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Introduction to “Schubert varieties, equivariant cohomology and characteristic classes, IMPANGA15 volume”

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Abstract. The conference IMPANGA15 took place in Będlewo (Poland) during the week 12–18 April 2015 and was organised by participants of the seminar IMPANGA. This volume is a collection of contributions by the participants of the conference as well as notes from the major lecture series of the seminar in the period 2010-2015. Both original research papers and self-contained expository surveys can be found in the volume. The articles circulate around a broad range of topics within algebraic geometry: vector bundles, Schubert varieties, degeneracy loci, homogeneous spaces, equivariant cohomology, Thom polynomials, characteristic classes, symmetric functions and polynomials, algebraic geometry in positive characteristic.

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Keywords. IMPANGA, vector bundles, Schubert varieties and degeneracy loci, homogeneous spaces, equivariant cohomology, Thom polynomials, characteristic classes, symmetric functions and polynomials, quasi-elliptic surfaces.

The volume

This volume is a conclusion of the activities of IMPANGA in the years 2010-2015, which celebrated 15 years of its existence in 2015. It is a follow up to previous books [1, 2, 3] and it contains contributions of the participants of a conference anniversary IMPANGA15.

In this introduction we briefly review what IMPANGA is, describe the conference and summarise the content of the volume.

Friedrich Hirzebruch passed away in 2012, during the aforementioned period. We dedicate this book to his memory.

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1. The seminar IMPANGA

IMPANGA is the name of the activities of Algebraic Geometers at the Institute of Mathematics, Polish Academy of Sciences, including one of the most important seminars in algebraic geometry in Poland. The head of the seminar is Piotr Pragacz, and the first seminar was held on 30 October 2000. The topics of the seminar lectures usually fit within the framework of complex algebraic geometry, although talks about positive characteristics, real algebraic geometry, symplectic geometry, complex analysis, singularity theory and other neighbouring areas of mathematics are also welcomed.

Figure 1. Professor Friedrich Hirzebruch at IMPAN, 2009.

The acronym IMPANGA stands for the Polish names of the Institute of Mathematics of the Polish Academy of Sciences (IMPAN, Instytut Matematyczny Polskiej Akademii Nauk) and Algebraic Geometry (GA, Geometria Algebraiczna). IMPAN is a Polish institute designated to mathematical research. Its headquarters are in Warsaw and it has branches in 6 other major Polish cities. The department of Algebra and Algebraic Geometry of the institute is one the most active research groups and it is chaired by Piotr Pragacz. Its members collaborate closely with other research groups of algebraic geometers in Poland, particularly those at the University of Warsaw, the Jagiellonian University and the Pedagogical University in Kraków.

The seminar IMPANGA usually runs two meetings per month on Fridays. During each meeting (typically in Warsaw) either two talks are delivered by two speakers, or a series of two lectures is presented by a single speaker. The meeting is attended by participants from Warsaw and Kraków and often also by visitors from other cities.

The seminar’s participants also organise meetings (schools, workshops, conferences) of various size, topics and impact, see https://www.impan.pl/~pragacz/impanga_miniszkoly.htm. The IMPANGA School 2010 https://www.impan.pl/~impanga/school/ and the IMPANGA15 conference https://www.impan.pl/
are two of the largest events. We hope to organise similar events every 5 years.

2. The conference IMPANGA15

The IMPANGA15 conference took place in Będlewo during the week 12-18 April 2015. In addition to hosting an excellent scientific program, we celebrated 15 years of IMPANGA and the 60th birthday of Professor Piotr Pragacz.

The conference exposed a rich variety of topics in algebraic geometry, especially: symmetric functions and polynomials, Schubert varieties and degeneracy loci, characteristic classes (particularly of singular varieties), Thom polynomials, characteristic $p$ problems, arithmetic algebraic geometry, moduli problems, tropical geometry.

IMPANGA15 gathered about 80 participants from Europe, United States, and Asia. Besides 17 plenary lectures, there were also 9 shorter talks delivered by the participants. During the conference there was a poster session with a range of interesting presentations. Numerous people contributed to the success of the conference. The organising committee consisted of J. Buczyński (Polish academy of Sciences; University of Warsaw), M. Donten-Bury (University of Warsaw; Freie Universität Berlin), G. Kapustka (Polish Academy of Sciences; Jagiellonian University), O. Kędzierski (University of Warsaw; Polish Academy of Sciences), M. Michałek (Polish Academy of Sciences; University of California, Berkeley), E. Postinghel (University of Leuven) and J. Szpond (Pedagogical University of Kraków). The scientific committee, that helped make the event exceptional, included: Paolo Aluffi
(Florida State University), Bernard Leclerc (Université de Caen), Richárd Rimányi (University of North Carolina at Chapel Hill), Matthias Schütt (Leibniz Universität Hannover) and Ravi Vakil (Stanford University).

The invited speakers were: David Anderson (Ohio State University), Anders Buch (Rutgers University), Hélène Esnault (Freie Universität Berlin), Gerard van der Geer (Universiteit van Amsterdam), June Huh (Princeton University and Institute for Advanced Study), Toshiyuki Katsura (Hosei University), Maxim Kazarian (Steklov Institute of Mathematics; Moscow Independent University), JongHae Keum (Korea Institute for Advanced Study), Allen Knutson (Cornell University), Adrian Langer (Polish Academy of Sciences; University of Warsaw), Laurentiu Maxim (University of Wisconsin, Madison), Toru Ohmoto (Hokkaido University), Sam Payne (Yale University), Piotr Pragacz (Polish Academy of Sciences), Steven Sam (University of California, Berkeley), Harry Tamvakis (University of Maryland) and Orsola Tommasi (Leibniz Universität Hannover).

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IMPANGA15 was also an opportunity to celebrate 60th birthday of Professor Piotr Pragacz. He was the special honorary guest of the meeting and the session on the afternoon of Tuesday, 14th April, was dedicated to his work. The organisers planned an excursion to Poznań during the afternoon of Wednesday, 15th April. It included a guided tour around the Old Market Square and a piano concert by Maria Wójcik, who played short pieces composed by F. Chopin. The concert took place in the Red Hall of the Działyński Palace near the Old Market Square.
3. Contributions in the volume

This section contains a short description of the chapters composing this volume.

The opening article of the volume is “Friedrich Hirzebruch—a handful of reminiscences” by Piotr Pragacz. It includes personal remarks by the author, some of which have never before been exposed to the public. The article also contains a brief overview of the famous Hirzebruch-Riemann-Roch theorem and other work of Professor Hirzebruch. A fraction of his inspiration and influence on Polish mathematics, and particularly on algebraic geometers, is certified by this historical note.

Soojin Cho and Takeshi Ikeda in “Pieri rule for the factorial Schur P-functions” derive formulas for the product of two factorial Schur P-functions as a linear combination of factorial Schur P-functions. In other words, they show how to compute generalisations of famous Littlewood-Richardson coefficients. Further, in the special case of the generalised Pieri rule, they obtain a positive formula for odd maximal orthogonal Grassmannians. Their method relies on a good understanding of the combinatorics of (generalisations of) shifted tableaux.

Izzet Coskun in “Restriction varieties and the rigidity problem” starts his survey from the basics of Schubert varieties and cohomologies of classical homogeneous spaces, exposing the similarities and differences between the cases A and B–D. A further topic is the restriction problem, which consists in computing an induced map from the cohomologies of full Grassmannian to the cohomologies of an isotropic Grassmannian (orthogonal or symplectic). This is obtained in terms of a combinatorial algorithm using sequences of brackets and braces. Finally, the survey concludes with rigidity problems: are Schubert varieties the only varieties that represent a given Schubert class in cohomology? This article contains lots of exercises and open problems in this area and it is based on a series of lectures by the author at the IMPANGA seminar in Warsaw in 2013.

In “On Plücker equations characterizing Grassmann cones”, Letterio Gatto and Parham Salehyan give a survey on the Kadomtsev-Petshiasvily hierarchy. This is a system of infinitely many PDE’s defining a universal family of deformations of an ordinary linear differential operator and whose rational solutions are parametrised by certain infinite Grassmann cones. An explicit construction of the solutions of the Kadomtsev-Petshiasvily equation in terms of the associated tau functions is given and the relation with the Plücker equations defining Grassmannians is showed. The authors phrase the latter both in the fermionic and in the bosonic coordinate systems and, furthermore, they provide an interpretation in terms of Schubert derivations.

Thomas Hudson and Tomoo Matsumura in “Kempf-Laksov Schubert classes for even infinitesimal cohomology theories” prove a generalisation of a determinantal formula of Schubert calculus on Grassmann bundles to the even infinitesimal cohomology theories of Grassmann bundles and Lagrangian Grassmann bundles. This extends and is built upon previous work of the two authors for other algebraic oriented cohomology theories, including K-theory (with Ikeda and Naruse) and Levine-Morel algebraic cobordism. The main tool employed is a formula for Segre classes of vector bundles in terms of Chern classes, that generalises the classical relation between Segre and Chern classes in the case of Chow rings.
Quasi-elliptic surfaces are complete two dimensional varieties admitting a map to a smooth curve, such that a generic fiber is a singular irreducible, geometrically reduced curve of arithmetic genus 1. These exist in characteristic 2 and 3 and exhibit a lot of phenomena that happen only in finite characteristics. In the article “On the multicanonical systems of quasi-elliptic surfaces in characteristic 3” Katsura studies quasi-elliptic surfaces $S$ of Kodaira dimension 1 in characteristic 3. The main theorem states that for any $m \geq 5$ the multicanonical system $mK_S$ provides the quasi-elliptic fibration and 5 is the best possible value. The analogous open problem in characteristic 2 of determining when the multicanonical system provides a quasi-elliptic fibration is also discussed.

In the review “Characteristic classes of mixed Hodge modules and applications” Laurentiu Maxim and Jörg Schürmann give a beautiful exposition of the topic in the title. They start by providing an introduction to mixed Hodge modules, Hodge-Chern and Hirzebruch classes including most important results and basic operations on them. Various applications and examples are presented, including generalisation of the classical Riemann-Roch theorem. In the last part, the theory of equivariant characteristic classes for singular varieties is discussed.

In “Thom polynomials in $A$-classification I: counting singular projections of a surface”, Takahisa Sasajima and Toru Ohmoto study Thom polynomials of map-germs $(\mathbb{C}^n,0) \to (\mathbb{C}^m,0)$. These are universal polynomials in Chern classes that are associated to ($A$-singularity types of) map-germs. They obtain explicit new results for unstable map-germs in low codimension. Furthermore, the formulas for Thom polynomials are applied to solve, both classical and new, enumerative problems for surfaces in $\mathbb{P}^3$ and $\mathbb{P}^4$.

The survey “On a certain family of $U(b)$-modules” by Piotr Pragacz contains an introduction on the theory of Kraśkiewicz-Pragacz modules. They are a functorial version of Schubert polynomials, whose existence was conjectured by Lascoux (Oberwolfach, June 1983) and proved by Kraśkiewicz and the author of this manuscript. This survey includes an account on the recent progress on Kraśkiewicz-Pragacz modules and filtrations that have such modules as their subquotients, made by Watanabe by means of the theory of highest weight categories. The manuscript ends with an account on two applications of Kraśkiewicz-Pragacz filtrations and of ample vector bundles to positivity of certain Schur functions, due to Fulton and Watanabe respectively.

The article “Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae” by Richárd Rimányi and Alexander Varchenko opens a bridge between characteristic classes and the weight functions. A rational function that appeared in the context of $q$-hypergeometric solutions of quantum Knizhnik-Zamolodchikov differential equations is shown to be equal to the torus equivariant Chern-Schwartz-MacPherson class of a Schubert cell in a partial flag variety. The key observation is that the interpolation and localisation conditions satisfied by the weight functions are also satisfied by the equivariant Chern-Schwartz-MacPherson classes. In addition, the authors provide explicit combinatorial formulas for the weight functions.

The survey “Schubert polynomials and degeneracy locus formulas” by Harry
Tamvakis is an exposition of the degeneracy loci problems from the point of view of the torus equivariant cohomology. It gives a good account of the bridge from the vector bundle language to the double Schubert polynomials. The author employs the nil-Coxeter algebra approach to construct the double Schubert polynomials introduced by Lascoux and Schützenberger for type $A$ groups, and Ikeda, Mihalcea, and Naruse for type $B$, $C$, and $D$ groups. A new proof of the author earlier result called splitting formula is given. In this approach, the cases $A$–$D$ can be treated more uniformly, and the splitting formulas are the only general ones available (that is, they do not treat only very special Weyl group elements, nor are they intrinsic to the associated $G/P$ space).

Hirzebruch $\chi_y$ genus, where $y$ is a parameter, generalises the Euler-Poincaré characteristic ($y = -1$), the Todd genus ($y = 0$) and the signature ($y = 1$). While the Euler-Poincaré characteristic is multiplicative in fiber bundles, this is not the case for $\chi_y$. Shoji Yokura explains the failure of multiplicativity in general in the article “Hirzebruch $\chi_y$-genera of complex algebraic fiber bundles – the multiplicativity of the signature modulo 4”. He provides an explicit formula for the difference between the $\chi_y$ genus of the fiber bundle and the product of $\chi_y$ for the base and for the fiber. In small dimensions these can be described in terms of the respective differences of signatures and/or Todd genera. As an application he proves that the signature is multiplicative in fiber bundles modulo 4.

Magdalena Zienkiewicz in “Pushing-forward Schur classes using iterated residues at infinity” presents her viewpoint on the equivariant Gysin homomorphism. It contains both a survey of author’s earlier work on residue formulas and iterated residues at infinity, and a presentation of how to apply these methods. As an example, a simplified proof of formulas of Pragacz and Ratajski for the push-forward of Schur polynomials over the Lagrangian Grassmannian is obtained.

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References


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