_6^{-1})^{n} \right) \left[ \frac{\theta_{16}}{2} \sigma_3 \left( \frac{\lambda_6^{-1}}{(1 - \lambda_6^{-1})^{-1} - 1} \right) - \frac{\theta_{16} + \theta_{\infty 6}}{2} \sigma_3 \left( \frac{1}{1 - (1 - \lambda_6^{-1})} \right) \right] \equiv \frac{1}{\lambda_6} \]

\[+ R_{16}^{-1} A_{t6} R_{16} \sum_{n=0}^{\infty} \left( (1 - t_6^{-1})^{-(n+1)} \right) \left[ - (1 - \lambda_6^{-1})^{n} + (1 - \lambda_6^{-1})^{n+1} \right] \sum_{n=0}^{\infty} \hat{G}_{n,1} (1 - \lambda_6^{-1})^{n} \equiv \frac{1}{\lambda_6 - t_6} \]

(4.82)

The term inside the large \([\ ]\) brackets on the left hand side of the equals sign comes from differentiating the term \(\lambda_6^{\theta_{\infty 6} + \theta_{t6} \sigma_3} (1 - \lambda_6)^{\frac{\theta_{t6}}{2} \sigma_3}\). The term inside the large \((\ )\) brackets on the right hand side of the equals sign is equivalently \(R_{16}^{-1} \left( \frac{A_{06}}{\lambda_6} + \frac{A_{t6}}{\lambda_6 - t_6} + \frac{A_{16}}{\lambda_6 - 1} \right) R_{16}\) after expanding in powers of \((1 - \lambda_6^{-1})\). 

Page 134 of 178
To solve for the terms $\hat{G}_{n,1}$, $n \geq 1$, we equate the coefficients of $(1 - \lambda_6^{-1})$ in (4.82):

\[
\mathcal{O}\left((1 - \lambda_6^{-1})^{-1}\right):
\]
\[
\frac{\theta_{16}}{2} \sigma_3 \hat{G}_{0,1} = \hat{G}_{0,1} \frac{\theta_{16}}{2} \sigma_3, \quad \text{which is satisfied since } \hat{G}_{0,1} := I. \quad (4.83)
\]

\[
\mathcal{O}(1):
\]
\[
\hat{G}_{1,1} + \left[ \hat{G}_{1,1}, \frac{\theta_{16}}{2} \sigma_3 \right] - \hat{G}_{0,1} \frac{\theta_{16}}{2} \sigma_3 - \hat{G}_{0,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3
\]
\[
= -\frac{\theta_{16}}{2} \sigma_3 \hat{G}_{0,1} + R_{16}^{-1} A_{06} R_{16} \hat{G}_{0,1} - R_{16}^{-1} A_{67} R_{16} \frac{t_{6}^{-1}}{1 - t_{6}^{-1}} \hat{G}_{0,1}, \quad (4.84)
\]

\[
\mathcal{O}(1 - \lambda_6^{-1}):
\]
\[
(2 \hat{G}_{2,1} - 2 \hat{G}_{1,1}) + \left[ \hat{G}_{2,1}, \frac{\theta_{16}}{2} \sigma_3 \right] - \hat{G}_{1,1} \frac{\theta_{16}}{2} \sigma_3 +
\]
\[
- \hat{G}_{1,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3 + \hat{G}_{0,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3
\]
\[
= -\frac{\theta_{16}}{2} \sigma_3 \hat{G}_{1,1} + R_{16}^{-1} A_{06} R_{16} \left( \hat{G}_{1,1} - \hat{G}_{0,1} \right) +
\]
\[
+ R_{16}^{-1} A_{67} R_{16} t_{6}^{-1} \left( (1 - t_{6}^{-1}) \hat{G}_{1,1} - (1 - t_{1}^{-1})^{-2} \hat{G}_{0,1} + \left(1 - t_{6}^{-1}\right)^{-2} \hat{G}_{0,1} \right), \quad (4.85)
\]

\[
\mathcal{O}\left((1 - \lambda_6^{-1})^{2}\right):
\]
\[
(3 \hat{G}_{3,1} - 4 \hat{G}_{2,1} + \hat{G}_{1,1}) + \left[ \hat{G}_{3,1}, \frac{\theta_{16}}{2} \sigma_3 \right] - \hat{G}_{2,1} \frac{\theta_{16}}{2} \sigma_3 +
\]
\[
- \hat{G}_{2,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3 + \hat{G}_{1,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3
\]
\[
= -\frac{\theta_{16}}{2} \sigma_3 \hat{G}_{2,1} + R_{16}^{-1} A_{06} R_{16} \left( \hat{G}_{2,1} - \hat{G}_{1,1} \right) +
\]
\[
+ R_{16}^{-1} A_{67} R_{16} t_{6}^{-1} \left( (1 - t_{6}^{-1}) \hat{G}_{2,1} - (1 - t_{1}^{-1})^{-2} \hat{G}_{1,1} - (1 - t_{6}^{-1})^{-3} \hat{G}_{0,1} +
\]
\[
+ (1 - t_{6}^{-1})^{-1} \hat{G}_{1,1} + (1 - t_{6}^{-1})^{-2} \hat{G}_{0,1} \right), \quad (4.86)
\]

\[
\mathcal{O}\left((1 - \lambda_6^{-1})^{n-1}\right):
\]
\[
(n \hat{G}_{n,1} - 2(n - 1) \hat{G}_{n-1,1} + (n - 2) \hat{G}_{n-2,1}) + \left[ \hat{G}_{n,1}, \frac{\theta_{16}}{2} \sigma_3 \right] - \hat{G}_{n-1,1} \frac{\theta_{16}}{2} \sigma_3 +
\]
\[
- \hat{G}_{n-1,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3 + \hat{G}_{n-2,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3
\]
\[
= -\frac{\theta_{16}}{2} \sigma_3 \hat{G}_{n-1,1} + R_{16}^{-1} A_{06} R_{16} \left( \hat{G}_{n-1,1} - \hat{G}_{n-2,1} \right) +
\]
\[
+ R_{16}^{-1} A_{67} R_{16} \left( - \sum_{l=0}^{n-1} (1 - t_{6}^{-1})^{l-n} \hat{G}_{l,1} + \sum_{l=0}^{n-2} (1 - t_{6}^{-1})^{l+1-n} \hat{G}_{l,1} \right). \quad (4.87)
\]
Now, we rearrange the terms on the left and right hand sides of (4.87) to find the following formula for $n \geq 3$:

\[
\begin{align*}
\left[ \hat{G}_{n,1}, \frac{\theta_{16}}{2} \sigma_3 \right] + n\hat{G}_{n,1} - (n-1)\hat{G}_{n-1,1} - \hat{G}_{n-1,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3 \\
- R_{16}^{-1} A_{06} R_{16} \hat{G}_{n-1,1} + R_{16}^{-1} A_{16} R_{16} t_6^{-1} \sum_{l=0}^{n-1} (1-t_6^{-1})^{l-n} \hat{G}_{l,1}
\end{align*}
\]

\[
= \left[ \hat{G}_{n-1,1}, \frac{\theta_{16}}{2} \sigma_3 \right] + (n-1)\hat{G}_{n-1,1} - (n-2)\hat{G}_{n-2,1} - \hat{G}_{n-2,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3 \\
- R_{16}^{-1} A_{06} R_{16} \hat{G}_{n-2,1} + R_{16}^{-1} A_{16} R_{16} t_6^{-1} \sum_{l=0}^{n-2} (1-t_6^{-1})^{l+1-n} \hat{G}_{l,1}.
\]

The crucial observation is that this formula is a simple difference equation of the form,

\[d_n = d_{n-1}, \quad n \geq 3,
\]

with the obvious solution $d_n = d_2$ for all $n \geq 3$. Using (4.84) and (4.85), we have,

\[
d_2 = \left[ \hat{G}_{2,1}, \frac{\theta_{16}}{2} \sigma_3 \right] + 2\hat{G}_{2,1} - \hat{G}_{1,1} - \hat{G}_{1,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3 - R_{16}^{-1} A_{06} R_{16} \hat{G}_{1,1} + \\
+ R_{16}^{-1} A_{16} R_{16} t_6^{-1} \left( (1-t_6^{-1})^{-2} \hat{G}_{0,1} + (1-t_6^{-1})^{-1} \hat{G}_{1,1} \right) \quad \text{by definition,}
\]

\[
= \left[ \hat{G}_{1,1}, \frac{\theta_{16}}{2} \sigma_3 \right] + \hat{G}_{1,1} - \hat{G}_{0,1} - \hat{G}_{0,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3 - R_{16}^{-1} A_{06} R_{16} \hat{G}_{0,1} + \\
+ R_{16}^{-1} A_{16} R_{16} (1-t_6^{-1}) \hat{G}_{0,1} \quad \text{by (4.84)},
\]

\[
= 0 \quad \text{by (4.85)}.
\]

We have shown for $n \geq 3$ that the following formula holds,

\[
\begin{align*}
\left[ \hat{G}_{n,1}, \frac{\theta_{16}}{2} \sigma_3 \right] + n\hat{G}_{n,1} - (n-1)\hat{G}_{n-1,1} - \hat{G}_{n-1,1} \frac{\theta_{16} + \theta_{\infty}}{2} \sigma_3 \\
- R_{16}^{-1} A_{06} R_{16} \hat{G}_{n-1,1} + R_{16}^{-1} A_{16} R_{16} t_6^{-1} \sum_{l=0}^{n-1} (1-t_6^{-1})^{l-n} \hat{G}_{l,1} = 0.
\end{align*}
\]

By inspection with (4.84) and (4.85), we see that this formula is in fact valid for $n \geq 1$. This concludes the derivation of formula (4.81).
freedom of multiplication on the right by a diagonal matrix. We now make a fixed choice once and for all of the matrix $R_{16}$ such that, under the substitutions (4.47)-(4.48) and the conditions of our Main Assumption 1, it has the following asymptotic behavior,

$$R_{16}(\varepsilon t_5) \sim I + \sum_{n=1}^{\infty} \varepsilon^n R^{(n)}_{16}(t_5), \quad \text{as } \varepsilon \to 0,$$

where $R^{(n)}_{16}(t_5)$ is some sequence of matrices. We note that it is always possible to make such a choice because $\theta_{16} = -\frac{4}{5} \varepsilon$ and, from the asymptotic behavior (4.52), $A_{16}$ has leading term $-\frac{\sigma_3}{2\varepsilon}$. Similar as in the case of the matrices $R_{06}$ and $R_{t6}$, this is a simple but cumbersome exercise in linear algebra which we omit. The asymptotic behavior of $R_{16}$ as above can also be confirmed by computing explicit formulae for $R^{(n)}_{16}$ and substituting (4.54)-(4.59). We note that the matrices $R^{(n)}_{16}$ can be found in terms of the matrices $A^{(n)}_{16}$ from the relation,

$$R_{16}^{-1} A_{16} R_{16} = \frac{\theta_{16}}{2} \sigma_3,$$

$$\Rightarrow \left( I - \varepsilon R^{(1)}_{16} + \ldots \right) \left( -\frac{\sigma_3}{2\varepsilon} - \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right) + \ldots \right) \left( I + \varepsilon R^{(1)}_{16} + \ldots \right) = -\frac{\sigma_3}{2\varepsilon}. \quad (4.88)$$

In particular, by comparing terms at order $\varepsilon^0$ we find,

$$\begin{bmatrix} R^{(1)}_{16} & \frac{\sigma_3}{2} \end{bmatrix} - \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right) = 0,$$

$$\Leftrightarrow \text{off}(R^{(1)}_{16}) = \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right) \sigma_3. \quad (4.88)$$

This will be used in the proof of Lemma 4.7 below.

**Lemma 4.7.** Let $\hat{G}_{1,1}$ and $H_{n,\infty}$ be defined by (4.81) and (4.38) respectively. Under the substitutions (4.47) - (4.48) and our Main Assumption 1, we have,

$$\lim_{\varepsilon \to 0} (-\varepsilon)^{-1} \hat{G}_{1,1} = H_{1,\infty}, \quad (4.89)$$

and, for $n \geq 2$, there exists a choice of $\hat{G}^{(n)}_{n-1,1}$ such that,

$$\lim_{\varepsilon \to 0} (-\varepsilon)^{-n} \hat{G}_{n,1} = H_{n,\infty}, \quad (4.90)$$
Proof. We first show (4.89) is correct by explicit computation. We look at (4.81) with \( n = 1 \), which determines \( \hat{G}_{1,1} \), remembering to make the substitutions (4.47) - (4.48) along with our Main Assumption 1, we have,

\[
\left[ \hat{G}_{1,1}, -\frac{\sigma_3}{2\varepsilon} \right] + \hat{G}_{1,1} = \frac{\theta_{\infty 5}}{2} \sigma_3 + R_{16}^{-1} A_{06} R_{16} + R_{16}^{-1} A_{16} R_{16} \frac{1}{1 - \varepsilon t_5}
\]

\[
\sim \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} + \varepsilon (A_{05}^{(1)} + A_{16}^{(1)} + [A_{05}, R_{16}^{(1)}] + [A_{t5}, R_{16}^{(1)}] + t_5 A_{t5}) + \ldots,
\]

(4.91) as \( \varepsilon \to 0 \). We note that the commutator term \( \left[ \hat{G}_{1,1}, -\frac{\sigma_3}{2\varepsilon} \right] \) has zero diagonal part and only depends on the off-diagonal part of \( \hat{G}_{1,1} \). Since the right hand side of (4.91) is of order \( \varepsilon^0 \), we must have that \( \text{off}(\hat{G}_{1,1}) = O(\varepsilon) \). Furthermore, since the diagonal part of the right hand side of (4.91) is of order \( \varepsilon \), recall (4.65), we conclude that \( \text{diag}(\hat{G}_{1,1}) = O(\varepsilon) \). We write,

\[
\hat{G}_{1,1} \sim \varepsilon \hat{G}_{1,1}^{(1)} + \varepsilon^2 \hat{G}_{1,1}^{(2)} + \ldots, \quad \varepsilon \to 0.
\]

Comparing the left and right hand sides of (4.91) at order \( \varepsilon^0 \), we have,

\[
\left[ \hat{G}_{1,1}^{(1)}, -\frac{\sigma_3}{2} \right] = \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5}, \quad \leftrightarrow \text{off} \left( -\hat{G}_{1,1}^{(1)} \right) = \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right) \sigma_3,
\]

from which we conclude that \( \text{off}(\hat{G}_{1,1}^{(1)}) = \text{off}(H_{1,\infty}) \), recall (4.72). Comparing the diagonal parts of the left and right hand sides of (4.91) at order \( \varepsilon \), we have,

\[
\text{diag} \left( \hat{G}_{1,1}^{(1)} \right) = \text{diag} \left( A_{06}^{(1)} + A_{16}^{(1)} + [A_{05} + A_{t5}, R_{16}^{(1)}] + t_5 A_{t5} \right).
\]

(4.92)

By the first assumption in the linear problem of \( P_{VI} \), we have,

\[
\text{diag} \left( A_{06}^{(1)} + A_{16}^{(1)} \right) = -\text{diag} \left( A_{16}^{(1)} \right)
\]

\[
= - \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right)^2 \sigma_3 \text{ by (4.60)}.
\]
Also note that,
\[
\text{diag} \left( \left[ A_{05} + A_{t5}, R_{16}^{(1)} \right] \right) = \text{diag} \left( \left[ \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5}, R_{16}^{(1)} \right] \right),
\]
\[
= \left[ \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5}, \text{off} \left( R_{16}^{(1)} \right) \right] \text{ by (4.65)},
\]
\[
= \left[ \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5}, \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right) \sigma_3 \right] \text{ by (4.88)},
\]
\[
= 2 \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right)^2 \sigma_3.
\]

Hence, combining these computations, (4.92) becomes,
\[
\text{diag} \left( \hat{G}_{1,1}^{(1)} \right) = \left( \frac{\theta_{\infty 5}}{2} \sigma_3 + A_{05} + A_{t5} \right)^2 \sigma_3 + t_5 A_{t5}.
\]

Comparing this with (4.74), we conclude that \( \text{diag}(-\hat{G}_{1,1}^{(1)}) = \text{diag}(H_{1,\infty}) \).

At the induction step, we now assume that,
\[
\hat{G}_{k,1} \sim (-1)^k \varepsilon^k H_{k,\infty} + \varepsilon^{k+1} \hat{G}_{k,1}^{(k+1)} + \ldots, \quad \varepsilon \to 0, \quad (4.93)
\]
for \( k = 1, \ldots, (n - 1) \) and \( n \geq 2 \), for some sequence of matrices \( \hat{G}_{k,1}^{(>k)} \). We aim to prove the same holds for \( k = n \). We look at the equation (4.81), which determines \( \hat{G}_{n,1} \), making the substitutions (4.47) - (4.48), we have,
\[
\left[ \hat{G}_{n,1}, \frac{\sigma_3}{2\varepsilon} \right] + n\hat{G}_{n,1} = (n - 1)\hat{G}_{n-1,1} + \hat{G}_{n-1,1} \frac{\theta_{\infty 5}}{2} \sigma_3 +
\]
\[
+ R_{16}^{-1} A_{06} R_{16} \hat{G}_{n-1,1} - R_{16}^{-1} A_{16} R_{16} (\varepsilon t_5)^{-1} \sum_{l=0}^{n-1} \left( 1 - (\varepsilon t_5)^{-1} \right)^{l-n} \hat{G}_{l,1}. \quad (4.94)
\]

Looking on the right hand side of this equation at order \( \varepsilon^{n-1} \), remembering the induc-
tive assumption (4.93) and our Main Assumption 1, we have,

\[
(-1)^{n-1}(n - 1)H_{n-1,\infty} + (-1)^{n-1}H_{n-1,\infty} \frac{\theta_{\infty}}{2} \sigma_3 + (-1)^{n-1}A_{05}H_{n-1,\infty} + \\
+ (-1)^{n-1} \sum_{l=0}^{n-1} t_5^{n-1-l}A_{t5}H_{l,\infty}
\equiv (-1)^{n-1} [H_{n,\infty}, \sigma_3^2] \text{ since this is (4.38).}
\]

In particular, we see that the right hand side of (4.94) at order \(\varepsilon^{n-1}\) has zero diagonal part. Therefore, from looking at the left hand side, we conclude that \(\widehat{G}_{n,1} = O(\varepsilon^n)\), in the same way as before. We write,

\[
\widehat{G}_{n,1} \sim \varepsilon^n \widehat{G}_{n,1}^{(n)} + \varepsilon^{n+1} \widehat{G}_{n,1}^{(n+1)} + \ldots, \quad \varepsilon \to 0.
\]

Looking at equation (4.94) at order \(\varepsilon^{n-1}\) and the diagonal part of order \(\varepsilon^n\), we have, respectively:

\[
\left[ \widehat{G}_{n,1}^{(n)}, -\frac{\sigma_3}{2} \right] = (-1)^{n-1}(n - 1)H_{n-1,\infty} \frac{\theta_{\infty}}{2} \sigma_3 + (-1)^{n-1}(A_{05} + A_{t5})H_{n-1,\infty} + \\
+ (-1)^{n-1} \sum_{l=0}^{n-2} t_5^{n-1-l}A_{t5}H_{l,\infty} \equiv (-1)^{n-1} [H_{n,\infty}, \sigma_3^2] \text{ by (4.38),}
\]

\[
\begin{align*}
\text{diag} \left( n\widehat{G}_{n,1}^{(n)} \right) &= \text{diag} \left( (n - 1)\widehat{G}_{n-1,1}^{(n)} + \left( \frac{\theta_{\infty}}{2} \sigma_3 + A_{05} + A_{t5} \right) \widehat{G}_{n-1,1}^{(n)} + \\
&\quad + \left[ A_{05}, R_{16}^{(1)} \right] + A_{t5}^{(1)} \right) H_{n-1,\infty} + \left( \left[ A_{t5}, R_{16}^{(1)} \right] + A_{t5}^{(1)} \right) \sum_{l=0}^{n-1} (-1)^{n-1-l}t_5^{n-1-l}H_{l,\infty} + \\
&\quad + A_{t5} \sum_{l=0}^{n-2} (-1)^{n-1-l}t_5^{l+1}G_{l,1} + \sum_{l=0}^{n-1} (n - l)t_5^{n-l}H_{l,\infty} \right) \quad (4.95)
\end{align*}
\]

From the first relation, we conclude that \(\text{off}((-1)^n\widehat{G}_{n,1}^{(n)}) = \text{off}(H_{n,\infty})\). In the second relation, we define \(\widehat{G}_{n-1,1}^{(n)}\) such that \(\text{diag}((-1)^n\widehat{G}_{n-1,1}^{(n)}) = \text{diag}(H_{n,\infty})\). We are free to do this, since \(\text{diag}(\widehat{G}_{n-1,1}^{(n)})\) has not been specified previously in the induction process. With this condition, we conclude that \((-1)^n\widehat{G}_{n,1}^{(n)} = H_{n,\infty}\) and the result (4.90) is proved for all \(n \geq 2\).

Henceforth, we assume that we have made such a choice of \(\widehat{G}_{n-1,1}^{(n)}\), for all \(n \geq 2\), so that the limits in Lemma 4.7 hold.
4.3.3 Obtaining the Solutions $Y_5^{(\infty,k)}(\lambda_5)$

In Subsections 4.3.1 and 4.3.2 we have understood how to take term-by-term limits of the solutions of equation (4.1) around $\lambda_6 = \infty$ and 1 respectively, to produce the formal series solution of equation (4.23) around $\lambda_5 = \infty$. We now apply Glutsyuk’s Theorem 2.5 to equations (4.1) and (4.23). Let $\eta \in \left(0, \frac{\pi}{2}\right)$ be some fixed value. We define the following sectors,

$$\mathcal{S}_k := \left\{ \lambda_5 : |\lambda_5| > \rho_\infty, \quad \arg(\lambda_5) - k\pi \in \left(\eta - \frac{\pi}{2}, \frac{3\pi}{2} - \eta\right) \right\}, \quad (4.96)$$

note that if $\lambda_5 \in \mathcal{S}_k$ then $\lambda_5 \in \Sigma_k$. These sectors will be the new domains of our fundamental solutions $Y_5^{(\infty,k)}(\lambda_5)$. The presence of $\eta$ is to ensure that the boundaries of the sectors $\mathcal{S}_k$ do not contain a Stokes’ ray, as is necessary in the hypothesis of Glutsyuk’s Theorem 2.5. We note that this condition is not satisfied by the sectors $\Sigma_k$ defined in Theorem 3.1, which have the maximal possible opening on which we can define single-valued analytic fundamental solutions.

We also define the following sectors,

$$\sigma_1(\varepsilon) := \left\{ \lambda_5 : \left|1 - \frac{1}{\varepsilon \lambda_5}\right| < \rho_{16}, \quad \arg(\varepsilon \lambda_5) \in (\eta - \pi, \pi - \eta), \right\}, \quad (4.97)$$

$$\sigma_\infty(\varepsilon) := \left\{ \lambda_5 : |\varepsilon \lambda_5| > \rho_{\infty6}, \quad \arg(\varepsilon \lambda_5) \in (\eta, 2\pi - \eta), \quad \arg(1 - \varepsilon \lambda_5) \in (\eta - \pi, \pi - \eta) \right\}, \quad (4.98)$$

note that if $\lambda_5 \in \sigma_1(\varepsilon)$ then $\lambda_6 \in \widehat{\Omega}_{16}$ and if $\lambda_5 \in \sigma_\infty(\varepsilon)$ then $\lambda_6 \in \widehat{\Omega}_{\infty6}$. These sectors will be the new domains of our fundamental solutions $Y_6^{(1)}(\varepsilon \lambda_5)$ and $Y_6^{(\infty)}(\varepsilon \lambda_5)$ respectively, they are illustrated below.

![Figure 23: Sectors $\sigma_1(\varepsilon)$ and $\sigma_\infty(\varepsilon)$](image-url)
As in Section 4.2, since our confluence procedure (4.47) rescales \( \lambda_6 \) by a factor of \( \varepsilon \) we must ensure that the domains on which our local fundamental solutions are defined do not vanish in the limit \( \varepsilon \to 0 \). Consequently, we now choose the radius \( \rho_{\infty 6} \) such that \( \lim_{\varepsilon \to 0} \frac{\rho_{\infty 6}}{\varepsilon} = \rho_{\infty 5} \). Notice that the maximal value \( \rho_{\infty 6} \) we could choose for the sector \( \sigma_{\infty} (\varepsilon) \), on which our fundamental solution \( Y_{6}^{(\infty)} (\varepsilon \lambda_5) \) is defined, is \( \rho_{\infty 6} = |\varepsilon t_5| \). This is the key reason to restrict our solutions to sectors rather than disks: the point \( \lambda_5 = \varepsilon^{-1} \) (corresponding to \( \lambda_6 = 1 \)) does not pose an obstruction to the radius of the sector \( \sigma_{\infty} (\varepsilon) \), see Figure 23 above. In this example, we would have \( \frac{\rho_{\infty 6}}{|\varepsilon|} = |t_5| \), which is indeed the maximal radius \( \rho_{\infty 5} \) we could choose for the sectors \( \Sigma_k \) in which our fundamental solutions \( Y_{5}^{(\infty,k)} (\lambda_5) \) are analytic. We say that, as \( \varepsilon \to 0 \) along a ray, the sector \( \sigma_{\infty} (\varepsilon) \) becomes in agreement with the sector,

\[
\{ \lambda_5 : |\lambda_5| > \rho_{\infty 5}, \ \eta < \arg (\varepsilon \lambda_5) < 2\pi - \eta \}.
\]

In the two limit directions we are concerned with, for \( \arg (\varepsilon) = \pm \frac{\pi}{2} \), we have,

\[
\eta < \arg (\varepsilon \lambda_5) < 2\pi - \eta \iff \eta \mp \frac{\pi}{2} < \arg (\lambda_5) < 2\pi \mp \frac{\pi}{2} - \eta.
\]

For the sector \( \sigma_{\frac{1}{2}} (\varepsilon) \), we choose \( \hat{\rho}_{16} \) such that \( \lim_{\varepsilon \to 0} \hat{\rho}_{16} |\varepsilon| = \rho_{\infty 5}^{-1} \). To explain this, we look at the first condition in \( \sigma_{\frac{1}{2}} (\varepsilon) \),

\[
\left| 1 - \frac{1}{\varepsilon \lambda_5} \right| < \hat{\rho}_{16} \iff \left| \frac{1}{\lambda_5} - \frac{1}{\varepsilon \lambda_5} \right| < \hat{\rho}_{16} |\varepsilon|,
\]

so that, if this condition is satisfied for all \( |\varepsilon| \) sufficiently small, then \( |\lambda_5| > \hat{\rho}_{16} \). For example, if we had chose the maximal value \( \hat{\rho}_{16} = |1 - (\varepsilon t_5)^{-1}| \), then \( \lim_{\varepsilon \to 0} \hat{\rho}_{16} |\varepsilon| = |t_5|^{-1} \); this gives the condition \( |\lambda_5| > |t_5| \) which indeed corresponds to the maximal radius \( \rho_{\infty 5} \) we could choose for the sectors \( \Sigma_k \) in which our fundamental solutions \( Y_{5}^{(\infty,k)} (\lambda_5) \) are analytic. In addition, we see that as \( \varepsilon \to 0 \) along a ray, the sector \( \sigma_{\frac{1}{2}} (\varepsilon) \) is translated along a ray tending to infinity. We illustrate this phenomenon in Figure 24 below.
Figure 24: As $\varepsilon \to 0$ along a ray, the sector $\sigma_1(\varepsilon)$ is translated along the branch cut and becomes in agreement with the sector $\Phi_5 := \{\lambda_5 : |\lambda_5| > \rho_{\infty 5}, |\arg(\varepsilon \lambda_5)| < \pi - \eta\}$.

For $\arg(\varepsilon) = \pm \frac{\pi}{2}$, we have,

$$\eta - \pi < \arg(\varepsilon \lambda_5) < \pi - \eta \iff \eta - \pi \mp \frac{\pi}{2} < \arg(\lambda_5) < 2\pi \mp \frac{\pi}{2} - \eta.$$

With all of these considerations in mind, we write:

$$\begin{align*}
\lim_{\varepsilon \to 0} \quad & \arg(\varepsilon) = \frac{\pi}{2} \quad \lim_{\varepsilon \to 0} \arg(\varepsilon) = \frac{\pi}{2} \\
\lim_{\varepsilon \to 0} \quad & \sigma_1(\varepsilon) = \mathcal{J}_0, \quad \lim_{\varepsilon \to 0} \arg(\varepsilon) = -\frac{\pi}{2} \\
\lim_{\varepsilon \to 0} \quad & \sigma_\infty(\varepsilon) = \mathcal{J}_1, \quad \lim_{\varepsilon \to 0} \arg(\varepsilon) = \frac{\pi}{2} \\
\lim_{\varepsilon \to 0} \quad & \sigma_1(\varepsilon) = \mathcal{J}_1, \quad \lim_{\varepsilon \to 0} \arg(\varepsilon) = -\frac{\pi}{2} \\
\lim_{\varepsilon \to 0} \quad & \sigma_\infty(\varepsilon) = \mathcal{J}_0.
\end{align*}$$

We apply Glutsyuk’s Theorem 2.5 with equation (4.1) in place of the perturbed equation and (4.23) in place of the non-perturbed equation. Glutsyuk’s theorem asserts the existence of invertible diagonal matrices $K_{\infty 6}(\varepsilon)$ and $K_{16}(\varepsilon)$ such that:

$$\begin{align*}
\lim_{\varepsilon \to 0} \quad & Y_{6}^{(\infty)}(\varepsilon \lambda_5) \bigg|_{\lambda_5 \in \sigma_\infty(\varepsilon)} \quad K_{\infty 6}^{-}(\varepsilon) = Y_{5}^{(\infty, 1)}(\lambda_5), \quad (4.99) \\
\lim_{\varepsilon \to 0} \quad & Y_{6}^{(1)}(\varepsilon \lambda_5) \bigg|_{\lambda_5 \in \sigma_1(\varepsilon)} \quad K_{16}^{-}(\varepsilon) = Y_{5}^{(\infty, 0)}(\lambda_5), \quad (4.100)
\end{align*}$$
uniformly for \( \lambda_5 \in \mathcal{S}_1 \) and \( \lambda_5 \in \mathcal{S}_0 \) respectively, and:

\[
\begin{align*}
\lim_{\varepsilon \to 0} \lim_{\arg(\varepsilon) = \frac{\pi}{2}} \left. Y^{(8)}_6(\varepsilon \lambda_5) \right|_{\lambda_5 \in \sigma_{\infty}(\varepsilon)} K^+_{\infty 6}(\varepsilon) &= Y^{(0)}_5(\infty,0)(\lambda_5), \quad (4.101) \\
\lim_{\varepsilon \to 0} \lim_{\arg(\varepsilon) = \frac{\pi}{2}} \left. Y^{(1)}_6(\varepsilon \lambda_5) \right|_{\lambda_5 \in \sigma_{\frac{1}{2}}(\varepsilon)} K^+_{16}(\varepsilon) &= Y^{(-1)}_5(\infty,-1)(\lambda_5), \quad (4.102)
\end{align*}
\]

uniformly for \( \lambda_5 \in \mathcal{S}_0 \) and \( \lambda_5 \in \mathcal{S}_{-1} \) respectively. We note that since we are considering two limits, namely one with \( \arg(\varepsilon) = \frac{\pi}{2} \) and another with \( \arg(\varepsilon) = -\frac{\pi}{2} \), we have distinguished the diagonal matrices in each case with a superscript + or − respectively.

Due to the asymptotics of the fundamental solutions \( Y^{(\infty,k)}_5(\lambda_5) \) as given in Theorem 4.3, each of these four limits is asymptotic to the formal fundamental solution \( Y^{(\infty)}_{5,J}(\lambda_5) \) as \( \lambda_5 \to \infty \) with \( \lambda_5 \) belonging to the corresponding sector.

Having applied Glutsyuk’s theorem to produce the true solutions \( Y^{(\infty,k)}_5(\lambda_5) \) in sectors, we now focus on understanding what we can deduce about the matrices \( K^\pm_{\infty 6}(\varepsilon) \) and \( K^\pm_{16}(\varepsilon) \). We are ready to state our first main theorem of this chapter.

**Main Theorem 3.** Let \( K^\pm_{\infty 6}(\varepsilon) \) and \( K^\pm_{16}(\varepsilon) \) be diagonal matrices satisfying (4.99)-(4.102). These matrices satisfy the following limits:

\[
\begin{align*}
\lim_{\varepsilon \to 0} \lim_{\arg(\varepsilon) = \pm \frac{\pi}{2}} K^\pm_{\infty 6}(\varepsilon) \varepsilon^{-\frac{\theta_{\infty 5}}{2}} \sigma_3 &= I, \\
\lim_{\varepsilon \to 0} \lim_{\arg(\varepsilon) = \pm \frac{\pi}{2}} K^\pm_{16}(\varepsilon) \varepsilon^{-\frac{\theta_{\infty 5}}{2}} \sigma_3 &= I.
\end{align*}
\]

As part of the proof of this theorem, we use Lemmas 3.13 and 3.14 from Section 3.3.

Note that Lemma 3.14 is written to fit with the hypergeometric equations, we should now make use of \( \varepsilon \equiv \alpha^{-1} \) so that \( \alpha \to \infty \iff \varepsilon \to 0 \).

**Proof of our Main Theorem 3.** To prove the first statement (4.103), let \( K^\pm_{\infty 6}(\varepsilon) \) be matrices satisfying (4.99) and (4.101). In either case \( \arg(\varepsilon) = \frac{\pi}{2} \) or \( -\frac{\pi}{2} \), let \( \mathcal{S}^* \) be a closed, proper subsector of \( \mathcal{S}_0 \) or \( \mathcal{S}_1 \) respectively. Combining the statements (4.99)
and (4.101), together with the asymptotic behavior (4.37), we have,

$$\lim_{\varepsilon \to 0} Y_6^{(\infty)}(\varepsilon \lambda_5) \bigg|_{\lambda_5 \in \sigma_\infty(\varepsilon)} K^\pm_\infty(\varepsilon) \sim Y_5^{(\infty)}(\lambda_5), \quad \text{as } \lambda_5 \to \infty, \ \lambda_5 \in \mathcal{I}. \quad (4.105)$$

We write the solution $Y_6^{(\infty)}(\lambda_6)$ as in (4.66), remembering to substitute the changes of variables (4.47) and parameters (4.48), we have,

$$\lim_{\varepsilon \to 0} Y_6^{(\infty)}(\varepsilon \lambda_5) \bigg|_{\lambda_5 \in \sigma_\infty(\varepsilon)} K^\pm_\infty(\varepsilon) =$$

$$\lim_{\varepsilon \to 0} \sum_{n=0}^\infty \hat{G}_{n,\infty} \varepsilon^{-n} \lambda_5^{-n} \bigg|_{\lambda_5 \in \sigma_\infty(\varepsilon)} (\varepsilon \lambda_5) - \frac{\delta_\infty}{2\pi} \sigma_3 (1 - \varepsilon \lambda_5)^{\frac{\sigma_3}{2}} K^\pm_\infty(\varepsilon).$$

We use Lemma 3.14 i) with,

$$f(\varepsilon) = \lambda_5^{-\frac{\delta_\infty}{2\pi} \sigma_3} (1 - \varepsilon \lambda_5)^{-\frac{\sigma_3}{2}},$$

and $g(\varepsilon) = \sum_{n=0}^\infty \hat{G}_{n,\infty} \varepsilon^{-n} \lambda_5^{-n} \bigg|_{\lambda_5 \in \sigma_\infty(\varepsilon)} K^\pm_\infty(\varepsilon) \varepsilon^{-\frac{\delta_\infty}{2\pi} \sigma_3}.$

Observe that the hypotheses of Lemma 3.14 hold: the limits,

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)g(\varepsilon)}{\arg(\varepsilon) = \pm \frac{\pi}{2}},$$

exist, since these are (4.99) and (4.101), and the limits

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\arg(\varepsilon) = \pm \frac{\pi}{2}}$$

exist and have determinant equal to one by (4.69). We therefore deduce,

$$\lim_{\varepsilon \to 0} Y_6^{(\infty)}(\varepsilon \lambda_5) \bigg|_{\lambda_5 \in \sigma_\infty(\varepsilon)} K^\pm_\infty(\varepsilon) =$$

$$\lim_{\varepsilon \to 0} \sum_{n=0}^\infty \hat{G}_{n,\infty} \varepsilon^{-n} \lambda_5^{-n} \bigg|_{\lambda_5 \in \sigma_\infty(\varepsilon)} K^\pm_\infty(\varepsilon) \varepsilon^{-\frac{\delta_\infty}{2\pi} \sigma_3} \lambda_5^{-\frac{\sigma_3}{2}} e^{\frac{\lambda_5}{2} \sigma_3}. \quad (4.106)$$
Combining this with (4.105) and writing \( Y^{(\infty)}(\lambda_5) \) as in (4.39), we have,

\[
\lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \hat{G}_{n,\infty} \varepsilon^{-n} \lambda_5^{-n} \mid_{\lambda_5 \in \sigma_{\infty}(\varepsilon)} \left[ K_{\infty 6}^{\pm}(\varepsilon) \varepsilon^{-\frac{\theta_{0,\infty}}{2}} \sigma_3 \right] \sim \sum_{n=0}^{\infty} H_{n,\infty} \lambda_5^{-n},
\]

as \( \lambda_5 \to \infty \) with \( \lambda_5 \in \mathcal{S}^* \).

We now define \( w = \lambda_5^{-1} \) and we apply Lemma 3.13 to find,

\[
H_{n,\infty} = \frac{1}{n!} \lim_{\varepsilon \to 0} \lim_{w \to 0} \frac{d^n}{dw^n} \sum_{l=0}^{\infty} \hat{G}_{l,\infty} \varepsilon^{-l} w^l \mid_{w^{-1} \in \sigma_{\infty}(\varepsilon)} K_{\infty 6}^{\pm}(\varepsilon) \varepsilon^{-\frac{\theta_{0,\infty}}{2}} \sigma_3.
\]

We first note that, due to the uniformity of the limits (4.99) and (4.101), we may interchange the limit in \( \varepsilon \) with the derivative and the limit in \( w \) as follows,

\[
H_{n,\infty} = \frac{1}{n!} \lim_{\varepsilon \to 0} \lim_{w \to 0} \frac{d^n}{dw^n} \sum_{l=0}^{\infty} \hat{G}_{l,\infty} \varepsilon^{-l} w^l \mid_{w^{-1} \in \sigma_{\infty}(\varepsilon)} K_{\infty 6}^{\pm}(\varepsilon) \varepsilon^{-\frac{\theta_{0,\infty}}{2}} \sigma_3.
\]

The next step is to notice that the series inside the limits on the right hand side represents an analytic function (or at least its analytic extension to the sector \( \sigma_{\infty}(\varepsilon) \) does). We may therefore interchange the derivative and series as follows,

\[
H_{n,\infty} = \frac{1}{n!} \lim_{\varepsilon \to 0} \lim_{w \to 0} \frac{d^n}{dw^n} \sum_{l=0}^{\infty} \hat{G}_{l,\infty} \varepsilon^{-l} w^l \mid_{w^{-1} \in \sigma_{\infty}(\varepsilon)} K_{\infty 6}^{\pm}(\varepsilon) \varepsilon^{-\frac{\theta_{0,\infty}}{2}} \sigma_3.
\]

Furthermore, due to the analyticity of the series on the right hand side, its limit as \( w \to 0 \) certainly exists and is simply equal to the first term of the series. We finally deduce,

\[
H_{n,\infty} = \frac{1}{n!} \lim_{\varepsilon \to 0} n! \hat{G}_{n,\infty} \varepsilon^{-n} K_{\infty 6}^{\pm}(\varepsilon) \varepsilon^{-\frac{\theta_{0,\infty}}{2}} \sigma_3. \tag{4.107}
\]
We use Lemma 3.14 once more, this time with,

\[ f(ε) = \hat{G}_{n,∞}e^{-n} \quad \text{and} \quad g(ε) = K_{∞6}^+(ε)e^{-\frac{θ_5}{2}+\sigma_3}. \]

Observe that the hypotheses of Lemma 3.14 hold: the limits,

\[ \lim_{ε \to 0} f(ε)g(ε), \]

exist, since these are (4.107) and, crucially, the limits,

\[ \lim_{ε \to 0} f(ε), \]

exist and are equal to \( H_{n,∞} \) by Lemma 4.5. We remark that, for all \( n \), the matrices \( H_{n,∞} \) must be invertible because they are the coefficients in the asymptotic expansion of the fundamental solutions \( Y_5^{(∞,k)}(λ_5) \), and fundamental solutions are invertible by definition. We therefore have,

\[ \lim_{ε \to 0} \hat{G}_{n,∞}e^{-n}K_{∞6}^+(ε)e^{-\frac{θ_5}{2}+\sigma_3} = H_{n,∞} \lim_{ε \to 0} K_{∞6}^+(ε)e^{-\frac{θ_5}{2}+\sigma_3}. \]

Comparing with the left hand side of (4.107) we deduce the desired result (4.103). The limit (4.104) can be proved following a similar procedure, using \( Y_6^{(1)}(ελ_5) \) as given by (4.80) on the sector \( σ_1^-(ε) \) and using Lemma 4.7 in place of Lemma 4.5.

\[ \square \]

### 4.4 Limits of Monodromy Data

We bring together the previous results of Sections 4.2 and 4.3 to prove our second main theorem of this chapter, concerned with producing the set of monodromy data \( M_5 \) from \( M_6 \).

**Main Theorem 4.** Define the monodromy data of the auxiliary linear system associated to \( P_{VI} \) as in (4.18)-(4.22) and to \( P_V \) as in (4.41)-(4.46). Under the substitutions (4.47)-(4.48) and the conditions in our Main Assumption 1, as stated in Section 4.1.3,
we have the following limits of connection matrices,

\[
\begin{align*}
\lim_{\varepsilon \to 0} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} C_6^{1\infty} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= S_0, \\
\lim_{\arg(\varepsilon) = -\frac{\pi}{2}} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} C_6^{1\infty} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= S_{-1}, \\
\lim_{\varepsilon \to 0} \varepsilon \frac{\theta_{16}^6 \sigma_3}{2} C_6^{0\infty} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= C_5^{0\infty}, \\
\lim_{\varepsilon \to 0} \varepsilon \frac{\theta_{16}^6 \sigma_3}{2} C_6^{1\infty} \left(C_6^{1\infty}\right)^{-1} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= C_5^{1\infty}, \\
\lim_{\varepsilon \to 0} \varepsilon \frac{\theta_{16}^6 \sigma_3}{2} C_6^{1\infty} \left(C_6^{1\infty}\right)^{-1} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= C_5^{1\infty}.
\end{align*}
\]

The limits above imply the following limits of monodromy matrices:

\[
\begin{align*}
\lim_{\arg(\varepsilon) = \frac{\pi}{2}} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} M_{06} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= M_{05}, \\
\lim_{\arg(\varepsilon) = \frac{\pi}{2}} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} M_{06} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= M_{15}, \\
\lim_{\arg(\varepsilon) = -\frac{\pi}{2}} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} C_6^{1\infty} M_{06} \left(C_6^{1\infty}\right)^{-1} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= M_{05}, \\
\lim_{\arg(\varepsilon) = -\frac{\pi}{2}} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} C_6^{1\infty} M_{06} \left(C_6^{1\infty}\right)^{-1} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= M_{15}, \\
\lim_{\arg(\varepsilon) = -\frac{\pi}{2}} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} M_{616} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= M_{5\infty}, \\
\lim_{\arg(\varepsilon) = -\frac{\pi}{2}} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} C_6^{1\infty} M_{616} \left(C_6^{1\infty}\right)^{-1} \varepsilon \frac{\theta_{16}^6 \sigma_6 \sigma_3}{2} &= M_{5\infty}.
\end{align*}
\]

As part of the proof of this theorem, we use Lemma 3.14 from Section 3.3.2, with the understanding that \(\varepsilon \equiv \alpha^{-1}\).

**Proof of our Main Theorem 4.** Let \(\sigma_1(\varepsilon)\) and \(\sigma_\infty(\varepsilon)\) be defined as in (4.97) and (4.98) respectively. As mentioned previously, if \(\lambda_5 \in \sigma_1(\varepsilon)\) then \(\lambda_6 \in \Omega_1\) and if \(\lambda_5 \in \sigma_\infty(\varepsilon)\) then \(\lambda_6 \in \Omega_\infty\), so that the connection matrix \(C_6^{1\infty}\) remains valid for the solutions \(Y_6^{(1)}(\varepsilon \lambda_5)\) and \(Y_6^{(\infty)}(\varepsilon \lambda_5)\) restricted to the sectors \(\sigma_1(\varepsilon)\) and \(\sigma_\infty(\varepsilon)\) respectively. Since...
the radii of these sectors do not diminish as \( \varepsilon \to 0 \), for \( |\varepsilon| \) sufficiently small we must have,

\[
\sigma_{\frac{1}{2}}(\varepsilon) \cap \sigma_{\infty}(\varepsilon) \neq \emptyset,
\]

recall Figure 23. Therefore, for \( |\varepsilon| \) sufficiently small, we have,

\[
Y_6(\infty) \frac{(-1)}{C_6^1} \lambda_5 \epsilon(\varepsilon) \cap \sigma_{\infty}(\varepsilon) \neq \emptyset, \quad \lambda_5 \in \sigma_{\frac{1}{2}}(\varepsilon).
\]

(4.120)

Let \( \mathcal{S}_k \) be the sectors as defined in (4.96). To prove the first limit (4.108), we first give a proof of Glutsyuk’s Corollary 2.1 in our case. We multiply by the matrices \( K_{\infty 6}(\varepsilon) \) and \( K_{16}(\varepsilon) \) and take the limit \( \varepsilon \to 0 \), with \( \arg(\varepsilon) = -\frac{\pi}{2} \), so that (4.120) above becomes,

\[
\lim_{\varepsilon \to 0} \arg(\varepsilon) = \frac{-\pi}{2} Y_6(\infty) \frac{(-1)}{C_6^1} \lambda_5 \epsilon(\varepsilon) \cap \sigma_{\infty}(\varepsilon) \neq \emptyset, \quad \lambda_5 \in \mathcal{S}_0 \cap \mathcal{S}_1.
\]

(4.121)

for \( \lambda_5 \in \mathcal{S}_0 \cap \mathcal{S}_1 \). We apply Lemma 3.14 i) with,

\[
f(\varepsilon) = Y_6(\varepsilon) \frac{(-1)}{C_6^1} \lambda_5 \epsilon(\varepsilon) \cap \sigma_{\frac{1}{2}}(\varepsilon) \quad \text{and} \quad g(\varepsilon) = (K_{16}^{-1}(\varepsilon))^{-1} C_6^1 K_{\infty 6}(\varepsilon).
\]

Observe that the hypotheses of Lemma 3.14 hold: the limit,

\[
\lim_{\varepsilon \to 0} \arg(\varepsilon) = \frac{-\pi}{2} f(\varepsilon) g(\varepsilon),
\]

exists and equals \( Y_5(\infty,1) \lambda_5 \), by (4.99), and the limit,

\[
\lim_{\varepsilon \to 0} \arg(\varepsilon) = \frac{-\pi}{2} f(\varepsilon),
\]

exists and equals \( Y_5(\infty,0) \lambda_5 \), by (4.100), which is clearly invertible because it is a fundamental solution. For all \( \varepsilon \), \( f(\varepsilon) \) is also clearly invertible because it is a fundamental solution. The limit,

\[
\lim_{\varepsilon \to 0} \arg(\varepsilon) = \frac{-\pi}{2} g(\varepsilon) = \lim_{\varepsilon \to 0} \arg(\varepsilon) = \frac{-\pi}{2} (K_{16}^{-1}(\varepsilon))^{-1} C_6^1 K_{\infty 6}(\varepsilon),
\]

Page 149 of 178
therefore exists and, from (4.121),

\[ Y_5^{(\infty,1)}(\lambda_5) = Y_5^{(\infty,0)}(\lambda_5) \lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = -\frac{\pi}{2}} (K_{16}(\varepsilon))^{-1} C_6^{1\infty} K_{\infty 6}(\varepsilon), \quad \lambda_5 \in \mathcal{S}_0 \cap \mathcal{S}_1. \]

Recall that if \( \lambda_5 \in \mathcal{S}_k \) then \( \lambda_5 \in \Sigma_k \) and recall Definition 4.3 of Stokes’ matrices, namely we have,

\[ Y_5^{(\infty,1)}(\lambda_5) = Y_5^{(\infty,0)}(\lambda_5) S_0, \quad \lambda_5 \in \Sigma_0 \cap \Sigma_1. \]

We conclude that,

\[ \lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = -\frac{\pi}{2}} (K_{16}(\varepsilon))^{-1} C_6^{1\infty} K_{\infty 6}(\varepsilon) = S_0, \]

which is precisely Glutsyuk’s Corollary 2.1 in our case. Combining this with (4.103) and (4.104) from our Main Theorem 3, we compute,

\[
S_0 = \lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = -\frac{\pi}{2}} (K_{16}^{-}(\varepsilon))^{-1} C_6^{1\infty} K_{\infty 6}^{-}(\varepsilon),
\]

\[
= \lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = -\frac{\pi}{2}} \left( K_{16}^{-}(\varepsilon)e^{-\frac{\lambda_5}{2} \sigma_3} e^{\frac{\lambda_5}{2} \sigma_1} \right)^{-1} C_6^{1\infty} K_{\infty 6}^{-}(\varepsilon)e^{-\frac{\lambda_5}{2} \sigma_3} e^{\frac{\lambda_5}{2} \sigma_1},
\]

\[
= \lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = -\frac{\pi}{2}} \varepsilon^{-\frac{\lambda_5}{2} \sigma_3} C_6^{1\infty} e^{\frac{\lambda_5}{2} \sigma_1},
\]

this proves the first limit (4.108) of the theorem. To prove the second limit (4.109), we multiply by the matrices \( K_{\infty 6}^{+}(\varepsilon) \) and \( K_{16}^{+}(\varepsilon) \) and take the limit \( \varepsilon \to 0 \), with \( \arg(\varepsilon) = \frac{\pi}{2} \), so that (4.120) becomes,

\[
\lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = \frac{\pi}{2}} Y_6^{(\infty)}(\varepsilon \lambda_5) \bigg|_{\lambda_5 \in \sigma_{\infty}(\varepsilon)} K_{\infty 6}^{+}(\varepsilon) = \lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = \frac{\pi}{2}} Y_6^{(1)}(\varepsilon \lambda_5) \bigg|_{\lambda_5 \in \sigma_{1}(\varepsilon)} K_{16}^{+}(\varepsilon) \left( K_{16}^{+}(\varepsilon) \right)^{-1} C_6^{1\infty} K_{\infty 6}^{+}(\varepsilon),
\]

(4.122)

for \( \lambda_5 \in \mathcal{S}_1 \cap \mathcal{S}_6 \). By following a similar procedure as above, using Lemma 3.14 and the relations (4.101) and (4.102), we deduce,

\[
\lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = \frac{\pi}{2}} (K_{16}^{+}(\varepsilon))^{-1} C_6^{1\infty} K_{\infty 6}^{+}(\varepsilon) = S_{-1}.
\]
Combining this with (4.103) and (4.104) from our Main Theorem 3, we compute,

\[
S_{-1} = \lim_{\varepsilon \to 0} \arg(\varepsilon) = \frac{\pi}{2} = \frac{\varepsilon_{65}}{2},
\]

where we have implicitly used Lemma 3.14, this proves the second limit (4.109) of the theorem.

To prove the third limit (4.110) we first note that the curve \( \gamma_{\infty 0} \) which defines the connection matrix \( C_{6}^{\infty} \) survives the confluent limit. In other words, after the change of variable \( \lambda_{6} = \varepsilon \lambda_{5} \), the curve does not diminish or become broken under the limit \( \varepsilon \to 0 \). This fact is expressed in the following limit,

\[
\lim_{\varepsilon \to 0} \arg(\varepsilon) = \frac{\pi}{2} = \frac{\varepsilon_{65}}{2},
\]

or equivalently, using the domains \( \omega_{05}^{+}(\varepsilon) \) and \( \Omega_{05}^{+} \) defined in subsections 4.2.1 and 4.1.2 respectively,

\[
\lim_{\varepsilon \to 0} \left. Y_{6}^{(0)}(\varepsilon \lambda_{5}) \right|_{\lambda_{5} \in \omega_{05}^{+}(\varepsilon)} = Y_{6}^{(0)}(\lambda_{5}) C_{6}^{\infty}, \quad \lambda_{5} \in \Omega_{05}^{+}.
\]

Combining this with the limits (4.62) in Theorem 4.5 and (4.103) in our Main Theorem 3, we deduce the required result (4.110) as follows,

\[
\lim_{\varepsilon \to 0} \left. Y_{6}^{(0)}(\varepsilon \lambda_{5}) \right|_{\lambda_{5} \in \omega_{05}^{+}(\varepsilon)} = Y_{5}^{(0)}(\lambda_{5}) C_{5}^{\infty}, \quad \lambda_{5} \in \Omega_{05}^{+}.
\]

where we have implicitly used Lemma 3.14.
The proof of the fourth limit (4.111) is analogous: the curve $\gamma_{10}$ which defines the connection matrix $C_6^{10 \equiv C_6^{0\infty}C_6^{\infty1}}$ survives the confluence limit. The substitution $\lambda_6 = \varepsilon \lambda_5$ and limit $\varepsilon \to 0$ certainly translates one of the base points of the curve, but not in such a way that the length of the curve vanishes or the homotopy of the curve is affected. This fact is expressed in the following limit,

$$
\lim_{\varepsilon \to 0} \text{arg} (\varepsilon) = -\frac{\pi}{2} \gamma_{10} \left[ Y_6^{(1)} K_{16}^{(1)} (\varepsilon \lambda_5) \right] = \gamma_{\infty 0} \left[ Y_5^{(\infty, 0)} \right],
$$

or equivalently,

$$
\lim_{\varepsilon \to 0} \text{arg} (\varepsilon) = -\frac{\pi}{2} \gamma_{10} \left[ Y_6^{(0)} (\varepsilon \lambda_5) \right]_{\lambda_5 \in \omega_{05}} (\varepsilon) C_6^{0\infty} (C_6^{1\infty})^{-1} K_{16}^{-} (\varepsilon) = Y_5^{(0)} (\lambda_5) C_5^{0\infty}, \quad \lambda_5 \in \Omega_{05}^{-}.
$$

Combining this with the limits (4.62) in Theorem 4.5 and (4.104) in our Main Theorem 3, we deduce the required result (4.111) as follows,

$$
\lim_{\varepsilon \to 0} \text{arg} (\varepsilon) = -\frac{\pi}{2} \gamma_{10} \left[ Y_6^{(0)} (\varepsilon \lambda_5) \right]_{\lambda_5 \in \omega_{05}} (\varepsilon) C_6^{0\infty} (C_6^{1\infty})^{-1} K_{16}^{-} (\varepsilon) \in C_6^{0\infty} \left( C_6^{1\infty} \right)^{-1} K_{16}^{-} (\varepsilon) \in C_6^{0\infty} \left( C_6^{1\infty} \right)^{-1} K_{16}^{-} (\varepsilon) = Y_5^{(0)} (\lambda_5) C_5^{0\infty}, \quad \lambda_5 \in \Omega_{05}^{-},
$$

The limits (4.112) and (4.113) are proved in the exact same way as the previous cases.

Having deduced the limits of connection matrices, the limits of monodromy matrices
follow as a direct result. For example, to prove (4.114),

\[
\lim_{\varepsilon \to 0} \varepsilon^{-\frac{\theta_{16}+\theta_\infty}{2}} \sigma_3 \ M_{06} \varepsilon^{\frac{\theta_{16}+\theta_\infty}{2}} \sigma_3
\]

\[
= \lim_{\varepsilon \to 0} \varepsilon^{-\frac{\theta_{16}+\theta_\infty}{2}} \sigma_3 \left( C_{06}^{0\infty} \right)^{-1} e^{i\pi \theta_0 \sigma_3} C_{06}^{0\infty} \varepsilon^{\frac{\theta_{16}+\theta_\infty}{2}} \sigma_3,
\]

\[
= \lim_{\varepsilon \to 0} \varepsilon^{-\frac{\theta_{16}+\theta_\infty}{2}} \sigma_3 \left( C_{06}^{0\infty} \right)^{-1} e^{i\pi \theta_0 \sigma_3} e^{i\pi \theta_0 \sigma_3} \varepsilon^{\frac{\theta_{16}+\theta_\infty}{2}} \sigma_3 C_{06}^{0\infty} \varepsilon^{\frac{\theta_{16}+\theta_\infty}{2}} \sigma_3,
\]

\[
= \left( C_{5}^{0\infty} \right)^{-1} e^{i\pi \theta_0 \sigma_3} C_{5}^{0\infty} = M_{05},
\]

as required. The limits (4.115)-(4.117) are proved in a similar manner.

Finally, the limits (4.118) and (4.119) are immediately found from (4.114)-(4.117) after using the cyclic relations (4.21) and (4.45) to write \( M_{\infty 6} M_{16} = M_{06}^{-1} M_{16}^{-1} \) and \( M_{\infty 5} = M_{05}^{-1} M_{t5}^{-1} \).

Remark 4.11. It is clear from (4.114)-(4.115) and (4.116)-(4.117) that the difference in the limits of monodromy data between choosing \( \arg(\varepsilon) = -\frac{\pi}{2} \) or \( \frac{\pi}{2} \) is a change of basis, note whether or not the monodromy matrices \( M_{16} \) are conjugated by \( C_{6}^{0\infty} \).
Chapter 5

The Fifth and Third Painlevé Equations

In this chapter we continue working with the auxiliary linear system for $P_V$ and consider another first order linear ODE whose monodromy preserving deformations are described by the third Painlevé equation, as found by Jimbo and Miwa [JM]. The linear system for $P^{D_6}_{III}$ consists of a differential equation with two irregular singularities, both of Poincaré rank one. It is well known that, by imposing certain rescalings on the parameters and variables, the fifth Painlevé equation tends to the third one. We give a new confluence procedure from the auxiliary linear system of $P_V$ to that of $P^{D_6}_{III}$, this involves merging two simple poles to create a double one. The formal limit passage among the auxiliary linear systems allows us to write the asymptotic behavior of the $P_V$ transcendent in terms of the $P_{III}$ transcendent and its derivative, this is stated in Theorem 5.3. The main result in this chapter is to explicitly show how the Stokes’ matrices of the $P^{D_6}_{III}$ linear system at the newly created double pole arise from the monodromy data around the merging simple poles of the $P_V$ linear system, this is stated as our Main Theorem 6. This chapter is briefer than the previous chapters; it is included to emphasise that we do not require the existence of surviving simple poles in the confluence limit and also to demonstrate our novel confluence procedure from the auxiliary linear system of $P_V$ to that of $P^{D_6}_{III}$. 

Page 154 of 178
5.1 Background

We begin by recalling the auxiliary linear system for $P_{III}$ (with $\gamma \delta \neq 0$), as found by Jimbo and Miwa [JM]. In subsection 5.1.1 we define the solutions of these equations and their monodromy data. In subsection 5.1.2 we then demonstrate our confluence procedure from the linear system of $P_{V}$ to that of $P_{III}$ on the formal level of the equations.

5.1.1 Auxiliary Linear System for $P_{D_6}^{III}$

The third Painlevé equation is derived as the compatibility equation (1.3) of the following first order differential equations,

\[
\frac{\partial Y_3}{\partial \lambda_3} = \left( \frac{\sigma_3}{2} + \frac{A_{03}(t_3)}{\lambda_3} + \frac{B_{03}(t_3)}{\lambda_3^2} \right) Y_3, \quad (5.1)
\]

\[
\frac{\partial Y_3}{\partial t_3} = -\frac{2B_{03}(t_3)}{t_3\lambda_3}, \quad (5.2)
\]

where,

\[
A_{03}(t_3) = \begin{pmatrix}
t_3 & -\frac{\theta_{\infty,3}}{2} \\
\frac{\theta_{03} - \theta_{\infty,3}}{2} - \frac{\theta_{\infty,3} z_3}{t_3} + y_3 z_3 - \frac{y_3 z_3 u_3}{t_3} & -y_3 z_3 u_3 
\end{pmatrix}, \quad (5.3)
\]

\[
B_{03}(t_3) = t_3 \begin{pmatrix}
z_3 - \frac{t_3}{2} & -z_3 u_3 \\
\frac{z_3 - t_3}{u_3} & -z_3 + \frac{t_3}{2}
\end{pmatrix}, \quad (5.4)
\]

where $y_3$, $z_3$ and $u_3$ are functions of $t_3$ and $\theta_{03}$ and $\theta_{\infty,3}$ are parameters. Given a solution of equation (5.1), the condition for this solution to satisfy equation (5.2) is precisely the condition that its monodromy data should be independent of $t_3$. We assume there exists $R_{03} \in \text{GL}_2(\mathbb{C})$ such that,

\[
R_{03}^{-1}B_{03}R_{03} = -\frac{t_3^2}{2}\sigma_3.
\]
By comparing terms at the poles \( \lambda_3 = 0 \) and \( \infty \), we have that the compatibility condition (1.3) of (5.1)-(5.2) is equivalent to the following system of first order ODEs:

\[
\begin{align*}
A_{03}' &= t_3^{-1} [\sigma_3, B_{03}], \quad (5.5) \\
B_{03}' &= 2t_3^{-1} (B_{03} + [A_{03}, B_{03}]). \quad (5.6)
\end{align*}
\]

This system of ODEs is equivalent to three first order ODEs:

\[
\begin{align*}
y_3' &= t_3^{-1} \left( 2t_3 + y_3(2\theta_{\infty 3} - 1) + 2y_3^2(2z_3 - t_3) \right), \quad (5.7) \\
z_3' &= t_3^{-1} \left( t_3(\theta_{03} + \theta_{\infty 3}) + z_3(4y_3(t_3 - z_3) - 2\theta_{\infty 3} + 1) \right), \quad (5.8) \\
u_3' &= -u_3z_3^{-1} \left( 2y_3z_3 + \theta_{03} + \theta_{\infty 3} \right). \quad (5.9)
\end{align*}
\]

In the Hamiltonian formulation of \( P_{III} \), \( q = y_3(t_3) \) is the coordinate and \( p = z_3(t_3) \) is the conjugate momentum. We note that the function \( u_3(t_3) \) represents the freedom of global conjugation of the system (5.1)-(5.2) by a diagonal matrix. The ODEs for \( y_3 \) and \( z_3 \) as above are equivalent to the third Painlevé equation \( P_{III} \). In other words, up to a global conjugation by a diagonal matrix, the compatibility condition (1.3) of (5.1)-(5.2) is equivalent to the Painlevé equation \( P_{III}^{D_6} \). This statement is embodied in the following theorem.

**Theorem 5.1 [JM].** Given a solution \( Y_3 \) of (5.1)-(5.2), then \( y_3(t_3) \) satisfies \( P_{III} \) (see Introduction), with \( \alpha = 4\theta_{03}, \beta = 4 - 4\theta_{\infty 3} \) and \( \gamma = -\delta = 4 \).

We are concerned with solving equation (5.1), which has two irregular singularities, each of Poincaré rank one, at \( \lambda_3 = 0 \) and \( \infty \). As such, following the theory of Section 2.1, the monodromy data of (5.1) consists of two monodromy matrices, each of which contains a product of two Stokes’ matrices. We summarise the Stokes’ phenomenon theory in this case in the following definitions and theorem.

**Definition 5.1.** The two components of each set,

\[
\{ \lambda_3 : \text{Re}(\lambda_3) = 0 \} \quad \text{and} \quad \left\{ \lambda_3 : \text{Re} \left( \frac{t_3^2}{\lambda_3} \right) = 0 \right\},
\]

are the Stokes’ rays at the points \( \lambda_3 = \infty \) and \( 0 \) respectively. At \( \lambda_3 = \infty \), the Stokes’ rays are the positive and negative imaginary negative axis, while, at \( \lambda_3 = 0 \), the Stokes’
The asymptotic behaviors of the true solutions at \( \lambda = \infty \) correspond to choosing \( \rho_0 = 0 \), and \( \rho_0 = \infty \), respectively. Each solution \( Y_n \) is uniquely specified by the relation (5.10) (resp. (5.11)).

**Theorem 5.2.** For some \( \rho_0, \rho_\infty > 0 \), let,

\[
\Sigma_k^{(\infty)} = \left\{ \lambda_3 : |\lambda_3| > \rho_\infty, \ -\frac{\pi}{2} < \arg(\lambda_3) - k\pi < \frac{3\pi}{2} \right\}
\]

and

\[
\Sigma_k^{(0)} = \left\{ \lambda_3 : |\lambda_3| < \rho_0, \ -\frac{\pi}{2} < \arg(\lambda_3) - \arg(t_3^2) + k\pi < \frac{3\pi}{2} \right\}.
\]

For all \( k \in \mathbb{Z} \), there exists a true (analytic) solution \( Y_3^{(\infty,k)}(\lambda_3) \) (respectively \( Y_3^{(0,k)}(\lambda_3) \)) of equation (5.1) defined in the sector \( \Sigma_k^{(\infty)} \) (resp. \( \Sigma_k^{(0)} \)) such that,

\[
Y_3^{(\infty,k)}(\lambda_3) \sim \left( \sum_{n=0}^{\infty} F_{n,\infty}^* \lambda_3^{-n} \right) \lambda_3^\frac{\rho_\infty}{2} e^{\frac{\lambda_3}{2} \sigma_3} \quad \text{as} \quad \lambda_3 \to \infty, \ \lambda_3 \in \Sigma_k^{(\infty)}, \quad (5.10)
\]

resp.

\[
Y_3^{(0,k)}(\lambda_3) \sim R_{03} \left( \sum_{n=0}^{\infty} F_{n,0}^* \lambda_3^{-n} \right) \lambda_3^\frac{\rho_0}{2} e^{\frac{t^2_3}{2} \sigma_3} \quad \text{as} \quad \lambda_3 \to 0, \ \lambda_3 \in \Sigma_k^{(0)}, \quad (5.11)
\]

where \( F_{0,0} = I = F_{0,0} \) and all other terms of the series are uniquely determined by the following recursive formulae, for \( n \geq 1 \),

\[
\begin{bmatrix}
F_{n,\infty} \\
F_{n,0}
\end{bmatrix} = (n-1) \begin{bmatrix}
F_{n-1,\infty} \\
F_{n-1,0}
\end{bmatrix} + \begin{bmatrix}
\frac{\theta_\infty}{2} & \sigma_3 \\
\frac{\theta_0}{2} & \sigma_3
\end{bmatrix} \begin{bmatrix}
F_{n-1,\infty} \\
F_{n-1,0}
\end{bmatrix} + A_{03} F_{n-1,\infty} + B_{03} F_{n-2,\infty}, \quad (5.12)
\]

\[
\begin{bmatrix}
F_{n,\infty} \\
F_{n,0}
\end{bmatrix} = (n-1) \begin{bmatrix}
F_{n-1,\infty} \\
F_{n-1,0}
\end{bmatrix} + \begin{bmatrix}
\frac{\theta_0}{2} & \sigma_3 \\
\frac{\theta_{03}}{2} & \sigma_3
\end{bmatrix} \begin{bmatrix}
F_{n-1,\infty} \\
F_{n-1,0}
\end{bmatrix} - R_{03}^{-1} A_{03} R_{03} F_{n-1,0} - R_{03} \frac{\sigma_3}{2} R_{03} F_{n-2,0}, \quad (5.13)
\]

respectively. Each solution \( Y_3^{(\infty,k)}(\lambda_3) \) (resp. \( Y_3^{(0,k)}(\lambda_3) \)) is uniquely specified by the relation (5.10) (resp. (5.11)).

**Remark 5.1.** Since equation (5.1) is a linear equation with only two poles at \( \lambda_3 = 0 \) and \( \infty \), we could choose the radii of the sectors \( \Sigma_k^{(0)} \) and \( \Sigma_k^{(\infty)} \) to be infinite. This would correspond to choosing \( \rho_0 = \infty \) and \( \rho_\infty = 0 \).

We denote the asymptotic behaviors of the true solutions at \( \lambda_3 = \infty \) and \( 0 \) by,

\[
Y_3^{(\infty)}(\lambda_3) = \left( \sum_{n=0}^{\infty} F_{n,\infty}^* \lambda_3^{-n} \right) \lambda_3^\frac{\rho_\infty}{2} e^{\frac{\lambda_3}{2} \sigma_3},
\]

and

\[
Y_3^{(0)}(\lambda_3) = R_{03} \left( \sum_{n=0}^{\infty} F_{n,0}^* \lambda_3^{-n} \right) \lambda_3^\frac{\rho_0}{2} e^{\frac{t^2_3}{2} \sigma_3},
\]
respectively. We call these functions formal solutions in the sense that both series diverge for general parameters $\theta_0$ and $\theta_\infty$. From the asymptotic relation (5.10) (respectively (5.11)), it is clear that the solutions,

\[
Y_3^{(\infty,k+2)}(\lambda_3) \quad \text{and} \quad Y_3^{(\infty,k)}(\lambda_3 e^{-2\pi i}) e^{-i\pi \theta_\infty \sigma_3},
\]

and

\[
y_3^{(0,k+2)}(\lambda_3) \quad \text{and} \quad y_3^{(0,k)}(\lambda_3 e^{2\pi i}) e^{-i\pi \theta_0 \sigma_3},
\]

have the same asymptotic behavior as $\lambda_3 \to \infty$ in the sector $\lambda_3 \in \Sigma_{k+2}^{(\infty)}$, resp. $\lambda_3 \to 0$ in the sector $\lambda_3 \in \Sigma_{k+2}^{(0)}$. By the last statement of Theorem 5.2, we therefore conclude that,

\[
Y_3^{(\infty,k+2)}(\lambda_3) \equiv Y_3^{(\infty,k)}(\lambda_3 e^{-2\pi i}) e^{-i\pi \theta_\infty \sigma_3},
\]

and

\[
y_3^{(0,k+2)}(\lambda_3) \equiv y_3^{(0,k)}(\lambda_3 e^{2\pi i}) e^{-i\pi \theta_0 \sigma_3}.
\]

In this sense, we have four fundamentally distinct solutions of (5.1), namely $Y_3^{(\infty,k)}(\lambda_3)$ and $Y_3^{(0,k)}(\lambda_3)$ when $k$ is even and when $k$ is odd.

**Definition 5.2.** Let,

\[
\Pi_k^{(\infty)} := \Sigma_k^{(\infty)} \cap \Sigma_k^{(\infty)} \equiv \left\{ \lambda_3 : \frac{\pi}{2} < \arg(\lambda_3) - k\pi < \frac{3\pi}{2} \right\}
\]

and

\[
\Pi_k^{(0)} := \Sigma_k^{(0)} \cap \Sigma_k^{(0)} \equiv \left\{ \lambda_3 : \frac{\pi}{2} < \arg(\lambda_3) - 2\arg(t_3) + k\pi < \frac{3\pi}{2} \right\}.
\]

We define Stokes’ matrices $S_k^{(\infty)}$, $S_k^{(0)} \in \text{SL}_2(\mathbb{C})$ as follows,

\[
Y_3^{(\infty,k+1)}(\lambda_3) = Y_3^{(\infty,k)}(\lambda_3) S_k^{(\infty)}, \quad \lambda_3 \in \Pi_k^{(\infty)},
\]

\[
y_3^{(0,k+1)}(\lambda_3) = y_3^{(0,k)}(\lambda_3) S_k^{(0)}, \quad \lambda_3 \in \Pi_k^{(0)}.
\]

**Remark 5.2.** From the asymptotic relations (5.10) and (5.11), we deduce,

\[
\lambda_3^{-\frac{a_{21} + a_{31}}{2}} e^{-\frac{a_{21} + a_{31}}{2} \sigma_3} S_k^{(\infty)} e^{-\frac{a_{21} + a_{31}}{2} \sigma_3} \sim I, \quad \text{as } \lambda_3 \to \infty, \lambda_3 \in \Pi_k^{(\infty)},
\]

and

\[
\lambda_3^{-\frac{a_{01} - a_{21}}{2}} e^{-\frac{a_{01} - a_{21}}{2} \sigma_3} S_k^{(0)} e^{-\frac{a_{01} - a_{21}}{2} \sigma_3} \sim I, \quad \text{as } \lambda_3 \to 0, \lambda_3 \in \Pi_k^{(0)},
\]

from which it is easy to see that the matrices $S_{2k}^{(\infty)}$ and $S_{2k}^{(0)}$ are upper triangular,
the matrices $S_{2k+1}^{(\infty)}$ and $S_{2k+1}^{(0)}$ are lower triangular and all Stokes’ matrices have unit diagonal.

We choose to normalise the monodromy data of equation (5.1) with respect to the fundamental solutions $Y_{3}^{(\infty,0)}(\lambda_{3})$ and $Y_{3}^{(0,0)}(\lambda_{3})$. Let $\gamma_{\infty 0} : [0,1] \to \mathbb{C}$ be an orientable curve with $\gamma_{\infty 0}(0) \in \Sigma_{0}^{(\infty)}$ and $\gamma_{\infty 0}(1) \in \Sigma_{0}^{(0)}$. We define the connection matrix $C_{3}^{0\infty} \in \text{GL}_{2}(\mathbb{C})$ as follows,

$$
\gamma_{\infty 0} \left[ Y_{3}^{(\infty,0)} \right](\lambda_{3}) = Y_{3}^{(0,0)}(\lambda_{3}) C_{3}^{0\infty}.
$$

Let $\gamma_{0} : [0,1] \to \mathbb{C}$ be a closed, orientable curve with $\gamma(0) = \gamma(1) \in \Sigma_{0}^{(\infty)}$ which encircles the singularity $\lambda_{3} = 0$ in the positive direction and define $\gamma_{\infty} := \gamma_{0}^{-1}$. We define monodromy matrices $M_{k3} \in \text{SL}_{2}(\mathbb{C})$ as follows,

$$
\gamma_{k} \left[ Y_{3}^{(\infty,0)} \right](\lambda_{3}) = Y_{3}^{(\infty,0)}(\lambda_{3}) M_{k3}, \quad k = 0, \infty. \quad (5.16)
$$

These matrices have the following form,

$$
M_{03} = \left( C_{3}^{0\infty} \right)^{-1} S_{-1}^{(0)} e^{i\pi \theta_{03} \sigma_{3}} S_{0}^{(0)} C_{3}^{0\infty} \quad \text{and} \quad M_{\infty 3} = S_{0}^{(\infty)} e^{i\pi \theta_{\infty 3} \sigma_{3}} S_{-1}^{(\infty)}, \quad (5.17)
$$

and satisfy the following cyclic relation,

$$
M_{\infty 3} M_{03} = I. \quad (5.18)
$$

**Definition 5.3.** We define the monodromy data of equation (5.1) as the set,

$$
\mathcal{M}_{3} := \left\{ \left( S_{0}^{(0)}, S_{-1}^{(0)}, S_{0}^{(\infty)}, S_{-1}^{(\infty)} \right) \in (\text{SL}_{2}(\mathbb{C}))^{4} \mid \begin{array}{c}
S_{0}^{(k)} = \begin{pmatrix}
1 & s_{0}^{(k)} \\
0 & 1
\end{pmatrix}, \quad k = 0, \infty, \\
S_{-1}^{(k)} = \begin{pmatrix}
1 & 0 \\
s_{-1}^{(k)} & 1
\end{pmatrix}, \quad k = 0, \infty, \\
2 \cos(\pi \theta_{\infty 3}) + e^{i\pi \theta_{\infty 3} s_{0}^{(\infty)} s_{-1}^{(\infty)}} = 2 \cos(\pi \theta_{03}) + e^{-i\pi \theta_{03} s_{0}^{(0)} s_{-1}^{(0)}}
\end{array} \right\} / \text{SL}_{2}(\mathbb{C}) \quad (5.19)
$$

where the quotient is by global diagonal conjugation.
5.1.2 A Confluence Procedure from \( P_5 \) to \( P_{III}^{D_6} \)

In this section we outline our confluence procedure from the auxiliary linear system (4.23)-(4.24) of \( P_5 \) to that (5.1)-(5.2) of \( P_{III}^{D_6} \). We make substitutions on the \( P_5 \) variables as follows,

\[
\lambda_5 = \lambda_3, \quad t_5 = \varepsilon t_3^2, \quad (5.20)
\]

and the following substitutions on parameters,

\[
\theta_{\infty 5} = \theta_{\infty 3}, \quad \theta_{05} = \theta_{03} + \varepsilon^{-1}, \quad \theta_{t5} = -\varepsilon^{-1}. \quad (5.21)
\]

Under the substitutions (5.20), the simple poles \( \lambda_5 = 0 \) and \( t_5 \) of the \( P_5 \) linear system merge as \( \varepsilon \to 0 \). In Proposition 5.1 below, we show that, under the substitutions (5.20)-(5.21) and the additional assumption below, the coalescence of these simple poles produces an irregular singularity of Poincaré rank one at \( \lambda_3 = 0 \).

**Main Assumption 2.** There exists an open sector \( E \subset \mathbb{C} \), with its base point at the origin, an open domain \( T \subset \mathbb{C} \) and sequences of matrices \( A_{05}^{(n)}(t_3) \) and \( A_{t5}^{(n)}(t_3) \), \( n \geq 1 \), such that,

\[
A_{05}(\varepsilon t_3^2) + A_{t5}(\varepsilon t_3^2) \sim A_{03}(t_3) + \sum_{n=1}^{\infty} \varepsilon^n \left( A_{05}^{(n)}(t_3) + A_{t5}^{(n)}(t_3) \right), \text{ as } \varepsilon \to 0, \varepsilon \in E, \quad (5.22)
\]

\[
\varepsilon t_3 A_{t5}(\varepsilon t_3^2) \sim t_3^{-1} B_{03}(t_3) + \varepsilon t_3 \sum_{n=0}^{\infty} \varepsilon^n A_{t5}^{(n)}(t_3), \text{ as } \varepsilon \to 0, \varepsilon \in E, \quad (5.23)
\]

uniformly for all \( t_3 \in T \).

With this assumption on the asymptotic behavior of \( A_{05}(t_5) \) and \( A_{t5}(t_3) \) as \( \varepsilon \to 0 \), we naturally extend their definitions to the point \( \varepsilon = 0 \) in the following way:

\[
(A_{05}(t_5) + A_{t5}(t_5))|_{\varepsilon=0} := \lim_{\varepsilon \to 0, \varepsilon \in E} (A_{05}(\varepsilon t_3^2) + A_{t5}(\varepsilon t_3^2)) = A_{03}(t_3),
\]

and

\[
\varepsilon t_3 A_{t5}(t_5)|_{\varepsilon=0} := \lim_{\varepsilon \to 0, \varepsilon \in E} \varepsilon t_3 (A_{t5}(\varepsilon t_3^2)) = t_3^{-1} B_{03}(t_3).
\]
The following proposition establishes how the above substitutions (5.20)-(5.21) and our Main Assumption 2 produce a limit passage from the auxiliary linear system for $P_V$ to that of $P_{l_{111}}^D$ by confluencing the simple poles $\lambda_5 = 0, t_5$ and leaving an irregular singularity at $\lambda_3 = 0$.

**Proposition 5.1.** Under the substitutions (5.20)-(5.21) and our Main Assumption 2, we have,

$$\lim_{\epsilon \to 0} \frac{\partial \lambda_5}{\partial \lambda_3} \left( \frac{\sigma_3}{2} + \frac{A_{05}(t_5)}{\lambda_5} + \frac{A_{15}(t_5)}{\lambda_5 - t_5} \right) = \left( \frac{\sigma_3}{2} + \frac{A_{03}(t_3)}{\lambda_3} + \frac{B_{03}(t_3)}{\lambda_3^2} \right),$$

and,

$$\lim_{\epsilon \to 0} \frac{\partial t_5 A_{15}(t_5)}{\partial t_3} \frac{\lambda_5 - t_5}{\lambda_5} = \frac{2B_{03}(t_3)}{t_3 \lambda_3},$$

with $\epsilon \in E$ and $t_3 \in T$.

**Proof.** Let $\epsilon \in E$ and $t_3 \in T$, as in our Main Assumption 2. For the first limit,

$$\lim_{\epsilon \to 0} \frac{\partial \lambda_5}{\partial \lambda_3} \left( \frac{\sigma_3}{2} + \frac{A_{05}(t_5)}{\lambda_5} + \frac{A_{15}(t_5)}{\lambda_5 - t_5} \right) = \lim_{\epsilon \to 0} \left( \frac{\sigma_3}{2} + \frac{A_{05}(\epsilon t_3^2)}{\lambda_3} + \frac{A_{15}(\epsilon t_3^2)}{\lambda_3 - \epsilon t_3^2} \right),$$

$$= \lim_{\epsilon \to 0} \left( \frac{\sigma_3}{2} + \frac{A_{05}(\epsilon t_3^2) + A_{15}(\epsilon t_3^2)}{\lambda_3} + \frac{\epsilon t_3^2 A_{15}(\epsilon t_3^2)}{\lambda_3^2} \sum_{n=0}^{\infty} \left( \frac{\epsilon t_3^2}{\lambda_3} \right)^n \right),$$

$$= \left( \frac{\sigma_3}{2} + \frac{A_{03}(t_3)}{\lambda_3} + \frac{B_{03}(t_3)}{\lambda_3^2} \right),$$

as required. We note that the second line above is valid for $|\lambda_3^{-1} \epsilon t_3^2| < 1$, it is without loss of generality that we may assume this is true since $\epsilon \to 0$; in particular, this means that the sum in the second line converges uniformly with respect to $\epsilon$, hence its limit is easily calculated as 1. For the second limit,

$$\lim_{\epsilon \to 0} \frac{\partial t_5 A_{15}(t_5)}{\partial t_3} \frac{\lambda_5 - t_5}{\lambda_5 - \epsilon t_3^2} = \lim_{\epsilon \to 0} 2\epsilon t_3 A_{15}(\epsilon t_3^2) = \frac{2B_{03}(t_3)}{t_3 \lambda_3},$$

as required. \qed

**Understanding the Confluence as a Limit Passage from $P_V$ to $P_{l_{111}}^D$**

We have established that the substitutions (5.20)-(5.21) and our Main Assumption
2 provide a limit passage from the isomonodromic deformation problem for $P_V$ to that of $P_{III}^{D_6}$. Since the isomonodromic deformation problems give rise to the Painlevé equations, we can understand our confluence procedure as a formal limit passage among the Painlevé transcendents $P_V$ and $P_{III}^{D_6}$.

**Theorem 5.3.** Let $Y_5(\lambda_5)$ be a solution of the linear system (4.23)-(4.24), hence $y_5(t_3)$ satisfies $P_V$ (recall Theorem 4.2). Under the substitutions (5.20)-(5.21) and the conditions of our Main Assumption 2, $y_5(t_3)$ has the following asymptotic behavior,

$$y_5(\varepsilon t_3^2) \sim \left(1 - \frac{t_3}{z_3(t_3)}\right) + \varepsilon \frac{t_3(\theta_{\infty 3} - \theta_{03} + 2y_3(t_3)(z_3(t_3) - t_3))}{2z_3(t_3)} + \sum_{n=2}^{\infty} \varepsilon^n y_5^{(n)}(t_3),$$

as $\varepsilon \to 0$, $\varepsilon \in E$, uniformly for all $t_3 \in T$, where $y_3(t_3)$ satisfies $P_{III}$, $z_3(t_3)$ is given in terms of $y_3(t_3)$ and $y'_3(t_3)$ by (5.8) and $y_5^{(n)}(t_3)$ are certain functions.

**Proof.** We first note that the asymptotic relation for $y_3(t_3)$ comes from the conditions in our Main Assumption 2. In fact, the conditions in this assumption are equivalent to the one for $y_3(t_3)$ written above and also,

$$z_5(\varepsilon t_3^2) \sim -\frac{z_3(t_3)}{\varepsilon} - \frac{\theta_{03} + \theta_{\infty 3}}{2} + \sum_{n=1}^{\infty} \varepsilon^n z_5^{(n)}(t_3),$$

$$u_5(\varepsilon t_3^2) \sim \frac{u_3(t_3)z_3(t_3)}{z_3(t_3) - t_3} + \sum_{n=1}^{\infty} \varepsilon^n u_5^{(n)}(t_3),$$

where $z_3^{(n)}(t_3)$ and $u_5^{(n)}(t_3)$ are certain functions. We need to show that our confluence procedure is able to produce the compatibility equations (5.5)-(5.6) of $P_{III}$ from the compatibility equations (4.27)-(4.28) of $P_V$. We first note that, due to the uniformity of the asymptotic (5.22) with respect to $t_3$, we have that,

$$\lim_{\varepsilon \to 0} \frac{d}{dt_3} \left(A_{05} (\varepsilon t_3^2) + A_{15} (\varepsilon t_3^2)\right) = \frac{d}{dt_3} A_{03}(t_3).$$

Now, using (4.27) and (4.28), we also have that,

$$\frac{d}{dt_3} (A_{05} + A_{15}) = \frac{dt_5}{dt_3} \left[\frac{\sigma_3}{2}, A_{15}\right] = [\sigma_3, \varepsilon t_3 A_{15}].$$
Combining these facts and applying the asymptotic behavior (5.23), we have,

\[ \frac{d}{dt_3} A_{03}(t_3) = \lim_{\varepsilon \to 0} [\sigma_3, \varepsilon t_3 A_{15}] = t_3^{-1} [\sigma_3, B_{03}(t_3)], \]

which is (5.1), as required. In order to derive (5.2), we similarly note that,

\[ \lim_{\varepsilon \to 0} \frac{d}{dt_3} (\varepsilon t_3^2 A_{15}) = \frac{d}{dt_3} B_{03}, \quad (5.24) \]

due to the uniformity of the asymptotic (5.23) with respect to \( t_3 \). Using (4.28), we have,

\[ \frac{d}{dt_3} (\varepsilon t_3^2 A_{15}) = \varepsilon \left( 2t_3 A_{15} + t_3^2 \frac{d}{dt_3} \left( \frac{\sigma_3}{2}, A_{15} \right) + \frac{1}{\varepsilon t_3^2} [A_{05}, A_{15}] \right), \]
\[ = 2\varepsilon t_3 A_{15} + [\sigma_3, \varepsilon t_3^2 A_{15}] + 2 [A_{05}, \varepsilon t_3 A_{15}]. \quad (5.25) \]

Note that the first term in this expression has limit \( 2t_3^{-1} B_{03} \) in the limit \( \varepsilon \to 0 \) and the second term tends to zero. In order to investigate the third term, we note that the asymptotics (5.22)-(5.23) are equivalent to the following:

\[ A_{05} (\varepsilon t_3^2) \sim -\frac{B_{03}(t_3)}{\varepsilon t_3^3} + A_{03}(t_3) + \ldots, \]
\[ A_{15} (\varepsilon t_3^2) \sim \frac{B_{03}(t_3)}{\varepsilon t_3^3} + \ldots, \]

recall that the asymptotic series of the sum of two functions is equal to the sum of their asymptotic series. Hence,

\[ [A_{05}, \varepsilon t_3 A_{15}] \sim \left( -\frac{B_{03}}{\varepsilon t_3^3} + A_{03} + \ldots \right) \left( \frac{B_{03}}{\varepsilon t_3^3} + \ldots \right) - \left( \frac{B_{03}}{\varepsilon t_3^3} + \ldots \right) \left( A_{03} - \frac{B_{03}}{\varepsilon t_3^3} + \ldots \right), \]
\[ \sim t_3^{-1} [A_{03}, B_{03}], \quad (5.26) \]
as \( \varepsilon \to 0 \). Combining (5.24)-(5.26), we have,

\[ \frac{d}{dt_3} B_{03} = \lim_{\varepsilon \to 0} \left( 2\varepsilon t_3 A_{15} + 2 [A_{05}, \varepsilon t_3 A_{15}] \right) = 2t_3^{-1} (B_{03}(t_3) + [A_{03}(t_3), B_{03}(t_3)]), \]

which is (5.6), as required.
For brevity, we state once and for all that $\varepsilon \in E$ and $t_3 \in T$ for the remainder of this chapter, where $E$ and $T$ are chosen to satisfy (5.22) and (5.23).

5.2 From Fuchsian Singularities to an Irregular Singularity

We now turn our attention to the behavior of the fundamental solutions $Y_5^{(0)}(\lambda_5)$ and $Y_5^{(t)}(\lambda_5)$, as defined in Section 4.1.2, under our confluence procedure. We continue our convention of denoting the coefficient of $\varepsilon^n$ in the asymptotic expansion of a matrix $R$ around $\varepsilon = 0$ as $R^{(n)}$.

5.2.1 Taking a Term-By-Term Limit of the Solution $Y_5^{(0)}(\lambda_5)$

We make a choice of the diagonalising matrix $R_{05}$ such that, under the substitutions (5.20)-(5.21) and the conditions of our Main Assumption 2,

$$R_{05}(\varepsilon t_3^2) \sim R_{03}(t_3) + \sum_{n=1}^{\infty} \varepsilon^n R^{(n)}_{05}(t_3), \quad \text{as } \varepsilon \to 0,$$

where $R^{(n)}_{05}(t_3)$ are certain matrices. We rewrite our fundamental solution in the following way,

$$Y_5^{(0)}(\lambda_5) = R_{05} \sum_{n=0}^{\infty} \hat{H}_{n,0} \lambda_5^{n} \left( \frac{\rho_{05}}{2} \sigma_3 (\lambda_5 - t_5) \frac{\rho_{05}}{2} \sigma_3, \lambda_5 \in \hat{\Omega}_{05}, \right. \quad \text{ (5.27)}$$

where, for some $\hat{\rho}_{05} > 0$, the new domain $\hat{\Omega}_{05}$ is defined below,

$$\left\{ \lambda_5 : |\lambda_5| < \hat{\rho}_{05}, \ 0 < \arg(\lambda_5) - \arg(t_5) < 2\pi, \right. \quad -\pi < \arg(\lambda_5 - t_5) - \arg(t_5) < \pi$$

$$\hat{H}_{0,0} := I \quad \text{ and the coefficients } \hat{H}_{n,1}, \ n \geq 1, \ \text{are determined by the recursion relation below,}$$

$$n \hat{H}_{n,0} + \left[ \hat{H}_{n,0}, \frac{\theta_{05}}{2} \sigma_3 \right] = \sum_{l=0}^{n-1} \hat{H}_{l,0} \frac{\theta_{05}}{2} \sigma_3 \sigma_3^{l-n} + R_{05}^{-1} \frac{\sigma_3}{2} R_{05} \hat{H}_{n-1,0},$$

$$- R_{05}^{-1} A_{05} R_{05} \sum_{l=0}^{n-1} \hat{H}_{l,0} \sigma_3^{l-n}. \quad \text{ (5.28)}$$
Observe how the final terms behave under the substitutions (5.20)-(5.21),
\[
\lambda_5^{\frac{\theta_5}{2\sigma_3}}(\lambda_5 - t_5)^{\frac{\theta_5}{2\sigma_3}} = \lambda_5^{\frac{\theta_5}{2\sigma_3}} \left(1 - \frac{t_5}{\lambda_5}\right)^{\frac{\theta_5}{2\sigma_3}},
\]
\[
= \lambda_3^{\frac{\theta_3}{2\sigma_3}} \exp \left(-\frac{\sigma_3}{2\epsilon} \log \left(1 - \frac{\epsilon t_3^2}{\lambda_3}\right)\right),
\]
therefore,
\[
\lim_{\epsilon \to 0} \lambda_5^{\frac{\theta_5}{2\sigma_3}}(\lambda_5 - t_5)^{\frac{\theta_5}{2\sigma_3}} = \lambda_3^{\frac{\theta_3}{2\sigma_3}} \exp \left(\frac{t_3^2}{2\lambda_3 \sigma_3}\right), \quad \text{as } \epsilon \to 0,
\]
which are precisely the terms found in the formal fundamental solution \(Y_{3,f}^{\text{(0)}}(\lambda_3)\). This shows how to asymptotically pass from power-like behavior to exponential behavior.

The terms of the new series satisfy the following lemma.

**Lemma 5.1.** Let \(\hat{H}_{n,0}\) and \(F_{n,0}\) be defined by (5.28) and (5.13) respectively. Under the substitutions (5.20)-(5.21) and the conditions of our Main Assumption 2,

\[
\lim_{\epsilon \to 0} \hat{H}_{1,0} = F_{1,0},
\]
and, for \(n \geq 2\), there exists a choice of \(\hat{H}_{n-1,0}^{(n)}\) such that,

\[
\lim_{\epsilon \to 0} \hat{H}_{n,0} = F_{n,0}.
\]

**Proof.** This lemma is proved in an analogous way to Lemmas 4.5 and 4.7. We note that the following observations are used in the proof of this lemma: from the forms of the matrices \(A_{03}\) and \(B_{03}\) given in (5.3) and (5.4), we have,

\[
\text{diag}(A_{03}) = -\frac{\theta_{03}}{2\sigma_3}.
\]

Furthermore, if we choose to diagonalise at \(\lambda_3 = 0\), in other words we conjugate equation (5.1) by a diagonalising matrix \(R_{03}\), then we would have,

\[
\text{diag}(R_{03}^{-1}A_{03}R_{03}) = \frac{\theta_{03}}{2\sigma_3}.
\]

\(\Box\)
5.2.2 Taking a Term-By-Term Limit of the Solution $Y_5^{(t)}(\lambda_5)$

We make a choice of the diagonalising matrix $R_{t5}$ such that, under the substitutions (5.20)-(5.21) and the conditions of our Main Assumption 2,

$$R_{t5}(\varepsilon t_3^2) \sim R_{03}(t_3) + \sum_{n=1}^{\infty} \varepsilon^n R_{t5}^{(n)}(t_3), \quad \text{as } \varepsilon \to 0,$$

where $R_{t5}^{(n)}(t_3)$ are certain matrices. We rewrite our fundamental solution in the following way,

$$Y_5^{(t)}(\lambda_5) = R_{t5} \sum_{n=0}^{\infty} \hat{H}_{n,t}(\lambda_5 - t_3)^n \lambda_5^{\frac{n}{2} \sigma_3} (\lambda_5 - t_3)^{\frac{n}{2} \sigma_3}, \quad \lambda_5 \in \hat{\Omega}_{t5}, \quad (5.30)$$

where, for some $\hat{\rho}_{t5} > 0$, the new domain $\hat{\Omega}_{t5}$ is defined as below,

$$\left\{ \lambda_5 : |\lambda_5 - t_3| < \hat{\rho}_{t5}, \quad 0 < \arg(\lambda_5) - \arg(t_3) < 2\pi, \quad -\pi < \arg(\lambda_5 - t_3) - \arg(t_3) < \pi \right\},$$

$\hat{H}_{0,t} := I$ and the coefficients $\hat{H}_{n,t}$, $n \geq 1$, are determined by the recursion relation below,

$$n\hat{H}_{n,t} + \left[ \hat{\Theta}_{t5} \frac{\sigma_3}{2} \right] = R_{t5}^{-1} \frac{\sigma_3}{2} R_{t5} \hat{H}_{n-1,t} + \sum_{l=0}^{n-1} (-t_3)^{l-n} \hat{H}_{l,t} \frac{\theta_05}{2} \sigma_3$$

$$- R_{t5}^{-1} A_{05} R_{t5} \sum_{l=0}^{n-1} (-t_3)^{l-n} \hat{H}_{l,t}. \quad (5.31)$$

Similarly as before, we have written the final terms in this solution so that they tend to the terms in the formal fundamental solution $Y_{3,f}^{(0)}(\lambda_3)$, recall (5.29). The terms of the new series satisfy the following lemma.

**Lemma 5.2.** Let $\hat{H}_{n,t}$ and $F_{n,0}$ be defined by (5.31) and (5.13) respectively. Under the substitutions (5.20)-(5.21) and the conditions of our Main Assumption 2,

$$\lim_{\varepsilon \to 0} \hat{H}_{1,t} = F_{1,t},$$

Page 166 of 178
and, for \( n \geq 2 \), there exists a choice of \( \hat{H}_{n-1}^{(n)} \) such that,

\[
\lim_{\epsilon \to 0} \hat{H}_{n,t} = F_{n,t}.
\]

**Proof.** This lemma is proved in an analogous way to Lemmas 4.5 and 4.7. \( \square \)

### 5.2.3 Obtaining the True Solutions \( Y_{3}^{(0,k)}(\lambda_3) \)

In Subsections 5.2.1 and 5.2.2 we have understood how to take term-by-term limits of the solutions of the \( P_V \) linear system at the merging simple poles \( \lambda_5 = 0 \) and \( t_5 \) to produce the formal series solution of the \( P_{111}^{D_6} \) linear system around \( \lambda_3 = 0 \). We now apply Glutsyuk’s Theorem 2.5 to equations (4.23) and (5.1). Let \( \eta \in (0, \frac{\pi}{2}) \) be some fixed value. We define the following sectors,

\[
\mathcal{S}_k^{(0)} := \left\{ \lambda_3 : |\lambda_3| < \rho_{03}, \quad \arg(\lambda_3) - \arg(t_3^2) + k\pi \in \left( \eta - \frac{\pi}{2}, \frac{3\pi}{2} - \eta \right) \right\}, \quad (5.32)
\]

note that if \( \lambda_3 \in \mathcal{S}_k^{(0)} \) then \( \lambda_5 \in \Omega_5 \). These sectors will be the domains of our fundamental solutions \( Y_{3}^{(\infty,k)}(\lambda_3) \). We also define the following sectors,

\[
\sigma_{\epsilon}(\epsilon) := \left\{ \lambda_3 : |\lambda_3 - \epsilon t_3^2| < \rho_{03}, \quad \arg(\lambda_3) - \arg(\epsilon t_3^2) \in (\eta, 2\pi - \eta), \quad \arg(\lambda_3 - \epsilon t_3^2) - \arg(\epsilon t_3^2) \in (\eta - \pi, \pi - \eta) \right\}, \quad (5.33)
\]

\[
\sigma_{0}(\epsilon) := \left\{ \lambda_3 : |\lambda_3| < \rho_{03}, \quad \arg(\lambda_3) - \arg(\epsilon t_3^2) \in (\eta, 2\pi - \eta), \quad \arg(\lambda_3 - \epsilon t_3^2) - \arg(\epsilon t_3^2) \in (\eta - \pi, \pi - \eta) \right\}, \quad (5.34)
\]

note that if \( \lambda_3 \in \sigma_{\epsilon}(\epsilon) \) is sufficiently close to \( \epsilon t_3^2 \) then \( \lambda_5 \in \Omega_{05} \) and if \( \lambda_3 \in \sigma_{0}(\epsilon) \) is sufficiently small then \( \lambda_5 \in \Omega_{05} \). These sectors will be the domains of our fundamental solutions \( Y_{5}^{(\epsilon)}(\lambda_3) \) and \( Y_{5}^{(0)}(\lambda_3) \) respectively, they are illustrated in Figure 25 below. Note that we use the notation \( \sigma_{\epsilon}(\epsilon) \) for the sector centred at \( \lambda_3 = \epsilon t_3^2 \), rather than using the subscript \( \epsilon t_3^2 \), for brevity.
As $\varepsilon \to 0$ along a ray, the sector $\sigma_\varepsilon$, which centred at $\lambda_3 = \varepsilon t_3^2$, is translated along a ray tending to zero. We illustrate this phenomenon in Figure 26 below.

![Figure 25: Sectors $\sigma_\varepsilon$ and $\sigma_0$.](image)

![Figure 26: As $\alpha \to \infty$ along a ray, the sector $\sigma_\varepsilon$ is translated along the branch cut and becomes in agreement with the sector $\Phi_3 := \{\lambda_3 : |\lambda_3| < \rho_{03}, |\lambda_3 - \varepsilon t_3^2| < \pi - \eta\}$.](image)

We are concerned in the two limit directions $\arg(\varepsilon) = \pm \frac{\pi}{2}$. In these cases, we have,

$$\eta - \pi < \arg(\lambda_3) - \arg(\varepsilon t_3^2) < \pi - \eta \iff \pm \frac{\pi}{2} + \eta - \pi < \arg(\lambda_3) - \arg(t_3^2) < \pm \frac{\pi}{2} + \pi - \eta.$$  

As $\varepsilon \to 0$ along a ray, the sector $\sigma_0$, whose base point is already fixed at zero, becomes in agreement with the sector,

$$\{\lambda_3 : |\lambda_3| < \rho_{03}, \eta < \arg(\lambda_3) - \arg(\varepsilon t_3^2) < +2\pi - \eta\}.$$

For $\arg(\varepsilon) = \pm \frac{\pi}{2}$, we have,

$$\eta < \arg(\lambda_3) - \arg(\varepsilon t_3^2) < 2\pi - \eta \iff \pm \frac{\pi}{2} + \eta < \arg(\lambda_3) - \arg(t_3^2) < \pm \frac{\pi}{2} + 2\pi - \eta.$$
With all of these considerations in mind, we write:

\[
\begin{align*}
\lim_{\varepsilon \to 0} \sigma_\varepsilon(\varepsilon) &= \mathcal{J}_1(0), \\
\lim_{\varepsilon \to 0} \sigma_0(\varepsilon) &= \mathcal{J}_0(0), \\
\lim_{\varepsilon \to 0} \sigma_\varepsilon(\varepsilon) &= \mathcal{J}_0^+(0), \\
\lim_{\varepsilon \to 0} \sigma_0(\varepsilon) &= \mathcal{J}_0^-(0).
\end{align*}
\]

We apply Glutsyuk’s Theorem 2.5 with equation (4.23) in place of the perturbed equation and equation (5.1) in place the non-perturbed equation. Glutsyuk’s theorem asserts the existence of invertible diagonal matrices \( K_{05}^\pm(\varepsilon) \) and \( K_{15}^\pm(\varepsilon) \) such that:

\[
\begin{align*}
\lim_{\varepsilon \to 0} \left| Y_5^{(0)}(\lambda_3) \right|_{\lambda_3 \in \sigma_0(\varepsilon)} K_{05}^+(\varepsilon) &= Y_3^{(0,0)}(\lambda_3), \\
\lim_{\varepsilon \to 0} \left| Y_5^{(1)}(\lambda_3) \right|_{\lambda_3 \in \sigma_0(\varepsilon)} K_{15}^+(\varepsilon) &= Y_3^{(0,1)}(\lambda_3), \tag{5.35}
\end{align*}
\]

uniformly for \( \lambda_3 \in \mathcal{J}_0^{(0)} \) and \( \mathcal{J}_1^{(0)} \) respectively, and:

\[
\begin{align*}
\lim_{\varepsilon \to 0} \left| Y_5^{(0)}(\lambda_3) \right|_{\lambda_3 \in \sigma_0(\varepsilon)} K_{05}^-(\varepsilon) &= Y_3^{(0,-1)}(\lambda_3), \\
\lim_{\varepsilon \to 0} \left| Y_5^{(1)}(\lambda_3) \right|_{\lambda_3 \in \sigma_0(\varepsilon)} K_{15}^-(\varepsilon) &= Y_3^{(0,0)}(\lambda_3), \tag{5.36}
\end{align*}
\]

uniformly for \( \lambda_3 \in \mathcal{J}_0^{(0)} \) and \( \mathcal{J}_1^{(0)} \) respectively. Due to the asymptotics of the fundamental solutions \( Y_3^{(0,k)}(\lambda_3) \) as given in Theorem 5.2, each of these four limits is asymptotic to the formal fundamental solution \( Y_3^{(0,0)}(\lambda_3) \) as \( \lambda_3 \to 0 \) with \( \lambda_3 \) belonging to the corresponding sector.

Having applied Glutsyuk’s theorem to produce the true solutions \( Y_3^{(0,k)}(\lambda_3) \) in sectors, we now focus on understanding what we can deduce about the matrices \( K_{05}^\pm(\varepsilon) \) and \( K_{15}^\pm(\varepsilon) \). We are ready to state our first main theorem of this chapter.

**Main Theorem 5.** Let \( K_{05}^\pm(\varepsilon) \) and \( K_{15}^\pm(\varepsilon) \) be diagonal matrices satisfying (5.35)-
These matrices satisfy the following limits:

\[
\lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = \pm \frac{\pi}{2}} K_{05}^{\pm}(\varepsilon) = I, \quad (5.39)
\]

\[
\lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = \pm \frac{\pi}{2}} K_{03}^{\pm}(\varepsilon) = I. \quad (5.40)
\]

**Proof of our Main Theorem 5.** To prove (5.39), we consider the solution \( Y_{5(0)}(\lambda_3) \) as in (5.27) restricted to the sector \( \sigma_0(\varepsilon) \). In either case \( \arg(\varepsilon) = \pm \frac{\pi}{2} \), let \( S^* \) denote a closed subsector of \( S_{-1}^{(0)} \) or \( S_0^{(0)} \) respectively. Using Lemma 3.14 and the limit in (5.29), we deduce,

\[
\lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = \pm \frac{\pi}{2}} \sum_{n=0}^{\infty} \hat{H}_{n,0}^{\lambda_3^0} \left|_{\lambda_3 \in \sigma_0(\lambda_3)} \right. K_{05}^{\pm}(\varepsilon) \sim \sum_{n=0}^{\infty} F_{n,0}^{\lambda_3^0},
\]

as \( \lambda_3 \to 0 \) with \( \lambda_3 \in S^* \). We now apply Lemma 3.13 to find,

\[
F_{n,0} = \frac{1}{n!} \lim_{\lambda_3 \to 0 \atop \lambda_3 \in S^*} \frac{d^n}{d\lambda_3^n} \lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = \pm \frac{\pi}{2}} \sum_{n=0}^{\infty} \hat{H}_{n,0}^{\lambda_3^0} \left|_{\lambda_3 \in \sigma_0(\varepsilon)} \right. K_{05}^{\pm}(\varepsilon).
\]

Now, due to the uniformity of the limits in (5.35) and (5.37), this expression becomes,

\[
F_{n,0} = \lim_{\varepsilon \to 0 \atop \arg(\varepsilon) = \pm \frac{\pi}{2}} \hat{H}_{n,0} K_{05}^{\pm}(\varepsilon).
\]

After recalling Lemma 5.1, we immediately find the desired result. The proof of (5.40) is similar, we use Lemma 5.2 in place of Lemma 5.1.

**5.3 Limits of Monodromy Data**

We prove our second main theorem of this chapter, concerned with producing the Stokes’ matrices of the \( P_{III}^{D_6} \) linear system around \( \lambda_3 = 0 \) from the monodromy data around the merging singularities of the \( PV \) linear system under our confluence procedure.

**Main Theorem 6.** Define the monodromy data of the auxiliary linear system associated to \( PV \) as in (4.41)-(4.46) and to \( P_{III}^{D_6} \) as in (5.14)-(5.19). Under the substitutions
(5.20)-(5.21) and the conditions of our Main Assumption 2, we have the following limits,

\[
\lim_{\varepsilon \to 0} C_5^{t\infty} (C_5^{t\infty})^{-1} = S_0^{(0)} \tag{5.41}
\]

\[
\lim_{\varepsilon \to 0} C_5^{t\infty} (C_5^{t\infty})^{-1} = S_{-1}^{(0)} \tag{5.42}
\]

**Proof of our Main Theorem 6.** Let \( \sigma_{\varepsilon}(\varepsilon) \) and \( \sigma_0(\varepsilon) \) be defined as in (5.33) and (5.34) respectively. As mentioned previously, if \( \lambda_3 \in \sigma_{\varepsilon}(\varepsilon) \) then \( \lambda_5 \in \Omega_{t5} \) and if \( \lambda_3 \in \sigma_0(\varepsilon) \) then \( \lambda_5 \in \Omega_{05} \), so that the connection matrix \( C_0^{t5} \equiv C_5^{t\infty} (C_5^{t\infty})^{-1} \) remains valid for the solutions \( Y_5^{(t)}(\lambda_3) \) and \( Y_5^{(0)}(\lambda_3) \) restricted to the sectors \( \sigma_{\varepsilon}(\varepsilon) \) and \( \sigma_0(\varepsilon) \) respectively. Since the radii of these sectors do not diminish as \( \varepsilon \to 0 \), for \( |\varepsilon| \) sufficiently small we must have,

\[
\sigma_{\varepsilon}(\varepsilon) \cap \sigma_0(\varepsilon) \neq \emptyset,
\]

recall Figure 25. Therefore, for \( |\varepsilon| \) sufficiently small, we have,

\[
Y_5^{(t)}(\lambda_3) = Y_5^{(0)}(\lambda_3)C_5^{t\infty} (C_5^{t\infty})^{-1}, \quad \lambda_3 \in \sigma_{\varepsilon}(\varepsilon) \cap \sigma_0(\varepsilon) \tag{5.43}
\]

Let \( \mathcal{S}_0^{(0)} \) be the sectors defined by (5.32). To prove the first limit, we multiply by the matrices \( K_{05}^{-}(\varepsilon) \) and \( K_{15}^{-}(\varepsilon) \) and take the limit \( \varepsilon \to 0 \), with \( \arg(\varepsilon) = -\frac{\pi}{2} \), so that (5.43) becomes,

\[
\lim_{\varepsilon \to 0} Y_5^{(t)}(\lambda_3) \bigg|_{\lambda_3 \in \sigma_0(\varepsilon)} K_{05}^{-}(\varepsilon) \tag{5.44}
\]

for \( \lambda_3 \in \mathcal{S}_0^{(0)} \cap \mathcal{S}_0^{(0)} \). Recall Definition 5.2 of Stokes’ matrices,

\[
Y_3^{(\infty,1)}(\lambda_3) = Y_3^{(0,0)}(\lambda_3)S_0^{(0)}, \quad \lambda_3 \in \Sigma_1^{(0)} \cap \Sigma_0^{(0)},
\]

and recall that if \( \lambda_3 \in \mathcal{S}_k^{(0)} \) then \( \lambda_3 \in \Sigma_k^{(0)} \). Using Lemma 3.14, we deduce from (5.44)
that,
\[ \lim_{\varepsilon \to 0} \left( K_{t_5}^{-} \right)^{-1} C_5^{0\infty} \left( C_5^{t\infty} \right)^{-1} K_{05}^{-}(\varepsilon) = S_{0}^{(0)}, \]

which is precisely Glutsyuk’s Corollary 2.1 in our case. Using (5.39) and (5.40) from our Main Theorem 5, we compute,

\[ S_{0}^{(0)} = \lim_{\varepsilon \to 0} \left( K_{t_5}^{-} \right)^{-1} C_5^{0\infty} \left( C_5^{t\infty} \right)^{-1} K_{05}^{-}(\varepsilon) = \lim_{\varepsilon \to 0} C_5^{0\infty} \left( C_5^{t\infty} \right)^{-1}, \]

as required. Similarly, to prove (5.42), we deduce,

\[ S_{-1}^{(0)} = \lim_{\varepsilon \to 0} \left( K_{t_5}^{+} \right)^{-1} C_5^{0\infty} \left( C_5^{t\infty} \right)^{-1} K_{05}^{+}(\varepsilon) = S_{-1}^{(0)}, \]

from which we calculate, using our Main Theorem 5,

\[ S_{-1}^{(0)} = \lim_{\varepsilon \to 0} \left( K_{t_5}^{+} \right)^{-1} C_5^{0\infty} \left( C_5^{t\infty} \right)^{-1} K_{05}^{+}(\varepsilon) = \lim_{\varepsilon \to 0} C_5^{0\infty} \left( C_5^{t\infty} \right)^{-1}, \]

and the theorem is proved. \qed
Chapter 6

Conclusions and Outlook

In this thesis we have analysed confluences of the hypergeometric equations and of the auxiliary linear systems associated to the Painlevé equations $P_{VI} \to P_V$ and $P_V \to P_{III}^{D_6}$. In each case we have demonstrated a procedure to explicitly calculate the Stokes’ matrices at the newly created double pole in terms of the monodromy data around the two merging simple poles. We have explained how to pass from solutions with power-like behavior which converge in neighbourhoods to solutions with exponential behavior which are analytic in sectors and have divergent asymptotic behavior. To achieve the explicit limits for Stokes’ matrices, we have combined our understanding of the behaviors of the solutions with a certain existence theorem of Glutsyuk. We also note that the confluence procedure we study from the auxiliary linear system of $P_V$ to that of $P_{III}^{D_6}$ is new.

As an outlook to future work, we expect that we will be able to generalise our procedure to the confluence of the many-variable Garnier system. We would also be interested to understand how to take limits of the monodromy data of the remaining auxiliary linear systems of the Painlevé equations in Figure 1. Not only would this involve understanding how to merge higher order poles, for example in the confluence $P_V \to P_{IV}$, but also how to pass from a diagonalisable matrix to a non-diagonalisable matrix, for example in the confluence $P_V \to P_V^{\deg}$.
References


[JM] Jimbo, M. and Miwa, T., Monodromy preserving deformation of linear ordinary


