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THE STABLE MANIFOLD THEOREM FOR SEMILINEAR
STOCHASTIC EVOLUTION EQUATIONS AND STOCHASTIC
PARTIAL DIFFERENTIAL EQUATIONS

II: EXISTENCE OF STABLE AND UNSTABLE MANIFOLDS

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Abstract. This article is a sequel to [M.Z.Z.1] aimed at completing the characterization of
the pathwise local structure of solutions of semi-linear stochastic evolution equations (see’s)
and stochastic partial differential equations (spde’s) near stationary solutions. The character-
ization is expressed in terms of the almost sure long-time behavior of trajectories of the
equation in relation to the stationary solution. More specifically, we establish local stable
manifold theorems for semi-linear see’s and spde’s (Theorems 4.1-4.4). These results give
smooth stable and unstable manifolds in the neighborhood of a hyperbolic stationary solu-
tion of the underlying stochastic equation. The stable and unstable manifolds are stationary,
live in a stationary tubular neighborhood of the stationary solution and are asymptotically
invariant under the stochastic semiflow of the see/spde. The proof uses infinite-dimensional
multiplicative ergodic theory techniques and interpolation arguments (Theorem 2.1).

1. Introduction. Hyperbolicity of a stationary trajectory.

In [M-Z-Z.1], we established the existence of perfect differentiable cocycles gener-
ated by mild solutions of a large class of semilinear stochastic evolution equations (see’s)
and stochastic partial differential equations (spde’s). The present article is a continuation
of the analysis in [M-Z-Z.1]. In this paper we introduce the concept of a stationary tra-
jectory for the see. Within the context of stochastic differential equations (with memory)
(sde’s and sfde’s), this concept has been used extensively in previous work of one of the authors with M. Scheutzow ([M-S.1], [M-S.2-4]). Our main objective is to characterize the pathwise local structure of solutions of semi-linear se’e’s and spde’s near stationary solutions. We introduce the concept of hyperbolicity for a stationary solution of an see. Hyperbolicity is defined by the non-vanishing of the Lyapunov spectrum of the linearized cocycle. The hyperbolic structure of the stochastic semiflow leads to local stable manifold theorems (Theorems 4.1-4.4) for semi-linear see’s and spde’s. For a hyperbolic stationary solution of the see, this gives smooth stable and unstable manifolds in a neighborhood of the stationary solution. The stable and unstable manifolds are stationary, live in a stationary tubular neighborhood of the stationary solution and are asymptotically invariant under the stochastic semiflow. The proof of the stable manifold theorem uses infinite-dimensional multiplicative ergodic theory techniques ([Ru.1], [Ru.2]) together with interpolation and perfection arguments ([Mo.1], [M-S.4]). In particular, we will assume that the reader is familiar with the results and the techniques in Ruelle’s articles [Ru.1] and [Ru.2]. Our results cover semilinear stochastic evolution equations, stochastic parabolic equations, stochastic reaction-diffusion equations, and Burgers equation with additive infinite-dimensional noise.

We recall below the definition of a cocycle in Hilbert space.

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. Suppose \(\theta: \mathbb{R} \times \Omega \to \Omega\) is a group of \(P\)-preserving ergodic transformations on \((\Omega, \mathcal{F}, P)\). Denote by \(\bar{\mathcal{F}}\) the \(P\)-completion of \(\mathcal{F}\).

Let \(H\) be a real separable Hilbert space with norm \(|\cdot|\) and Borel \(\sigma\)-algebra \(\mathcal{B}(H)\).

Take \(k\) to be any non-negative integer and \(\epsilon \in (0, 1]\). Recall that a \(C^{k,\epsilon}\) perfect cocycle \((U, \theta)\) on \(H\) is a \((\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))\)- measurable random field \(U: \mathbb{R}^+ \times H \times \Omega \to H\) with the following properties:
(i) For each $\omega \in \Omega$, the map $\mathbb{R}^+ \times H \ni (t, x) \mapsto U(t, x, \omega) \in H$ is continuous; for fixed $(t, \omega) \in \mathbb{R}^+ \times \Omega$, the map $H \ni x \mapsto U(t, x, \omega) \in H$ is $C^{k,\epsilon}$ ($D^k U(t, x, \omega)$ is $C^\epsilon$ in $x$ on bounded subsets of $H$).

(ii) $U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$ for all $t_1, t_2 \in \mathbb{R}^+$, all $\omega \in \Omega$.

(iii) $U(0, x, \omega) = x$ for all $x \in H, \omega \in \Omega$.

We now introduce the concept of a stationary point for a cocycle $(U, \theta)$. Stationary points play the role of stochastic equilibria for the stochastic dynamical system.

**Definition 1.1.**

An $\mathcal{F}$-measurable random variable $Y : \Omega \to H$ is said be a stationary random point for the cocycle $(U, \theta)$ if it satisfies the following identity:

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$  \hspace{1cm} (1.1)

for all $(t, \omega) \in \mathbb{R}^+ \times \Omega$.

The reader may note that the above definition is an infinite-dimensional analogue of a corresponding concept of invariance that was used by one of the authors in joint work with M. Scheutzow to give a proof of the stable manifold theorem for stochastic ordinary differential equations (Definition 3.1, [M-S.3]). The concept essentially gives a useful realization of the idea of an invariant measure for the stochastic dynamical system, and allows us to analyze the local almost sure stability properties of the stochastic semiflow in the neighborhood of the stationary point.

The existence (and uniqueness) of a stationary random point for the stochastic Burgers equation with additive noise was established by Weinan E., K. Khanin, A. Mazel and Ya. Sinai ([E-K-M-S], [Si]). In this article, we will apply our stable/unstable theorem to examine further the almost sure asymptotic structure of Burgers flow, and establish the existence of local stable and unstable manifolds near the stationary point.
The main objective of this section is to define the concept of hyperbolicity for a stationary point $Y$ of the cocycle $(U, \theta)$.

First, we linearize the $C^{k,\epsilon}$ cocycle $(U, \theta)$ along a stationary random point $Y$. By taking Fréchet derivatives at $Y(\omega)$ on each side of the cocycle identity (ii) above, using the chain rule and the definition of $Y$, we immediately see that $(DU(t, Y(\omega), \omega), \theta(t, \omega))$ is an $L(H)$-valued perfect cocycle. Secondly, we appeal to the following classical result which goes back to Oseledec in the finite-dimensional case, and to D. Ruelle in infinite dimensions.

**Theorem 1.1.** *(Oseledecc-Ruelle)*

Let $T : \mathbb{R}^+ \times \Omega \to L(H)$ be strongly measurable, such that $(T, \theta)$ is an $L(H)$-valued cocycle, with each $T(t, \omega)$ compact. Suppose that

$$
E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\|_{L(H)} + E \sup_{0 \leq t \leq 1} \log^+ \|T(1 - t, \theta(t, \cdot))\|_{L(H)} < \infty.
$$

Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbb{R}^+$, and for each $\omega \in \Omega_0$, the limit

$$
\Lambda(\omega) := \lim_{t \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)}
$$

exists in the uniform operator norm. Each linear operator $\Lambda(\omega)$ is compact, non-negative and self-adjoint with a discrete spectrum

$$
e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \ldots
$$

where the $\lambda_i$’s are distinct and non-random. Each eigenvalue $e^{\lambda_i} > 0$ has a fixed finite non-random multiplicity $m_i$ and a corresponding eigen-space $F_i(\omega)$, with $m_i := \dim F_i(\omega)$. Set $i = \infty$ when $\lambda_i = -\infty$. Define

$$
E_1(\omega) := H, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^\perp, \quad i > 1, \quad E_\infty := \ker \Lambda(\omega).
$$
Then
\[ E_\infty \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = H, \]
\[
\lim_{t \to \infty} \frac{1}{t} \log |T(t, \omega)x| = \begin{cases} 
\lambda_i & \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega), \\
-\infty & \text{if } x \in E_\infty(\omega), 
\end{cases}
\]
and
\[ T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega)) \]
for all \( t \geq 0, \ i \geq 1. \)

The following figure illustrates the Oseledec-Ruelle theorem.

The Spectral Theorem
Proof of Theorem 1.1.

The proof is based on a discrete version of Oseledec’s multiplicative ergodic theorem and the perfect ergodic theorem ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. [O], [Mo.1], Lemma 5. See also Lemma 3.1 (ii) of this article). Details of the extension to continuous time are given in [Mo.1] within the context of linear stochastic functional differential equations. The arguments in [Mo.1] extend directly to general linear cocycles in Hilbert space. □

Definition 1.2.

The sequence \{⋯ < λ_{i+1} < λ_i < ⋯ < λ_2 < λ_1\} in the Oseledec-Ruelle theorem (Theorem 1.1) is called the Lyapunov spectrum of the linear cocycle \((T, \theta)\).

Hyperbolicity of a stationary point \(Y : \Omega \to H\) of the non-linear cocycle \((U, \theta)\) may now be defined in terms of a spectral gap in the Lyapunov spectrum of the linearized cocycle \((DU(t, Y(\omega), \omega), \theta(t, \omega))\).

Definition 1.3.

Let \((U, \theta)\) be a \(C^{k,\epsilon}\) \((k \geq 1, \epsilon \in (0, 1])\) perfect cocycle on a separable Hilbert space \(H\) such that \(U(t, \cdot, \omega) : H \to H\) takes bounded sets into relatively compact sets for each \((t, \omega) \in \mathbb{R}^+ \times \Omega\). A stationary point \(Y(\omega)\) of the cocycle \((U, \theta)\) is hyperbolic if

(a) For any \(T \in (0, \infty)\),

\[
\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|DU(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega))\|_{L(H)} \, dP(\omega) < \infty.
\]

(b) The linearized cocycle \((DU(t, Y(\omega), \omega), \theta(t, \omega))\) has a non-vanishing Lyapunov spectrum \{⋯ < λ_{i+1} < λ_i < ⋯ < λ_2 < λ_1\}, viz. \(λ_i \neq 0\) for all \(i \geq 1\).
By the Oseledec theorem (Theorem 1.1), the integrability condition in Definition 1.2 (a) implies the existence of a discrete Lyapunov spectrum for the linearized cocycle \((DU(t, Y(\omega), \omega), \theta(t, \omega))\) in Definition 1.2 (b) above.

The following result is a random version of the saddle point property for hyperbolic linear cocycles. A proof is given in ([Mo.1], Theorem 4, Corollary 2; [M-S.1], Theorem 5.3) within the context of stochastic differential systems with memory; but the arguments therein extend immediately to linear cocycles in Hilbert space.

**Theorem 1.2. (Stable and unstable subspaces)**

Let \((T, \theta)\) be a linear cocycle on a Hilbert space \(H\). Assume that \(T(t, \omega) : H \to H\) is a compact linear operator for each \(t > 0\) and a.a. \(\omega \in \Omega\). Suppose that

\[
\mathbb{E} \log^+ \sup_{0 \leq t_1, t_2 \leq 1} \|T(t_2, \theta(t_1, \cdot))\|_{L(H)} < \infty,
\]

and let the cocycle \((T, \theta)\) have a non-vanishing Lyapunov spectrum \(\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}\). Pick \(i_0 > 1\) such that \(\lambda_{i_0} < 0 < \lambda_{i_0-1}\).

Then there is a sure event \(\Omega^* \in \mathcal{F}\) and stable and unstable subspaces \(\{S(\omega), U(\omega) : \omega \in \Omega^*\}\), \(\mathcal{F}\)-measurable (into the Grassmanian), such that for each \(\omega \in \Omega^*\), the following is true:

(i) \(\theta(t, \cdot)(\Omega^*) = \Omega^*\) for all \(t \in \mathbb{R}\).

(ii) \(H = U(\omega) \oplus S(\omega)\). The subspace \(U(\omega)\) is finite-dimensional with a fixed non-random dimension, and \(S(\omega)\) is closed with a finite non-random codimension.

(iii) (Invariance)

\[
T(t, \omega)(U(\omega)) = U(\theta(t, \omega)), \quad T(t, \omega)(S(\omega)) \subseteq S(\theta(t, \omega)),
\]

for all \(t \geq 0\),

(iv) (Exponential dichotomies)

\[
|T(t, \omega)(x)| \geq |x|e^{\delta_1 t} \quad \text{for all} \quad t \geq \tau_1^*, x \in U(\omega),
\]
|T(t, ω)(x)| ≤ |x|e^{−δ_2 t} for all  \ t ≥ τ_i^*, \ x \in S(ω),

where τ_i^* = τ_i^*(x, ω) > 0, i = 1, 2, are random times and δ_i > 0, i = 1, 2, are fixed.

**Remark.**

It is not clear to us how our concept of hyperbolicity (Definition 1.3) relates to the one introduced by E, Khanin, Mazel and Sinai for the stochastic Burgers equation within the context of one sided-minimizers ([E-K-M-S]). On the other hand, our concept of hyperbolicity implies the existence of local stable and unstable manifolds near the stationary solution of the stochastic one-dimensional Burgers equation

\[
\frac{∂u}{∂t} + u \frac{∂u}{∂ξ} = ν \frac{∂^2 u}{∂ξ^2} + b(ξ, t) \tag{1.2}
\]
on $S^1$, with space-periodic (white in $t$) random forcing

$$b(\xi, t) := \sum_{k=1}^{\infty} \sigma_k(\xi) W_k(t),$$

where $\{W_k\}_{k=1}^{\infty}$ is a sequence of independent standard one-dimensional Brownian motions, and the $\sigma_k$’s are sufficiently regular functions on $S^1$. See [E-K-M-S] for precise regularity conditions on the coefficients of (1.2). We associate with (1.2) a periodic boundary condition and a sufficiently regular initial function $S^1 \to \mathbb{R}$ on the unit circle $S^1$. The functions $\sigma_k$ are periodic with period 1 and such that $\sigma'_k \in C^r(S^1)$. Furthermore, assume that $\|\sigma'_k\|_{C^r} \leq C k^{-2}$ for all $k \geq 1$ and some $r \geq 3$, where $C$ is a constant ([E-K-M-S]).

Needless to say the above stochastic Burgers equation has been studied extensively in the physics literature. See [E-K-M-S] and the references therein. Using the existence result of a $C^1$-cocycle for (1.2) ([M-Z-Z.1], Theorem 4.3), we will establish local $C^1$ stable/unstable manifolds of equation (1.2) near a hyperbolic stationary random point in $L^2(S^1)$. In [Si] it was shown that for any $C^2$ initial function, the solution of (1.2) converges to the stationary solution as $t \to \infty$. Comparing Sinai’s results with our Theorem 4.4, it is not difficult to conclude that all $C^2$ functions form part of the stable manifold, but it is not clear if the unstable manifold is a single point.

2. The non-linear ergodic theorem.

The main objective of this section is to refine and extend discrete-time results of D. Ruelle to the continuous-time setting in Theorem 2.1 below. This setting underlies the dynamics of the semilinear see’s and spde’s studied by the authors in [M-Z-Z.1]. As will be apparent later, the extension of Ruelle’s results to continuous-time is non-trivial. Indeed, Section 3 in its entirety is devoted to the proof of Theorem 2.1. The main difficulties in the analysis are outlined after the statement of the theorem.

In the following, denote by $B(x, \rho)$ the open ball, radius $\rho$ and center $x \in H$, and by $\bar{B}(x, \rho)$ the corresponding closed ball.
Theorem 2.1. (The local stable manifold theorem)

Let \((U, \theta)\) be a \(C^{k, \epsilon}\) \((k \geq 1, \epsilon \in (0, 1])\) perfect cocycle on a separable Hilbert space \(H\) such that for each \((t, \omega) \in \mathbb{R}^+ \times \Omega, U(t, \cdot, \omega) : H \to H\) takes bounded sets into relatively compact sets. For any \(\rho \in (0, \infty)\), denote by \(\|\cdot\|_{k, \epsilon}\) the \(C^{k, \epsilon}\)-norm on the space \(C^{k, \epsilon}(\overline{B}(0, \rho), H)\). Let \(Y\) be a hyperbolic stationary point of the cocycle \((U, \theta)\) satisfying the following integrability property:

\[
\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|U(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega))\|_{k, \epsilon} \, dP(\omega) < \infty
\]

for any fixed \(0 < \rho, T < \infty\) and \(\epsilon \in (0, 1]\). Denote by \(\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}\) the Lyapunov spectrum of the linearized cocycle \((DU(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)\). Define \(\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}\) if at least one \(\lambda_i < 0\). If all finite \(\lambda_i\) are positive, set \(\lambda_{i_0} := -\infty\). (Thus \(\lambda_{i_0-1}\) is the smallest positive Lyapunov exponent of the linearized cocycle, if at least one \(\lambda_i > 0\); in case all the \(\lambda_i\)'s are negative, set \(\lambda_{i_0-1} := \infty\).)

Fix \(\epsilon_1 \in (0, -\lambda_{i_0})\) and \(\epsilon_2 \in (0, \lambda_{i_0-1})\). Then there exist

(i) a sure event \(\Omega^* \in \mathcal{F}\) with \(\theta(t, \cdot)(\Omega^*) = \Omega^*\) for all \(t \in \mathbb{R}\),

(ii) \(\mathcal{F}\)-measurable random variables \(\rho_i, \beta_i : \Omega^* \to (0, 1), \beta_i > \rho_i > 0, i = 1, 2,\) such that for each \(\omega \in \Omega^*\), the following is true:

There are \(C^{k, \epsilon}\) \((\epsilon \in (0, 1])\) submanifolds \(\tilde{S}(\omega), \tilde{U}(\omega)\) of \(\tilde{B}(Y(\omega), \rho_1(\omega))\) and \(\tilde{B}(Y(\omega), \rho_2(\omega))\) (resp.) with the following properties:

(a) For \(\lambda_{i_0} > -\infty\), \(\tilde{S}(\omega)\) is the set of all \(x \in \tilde{B}(Y(\omega), \rho_1(\omega))\) such that

\[
|U(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}
\]

for all integers \(n \geq 0\). If \(\lambda_{i_0} = -\infty\), then \(\tilde{S}(\omega)\) is the set of all \(x \in \tilde{B}(Y(\omega), \rho_1(\omega))\) such that

\[
|U(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{\lambda n}
\]
for all integers \( n \geq 0 \) and any \( \lambda \in (-\infty, 0) \). Furthermore,
\[
\limsup_{t \to \infty} \frac{1}{t} \log |U(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0} \tag{2.1}
\]
for all \( x \in \mathcal{S}(\omega) \). Each stable subspace \( S(\omega) \) of the linearized cocycle
\[(DU(t, Y(\cdot, \cdot), \theta(t, \cdot))) \text{ is tangent at } Y(\omega) \text{ to the submanifold } \mathcal{S}(\omega) \]
\( \text{viz. } T_{Y(\omega)} \mathcal{S}(\omega) = S(\omega) \). In particular, \( \text{codim } \mathcal{S}(\omega) = \text{codim } S(\omega) \), is fixed and finite.

\[ \limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup \left\{ \frac{|U(t, x_1, \omega) - U(t, x_2, \omega)|}{|x_1 - x_2|} : x_1 \neq x_2, x_1, x_2 \in \mathcal{S}(\omega) \right\} \right] \leq \lambda_{i_0}. \]

(c) (Cocycle-invariance of the stable manifolds):

There exists \( \tau_1(\omega) \geq 0 \) such that
\[
U(t, \cdot, \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)) \tag{2.2}
\]
for all \( t \geq \tau_1(\omega) \). Also
\[
DU(t, Y(\omega), \omega)(S(\omega)) \subseteq S(\theta(t, \omega)), \quad t \geq 0. \tag{2.3}
\]

(d) For \( \lambda_{i_0-1} < \infty \), \( \tilde{U}(\omega) \) is the set of all \( x \in \tilde{B}(Y(\omega), \rho_2(\omega)) \) with the property that there is a discrete-time “history” process \( y(\cdot, \omega) : \{-n : n \geq 0\} \to H \) such that \( y(0, \omega) = x \) and for each integer \( n \geq 1 \), one has \( U(1, y(-n, \omega), \theta(-n, \omega)) = y(-(n - 1), \omega) \) and
\[
|y(-n, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega)e^{-(\lambda_{i_0-1} - \epsilon_2)n}. \]

If \( \lambda_{i_0-1} = \infty \), \( \tilde{U}(\omega) \) is the set of all \( x \in \tilde{B}(Y(\omega), \rho_2(\omega)) \) with the property that there is a discrete-time “history” process \( y(\cdot, \omega) : \{-n : n \geq 0\} \to H \) such that \( y(0, \omega) = x \) and for each integer \( n \geq 1 \),
\[
|y(-n, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega)e^{-\lambda n},
\]
for any \( \lambda \in (0, \infty) \). Furthermore, for each \( x \in \tilde{U}(\omega) \), there is a unique continuous-time “history” process also denoted by \( y(\cdot, \omega) : (-\infty, 0] \to H \) such that \( y(0, \omega) = x \),
\[
U(t, y(s, \omega), \theta(s, \omega)) = y(t + s, \omega) \text{ for all } s \leq 0, 0 \leq t \leq -s, \text{ and }
\]
\[
\limsup_{t \to -\infty} \frac{1}{t} \log |y(-t, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}. \]
Each unstable subspace $\mathcal{U}(\omega)$ of the linearized cocycle $(DU(t, Y(\cdot, \cdot), \theta(t, \cdot)))$ is tangent at $Y(\omega)$ to $\tilde{U}(\omega)$, viz. $T_Y(\omega)\tilde{U}(\omega) = \mathcal{U}(\omega)$. In particular, $\dim \tilde{U}(\omega)$ is finite and non-random.

(e) Let $y(\cdot, x_i, \omega), i = 1, 2$, be the history processes associated with $x_i = y(0, x_i, \omega) \in \tilde{U}(\omega), i = 1, 2$. Then

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup \left\{ \frac{|y(-t, x_1, \omega) - y(-t, x_2, \omega)|}{|x_1 - x_2|} : x_1 \neq x_2, x_i \in \tilde{U}(\omega), i = 1, 2 \right\} \leq -\lambda_{i_0-1}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\tilde{U}(\omega) \subseteq U(t, \cdot, \theta(-t, \omega))(\tilde{U}(\theta(-t, \omega)))$$

for all $t \geq \tau_2(\omega)$. Also

$$DU(t, \cdot, \theta(-t, \omega))(U(\theta(-t, \omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$DU(t, \cdot, \theta(-t, \omega))|U(\theta(-t, \omega)) : U(\theta(-t, \omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

(g) The submanifolds $\tilde{U}(\omega)$ and $\tilde{S}(\omega)$ are transversal, viz.

$$H = T_{Y(\omega)}\tilde{U}(\omega) \oplus T_{Y(\omega)}\tilde{S}(\omega).$$

Assume, in addition, that the cocycle $(U, \theta)$ is $C^\infty$. Then the local stable and unstable manifolds $\tilde{S}(\omega), \tilde{U}(\omega)$ are also $C^\infty$.

The figure on the next page summarizes the essential features of the stable manifold theorem:
The Stable Manifold Theorem

Before we give a detailed proof of Theorem 2.1, we will outline below its basic ingredients.

An outline of the proof of Theorem 2.1:

• Since $Y$ is a hyperbolic stationary point of the cocycle $(U, \theta)$ (Definition 1.2), then the linearized cocycle satisfies the hypotheses of “perfect versions” of the ergodic theorem and Kingman’s subadditive ergodic theorem (Lemma 3.1 (ii), (iii) in Section 3). These refined versions of the ergodic theorems give invariance of the Oseledec spaces under the continuous-time linearized cocycle (Theorem 1.2). Thus the
stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear cocycle $(U, \theta)$.

• We introduce the auxiliary perfect cocycle $(Z, \theta)$ where

$$ Z(t, \cdot, \omega) := U(t, \cdot) + Y(\omega) - Y(\theta(t, \omega)), \quad t \in \mathbb{R}^+, \omega \in \Omega. $$

We then refine the arguments in ([Ru.2], Theorems 5.1 and 6.1) to construct local stable/unstable manifolds for the discrete cocycle $(Z(nr, \cdot, \omega), \theta(nr, \omega))$ near 0 and hence (by translation) for $U(nr, \cdot, \omega)$ near $Y(\omega)$ for all $\omega$ sampled from a $\theta(t, \cdot)$-invariant sure event in $\Omega$. This is possible because of the continuous-time integrability estimate in Theorem 2.1, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between discrete times and further refining the arguments in [Ru.2], one can show that the above manifolds also serve as local stable/unstable manifolds for the continuous-time cocycle $(U, \theta)$ near $Y$.

• The final key step is to establish the asymptotic invariance of the local stable manifolds under the cocycle $(U, \theta)$. We use arguments underlying the proofs of Theorems 4.1 and 5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability property of Theorem 2.1, coupled with a refined version of the perfect subadditive ergodic theorem (Lemma 3.2, Section 3). The asymptotic invariance of the local unstable manifolds follows by employing the concept of a stochastic history process for $U$ coupled with similar arguments to the above. The existence of the history process compensates for the lack of invertibility of the semiflow. This completes the outline of the proof of Theorem 2.1.

A full proof of Theorem 2.1 will be given in the next section. The proof is based on a discrete-time version of the theorem given in theorems 5.1, 6.1 [Ru.2]. The extension to continuous-time is done via perfection techniques and interpolation between discrete times.
3. Proof of the local stable manifold theorem.

We devote this section to the proof of Theorem 2.1 in continuous time. A large part of the computations are directed towards perfection arguments, whereby we show that the local stable/unstable manifolds are parametrized by sure events which are invariant under the continuous-time shift $\theta(t, \cdot) : \Omega \to \Omega$. The integrability hypothesis on the cocycle $(U, \theta)$ (Theorem 2.1) plays a crucial role in controlling the excursions of the cocycle between discrete times.

Our first lemma gives “perfect versions” of the ergodic theorem and Kingman’s subadditive ergodic theorem. These results are needed in order to construct the shift-invariant sure events appearing in the statement of the local stable manifold theorem (Theorem 2.1), probability space $(\Omega, \mathcal{F}, P)$,

For convenience, we shall frequently adopt the following convention:

**Definition 3.1.**

Let $\{P(\omega) : \omega \in \Omega\}$ be a family of propositions. We say that $P(\omega)$ holds perfectly in $\omega$ if there is a sure event $\Omega^* \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$ and $P(\omega)$ is true for every $\omega \in \Omega^*$.

**Lemma 3.1.**

(i) Let $\Omega_0 \in \bar{\mathcal{F}}$ be a sure event such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \geq 0$. Then there is a sure event $\Omega_0^* \in \mathcal{F}$ such that $\Omega_0^* \subseteq \Omega_0$ and $\theta(t, \cdot)(\Omega_0^*) = \Omega_0^*$ for all $t \in \mathbb{R}$.

(ii) Let $h : \Omega \to \mathbb{R}^+$ be any function such that there exists an $\bar{\mathcal{F}}$-measurable function $g_1 \in L^1(\Omega, \mathbb{R}^+; P)$ and a sure event $\Omega_1 \in \bar{\mathcal{F}}$ such that $\sup_{0 \leq u \leq 1} h(\theta(u, \omega)) \leq g_1(\omega)$ for all $\omega \in \Omega_1$. Then

$$\lim_{t \to \infty} \frac{1}{t} h(\theta(t, \omega)) = 0$$

perfectly in $\omega$. 
(iii) Suppose $f : \mathbb{R}^+ \times \Omega \to \mathbb{R} \cup \{-\infty\}$ is a process such that for each $t \in \mathbb{R}^+$, $f(t, \cdot)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R} \cup \{-\infty\}))$-measurable and the following conditions hold:

(a) There is an $\mathcal{F}$-measurable function $g_2 \in L^1(\Omega, \mathbb{R}^+; P)$ and a sure event $\tilde{\Omega}_1 \in \mathcal{F}$ such that
\[
\left[ \sup_{0 \leq u \leq 1} f^+(u, \omega) + \sup_{0 \leq u \leq 1} f^+(1-u, \theta(u, \omega)) \right] \leq g_2(\omega) \quad \text{for all } \omega \in \tilde{\Omega}_1.
\]

(b) $f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ for all $t_1, t_2 \geq 0$ and all $\omega \in \Omega$.

Then there is a fixed (non-random) number $f^* \in \mathbb{R} \cup \{-\infty\}$ such that
\[
\lim_{t \to \infty} \frac{1}{t} f(t, \omega) = f^*
\]
perfectly in $\omega$.

**Proof.**

The proof of assertion (i) of the lemma is given in Proposition 2.3 ([M-S.4]).

Assertions (ii) and (iii) of the lemma follow from assertion (i) and easy adaptations of the arguments in the proofs of Lemmas 5 and 7 in [Mo.1]. See also Lemma 5.1 in [M-S.4]. □

Lemma 3.2 below is used to construct the continuous-time shift-invariant sure events which appear in the statement of Theorem 2.1. In essence, the lemma is a continuous-time “perfect version” of Ruelle’s Corollary A.2 ([Ru.2], p. 288).

**Lemma 3.2.**

Suppose $f : \mathbb{R}^+ \times \Omega \to \mathbb{R} \cup \{-\infty\}$ is a $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R} \cup \{-\infty\}))$-measurable process satisfying the following conditions:

(a) $\int_{\Omega} \left[ \sup_{0 \leq t_1, t_2 \leq T} f^+(t_1, \theta(t_2, \omega)) \right] dP(\omega) < \infty$ for all $T \in (0, \infty)$.

(b) $f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ for all $t_1, t_2 \geq 0$ and all $\omega \in \Omega$. 
Then there exists (a non-random) \( f^* \in \mathbb{R} \cup \{-\infty\} \) such that the following statements hold perfectly in \( \omega \):

(i) \[
\lim_{t \to \infty} \frac{1}{t} f(t, \omega) = f^*.
\]

(ii) If \( g^* \in \mathbb{R} \) is a finite number such that \( f^* \leq g^* \), then for every \( \epsilon > 0 \), there exists an \( \mathcal{F} \)-measurable function \( K_\epsilon : \Omega \to [0, \infty) \) with the property that

\[
f(t - s, \theta(s, \omega)) \leq (t - s)g^* + \epsilon t + K_\epsilon(\omega)
\]

whenever \( 0 \leq s \leq t < \infty \). Furthermore, \( K_\epsilon \) may be chosen such that \( K_\epsilon(\theta(l, \omega)) \leq K_\epsilon(\omega) + \epsilon l \) for all \( l \in [0, \infty) \).

Proof.

By Lemma 3.1 (iii), there exists \( f^* \in \mathbb{R} \cup \{-\infty\} \) and a sure event \( \Omega_2 \in \mathcal{F} \) such that \( \theta(t, \cdot)(\Omega_2) = \Omega_2 \) for all \( t \in \mathbb{R} \) and assertion (i) holds for all \( \omega \in \Omega_2 \). By hypotheses (a) and Lemma 3.1 (i), there is a sure event \( \Omega_0 \subseteq \Omega_2 \) such that \( \Omega_0 \in \mathcal{F} \), \( \theta(t, \cdot)(\Omega_0) = \Omega_0 \) for all \( t \in \mathbb{R} \), and \( \sup_{0 \leq t_1, t_2 \leq T} f^+(t_1, \theta(t_2, \omega)) < \infty \) for all \( T \geq 0 \) and all \( \omega \in \Omega_0 \). Suppose \( g^* \) is a finite real number such that \( f^* \leq g^* \). Define the process \( g : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+ \) by

\[
g(t, \omega) := \begin{cases} 
\max\{f(t, \omega) - tg^*, 0\}, & t \geq 0, \ \omega \in \Omega_0, \\
0, & t \geq 0, \ \omega \notin \Omega_0.
\end{cases}
\]

It is easy to check that \( g \) is non-negative, \( (\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}^+)) \)-measurable and satisfies conditions (a) and (b).

Define the process \( g' : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+ \) by

\[
g'(t, \omega) := \sup_{0 \leq s \leq t} [g(s, \omega) + g(t - s, \theta(s, \omega))], \quad t \geq 0, \ \omega \in \Omega.
\]

Using the fact that the projection of a \( (\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}) \)-measurable set is \( \mathcal{F} \)-measurable ([Co], p. 281), it follows that \( g' \) satisfies the hypotheses of Lemma 3.1 (iii). Therefore, there exists \( g'^* \geq 0 \), a sure event \( \Omega_3 \in \mathcal{F} \) such that \( \theta(t, \cdot)(\Omega_3) = \Omega_3 \) for all \( t \in \mathbb{R} \) and

\[
\lim_{t \to \infty} \frac{1}{t} g'(t, \omega) = g'^* \quad \text{for all } \omega \in \Omega_3.
\]
Next, we claim that
\[
\lim_{t \to \infty} \frac{1}{t} \sup_{0 \leq s \leq t} g(t - s, \theta(s, \cdot)) = 0 \tag{3.1}
\]
in probability. This claim easily implies \(g'' = 0\). Hence there is a sure event \(\Omega_4 \in \mathcal{F}\) such that \(\Omega_4 \subseteq \Omega_0 \cap \Omega_3\), \(\theta(t, \cdot)(\Omega_4) = \Omega_4\) for all \(t \in \mathbb{R}\) and assertion (i) holds for all \(\omega \in \Omega_4\). The proof of assertion (ii) is completed by setting

\[
K_\epsilon(\omega) := \sup_{0 \leq s \leq t < \infty} \left[ g(t - s, \theta(s, \omega)) - \epsilon t \right]
\]
for all \(\omega \in \Omega_4\) and a fixed \(\epsilon > 0\). It is easy to see from the above definition that \(K_\epsilon : \Omega_4 \to [0, \infty)\) is \((\bar{\mathcal{F}}, \mathcal{B}(\mathbb{R}^+))\)-measurable and \(K_\epsilon(\theta(l, \omega)) \leq K_\epsilon(\omega) + \epsilon l\) for all \(l \in [0, \infty)\) and all \(\omega \in \Omega_4\).

It remains to establish our claim (3.1). The process \(h : \mathbb{R}^+ \times \Omega \to \mathbb{R}\)
\[
h(t, \omega) := g(t, \theta(-t, \omega)), \quad t \in \mathbb{R}^+, \ \omega \in \Omega
\]
satisfies the conditions of Lemma 3.1 (iii). Therefore
\[
\lim_{t \to \infty} \frac{1}{t} h(t, \omega) = 0
\]
for almost all \(\omega \in \Omega_3\) and hence in probability. Fix \(\delta > 0\) and \(t_0 > 0\) such that
\[
P\left(\frac{1}{t} h(t, \cdot) \geq \delta\right) < \delta \text{ for all } t \geq t_0. \text{ Suppose } t \geq t_0, \text{ and consider}
\]
\[
\sup_{0 \leq s \leq t} \frac{1}{t} g(t - s, \theta(s, \omega)) \leq \sup_{0 \leq s \leq t - t_0} \frac{1}{t} g(t - s, \theta(s, \omega)) + \sup_{t - t_0 \leq s \leq t} \frac{1}{t} g(t - s, \theta(s, \omega))
\]
\[
\leq \sup_{0 \leq s \leq t - t_0} \frac{1}{t} g(t - s, \theta(-(t - s), \theta(t, \omega))) + \sup_{t - t_0 \leq s \leq t} \frac{1}{t} g(t - s, \theta(s, \omega)).
\]
The first term in the right hand side of the last inequality is less than or equal to \(\delta\) with probability at least \(1 - \delta\). The second term converges to 0 in probability by assumption (a). Hence equality (3.1) holds and the proof of the lemma is complete. □

Our next result is basically a “perfect version” of Proposition 3.2 in [Ru.2], p. 257. The proof uses Lemma 3.2. We denote by \(\mathcal{B}_s(L(H))\) the Borel \(\sigma\)-algebra on \(L(H)\) generated by the strong topology on \(L(H)\), viz. the smallest topology on \(L(H)\) for which all evaluations \(L(H) \ni A \mapsto A(z) \in H, z \in H\), are continuous.
Lemma 3.3.

Suppose \((T^t(\omega), \theta(t, \omega)), \ t \geq 0,\) is a perfect cocycle of bounded linear operators in \(H\) satisfying the following hypotheses:

(i) The process \(R^+ \times \Omega \ni (t, \omega) \mapsto T^t(\omega) \in L(H)\) is \((\mathcal{B}(R^+) \otimes \mathcal{F}, B_s(L(H)))\)-measurable.

(ii) The map \(R^+ \times \Omega \ni (t, \omega) \mapsto \theta(t, \omega) \in \Omega\) is \((\mathcal{B}(R^+) \otimes \mathcal{F}, \mathcal{F})\)-measurable, and is a group of ergodic \(P\)-preserving transformations on \((\Omega, \mathcal{F}, P)\).

(iii) \(E \cdot \sup_{0 \leq t_1, t_2 \leq a} \log^+ \|T^{t_2}(\theta(t_1, \cdot))\|_{L(H)} < \infty\) for any finite \(a > 0\).

(iv) For each \(t > 0\), \(T^t(\omega)\) is compact, perfectly in \(\omega\).

(v) For any \(u \in H\), the map \([0, \infty) \ni t \mapsto T^t(\omega)(u) \in H\) is continuous, perfectly in \(\omega\).

Let \(\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}\) be the Lyapunov spectrum of \((T^t(\omega), \theta(t, \omega))\), with Oseledec spaces

\[
\cdots E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = H.
\]

Let \(j_0 \geq 1\) be any fixed integer with \(\lambda_{j_0} > -\infty\). Let the integer function \(r : \{1, 2, \cdots, Q\} \rightarrow \{1, 2, \cdots, j_0\}\) “count” the multiplicities of the Lyapunov exponents in the sense that \(r(1) = 1, \ r(Q) = j_0, \) and for each \(1 \leq i \leq j_0\), the number of integers in \(r^{-1}(i)\) is the multiplicity of \(\lambda_i\). Set \(V_n(\omega) := E_{j_0+1}(\theta(nt_0, \omega)), \ n \geq 0,\)

Then the sequence \(T_n(\omega) := T^1(\theta((n-1), \omega)), \ n \geq 1,\) satisfies Condition (S) of ([Ru.2], pp. 256-257) perfectly in \(\omega\) with \(Q = \text{codim} E_{j_0+1}(\omega)\). In particular, there is an \(\mathcal{F}\)-measurable set of \(Q\) orthonormal vectors \(\{\xi_0^{(1)}(\omega), \cdots, \xi_0^{(Q)}(\omega)\}\) such that \(\xi_0^{(k)}(\omega) \in [E_{r(k)}(\omega) \setminus E_{r(k)+1}(\omega)]\) for \(k = 1, \cdots, Q,\) perfectly in \(\omega,\) and satisfying the following properties:

Set \(\xi_t^{(k)}(\omega) := \frac{T^t(\omega)(\xi_0^{(k)}(\omega))}{|T^t(\omega)(\xi_0^{(k)}(\omega))|},\) and for any \(u \in H,\) write

\[
u = \sum_{k=1}^{Q} u_t^{(k)}(\omega) \xi_t^{(k)}(\omega) + u_t^{(Q+1)}(\omega), \quad u_t^{(Q+1)}(\omega) \in V_0(\theta(t, \omega)), \ \omega \in \Omega.
\]
Then for any $\epsilon > 0$, there is an $\mathcal{F}$-measurable random constant $D_{\epsilon}(\omega) > 0$ such that the following inequalities hold perfectly in $\omega$:

$$
|u_t^{(k)}(\omega)| \leq D_{\epsilon}(\omega)e^{\epsilon t}|u|
$$

$$
|u_t^{(Q+1)}(\omega)| \leq D_{\epsilon}(\omega)e^{\epsilon t}|u|
$$

$$
D_{\epsilon}(\theta(l, \omega)) \leq D_{\epsilon}(\omega)e^{\epsilon l}
$$

for all $t \geq 0$, $1 \leq k \leq Q$ and for all $l \in [0, \infty)$.

Furthermore, all the random constants in Ruelle’s condition (S) ([Ru.2], pp. 256-257) may be chosen to be $\mathcal{F}$-measurable in $\omega$.

Proof.

We will follow the proof of Proposition 3.2 in [Ru.2], ensuring that the relevant parts of the argument hold perfectly in $\omega$. Ruelle’s conditions (S) will be referred to as (S1)-(S4) ([Ru.2, pp. 256-257]).

First note that in view of (iii), the perfect cocycle property, Lemma 3.1 and the argument in Theorem 4 ([Mo.1]), it follows that $T_n(\omega)$ satisfies (S1) perfectly in $\omega$. (Observe that Condition 3.4 in [Ru.2] holds perfectly by the ordering of the fixed Lyapunov spectrum.) Let $\Omega^*$ be the perfect event where (S1) holds. Let $\text{codim} V_0(\omega) = Q$, for all $\omega \in \Omega^*$; then, by ergodicity, $\text{codim} V_n(\omega) = \text{codim} E_{j_{n+1}}(\theta(n, \omega)) = Q$. Hence (S2) holds for all $\omega \in \Omega^*$.

To establish a perfect version of (S3), we will prove the stronger statement that $(T^t(\omega), \theta(t, \omega))$ satisfies (S3) perfectly in $\omega$. Define $\hat{T}^t(\omega) := T^t(\omega)|V_0(\omega), \omega \in \Omega^*, t \geq 0$. Then $\hat{T}^t(\omega)(V_0(\omega)) \subseteq V_0(\theta(t, \omega))$, and the following cocycle identity

$$
\hat{T}^{t_1+t_2}(\omega) = \hat{T}^{t_2}(\theta(t_1, \omega)) \circ \hat{T}^{t_1}(\omega)
$$

holds for all $\omega \in \Omega^*, t \geq 0$. Define $F_t(\omega) := \log \|\hat{T}^t(\omega)\|$, $\omega \in \Omega^*, t \geq 0$. Then the above cocycle identity implies that $(F_t(\omega), \theta(t, \omega))$ is perfectly subadditive, and (iii) implies that
sup \(0 \leq t_1, t_2 \leq T\), \(F_t^+ (\theta(t_1, \cdot))\) is integrable for any finite \(T > 0\). Hence Lemma 3.1 applies, and we get a fixed number \(F^* \in \mathbb{R} \cup \{-\infty\}\) such that

\[
\lim_{t \to \infty} \frac{1}{t} F_t(\omega) = F^*
\]

perfectly in \(\omega\). Let \(S = j_0\), and \(\mu^{(S+1)} := \lambda_{j_0+1}\), when \(\lambda_{j_0+1} > -\infty\); if \(\lambda_{j_0+1} = -\infty\), we set \(\mu^{(S+1)}\) to be any fixed finite number in \((-\infty, \lambda_{j_0})\). From (3.5), p. 257 in [Ru.2], we see that \(F^* \leq \mu^{(S+1)}\). Let \(\epsilon > 0\). If \(\lambda_{j_0+1} > -\infty\), then by Lemma 3.2(ii), we get

\[
\log \|\hat{T}^{t-s}(\theta(s, \omega))\| \leq (t-s)\mu^{(S+1)} + \epsilon t + K_\epsilon(\omega), \quad 0 \leq s \leq t < \infty,
\]

perfectly in \(\omega\), with \(K_\epsilon \mathcal{F}\)-measurable. Note that by Lemma 3.2(ii), \(K_\epsilon(\omega)\) is finite (perfectly in \(\omega\)) and satisfies the inequality

\[
K_\epsilon(\theta(l, \omega)) \leq K_\epsilon(\omega) + \epsilon l
\]

perfectly in \(\omega\) for all \(l \in [0, \infty)\). (In case \(\lambda_{j_0+1} = -\infty\), the inequality (3.2) is valid when \(\mu^{(S+1)}\) is replaced by any finite number in \((-\infty, \lambda_{j_0})\).) Putting \(t = n, s = m + 1\) in (3.2) where \(0 < m < n\) are integers, shows that \(T_n(\omega)\) satisfies (S3) perfectly in \(\omega\).

Finally, we show that the above sequence also satisfies (S4) perfectly in \(\omega\). In the spirit of the preceding analysis, it is sufficient to prove that the continuous-time cocycle \((T^t(\omega), \theta(t, \omega))\) satisfies (S4) perfectly in \(\omega\). Define the family of operators \(\hat{T}^t(\omega) : H \to V_0(\theta(t, \omega)) \subseteq H\), \(\tilde{T}^t(\omega) : H \to V_0(\theta(t, \omega)) \subseteq H\) via the orthogonal decomposition

\[
T^t(\omega)(\xi) = \hat{T}^t(\omega)(\xi) + \tilde{T}^t(\omega)(\xi)
\]

for all \(\xi \in H, t \geq 0, \omega \in \Omega^*\), where \(\hat{T}^t(\omega)(\xi) \in V_0(\theta(t, \omega))\), \(\tilde{T}^t(\omega)(\xi) \in V_0(\theta(t, \omega))\) are the orthogonal projections of \(T^t(\omega)(\xi)\) on \(V_0(\theta(t, \omega))\) and \(V_0(\theta(t, \omega))\), respectively. We
claim that \((\hat{T}^t(\omega), \theta(t, \omega))\) satisfies the perfect cocycle property in \(L(H)\). To see this, fix \(\omega \in \Omega, t_1, t_2 \geq 0, \xi \in H\) and consider

\[
T^{t_1+t_2}(\omega)(\xi) = T^{t_2}(\theta(t_1, \omega))[T^{t_1}(\omega)(\xi)]
\]

\[
= \hat{T}^{t_2}(\theta(t_1, \omega))[\hat{T}^{t_1}(\omega)(\xi)] + \hat{T}^{t_2}(\theta(t_1, \omega))[\hat{T}^{t_1}(\omega)(\xi)] + \hat{T}^{t_2}(\theta(t_1, \omega))[\hat{T}^{t_1}(\omega)(\xi)]
\]

\[+ \hat{T}^{t_2}(\theta(t_1, \omega))[\hat{T}^{t_1}(\omega)(\xi)].
\]

(3.4)

Now by the cocycle invariance of \(V_0(\omega)\) under \(T^t(\omega)\), it follows that \(\hat{T}^t(\omega)(\xi) = 0\) whenever \(\xi \in V_0(\omega)\). Therefore \(\hat{T}^{t_2}(\theta(t_1, \omega))[\hat{T}^{t_1}(\omega)(\xi)] = 0\). Thus (3.4) gives

\[
T^{t_1+t_2}(\omega)(\xi) = \hat{T}^{t_2}(\theta(t_1, \omega))[\hat{T}^{t_1}(\omega)(\xi)] + \hat{T}^{t_2}(\theta(t_1, \omega))[\hat{T}^{t_1}(\omega)(\xi)] + \hat{T}^{t_2}(\theta(t_1, \omega))[\hat{T}^{t_1}(\omega)(\xi)].
\]

(3.5)

That is

\[
T^{t_1+t_2}(\omega)(\xi) = \hat{T}^{t_1}(\omega)(\xi) + \hat{T}^{t_1+t_2}(\omega)(\xi)
\]

(3.6)

for all \(\xi \in H\). The first term on the right-hand side of (3.5) belongs to \(V_0(\theta(t_1 + t_2, \omega))^\perp\) and the second two terms belong to \(V_0(\theta(t_1 + t_2, \omega))\). Therefore by uniqueness of the direct-sum representation on the right-hand side of (3.6), it follows that

\[
\hat{T}^{t_1+t_2}(\omega)(\xi) = \hat{T}^{t_2}(\theta(t_1, \omega))[\hat{T}^{t_1}(\omega)(\xi)]
\]

(3.7)

for all \(\xi \in H\). This proves that \((\hat{T}^t(\omega), \theta(t, \omega))\) satisfies the perfect cocycle property in \(L(H)\). To complete the proof of (S4), note first that the integrability property (iii) of the lemma implies that

\[
E \sup_{0 \leq t_1, t_2 \leq a} \log^+ \|\hat{T}^{t_2}(\theta(t_1, \cdot))\|_{L(H)} < \infty
\]

(3.8)

for any finite \(a > 0\). Applying the perfect Oseledec theorem to \((T^t(\omega), \theta(t, \omega))\) and \((\hat{T}^t(\omega), \theta(t, \omega))\) shows that the following limits exist perfectly in \(\omega\) for all \(\xi \in H\):

\[
\lim_{t \to \infty} \frac{1}{t} \log |T^t(\omega)(\xi)| = \tilde{l}_\xi, \quad \lim_{t \to \infty} \frac{1}{t} \log |\hat{T}^t(\omega)(\xi)| = l_\xi
\]
where \( l_\xi, \hat{l}_\xi \) are fixed numbers in \( \mathbb{R} \cup \{-\infty\} \). Now from (3.6) in ([Ru.2], p. 259), we know that

\[
\hat{l}_\xi = \lim_{n \to \infty} \frac{1}{n} \log |\hat{T}^n(\omega)(\xi)| = \lim_{n \to \infty} \frac{1}{n} \log |T^n(\omega)(\xi)| = l_\xi
\]

for a.a. \( \omega \) and for all \( \xi \in H \setminus V_0(\omega) \). Therefore the equality

\[
\lim_{t \to \infty} \frac{1}{t} \log |\hat{T}^t(\omega)(\xi)| = \lim_{t \to \infty} \frac{1}{t} \log |T^t(\omega)(\xi)|
\]

holds perfectly in \( \omega \) for all \( \xi \in H \setminus V_0(\omega) \). Hence, relation (3.6) in ([Ru.2], p. 259) may be replaced by the continuous-time “perfect” relation

\[
\lim_{t \to \infty} \frac{1}{t} \log \left| \frac{\hat{T}^t(\omega)(\xi)}{|T^t(\omega)(\xi)|} \right| = 0
\] \hspace{1cm} (3.9)

for all \( \xi \in H \setminus V_0(\omega) \).

We now complete the proof of the lemma by following the rest of the argument in the proof of Proposition 3.2 in ([Ru.2], p. 259). By ([C-V], Theorem III.6, p. 65) and Gram-Schmidt orthogonalization, we may select a set of \( Q, \mathcal{F} \)-measurable, orthonormal vectors \( \{\xi_0^{(1)}(\omega), \ldots, \xi_0^{(Q)}(\omega)\} \) such that \( \xi_0^{(k)}(\omega) \in [E_{r(k)}(\omega) \setminus E_{r(k)+1}(\omega)] \cap V_0(\omega) \) for \( k = 1, \ldots, Q \), perfectly in \( \omega \). In the argument in [Ru.2], p. 259, replace (3.6) by (3.9) above, \( n \) by \( t \), \( \xi_n^{(k)} \) by \( \xi_t^{(k)}(\omega) := \frac{T^t(\omega)(\xi_0^{(k)}(\omega))}{|T^t(\omega)(\xi_0^{(k)}(\omega))|} \), \( V_n \) by \( V_0(\theta(t, \omega)) \), and \( \eta_n^{(k)} \) by \( \eta_t^{(k)}(\omega) := \frac{T^t(\omega)(\xi_0^{(k)}(\omega))}{|T^t(\omega)(\xi_0^{(k)}(\omega))|} \).

Therefore for \( u \in H \), we write

\[
u = \sum_{k=1}^{Q} u_t^{(k)}(\omega) \xi_t^{(k)}(\omega) + u_t^{(Q+1)}(\omega), \quad u_t^{(Q+1)}(\omega) \in V_0(\theta(t, \omega)),
\]

perfectly in \( \omega \) for all \( t \geq 0 \). Furthermore, as in [Ru.2], p. 259, (3.9) implies that

\[
\lim_{t \to \infty} \frac{1}{t} \log |\det(\eta_t^{(1)}(\omega), \ldots, \eta_t^{(Q)}(\omega))| = 0,
\]

perfectly in \( \omega \).
Finally, we will show that for any \( \epsilon > 0 \), there is a \( \mathcal{F} \)-measurable non-negative function \( D_\epsilon : \Omega \to (0, \infty) \) such that the following inequalities hold perfectly in \( \omega \):

\[
\begin{align*}
|u^{(k)}_t(\omega)| &\leq D_\epsilon(\omega)e^{\epsilon t}|u| \\
|u^{(Q+1)}_t(\omega)| &\leq D_\epsilon(\omega)e^{\epsilon t}|u| \\
D_\epsilon(\theta(l, \omega)) &\leq D_\epsilon(\omega)e^{\epsilon l}
\end{align*}
\]

for all \( t \geq 0 \), \( 1 \leq k \leq Q \) and for all \( l \in [0, \infty) \).

To prove the above inequalities, define

\[
D_\epsilon(\omega) := 1 + Q \sup_{0 \leq s \leq t \leq \infty} e^{-\epsilon t} |\det(\eta^{-1}_{t-s}(\theta(s, \omega)), \eta^{(2)}_{t-s}(\theta(s, \omega)), \ldots, \eta^{(Q)}_{t-s}(\theta(s, \omega)))|^{-1}
\]

perfectly in \( \omega \). We will first show that \( D_\epsilon(\omega) < \infty \) perfectly in \( \omega \). Let \( 0 \leq s \leq t \).

Using the fact that the determinant of the linear operator \( \bar{T}^{t-s}(\theta(s, \omega)) \) is given by

\[
|\det(\eta^{-1}_{t-s}(\theta(s, \omega)), \ldots, \eta^{(Q)}_{t-s}(\theta(s, \omega)))|^{-1}
\]

for any choice of basis \( \{v_1, \ldots, v_Q\} \) in \( V_0(\theta(s, \omega))^\perp \), it is easy to see that

\[
\begin{align*}
&= \prod_{k=1}^{Q} |[T^{t-s}(\theta(s, \omega))(\xi^{(k)}_0)(\theta(s, \omega))]| \\
&\leq \prod_{k=1}^{Q} \frac{|\det(\bar{T}^{t-s}(\theta(s, \omega))(\xi^{(1)}_0)(\theta(s, \omega)), \ldots, \bar{T}^{t-s}(\theta(s, \omega))(\xi^{(Q)}_0)(\theta(s, \omega)))|}{|[\bar{T}^{t-s}(\theta(s, \omega))\xi^{(1)}_0](\theta(s, \omega))+\cdots+\bar{T}^{t-s}(\theta(s, \omega))\xi^{(Q)}_0(\theta(s, \omega))]|}
\end{align*}
\]

perfectly in \( \omega \). The integrability condition (iii) implies that

\[
\sup_{0 \leq s \leq t \leq a} |[T^{t-s}(\theta(s, \omega))]|^Q \cdot |\bar{T}^{s}(\omega)|^Q < \infty
\]
perfectly in $\omega$ for any finite $a > 0$. We next show that
\[
\sup_{0 \leq s \leq t \leq a} |\det(\eta_{t-s}(\theta(s, \omega)), \ldots, \eta_{t-s}^{(Q)}(\theta(s, \omega)))|^{-1} < \infty \tag{3.16}
\]
perfectly in $\omega$ for any finite $a > 0$. To prove (3.16), it suffices to show that
\[
\inf_{(t, v_1, \ldots, v_Q) \in S(\omega)} |\wedge_{k=1}^{Q} [\tilde{T}^t(\omega)(v_k)]| > 0 \tag{3.17}
\]
perfectly in $\omega$, where $S(\omega)$ stands for the compact set
\[
S(\omega) := \{(t, v_1, \ldots, v_Q) : t \in [0, a], v_k \in V_0(\omega)^{\perp}, |v_k| = 1, < v_k, v_l> = 0, 1 \leq k < l \leq Q\}.
\]
To establish (3.17), note that each map $\tilde{T}^t(\omega)|V_0(\omega)^{\perp} : V_0(\omega)^{\perp} \rightarrow V_0(\theta(t, \omega))^\perp$ is injective for each $t \geq 0$ perfectly in $\omega$. This follows easily from the cocycle property and the fact that $\lambda_{j_0} > -\infty$. Indeed,
\[
|\wedge_{k=1}^{Q} [\tilde{T}^t(\omega)(v_k)]| > 0 \tag{3.18}
\]
for all $(t, v_1, \ldots, v_Q) \in S(\omega)$. From hypothesis (v) of the lemma, the map
\[
[0, a] \times [V_0(\omega)^{\perp}]^{Q} \ni (t, v_1, \ldots, v_Q) \mapsto |\wedge_{k=1}^{Q} [\tilde{T}^t(\omega)(v_k)]| \in [0, \infty)
\]
is jointly continuous. Hence by (3.18) and the compactness of $S(\omega)$, (3.17) follows. In view of (3.15) and (3.17), one gets (3.16).

Next, we claim that
\[
\lim_{t \to \infty} \frac{1}{t} \log \sup_{0 \leq s \leq t} |\det(\eta_{t-s}^{(1)}(\theta(s, \omega)), \ldots, \eta_{t-s}^{(Q)}(\theta(s, \omega)))|^{-1} = 0 \tag{3.19}
\]
perfectly in $\omega$. To prove (3.19), use (3.14) to obtain the estimate
\[
|\det(\eta_{t-s}^{(1)}(\theta(s, \omega)), \ldots, \eta_{t-s}^{(Q)}(\theta(s, \omega)))|^{-1} \leq \frac{\Pi_{k=1}^{Q} [[\tilde{T}^{t-s}(\theta(s, \omega))|E_{r(k)}(\theta(s, \omega))] \cdot ||\tilde{T}^{s}(\omega)|E_{r(k)}(\omega)||]}{||\tilde{T}^t(\omega)|V_0(\omega)^{\perp}]^{\wedge Q}||}
\]
for $0 \leq s \leq t$ perfectly in $\omega$. Take $\frac{1}{t} \log \sup_{0 \leq s \leq t}$ on both sides of the above inequality and use Lemma 3.2(ii) to obtain

$$\frac{1}{t} \log \sup_{0 \leq s \leq t} | \det(\eta_{t-s}^{(1)}(\theta(s, \omega)), \cdots, \eta_{t-s}^{(Q)}(\theta(s, \omega))) |^{-1}$$

$$\leq \frac{1}{t} \sup_{0 \leq s \leq t} \left\{ \sum_{k=1}^{Q} (\log \| [T_{t-s}^{1}((\theta(s, \omega))|E_{r(k)}(\theta(s, \omega))] + \log \| [\hat{T}_{s}^T(\omega)|E_{r(k)}(\omega)] \| \right\}$$

$$- \frac{1}{t} \log \| [\hat{T}_{t}^T(\omega)|V_{0}(\omega)]^{\perp} \|$$

$$\leq \frac{1}{t} \sup_{0 \leq s \leq t} \left\{ \sum_{k=1}^{Q} (t-s)\lambda_{r(k)} + et + K_{e}^{1}(\omega) + \sum_{k=1}^{Q} s\lambda_{r(k)} + es + K_{e}^{2}(\omega) \right\}$$

$$- \frac{1}{t} \log \| [\hat{T}_{t}^T(\omega)|V_{0}(\omega)]^{\perp} \|$$

$$= \sum_{k=1}^{Q} \lambda_{r(k)} + 2e \left[ K_{e}^{1}(\omega) + K_{e}^{2}(\omega) \right] - \frac{1}{t} \log \| [\hat{T}_{t}^T(\omega)|V_{0}(\omega)]^{\perp} \|, \quad t > 0,$$

for arbitrary $\epsilon > 0$, where $K_{i}^{j}(\omega), i = 1, 2,$ are finite positive constants (independent of $t$). The above inequality holds perfectly in $\omega$. Letting $t \to \infty$ in the above inequality, we obtain

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{0 \leq s \leq t} | \det(\eta_{t-s}^{(1)}(\theta(s, \omega)), \cdots, \eta_{t-s}^{(Q)}(\theta(s, \omega))) |^{-1}$$

$$\leq \sum_{k=1}^{Q} \lambda_{r(k)} + 2e - \liminf_{t \to \infty} \frac{1}{t} \log \| [\hat{T}_{t}^T(\omega)|V_{0}(\omega)]^{\perp} \|$$

$$= \sum_{k=1}^{Q} \lambda_{r(k)} + 2e - \sum_{k=1}^{Q} \lambda_{r(k)}$$

$$= 2e.$$ 

Since $\epsilon > 0$ is arbitrary, the above inequality implies

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{0 \leq s \leq t} | \det(\eta_{t-s}^{(1)}(\theta(s, \omega)), \cdots, \eta_{t-s}^{(Q)}(\theta(s, \omega))) |^{-1} \leq 0 \quad (3.20)$$

perfectly in $\omega$. The inequality

$$\liminf_{t \to \infty} \frac{1}{t} \log \sup_{0 \leq s \leq t} | \det(\eta_{t-s}^{(1)}(\theta(s, \omega)), \cdots, \eta_{t-s}^{(Q)}(\theta(s, \omega))) |^{-1}$$

$$\geq \liminf_{t \to \infty} \frac{1}{t} \log | \det(\eta_{t}^{(1)}(\omega), \cdots, \eta_{t}^{(Q)}(\omega))) |^{-1} = 0 \quad (3.21)$$
follows immediately from (3.11). Combining (3.20) and (3.21) yields (3.19).

Using (3.16), (3.19) and (3.13), it is now easy to see that $D_\epsilon(\omega)$ is finite perfectly in $\omega$.

The reader may check that the last inequality in (3.12) follows directly from (3.13).

We next prove the first two inequalities in (3.12). Consider the equation

$$\tilde{u}(\omega) = \sum_{k=1}^{Q} u_t^{(k)}(\omega) \eta_t^{(k)}(\omega), \quad u \in H, t \geq 0.$$ 

View $\tilde{u}(\omega), \eta_t^{(k)}(\omega), 1 \leq k \leq Q$, as column vectors in $\mathbb{R}^Q$ with respect to the basis $\{\xi_0^{(k)}(\theta(t, \omega)) : 1 \leq k \leq Q\}$. Solving the above equation for each $u_t^{(k)}(\omega)$ gives

$$|u_t^{(k)}(\omega)| = \left| \frac{\det(\eta_t^{(1)}(\omega), \ldots, \eta_t^{(k-1)}(\omega), \tilde{u}(\omega), \eta_t^{(k+1)}(\omega), \ldots, \eta_t^{(Q)}(\omega))}{\det(\eta_t^{(1)}(\omega), \ldots, \eta_t^{(Q)}(\omega))} \right|$$

$$\leq \frac{|\tilde{u}(\omega)|}{|\det(\eta_t^{(1)}(\omega), \ldots, \eta_t^{(Q)}(\omega))|}$$

$$\leq \frac{[D_\epsilon(\omega) - 1]}{Q} |u| e^{\epsilon t}$$

$$\leq D_\epsilon(\omega)|u| e^{\epsilon t}, \quad 1 \leq k \leq Q, t \geq 0,$$

perfectly in $\omega$, by Cramer’s rule and (3.13). Using (3.10), the triangle inequality and (3.22), we obtain

$$|u_t^{(Q+1)}(\omega)| \leq |u| + \sum_{k=1}^{Q} |u_t^{(k)}(\omega)| \leq D_\epsilon(\omega)|u| e^{\epsilon t}, \quad t \geq 0,$$

perfectly in $\omega$. This proves that $T_n(\omega)$ satisfies (S4) perfectly in $\omega$, and completes the proof of the proposition. □

The following lemma is used in the discretization argument underlying the proof of the local stable-manifold theorem (Theorem 2.1).
Lemma 3.4.

Let \( Y : \Omega \to H \) be a stationary point of the cocycle \((U, \theta)\) satisfying the integrability condition

\[
\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|U(t_2, Y(\theta(t_1, \omega)) + (\cdot, \theta(t_1, \omega)))\|_{k, \epsilon} \, dP(\omega) < \infty
\]

for any fixed \( 0 < \rho, T < \infty \) and \( \epsilon \in (0, 1] \).

Define the random field \( Z : \mathbb{R}^+ \times H \times \Omega \to H \) by

\[
Z(t, x, \omega) := U(t, x + Y(\omega), \omega) - Y(\theta(t, \omega))
\]

for \( t \geq 0, x \in H, \omega \in \Omega \). Then \((Z, \theta)\) is a \( C^{k, \epsilon} \) perfect cocycle. Furthermore, there is a sure event \( \Omega_5 \in \mathcal{F} \) with the following properties:

(i) \( \theta(t, \cdot)(\Omega_5) = \Omega_5 \) for all \( t \in \mathbb{R} \),

(ii) For every \( \omega \in \Omega_5 \) and any \( x \in H \), the statement

\[
\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| < 0
\]

implies

\[
\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)|.
\]

Proof.

Note that, by definition, \( Z \) is a “centering” of the cocycle \( U \) with respect to the stationary trajectory \( \{Y(\theta(t, \cdot) : t \geq 0)\} \) in the sense that \( Z(t, 0, \omega) = 0 \) for all \((t, \omega) \in \mathbb{R}^+ \times \Omega \). Furthermore, \((Z, \theta)\) is a \( C^{k, \epsilon} \) perfect cocycle. To see this let \( t_1, t_2 \geq 0, \omega \in \Omega, x \in H \). Then by the perfect cocycle property for \( U \), it follows that

\[
Z(t_2, Z(t_1, x, \omega), \theta(t_1, \omega)) = U(t_2, Z(t_1, x, \omega) + Y(\theta(t_1, \omega)), \theta(t_1, \omega)) - Y(\theta(t_2, \theta(t_1, \omega)))
\]

\[
= U(t_2, U(t_1, x + Y(\omega), \omega), \theta(t_1, \omega)) - Y(\theta(t_2 + t_1, \omega))
\]

\[
= Z(t_1 + t_2, x, \omega).
\]
Using the integrability condition of the lemma, the proofs of assertions (i) and (ii) follow exactly in the same manner as for the corresponding assertions in Lemma 3.4 ([M-S.3]). □

Proof of Theorem 2.1.

All real-valued random variables in this proof will be taken to be $\bar{\mathcal{F}}$-measurable.

Our method of proof of Theorem 2.1 will be based on first establishing the assertions of the theorem for discrete time following Ruelle [Ru.2], and then extending the results to continuous time using perfection arguments.

Consider the auxiliary cocycle $(Z, \theta)$ as defined in Lemma 3.4. Define the family of maps $F_\omega : \bar{B}(0,1) \to H$, $\omega \in \Omega$, by $F_\omega(x) := Z(1,x,\omega)$, and let $\tau := \theta(1,\cdot) : \Omega \to \Omega$. Following Ruelle ([Ru.2], p. 272), define $F^n_\omega := F_{\tau}^{n-1}(\omega) \circ \cdots \circ F_{\tau}(\omega) \circ F_\omega$. Then by the cocycle property for $Z$, we get $F^n_\omega = Z(n,\cdot,\omega)$ for each $n \geq 1$. Clearly, each $F_\omega$ is $C^{k,\epsilon}$ ($\epsilon \in (0,1]$) on $\bar{B}(0,1)$ and $(DF_\omega)(0) = DU(1,Y(\omega),\omega)$. By the integrability hypothesis of the theorem, it follows that $\log^+ \|DU(1,Y(\cdot),\cdot)\|_{L(H)}$ is integrable. Furthermore, the discrete-time cocycle $((DF^n_\omega)(0), \theta(n,\omega))$ has a Lyapunov spectrum which coincides with that of the linearized continuous-time cocycle $(DU(t,Y(\omega),\omega), \theta(t,\omega))$, viz. $\{-\infty < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Suppose $\lambda_{i_0} \in (-\infty,0)$. We then apply Theorem 5.1 of Ruelle ([Ru.2], p. 272) under his hypotheses (I). This gives a sure event $\Omega^*_1 \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega^*_1) = \Omega^*_1$ for all $t \in \mathbb{R}$, $\bar{\mathcal{F}}$-measurable positive random variables $\rho_1, \beta_1 : \Omega^*_1 \to (0,1)$, and a random family of $C^{k,\epsilon}$ ($k \geq 1, \epsilon \in (0,1]$) stable submanifolds $\tilde{S}_d(\omega)$ of $\tilde{B}(0,\rho_1(\omega))$ satisfying the following properties for each $\omega \in \Omega^*_1$:

$$\tilde{S}_d(\omega) = \{x \in \tilde{B}(0,\rho_1(\omega)) : |Z(n,x,\omega)| \leq \beta_1(\omega)e^{(\lambda_{i_0}+\epsilon_1)n} \text{ for all integers } n \geq 0\}. \quad (3.25)$$

If $\lambda_{i_0} = -\infty$, $\tilde{S}_d(\omega)$ is defined by a similar relation to (3.25) where $\lambda_{i_0} + \epsilon_1$ is replaced by any $\lambda \in (-\infty,0)$. Each $\tilde{S}_d(\omega)$ is tangent at 0 to the stable subspace $S(\omega)$ of the linearized
cocycle $(DU(t, Y(\omega), \theta(t, \omega)), \text{viz. } T_0 \tilde{S}_d(\omega) = S(\omega))$. In particular, \text{codim } \tilde{S}_d(\omega) \text{ is finite and non-random. Furthermore, according to (Theorem 5.1, [Ru.2]), one has:}

$$\limsup_{n \to \infty} \frac{1}{n} \log \left[ \sup_{x_1 \neq x_2 \in \tilde{S}_d(\omega)} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right] \leq \lambda_{i_0}.$$  \hfill (3.26)

At this point we will outline the construction of the $\theta(t, \cdot)$-invariant sure event $\Omega_1^\ast$ referred to in the above paragraph. This will follow from the proof of Theorem 5.1 ([Ru.2], p. 272) coupled with additional perfection arguments. More specifically, and in the notation of [Ru.2], let $T^t(\omega) := DZ(t, 0, \omega), f(\omega) := \theta(1, \omega), T_n(\omega) := DZ(1, 0, \theta((n - 1), \omega)), t \in \mathbb{R}^+, n \in \mathbb{Z}^+$. By the integrability hypothesis of the theorem and the perfect ergodic theorem (Lemma 3.1 (ii)), one may replace (5.3) in [Ru.2], p. 274) by its continuous-time analogue

$$\lim_{t \to \infty} \frac{1}{t} \log^+ \|Z(1, \cdot, \theta(t, \omega))\|_{1, \epsilon} = 0.$$ \hfill (3.27)

The above relation holds perfectly in $\omega$, \text{viz.} there is a sure event $\Omega_1^\ast \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_1^\ast) = \Omega_1^\ast$ for all $t \in \mathbb{R}$ and (3.27) holds for all $\omega \in \Omega_1^\ast$. In the notation of Theorem 1.1 ([Ru.2], p. 248), set $S = i_0 - 1$, fixed, and $\mu^{(S+1)} = \lambda_{i_0}$, when $\lambda_{i_0} > -\infty$; if $\lambda_{i_0} = -\infty$, we replace $\mu^{(S+1)}$ by any fixed number in $(-\infty, 0)$. In view of the integrability hypothesis of the theorem and Lemma 3.3 (with $j_0 = i_0 - 1$), it follows that there is a sure event $\Omega_2^\ast \in \mathcal{F}$ such that $\Omega_2^\ast \subseteq \Omega_1^\ast, \theta(t, \cdot)(\Omega_2^\ast) = \Omega_2^\ast$ for all $t \in \mathbb{R}$, and the sequence $T_n(\omega), V_n(\omega) := E_{i_0}(\tau^n(\omega)), n \geq 1$, satisfies Conditions (S) of [Ru.2], p. 256) for every $\omega \in \Omega_2^\ast$. Fixing any $\omega \in \Omega_2^\ast$, we continue to follow the proof of Theorem 5.1 in [Ru.2], pp. 274-278. In particular, the “perturbation theorem” (Theorem 4.1, [Ru.2], pp. 262-263) holds for the sequence $T_n(\omega), n \geq 1$, and therefore the results quoted in the previous paragraph hold for $k = 1, \epsilon \in (0, 1]$. To see that the $\tilde{S}_d(\omega)$ are $C^{k, \epsilon}$ manifolds ($k > 1, \epsilon \in (0, 1]$) \textit{perfectly in } $\omega$, we follow the inductive argument in [Ru.2], pp. 278-279, by applying the previous analysis to the following perfect cocycle on $H \oplus H$:

$$\left( \tilde{Z}(t, x, x_1, \omega) := (Z(t, x, \omega), DZ(t, x, \omega)x_1), \theta(t, \omega) \right),$$
for \( x, x_1 \in H, \ t \geq 0 \). The inductive argument yields that \( \tilde{S}_d(\omega) \) is a \( C^{k,\epsilon} \) manifold perfectly in \( \omega \).

Consider the set \( \tilde{S}(\omega), \omega \in \Omega_1^* \), defined in part (a) of the theorem. Using (3.25) and the definition of \( Z \), it follows immediately that

\[
\tilde{S}(\omega) = \tilde{S}_d(\omega) + Y(\omega)
\]

(3.28)

for all \( \omega \in \Omega_1^* \). Hence \( S(\omega) \) is a \( C^{k,\epsilon} \) manifold \((k > 1, \epsilon \in (0, 1])\). Furthermore, \( T_Y(\omega)\tilde{S}(\omega) = T_0\tilde{S}_d(\omega) = S(\omega) \). In particular, \( \text{codim} \tilde{S}(\omega) = \text{codim} S(\omega) \) is finite and non-random.

We next show the assertion (2.1) in part (a) of the theorem. By (3.26), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \lambda_{i_0}
\]

perfectly in \( \omega \) for all \( x \in \tilde{S}_d(\omega) \). Therefore by Lemma 3.4, there is a sure event \( \Omega_3^* \subseteq \Omega_2^* \), \( \Omega_3^* \in \mathcal{F} \), such that \( \theta(t, \cdot)(\Omega_3^*) = \Omega_3^* \) for all \( t \in \mathbb{R} \), and

\[
\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \lambda_{i_0}
\]

(3.29)

for all \( \omega \in \Omega_3^* \) and all \( x \in \tilde{S}_d(\omega) \). Now inequality (1) of the theorem follows directly from (3.29) and the definition of \( Z \).

We next prove assertion (b) of the theorem. Take any \( \omega \in \Omega_1^* \). By (3.29), there is a positive integer \( N_0 := N_0(\omega) \) (independent of \( x \in \tilde{S}_d(\omega) \)) such that \( Z(n, x, \omega) \in \bar{B}(0, 1) \) for all \( n \geq N_0 \). Let \( \Omega_3 \) be a \( \theta(t, \cdot) \)-invariant sure event such that

\[
\lim_{t \to \infty} \frac{1}{t} \log^+ \sup_{0 \leq u \leq 1, (v^*, \eta^*) \in \bar{B}(0, 1)} \|DZ(u, (v^*, \eta^*), \theta(t, \omega))\|_{L(H)} = 0
\]

for all \( \omega \in \Omega_3 \) (Lemma 3.1 (ii)). Let \( \Omega_4^* := \Omega_3^* \cap \Omega_3 \). Then \( \Omega_4^* \in \mathcal{F} \), is a sure event and \( \theta(t, \cdot)(\Omega_4^*) = \Omega_4^* \) for all \( t \in \mathbb{R} \). By a similar argument to the one used in the proof of
Lemma 3.4 in [M-S.3], it follows that

\[
\sup_{n \leq t \leq n+1} \frac{1}{t} \log \left[ \sup_{x_1 \neq x_2, (x_1, n_1), x_2 \in S(\omega)} \frac{|U(t, x_1, \omega) - U(t, x_2, \omega)|}{|x_1 - x_2|} \right]
\]

\[
= \sup_{n \leq t \leq n+1} \frac{1}{t} \log \left[ \sup_{x_1 \neq x_2, x_1, x_2 \in S(\omega)} \frac{|Z(t, x_1, \omega) - Z(t, x_2, \omega)|}{|x_1 - x_2|} \right]
\]

\[
\leq \frac{1}{n} \log^+ \sup_{0 \leq u \leq 1, (v^*, \eta^*) \in B(0, 1)} \|DZ(u, (v^*, \eta^*), \theta(n, \omega))\|_{L(H)}
\]

\[
+ \frac{n}{n+1} \frac{1}{n} \log \left[ \sup_{x_1 \neq x_2, x_1, x_2 \in S(\omega)} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right]
\]

for all \(\omega \in \Omega_1^*, \) all \(n \geq N_0(\omega)\) and sufficiently large. Taking \(\limsup_{n \to \infty}\) in the above inequality and using (3.26), immediately gives assertion (b) of the theorem.

To prove the cocycle-invariance statements (c), we begin by the inclusion (2.3) in the theorem. This is proved by applying the Oseledec-Ruelle theorem (Theorem 1.1) to the linearized cocycle \((DU(t, Y(\omega), \omega), \theta(t, \omega))\). Hence there is a sure \(\theta(t, \cdot)\)-invariant event, also denoted by \(\Omega_1^* \in \mathcal{F}\), such that \(DU(t, Y(\omega), \omega)(S(\omega)) \subseteq S(\theta(t, \omega))\) for all \(t \geq 0\) and all \(\omega \in \Omega_1^*\).

We next prove the asymptotic invariance property (2.2) of the theorem. To this end, we will need to modify the proofs of Theorems 5.1 and 4.1 in [Ru.2], pp. 262-279. We will first show that two random variables \(\rho_1, \beta_1\) and a sure event (also denoted by) \(\Omega_1^*\) may be chosen such that \(\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*\) for all \(t \in \mathbb{R}\), and

\[
\rho_1(\theta(t, \omega)) \geq \rho_1(\omega)e^{(\lambda_{i_0} + \epsilon_1)t}, \quad \beta_1(\theta(t, \omega)) \geq \beta_1(\omega)e^{(\lambda_{i_0} + \epsilon_1)t}
\] (3.30)

for every \(\omega \in \Omega_1^*\) and all \(t \geq 0\). For the given choice of \(\epsilon_1\), fix \(0 < \epsilon_3 < -\epsilon(\lambda_{i_0} + \epsilon_1)/4\). The above inequalities hold in the discrete case (when \(t = n\), a positive integer) from Theorem 5.1 (c) ([Ru.2], p. 274). We claim that \(\rho_1\) and \(\beta_1\) may be redefined so that the relations (3.30) hold for continuous time. To see this, we will modify the definitions of these random variables in the proofs of Theorems 5.1 and 4.1 in [Ru.2]. In the notation of the proof of
Theorem 5.1 ([Ru.2], p. 274), we replace the random variable $G$ in (5.4) ([Ru.2], p. 274) by the larger one

$$
\tilde{G}(\omega) := \sup_{t \geq 0} \|Z(1, \cdot, \theta(t, \omega))\|_{1, \epsilon} e^{(-t\epsilon_3 - \lambda \epsilon)}.
$$

In (3.31), $\epsilon \in (0, 1]$ stands for the Hölder exponent of $U$. By (3.27) and Lemma 3.2, it is easy to see that $\tilde{G}(\omega) < \infty$ perfectly in $\omega$. Following ([Ru.2], pp. 266, 274), the random variables $\rho_1, \beta_1$ may be chosen according to the relations

$$
\beta_1 := \left[ \frac{\delta_1 \wedge \left( \frac{1}{\sqrt{2}A} \right)}{2\tilde{G}} \right]^{\frac{1}{2}} \wedge 1
$$

$$
\rho_1 := \frac{\beta_1}{B_{\epsilon_3}}
$$

where $A, \delta_1$ and $B_{\epsilon_3}$ are random positive constants that are defined via continuous-time analogues of the relations (4.26), (4.18)-(4.21), (4.24), (4.25) in [Ru.2], pp. 265-267, with $\eta$ replaced by $\epsilon_3$. In particular, the “ancestry” of $A, \delta_1$ and $B_{\epsilon_3}$ in Ruelle’s argument may be traced back to the constants $D_{\epsilon_3}, K_{\epsilon_3}$ which appear in Lemmas 3.3 and 3.2 of this article. Thus, in order to establish (3.30), it suffices to observe that, for sufficiently small $\epsilon_3 > 0$, the following inequalities

$$
\begin{align*}
K_{\epsilon_3}(\theta(l, \omega)) &\leq K_{\epsilon_3}(\omega) + \frac{\epsilon_3 l}{2} \\
D_{\epsilon_3}(\theta(l, \omega)) &\leq e^{\epsilon_3 l} D_{\epsilon_3}(\omega) \\
\tilde{G}(\theta(l, \omega)) &\leq e^{\epsilon_3 l} \tilde{G}(\omega)
\end{align*}
$$

hold perfectly in $\omega$ for all real $l \geq 0$. The first inequality in (3.34) follows from Lemma 3.2(ii), while the second inequality is a consequence of Lemma 3.3. The third inequality in (3.34) follows directly from (3.31). In view of (3.32) and (3.33), (3.30) holds. This completes the proof of (3.30).

We are now ready to prove the asymptotic invariance property (2.2) in (c) of the theorem. Use assertion (b) of the theorem to obtain a sure event $\Omega_5^* \subseteq \Omega_4^*$ such that
\[ \Theta(t, \cdot)(\Omega^*_x) = \Omega^*_x \text{ for all } t \in \mathbb{R}, \text{ and for any } 0 < \epsilon' < \epsilon_1 \text{ and } \omega \in \Omega^*_4, \text{ there exists } \beta'(\omega) > 0 \]

(independent of \( x \)) with

\[ |U(t, x, \omega) - Y(\theta(t, \omega))| \leq \beta'(\omega)e^{(\lambda_{i_0} + \epsilon')t} \tag{3.35} \]

for all \( x \in \tilde{S}(\omega), t \geq 0 \). Fix any real \( t \geq 0, \omega \in \Omega^*_5 \) and \( x \in \tilde{S}(\omega) \). Let \( n \) be a non-negative integer. Then the cocycle property and (3.35) imply that

\[ |U(n, U(t, x, \omega), \theta(t, \omega)) - Y(\theta(n, \theta(t, \omega)))| = |U(n + t, x, \omega) - Y(\theta(n + t, \omega))| \leq \beta'(\omega)e^{(\lambda_{i_0} + \epsilon')(n+t)} \leq \beta'(\omega)e^{(\lambda_{i_0} + \epsilon')t}e^{(\lambda_{i_0} + \epsilon_1)n}. \tag{3.36} \]

If \( \omega \in \Omega^*_5 \), then it follows from (3.30),(3.35), (3.36) and the definition of \( \tilde{S}(\theta(t, \omega)) \) that there exists \( \tau_1(\omega) > 0 \) such that \( U(t, x, \omega) \in \tilde{S}(\theta(t, \omega)) \) for all \( t \geq \tau_1(\omega) \). This proves the invariance property (2.2) and completes the proof of assertion (c) of the theorem.

We now prove assertion (d) of the theorem, regarding the existence of the local unstable manifolds \( \hat{U}(\omega) \) perfectly in \( \omega \). Define the random field \( \hat{Z} : \mathbb{R}^+ \times H \times \Omega \rightarrow H \) by

\[ \hat{Z}(t, x, \omega) := U(t, x + Y(\theta(-t, \omega)), \theta(-t, \omega)) - Y(\omega) \tag{3.37} \]

for all \( t \geq 0, x \in H, \omega \in \Omega \). Observe that \( \hat{Z}(t, \cdot, \omega) = Z(t, \cdot, \theta(-t, \omega)), t \geq 0, \omega \in \Omega; \text{ and } \hat{Z} \) is \((\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))\)-measurable. From (1.1) (with \( \omega \) replaced by \( \theta(-t, \omega) \)), it follows immediately that \( \hat{Z}(t, 0, \omega) = 0 \) for all \( t \geq 0, \omega \in \Omega \). We claim that \((\hat{D}\hat{Z}(t, 0, \omega))^*, \theta(-t, \omega), t \geq 0)\) is a perfect linear cocycle (in \( L(H) \)). To see this we argue as follows. Note first that \((DU(t, Y(\omega), \theta(t, \omega)))\) is an \( L(H) \)-valued perfect cocycle:

\[ DU(t_1 + t_2, Y(\omega), \omega) = DU(t_1, Y(\theta(t_2, \omega)), \theta(t_2, \omega)) \circ DU(t_2, Y(\omega), \omega) \]

for all \( \omega \in \Omega, t_1, t_2 \geq 0 \). Taking adjoints in the above identity and replacing \( \omega \) by \( \theta(-t_1 - t_2, \omega) \) gives

\[ [DU(t_1 + t_2, Y(\theta(-t_1 - t_2, \omega)), \theta(-t_1 - t_2, \omega))]^* \]

\[ = [DU(t_2, Y(\theta(-t_1 - t_2, \omega)), \theta(-t_1 - t_2, \omega))]^* \circ [DU(t_1, Y(\theta(-t_1, \omega)), \theta(-t_1, \omega))]^* \]

\[ = [DU(t_2, Y(\theta(-t_1 - t_2, \omega)), \theta(-t_1 - t_2, \omega))]^* \circ [DU(t_1, Y(\theta(-t_1, \omega)), \theta(-t_1, \omega))]^* \]
for all $\omega \in \Omega, t_1, t_2 \geq 0$. Hence

$$[D\hat{Z}(t_1 + t_2, 0, \omega)]^* = [D\hat{Z}(t_2, 0, \theta(-t_1, \omega))]^* \circ [D\hat{Z}(t_1, 0, \omega)]^*$$

for all $\omega \in \Omega, t_1, t_2 \geq 0$. This proves that $([D\hat{Z}(t, 0, \omega)]^*, \theta(-t, \omega), t \geq 0)$ is a perfect cocycle in $L(H)$, as claimed.

We next show that the cocycles $(DU(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ and $([D\hat{Z}(t, 0, \omega)]^*, \theta(-t, \omega), t \geq 0)$ have the same Lyapunov spectrum with multiplicities. First, we need to verify the integrability condition

$$\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|[D\hat{Z}(t_2, 0, \theta(-t_1, \omega))]^*\|_{L(H)} dP(\omega) < \infty \quad (3.38)$$

for any fixed $T \in (0, \infty)$. To prove (3.38), use the integrability hypothesis of Theorem 2.1 and the $P$-preserving property of $\theta(t, \cdot)$ in order to obtain the following relations:

$$\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|[D\hat{Z}(t_2, 0, \theta(-t_1, \omega))]^*\|_{L(H)} dP(\omega)$$

$$= \int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|[DU(t_2, Y(\theta(-t_2 - t_1, \omega)), \theta(-t_2 - t_1, \omega))]\|_{L(H)} dP(\omega)$$

$$\leq \int_{\Omega} \log^+ \sup_{0 \leq t_1, 2T, 0 \leq t_2 \leq T} \|[DU(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega))]\|_{L(H)} dP(\omega)$$

$$\leq \int_{\Omega} \log^+ \sup_{0 \leq t_1 \leq T, 0 \leq t_2 \leq T} \|[DU(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega))]\|_{L(H)} dP(\omega)$$

$$+ \int_{\Omega} \log^+ \sup_{T \leq t_1 \leq 2T, 0 \leq t_2 \leq T} \|[DU(t_2, Y(\theta(t_1 - T, \omega)), \theta(t_1 - T, \omega))]\|_{L(H)} dP(\omega)$$

$$= 2 \int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|[DU(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega))]\|_{L(H)} dP(\omega) < \infty.$$

In view of the integrability property (3.38), it follows that the linear cocycle $([D\hat{Z}(t, 0, \omega)]^*, \theta(-t, \omega), t \geq 0)$ has a fixed discrete Lyapunov spectrum which coincides with that of $(DU(t, Y(\omega, \omega), \theta(t, \omega)))$, viz. $\{\cdots \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ where $\lambda_i \neq 0$ for all $i \geq 1$, by hyperbolicity. See [Ru.2], Section 3.5, p. 261.
To establish a perfect version of the local unstable manifolds \( \tilde{U}(\omega) \), we begin with the estimate
\[
\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq 1} \| \tilde{Z}(t_2, \cdot, \theta(-t_1, \omega)) \|_{k, \epsilon} dP(\omega) < \infty,
\]
which follows from the \( P \)-preserving property of \( \theta(t, \cdot), t \in \mathbb{R} \), and the integrability hypothesis of the theorem. Define \( i_0 \) as before, so that \( \lambda_{i_0-1} \) is the smallest positive Lyapunov exponent of the linearized cocycle. Fix \( 0 < \epsilon_2 < \lambda_{i_0-1} \). In view of the above integrability property, it follows from Lemma 3.3 that the sequence \( \tilde{T}_n(\omega) \) as before satisfies Condition (S) of \([Ru.2]\) perfectly in \( \omega \). Therefore Proposition 3.3 in \([Ru.2]\) implies that the sequence \( \tilde{T}_n(\omega), n \geq 1 \), satisfies Corollary 3.4 \((Ru.2), p. 260\) perfectly in \( \omega \). Now one can adapt the proof of Theorem 6.1 \((Ru.2), p. 280\) along similar lines to the preceding arguments in this proof. This yields a \( \theta(-t, \cdot) \)-invariant sure event \( \hat{\Omega}_1 \in \mathcal{F} \) and \( \mathcal{F} \)-measurable random variables \( \rho_2, \beta_2 : \hat{\Omega}_1 \rightarrow (0, 1) \) with the following properties. For \( \lambda_{i_0-1} < \infty \), let \( \tilde{U}_d(\omega) \) be the set of all \( x_0 \in \tilde{B}(0, \rho_2(\omega)) \) with the property that there is a discrete “history” process \( u(-n, \cdot) : \Omega \rightarrow H, n \geq 0 \), such that \( u(0, \omega) = x_0, \tilde{Z}(1, u(-(n+1), \omega), \theta(-n, \omega)) = u(-n, \omega) \) and \( |u(-n, \omega)| \leq \beta_2(\omega)e^{-n(\lambda_{i_0-1}-\epsilon_2)} \) for all \( n \geq 0 \). When \( \lambda_{i_0-1} = \infty \), take \( \tilde{U}_d(\omega) \) to be the set of all \( x_0 \in H \) with the property that there is a discrete history process \( u(-n, \cdot) : \Omega \rightarrow H, n \geq 0 \), such that \( u(0, \omega) = x_0 \), and \( |u(-n, \omega)| \leq \beta_2(\omega)e^{-\lambda n} \) for all \( n \geq 0 \) and arbitrary \( \lambda > 0 \). The history process \( u(-n, \cdot) \) is uniquely determined by \( x_0 \) \((Ru.2), p. 281\). Furthermore, for every \( \omega \in \hat{\Omega}_1 \), \( \tilde{U}_d(\omega) \) is a \( C^{k,\epsilon} \) \((k \geq 1, \epsilon \in (0, 1)) \) finite-dimensional submanifold of \( \tilde{B}(0, \rho_2(\omega)) \) with tangent space \( U(\omega) \) at 0. Also \( \dim \tilde{U}_d(\omega) \) is fixed independently of \( \omega \) and \( \epsilon_2 \); and the following estimates hold perfectly in \( \omega \) for all \( t \geq 0 \):
\[
\rho_2(\theta(-t, \omega)) \geq \rho_2(\omega)e^{-(\lambda_{i_0-1}-\epsilon_2)t}, \quad \beta_2(\theta(-t, \omega)) \geq \beta_2(\omega)e^{-(\lambda_{i_0-1}-\epsilon_2)t}. \tag{3.39}
\]
We claim that the set $\tilde{U}(\omega)$ defined in (d) of Theorem 2.1 coincides with $\tilde{U}_d(\omega) + Y(\omega)$ for each $\omega \in \hat{\Omega}_1^*$. We first show that $\tilde{U}_d(\omega) + Y(\omega) \subseteq \tilde{U}(\omega)$. Let $x_0 \in \tilde{U}_d(\omega)$ and $u$ be as above. Set

$$y_0(-n, \omega) := u(-n, \omega) + Y(\theta(-n, \omega)), \quad n \geq 0. \quad (3.40)$$

It is easy to check that $y_0$ is a discrete history process satisfying the first and second assertions in (d) of the theorem. Hence $x_0 + Y(\omega) \in \tilde{U}(\omega)$. Similarly, $\tilde{U}(\omega) \subseteq \tilde{U}_d(\omega) + Y(\omega)$ for all $\omega \in \hat{\Omega}_1^*$. Hence $\tilde{U}(\omega) = \tilde{U}_d(\omega) + Y(\omega)$ for all $\omega \in \hat{\Omega}_1^*$. This immediately implies that $\tilde{U}(\omega)$ is a $C^{k,\epsilon}$ ($k \geq 1, \epsilon \in (0, 1]$) finite-dimensional submanifold of $\bar{B}(Y(\omega), \rho_2(\omega))$ and

$$T_{Y(\omega)} \tilde{U}(\omega) = T_{0} \tilde{U}_d(\omega) = U(\omega).$$

for all $\omega \in \hat{\Omega}_1^*$.

To prove the third assertion in part (d) of the theorem, let $x \in \tilde{U}(\omega)$ and write $x = x_0 + Y(\omega)$ where $x_0 \in \tilde{U}_d(\omega)$. We will prove that the discrete process $y_0$ given by (40) extends to a continuous-time history process $y(\cdot, \omega) : (-\infty, 0] \to H$ such that $y(0, \omega) = x$, and $y(\cdot, \omega)$ satisfies the third assertion in (d) of the theorem. To do this, we use the cocycle property of $U$ to interpolate within the periods $[-(n+1), -n]$, $n \geq 0$. Let $s \in (-n+1, -n)$ and write $s = \alpha - (n+1)$ for some $\alpha \in (0, 1)$. Define

$$y(s, \omega) := U(s + n + 1, y_0(-(n+1), \omega), \theta(-(n+1), \omega)).$$

Clearly $y(0, \omega) = x_0 + Y(\omega) = x$. Fix $s \in (-n+1, -n)$ as above and let $0 < t \leq -s$. Then there is a positive integer $m < n$ such that $s + t \in [-m+1, -m]$. Using the perfect cocycle property for $U$ and the above definition of $y$, the reader may check that

$$y(t + s, \omega) = U(t, y(s, \omega), \theta(s, \omega)). \quad (3.41)$$

(Note that if we put $s = -t$ in (3.41), we get $U(t, y(-t, \omega), \theta(-t, \omega)) = x$ for all $t \geq 0$.)
Next we show that
\[
\limsup_{t \to \infty} \frac{1}{t} \log |y(-t, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}
\]
perfectly in \( \omega \). From Theorem 6.1 (b) in [Ru.2], we have
\[
\limsup_{n \to \infty} \frac{1}{n} \log |y(-n, \omega) - Y(\theta(-n, \omega))| \leq -\lambda_{i_0-1}
\]
perfectly in \( \omega \). For each \( t \in (n, n+1) \), write \(-t = \alpha - (n+1)\) for some \( \alpha \in (0, 1) \). Then by the definition of \( y \) and the Mean Value Theorem, we have
\[
|y(-t, \omega) - Y(\theta(-t, \omega))| = |U(\alpha, y(-n+1, \omega), \theta(-(n+1), \omega)) - U(\alpha, Y(\theta(-(n+1), \omega), \theta(-(n+1), \omega))| \\
\leq \sup_{(v^*, \eta^*) \in B(0,1), \alpha \in (0,1)} \|DU(\alpha, (v^*, \eta^*) + Y(\theta(-(n+1), \omega)), \theta(-(n+1), \omega))\|_{L(H)} \\
\times |y(-(n+1), \omega) - Y(\theta(-(n+1), \omega))|
\]
perfectly in \( \omega \). Therefore
\[
\limsup_{t \to \infty} \frac{1}{t} \log |y(-t, \omega) - Y(\theta(-t, \omega))| \\
\leq \limsup_{n \to \infty} \frac{1}{n} \log^+ \sup_{(v^*, \eta^*) \in B(0,1), \alpha \in (0,1)} \|DU(\alpha, (v^*, \eta^*) + Y(\theta(-(n+1), \omega)), \theta(-(n+1), \omega))\|_{L(H)} \\
+ \limsup_{n \to \infty} \frac{1}{n} \log |y(-(n+1), \omega) - Y(\theta(-(n+1), \omega))|.
\]
The first term on the right hand side of the above inequality is zero, perfectly in \( \omega \in \Omega \), because of Lemma 3.1 (ii) and the integrability condition of the theorem. The second term is less than or equal to \(-\lambda_{i_0-1}\) because \( y(0) \in \tilde{U}(\omega) \). The uniqueness of the continuous-time history process for a given \( x \in \tilde{U}(\omega) \) follows from that of the discrete-time process and (3.41). Hence the proof of assertion (d) of the theorem is complete.

The proof of assertion (e) of the theorem uses an interpolation argument similar to the above. The reader may check the details.
We will now verify the asymptotic invariance property in (f), that is
\[ \tilde{U}(\omega) \subseteq U(t, \cdot, \theta(-t, \omega))(\tilde{U}(\theta(-t, \omega))), \quad t \geq \tau_2(\omega) \tag{3.44} \]
perfectly in \( \omega \) for some \( \tau_2(\omega) > 0 \). To do this, let \( x \in \tilde{U}(\omega) \). Then by assertions (d), (e) of the theorem and inequalities (3.39), there exists a (unique) history process \( y(-t, \omega), t \geq 0 \), and a random time \( \tau_2(\omega) > 0 \) such that \( y(0, \omega) = x, y(-t, \omega) \in \tilde{B}(Y(\theta(-t, \omega)), \rho_2(\theta(-t, \omega))) \) for all \( t \geq \tau_2(\omega) \), and
\[ y(t' - t, \omega) = U(t', y(-t, \omega), \theta(-t, \omega)), \quad 0 < t' \leq t, \tag{3.45} \]
perfectly in \( \omega \). Fix \( t_1 \geq \tau_2(\omega) \). Note that by (3.45) (for \( t = t' = t_1 \)), we have \( x = U(t_1, y(-t_1, \omega), \theta(-t_1, \omega)) \). We claim that \( y(-t_1, \omega) \in \tilde{U}(\theta(-t_1, \omega)) \). To see this, define the process \( y_1(-t, \omega) := y(-t - t_1, \omega), \quad t \geq 0 \) Then \( y_1(-t_1, \omega) \) is a history process with
\[ y_1(0, \omega) = y(-t_1, \omega) \in \tilde{B}(Y(\theta(-t_1, \omega)), \rho_2(\theta(-t_1, \omega))). \]
Therefore \( y(-t_1, \omega) \in \tilde{U}(\theta(-t_1, \omega)) \). Since \( t_1 \geq \tau_2(\omega) \) is arbitrary, (3.44) follows. The invariance assertion (4) in (f) of the theorem and the fact that
\[ DU(t, \cdot, \theta(-t, \omega))(U(\theta(-t, \omega)) : U(\theta(-t, \omega)) \to U(\omega), \quad t \geq 0, \]
is a linear homeomorphism onto, are consequences of the Oseledec theorem and the cocycle property for the linearized semiflow; cf. [Mo.1], Corollary 2 (v) of Theorem 4.

The transversality assertion in (g) of the theorem follows immediately from the relations
\[ T_{Y(\omega)} \tilde{U}(\omega) = U(\omega), \quad T_{Y(\omega)} \tilde{S}(\omega) = S(\omega), \quad H = U(\omega) \oplus S(\omega) \]
which hold perfectly in \( \omega \).

Taking \( \Omega^* := \Omega_1^* \cap \hat{\Omega}_1^* \), completes the proof of assertions (a)-(g) of the theorem.

Suppose \( U \) is a \( C^\infty \) cocycle. Then a simple adaptation of the argument in [Ru.2], section (5.3) (p. 297) gives a \( \theta(t, \cdot) \)-invariant sure event in \( \mathcal{F} \), also denoted by \( \Omega^* \), such that \( \tilde{S}(\omega) \) and \( \tilde{U}(\omega) \) are \( C^\infty \) for all \( \omega \in \Omega^* \). The proof of Theorem 2.1 is now complete. \( \square \)
4. The local stable manifold theorem for see’s and spde’s.

In this section, we discuss several classes of semilinear stochastic evolutions equations and spde’s. The objective is to establish sufficient conditions for a local stable manifold theorem for each class.

(a) Stochastic semilinear evolution equations: Additive noise.

Let $K, H$ be two separable real Hilbert spaces. Let $A$ be a self-adjoint operator on $H$ such that $A \geq cI_H$, where $c$ is a real constant and $I_H$ is the identity operator on $H$. Assume that $A$ admits a discrete non-vanishing spectrum $\{\mu_n, n \geq 1\}$ which is bounded below. Let $\{e_n, n \geq 1\}$ denote a basis for $H$ consisting of eigen vectors of $A$, viz. $Ae_n = \mu_n e_n, n \geq 1$. Assume further that $A^{-1}$ is trace-class. Suppose $B_0 \in L_2(K, H)$. Let $W(t), t \in \mathbb{R}$, be cylindrical Brownian motion on the canonical filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and with a separable covariance Hilbert space $K$ ([M.Z.Z.1], section 2). Let $T_t = e^{-At}$ stand for the strongly continuous semigroup generated by $-A$.

Denote by $\mu_m$ the largest negative eigenvalue of $A$ and by $\mu_{m+1}$ its smallest positive eigenvalue. Thus there is an orthogonal $\{T_t\}_{t \geq 0}$-invariant splitting of $H$ using the negative eigenvalues $\{\mu_1, \mu_2, \cdots, \mu_m\}$ and the positive eigenvalues $\{\mu_n : n \geq m + 1\}$ of $A$:

$$H = H^+ \oplus H^-$$

where $H^+$ is a closed linear subspace of $H$ and $H^-$ is a finite-dimensional subspace. Denote by $p^+: H \to H^+$ and $p^- : H \to H^-$ the corresponding projections onto $H^+$ and $H^-$ respectively. Since $H^-$ is finite-dimensional, then $T_t|H^-$ is invertible for each $t \geq 0$. Therefore, we can set $T_{-t} := [T_t|H^-]^{-1} : H^- \to H^-$ for each $t \geq 0$.

Consider the following semi-linear stochastic evolution equation on $H$:

$$du(t) = [-Au(t) + F(u(t))] dt + B_0 dW(t), \quad t \geq 0,$$

$$u(0) = x \in H.$$
In the above equation, let $F : H \to H$ be a globally Lipschitz map with Lipschitz constant $L$:

$$|F(v_1) - F(v_2)| \leq L|v_1 - v_2|, \quad v_1, v_2 \in H.$$  

Then (4.1) has a unique mild solution given by

$$u(t, x) = T_t x + \int_0^t T_{t-s} F(u(s, x))ds + \int_0^t T_{t-s} B_0 dW(s), \quad t \geq 0 \quad (4.2)$$

Furthermore, if $F : H \to H$ is $C^{k,\epsilon}$, the mild solution of (4.2) generates a $C^{k,\epsilon}$ perfect cocycle also denoted by $u : \mathbb{R}^+ \times H \times \Omega \to H$.

Suppose that $F : H \to H$ is globally bounded, and its Lipschitz constant $L$ satisfies

$$L[\mu_{m+1}^{-1} - \mu_m^{-1}] < 1. \quad (4.3)$$

Note that the above condition is automatically satisfied in the affine linear case $F \equiv 0$.

The next proposition is key to the existence and uniqueness of a stationary random point for the cocycle $(u, \theta)$ in the sense of Definition 1.1.

**Proposition 4.1.**

Assume the above conditions on $A, B_0, F$ together with (4.3). Then there is a unique $\mathcal{F}$-measurable map $Y : \Omega \to H$ satisfying

$$Y(\omega) = \int_{-\infty}^0 T_{-s} p^+ F(Y(\theta(s, \omega)))ds - \int_0^\infty T_{-s} p^- F(Y(\theta(s, \omega)))ds$$

$$+ (\omega) \int_{-\infty}^0 T_{-s} p^+ B_0 dW(s) - (\omega) \int_0^\infty T_{-s} p^- B_0 dW(s) \quad (4.4)$$

for all $\omega \in \Omega$.

**Proof.**

We use a contraction mapping argument to show that the integral equation (4.4) has an $\mathcal{F}$-measurable solution $Y : \Omega \to H$. 
Define the $\mathcal{F}$-measurable map $Y_1 : \Omega \to H$ by

$$Y_1(\omega) := (\omega) \int_{-\infty}^{0} T_s p^+ B_0 \, dW(s) - (\omega) \int_{0}^{\infty} T_s p^- B_0 \, dW(s), \quad \omega \in \Omega.$$ 

Denote by $B(\Omega, H)$ the Banach space of all (surely) bounded $\mathcal{F}$-measurable maps $Z : \Omega \to H$ given the supremum norm $\|Z\|_\infty := \sup_{\omega \in \Omega} |Z(\omega)|$. Define the map $M : B(\Omega, H) \to L^0(\Omega, H)$ by

$$M(Z)(\omega) := \int_{-\infty}^{0} T_s p^+ F(Z(\theta(s, \omega)) + Y_1(\theta(s, \omega))) \, ds$$

$$- \int_{0}^{\infty} T_s p^- F(Z(\theta(s, \omega)) + Y_1(\theta(s, \omega))) \, ds$$

for all $Z \in B(\Omega, H)$ and all $\omega \in \Omega$.

Note first that $M$ maps $B(\Omega, H)$ into itself. To see this let $Z \in B(\Omega, H)$ and $\omega \in \Omega$. Then

$$|M(Z)(\omega)| \leq \|F\|_\infty \left[ \int_{-\infty}^{0} \|T_s p^+\| \, ds + \int_{0}^{\infty} \|T_s p^-\| \, ds \right]$$

$$\leq \|F\|_\infty \left[ \int_{-\infty}^{0} e^{s\mu_{m+1}} \, ds + \int_{0}^{\infty} e^{s\mu_m} \, ds \right]$$

$$\leq \|F\|_\infty [\mu_{m+1}^{-1} - \mu_m^{-1}] < \infty$$

where $\|F\|_\infty := \sup_{v \in H} |F(v)|$. Hence $M(Z) \in B(\Omega, H)$ for all $Z \in B(\Omega, H)$.

Secondly, $M$ is a contraction. To prove this, take any $Z_1, Z_2 \in B(\Omega, H)$ and $\omega \in \Omega$. Then from the definition of $M$, we get

$$|M(Z_1)(\omega) - M(Z_2)(\omega)| \leq L \int_{-\infty}^{0} \|T_s p^+\| \cdot |Z_1(\theta(s, \omega)) - Z_2(\theta(s, \omega))| \, ds$$

$$+ L \int_{0}^{\infty} \|T_s p^-\| \cdot |Z_1(\theta(s, \omega)) - Z_2(\theta(s, \omega))| \, ds$$

$$\leq L \|Z_1 - Z_2\|_\infty \left[ \int_{-\infty}^{0} \|T_s p^+\| \, ds + \int_{0}^{\infty} \|T_s p^-\| \, ds \right]$$

$$\leq L \|Z_1 - Z_2\|_\infty \left[ \int_{-\infty}^{0} e^{s\mu_{m+1}} \, ds + \int_{0}^{\infty} e^{s\mu_m} \, ds \right]$$

$$= L [\mu_{m+1}^{-1} - \mu_m^{-1}] \|Z_1 - Z_2\|_\infty$$

$$= \mu \|Z_1 - Z_2\|_\infty$$
where $\mu := L[\mu_{m+1}^{-1} - \mu_m^{-1}] < 1$. This proves that $\mathcal{M} : B(\Omega, H) \to B(\Omega, H)$ is a contraction, and hence has a unique fixed point $Z_0 \in B(\Omega, H)$. That is

$$Z_0(\omega) := \int_{-\infty}^{0} T_{-s} p^+ F(Z_0(\theta(s, \omega)) + Y_1(\theta(s, \omega))) ds - \int_{0}^{\infty} T_{-s} p^- F(Z_0(\theta(s, \omega)) + Y_1(\theta(s, \omega))) ds$$

for all $\omega \in \Omega$. Now define $Y : \Omega \to H$ by

$$Y(\omega) := Z_0(\omega) + Y_1(\omega), \quad \omega \in \Omega.$$ 

It is easy to check that $Y$ satisfies the identity (4.4).

Since $Z_0$ is uniquely determined, then so is $Y$. \qed

The following proposition gives existence and uniqueness of a stationary point for the see (4.1).

**Proposition 4.2.**

Assume all the conditions on $A, B_0, F$ stated in Proposition 4.1. Suppose that $F$ is globally bounded, globally Lipschitz and satisfies condition (4.3). Then the semilinear see (4.1) has a unique stationary point $Y : \Omega \to H$, i.e. $u(t, Y(\omega), \omega) = Y(\theta(t, \omega))$ for all $t \geq 0$ and $\omega \in \Omega$. Furthermore, $Y \in L^p(\Omega, H)$ for all $p \geq 1$.

**Proof.**

By hypotheses and Proposition 4.1, the integral equation (4.4) has a unique $\mathcal{F}$-measurable solution $Y : \Omega \to H$. Let $t \geq 0$. Using (4.4), it follows that

$$Y(\theta(t, \omega)) = \int_{-\infty}^{0} T_{-s} p^+ F(Y(\theta(t + s, \omega))) ds - \int_{0}^{\infty} T_{-s} p^- F(Y(\theta(t + s, \omega))) ds$$

$$+ (\omega) \int_{-\infty}^{0} T_{-s} p^+ B_0 \, dW(s + t) - (\omega) \int_{0}^{\infty} T_{-s} p^- B_0 \, dW(s + t)$$

$$= \int_{-\infty}^{t} T_{t-s} p^+ F(Y(\theta(s, \omega))) ds - \int_{t}^{\infty} T_{t-s} p^- F(Y(\theta(s, \omega))) ds$$
\[
+ (\omega) \int_{-\infty}^{t} T_{t-s} p^+ B_0 \, dW(s) - (\omega) \int_{t}^{\infty} T_{t-s} p^- B_0 \, dW(s)
\]
\[
= T_t \left[ \int_{-\infty}^{0} T_{-s} p^+ F(Y(\theta(s, \omega))) \, ds - \int_{0}^{\infty} T_{-s} p^- F(Y(\theta(s, \omega))) \, ds \right]
\]
\[
+ (\omega) \int_{-\infty}^{0} T_{-s} p^+ B_0 \, dW(s) - (\omega) \int_{0}^{\infty} T_{-s} p^- B_0 \, dW(s) \right]
\]
\[
+ \int_{0}^{t} T_{t-s} p^+ F(Y(\theta(s, \omega))) \, ds + \int_{0}^{t} T_{t-s} p^- F(Y(\theta(s, \omega))) \, ds
\]
\[
+ (\omega) \int_{0}^{t} T_{t-s} p^+ B_0 \, dW(s) + (\omega) \int_{0}^{t} T_{t-s} p^- B_0 \, dW(s)
\]
\[
= T_t Y(\omega) + \int_{0}^{t} T_{t-s} F(Y(\theta(s, \omega))) \, ds + (\omega) \int_{0}^{t} T_{t-s} B_0 \, dW(s).
\]

This gives
\[
Y(\theta(t, \omega)) = T_t Y(\omega) + \int_{0}^{t} T_{t-s} F(Y(\theta(s, \omega))) \, ds + (\omega) \int_{0}^{t} T_{t-s} B_0 \, dW(s)
\]
for all \( t \geq 0 \). Therefore, \( Y(\theta(t, \omega)), t \geq 0, \omega \in \Omega, \) is a stationary solution of (4.2) (with \( x = Y(\omega) \)). Since \( u(t, Y(\omega), \omega), t \geq 0, \omega \in \Omega, \) is also a solution of (4.2), then by uniqueness of the solution to (4.2), we must have
\[
u(t, Y(\omega), \omega) = Y(\theta(t, \omega))
\]
for all \( t \geq 0 \) and all \( \omega \in \Omega \). Hence \( Y \) is a stationary point for the see (4.1).

The stationary point for (4.1) is unique (within the class of \( \mathcal{F} \)-measurable maps \( \Omega \to H \)). To see this, it is sufficient to observe that the above computation shows that every stationary point of (4.1) is a solution of the integral equation (4.4). Uniqueness of the stationary solution then follows from Proposition 4.1.

In view of the proof of Proposition 4.1, the last assertion of Proposition 4.2 follows from the fact that \( Y_1 \in L^p(\Omega, H) \) for all \( p \geq 1 \) and \( Z_0 \in L^\infty(\Omega, H) \). □

The existence of local stable and unstable manifolds near a stationary point of the affine stochastic evolution equation (4.1) follows from a straightforward modification of the proof of Theorem 4.1 in the next section.
(b) Semilinear stochastic evolution equations: Linear noise

Here we recall the setting leading to Theorem 2.6 in [M-Z-Z.1].

We will prove the existence of local stable and unstable manifolds for semiflows generated by mild solutions of semilinear stochastic evolution equations of the form:

\[
\begin{aligned}
&du(t) = -Au(t)dt + F(u(t))dt + Bu(t)dW(t), \quad t > 0, \\
&u(0) = x \in H.
\end{aligned}
\] (4.5)

In the above equation \(A : D(A) \subset H \to H\) is a closed linear operator on a separable real Hilbert space \(H\). Assume that \(A\) has a complete orthonormal system of eigenvectors \(\{e_n : n \geq 1\}\) with corresponding positive eigenvalues \(\{\mu_n, n \geq 1\}\); i.e., \(Ae_n = \mu_n e_n, n \geq 1\).

Suppose \(-A\) generates a strongly continuous semigroup of bounded linear operators \(T_t : H \to H, t \geq 0\). Let \(W(t), t \geq 0\), be cylindrical Brownian motion defined on the canonical filtered Wiener space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) and with a separable covariance Hilbert space \(K\).

That is, \(\Omega\) is the space of all continuous paths \(\omega : \mathbb{R} \to K\) such that \(\omega(0) = 0\) with the compact open topology, \(\mathcal{F}\) is its Borel \(\sigma\)-field, \(\mathcal{F}_t\) is the sub-\(\sigma\)-field generated by all evaluations \(\Omega \ni \omega \mapsto \omega(u) \in K, u \leq t\), and \(P\) is Wiener measure on \(\Omega\). The Brownian motion is given by

\[W(t, \omega) := \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R},\]

and may be represented by

\[W(t) = \sum_{k=1}^{\infty} W^k(t)f_k, \quad t \in \mathbb{R},\]

where \(\{f_k : k \geq 1\}\) is a complete orthonormal basis of \(K\), and the \(W^k, k \geq 1\), are standard independent one-dimensional Wiener processes ([D-Z], Chapter 4).

Suppose \(B : H \to L_2(K, H)\) is a bounded linear operator. The stochastic integral in (4.5) is defined in the sense of ([D-Z], Chapter 4).
We will denote by \( \theta : \mathbb{R} \times \Omega \to \Omega \) the standard \( P \)-preserving ergodic Wiener shift on \( \Omega \):
\[
\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}.
\]

Let \( L(H) \) be the Banach space of all bounded linear operators \( H \to H \) given the uniform operator norm \( \| \cdot \| \). Denote by \( L_2(H) \subset L(H) \) the Hilbert space of all Hilbert-Schmidt operators \( S : H \to H \).

Suppose \( F : H \to H \) is a (Fréchet) \( C^{k, \epsilon} \) \( (k \geq 1, \epsilon \in (0, 1]) \) non-linear map satisfying the following Lipschitz and linear growth hypotheses:
\[
\begin{align*}
|F(v)| &\leq C(1 + |v|), \quad v \in H \\
|F(v_1) - F(v_2)| &\leq L_n|v_1 - v_2|, \quad v_i \in H, |v_i| \leq n, i = 1, 2,
\end{align*}
\]
for some positive constants \( C, L_n, n \geq 1 \).

The mild solutions of the see (4.5) generate a \( C^{k, \epsilon} \) \( (k \geq 1, \epsilon \in (0, 1]) \) perfect cocycle \((U, \theta)\) on \( H \), satisfying all the assertions of Theorem 2.6 of [M-Z-Z.1].

Under the above conditions, one gets the following stable manifold theorem for hyperbolic stationary trajectories of the see (4.5).

**Theorem 4.1.**

Assume the above hypotheses on the coefficients of the see (4.5). Assume that the stochastic semiflow \( U : \mathbb{R}^+ \times H \times \Omega \to H \) generated by mild solutions of (4.5) has a hyperbolic stationary point \( Y : \Omega \to H \) such that \( E\log^+ |Y| < \infty \). Then \((U, \theta)\) has a perfect family of \( C^k \) local stable and unstable manifolds satisfying all the assertions of Theorem 2.1.

**Proof.**

One first checks the estimate
\[
\int_{\Omega} \log^+ \sup_{0 \leq t_1, t_2 \leq T} \|U(t_2, Y(\theta(t_1, \omega)) + (\cdot, \theta(t_1, \omega)))\|_{k, \epsilon} dP(\omega) < \infty \quad (4.7)
\]
for any fixed $0 < \rho, T < \infty, k \geq 1$ and $\epsilon \in (0, 1]$. This estimate follows from the integrability condition on $Y$ and assertion (vi) of Theorem 2.6 in [M-Z-Z.1]. The conclusion of Theorem 4.1 now follows immediately from Theorem 2.1. □

(c) Semilinear parabolic spde’s: Lipschitz nonlinearity

Laplacian

$$\Delta := \frac{1}{2} \sum_{i,j=1}^{d} \partial_{\xi_i}^2$$

(4.8)

defined on a smooth bounded domain $D$ in $\mathbb{R}^d$ (with a smooth boundary $\partial D$ with zero Dirichlet boundary conditions). Assume that $f : \mathbb{R} \to \mathbb{R}$ is a $C_0^\infty$ function and let $d\xi$ be Lebesgue measure on $\mathbb{R}^d$. Let $W_n, n \geq 1$, be independent one-dimensional standard Brownian motions with $W_n(0) = 0$ defined on the canonical filtered Wiener space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in \mathbb{R}})$. Let $\theta$ denote the Brownian shift on $\Omega := C(\mathbb{R}, \mathbb{R}^\infty)$. are smooth (bounded) functions, and $A$ is uniformly elliptic. Recall that the Sobolev space $H_0^k(D)$ is the completion of $C_0^\infty(D, \mathbb{R})$ under the Sobolev norm

$$||u||_{H_0^k(D)}^2 := \sum_{|\alpha| \leq k} \int_D |D^\alpha u(\xi)|^2 d\xi.$$

Suppose further that $\sigma_n \in H_0^s(D), n \geq 1$, and the series $\sum_{n=1}^\infty \sigma_n$ converges absolutely in $H_0^s(D)$ where $s > k + \frac{d}{2} > d$.

By Theorem 3.5 ([M-Z-Z.1]), weak solutions of the initial-value problem:

$$\begin{align*}
du(t) &= \frac{1}{2} \Delta u(t) dt + f(u(t)) dt + \sum_{n=1}^\infty c_n(\xi) u(t) dW_n(t), \quad t > 0 \\
u(0) &= \psi \in H_0^k(D)
\end{align*}$$

(4.9)

give a perfect smooth cocycle $(U, \theta)$ on the Sobolev space $H_0^k(D)$ which satisfies all the assertions of Theorem 3.5 in [M.Z.Z.1]. Applying Theorem 2.1, we get the following stable manifold theorem for the spde (4.9):
Theorem 4.2.

Assume the above hypotheses on the coefficients of the spde (4.9). Assume that the stochastic semiflow $U : \mathbb{R}^+ \times H_0^k(D) \times \Omega \to H_0^k(D)$ generated by weak solutions of (4.9) has a hyperbolic stationary point $Y : \Omega \to H_0^k(D)$ such that $E \log^+ \|Y\|_{H_0^k} < \infty$. Then $(U, \theta)$ has a perfect family of $C^\infty$ local stable and unstable manifolds in $H_0^k(D)$ satisfying all the assertions of Theorem 2.1.

(d) Stochastic reaction diffusion equations: dissipative nonlinearity

In section 4 (a) of [M.Z.Z.1], we constructed a $C^1$ stochastic semiflow on the Hilbert space $H := L^2(D)$ for a stochastic reaction-diffusion equation

$$
\dot{u} = \nu \Delta u \, dt + u(1 - |u|^\alpha) \, dt + \sum_{i=1}^{\infty} \sigma_i(\xi)u(t) \, dW_i(t), \quad (4.10)
$$

defined on a bounded domain $D \subset \mathbb{R}^d$ with a smooth boundary $\partial D$. In (4.10) the Laplacian on $D$ is denoted by $\Delta$, and we impose Dirichlet boundary conditions on $\partial D$. In (4.10), the $W_i, i \geq 1$, are independent one-dimensional standard Brownian motions and $\sum_{i=1}^{\infty} \sigma_i$ is absolutely convergent in $H^s(D)$, for $s > 2 + \frac{d}{2}$. The dissipative term yields the existence of a unique stationary solution of (4.10) under suitable choice of the parameter $\nu$ ([D-Z.2]).

In view of the estimates in Theorem 4.1 ([M.Z.Z.1]) and Theorem 2.1, one gets the following:

Theorem 4.3.

Assume the above hypotheses on the coefficients of the spde (4.10). Let $\alpha < \frac{4}{d}$. Assume that the stochastic semiflow $U : \mathbb{R}^+ \times L^2(D) \times \Omega \to L^2(D)$ generated by mild solutions of (4.10) has a hyperbolic stationary point $Y : \Omega \to L^2(D)$ such that $E \log^+ \|Y\|_{L^2} < \infty$. Then $(U, \theta)$ has a perfect family of $C^1$ local stable and unstable manifolds in $L^2(D)$ satisfying all the assertions of Theorem 2.1.
Remarks.

(i) The results in Sections (c) and (d) hold if the Euclidean domain $\mathcal{D}$ is replaced by a compact smooth $d$-dimensional Riemannian manifold $M$ (possibly with a smooth boundary $\partial M$).

(ii) We conjecture that Theorem 4.3 still holds (but with Lipschitz stable/unstable manifolds) if the dissipative term $u(1 - |u|^\alpha)$ is replaced by a more general one of the form $F(u) := f \circ u$, where $f : \mathbb{R} \to \mathbb{R}$ is a $C^1$ function satisfying the following classical estimates:

$$-c_1 - \alpha_1 |x|^p \leq f(x)x \leq c_1 - \alpha_2 |x|^p, \quad f'(x) \leq c_2,$$

for all $x \in \mathbb{R}$, with $c_1, c_2, \alpha_1, \alpha_2$ positive constants, and $p$ any integer greater than 2.

(iii) Is it true that the stochastic flow and the local stable/unstable manifolds in Theorem 4.3 are of class $C^2$?

(e) Stochastic Burgers equation: additive noise

The existence of a $C^1$ stochastic semiflow on $L^2(S^1)$ for Burgers equation

$$du + u \frac{\partial}{\partial \xi} u dt = \nu \Delta u dt + \sum_{i=1}^{\infty} \sigma_i(\xi) dW_i(t) \quad (4.11)$$

was established in Part I of this work under the regularity conditions of [E-K-M-S]. See [M-Z-Z.1], Theorem 4.3.

Note that the existence of a unique stationary point for Burgers equation (4.11) was established in [E-K-M-S] and [Si]. If (4.11) has a hyperbolic stationary point, then one gets the following result:
Theorem 4.4.

Assume the hypotheses of Theorem 4.2 of [M.Z.Z.1] on the coefficients of Burgers spde (4.11). Assume that the stochastic semiflow $U : \mathbb{R}^+ \times H^1(S^1) \times \Omega \rightarrow L^2(S^1)$ generated by mild solutions of (4.11) has a hyperbolic stationary point $Y : \Omega \rightarrow L^2(S^1)$ such that $E \log^+ \|Y\|_{L^2} < \infty$. Then $(U, \theta)$ has a perfect family of $C^1$ local stable and unstable manifolds in $L^2(S^1)$ satisfying all the assertions of Theorem 2.1.

Note that hyperbolicity of the stationary point in Theorem 4.4 is in the sense of Definition (1.3). The relationship between the notion of hyperbolicity in Theorem 4.4 and that in [E-K-M-S] is unclear. Note that Theorems 1.1 and 4.3 ([M.Z.Z.1]) imply that the Lyapunov spectrum for (4.11) exists and is discrete. However, an analysis of the Lyapunov spectrum for (4.11) is postponed to a future project. We conjecture here that for almost all $\nu > 0$, the stationary solution of the Burgers equation (4.11) is hyperbolic.

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