An integral equation approach to continuous system identification and model reduction

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AN INTEGRAL EQUATION APPROACH TO CONTINUOUS SYSTEM IDENTIFICATION AND MODEL REDUCTION

by

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A Doctoral Thesis

Submitted in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy of Loughborough University of Technology

Date: June, 1988
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Dedicated to the memory of my uncle who
died just before the completion of this work
Acknowledgements

I would like to express my deepest gratitude to Dr. A. H. Whutfield, for his continuous guidance, his invaluable suggestions and useful recommendations, for his patience and his help in many other aspects throughout the course of research and the preparation of this thesis.

I would also like to thank Professor A. C. Bajpai, the head of the Engineering Mathematics Department, and all departmental staff members.

My gratitude also goes to my wife Joanne for her support and interest that she has shown in this work.
SYNOPSIS

An integral equation description for linear systems is developed and used as the basis for the development of various system identification, model reduction and order determination methods.

The system integral equation is utilized in the problem of parameter identification in continuous linear single-input single-output, multi-input multi-output and linear in parameters nonlinear systems. The approach is developed in the time domain where the effect of non-zero initial conditions and additive disturbances occurs naturally. Parameter estimates are deduced using several weighted residual concepts which have previously been used to produce approximate solutions to differential equations.

The problems of model order reduction and lumped parameter approximation of distributed parameter systems are both reduced to a continuous system parameter identification problem.

Two different techniques for model order determination are developed for use with linear continuous-time systems. The first is based on the behaviour of singular values of a specific non-square matrix the elements of which are calculated from the evaluation of the integral equation at different points in time. The second uses the shifted Legendre polynomials to produce a square matrix whose non-zero eigenvalues then identify the order of the system under consideration.

Several numerical examples illustrate the application of the methods to the problems of continuous system parameter identification, model reduction, lumped parameter approximation of distributed parameter systems and model order determination.
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1. AN OVERVIEW OF SYSTEM IDENTIFICATION

1.1. Introduction

In the field of control engineering, the development of a model of a prescribed physical phenomenon is of fundamental importance when seeking to synthesise control algorithms and make analytical predictions about the behaviour of the system under the effect of control efforts and disturbances. This model can be either physical, i.e. a full or reduced size copy of the original plant, or mathematical where the relationship between the physical variables in the system are mapped onto mathematical structures by means of known physical laws. Physical model building may be a costly process and the costs of producing and testing models increase as components become more complex and higher performances are required. Mathematical modelling came into being as a response to the pressure of these costs. Mathematical models can be divided into two classes: lumped and distributed parameter. The difference between the classes is that the variables of interest for lumped parameter systems are functions of time alone whilst the distributed parameter model accounts for both time and spatial variations in the physical process. The dynamic behaviour of lumped parameter systems is very often described by either ordinary differential equations for continuous systems or difference equations for discrete systems. Distributed parameter systems are those that are modelled by partial differential equations, integro-differential equations or delay-differential equations. For such systems the state space is infinite-dimensional. Models can be either stochastic, where the relation between variables is given in terms of statistical distributions, or deterministic where the probability of events does not appear in the equations. One must also distinguish between linear and nonlinear, fixed and time-varying models. Linear models of physical systems are described by linear differential equations and are unified by the powerful superposition principle. Correspondingly nonlinear models are described by nonlinear differential equations but there is no sufficiently powerful principle to unify them. The difference between fixed and time-varying models is that in fixed models the parameters of the equations describing the system are constant whereas in the time-varying case these parameters vary with the time.

In practical situations, mathematical models may be difficult to formulate from physical arguments because of either some unknown phenomenon (such as unknown chemical reactions and unknown boundary conditions) or the extreme complexity of the physical plant. In such instances we may resort to the system identification techniques where input and output measurements from
the plant are used to determine an appropriate mathematical model. Parameter identification may be
defined as the determination, from experimental input/output data, of a set of unknown parameters
in a mathematical model of a physical process such that, over a desired range of operating
conditions, the model outputs are close, in some well-defined sense, to the process outputs when
the two are subject to analogous inputs and initial conditions. Very often engineering experience
can provide valuable information about the most suitable form for the mathematical model whose
parameters are to be identified from observed input/output data. In an engineering environment, the
experimental data may be corrupted with noise which may result in severe degradation of the
parameter estimates. In such situations special identification methods are used to eliminate the
influence of all disturbances thereby yielding adequate estimates. In many practical applications the
model order is not known 
 and the overall system identification problem includes the
determination of the model order as well as the parameters of the model. In such situations, the
model order is usually determined before the rest of the parameters, since it specifies the total
number of unknowns to be identified (or estimated) in the mathematical model.

Mathematical modelling of physical systems or processes can sometimes lead to very large
models. The practical application of modern concepts such as state estimation, optimal state
feedback and numerical simulation is limited by the capacity of the available digital computer. It is
therefore often desirable, and sometimes absolutely necessary, to reduce the order of the system
model to simplify the analysis, simulation and computational effort in the design of controllers.
This process is known as model order reduction and is loosely defined as follows: given a
mathematical model which is considered to be complex, find a simpler mathematical model which
adequately approximates the original model. For distributed parameter systems, the approximation
(order reduction) of the infinite dimensional space by a finite dimensional one is of primary
concern since only a finite number of actuators and sensors can be used in practice. The
approximation is achieved by means of discretisation methods (e.g. finite differences or finite
elements) and the reduced order model thereby deduced is a lumped parameter model. The finite-
dimensional controller can be synthesized either by a direct model reduction, which consists of
reducing the distributed parameter model and then synthesizing the controller via the established
theory of lumped parameter systems, or by indirect model reduction which consists of synthesizing
an infinite-dimensional controller and then obtaining a finite dimensional approximation.

In the following section of this chapter we give a brief overview of the work done over the
last few years on the subjects of system identification and model order reduction.
1.2. Literature survey

1.2.1 Model order determination

Considerable interest has been paid to methods for order determination and a variety of
techniques have been proposed for this purpose. Almost all existing methods are designed for
discrete system order estimation. Some of these techniques are based on the minimization of
Akaike's criteria (the final prediction error "FPE" and the Akaike information criterion "AIC")
(Chan et al [1], Edmunds [2]). Both AIC and FPE appear to be very powerful practical approaches
([3]) for stochastic system (mainly auto regressive moving-average process) order determination.
An extension of Akaike's final predictor error (FPE) criterion, based on the analysis of the
estimated variance of the white noise in a SISO auto-regressive moving average process, is
presented by Chan et al [1]. This modified FPE was found to be more consistent than the
previous procedures in predicting the correct model structure. A modified version of the AIC test
suggested by Edmunds [2] is shown to be more accurate and more useful than the normal AIC test
in testing model order for control system design purposes. The relationship between the model
order and the confidence limits in the estimated parameters has been used to show that this method
can decrease the chance of overestimating the order.

Among methods based on the system information matrix are those designed for discrete
models which investigate the near singularity of the product moment matrix, its condition number,
determinant ratio and the behaviour of its determinant (Sagara et al [3], Chow [4], Wellstead [5]
and Soderstrom [6]). A recursive algorithm for the determinant ratio, associated with an
instrumental product moment (IPM) matrix, is described by Sagara et al [3] as a useful order test.
The algorithm provides a fitting error explained in the context of the instrumental variable method,
called the "instrumental residual" (IR), and the parameter estimates of the model. By exploiting the
concept of IR and using the parameter estimates, a more robust statistic is obtained for noisy
situations. Chow [4] has proposed a method for ARMA processes where the order is determined by
testing the singularity of the correlation matrix. The near singularity test associated with an
instrumental variable modification to the product moment technique, which overcomes the
problem of additional computational effort required for noisy situations, is described by Wellstead
[5]. Most of these techniques use iterative schemes and natural observed input/output data for
testing the order of linear systems. Soderstrom [6] has investigated how the model order influences
the singularity of the information matrix and deduced that this singularity is equivalent to
parameter identifiability of the system.

Other methods developed for the purpose of discrete system order determination include the
pole-zero cancellation technique (Sodestrom [7]), the minimization of the residual error (Young et
a/ al [8]) and methods designed for adaptive systems (Shinnaka et al [9]). Application of the pole/zero cancellation test to models obtained from least-squares identification is considered by Soderstrom [7]. A technique presented by Young et al [8] is based on properties of the parametric error-covariance matrix generated by instrumental variable estimators. The algorithm is recursive and applicable to a wide variety of discrete and continuous systems.

A technique designed to identify both the structure (order) and the parameters of a general SISO lumped continuous system from observed input/output data is presented by Saha [10]. The method is called the 'Poisson moment functional approach'. It was tested with noisy data and found superior to the methods based on evaluation of rank of certain information matrices.

There have been many attempts to compare the various existing methods. Woodside [11] described three procedures based on a measured product moment matrix for testing model orders: 1- the near singularity of the information matrix, 2- comparison of residuals and 3- the likelihood ratio for system order. A brief review of some methods for model structure testing is given in Soderstrom [12], where Akaike's criteria (FPE, AIC) and the F-test are compared and shown to be asymptotically equivalent. Unfortunately the results were erroneous; corrections were made by Akaike in [13]. Seven order determination techniques for discrete models are investigated and the rules for their application are formulated by Unbehauen and Gohrin [14]. The tests are those of :1- determinant ratio test, 2- condition number, 3- polynomial test, 4- test for independence, 5- test for normality, 6- test of signal errors, 7- the F-test. Unbehauen and Gohrin concluded that there is no universal test for estimating the accurate order and a single test can cause incorrect decisions whereas the application of several testing methods will specify the most accurate order of the model. A similar report was written by Van Den Boom and Van Den Eden [15] where the description of 5 tests are given: 1- the behaviour of the error function, 2- the whiteness of the residual, 3- the statistical independency of the loss function (i.e. F-test), 4- the behaviour of the determinant and 5- the pole/zero cancellation effect. Simulated results show that an acceptable test of order is possible even at a signal-to-noise ratio of -15 dB. Although the behaviour of the determinant was the method preferred by the authors, they advised the use of the different tests in parallel for an accurate order determination.

**1.2.2 Lumped parameter identification**

The parameter identification problem received much attention in the last twenty years and a multitude of techniques, ranging from frequency response analysis to various sophisticated time domain parameter identification approaches, have been proposed. When formulating an identification problem under the assumption of a given form of system equations, an error criterion
(usually of quadratic form) is introduced to give a measure of how well a model fits the experimental data; minimisation of this criterion with respect to the parameters then yields model parameter estimates. This formulation leads to an optimization problem and the different parametric methods used to solve it are classified with respect to the type of model under study and to the type of signals used (continuous or discrete, deterministic or stochastic). They are also classified with respect to the specific algorithm used to minimize the chosen criterion. Thus there are many possibilities for combining experimental conditions, model classes and criteria. A broad distinction can be made between on-line and off-line schemes. On-line methods give estimates recursively as the measurements are obtained and must be used if the identification scheme is to be employed in an adaptive controller. For off-line methods the data is collected and processed as a batch rather than sequentially.

The following description of the existing techniques will be based on the type of signals used and the specific algorithm of optimization. Thus no distinction shall be made between discrete and continuous system models when discussing the various identification methods. For those who wish to make that distinction, the relevant references are:

Continuous models: [30], [35], [37], [38], [39], [44], [51].
Discrete models: [20], [21], [22], [31], [36], [40], [42], [43], [45], [46], [50].

Several attempts to compare various identification methods have been made (Ljung and Glover [19], Ljung [20], Iserman et al [21] and Saridis [22]) but the comparisons are largely inconclusive in the sense that there is no method that is universally best. Ljung and Glover [19] present connections and distinctions between some basic identification techniques in the time domain and those in the frequency domain together with a discussion of their ease of use in different experimental conditions; the conclusion was that time and frequency domain techniques are complementary rather than competing. Ljung [20] presents the close conceptual relationships between basic approaches to the estimation of the transfer function of linear systems where the classical methods of frequency and spectral analysis are shown to be related to the time domain methods of predictor error type. The asymptotic properties of the estimates obtained by the respective methods are described and discussed and asymptotic expressions for the estimate of mean square error are shown to be similar for both types of method. Performance, computational effort and overall reliability of six recursive identification and parameter estimation methods are compared in Isermann et al [21]. The methods are those of least-squares, generalized least-squares, instrumental variables, stochastic approximation, correlation analysis in the least-squares parameter estimation and Fourier analysis. The type of processes to which the techniques can be applied and the type of possible input signals are given and the overall reliability of methods is tested by the
percentage of successful runs for a given total number of runs. It is suggested that identification methods using the same \textit{a priori} knowledge of the process model result in about the same performance. The main differences between the identification methods are therefore to be seen in the kind of input signals, computational expense, overall reliability and the \textit{a priori} known factors. The computational and the convergence properties of six other identification algorithms for linear discrete-time dynamic systems (cross-correlation method, the first and second stochastic approximation techniques, the maximum likelihood, the maximum \textit{a posteriori} filter and the extended Kalman filter) are also compared by Saridis [22]. The conclusion drawn was that very poor results were demonstrated by the application of the maximum \textit{a posteriori} filter.

An explicit description of all the possible methods of parameter identification is beyond the scope of this survey. We shall restrict ourselves to discussing only the major techniques used in parametric identification problems. Explicit descriptions of all existing procedures can be found in Eykhoff [23] and Norton [24]. Non-parametric methods, where the model of dynamic process can be obtained by means of simple methods in terms of non-parameterized representations such as the impulse response, the step response and the frequency response are discussed by Wellstead [25] and Rake [26]. Further details about system identification are given in surveys by (Astrom and Eykhoff [27], Balakrishnan and Peterka [28], Nieman [29], Young [30], Saridis [31] and Mehra [32]). The survey written by Nieman [29] discusses the work on identification and estimation techniques applied to lumped parameter deterministic dynamical systems before 1971. It also outlines the general characteristics of the different methods and provides a comparison of the techniques and a guide in their application to specific problems. Young [30] reviews the progress of research on parameter estimation for continuous-time models over the period 1958–1980. An exposition of the stochastic approximation algorithms and their application to various parameter identification and self-organizing control algorithms is presented by Saridis [31]. The design of optimal inputs for system identification in linear dynamic systems is the subject of the survey written by Mehra [32] where several different representations of the optimal input and several methods for its numerical computation are considered. A most recent survey on the subject of system identification was reported by Unbehauen [33] where an introduction to identification methods using parameter estimation was given. The author discusses the four steps into which the identification procedure can be divided: 1- planning and analyzing the experimental conditions (state of the process, selection of input signals and sampling time); 2- selection of an approximate model structure; 3- estimation of the parameter models (direct solution, recursive solution and iterative solution), 4- model validation (including order testing).

In the deterministic case, it is assumed that the noise is either not present in the system or is
negligible. Some deterministic approaches admit zero mean noise but cannot express the uncertainty of the estimates caused by the noise (Strejc [34]). The most important class of deterministic parameter identification is the least squares technique because of its simplicity and ease of practical implementation. Here the index of performance consists of the sum (integral) of weighted error squares and the error is defined as the difference between the model and the system outputs (Golubev and Horowitz [35] and Strejc [36]). In Golubev and Horowitz [35] the transfer function of continuous stable or unstable models is calculated using repeated integration of the data. The technique was found suitable for problems where existing techniques failed. The algorithm has in practice been found applicable for approximating model orders not greater than ten. Systems with larger orders yield ill-conditioned normal equations. In Strejc [36] analytical as well as numerical approaches to least-squares methods (instrumental variable method, the generalized least squares and the extended least squares) for the estimation of system dynamics of regression model form are described. White noise and correlated noise are also discussed. The methods can be used either on-line (recursive least-squares) or off-line. It was shown that the most important factors influencing the success of identification schemes are the excitation of the system to be identified, the choice of output variables which can be measured, the precision of measurements, the signal-to-noise ratio, the properties of noise, the method of identification, the numerical procedure of calculation, the redundancy of parameters in the regression model, the sampling period and the length of computer word. Other techniques used for the same purpose include gradient search methods which are recursive schemes for finding the minimum of the error criterion (Eykhoff [23]), adaptive algorithms (Ichikawa [37]) where the algorithm has an excellent (exponential) convergence property, and other methods based on the property of orthogonal polynomials such as shifted Legendre polynomials (Hwang and Guo [38], Chang and Wang [39]). In [37] the problem is assumed to be noiseless and the plant is asymptotically stable. An error equation is formed and the adaptive law for parameter identification is proposed. The problem reduces to solving a set of ordinary differential equations to which suitable initial conditions must be supplied. The technique proposed by Hwang and Guo [38] identifies the parameters of a linear multi-input multi-output system. The method involves multiple integration of the ordinary differential equations and the system input/output data is expanded in shifted Legendre series. An overdetermined set of linear algebraic equations in the parameters is generated by means of an operational matrix which relates shifted Legendre polynomials to their integrals. This operational matrix was first used by Chang and Wang [39] for single-input single-output parameter identification.

Stochastic methods comprise statistical and probabilistic approaches. It is assumed that the
A dynamical system is subject to random disturbances which satisfy some general statistical properties (belonging to a certain type of distribution with known statistical moments e.g. mean value) and techniques which account for the influence of disturbance components are used to estimate the system parameters. The most important class of parameter estimation methods are based on the following principles: least squares parameter estimation (Janiszowski [40]) where the effect of the identification errors is analysed, maximum likelihood and prediction error methods (Astrom [41], Puthenpura and Sinha [42]), Bayesian approaches (Peterka [43]), correlation methods (Godfrey [44]) and non-linear filtering techniques (Halme et al [45], Saridis [46]). Astrom [41] reviews the maximum likelihood method and the closely related prediction error method and their application to system parameter estimation is given. The problem of robustness (sensitivity of the estimates to variation in data) is discussed. Various ways of minimizing the likelihood function are also discussed together with alternative methods for computing the likelihood function, its gradient and its Hessian. The basic ideas behind the parameter estimation methods are also discussed together with computational aspects, theoretical results, model validation and selection of model structures. The modified maximum likelihood method presented by Puthenpura and Sinha [42] is similar to the ordinary maximum likelihood method, possessing its important properties but with an additional advantage of better convergence and robustness. The method is based on the censoring of the observed data and has been found useful in cases where generalized least squares and ordinary maximum likelihood methods fail (e.g. in situations with gross errors in the data due to large disturbances). An introduction to the main features of the Bayesian method are briefly summarized by Peterka [43] where an attempt is made to build a consistent theory of system identification on the Bayesian basis; real-time identification, estimation and prediction in the closed loop, redundant and unidentifiable parameters, time-varying parameters and adaptivity problems are also discussed. The theory and application of correlation methods are displayed in Godfrey [44] where the use of cross-correlation functions to determine weighting functions of linear systems is emphasised and the corresponding frequency domain expressions are derived. In Halme et al [45] a non-linear filter is introduced and its relationship with the well known extended Kalman filter is derived. The authors state that this non-linear filter suffers less computational complexities than the extended Kalman filter. A convergence analysis of the extended Kalman filter is given by Ljung [46] and it is shown that the estimates may be biased or even divergent if no term is added to the filter. The scheme can be interpreted as the maximization of the likelihood function for estimation of the parameters or as a recursive prediction error algorithm.

The maximum likelihood procedure and the prediction error methods are regarded as the most general and powerful methods of parameter estimation since they have the advantage that they can
be applied to a wide variety of model structures and experimental conditions. They have good asymptotic properties but a draw back is that they require substantial computational effort. The non-linear filtering method is a formulation which uses the state-space (joint state/parameter) approach where the solution is obtained recursively. There is no general solution to the non-linear filtering problem. Therefore most techniques rely on local linearization to yield a sub-optimal filter such as the "extended Kalman filter" whose estimates are biased and there is no guarantee of stability of the filter.

In the following section we give a broad classification of distributed parameter system identification techniques and the contributions of several authors to the development of each technique.

1.2.3 Distributed parameter system identification

The distributed parameter system (DPS) identification problem has been the subject of recent surveys (Polis and Goodson [52], Polis [53] and Kubrusly [54]). Polis and Goodson [52] have presented a thorough overview where a step by step approach for distributed parameter system identification is given, major existing techniques are discussed and their applications are illustrated. The problem is divided into eight largely independent subproblems: 1- write the mathematical description containing unknown parameters of the physical plant; 2- choose a method for solving the mathematical description; 3- decide on measurement type and location in the spatial domain; 4- choose a criterion of performance; 5- perform a sensitivity analysis; 6- perform the physical experiments to obtain data; 7- choose an optimization scheme; 8- perform an error analysis. The survey reveals the contribution of various authors to each step in the solution to the distributed parameter system identification problem. Polis [53] extended the work with his survey in which results concerning the distributed parameter system identification problem over the period 1977-1982 are outlined. An interesting classification of the different methods used to identify the parameters for distributed parameter systems is given by Kubrusly [54] where the major techniques are classified according to three main classes as shown by Fig 1.1. Methods belonging to the first class are those based on the optimization schemes which are directly applied to the model in its original form (distributed or infinite-dimensional) ([55], [59], [61], [62], [63], [66], [67], [69] and [74]). The second class consists of those methods that reduce the distributed parameter model to a lumped parameter model (described by ordinary differential equations) by means of techniques such as finite differences, finite elements and Galerkin's method ([56], [61], [64], [68], [71], [73] and [75]). Techniques available for lumped parameter systems are used to estimate (or identify) the unknown parameters of the original model. The third and last class comprises techniques using
finite differences or integral transformations to reduce the partial differential equations to a set of algebraic equations ([57], [58], [60]).

Paths (2-A) and (2-B) in Figure 1.1 characterize the routes to reducing the distributed parameter system to a discrete or continuous-time lumped parameter model. The link (2-3) represents the possibility of reduction to an algebraic equation via an ordinary differential equation. Among techniques for reducing distributed parameter model into lumped parameter model or into algebraic equations are:

Path (2-A)  Finite-differences [75]
Path (2-B)  Method of lines (finite-differences over spatial domain) [61]
            Method of Characteristic [56], [73]
            Galerkin method [64]
            Cubic splines [71]
            Finite elements [68]
Path (2-3)  Method of lines+integral transformation [60]
Path (3)    Finite-differences [58]
            Integral transformation [57]

For each of the above classes, the optimization algorithms used to minimize the performance
criterion are those used for the lumped parameter identification problems: gradient methods ([56], [59], [61], [62]), least-squares [62], non-linear programming [63], non-linear filtering [64] and stochastic approaches ([73], [74], [75]).

Perdreaule and Goodson [57] is one of the early works in distributed system parameter identification. They reduced the higher derivatives to those corresponding to the available data by multiplying each side of each partial differential equation by a suitable function (usually powers of sines which depend upon the form of the PDE and the measured data) in order that, after integrating by parts, non-integral terms evaluated on the boundaries are all zero, and, hence simplify subsequent computation. The result of this operation is a set of algebraic equations in the unknown parameters. The method applies equally well to linear and nonlinear models (where extraneous terms may be included). The case of spatially varying coefficients is also considered and normal operating records may be used. However this method has some limitations: 1- results are significantly affected by noise; 2- on-line applications are not convenient; 3- the PDE's may contain terms that cannot be treated since it is not always easy to find the appropriate matrix function; 4- each of the PDE's has to be considered separately.

Collins and Khatri [58] assumed a deterministic class of PDE's described by time varying models which can be nonlinear in the dynamics but must be linear in the parameters. They approximated the partial derivatives using finite differences and expressed the q unknown parameters in the differential equations and in the boundary conditions by these variables at the mesh points, an operation which reduces the identification problem to that of solving a q-dimensional linear algebraic equation. The use of finite difference formulae implies that observations are taken at a finite number (>q) of points in time and/or space. The method then requires the inversion of a q\times q matrix which is frequently ill-conditioned in practice. The authors suggest that least-squares estimation and sufficient separation in time and/or space of the q points can reduce this ill-conditioning problem. Normal operating data and on-line identification may be used but, due to the necessity of taking all measurements in the spatial domain simultaneously and due to the sensitivity of the method to the level of measurement error, the authors have shown that this technique is not generally useful particularly for systems with time-varying parameters.

Seinfeld [59] has developed a method for nonlinear distributed parameter system identification where the parameters may enter in the boundary conditions as well as in the partial differential equations. The method uses the steepest descent algorithm which is based on the minimization of the difference between the observed output and that predicted by the model. Seinfeld referred to the relation between observability and identifiability and suggested that observability and sensor locations might be based on sensitivity considerations.
Fairman and Shen [60] developed a method called the 'moment functional method' which is a modification of Perdreaeville and Goodson's method. They avoid spatial integrations and time integration was accomplished after multiplying the differential equation by a modulation function of the form: $P_k(t_2-t) = (t_2-t)^k / k! \cdot \exp(-c(t_2-t))$, whose derivatives vanish at $t_2$ and when $t \to -\infty$. The result of this procedure is a set of algebraic equations in the unknown system parameters. Observations were taken at a finite number of points in space and assumed to be noiseless. Application was made to the one-dimensional wave and diffusion equations and was extended to include systems characterized by the one-dimensional diffusion equation with a coefficient which is a polynomial in time.

Seinfeld and Chen [61] used noisy experimental data and algorithms based on steepest descent, quasilinearization and collocation methods for an off-line estimation of constant parameters in nonlinear PDE's and BC's. Optimal location of measurements were also investigated and were determined by minimizing the variances of the parameter estimates. Estimation of diffusivity in the heat equation, the activation energy for a single reaction from dynamic plug flow reactor data and the permeabilities in a two-region reservoir model were included.

The identification of a vector of unknown spatially-varying parameters in nonlinear partial differential equations from noisy observations was examined by Chen and Seinfeld [62]. Two techniques were presented: steepest descent and least squares filtering. Diffusivity in the heat equation was estimated.

The finite difference method was used by Hamza and Sheirah [63] to approximate the partial derivatives in the system equation. To identify time-varying parameters in DPS, they minimized the integral of the weighted error squared over an observation interval by means of a nonlinear programming technique. In the case of constant parameters the method reduces to that proposed by Collin and Khatri (matrix inversion). The technique is suitable for on-line application and uses a limited number of sensors. Noisy measurements, extraneous terms and experimental data were considered. It was found that satisfactory results can be obtained if the signal-to-noise ratio is greater than twenty to one.

An approach to the distributed parameter identification problem is proposed in a step by step procedure by Polis, Goodson and Wozny [64]. An approximate solution to the distributed model, based on a finite set of orthogonal functions over the spatial domain, is assumed and Galerkin's criterion is used to reduce the PDE to a set of ODE's. The constant parameters are identified using standard methods for ODE's. Three optimization schemes are discussed to minimize the criterion; a steepest descent method, a search technique and non-linear filtering. In the nonlinear filtering technique, the estimation equations are derived by embedding the original process involving fixed
duration T and the final observation c in a class of process for which 0≤T and →<c<+∞ and interconnecting the costs for these processes. The form of the resulting sequential estimation equations is analogous to the well known extended Kalman filter equations.

Ward and Goodson [65] proposed a method for identifying unknown parameters in nonlinear boundary conditions. The method required as many measurement sensors as there are unknown boundary conditions which must be linear in the parameters. Applications to the heat conduction with radiation boundary and to the cantilever beam with a nonlinear moment at the boundary were presented.

Kobayachi and Hitotsuya [66] estimated the unknown initial states in DPS of parabolic and hyperbolic types with continuous-time input/output data. They constructed finite dimensional state observers to perform the state estimation. Using the state estimates, they determined the unknown input distribution functions. A parallel theory was presented by Kobayachi [67] in which discrete-time observers were considered.

Sehitoglu [68] has proved that a Lyapunov design technique combined with the finite element method can be used to obtain efficient real-time parameter identification algorithms for DPS's governed by parabolic and hyperbolic PDE's. The finite element technique reduces the problem into a set of ODE's. A suitable quadratic performance index of Lyapunov type was then performed in connection with Lyapunov's second method to derive a proportional and a proportional plus integral type identification scheme. The method required output data of the dependent variable at a number of points in the solution domain but knowledge of the process initial condition is not required. The application was successfully carried out on time-invariant and space dependent parameters in wave and diffusion equations.

Kitamura and Nakagun [69] studied the identifiability of spatially varying parameters in DPS's described by a linear one-dimensional parabolic partial differential equation. Several results about the identifiability and non-identifiability of the parameters were given. Courdesses [76] corrected errors in Kitamura and Nakagun's formulation and gave necessary and sufficient conditions for identifiability of system parameters.

A second order, linear one-dimensional, parabolic partial differential equation with spatially-varying parameters was considered by Pierce [70]. He has shown that, under certain conditions, the eigenvalues of the PDE are uniquely identifiable by observations at one point in space. In the case where the equation is in normal form and in which one coefficient appears, this coefficient can also be determined by observations at one point in space.

A technique, based on cubic spline approximations for estimating variable coefficients (in
space and time domain) in parabolic distributed systems was presented by Banks [71]. Convergence results and a summary of numerical performance of the algorithms are given.

Dale and Cohen [72] used steady state frequency response data near resonance frequencies to identify unknown parameters in linear vibratory systems described by a hyperbolic partial differential equations where the steady state equations may be reduced to a set of ordinary differential equations in the spatial variables. Neither analytic solution to the system nor multiple response transducers are required.

Carpenter, Womy and Goodson [73] reduced a linear first order partial differential equation to a set of ordinary differential equations using a method based upon determining a characteristic equation for the system, and chose stochastic approximation algorithms as a recursive search scheme for finding the estimates by minimizing a quadratic performance criterion. The method is designed for DPS's described by a set of partial differential equations of the form: 
\[ \frac{\partial u}{\partial t} + A(u,x,t)\frac{\partial u}{\partial x} + b(u,x,t) = 0. \]
Where \( u(x,t) \) is an \( n \times 1 \) vector, and one or both functions \( A(u,x,t) \) and \( b(u,x,t) \) were assumed to be unknown and \( C^1 \) in \( u \). Noisy measurements and limited available measurement transducers were discussed with the following conclusions: 1- estimates are shown to converge for noisy measurements. 2- the noise has no effect on the estimation of \( b \) but introduces a bias term in the estimation of the elements of \( A \).

Sunahara, Ohsumi and Imamura [74] invoked the Bayesian theoretic approach and filtering techniques in the Markovian framework for identifying the unknown parameters in stochastic DPS's. A linear stochastic PDE was analysed, where the unknown parameters which were assumed to be a set of random constants with known probabilities were contained in the exciting terms. Kubrusly and Curtain [75] applied stochastic algorithms based on noisy observations at a finite number of discrete points for identification of space varying parameters in DSP's. The model was a second-order linear parabolic PDE, excited by random inputs with non specific restriction about probabilities, a finite difference technique being used to reduce the DPS to a discrete-time lumped parameter system.

Parameter identification for distributed parameter systems may not be helpful to the control engineer at the present time since there are not many associated design tools. The most natural way to obtain controllers for such systems is to base them on a lumped parameter (finite dimensional) approximation of distributed parameter models. Such an approximation involves reduction of the infinite dimensional space over which distributed systems are defined, to an approximate finite dimensional subspace.
1.2.4 Model order reduction

Model order reduction applied to high order systems is another subject which has generated a
great deal of interest in the last twenty years. The methods proposed for this purpose can be divided
into three groups: 1- aggregation methods; 2- Padé-type and partial realization; 3- error
minimisation approaches. The methods are equally applicable to both discrete and continuous time
systems.

The first group of methods is based on aggregation theory where the reduced order model is
directly derived from the high order model under consideration through algebraic relationships
(Hickin and Sinha [80]). Using a canonical form for the original system, the authors combined two
techniques to obtain the aggregated reduced order model. They retain the desired eigenvalues of the
original system to ensure the stability and match certain of the generalized Markov parameters to
provide a good fit to the steady state response. Among techniques which can be regarded as
aggregation methods are the dominant pole approach which consists of retaining dominant
eigenvalues (corresponding to slow modes) of the original system (Bonvin and Mellichamp [81]
and Mahapatra [82]) and the perturbation method where fast phenomena are neglected (Kokotovic
\textit{et al} [83], Kokotovic \textit{et al} [84], Litz and Roth [85] and Lastman \textit{et al} [86]). These approaches
preserve the stability of the system but may not preserve the static gain. In [81] six of the most
commonly used modal reduction techniques are analysed and their performances are compared by
determining the characteristics of each method, i.e. 1- steady state agreement; 2- initial value
agreement; 3- independence of the reduced model on the choice of retained state variables and
inputs; 4- eigenvector orientation. Weaknesses and strengths of the approaches such as the steady
state agreement, the transient state agreement and the dependence of the reduced model on the
choice of inputs are thereby revealed. A criterion for selecting the size of the low order model after
simplifying a higher order plant which improves the steady state response of the reduced model
obtained by Davison and Chidambara's technique is presented by Mahapatra [82]. An iterative
scheme for the separation of slow and fast modes which also improves the accuracy of the lower
order model in the steady state response is given in [83]. Results on singular perturbations used for
model order reduction and separation of time scales in control system design are surveyed by
Kokotovic \textit{et al} [84]. Litz and Roth [85] describe the state decomposition for singular perturbation
order reduction and introduces a measure for the degree of dependency of state variables on the high
order dominant eigenvalues. The contribution of each state to the total impulse energy is evaluated
to determine the most important states to be retained in a reduced model approximation [86]. The
method may be regarded as a combination of singular perturbations and aggregation.

The second approach to the problem is based on the expansion of the original transfer
function. In the Padé approximation method, the approximant and the approximated transfer functions are expanded into a Taylor or Laurent series and the coefficients of the reduced model are found by matching the Markov parameters (or time moments) of the two expansions (Daly and Colebourn [87] and Shamash [89]). Although the method of Padé approximation was originally designed for transfer function models it has been implemented for systems in the state space form [87]. Elementary equivalence transformations were used to progressively tridiagonalise the state matrix, hence generating approximations of increasing order. Warwick [88] has defined an error polynomial in the z domain in terms of the difference between the step response of the original model and that of the reduced model. He has shown, by setting the coefficients of this polynomial to zero, that the Markov parameters and the time series proportionals of the reduced model can be matched with those of the original system model. This second approach to model reduction preserves the static gain of the original model but may give an unstable reduced order model. The algorithm introduced by Shamash [89] ensures that the reduced order model derived by equating time moments is stable if the original plant is stable. The continued fraction expansion (Chen and Shieh [90]) and the time moment approach (Zakian [91]) are related to the Padé approximation (Paraskevopoulos [111]). The methods proposed by Chen and Shieh [90] are applicable to both state space and transfer function representations. If the system is in transfer function form then the technique expands the function into a continued fraction and simply ignores certain quotients. If the system is in state space form, the state matrix is partitioned and some parts of it are discarded. The time moment approximation method [91] is based on matching the step responses of the reduced model and that of the process; its relationship to Padé approximation is stated in the paper. A similar method which preserves stability is the Routh approximation technique where the two transfer functions are expanded in a canonical form called the "alpha-beta" expansion and the reduced model is derived so that the coefficients of its "Routh table" agree up to a given order with those of the original system ([92], [93], [94], [95] and [96]). Hutton and Friedland [92] used the Routh approximation method to reduce the order of linear time-invariant systems. The technique guarantees the stability of the reduced model if the original plant is stable. The sequence of approximations converges monotonically to the original in terms of "impulse response" energy, and the poles and zeros of the approximant move towards the poles and zeros of the process as the order of the approximation increases. The application of Routh approximations and the simplification of two distributed parameter mechanical systems (DPS) are described by Hutton and Rabin [93]. First, using the finite element model of the structure, a low order transfer function is obtained for studying vibrations. Second the Routh method is used in the design of a thermal control system where the heat conduction is modelled as a distributed parameter system whose
transfer function has infinite order. The accuracy of the approximation is determined by comparing
the approximate poles, residues, time and/or frequency responses as function of the reduced order.
The results obtained indicate rapid convergence of the approximation accuracy as the order of the
reduced model increases. An improvement to the Routh-Padé approximation techniques is presented
by Lepschy and Viaro [94]. The method contains two free parameters and a systematic procedure to
evaluate the coefficients of the reduced model is shown to give a good fit in the Padé sense (good
approximation in the neighbourhood of s=0), and a suitable stability margin. A frequency domain
approach is given by Lucas and Davidson [95] where the method is based upon the Schwarz
canonical form. The approach is shown to have a continued fraction representation and is related to
the Routh method described by Hutton and Friedland [92]. It is also shown that this technique can
be combined with a time-moment matching method to improve polynomial and step response. The
approach reported by Farsi et al [96] is designed to reduce the model order of linear discrete
systems. The Routh stability criterion is used to determine both the numerator and the
denominator of the transfer function. The numerator is also determined by an alternative mixed
method of Markov-Padé approximant matching.

The last type of approximation method is the optimization approach which leads to the
minimization of some cost function involving the error between the output of the reduced model
and that of the approximated system. Either direct schemes involving the state space representation
(Wilson [97], Riggs and Edgar [98], Wilson and Mishra [99] and Ettelberg [100]) or the observed
input/output data (Sinha and Berezni [101], Sinha and Pille [102], Genesio and Pompe [103]) are
generally used. The algorithm presented by Wilson [97] finds the optimum reduced order model for
continuous linear multivariable and multi-output systems. The least-squares reduction method
which is applicable to multi-input single output processes and based on the impulse response is
reported by Riggs and Edgar [98] where necessary conditions for the optimum linear dynamic
model reduction are also derived. The mean-square error is used in [99] for obtaining the reduced
models for multi-input multi-output systems with a step and other forms of input. The mean
square error of the transient portion of the system response is minimized, while the steady state
portion is matched exactly. The method proposed by Ettelberg [100] minimizes the equation error
and yields a stationary exact model via a set of algebraic equations. The technique is also used to
reduce the order of unstable time-invariant models by introducing a time varying weighting matrix.
Sinha and Berezni [101] presented an iterative algorithm for finding the optimum reduced order
model. Starting from an approximate first or second order model, the optimum is determined and
the process is continued, increasing the order progressively. A method which requires less
computational effort and which is based on the use of the matrix pseudo-inverse to estimate the
lower order model parameters is given by Sinha and Pille [102]. It minimizes the sum of the square errors between the response of the process and that of the model at sampling instants. The algorithm is also iterative. Genesio and Pome [103] described the problem of reducing the model order of time-invariant dynamic systems from noisy data. The reduced model is such that the 'worst case' error with respect to the unknown system is minimized when the input runs over a given set. An algorithm is given for the computation of the optimal reduced model and the error bounds according to the disturbances.

Several other approaches have been reported such as the Chebyshev polynomial techniques (Bistritz [104]), the Walsh series approach (Subbayan and Vauthilingam [105]), the $L_1$ and $L_\infty$ norm minimization (El-Attar and Vidyasagar [106]), the reduction over a frequency interval (Langholz and Bistritz [107], the optimal Hankel-norm approximation (Kung and Lin [108], Glover [109]), and a model following technique (Hassan and Cook [110]).

A brief review of the most common order reduction methods was recently presented by Paraskevopoulos [111]. The different techniques are grouped under six main approaches: 1- the theory of aggregation, 2- the dominant-pole or eigenvalue approach, 3- Padé-type approximant and partial realization, 4- Routh approximation, 5- perturbation methods and 6- the error minimization approach. These approaches are briefly described and compared and a comprehensive set of references is given in the paper. The conclusion was that the philosophy of most of the model order reduction techniques presented is based on the idea of neglecting the fast phenomena involved in the original mathematical model.

Although model reduction has received considerable attention, there is still no universal model reduction approach. This thesis presents a new integral equation derived approach to the problem of model reduction for continuous time systems. The approach is derived in the time domain and based on the system input/output data measurements where the effect of non-zero initial conditions and additive disturbances occurs naturally. The main advantage of using input/output data is that the original model of the system need not be available.

1.3. Outline of the thesis

Although continuous system dynamics are classically described in terms of the system ordinary differential equation, the direct use of this mathematical model for system identification and model order reduction purposes may not lead to satisfactory numerical results. In practical engineering problems the observed input/output data is often subject to unknown disturbances and successive numerical differentiation of such data yields meaningless values.

An alternative mathematical representation, called the system integral equation, of continuous
time-invariant system dynamics is introduced in chapter two. The formulation is first derived in the context of linear SISO systems where the presence of additive disturbances at both system input and output is included in the analysis. The formulation is then extended to cover linear MIMO and nonlinear systems. The effect of initial conditions is explicitly reflected in the integral equation and a formulation to recover such initial conditions is given in the context of linear SISO systems.

The concept of weighted residual methods, which have been used classically to produce approximate solutions to ordinary differential equations (Finalyson [112]), is extended to the two problems of continuous linear system identification and model reduction in chapter three. The three individual areas of equation solution, system identification and model reduction are classified as fundamental problems and several individual approaches arise from the general weighted residual framework. The application of some weighted residual methods to the problem of parameter identification in continuous linear SISO, MIMO and nonlinear systems is illustrated by several numerical examples. The integral equation is used as a basis for identification (initial state and parameter identification) problems. The approach is developed in the time domain and the effect of non-zero initial conditions and deterministic additive disturbances at system input and output are included in the identification. In the nonlinear parameter identification problem, the parameters of a nonlinear ordinary differential equation describing the unforced rolling motion of a ship (Roberts [49]) are identified.

The problem of continuous linear system order reduction from observed input/output data is posed as an identification problem in chapter four. The advantage of this approach is that the original system model need not be known. The numerical examples include linear SISO and MIMO model reduction. An example is included to illustrate some of the potential difficulties that can arise from use of the signal residual and the subsequent nonlinear optimization. An application to a practical problem where a high order model of an automatic voltage regulator is reduced to a lower order model is given at the end of the chapter.

Two new techniques for continuous system order determination in continuous SISO systems are presented in chapter five. The first method is based on the behaviour of singular values of a specific non-square matrix, the elements of which are calculated from multiple integrals of input and output data. The second method uses shifted Legendre polynomials to produce a square matrix whose non-zero eigenvalues then identify the order of the system under consideration.

Chapter six discusses the approximation of a distributed parameter model by a lumped parameter model, i.e. given a distributed parameter model described by a partial differential equation, boundary and initial conditions and its input/output data at specified points in the spatial domain, the problem is to find an approximate ordinary differential equation which if subject to
an input signal and initial conditions similar to those of the partial differential equation, gives rise to an output similar to that of the distributed model. The problem is therefore that of identifying the parameters of the ordinary differential equation or of its equivalent integral equation. The numerical examples include models of both hyperbolic and parabolic type.

Chapter seven presents the conclusions of the thesis and relates the use of the integral equation representation of continuous system dynamics in the problems of system parameter identification, model reduction, order determination and lumped approximation of distributed parameter systems.

A brief description of another technique for identification and model reduction of continuous linear systems which also avoids numerical differentiation of the data is given in Appendix A. The method is based on an appropriate choice of a set of functions of time whose derivatives vanish on both the lower and upper limit of the observation time interval. The effect of the initial conditions in the final equations which are obtained after multiplying the ordinary differential equation by chosen functions and integrating the product over the observation time interval is eliminated.

Appendix B briefly outlines a new technique for reducing high order models of continuous linear systems. The original model must be known since the method is based on some carefully selected points in s domain at which the original transfer function must be evaluated. The results given by this method are exceptionally good. However, the choice of the points in s domain to ensure stability of the reduced model is based on a heuristic analysis and further research is needed to theoretically justify this choice.

Some properties of the shifted Legendre and Chebyshev polynomials used in connection with Galerkin technique in the identification, order determination and order reduction problems are discussed in Appendix C.

Appendix D contains work published by the author of this thesis in conjunction with the supervisor, Dr A. H. Whitfield.
2. AN INTEGRAL EQUATION FORMULATION FOR SYSTEM DYNAMICS

There are two modelling and analysis approaches in customary use for linear systems: the frequency domain approach and the time domain approach. Both approaches are derived from the ordinary differential equation which describes the dynamic behaviour of the physical system under consideration. The fundamental vehicle for frequency domain analysis is the transfer function which expresses the relationship between the Laplace transform of the system output and Laplace transform of the system input when all the initial conditions of the ordinary differential equation are zero. It is often referred to as an input/output description. Perhaps the most time domain system representation is the state space description in which the plant dynamics are described as a set of first-order differential equations; this description is equally applicable to SISO and MIMO systems.

In the following section we shall derive an alternative description of the time domain behaviour of a system: the system "integral equation" representation. The advantages of the integral equation description become obvious in the area of system identification (order and parameter identification) and order reduction as will be seen in the following chapters. We shall derive the integral equation formulation for linear time-invariant single-input single-output (SISO) systems. The formulation will then be extended to include linear time-invariant multi-input multi-output and nonlinear systems. Although the analysis can be further extended to include time-varying systems, such systems shall not be considered in this thesis.

2.1. An integral equation for linear SISO systems

A non-delay linear SISO system is shown in block diagram form in Fig.2.1.

![Block diagram of a simple linear system](image_url)

Figure 2.1. Block diagram of a simple linear system

Continuous systems of this form may be expressed as an ordinary differential equation
\[
x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_n x(t) = b_1 u^{(n-1)}(t) + \ldots + b_n u(t)
\]  

(2.1)

subject to initial conditions \(x^{(i)}(0) = x_{i0}\) (\(i = 0, 1, \ldots, n-1\)), where \(x^{(i)}(t) = \frac{d^i x(t)}{dt^i}\), or by the equivalent transfer function

\[
\frac{X(s)}{U(s)} = \frac{b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}
\]  

(2.2)

in which the input to the system \(u(t)\) produces a corresponding output \(x(t)\). Equations (2.1) and (2.2) are the most common expressions for non-delay continuous linear SISO system dynamics, with (2.1) providing a time-domain description and (2.2) giving rise to a frequency-domain description. An alternative time-domain description will now be presented in the form of a multiple integral equation.

Firstly we introduce some notation that will be used to concisely express a multiple integral. We define

\[
x^{(k)}(t) = \int_0^t \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{k-1}} x(t) \, dt_{k} \ldots dt_2 dt_1 \quad \text{For } k \geq 2
\]

\[
x^{(1)}(t) = \int_0^t x(t) \, dt_1
\]

For a linear SISO system described by (2.1), consider evaluation at an arbitrary time \(t_1\) and integrate from 0 to \(t\) with respect to \(t_1\):

\[
[x^{(n-1)}(t) - x^{(n-1)}(0)] + a_1 [x^{(n-2)}(t) - x^{(n-2)}(0)] + \ldots + a_n [x(t) - x(0)] + a_n x^{(1)}(t)
\]

\[
= b_1 [u^{(n-2)}(t) - u^{(n-2)}(0)] + \ldots + b_{n-1} [u(t) - u(0)] + b_n u^{(1)}(t)
\]

The latter equation holds at all points in time, and hence consideration of its evaluation at \(t_2\) followed by integration from 0 to \(t\) with respect to \(t_2\) gives
Continuing this process to the point at which (2.1) has been integrated \( n \) times and collecting terms, we have

\[
\sum_{k=1}^{n} a_k x^{(k)}(t) - \sum_{k=1}^{n} b_k U^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} = x(t)
\]

(2.3)

where

\[
c_1 = -x(0)
\]

\[
c_k = -x^{(k-1)}(0) - \sum_{j=1}^{k-1} a_j x^{(k-j-1)}(0) + \sum_{j=1}^{k-1} b_j U^{(k-j-1)}(0) \quad (k = 2, 3, \ldots, n)
\]

The effect of input and output derivative initial conditions is contained entirely in the terms

\[
\sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!}
\]

and these terms can only be omitted from the integral equation formulation of (2.3) if all initial conditions are zero.

Equation (2.3) is obtained after repeated \( n \) times integration of input and output data over a defined time interval. Indefinite integration can also be used to obtain an expression similar in structure to (2.3). Thus if we denote

\[
\tilde{X}^{(n)}(t) = \int_{t_0}^{t} \int_{t_0}^{t_0} \cdots \int_{t_0}^{t_0} x(t) dt
\]

\( n \)-tuple

then after \( n \) indefinite integrations of equation (2.1) we obtain

\[
\sum_{k=1}^{n} a_k \tilde{X}^{(k)}(t) - \sum_{k=1}^{n} b_k \tilde{U}^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} = x(t)
\]
However this latter equation is of no computational use precisely because the elements \( \bar{X}^{(k)}(t) \), \( \bar{U}^{(k)}(t) \); \((k=1,2,...,n)\) represent the \( k \)th indefinite integrals of \( x(t) \) and \( u(t) \) respectively.

The major motivation for selecting the integral equation as a vehicle for system identification and model reduction is that reconstruction of signal integrals is more accurate than the reconstruction of signal derivatives particularly from noise corrupted signals. Thus integral equation description of a system will be central to much of the analysis of this thesis.

2.2. Additive disturbances

Physical systems, under normal operating conditions, are often subject to some additive disturbances. In the following sections we shall extend the integral equation formulation for continuous linear SISO systems to include the effect of such disturbances at system input and output. The formulation will then be extended to include continuous linear multi-input multi-output (MIMO) and nonlinear systems.

2.2.1 Linear SISO system formulation

Consider a continuous linear SISO system under normal operating conditions and whose block diagram is shown in Fig. 2.2. The observed system input and output, denoted \( r(t) \) and \( y(t) \) respectively, are related to the actual system input and output, \( u(t) \) and \( x(t) \), by

\[
\begin{align*}
  u(t) &= r(t) + p(t) \\
  y(t) &= x(t) + q(t)
\end{align*}
\]

where \( p(t) \), \( q(t) \) are deterministic disturbances.

![Figure 2.2. SISO system with disturbances.](image)

We shall now derive the integral equation representation for continuous linear SISO systems under these conditions. Since \( x(t) = y(t) - q(t) \) and \( u(t) = r(t) + p(t) \), the integral equation (2.3) governing \( x(t) \) for a given \( u(t) \) can be rewritten as an integral equation governing \( y(t) \) for a given \( r(t) \) as
\[ \sum_{k=1}^{n} a_k Y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t^{(k-1)}}{(k-1)!} - \sum_{k=1}^{n} a_k Q^{(k)}(t) - \sum_{k=1}^{n} b_k P^{(k)}(t) = -y(t) \]  

(2.4)

where \( P^{(k)}(t) \) and \( Q^{(k)}(t) \) are integrals of the disturbances defined as follows:

\[
Q^{(k)}(t) = \int_{0}^{t} \int_{0}^{\tau_1} \int_{0}^{\tau_2} \ldots \int_{0}^{\tau_{k-1}} q(\tau_k) d\tau_k \ldots d\tau_2 d\tau_1 d\tau_0
\]

for \( k \geq 2 \)

\[
P^{(k)}(t) = \int_{0}^{t} \int_{0}^{\tau_1} \int_{0}^{\tau_2} \ldots \int_{0}^{\tau_{k-1}} p(\tau_k) d\tau_k \ldots d\tau_2 d\tau_1 d\tau_0
\]

and

\[
Q^{(1)}(t) = \int_{0}^{t} q(\tau) d\tau
\]

\[
P^{(1)}(t) = \int_{0}^{t} p(\tau) d\tau
\]

Equation (2.4) holds at any point in time \( t \), and the system parameters \( \{a_k, b_k; k=1,2,\ldots,n\} \) can be identified from measurements of the system input/output data over a certain observation time interval.

2.2.2 Quantitative disturbance descriptions available

In this situation \( p(t) \) and \( q(t) \) are assumed to be known precisely; therefore the case when disturbances are known to be zero is covered by the analysis of this section. Since \( p(t) \) and \( q(t) \) are known at all points in time \( t \), the actual system inputs and outputs can be recovered via

\[
u(t) = r(t) + p(t)
\]

\[
x(t) = y(t) - q(t)
\]

and (2.3), from which the equation (2.4) arose, can be considered at any time \( t \):

\[ \sum_{k=1}^{n} a_k X^{(k)}(t) - \sum_{k=1}^{n} b_k U^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t^{(k-1)}}{(k-1)!} = -x(t) \]  

(2.5)
Equation (2.5) provides an exact description of the system. It is essential that the initial condition parameters are included in any estimation process when the system has non-zero initial conditions, which is the case in most physical situations.

2.2.3 Qualitative disturbance descriptions available

In certain situations we may only have qualitative knowledge of the deterministic disturbances that are present in a system, and quantifying such disturbances may require additional work. Typically we may be aware that a measurement transducer incorporates an offset and a calibration exercise would be required to evaluate this offset. Such qualitative knowledge can be included in the preceding analysis when the disturbance is linear in its unknown parameters. Suppose that the disturbances have the form

\[ p(t) = \sum_{j=1}^{n_p} \alpha_j g_j(t), \quad q(t) = \sum_{j=1}^{n_q} \beta_j h_j(t) \]

in which the functions \( \{g_j(t); j=1,2,...,n_p\} \) \( \{h_j(t); j=1,2,...,n_q\} \) are known, although the parameters \( \{\alpha_j; j=1,2,...,n_p\} \) \( \{\beta_j; j=1,2,...,n_q\} \) are unknown. Then (2.4) gives

\[ \sum_{k=1}^{n_a} a_k Y^{(k)}(t) - \sum_{k=1}^{n_b} b_k R^{(k)}(t) + \sum_{k=1}^{n_c} c_k \frac{t^{k-1}}{(k-1)!} - \sum_{j=1}^{n_q} \beta_j h_j(t) + \sum_{k=1}^{n_q} a_k H_j^{(k)}(t) - \sum_{j=1}^{n_p} \alpha_j \sum_{k=1}^{n_p} b_k G_j^{(k)}(t) = -y(t) \]

or

\[ \sum_{k=1}^{n_a} a_k Y^{(k)}(t) - \sum_{k=1}^{n_b} b_k R^{(k)}(t) + \sum_{k=1}^{n_c} c_k \frac{t^{k-1}}{(k-1)!} - \sum_{j=1}^{n_q} \beta_j h_j(t) - \sum_{j=1}^{n_q} \sum_{k=1}^{n_q} \beta_j H_j^{(k)}(t) - \sum_{j=1}^{n_p} \sum_{k=1}^{n_p} \alpha_j G_j^{(k)}(t) = -y(t) \]

where

\[ \bar{\beta}_{jk} = \beta_j a_k \text{ and } \bar{\alpha}_{jk} = \alpha_j b_k \]

and \( H_j^{(k)}(t) \), \( G_j^{(k)}(t) \) are the \( k \)th integrals of \( h_j(t) \) and \( g_j(t) \) as defined by \( X^{(k)}(t) \) for \( x(t) \). The effect of the unquantified disturbances \( p(t) \) and \( q(t) \) has now been included in the analysis.
The effect of polynomial disturbances (e.g., offsets and ramps etc...) can be included in the analysis without explicitly introducing the \( n_p + n_q \) unknown parameters that would generally be required. By way of an example, consider a system subject to unknown polynomial disturbances at both the input and the output, i.e.

\[
p(t) = \sum_{j=1}^{n_p} \alpha_j t^{j-1}, \quad q(t) = \sum_{j=1}^{n_q} \beta_j t^{j-1}
\]

Then after collecting the powers of \( t \) due to multiple integrations of \( p(t) \) and \( q(t) \) (2.6) becomes

\[
\sum_{k=1}^{n_p} a_k Y^{(k)}(t) - \sum_{k=1}^{n_q} b_k R^{(k)}(t) + \sum_{k=1}^{n_p+n_q} c_k \frac{t^{k-1}}{(k-1)!} - \sum_{k=1}^{n_p+n_q} \alpha_k \frac{t^{k-1}}{k!} - \sum_{k=1}^{n_p+n_q} \beta_k \frac{t^{k-1}}{k!} = -y(t) \tag{2.8}
\]

where \( (\alpha_k; k=2,3,...,n_p+n) \) and \( (\beta_k; k=1,2,...,n_q+n) \) can be determined from \( (\alpha_j; j=1,2,...,n_p) \) and \( (\beta_j; j=1,2,...,n_q) \) respectively. After collecting all the powers of \( t \) in (2.8) we obtain

\[
\sum_{k=1}^{n_p} a_k Y^{(k)}(t) - \sum_{k=1}^{n_q} b_k R^{(k)}(t) + \sum_{k=1}^{n_p+n_q} d_k \frac{t^{k-1}}{(k-1)!} = -y(t) \tag{2.9}
\]

where \( n_s \) is the larger of the two integers \( n_p \) and \( n_q \), i.e. \( n_s = \max(n_p, n_q) \).

For the particular case where both disturbances \( p(t) \) and \( q(t) \) are simply offsets, i.e. \( u(t) = r(t) + p_0 \) and \( y(t) = x(t) + q_0 \), the final equation becomes

\[
\sum_{k=1}^{n_p} a_k Y^{(k)}(t) - \sum_{k=1}^{n_q} b_k R^{(k)}(t) + \sum_{k=1}^{n_p+n_q} d_k \frac{t^{k-1}}{(k-1)!} - q_0 - \sum_{k=1}^{n_p} a_k q_0 \frac{t^k}{k!} - \sum_{k=1}^{n_q} b_k p_0 \frac{t^k}{k!} = -y(t)
\]

or

\[
\sum_{k=1}^{n_p} a_k Y^{(k)}(t) - \sum_{k=1}^{n_q} b_k R^{(k)}(t) + \sum_{k=1}^{n_p+n_q} d_k \frac{t^{k-1}}{(k-1)!} = -y(t) \tag{2.10}
\]

where the parameters \( (d_k; k=1,2,...,n_s+1) \) include the effects of both the system initial conditions and the offsets on the governing integral equation.
2.3. Initial state recovery

The integral equation (2.3) explicitly contains the parameters \( (c_k; k=1,2,\ldots,n) \) which are dependent upon system initial conditions. In the following section we outline a formulation for linear time-invariant SISO systems which utilizes this fact and thereby yields explicit identification of initial state variable values in the companion observable state space formulation.

The concept of observability arises when we consider the following problem: we are given a representation of a linear time-invariant SISO system in the form

\[
\begin{align*}
\dot{z}(t) &= Az(t) + bu(t) \\
x(t) &= d^T z(t)
\end{align*}
\]

(2.11)

where the matrix \( A \), the vectors \( b \) and \( d \) are of appropriate dimensions and \( z \) is the \((n+1)\) state vector, the input \( u(t) \) and the output \( x(t) \) are observed over an interval of time \([0,t]\); the observability problem is to find the initial state \( z_0 = z(0) \) given both the input and the output over an interval of time \([0,t]\).

The following theorem will enable us to determine the initial state \( z_0 \) of companion observable state space vector from observations made on the system input and output data.

**Theorem:**

If the state space representation (2.11) is in its companion observable form, i.e.

\[
A = \begin{bmatrix}
0 & \ldots & -a_n \\
1 & \ldots & -a_{n-1} \\
0 & 1 & \ldots & -a_2 \\
\vdots & \vdots & \ddots & \ddots \\
0 & \ldots & 1 & -a_1 \\
\end{bmatrix} \quad b = \begin{bmatrix}
b_n \\
b_{n-1} \\
b_{n-2} \\
\vdots \\
b_1 \\
\end{bmatrix} \quad d = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

then the initial state values \( \{z_k(0); k=1,2,\ldots,n\} \) of \( z(t) \) are equal to the constants in the integral equation (2.3) \((-c_k; k=n,n-1,\ldots,1)\), i.e.

\[
\begin{align*}
c_1 &= -z_n(0) = x(0) \\
c_2 &= -z_{n-1}(0) \\
\vdots \\
c_n &= -z_1(0)
\end{align*}
\]
Proof:
If we let :
\[ f_j(t) = -a_{n+1-j}z_n(t) + b_{n+1-j}u(t), \quad j = 1, 2, \ldots, n. \]
then equation (2.11) can be written as:
\[ \begin{align*}
    \dot{z}_1 &= f_1(t) \\
    \dot{z}_2 &= z_1 + f_2(t) \\
    & \vdots \\
    \dot{z}_n &= z_{n-1} + f_n(t) \\
x(t) &= z_n(t)
\end{align*} \]

The integration from 0 to t of the above set of first order differential equations gives:
\[ \begin{align*}
    z_1(t) &= z_1(0) + \frac{z_1(0)t}{1!} + \frac{z_1(0)t^2}{2!} + \cdots + \frac{z_1(0)t^n}{n!} + F_1(t) + F_2(t) + \cdots + F_n(t) \\
    z_2(t) &= z_2(0) + z_1(0)t + \frac{z_2(0)t^2}{2!} + \cdots + \frac{z_2(0)t^n}{n!} + F_2(t) + F_3(t) + \cdots + F_n(t) \\
    & \vdots \\
    z_n(t) &= z_n(0) + z_{n-1}(0)t + \frac{z_n(0)t^2}{2!} + \cdots + \frac{z_n(0)t^n}{n!} + F_n(t) + F_{n-1}(t) + \cdots + F_1(t) \\
x(t) &= z_n(t)
\end{align*} \]

where
\[ F_j^{(i)}(t) = -a_{n+1-j}z_n^{(i)} + b_{n+1-j}u^{(i)} \quad \text{for } j = 1, 2, \ldots, n \]

Replacing \( F_j^{(i)}(t) \) and \( z_n(t) \) in the \( n \)th equation:
\[ x(t) = z_n(0) + z_{n-1}(0)t + \cdots + \frac{z_1(0)t^n}{n!} + \sum_{k=1}^{n} a_k X^{(k)}(t) + \sum_{k=1}^{n} b_k U^{(k)}(t) \]

Comparing this equation to equation (2.3) we obtain:
\[ \begin{align*}
    z_1(0) &= -c_n \\
    z_2(0) &= -c_{n-1} \\
    & \vdots \\
    z_n(0) &= -c_1 = x(0)
\end{align*} \]
We have proven that the initial state \( z_0 \) of a system under its companion observable form is equal to the vector \( \{- c_k; k=n,n-1,\ldots,1\} \) in the integral equation (2.3).

In the following chapters the integral equation (2.3) is used to identify the parameters \( \{a_k, b_k; k=1,2,\ldots,n\} \) and the initial conditions \( \{c_k; k=1,2,\ldots,n\} \) of a linear system from observed input/output data. The parameters \( \{c_k; k=1,2,\ldots,n\} \) are then used as initial conditions for the observable form of state space representation and this latter form is simulated using the observed input to yield the corresponding model output behaviour. The results of the identification are assessed by comparing the observed output and the computed model output.

### 2.4. Linear MIMO system formulation

The preceding SISO linear system formulation scheme is now extended to cover linear MIMO systems where \( r \) measured inputs \( g(t) \) give rise to \( m \) measured outputs \( y(t) \). Such measured signals may differ from the actual system inputs \( u(t) \) and outputs \( x(t) \) by additive disturbances, i.e.

\[
\begin{align*}
    u(t) &= g(t) + d(t) \\
    x(t) &= z(t) + q(t)
\end{align*}
\]

where \( d(t) \) and \( q(t) \) are \( r \)- and \( m \)-dimensional vectors respectively, as shown in Fig. 2.3.

For a proper MIMO linear system the \( m \) ordinary differential equations relating the \( r \) inputs \( u(t) \) to the \( m \) outputs \( x(t) \) can be written as

\[
\begin{align*}
    x^{(n)}(t) + \sum_{k=1}^{r} a_k x^{(n-k)}(t) &= \sum_{j=1}^{r} \sum_{k=1}^{n} b_{kj} u^{(n-k)}(t) (\ell = 1, 2, \ldots, m) \quad (2.12)
\end{align*}
\]

Not all \( n \) modes of the system may be observable in each of the outputs, and therefore one or more...
of the ordinary differential equations in (2.12) may be of order less than \( n \). Following the same multiple integration arguments used in the SISO case, the equivalent set of system integral equations

\[
\sum_{k=1}^{n} a_k X^{(k)}_z(t) - \sum_{j=1}^{r} \sum_{k=1}^{n} b_{ij} U^{(k)}_j(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} = -x_\ell(t) \quad (\ell = 1, 2, \ldots, m) \quad (2.13)
\]

where

\[
c_k^\ell = -x_\ell(0)
\]

\[
c_k^\ell = -x_\ell(0) - \sum_{j=1}^{r} \sum_{i=1}^{k-1} b_{ij} U^{(k-j-1)}_j(0) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} \quad k \geq 2
\]

and the associated set of integral equations for measured inputs and outputs is

\[
\sum_{k=1}^{n} a_k Y^{(k)}_z(t) - \sum_{j=1}^{r} \sum_{k=1}^{n} b_{ij} R^{(k)}_j(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} - q_\ell(t) - \sum_{k=1}^{n} a_k Q^{(k)}_z(t) - \sum_{j=1}^{r} \sum_{k=1}^{n} b_{ij} P^{(k)}_j(t) = -y_\ell(t) \quad (\ell = 1, 2, \ldots, m) \quad (2.14)
\]

As in the SISO case, this latter set of integral equations can be used to describe the system dynamics when the disturbances \( p(t) \) and \( q(t) \) are known precisely and when such disturbances are known only qualitatively but are linear in their unknown parameters.

When \( p(t) \) and \( q(t) \) are known exactly the actual system inputs and outputs can be recovered at any point in time \( t \) via

\[
u(t) = z(t) + p(t)
\]

\[
x(t) = y(t) - q(t)
\]

and the equation to be considered at all time is:

\[
\sum_{k=1}^{n} a_k X^{(k)}_z(t) - \sum_{j=1}^{r} \sum_{k=1}^{n} b_{ij} U^{(k)}_j(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} = -x_\ell(t) \quad (\ell = 1, 2, \ldots, m) \quad (2.15)
\]

The procedure for including only linear-in-parameters qualitative effects is identical with the
SISO case, i.e. for

\[ p_j(t) = \sum_{i=1}^{n_2} a_{ij} g_{ij}(t) \quad (j = 1, 2, \ldots, r) \]

\[ q_{\ell}(t) = \sum_{i=1}^{n_2} \beta_{i \ell} h_{i \ell}(t) \quad (\ell = 1, 2, \ldots, m) \]

equation (2.14) becomes

\[
\sum_{k=1}^{n} a_k y^{(k)}(t) - \sum_{j=1}^{r} \sum_{k=1}^{n} b_{j k} R_j^{(k)}(t) + \sum_{k=1}^{n} \alpha_k t^{k-1} \frac{G^{(k)}(t)}{(k-1)!} - \sum_{i=1}^{n_2} \beta_{i \ell} h_{i \ell}(t) - \sum_{k=1}^{n} \sum_{i=1}^{n_2} b_k \alpha_{ij} G_{ij}^{(k)}(t) = -y_{\ell}(t) \quad (\ell = 1, 2, \ldots, m) \quad (2.16)
\]
or

\[
\sum_{k=1}^{n} a_k y^{(k)}(t) - \sum_{j=1}^{r} \sum_{k=1}^{n} b_{j k} R_j^{(k)}(t) + \sum_{k=1}^{n} \alpha_k t^{k-1} \frac{G^{(k)}(t)}{(k-1)!} - \sum_{i=1}^{n_2} \beta_{i \ell} h_{i \ell}(t) - \sum_{k=1}^{n} \sum_{i=1}^{n_2} b_k \alpha_{ij} G_{ij}^{(k)}(t) = -y_{\ell}(t) \quad (\ell = 1, 2, \ldots, m) \quad (2.17)
\]

where

\[ \bar{\beta}_{i \ell} = a_k \beta_{i \ell} \quad \text{and} \quad \bar{\alpha}_{ij} = b_k \alpha_{ij} \]

In the case of polynomial disturbances, i.e.

\[ p_j(t) = \sum_{i=1}^{n_2} \alpha_{ij} t^{i-1} \quad \text{and} \quad q_{\ell}(t) = \sum_{i=1}^{n_2} \beta_{i \ell} t^{i-1} \quad (j = 1, 2, \ldots, r; \ell = 1, 2, \ldots, m) \]

the final expression of (2.14) is

\[
\sum_{k=1}^{n} a_k y^{(k)}(t) - \sum_{j=1}^{r} \sum_{k=1}^{n} b_{j k} R_j^{(k)}(t) + \sum_{k=1}^{n+n_2} d_k t^{k-1} \left( \frac{t}{(k-1)!} \right)^{k-1} = -y_{\ell}(t) \quad (\ell = 1, 2, \ldots, m) \quad (2.18)
\]
where \( n_x \) = \( \text{max} (n_x \varepsilon, n_p j, j = 1, 2, ..., r) \). For simple offsets \( (n_x \varepsilon = 1; n_p = 1) \) we have

\[
\sum_{k=1}^{n} a_k x^{(k)}(t) - \sum_{j=1}^{n} \sum_{k=1}^{n} b_{k,j} R^{(k)}(t) + \sum_{k=1}^{n+1} \beta_k \frac{1}{(k-1)!} = -y_x(t) \quad (k = 1, 2, ..., m) \tag{2.19}
\]

Once again the effect of all the initial conditions and the offset is entirely contained in the terms \( \beta_k \frac{1}{(k-1)!}; (k = 1, 2, ..., n+1) \).

### 2.5. Nonlinear system formulation

The preceding integral equation approach can also be extended to cover the description of certain nonlinear system dynamics. Initially we shall consider a SISO system governed by an ordinary differential equation of the form

\[
x^{(n)}(t) + a_1 x^{(n-1)}(t) + ... + a_n x(t) + \sum_{s=1}^{n'} f_s[x(t), u(t), t] = b_1 u^{(n-1)}(t) + ... + b_n u^{(n)}(t) \tag{2.20}
\]

where the non-linearity can be expressed by the functions \( f_s[x(t), u(t), t]; s = 1, 2, ..., n' \). The integral equation equivalent is

\[
x(t) + a_1 x^{(1)}(t) + ... + a_n x^{(n)}(t) + \sum_{s=1}^{n'} \gamma_s F_s^{(n)}[x(t), u(t), t] + \sum_{k=1}^{n} \gamma_k \frac{1}{(k-1)!} = b_1 U^{(1)}(t) + ... + b_n U^{(n)}(t) \tag{2.21}
\]

where the parameters \( \{c_k; k=1, 2, ..., n\} \) cover the effect of non-zero initial conditions and

\[
F_s^{(n)}[x(t), u(t), t] = \int_0^t ... \int_0^{\tau_{n-1}} f_s[x(\tau_1), u(\tau_2), ..., \tau_n] d\tau_1 ... d\tau_{n-1} d\tau_n
\]

When input and output disturbances are known, \( u(t) \) and \( x(t) \) are again recoverable from the measured input and output and the equation to be used for system identification is

\[
\sum_{k=1}^{n} a_k x^{(k)}(t) - \sum_{k=1}^{n} \beta_k U^{(k)}(t) + \sum_{s=1}^{n'} \gamma_s F_s^{(n)}[x(t), u(t), t] + \sum_{k=1}^{n} \beta_k \frac{1}{(k-1)!} = -x(t) \tag{2.22}
\]
Linear-in-parameters nonlinear systems may also lead to a similar analysis as that given for linear systems when the disturbances are only known qualitatively but are linear in their unknown parameters. Specific investigation of the particular nonlinear system is needed to ascertain whether or not this is the case. Consider an example in which a system is governed by a nonlinear ordinary differential equation

\[ \ddot{x} + a_1 \dot{x} + \gamma x^2 = b_1 u + b_2 u \]

The equivalent integral equation is

\[ a_1 X^{(1)}(t) - b_1 U^{(1)}(t) - b_2 U^{(2)}(t) + \gamma \int_0^t \int_0^t (x(t_1) - x(t_1 - t_2)) \, dt_1 \, dt_2 + c_1 + c_2 t = -x(t) \]

If the measured output is the actual output plus a constant offset and the measured input is equal to the actual input, i.e. \( y(t) = x(t) + q_0, r(t) = u(t) \), then the associated integral equation for \( y(t) \) and \( r(t) \) is

\[ a_1 Y^{(1)}(t) - q_0 Y^{(2)}(t) - b_1 R^{(1)}(t) - b_2 R^{(2)}(t) + \gamma \int_0^t \int_0^t (y(t_1) - q_0)^2 \, dt_1 \, dt_2 + c_1 + c_2 t = -y(t) + q_0 \]

or

\[ a_1 Y^{(1)}(t) - 2q_0 Y^{(2)}(t) - b_1 R^{(1)}(t) - b_2 R^{(2)}(t) + \gamma \int_0^t \int_0^t y(t_1)^2 \, dt_1 \, dt_2 + [c_1 - q_0] + [c_2 - a_1 q_0] t + \gamma q_0 t^2 \frac{t^2}{2!} = -y(t) \]

or

\[ \alpha_1 Y^{(1)}(t) + \alpha_2 Y^{(2)}(t) - \beta_1 R^{(1)}(t) - \beta_2 R^{(2)}(t) + \gamma \int_0^t \int_0^t y(t_1)^2 \, dt_1 \, dt_2 + d_1 + d_2 t + d_3 t^2 \frac{t^2}{2!} = -y(t) \]

The relations between the parameters \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, d_1, d_2, d_3 \), the original system parameters and the unknown offset are
Thus this particular nonlinear system with output subject to a constant offset yields an integral equation which is linear in its unknown parameters and which can therefore be used as a vehicle for identification. The governing differential equation for a more general linear-in-parameters nonlinear system may be written

\[
\sum_{s=1}^{n'} \gamma_s g_s \left[ x^{(n)}(t), ..., x^{(1)}(t), x(t), u^{(n-1)}(t), ..., u^{(1)}(t), u(t), t \right] = 0
\]

and the equivalent integral equation can only be written as

\[
\sum_{s=1}^{n'} \gamma_s G_s^{(n)} \left[ x^{(n)}(t), ..., x^{(1)}(t), x(t), u^{(n-1)}(t), ..., u^{(1)}(t), u(t), t \right] = 0
\]

where

\[
G_s^{(n)} = \int_0^{\tau_n} \int_0^{\tau_{n-1}} ... \int_0^{\tau_1} d\tau_{n-1} d\tau_{n-2} ... d\tau_1 d\tau_0
\]

With the inclusion of input and output disturbances, the associated integral equation for \(y(t)\) and \(r(t)\) is

\[
\sum_{s=1}^{n'} \gamma_s G_s^{(n)} \left[ y^{(n)}(t) - q^{(n)}(t), ..., y(t) - q(t), r^{(n-1)}(t) + p^{(n-1)}(t), ..., r(t) + p(t), t \right] = 0
\]

which may or may not provide a useful description for identification purposes.

2.6. Summary

We have an alternative mathematical formulation (the integral equation formulation) for
describing the dynamics of continuous systems. The integral equation for linear SISO and MIMO systems was formulated in generality whereas nonlinear systems require careful individual formulation. The effect of additive deterministic disturbances at system input and output is included in the analysis. Particular attention was given to unknown polynomial disturbances, amongst which the offset is the most common.

The integral equation explicitly preserves the effect of initial conditions and a formulation to recover these conditions was given in the context of linear SISO systems, although the same analysis is applicable to MIMO systems.

The importance of the integral equation formulation will be seen in the following chapters when the problems of system identification and model reduction will be considered in some detail.
CHAPTER 3.
3. WEIGHTED RESIDUAL APPROACHES TO LUMPED PARAMETER SYSTEM IDENTIFICATION

In this chapter we recall the descriptions of linear and nonlinear continuous system dynamics and express them in a unified context by a general operator statement. Three individual areas of interest, namely equation solution, system identification and model reduction, are defined and classified as fundamental problems. Although the definitions are given in the context of linear SISO systems they remain valid for linear MIMO and nonlinear systems. The application of many existing weighted residual methods, classically applied to the equation solution problem, are extended to cover the problems of system identification and model reduction. The significance of the choice of input signals in the identification and model reduction problems is considered and the application of several weighted residual methods are illustrated by several numerical examples where the effect of disturbances at system input and output is included.

3.1. Operator representations of system dynamics

A general operator statement describing the dynamics of continuous linear SISO, linear MIMO and nonlinear systems will be presented in the next three sections. The operator will be explicitly formulated for linear systems whereas careful individual formulation will be required for nonlinear systems.

3.1.1 Linear SISO systems

As previously stated, the most common time domain expression for the dynamics of a non-delay time-invariant continuous linear SISO system is an ordinary differential equation of the form

\[ x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_n x(t) = b_1 u^{(n-1)}(t) + \ldots + b_n u(t) \]  \hspace{1cm} (3.1)

subject to initial conditions \( x^{(i)}(0) = x_{i0} \) \((i=0,1,\ldots,n-1)\). We can write such a general ordinary differential equation as a linear differential operator equation. Thus if we define an operator \( \mathcal{L}_d \) such that

\[ \mathcal{L}_d^{(n)} [g(t)] = \left[ \frac{d^n g(t)}{dt^n} \frac{d^{n-1} g(t)}{dt^{n-1}} \ldots \frac{d g(t)}{dt} 1 \right]^T \]  \hspace{1cm} (3.2)

\[ \mathcal{L}_d [g_1(0), g_2(0), \ldots, g_s(0)] = \beta_1^T \mathcal{L}_d^{(n)} [g_1(0)] + \beta_2^T \mathcal{L}_d^{(n-1)} [g_2(0)] + \ldots + \beta_s^T \mathcal{L}_d^{(n-1)} [g_s(0)] \]
where \((\beta_i; \ i=1,2,\ldots,s)\) are constant vectors of appropriate dimensions, then (3.1) may be written as a linear differential operator \(\mathcal{L}_d[x(t), u(t)] = 0\), subject to the same initial conditions on \(x(t)\) and its derivatives, where \(\beta_1 = [1, a_1, \ldots, a_m]^T\) and \(\beta_2 = [b_1, \ldots, b_n]^T\).

As shown in chapter 2, an alternative description of this system is provided by the integral equation expression

\[
x(t) + \sum_{k=1}^{n} a_k x^{(k)}(t) - \sum_{k=1}^{n} b_k U^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} = 0
\]

(3.3)

where

\[
c_1 = -a(0)
\]

\[
c_k = -a^{(k-1)}(0) - \sum_{j=1}^{k-1} a_j x^{(k-j)}(0) + \sum_{j=1}^{k-1} b_j u^{(k-j)}(0) \quad (k=2,3,\ldots,n)
\]

and

\[
x^{(1)}(t) = \int_{0}^{t} x(\tau) d\tau
\]

\[
x^{(2)}(t) = \int_{0}^{t} \int_{0}^{\tau_2} x(\tau_1) d\tau_1 d\tau_2
\]

\[
x^{(k)}(t) = \int_{0}^{t} \int_{0}^{\tau_k} \int_{0}^{\tau_{k-1}} \cdots \int_{0}^{\tau_2} x(\tau_1) d\tau_1 d\tau_2 \cdots d\tau_k
\]

for \(k \geq 2\)

Equation (3.3) can also be replaced by a more general expression, i.e. \(\mathcal{L}_i[x(t), u(t)] = 0\) if the linear integral operator \(\mathcal{L}_i\) is defined as

\[
\mathcal{L}_i[g_1(t), g_2(t), \ldots, g_s(t)] = \int_{0}^{\tau_n} \int_{0}^{\tau_{n-1}} \cdots \int_{0}^{\tau_2} \int_{0}^{\tau_1} \mathcal{L}_d[g_1(\tau_1), g_2(\tau_2), \ldots, g_s(\tau_s)] d\tau_1 d\tau_2 \cdots d\tau_n
\]

(3.4)

Thus we can express the dynamics of such a linear system by a general operator statement \(\mathcal{L}[x(t), u(t)] = 0\), where \(\mathcal{L}\) includes the system initial conditions if it is an integral operator or is subject to such initial conditions if it is a differential operator.

3.1.2 Linear MIMO systems

The preceding SISO linear operator representation is now extended to cover MIMO systems where \(r\) inputs give rise to \(m\) outputs. The ordinary differential equation representation of MIMO systems is given by a set of ordinary differential equations:
\[
\begin{align*}
(x_{k})^{(n)}(t) + \sum_{k=1}^{n} a_{k} x_{k}^{(n-k)}(t) = \sum_{j=1}^{r} \sum_{k=1}^{n} b_{ij} u_{j}^{(n-k)}(t) \\
& \quad \quad (k = 1, 2, \ldots, m)
\end{align*}
\] (3.5)

with the initial conditions \( x_{k}^{(i)}(0) = x_{k,10} \) \((i=0,1,\ldots,n-1; k=1,2,\ldots,m)\). Using the differential operator defined by (3.2) we can represent linear MIMO systems by the set of linear differential operators \( \mathcal{L}_{k} \{ x_{k}(t), u_{1}(t), \ldots, u_{r}(t) \} = 0 \) \((k=1,2,\ldots,m)\), subject to the same initial conditions on \( \{ x_{k}(0); k=1,2,\ldots,m \} \) and their derivatives, where \( \beta_{1} = [a_{1},a_{2},\ldots,a_{n}]^{T} \) and \( \beta_{2} = [b_{1}^{T},b_{2}^{T},\ldots,b_{m}^{T}]^{T} \) \((i=1,2,\ldots,r; k=1,2,\ldots,m)\).

Similarly the integral equation representation for MIMO systems is

\[
\sum_{k=1}^{n} a_{k} x_{k}^{(k)}(0) - \sum_{j=1}^{r} \sum_{k=1}^{n} b_{ij} u_{j}^{(k)}(0) + \sum_{k=1}^{n} c_{k} \frac{(k-1)!}{(k-1)!} = -x_{k}(0) \quad (k = 1, 2, \ldots, m)
\] (3.6)

where

\[
\begin{align*}
& c_{1} = -x_{k}(0) \\
& c_{k} = -x_{k}^{(k-1)}(0) - \sum_{j=1}^{r} a_{ij} x_{j}^{(k-1)}(0) + \sum_{i=1}^{r} \sum_{j=1}^{n} b_{ij} u_{j}^{(k-1)}(0) \quad k \geq 2
\end{align*}
\]

can be replaced by a set of general integral operator statement \( \mathcal{L}_{1} \{ x_{k}(t), u_{1}(t), \ldots, u_{r}(t) \} = 0 \) \((k=1,2,\ldots,m)\), where \( \mathcal{L}_{1} \) is defined by (3.4).

Thus we can express the dynamics of linear MIMO systems by a set of general operator statement \( \mathcal{L} \{ x_{k}(t), u_{1}(t), \ldots, u_{r}(t) \} = 0 \) \((k=1,2,\ldots,m)\), where \( \mathcal{L} \) includes the effect of initial conditions if it is an integral operator or is subject to initial conditions if it is a differential operator.

### 3.1.3 Nonlinear systems

As there is no explicit general ordinary differential equation describing nonlinear system dynamics there is no explicit general operator statement for such systems. For a general nonlinear SISO system represented by a general ordinary differential equation of the form
or by the equivalent integral equation

\[ \sum_{s=1}^{n'} \gamma_s g_s^n [x^{(n)}(t), ..., x^{(1)}(t), x(t), u^{(n-1)}(t), ..., u^{(1)}(t), u(t), t] = 0 \]  

(3.8)

where

\[ G_s^{(n)} = \int_0^t \int_0^{\tau_n} \int_0^{\tau_{n-1}} ... \int_0^{\tau_1} g_s \, d\tau_n ... d\tau_1 \, d\tau_0 \]

a general nonlinear operator statement of the form \( N[x(t), u(t)] = 0 \) can be used to replace (3.7) or (3.8), where \( N \) contains the effect of initial conditions if it replaces (3.8) or subject to initial conditions if it replaces (3.7).

### 3.2. Three fundamental problems

Although the definitions of the problems of interest are given in the context of linear SISO systems (i.e. using a single linear operator statement), the definitions remain valid for linear MIMO and nonlinear systems. Having posed the dynamics of a linear system as

\[ \mathcal{L}[x(t), u(t)] = 0 \]  

(3.9)

we shall now distinguish between three basic problems: 1- equation solution, 2- system identification, 3- model order reduction.

In the problem of equation solution we seek to find the function \( x(t) \) which solves equation (3.9) for a prescribed input \( u(t) \). The precise structure of the operator \( \mathcal{L} \) and all the associated parameters of the system are assumed known i.e. the system order \( n \), the constant coefficients \( \{a_k, b_k; k=1,2,...,n\} \) and either the initial conditions \( x^{(i)}(0) = x_{i0} (i=0,1,...,n-1) \) if \( \mathcal{L} \) is a differential operator as in (3.1) or the constants \( \{c_k; k=1,2,...,n\} \) if \( \mathcal{L} \) is an integral operator as in (3.3) are specified. We note that although the integral equation (3.3) is an entirely valid description of the linear system, the ordinary differential equation (3.1) is the standard vehicle to calculate \( x(t) \) both analytically and numerically.

In the second situation i.e. the problem of system identification, it is generally presumed that
the order of the system, \( n \), is known and that the values of the output \( x(t) \) when the system is subject to a prescribed input \( u(t) \) are also known over a certain interval of time. In practical situations, the number of numerical values of the input/output data over a given time interval is finite and generally available at discrete points in time i.e. as time series \( \{x(t_i); i=0,1,...,N\} \), \( \{u(t_i); i=0,1,...,N\} \). The problem then is to identify the parameters \( \{a_k, b_k; k=1,2,...,n\} \) in the system ordinary differential equation (3.1). Given the input/output data, its successive derivatives can be numerically calculated and the ordinary differential equation (3.1) may be used to identify the system parameters. However the integral equation (3.3) may act as a more robust vehicle for this purpose since, if either the input or output signals are corrupted by noise, forming the derivative components of (3.1) may yield numerically unacceptable approximations. Since the parameters \( \{c_k; k=1,2,...,n\} \) are fundamental to the integral equation expression of system dynamics, they must also be identified in this latter formulation.

A third and final problem is that of model reduction. Here it is presumed that either a high order model has been identified or that input/output data from a high order system is available. The problem is to identify parameters \( \{a_k, b_k; k=1,2,...,n\} \) in a reduced model which still convey the important dynamics of the higher order system with order \( n_R \gg n \).

### 3.3. Weighted residual methods

#### 3.3.1 A residual formulation for SISO systems

Weighted residual methods (Finlayson [112]) provide a class of approaches to the problem of solving an ordinary differential equation of which (3.1) is a typical example. Such methods could also be used to solve an integral equation of the form (3.3) and we shall therefore present such methods in the context of general linear operator equation (3.9).

In the context of equation solution, weighted residual methods calculate a set of unknown parameter values \( \{\alpha_i; i=1,2,...,N_{\alpha}\} \) in an approximate solution \( \hat{x}(t) \) to the operator equation (3.9) where:

\[
\hat{x}(0) = \sum_{i=1}^{N_{\alpha}} \alpha_i \varphi_i(t)
\]  

(3.10)

and \( \{\varphi_i(t); i=1,2,...,N_{\alpha}\} \) are combinations of preselected basis functions such as monomials \( \{t^{i-1}; i=1,2,...,N_{\alpha}\} \) or orthogonal polynomials \( \{P_{i-1}(t); i=1,2,...,N_{\alpha}\} \), chosen so that \( \hat{x}(t) \) satisfies the given initial conditions on \( x(t) \). The technique for determining the parameters \( \{\alpha_i; i=1,2,...,N_{\alpha}\} \) is dictated by the particular method chosen. For linear operator equation...
solution problems, a residual $R(t)$ arises from the application of the known linear operator to the approximate solution $\hat{x}(t)$. Thus

$$R(t) = L[\hat{x}(t), u(t)]$$

(3.11)

and the weighted residual methods calculate the solution parameters \{\alpha; i=1,2,\ldots,N_{\alpha}\} by forcing $R(t)$ to zero in various ways.

In the following we seek to investigate various system identification and model reduction techniques which also arise from the application of weighted residual principles. In the context of system identification an equation residual is formed by application of a linear operator with correct structure but potentially incorrect model parameters while in the model reduction problem an equation residual is formed by application of a linear operator with incorrect structure and consequently incorrect parameters. In both situations an equation residual $R(t)$ arises from the application of an approximate operator $L$ to the actual system input and output, i.e.

$$R(t) = L[x(t), u(t)]$$

(3.12)

In the case of system identification and model reduction via the ordinary differential equation (3.1), we seek the parameters \{a_k, b_k; k=1,2,\ldots,n\} while via the integral equation (3.3) we seek the parameters \{a_k, b_k, c_k; k=1,2,\ldots,n\} which suitably force the residual to zero.

An alternative range of approaches to system identification and model reduction may be generated by consideration of a signal residual. This latter residual is defined as:

$$R(t) = x(t) - \hat{x}(t)$$

where $x(t)$ is the output of the system and $\hat{x}(t)$ is the output predicted by a model when subject to the same input and initial conditions.

We can now encompass all three basic problems in one general weighted residual formulation. We define the general problem as that of finding a parameter vector $\beta$ which suitably forces a residual $R(\beta, t)$ to zero, noting that in the case of operator equation solution the parameter vector $\beta$ has components \{\alpha; i=1,2,\ldots,N_{\beta}\} and the residual is formed by (3.11) while in the case of system identification $\beta$ has components \{a_k, b_k, c_k; k=1,2,\ldots,n\} for the ordinary differential equation (3.1), \{a_k, b_k, c_k; k=1,2,\ldots,n\} for the integral equation (3.3) and the residual is formed by (3.12).
general we denote the number of parameters in the vector $\mathbf{a}$ by $N_p$. For $t \in [0, T_f]$ we write an inner product of two functions $g_1(t)$ and $g_2(t)$ as $\langle g_1(t), g_2(t) \rangle$; for example the inner product generally used in this thesis is

$$\langle g_1(t), g_2(t) \rangle = \int_0^{T_f} g_1(t)g_2(t) dt$$

(3.13)

The standard weighted residual methods for operator equation solution will now be written in terms of the latter general formulation and therefore all such methods are equally applicable to the problems of system identification and model reduction. Having defined the residual, the general formulation of weighted residual methods is given by

$$\text{solve } \langle R (\mathbf{a}, t), \alpha_i(t) \rangle = 0 \quad i=1,2,...$$

(3.14)

where the $\{\alpha_i(t); i=1,2,...\}$ are called weighting functions. These functions can be chosen in many ways and each choice corresponds to a different criterion which gives the name of the particular weighted residual method.

In the context of nonlinear operator equation solution, a residual $R(t)$ arises from the application of the known nonlinear operator to the approximate solution $\hat{x}(t)$. Thus

$$R (\mathbf{a}, t) = N [\hat{x}(t), u(t)]$$

(3.15)

and the weighting residual methods calculate the solution parameters $\{\alpha_i; i=1,2,...N_p\}$ by forcing the residual $R (\mathbf{a}, t)$ to zero.

In the case of nonlinear system identification and model reduction an equation residual arises from the application of an approximate nonlinear operator $\hat{N}$ to the actual system input and output, i.e.

$$R (\mathbf{a}, t) = \hat{N} [x(t), u(t)]$$

(3.16)

and the parameters $\{\alpha_i; i=1,2,...N_p\}$ are found by forcing the residual $R (\mathbf{a}, t)$ to zero using standard weighted residual methods (3.14).

The most common techniques applied to solution equation, but equally applicable to system
identification and model reduction are defined and described below.

**Method of moments**

\[ \text{solve } \sum_{i} < R(\Theta, t), t^i > = 0 \quad i=1,2,\ldots,N_p \]  (3.17)

Here the weighting functions are the monomials \( t^i; i=1,2,\ldots,N_p \), thus successively higher moments of the residual are required to be zero. The residual is forced to zero by making it orthogonal to each of the monomials thereby selected.

**Galerkin**

\[ \text{solve } \sum_{i} < R(\Theta, t), \psi_i(t) > = 0 \quad i=1,2,\ldots,N_p \]  (3.18)

where \( \{ \psi_i(t); i=1,2,\ldots \} \) form a complete (orthogonal) set of functions on \([0,T_f]\), i.e.

\[ <\psi_i(t), \psi_j(t)> = \begin{cases} 0 & \text{for } i \neq j \\ \neq 0 & \text{for } i = j \end{cases} \]

The residual is forced to zero by making it orthogonal to each of the complete set of chosen functions.

**Least squares**

\[ \min_{\Theta} \quad < R(\Theta, t), R(\Theta, t) > \]

i.e.

\[ \text{solve } \sum_{i} < R(\Theta, t), \frac{\partial R(\Theta, t)}{\partial \Theta} > = 0 \]  (3.19)

Here, the weighting functions are \( \partial R(\Theta, t)/\partial \Theta \). The interpretation is that the mean square residual is being minimized with respect to the parameter vector \( \Theta \).

**Point collocation**

\[ \text{solve } \sum_{i} < R(\Theta, t), t^i > = 0 \quad i=0,1,\ldots,N_p-1 \]  (3.20)

i.e.

\[ \sum_{i} R(\Theta, t^i) = 0 \quad i=0,1,\ldots,N_p-1 \]
where \( t_i \) are values of \( t, t \in [0,T_f] \). The displaced Dirac delta functions are the chosen weighting functions. The residual is required to be zero at specified points \( t_i \).

**Orthogonal collocation**

\[
\text{solve } \min_{\Omega} \sum_{i=0}^{N_q-1} \mathcal{R}(\Omega, t_i) = 0 \quad i=0,1,...,N_p-1
\]

i.e. \( \mathcal{R}(\Omega, t_i) = 0 \quad i=0,1,...,N_p-1 \)  \hspace{1cm} (3.21)

where \( t_i \) are roots of polynomials which are orthogonal with respect to the inner product defined on \([0,T_f]\).

**Least squares point collocation**

\[
\text{solve } \min_{\Omega} \sum_{i=0}^{N_q-1} \mathcal{R}(\Omega, t_i) = 0 \quad i=0,1,...,N_p-1
\]

i.e. \( \sum_{i=0}^{N_q-1} \mathcal{R}(\Omega, t_i) = 0 \)

where \( N_c > N_p \) and \( t_i \) are values of \( t, t \in [0,T_f] \).

**Least squares quadrature**

\[
\text{solve } \min_{\Omega} \sum_{i=0}^{N_q-1} w_i \mathcal{R}(\Omega, t_i) = 0 \quad i=0,1,...,N_p-1
\]

i.e. \( \sum_{i=0}^{N_q-1} w_i \mathcal{R}(\Omega, t_i) = 0 \)

where \( N_q > N_p \) and \( (t_i, w_i : i=0,1,...,N_q-1) \) are suitable quadrature points (on \([0,T_f]\)) and weights which approximate the least squares inner product problem defined by (3.19).

We briefly note a similarity between application of the trapezoidal rule in least squares quadrature and the least squares point collocation formulation. Thus when \( N_q = N_c \) and the quadrature points \( \{t_i : i=0,1,...,N_q-1\} \) are equally spaced with \( \Delta T = t_{i+1} - t_i \) then using the trapezoidal
rule in (3.23) leads to an equation set

\[
\min_{\mathbf{\theta}} \Delta T \left[ \frac{1}{2} \sum_{i=0}^{N_q-2} \mathcal{J}_l(\mathbf{\theta}, t_i) + \frac{1}{2} \mathcal{J}_l(\mathbf{\theta}, t_{N_q-1}) \right]
\]

i.e. solve

\[
\frac{1}{2} \mathcal{J}_l(\mathbf{\theta}, t_0) \frac{\partial \mathcal{J}_l(\mathbf{\theta}, t_0)}{\partial \mathbf{\theta}} + \sum_{i=1}^{N_q-2} \mathcal{J}_l(\mathbf{\theta}, t_i) \frac{\partial \mathcal{J}_l(\mathbf{\theta}, t_i)}{\partial \mathbf{\theta}} + \frac{1}{2} \mathcal{J}_l(\mathbf{\theta}, t_{N_q-1}) \frac{\partial \mathcal{J}_l(\mathbf{\theta}, t_{N_q-1})}{\partial \mathbf{\theta}} = 0
\]

(3.24)

while the least squares point collocation scheme (3.22) produces a set of equations

\[
solve_{\mathbf{\theta}} \sum_{i=0}^{N_q-1} \mathcal{J}_l(\mathbf{\theta}, t_i) \frac{\partial \mathcal{J}_l(\mathbf{\theta}, t_i)}{\partial \mathbf{\theta}} = 0
\]

(3.25)

We note that (3.24) and (3.25) only differ by the factor 1/2 in the contributions from \( \mathcal{J}_l(\mathbf{\theta}, t_0) \) and \( \mathcal{J}_l(\mathbf{\theta}, t_{N_q-1}) \). Thus for \( N_q \) relatively large, which will generally be the case in the identification problem and can be forced in the operator equation solution and model reduction problems, (3.24) will only differ marginally from (3.25) and we can expect very similar estimates of \( \mathbf{\theta} \) from both methods.

3.3.2 Comparison of methods

As previously commented, application of the various weighted residual methods is well developed for the problem of operator equation solution. We shall now contrast the various methods when applied to this latter problem and when applied to the problems of system identification and model reduction.

One fundamental difference between the problems is that when considering equation solution the number of unknowns, \( N_p \), in the approximate expansion can be increased. Hence in the integral non-least squares methods (method of moments and Galerkin) the residual can be made orthogonal to an increasing number of functions while in the collocation non-least squares methods (point collocation, orthogonal collocation) the residual can be forced to zero at an increasing number of points, both strategies ultimately producing a more accurate solution. However the number of unknowns is fixed in both the identification and model reduction problems and the number of equations produced by (3.17), (3.18), (3.20) and (3.21) may not be sufficient to adequately force the residual to zero in a global sense. This is particularly true in the model
reduction problem where the number of free parameters is generally small. Hence the problems of system identification and model reduction may well be best addressed by least squares approaches.

A second difference between the three problems results from the nature of the input/output data available for system identification and model reduction in that such data is available at only discrete, usually equidistant, points in time. Thus the inner product integrals required by the method of moments, Galerkin and least squares must be evaluated numerically using the available data. This is in contrast to the equation solution problem where such integrals can often be performed analytically following application of the equation operator to the continuous approximation of the solution.

Several other distinctions will now be drawn between the three fundamental problems when addressed by the various weighted residual methods.

Method of moments

The inner products involving the monomial basis functions that are required by this method can yield a rather ill-conditioned set of equations in each of the fundamental problems. This phenomenon is particularly apparent for larger values of \( N_p \) and is therefore less significant in the system identification and model reduction problems when \( N_p \) tends to be relatively small. Nonetheless use of extended precision arithmetic during the solution of (3.17) is recommended when applying this technique to all three fundamental problems.

Galerkin

For the operator equation solution problem the basis functions \( \{ \psi_i(t); i=1,2,\ldots,N_p \} \) used in the inner products (3.18) are fixed as the functions defined in the approximate solution expansion (3.10) and are therefore constrained by the need to impose any initial conditions upon this expansion. However there is no such restriction in the system identification and model reduction problems and a wide choice of complete function sets \( \{ \psi_i(t); i=1,2,\ldots \} \) is available. Typically we could choose the standard inner product (3.13) and use

\[
\psi_i(t) = P_{i-1}(t) \quad i=1,2,\ldots
\]

(3.26)

where \( P_{i-1}(t) \) is the shifted Legendre polynomial of degree \( (i-1) \), such polynomials forming an orthogonal basis on \([0,T]\) (Hwang and Guo [38]). Since such polynomials are simply linear combinations of the monomial basis functions, the set of equations (3.18) formed by this choice of \( \{ \psi_i(t); i=1,2,\ldots,N_p \} \) should yield identical solutions to that given by the equation set (3.17) produced by the method of moments. However the orthogonality of the shifted Legendre
polynomials on \([0,T_f]\) produces a better conditioned set of equations and solutions produced numerically from (3.18) are generally more accurate than those produced numerically from (3.17) and are therefore to be preferred.

Another useful approach within the Galerkin classification is given by defining the inner product as

\[
\langle g_1(t), g_2(t) \rangle = \frac{1}{2} \int_0^{T_f} \frac{g_1(t) g_2(t)}{\sqrt{(t/T_f)(1-t/T_f)}} \, dt
\]  

(3.27)

and choosing the basis functions as \(\psi_i(t) = T_{i-1}^*(t)\), the shifted Chebyshev polynomials of degree \((i-1)\) on \([0,T_f]\), such polynomials forming an orthogonal basis with respect to the latter inner product on \([0,T_f]\). Solution of the Galerkin equations (3.18) then provides a model which approximately minimises the maximum value of the residual on \([0,T_f]\).

Alternative sets of basis functions are provided by considering the Fourier series of \(R(t)\). Thus choosing the standard inner product (3.13) and

\[
\psi_i(t) = \begin{cases} 
\cos \pi (i-1)t/T_f & i = 1,3,5,... \\
\sin \pi i t/T_f & i = 2,4,6,...
\end{cases}
\]  

(3.28)

and solving (3.18) forces the first \(N_p\) coefficients of the general Fourier series expansion of \(R(t)\) with period \(T_f\) to zero. Further alternative choices within this category are

\[
\psi_i(t) = \cos \pi (i-1)t/T_f \quad i = 1,2,....
\]  

(3.29)

which forces the first \(N_p\) coefficients of \(R(t)\), considered as an even function with period \(2T_f\), to zero and

\[
\psi_i(t) = \sin \pi i t/T_f \quad i = 1,2,....
\]  

(3.30)

which performs the corresponding role when \(R(t)\) is considered as an odd function with period \(2T_f\).

\textit{Least squares}

Since the inner product integrals which arise in this method have to be computed by a quadrature scheme, comparison of the fundamental problem formulations will be treated under the
least squares quadrature heading.

**Point and orthogonal collocation**

The limited number of free parameters in both the system identification and model reduction problems severely restricts the application of the point and orthogonal collocation methods since the residual is forced to zero at a minimal number of points. The advantage of the orthogonal collocation method is that the collocation points are picked automatically thus avoiding the arbitrary (and possibly poor) choice by the user.

**Least squares point collocation and least squares quadrature**

These methods are equally applicable to each of the three fundamental problems. The input/output data points are restricted in the system identification and model reduction problems and hence the values of input and output at quadrature rule collocation points will almost certainly need to be derived from interpolation of the measured data points. As suggested previously, there is a close similarity between least squares point collocation and least squares quadrature when the trapezoidal rule is used as the quadrature scheme in the latter technique. The least squares point collocation method has been previously considered in isolation (Whitfield and Messali [51]).

**3.3.3 Comparison of residuals**

At this stage we emphasise that problems of entirely different structure result from the use of equation and signal residuals when performing system identification and model reduction. The signal residual is derived from a simulation of the system subject to a prescribed input and initial conditions. Thus, even for a linear system, the signal residual will depend non-linearly upon the model parameters and all the weighted residual methods will require either the solution of a set of non-linear equations or a complex non-linear optimisation. In either case the resulting problem is generally of considerable magnitude. By contrast a linear system, or even linear-in-parameters non-linear system, yields an equation residual which is linear in the unknown model parameters and the weighted residual methods produce either a set of linear equations or a linear least-squares problem. In both cases the solution is unique and easily computed.

**3.3.4 Residual formulations for linear MIMO systems**

Weighted residual methods are now extended to include the MIMO system formulation of the three fundamental problems.

In the context of linear operator equation \( \mathbf{L}_d \{ x_2(t), u_1(t), ..., u_r(t) \} = 0, (\ell = 1, 2, ..., m) \), all
the weighted residual methods calculate the parameters \( \{a_i; i=1,2,...,Na; \ell=1,2,...,m\} \) in an approximate solution \( \hat{x}_G(t) \) where

\[
\hat{x}_G(t) = \sum_{i=1}^{N_a} \alpha_i \psi_i(t) \quad \ell = 1,2,...,m
\] (3.31)

and \( \{\psi_i(t); i=1,2,...,N_a\} \) are combinations of preselected basis functions such as monomials \( \{i^{i-1}; i=1,2,...,N_a\} \) or orthogonal polynomials \( \{P_{i-1}(t); i=1,2,...,N_a\} \), chosen so that \( \hat{x}_G(t) \) satisfies the initial conditions on \( x_G(t) \). The parameters \( \{a_i; i=1,2,...,Na; \ell=1,2,...,m\} \) are calculated by simultaneously forcing the residuals \( \{R_G(\ell); \ell=1,2,...,m\} \), which arise from the application of known linear operator to the approximate solution \( \hat{x}_G(t) \), i.e.

\[
R_G(\ell) = L(\hat{x}_G(t), u_1(t), ..., u_r(t)) \quad \ell = 1,2,...,m
\] (3.32)

close to zero over a prescribed time interval.

In the problems of system identification and model reduction the equation residuals \( \{R_G(\ell); \ell=1,2,...,m\} \) arise from the application of an approximate operator \( \hat{L} \) to the actual system inputs and outputs, i.e.

\[
R_G(\ell) = \hat{L}(x_G(t), u_1(t), ..., u_r(t)) \quad \ell = 1,2,...,m
\] (3.33)

and the parameters \( \{a_k, b_k; k=1,2,...,n; \ell=1,2,...,m; j=1,2,...,r\} \) in (3.5) or \( \{a_k, b_k, c_k; k=1,2,...,n; \ell=1,2,...,m; j=1,2,...,r\} \) in (3.6) are identified by forcing all the residuals \( \{R_G(\ell); \ell=1,2,...,m\} \) simultaneously close to zero.

As in the linear SISO system identification and model reduction problems, other approaches may be developed by considering the signal residual

\[
R_G(\ell) = x_G(\ell) - \hat{x}_G(\ell) \quad \ell = 1,2,...,m
\] (3.34)

where \( \{x_G(\ell); \ell=1,2,...,m\} \) are the system outputs and \( \{\hat{x}_G(\ell); \ell=1,2,...,m\} \) are the outputs predicted by the model when subject to same inputs and initial conditions.

In order to encompass the three basic problems in a one general formulation as for SISO systems where only a single residual is minimized by various weighted residual methods, we need
to formulate a single residual expression. One way is to square all the residuals and take the sum, i.e.

\[ R(\Omega, t) = \sum_{i=1}^{m} R_i^2(t) \]  \hspace{1cm} (3.35)

where \( \Omega \) has the components \( \{a_i; i=1,2,\ldots,N_\Omega; \ell=1,2,\ldots,m\} \) if the problem is that of finding an approximate solution to (3.5). In the case of system identification and model reduction problems \( \Omega \) has the components \( \{a_k, b_k, c_k; k=1,2,\ldots,n; \ell=1,2,\ldots,m; j=1,2,\ldots,r\} \) if the residual in (3.33) is formed by an approximate differential operator or \( \{a_k, b_k, c_k; k=1,2,\ldots,n; \ell=1,2,\ldots,m; j=1,2,\ldots,r\} \) if the residual is formed by an approximate integral operator.

Having chosen a single residual expression, the general formulation of the weighted residual methods is given by

\[ \text{solve} \quad <R(\Omega, t), \psi_i(t)> = 0 \quad i=1,2,\ldots \]  \hspace{1cm} (3.36)

where \( \{\psi_i(t); i=1,2,\ldots\} \) are weighting functions. Expression (3.36) is the same as (3.14) which was formulated for linear SISO systems, therefore all the previously weighted residual methods discussed in the context of SISO systems are equally applicable to MIMO systems.

We note that the single residual in (3.35) expression can be formed from either signal residual or equations residual.

### 3.4. Choice of input signal

Before we present the practical application of integral equation based weighted residual approaches to system identification, we raise an important question on the choice of input signals which are to be used for the identification purpose.

An interesting feature of the integral equation approach to system identification is that the set of input signals that can be applied to the system is limited and therefore must be carefully selected. Referring to the SISO linear system and its governing integral equation (3.3), the parameters \( \{c_k; k=1,2,\ldots,n\} \) arise solely from initial conditions and if all these conditions are known to be zero then we have \( c_k = 0 \) \((k=1,2,\ldots,n)\) and we need only estimate the parameters \( \{a_k, b_k; k=1,2,\ldots,n\} \). In such a situation the input \( u(t) \) must be chosen such that \( \{X(k)(t), U(k); k=1,2,\ldots,n\} \) form a set of \( 2n \) linearly independent functions of time in order to
avoid singularity problems. As a consequence, inputs such as a simple exponential function 
\[ u(t) = \exp(-\lambda t) \] may be ruled out, since \[ U^{(2)}(t) = -U^{(1)}(t)/\lambda, \] as may a simple sinusoid 
\[ u(t) = \sin(\omega t), \] since \[ U^{(3)}(t) = -U^{(1)}(t)/\omega^2. \]

In the most general identification problem we may not have zero initial conditions and the 
system integral equation demands that we must identify the full parameter set \( \{a_k, b_k, c_k; k=1,2,\ldots,n\} \). In such situations \( u(t) \) must be chosen such that 
\( \{X^{(k)}(t), U^{(k)}(t), t^{k-1}; k=1,2,\ldots,n\} \) forms a set of \( 3n \) linearly independent functions of time. 
Thus, while a step input is useful in the case of zero initial condition situation, it is actually 
eliminated from consideration as an input in the unknown initial condition situation since for 
\( u(t) = \gamma \) (a constant), \( U^{(0)}(t) = \gamma t^0/0! \) and the functions \( \{X^{(k)}(t), U^{(k)}(t), t^{k-1}; k=1,2,\ldots,n\} \) do not 
possess the requisite linear independence. Similarly one must also be careful not to choose input 
signals of polynomial form such as 
\[ u(t) = \sum_{j=0}^{\ell} \alpha_j t^j \quad (\ell = 1, 2, \ldots, n-2) \]
when the initial conditions are not zero for some integrals \( \{U^{(k)}(t); k=1,2,\ldots,n\} \) can be a linear 
combination of \( \{t^{k-1}; k=1,2,\ldots,n\} \). Inputs of the form 
\[ u(t) = \sum_{j=0}^{\ell} \alpha_j t^j \quad (\ell = 1, 2, \ldots, n+n_2-1) \]
where \( n_2 \) is the largest of the two polynomial disturbance orders (input disturbance and output 
disturbances) of the system are also prohibited. Some integrals of these signals can be a linear 
combination of some elements of the set \( \{t^{k-1}; k=1,2,\ldots,n+n_2\} \).

3.5. Numerical results

Before we document the results of applying several of the previously defined weighted residual 
techniques to various problems of system identification, we shall firstly identify the appropriate 
residual and then explain the numerical implementation of these methods.

3.5.1 Numerical implementation

The procedure for implementing MIMO system identification techniques is a simple extension 
of SISO identification techniques and therefore only SISO numerical implementation will be 
detailed here.
Since the system integral equation (2.4) holds at any point $t = t_i$ and the measured input and output are available at discrete points in time, we have:

$$\sum_{k=1}^{n} a_k Y^{(k)}(t_i) - \sum_{k=1}^{n} b_{k-1} R^{(k)}(t_i) + \sum_{k=1}^{n} c_k Q^{(k)}(t_i) - q(t_i) = \sum_{k=1}^{n} a_k P^{(k)}(t_i) = -y(t_i)$$

\[ (i = 0, 1, \ldots) \quad (3.37) \]

From the measured input $r(t_i)$ and the measured output $y(t_i)$ ($i=0,1,2,\ldots$), their respective multiple integrals $R^{(k)}(t_i)$ and $Y^{(k)}(t_i)$ can be evaluated by a suitable quadrature scheme. A particularly simple and computationally efficient formulation is available if the trapezoidal integration rule is used with equidistant sampling in time; this is usually the case in digital sampling. Thus if $t_i = i\Delta T (i=0,1,2,\ldots)$ then we have:

$$R^{(k)}(t_i) = R^{(k)}(t_{i-1}) + [R^{(k-1)}(t_i) + R^{(k-1)}(t_{i-1})] \Delta T/2$$

$$Y^{(k)}(t_i) = Y^{(k)}(t_{i-1}) + [Y^{(k-1)}(t_i) + Y^{(k-1)}(t_{i-1})] \Delta T/2$$

\[ \text{for } i \geq 1 \text{ and } k \geq 1 \]

where

$$R^{(k)}(t_0) = 0, \quad Y^{(k)}(t_0) = 0 \quad \text{for } k > 0$$

$$R^{(0)}(t_i) = r(t_i), \quad Y^{(0)}(t_i) = y(t_i) \quad \text{for } i \geq 0$$

In (3.37) the values of the multiple integrals $(Y^{(k)}(t_i), R^{(k)}(t_i); k = 1,2,\ldots,n; i = 0,1,\ldots)$ are therefore quickly computed and the values $(t_i^{(k-1)}/(k-1)!; k = 1,2,\ldots,n)$ are known via the sampling time instants. Once a residual has been created identification of the system parameters of the system parameters $[a_k, b_k, c_k; k = 1,2,\ldots,n]$ can be carried out using the previously described weighted residual techniques in situations which are distinguished by the degree to which knowledge of the disturbances is available (qualitative or quantitative knowledge as previously described).

Equation (3.37) is a valid description of our system (as shown by Fig. 2.2.) assuming that there is no noise in the measured input/output data and there is no error due to the quadrature scheme used in the integration formulation. However in a practical situation the measurements may be corrupted by noise and the quadrature scheme of integration will be subject to errors and therefore a residual is generated:
where $\Omega$ is the parameter vector $(a_k, b_k, c_k; k=1,2,\ldots,n)^T$ to be identified.

We are now in a position to document the implementation of the weighted residual techniques used to identify the vector $\Omega$. In order to simplify the analysis we shall consider the case where the disturbances are unknown polynomials. A similar analysis can be performed for other disturbances. With polynomial disturbances equation (3.38) becomes:

$$
R(\Omega, t) = y(t) + \sum_{k=1}^{n} a_k y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t^{k-l}}{(k-l)!} - q(t) - \sum_{k=1}^{n} a_k Q^{(k)}(t) - \sum_{k=1}^{n} b_k P^{(k)}(t)
$$

(3.39)

Where $n_s$ is the largest of the two polynomial $q(t)$ and $p(t)$ orders, as in equation (2.9), and $\Omega$ is the parameter vector $\Omega = [a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n, d_1, \ldots, d_{n+n_s}]^T$.

Least squares:

As described by (3.19), the least squares method entails minimizing of the sum of the residuals squared over the observation time interval $[0,T_f]$ i.e.

$$
\text{min}_{\Omega} \langle R(\Omega, t), R(\Omega, t) \rangle
$$

Thus using the standard inner product previously defined by (3.13) as:

$$
\langle g_1(t), g_2(t) \rangle = \int_0^{T_f} g_1(t) g_2(t) dt
$$

the least squares method is equivalent to:

$$
\text{min}_{\Omega} \int_0^{T_f} R^2(\Omega, t) dt
$$
Since the data is observed at discrete points in time $t_i$ ($i = 0,1,\ldots,N$) and the integral equation residual is therefore known only at such points in time we may approximate the least squares method by the least squares point collocation scheme (3.22), i.e.

$$
\min_\Omega \int_0^{T_f} R(\Omega, t) \, dt = \min_\Omega \sum_{i=0}^{N} R(\Omega, t_i)
$$

(3.40)

where $t_N = T_f$, the approximation arising when the trapezoidal rule is used as a quadrature scheme to evaluate the integral of $R(\Omega, t)$ between 0 and $T_f$ as shown in § 3.3.1. The $(N+1)$ residuals are given by imposing equation (3.39) at $(N+1)$ points in time $t_i$ ($i = 0,1,\ldots,N$), thereby giving rise to a set of $(N+1)$ algebraic equations in the system parameters.

\[
\begin{bmatrix}
R(\Omega, t_0) & y(t_0) \\
R(\Omega, t_1) & y(t_1) \\
\vdots & \vdots \\
R(\Omega, T_f) & y(T_f)
\end{bmatrix}
\begin{bmatrix}
_Y(1)(t_0) \ldots -Y(n)(t_0) R(1)(t_0) \ldots R(n)(t_0) t_0^m' \ldots 1 \\
_Y(1)(t_1) \ldots -Y(n)(t_1) R(1)(t_1) \ldots R(n)(t_1) t_1^m' \ldots 1 \\
\vdots & \vdots \\
_Y(1)(T_f) \ldots -Y(n)(T_f) R(1)(T_f) \ldots R(n)(T_f) T_f^m' \ldots 1
\end{bmatrix}
\begin{bmatrix}
a_1 \\
\vdots \\
a_n \\
b_1 \\
\vdots \\
b_n \\
d'_1 \\
\vdots \\
d'_{m'+1}
\end{bmatrix}
\]

(3.41)

Where $m'=n+n_2-1$ and $d_k' = -d_k/(k-1)!$; $(k=1,2,\ldots,m'+1)$. In the matrix form we have:

$$
R(\Omega) = y - M\Omega
$$

(3.42)

Replacing the residual in (3.40) we get:

$$
\min_\Omega \sum_{i=0}^{N} R(\Omega, t_i) = \min_\Omega R^T(\Omega) R(\Omega) = \min_\Omega (y - M\Omega)^T(y - M\Omega)
$$

or

$$
\min_\Omega \sum_{i=0}^{N} R(\Omega, t_i) = \min_\Omega y^T y - 2y^T M y + y^T M^2 y
$$

(3.43)

The solution of this equation is given by (3.19), i.e.
\[
\text{solve } \frac{\partial}{\partial \Theta} [y^T (\chi - \Theta M \chi - y^T M \Theta + \Theta M^T M \Theta) \theta = 0
\]

which has analytic solution

\[
\Theta = (M^T M)^{-1} M^T y
\]

Such analytical solution is available for any model which is linear in parameters and an application to a linear-in-parameters nonlinear model will be given in the examples.

Method of moments:

The method of moments is the solution of the set of algebraic equations in the \(N_p\) (\(N_p = 3n + n_s\)) parameters as defined by (3.8), i.e.

\[
\text{solve } \langle \Phi, i \rangle, t^i - 1 = 0 \quad i = 1, 2, ..., N_p
\]

which, using the standard inner product (3.13) and the residual (3.39) is equivalent to:

\[
\int_0^T \left[ \sum_{k=1}^n a_k y^{(k)}(t) - \sum_{k=1}^{n+n_s} b_k R^{(k)}(t) + \sum_{k=1}^{n+n_s} d_k \frac{t^{(k-1)}}{(k-1)!} \right] t^i - 1 \, dt = 0 \quad (i=1, 2, ..., N_p)
\]

or

\[
\int_0^T \left[ -\sum_{k=1}^n a_k y^{(k)}(t) + \sum_{k=1}^{n+n_s} b_k R^{(k)}(t) - \sum_{k=1}^{n+n_s} d_k \frac{t^{(k-1)}}{(k-1)!} \right] t^i - 1 \, dt = \int_0^T y(t) t^i - 1 \, dt \quad (i=1, 2, ..., N_p)
\]

The result of the operation is the solution of the following set of equations in the matrix form:

\[
M \Phi = \xi
\]

Where \(M\) is the partitioned matrix \(M = [M_1, M_2, M_3]\) and
\[ M_1 = \begin{bmatrix}
<y(1), 1> & <y(2), 1> & \ldots & <y(n), 1> \\
<y(1), t> & <y(2), t> & \ldots & <y(n), t> \\
\vdots & \vdots & \ddots & \vdots \\
<y(1), t^{N_p-1}> & <y(2), t^{N_p-1}> & \ldots & <y(n), t^{N_p-1}>
\end{bmatrix} \]

\[ M_2 = \begin{bmatrix}
<R(1), 1> & <R(2), 1> & \ldots & <R(n), 1> \\
<R(1), t> & <R(2), t> & \ldots & <R(n), t> \\
\vdots & \vdots & \ddots & \vdots \\
<R(1), t^{N_p-1}> & <R(2), t^{N_p-1}> & \ldots & <R(n), t^{N_p-1}>
\end{bmatrix} \]

\[ M_3 = \begin{bmatrix}
<-1, 1> & <-t, 1> & \ldots & <-t^{m'}, 1> \\
<-1, t> & <-t, t> & \ldots & <-t^{m'}, t> \\
\vdots & \vdots & \ddots & \vdots \\
<-1, t^{N_p-1}> & <-t, t^{N_p-1}> & \ldots & <-t^{m'}, t^{N_p-1}>
\end{bmatrix} \]

\[ \xi = \{ <y(t), 1> <y(t), t> \ldots <y(t), t^{N_p-1}> \}^T, \quad \Omega = [a_1 \ a_2 \ \ldots \ a_n \ b_1 \ b_2 \ \ldots \ b_n \ d_1' \ d_2' \ \ldots \ d_{m'+1}']^T \]

\[ d'_k = d_k/(k-1)!, \quad (k=1, 2, \ldots, m'+1). \]

Knowing \( y(t) \) and \( r(t) \) and their respective multiple integrals at equidistant discrete points in time on the interval \( [0, T_f] \) the inner products are numerically evaluated using the seventh order Newton-Cotes scheme, i.e.

\[ <R(\xi, t), t^{i-1}> = \int_0^{T_f} R(\xi, t) \cdot t^{i-1} dt \]

\[ = \int_0^{6\Delta T} R(\xi, t) \cdot t^{i-1} dt + \int_0^{12\Delta T} R(\xi, t) \cdot t^{i-1} dt + \ldots + \int_0^{T_f} R(\xi, t) \cdot t^{i-1} dt \]

\[ (i=1, 2, \ldots, N_p). \]

Where
\[
\mathcal{T} = 6 \Delta T \\
\int_{t_k}^{t_{k+1}} R(\zeta, t) \, dt = \frac{6 \Delta T}{840} [41 \mathcal{R} \left( \theta, t_{k+3} \right) t_{k+3}^{i-1} + 216 \mathcal{R} \left( \theta, t_{k+4} \right) t_{k+4}^{i-1} + 27 \mathcal{R} \left( \theta, t_{k+5} \right) t_{k+5}^{i-1} + \\
272 \mathcal{R} \left( \theta, t_{k+6} \right) t_{k+6}^{i-1} + 27 \mathcal{R} \left( \theta, t_{k+6} \right) t_{k+7}^{i-1} + 216 \mathcal{R} \left( \theta, t_{k+7} \right) t_{k+7}^{i-1} + 41 \mathcal{R} \left( \theta, t_{k+8} \right) t_{k+8}^{i-1} ]
\]

with \( t_{k+j} = t_k + j \Delta T \).

**Galerkin Method:**

The implementation of Galerkin's method is similar to that of the time moments except that the monomials \( \{ t_{i-1}; i=1,2,\ldots,N \} \) are replaced by a complete set of functions \( \{ \psi_i(t); i=1,2,\ldots,N \} \) on \([0, T]\). The matrices \( M_1, M_2 \) and \( M_3 \) of the partitioned matrix \( M \) of equation (3.48) and the right hand side \( \xi \) become:

\[
\begin{bmatrix}
-<Y^{(1)}(t), \psi_1(t)> & <Y^{(2)}(t), \psi_1(t)> & \cdots & <Y^{(n)}(t), \psi_1(t)> \\
-<Y^{(1)}(t), \psi_2(t)> & <Y^{(2)}(t), \psi_2(t)> & \cdots & <Y^{(n)}(t), \psi_2(t)> \\
\vdots & \vdots & \ddots & \vdots \\
-<Y^{(1)}(t), \psi_N(t)> & <Y^{(2)}(t), \psi_N(t)> & \cdots & <Y^{(n)}(t), \psi_N(t)>
\end{bmatrix}
\]

\[
\begin{bmatrix}
-<R^{(1)}(t), \psi_1(t)> & <R^{(2)}(t), \psi_1(t)> & \cdots & <R^{(n)}(t), \psi_1(t)> \\
-<R^{(1)}(t), \psi_2(t)> & <R^{(2)}(t), \psi_2(t)> & \cdots & <R^{(n)}(t), \psi_2(t)> \\
\vdots & \vdots & \ddots & \vdots \\
-<R^{(1)}(t), \psi_N(t)> & <R^{(2)}(t), \psi_N(t)> & \cdots & <R^{(n)}(t), \psi_N(t)>
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1, \psi_1(t) & -t, \psi_1(t) & \cdots & -t^{m'}, \psi_1(t) \\
-1, \psi_2(t) & -t, \psi_2(t) & \cdots & -t^{m'}, \psi_2(t) \\
\vdots & \vdots & \ddots & \vdots \\
-1, \psi_N(t) & -t, \psi_N(t) & \cdots & -t^{m'}, \psi_N(t)
\end{bmatrix}
\]
\[ \xi = [\langle y(t), \psi_1(t) \rangle \langle y(t), \psi_2(t) \rangle \ldots \langle y(t), \psi_{N_p}(t) \rangle]^T. \]

The orthogonal polynomials used in the numerical applications are the shifted Legendre polynomials \( \psi_i(t) = P_i(t); \ i = 1, 2, \ldots, N_p \) and the shifted Chebyshev polynomials \( \psi_i(t) = T_{i-1}(t); \ i = 1, 2, \ldots, N_p \) described in Appendix C. For the results quoted Galerkin (general Fourier), Galerkin (even Fourier) and Galerkin (odd Fourier) the functions \( \psi_i(t) \) are given by equations (3.28), (3.29) and (3.30) respectively.

We note that in the case of Galerkin (Chebyshev) method, the inner product

\[
\langle \mathfrak{R}((\Omega, t), T_{i-1}^*)(t) = \frac{1}{2} \int_0^T \mathfrak{R}((\Omega, t), T_{i-1}^*)(t) \, dt
\]

cannot be evaluated using the seventh order Newton-Cotes formula since the integral is singular at both its limits, i.e. at \( t = 0 \) and \( t = T_f \); a change of variable is therefore necessary. For our numerical results we substitute:

\[
\cos \tau = \frac{2t}{T_f} - 1 \quad \text{or} \quad \tau = \arccos \left( \frac{2t}{T_f} - 1 \right)
\]

and the inner product becomes

\[
\langle \mathfrak{R}((\Omega, t), T_{i-1}^*)(t) = \frac{T_f}{4} \int_0^T \mathfrak{R}((\Omega, t), T_{i-1}^*)(t) \, dt
\]

Since the data is given at equidistant points in time \( \Delta T = \text{constant} \), the quadrature scheme used perform the integral is the trapezoidal rule with a variable step size \( \Delta \tau \) given by

\[
\Delta \tau = \arccos \left( \frac{2(t+\Delta T)}{T_f} - 1 \right) - \arccos \left( \frac{2t}{T_f} - 1 \right)
\]

The numerical results given by the Galerkin (Chebyshev) method and those given by Galerkin (Legendre) are expected to be potentially different because of the difference in the integration
scheme and the different residual weighting.

For this example of polynomial disturbances, if only the system parameters \( \{a_k, b_k: k=1,2,...,n\} \) need to be identified, i.e. neither the system initial conditions nor the quantitative nature of the disturbances are of interest, equation (3.48) can be simplified by only higher order shifted Legendre or Chebyshev polynomials as weighting functions. Specially we use the formulation

\[
\text{solute } <R(\theta, t), \psi_i(t)> = 0 \quad i = m'+2, m'+3, \ldots, m'+N_p + 1
\]

where \( N_p = 2n \). The elements of matrix \( M_3 \) become identically zero since

\[
<P^*_k(t), t> = 0 \quad \forall \, \ell < k
\]

and

\[
<T^*_k(t), t> = 0 \quad \forall \, \ell < k
\]

where \( P^*_k(t) \) and \( T^*_k(t) \) are the shifted Legendre and Chebyshev polynomials respectively. For a proof see Appendix C.

3.5.2 Numerical data

Unless otherwise stated the data used for identification purposes is generated by subjecting the system model under consideration to an input and prescribed initial conditions and integrating the corresponding first ordinary differential equations using the well known Runge-Kutta fourth order scheme. The form chosen for the set of first ordinary differential equations is the canonical observable form which using the analysis of §2.3, allows us to directly compare the initial conditions under which the model was simulated and the initial conditions \( \{c_k: k=1,2,\ldots,n\} \) identified by the various methods.

Where analogue to digital (A/D) conversion is quoted, the signal minima and maxima have been scaled to full range on \([0,4095]\) with intermediate values being taken from the nearest integer of the scaled signal within that range. Such 12-bit A/D conversion incurs a measurement error of approximately 0.025%. When an offset value is quoted such an offset was imposed on the true system output but was presumed unknown as far as the identification was concerned. No offset indicates a knowledge of zero offset and therefore identification of any offset value was unnecessary.
3.5.3 Linear SISO systems

Example 1.

A first-order system with \(a_1 = b_1 = 1\), i.e. \(\dot{x} + x = u\), was used as a vehicle to show the effectiveness of the trapezoidal integration scheme for calculating the values of the multiple integrals and the influence of integration step size \(\Delta T\) upon parameter estimation. With initial conditions and disturbances at input and output known to be zero, only the parameters \(a_1\) and \(b_1\) need be estimated. Figure 3.1 shows the variation of those estimates with step size \(\Delta T\) time units for a total observation time of \(T_f = 5\) units when the input to the plant is \(r(t) = 2t/(1+2t)\).

For the given system, which has a unit time constant, acceptable estimation occurs for \(\Delta T = 0.1\) with improved accuracy being attained by even shorter sampling intervals. The same first-order system is also used to illustrate the effect of 12-bit A/D quantization on parameter estimation. With both input and output being passed into a 12-bit A/D (full scale range -5.12 V to 5.12 V) without rescaling and sampling parameters \(\Delta T = 0.01\), \(N=300\), \(T=3\), the results shown in Table 3.1 were obtained. Quantization clearly reduces the accuracy of the estimates, though by a tolerable degree.

Identified model

<table>
<thead>
<tr>
<th>System</th>
<th>Without A/D</th>
<th>With A/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>(b_1)</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 3.1.

Example 2.

Here we consider a system with transfer function

\[
F(s) = \frac{X(s)}{U(s)} = \frac{s + 15}{s^2 + 10s + 21}
\]

which has an equivalent state space description.
Fig. 3.1. Effect of $\Delta T$ on parameter estimation.

(System $\dot{x} + x = u$, i.e. $a_1=1$, $b_1=1$.)
\[ \dot{z}_1(t) = -21 z_2(t) + 15 u(t) \]
\[ \dot{z}_2(t) = z_1(t) - 10 z_2(t) + u(t) \]
\[ x(t) = z_2(t) \]

We firstly consider the effect of initial conditions on the system parameter estimates. The system model was subject to initial conditions \( z_1(0) = 1, z_2(0) = 2 \) and an input \( u(t) = \sin t \). Discrete data was collected on the time interval \([0, 2] \) with \( N = 200 \) sample points (i.e. \( \Delta T = 0.01 \)). With no noise contamination the methods of least squares, time moments and Galerkin (i.e. Legendre, Chebyshev, even, general and odd Fourier) all identified the system transfer function as

\[
\frac{1.000s + 14.992}{s^2 + 9.996s + 20.989}
\]

and the parameters reflecting the initial conditions were identified as \( c_1 = -2, c_2 = -1.002 \). The same system model was simulated with different initial conditions which were: \( z_1(0) = 4, z_2(0) = 5 \), but with the same input \( u(t) = \sin t \) and same number of sampling points (\( N = 200 \)). The transfer function identified by all the methods was

\[
\frac{1.000s + 14.992}{s^2 + 9.996s + 20.989}
\]

and the parameters \( c_1, c_2 \) were identified as \( c_1 = -4.006, c_2 = -5.000 \). The above two examples indicate that the initial conditions have no effect on the system parameter estimates. Since in both situation the parameters and the initial conditions were accurately identified. The small error in the estimated parameters occurs as a result of sampling the continuous input/output signals at discrete points in time.

For the same system and under the same conditions as above (i.e. \( z_1(0) = 1, z_2(0) = 2 \)), we consider the effect on the parameter estimates of eliminating the initial conditions in the final set of algebraic equations produced by choosing the Galerkin weighted residual method

\[
\text{solve } \langle R(\phi, t), \psi_i(t) \rangle = 0 \quad \text{for } i = 3,4,5,6
\]

where \( \psi_i(t) = \Phi_{i-1}(t) \) are the shifted Legendre polynomials. The elements of matrix \( M_3 \) of equation (3.48) are identically zero and the final set of equations to be solved is
The transfer function was identified as

\[
\frac{1.000s + 14.992}{s^2 + 9.996s + 20.989}
\]

This suggests that computational effort can be simplified by selecting high order shifted orthogonal polynomials without loss of accuracy in the parameter estimates. We note that the shifted Chebyshev polynomials could have been used for the same purpose.

We now examine the effect of the input signal on the estimated parameters. The same system model was subjected to initial conditions \( z_1(0) = 1, z_2(0) = 2 \), but this time to a different type of input signal, i.e. \( u(t) = 2t/(1 + 2t) \). The number of sampling points was also the same number of data points (\( N = 200 \)). Under similar conditions (i.e. no noise contamination) the same methods identified the model as

\[
\frac{1.000s + 14.993}{s^2 + 9.997s + 20.990}
\]

and the rest of the parameters as \( c_1 = -2, c_2 = -1.003 \). We notice that the parameter estimates and the initial conditions identified using the two different inputs \( u(t) = \sin t \) and \( u(t) = 2t/(1+2t) \) are almost identical.

The latter system was also used to account for unquantified deterministic disturbances in the identification procedure. The system, subject to initial conditions \( x(0) = 2, \dot{x}(0) = -19 \) (i.e. \( z_1(0) = 1, z_2(0) = 2 \)), was stimulated by an input \( r(t) = \sin t \), and the measured output \( y(t) \) was taken as \( y(t) = x(t) + 3 \). The identification scheme (2.10) was implemented wherein it is assumed that the system has a constant offset at the output though the value of the offset is unknown. The choice of \( r(t) = \sin t \) does not cause any problems in the identification of second-order systems since only the integrals \( R^{(1)}(t) = 1 - \cos t \) and \( R^{(2)}(t) = t \sin t \) are involved in the overdetermined equations columns, and together with the columns \( Y^{(1)}(t), Y^{(2)}(t), 1, t, t^2/2! \) they form a linearly
independent set of functions. Such an input could not, however, be used for a third-order system identification since \( R^{(3)}(t) = t^2/2! + \cos t - 1 \) is a linear combination of \( R^{(1)}(t) \) and \( t^2/2! \). However, an input of the form \( r(t) = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t \) would suffice. Following the notation of (2.10) with \( d_3 \) as the coefficient of \( t^2/2! \) and sampling with \( \Delta T = 0.01 \), \( N = 700 \), \( T = 7 \) we obtain the results shown in Table 3.2. The least squares identification procedure (3.45) was used to estimate the parameters. The value of the unquantified offset is evaluated from \( d_3 = a_2 q_0 \) as \( q_0 = 3.001 \) with full floating-point data and \( q_0 = 3.000 \) with 12-bit digitized data. Again the identification procedure yields accurate parameter estimates.

<table>
<thead>
<tr>
<th>Identification model</th>
</tr>
</thead>
<tbody>
<tr>
<td>System</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>( n )</td>
</tr>
<tr>
<td>( a_1 )</td>
</tr>
<tr>
<td>( a_2 )</td>
</tr>
<tr>
<td>( b_1 )</td>
</tr>
<tr>
<td>( b_2 )</td>
</tr>
<tr>
<td>( d_3 )</td>
</tr>
</tbody>
</table>

Table 3.2.

When both input \( r(t) = 2t/(1 + 2t) \) and output signals were subject to 12-bit A/D conversion the results shown in table 3.3 were obtained. The system was subject to initial conditions \( x(0) = 2 \), \( \dot{x}(0) = -19 \), and data was collected on the time interval \([0,7]\) with \( N=700 \) sample points. Clearly the presence of quantisation error and unknown observation offset have little effect on parameter estimation for the least squares approach and the polynomial based approaches of the method of moments and Galerkin (Legendre and Chebyshev) though the Fourier approaches are somewhat less robust. This will often be the case since Legendre and Chebyshev polynomial expansions of the integral equation residual will generally converge quicker than a Fourier series expansion.
### 3.5.4 A linear MIMO system

A three-output two-input linear system with transfer function matrix

\[
G(s) = \begin{bmatrix}
\frac{3}{s+4} & \frac{2.7(s+5)}{(s+4)(s+0.6)} \\
\frac{4}{(s+4)(s+0.6)} & \frac{1.2}{s+0.6} \\
\frac{3.5(s+1.0)}{(s+4)(s+0.6)} & \frac{0.5(s+10)}{(s+4)(s+0.6)}
\end{bmatrix}
\]
was stimulated by inputs $u_1(t) = 20 \sin 0.5t, \ u_2(t) = 25 \sin t$ and subject to initial conditions $x_1(0) = 4.0, \ \dot{x}_1(0) = -15.3, \ x_2(0) = 2, \ \dot{x}_2(0) = -7.8, \ x_3(0) = 1.0, \ \dot{x}_3(0) = -4.0$. Offsets of 5, 1 and 0 units were added to the actual outputs to create three measurements $y_1(t), \ y_2(t), \ y_3(t)$. The extension of the least squares scheme (3.42) to MIMO integral equation (3.6) was implemented to identify a total of 23 unknowns, and using $\Delta T = 0.01, \ N = 900, \ T_f = 9$ for each measurement the pertinent parameters lead to the following identified transfer function matrices:

**exact data:**

$$\frac{1}{(s + 3.999)(s + 0.6000)} \begin{bmatrix} 2.999(s + 0.6000) & 2.7(s + 4.999) \\ 0.00015(s + 2666.0) & 1.2(s + 3.999) \\ 3.4996(s + 1.0000) & 0.5001(s + 9.996) \end{bmatrix}$$

**12-bit A/D data**

$$\frac{1}{(s + 4.013)(s + 0.5998)} \begin{bmatrix} 3.003(s + 0.601) & 2.700(s + 5.015) \\ -0.003(s - 1337.0) & 1.199(s + 4.014) \\ 3.507(s + 1.001) & 0.498(s + 10.07) \end{bmatrix}$$

Clearly the exact data yield virtually exact identification once the appropriate pole-zero cancellations and the negligible term 0.00015s are accounted for. The 12-bit A/D sampled data yields slightly less accurate identification.

3.5.5 A linear-in-parameters non-linear system

The unforced rolling motion of a ship may be approximately described (Roberts [49] a,b) by the normalized equation

$$\ddot{x} + (a_1 + n_1 x) \dot{x} + (a_2 + n_2 x^2) x = 0 \quad (3.49)$$

Estimation of the parameters $a_1, \ a_2, \ n_1, \ n_2$ is to be considered when the roll angle $x(t)$ is measured directly and in the presence of a unknown offset; Gawthrop [48] has previously considered this problem.
Double integration of (3.49) gives the integral equation for \( x(t) \) as

\[
a_1 x^{(1)}(t) + a_2 x^{(2)}(t) + n_1 \int_0^t \int \dot{x}(\tau_1) \dot{x}(\tau_1) d\tau_1 d\tau_2 + n_2 \int_0^t x^3(\tau_1) d\tau_1 d\tau_2 - x(0) - t\dot{x}(0) - a_1 x(0) = -x(t)
\]

In the presence of an unknown offset \( q_0 \), the measured roll angle is \( y(t) = x(t) + q_0 \), and the associated integral equation for \( y(t) \) is

\[
a_1 y^{(1)}(t) + a_2 y^{(2)}(t) + n_1 \int_0^t \int \dot{y}(\tau_1) \dot{y}(\tau_1) d\tau_1 d\tau_2 + n_2 \int_0^t y^3(\tau_1) d\tau_1 d\tau_2 + y(0) = \gamma(t)
\]

where

\[
\begin{align*}
\alpha_1 &= a_1, \\
\alpha_2 &= a_2 + 3q_0n_2 \\
\gamma_1 &= n_1, \\
\gamma_2 &= n_2, \\
\gamma_3 &= -3q_0n_2 \\
d_1 &= -x(0) - q_0, \\
d_2 &= -a_1 x(0) - \dot{x}(0) - a_1 q_0, \\
d_3 &= -a_2 q_0 - n_2 q_0^3
\end{align*}
\]

It is noticeable that the cubic non-linearity in \( x(t) \) gives rise to cubic and quadratic nonlinear expressions for \( y(t) \), together with contributions to linear and initial condition coefficients. Equation (3.50) can be imposed at points in time \( t_i \) \((i=0,1,...,N)\) to form a set of \( N+1 \) equations in the unknowns \([a_1, a_2, \gamma_1, \gamma_2, d_1, d_2, d_3]\) and least squares solutions then yield values of these unknowns. Parameter and initial-condition identification follows from equations such as

\[
\begin{align*}
a_1 &= \alpha_1, \\
n_1 &= \gamma_1, \\
n_2 &= \gamma_2 \\
q_0 &= \gamma_3/3n_2 \\
a_2 &= \alpha_2 - 3q_0n_2 \\
x(0) &= -d_1 - q_0, \\
x(0) &= -d_1 - q_0
\end{align*}
\]

though this latter equation set is not unique (e.g. \( a_2 \) could be calculated from \( a_2 = (-d_3 - a_2 q_0^3)/q_0)\).
With parameter values $a_1 = 0.02, a_2 = 1.00, n_1 = 0.01, n_2 = 0.0002$ and initial conditions $x(0) = 40, \dot{x}(0) = 0.1$, the latter system produces a transient as shown in Fig. 3.2. For identification purposes, discrete samples were taken with $\Delta T = 0.01, N = 900$, $T = 9$ in situations of zero and 5 units offset. Two further data sets were

<table>
<thead>
<tr>
<th>System</th>
<th>With exact data</th>
<th>With A/D sampled data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Offset = 0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_1$</td>
<td>0.02</td>
<td>0.0199</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1.00</td>
<td>0.9999</td>
</tr>
<tr>
<td>$n_1$</td>
<td>0.01</td>
<td>0.0100</td>
</tr>
<tr>
<td>$n_2$</td>
<td>0.0002</td>
<td>0.0002</td>
</tr>
<tr>
<td>$n_3$</td>
<td>0</td>
<td>0.0000</td>
</tr>
<tr>
<td>$d_1$</td>
<td>-40.0</td>
<td>-40.00</td>
</tr>
<tr>
<td>$d_2$</td>
<td>-0.9</td>
<td>-0.8976</td>
</tr>
<tr>
<td>$d_3$</td>
<td>0</td>
<td>-0.0007</td>
</tr>
</tbody>
</table>

Table 3.3

created by 12-bit full range A/D sampling of the latter data sets. The results shown in Table 3.3. and table 3.4. were obtained.

The exact data give virtually exact identification, while the 12-bit A/D sampled data are marginally worse, the principle source of error in the latter case being the numerical differentiation of $y$ required to form $\dot{y}$. Gawthrop [48] has reported reasonable parameter estimation (e.g. $a_1 = -0.0232$) using exact data with $\Delta T = 0.01, N = 5000$ and $T = 50$, and concluded that inaccurate initial condition identification (e.g. $x(0) = 43.608$) may be due to an identifiability problem. However, even the severely limited data set used above gives accurate parameter and initial condition identification and there is apparently no identifiability problem inherent in the system.
Fig. 3.2. Unforced rolling motion of a ship.
3.6. Summary

A unified operator representation of continuous systems dynamics has been presented. Three fundamental problems of differential equation solution, system identification and model reduction have been defined and several weighted residual methods for solving these problems have been presented. The integral equation formulation has been used with various weighted residual methods for parameter identification in continuous linear SISO, MIMO and linear-in-parameters nonlinear systems. The integral equation for linear systems can be formulated in generality which is not the case for linear-in-parameters nonlinear systems which require careful individual formulation. Effects of disturbances at both input and output can be included in the analysis. Parameters that arise naturally from non-zero initial conditions must also be estimated in the procedure. However, if Galerkin method with high order shifted Legendre or Chebychev polynomials is used the effect of non-zero initial conditions will be eliminated from the equation residual and therefore the corresponding parameters need not be identified. This also serves the purpose of eliminating the effect of additive polynomial disturbances from the integral equation residual. Thus large simplification in computational effort can be made by using Galerkin method for solving identification problems. Excellent results were obtained following the application of various weighted residual methods to problems of identifying SISO, MIMO and nonlinear system
parameters in the presence of offsets and measurement errors caused by 12-bit analogue to digital conversion of data. A feature of the procedure is that care must be taken in the choice of test input signal, with certain standard inputs such as impulses, steps and ramps being excluded from consideration in the presence of non-zero initial conditions. Care must also be taken in selecting the step size when using the trapezoidal quadrature scheme to compute the multiple integrals of data. A small step size would result in better estimates of the system parameters since using such steps enables us to pick up more information from the fast decaying modes of the system response. However this would result in using even larger amount of data points since information is also needed about the slow transient modes and the steady state part of the system response.
CHAPTER 4.
4. MODEL ORDER REDUCTION

The integral equation based weighted residual approaches to system identification discussed in the previous chapter will now be extended to the problem of continuous linear system model reduction. Using observed input/output data of the original linear high order system, a lower order model, which conveys the important dynamics of the system under consideration, will be determined. The various weighted residual methods which identify the reduced model are illustrated by numerical examples.

4.1. Introduction and problem statement

In many engineering investigations the complexity of a control system or a filter is often a primary consideration. Any solution designed without regard for its practical implementation may often be judged as unsatisfactory because the required calculations cannot always be made in the time available. Satisfactory solutions can be achieved by simplifying the mathematical model under study, i.e. by reducing the order of the ordinary differential equation describing the system or its components, or by reducing the equivalent dimension of its state vector. The problem can be addressed in two different ways. The first is by defining a desired structure (i.e. a structure of predefined order) for the controller or filter and then optimizing the free parameters with respect to a given performance index. This method has been used to design filters having less complexity than say Kalman filters and is known as reduced-order-filtering. The second way of addressing the problem is by firstly reducing the model of the physical system that is actually of high order, and then designing a filter or controller based on the reduced model thereby obtained. This method is referred to as reduced-order-modelling. Our concern is not to discuss which of the methods leads to an easy and better solution to the design problem, but is to consider how a reduced model of a plant which could be a filter, a controller or a system to be controlled or simulated can be obtained.

One may construct a reduced model using several techniques as said in §1.2.4 but most of those approaches require the availability of the high order model and large amount of computational effort. The approach which will be used here is based on the measured input/output data and the integral equation description of the system. First, a desired structure of the reduced model is fixed then the model parameters are determined by forcing an equation residual to zero on the observation time interval. The advantage of using input/output data is that a reduced order model can be obtained without explicitly identifying the high order system. Here it is assumed that original
system is of high order \( n_s \) and is described by the ordinary differential equation

\[
\begin{align*}
x^{(n_s)}(t) + a_1 x^{(n_s-1)}(t) + \cdots + a_{n_s} x^{(1)}(t) + b_1 u^{(n_s-1)}(t) + \cdots + b_{n_s} u(t) &= 0 \tag{4.1}
\end{align*}
\]

or by the equivalent transfer function

\[
\frac{X(s)}{U(s)} = \frac{b_1 s^{n_s-1} + \cdots + b_{n_s}}{s^{n_s} + a_1 s^{n_s-1} + \cdots + a_{n_s}}
\]

This system is to be approximated by a model of lower order \( n < n_s \):

\[
\begin{align*}
\hat{x}^{(n)}(t) + a_1 \hat{x}^{(n-1)}(t) + \cdots + a_n \hat{x}(t) &= b_1 u^{(n-1)}(t) + \cdots + b_n u(t) \tag{4.2}
\end{align*}
\]

or

\[
\frac{\hat{X}(s)}{U(s)} = \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}
\]

such that the residual \( R(t) = x(t) - \hat{x}(t) \) is forced close to zero on the observation time interval \([0, T_f]\), where \( x(t) \) is the output of the original model, \( \hat{x}(t) \) is the output predicted by the reduced model when subject to the same input \( u(t) \), and \( \Theta \) is the vector parameter \( [a_k, b_k; k=1,2,\ldots,n]^T \) to be determined.

4.2. Reduction to identification problem

As stated previously, under normal operating conditions a linear SISO system shown in Fig.2.2 can be represented by the equivalent integral equation

\[
y(t) + \sum_{k=1}^{n} a_k Y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} d_k (q(t) - \sum_{k=1}^{n} a_k Q^{(k)}(t) - \sum_{k=1}^{n} b_k P^{(k)}(t)) = 0 \tag{4.3}
\]

where \( P^{(k)}(t) \) and \( Q^{(k)}(t) \) are the \( k \)th integrals of the disturbances \( p(t) \) and \( q(t) \) at the input and output respectively, i.e.

\[
u(t) = r(t) + p(t)
\]

\[
y(t) = x(t) + q(t)
\]
and \( r(t) \), \( y(t) \) are measured input and output of the high order system. Suppose that the desired reduced order model which conveys the important dynamics of the system has the following form

\[
\dot{x}(t) + \sum_{k=1}^{n} a_k X^{(k)}(t) - \sum_{k=1}^{n} b_k U^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t}{(k-1)!} = 0 \tag{4.4}
\]

where \( n < n_s \) and \( \hat{x}(i) \) is the output predicted by the reduced model. The parameter vector \( \Theta = (a_k, b_k, c_k; k=1,2,\ldots,n)^T \) has to be identified using the system measured input \( r(t) \) and output \( y(t) \). If the reduced model input \( u(t) \) and output \( \hat{x}(t) \) were replaced by the system measured input and output in (4.4) the result would be an equation residual which reflects the effect of additive disturbances in the measured data, i.e.

\[
y(t) + \sum_{k=1}^{n} a_k Y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t}{(k-1)!} - q(t) - \sum_{k=1}^{n} a_k Q^{(k)}(t) - \sum_{k=1}^{n} b_k P^{(k)}(t) = R(\Theta, t) \tag{4.5}
\]

The parameter vector \( \Theta = (a_k, b_k, c_k; k=1,2,\ldots,n)^T \) can now be determined by forcing the residual \( R(\Theta, t) \) close to zero using weighted residual methods described in the previous chapter. We note that if the residual \( R(\Theta, t) \) is zero equation (4.5) will be the same as (2.4) used previously for system identification where \( y(t) \) was the output of the system whose parameter vector \( \Theta = (a_k, b_k, c_k; k=1,2,\ldots,n)^T \) were identified from input/output data. Hence from now on, the problem of model reduction will be treated in the same manner as that of system parameter identification.

When a quantitative description of disturbances is available the model order reduction problem is be formulated using the following integral equation

\[
\sum_{k=1}^{n} a_k X^{(k)}(t) - \sum_{k=1}^{n} b_k U^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t}{(k-1)!} = -x(t) \tag{4.6}
\]

However, if the disturbances are unquantified, i.e.

\[
p(t) = \sum_{j=1}^{n_p} \alpha_j g_j(t), \quad q(t) = \sum_{j=1}^{n_q} \beta_j h_j(t)
\]

where \( \{g_j(t); j=1,2,\ldots,n_p\}, \{h_j(t); j=1,2,\ldots,n_q\} \) are known, but the parameters \( \{\alpha_j; j=1,2,\ldots,n_p\}, \{\beta_j; j=1,2,\ldots,n_q\}\)
\[
\begin{align*}
\sum_{k=1}^{n} a_k Y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{j=1}^{n} c_j \frac{t^{k-1}}{(k-1)!} - \sum_{j=1}^{n} \frac{d_j}{(k-1)!} & = -y(t) \\
\sum_{j=1}^{n} \frac{d_j}{(k-1)!} & = -x(t)
\end{align*}
\] (4.7)

and \( \beta_{jk} = \beta_{jk}a_k \) and \( \alpha_{jk} = \alpha_{jk}b_k \). In particular, if the disturbances are polynomials of unknown coefficients, i.e.

\[
p(t) = \sum_{j=1}^{n_p} \alpha_j t^j, \quad q(t) = \sum_{j=1}^{n_q} \beta_j t^j
\]

then (4.7) becomes

\[
\sum_{k=1}^{n} a_k Y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} \frac{n_{pq} + n_q - n_p + 1}{(k-1)!} t^{k-1} = -y(t)
\] (4.8)

where \( n_{pq} = \max(n_p, n_q) \). When \( p(t) \) and \( q(t) \) are simply offsets we obtain

\[
\sum_{k=1}^{n} a_k Y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} \frac{n_{pq} + n_q - n_p + 1}{(k-1)!} t^{k-1} = -y(t)
\] (4.9)

The MIMO model reduction problem can be formulated in a similar way as in §2.4. of linear MIMO system formulation.

Notice that although the parameters \( \{c_k; k=1,2,\ldots,n\} \) in (4.6) have to be identified, they no longer identify the initial state values of the original model for the dimensions are not the same. They identify the initial state values of the reduced model. These initial state values are important if simulation the reduced order model is needed for the purpose of comparing the measured and the predicted outputs. However, if the original model is available then zero initial conditions can be imposed in generating the input/output data which are needed to find the reduced order model and the parameters \( \{c_k; k=1,2,\ldots,n\} \) need not be identified. The integral equation to be used when the original model is available is therefore simply

\[
\sum_{k=1}^{n} a_k X^{(k)}(t) - \sum_{k=1}^{n} b_k U^{(k)}(t) = -x(t)
\] (4.10)
and standard signals such as steps, impulses and ramps are allowed as inputs for generating the high order system output data.

4.3. The integral equation and system modes

In this section we shall analyse the effect of the integral equation representation and the original system modes on the the order of the reduced model. To simplify the analysis we shall consider only the problem with zero initial conditions and with no additive disturbances. In such a situation the integral equation residual to be considered is

\[ x(t) + \sum_{k=1}^{n} a_k x(k)(t) - \sum_{k=1}^{n} b_k U(k)(t) = \mathcal{R}(\Omega, t) \]  

and the problem of model reduction is that of finding the vector parameter \( \Omega = [a_k, b_k; k=1,2,\ldots,n]^T \) by forcing the residual \( \mathcal{R}(\Omega, t) \) close to zero on \([0, T_f]\) using weighted residual methods. We shall now investigate the effect that the various system modes have on the integral equation residual and shall seek to convey the reason why the system integral equation is a useful vehicle for the purpose of model reduction. To simplify the analysis we shall assume that the original system has distinct modes, i.e. the transient part of the solution to the ordinary differential equation (4.1) is a linear combination of \( n_s \) distinct exponentials. We can therefore write the solution as

\[ x(t) = \sum_{j=1}^{n_s} \alpha_j \exp(y_j t) + h(t) \]  

where \( h(t) = \rho u(t+T), \rho \) being the system gain. Substituting this solution into (4.11) we obtain

\[ \sum_{j=1}^{n_s} \alpha_j \exp(y_j t) + h(t) + \sum_{k=1}^{n_s} a_k \left[ \sum_{j=1}^{n_s} \frac{\alpha_j}{y_j} \exp(y_j t) + H^{(k)}(t) \right] - \sum_{k=1}^{n_s} b_k U^{(k)}(t) = \mathcal{R}(\Omega, t) \]  

where \( H^{(k)} \) is the \( k^{th} \) integral of \( h(t) \). In the equation residual (4.13) the modes of the original system which are affected by the reduced model order \( n \) are reflected by the expression

\[ \mathcal{P}_n = \sum_{k=1}^{n} a_k \left[ \sum_{j=1}^{n_s} \frac{\alpha_j}{y_j} \exp(y_j t) \right] \]  

(4.14)
It is evident that the effect of fast modes (those with large $\gamma_f$) on the integral equation residual is naturally constrained by division by $\gamma_f$, whilst the same process naturally amplifies the effect of the slow modes. Therefore forcing the integral equation residual to be small will naturally identify its principle components, i.e. the slow modes of the system.

We conclude that the integral equation serves the purpose of model reduction problem well because it naturally restrains the effect of the original system fast modes and increases the effect of slow modes and any residual thereby formed is primarily dependent on the system gain and slow transient modes. Choosing the ordinary differential equation and differentiating the data would have exactly the opposite effect, i.e. it would restrain the effect of slow modes and increasingly emphasize the effect of fast mode components in the equation residual.

4.4. Numerical results

Since the model reduction problem has been reconstituted as one of parameter identification, the numerical implementation of the weighted residual methods applied to model reduction is the same as in the previous chapter, and therefore will not be discussed here. We shall document the results of applying the techniques to various problems of model reduction. Unless otherwise stated the residual used in each method arises from the system integral equation, the formulation following that given in §3.3. The input/output data required to produce low order models is in the form of discrete time series \( \{u(t_i), x(t_i); i=0,1,\ldots,N\} \) where the points \( t_i \) are equidistant in time and span the interval \( [0,T_f] \), i.e. \( t_N = T_f \). Thus the inner product integrals required by the method of moments and the Galerkin technique (Legendre and Fourier) must be evaluated by a suitable quadrature rule; here the seventh order Newton-Cotes scheme is used. The trapezoidal scheme with variable step size is used in the case of Galerkin (Chebyshev) technique for the same reasons as previously, i.e. to avoid the singularity of the integrand at both limits of the observation time interval.

Results quoted as 'least squares' have come from implementation of the least squares point collocation equations (3.22); such results differ only negligibly from those produced by the least squares quadrature technique. In documenting the Galerkin technique, the results denoted 'Legendre' have been produced by choosing the shifted Legendre polynomials on \([0,T_f]\), (3.26), as basis functions while those denoted 'general Fourier', 'even Fourier' and 'odd Fourier' have been produced by basis function choices (3.28), (3.29) and (3.30) respectively. The inner product used for all the latter basis functions is the unweighted inner product (3.13). Results denoted 'Chebyshev' have been produced by the Galerkin technique using shifted Chebyshev polynomials on \([0,T_f]\) as basis functions in association with the weighted inner product (3.27).
4.4.1 Linear SISO model reduction

Example 1.

The linear system with transfer function

\[ G(s) = \frac{375000(s+0.08333)}{s^7 + 83.635s^6 + 4097s^5 + 70342s^4 + 853703s^3 + 2814271s^2 + 3310875s + 281250} \]

has been used previously (Sinha and Berezai [101], Sinha and Pille [102], Genesio and Pompe [103]) to illustrate model reduction techniques. The response of this system was simulated for a unit step input, zero initial conditions and data was collected at \( N = 100 \) discrete equidistant points in time on the interval \([0, 10]\). Application of the various weighted residual methods produced the second and third order reduced models shown in table 4.1 below.

<table>
<thead>
<tr>
<th>Identification method</th>
<th>Second order</th>
<th>Third order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least squares</td>
<td>( \frac{0.029s + 0.296}{s^2 + 2.201s + 2.512} )</td>
<td>( \frac{0.046s + 0.589s + 0.039}{s^3 + 4.013s^2 + 5.140s + 0.364} )</td>
</tr>
<tr>
<td>Method of moments, and</td>
<td>( \frac{0.023s + 0.282}{s^2 + 2.093s + 2.399} )</td>
<td>( \frac{0.042s + 0.576s + 0.034}{s^3 + 3.924s^2 + 4.991s + 0.313} )</td>
</tr>
<tr>
<td>Galerkin (Legendre)</td>
<td>( \frac{0.0084s + 0.3229}{s^2 + 2.2420s + 2.743} )</td>
<td>( \frac{0.038s + 0.558s + 0.018}{s^3 + 3.778s^2 + 4.737s + 0.176} )</td>
</tr>
<tr>
<td>Galerkin (Chebyshev)</td>
<td>( \frac{-0.031s + 0.592}{s^2 + 3.919s + 5.001} )</td>
<td>( \frac{0.048s + 0.592s + 0.077}{s^3 + 4.131s^2 + 5.408s + 0.680} )</td>
</tr>
<tr>
<td>Galerkin (General Fourier)</td>
<td>( \frac{0.034s + 0.250}{s^2 + 1.969s + 2.126} )</td>
<td>( \frac{0.045s + 0.588s + 0.045}{s^3 + 4.026s^2 + 5.168s + 0.407} )</td>
</tr>
<tr>
<td>Galerkin (Even Fourier)</td>
<td>( \frac{0.057s + 0.195}{s^2 + 1.799s + 1.663} )</td>
<td>( \frac{0.048s + 0.599s + 0.049}{s^3 + 4.093s^2 + 5.284s + 0.442} )</td>
</tr>
</tbody>
</table>

Table 4.1.
The outputs produced by applying a unit step to the actual system and the second and third order reduced models produced by the least squares technique are shown in Fig.4.1. Clearly a third order model adequately describes the system dynamics though it is arguable whether or not the second order model does so. Figure 4.2 shows similar outputs for second order and third order models produced by other authors. The second order model was identified by Sinha and Pille [102] as

\[ G_a(s) = \frac{0.341}{s^2 + 2.2412s + 2.9316} \]

while the third order model was given by Sinha and Bereznai [101] as

\[ G_a(s) = \frac{-0.1142s + 0.854s + 0.40}{s^3 + 6.6677s^2 + 9.605s + 3.4836} \]

Clearly the least squares weighted residual approach improves upon the step response transient approximation of the latter two models. The step responses from the other weighted residual techniques differ only marginally from those in Fig.4.1 produced by the least squares point collocation approach. Figure 4.3 shows the step responses of the second and third order models derived by using shifted Chebyshev polynomials and the weighted inner product (3.27) within the Galerkin technique. As expected, the second order model is inferior to the third order model though its transient clearly illustrates an almost even distribution of error over the sampling period.

The above seventh order model output was also corrupted by additive Gaussian noise \( N(0,0.006) \) to simulate the effect of collecting noisy data from a high order system. The third order models identified by the various techniques are shown in table 4.2. At first sight it appears that the least squares method produces a significantly different input/output model following the visible pole/zero cancellation. In fact the third order models derived from noiseless data, shown in shown in table 4.1, differ only negligibly in their response since they identify one very fast mode together with a second order mode whose real and imaginary parts are very similar to that produced by the least squares method when applied to the noisy data set. Figure 4.4 shows a visual comparison of the output of the third order reduced model identified via the Galerkin (odd Fourier) technique. Clearly the noise level is significant (signal to noise ratio = twenty to one) and has a consequent, though not disastrous, effect upon parameter identification in this latter scheme which is one of the weakest implementations of the Galerkin technique.
Fig. 4.1. Least squares reduced model responses.
Fig. 4.2. Previous second and third order model responses.
Fig. 4.3. Step responses from Galerkin (Chebyshev) models.
Fig. 4.4. The effect of observation noise upon reduced model identification.
Identification method | Third order
----------------------|----------------------
Least squares         | $\frac{0.0255s^2 + 0.331s - 0.000418}{s^3 + 2.333s^2 + 2.808s - 0.0036} = \frac{(0.0255s + 0.332)(s - 0.0012)}{(s^2 + 2.334s + 2.811)(s - 0.0012)}$
Method of moments and Galerkin (Legendre) | $- \frac{0.0036s^2 + 0.5488s + 1.0339}{s^3 + 6.2133s^2 + 11.188s + 8.766}$
Galerkin (Chebyshev)  | $\frac{0.0418s^2 + 0.2913s + 0.3217}{s^3 + 2.978s^2 + 4.540s + 2.720}$
Galerkin (General Fourier) | $- \frac{0.0201s^2 + 0.9667s + 1.331}{s^3 + 9.363s^2 + 16.870s + 11.224}$
Galerkin (Even Fourier) | $- \frac{0.0425s^2 + 0.7058s + 0.8267}{s^3 + 6.510s^2 + 11.131s + 7.0229}$
Galerkin (Odd Fourier) | $- \frac{0.1185s^2 + 0.9582s + 0.5023}{s^3 + 6.840s^2 + 11.113s + 4.2942}$

Table 4.2.

To illustrate the effectiveness of the techniques in presence of unquantified polynomial disturbances, the actual input/output data of the original seventh order model was subject to the following additive disturbances when the measured input was $r(t) = 2\sin(0.5t)$ and the initial value $x(0) = 0.2$:

$p(t)=1+t$ \hspace{1cm} $q(t)=3+2t+t^2$

The second order model transfer function identified using the Galerkin technique (with Legendre polynomials $P_{i-1}(t)$ for $i = 6, 7, 8$) was

$$G_a(s) = \frac{1.058}{s^2 + 7.043s + 8.530}$$

Figure 4.5 shows the actual output $x(t)$ of the seventh order model together with the actual output of the latter second order approximation when both models are subject to the same input $r(t) = 2\sin(0.5t)$.  

Fig. 4.5. Model simulation following identification in presence of unknown polynomial disturbances.
Example 2.

The major emphasis in this section has thus far been placed on the residual formed from the system integral equation where application of the various weighted residual methods leads to either a set of linear equations or a linear least squares problem. In this example we illustrate some of the potential difficulties that can arise from use of the signal residual and the subsequent nonlinear optimisation. The specific problem is to find a second order approximation of the form

\[
\frac{\theta_2}{s^2 + \theta_1 s + \theta_2}
\]

to the fourth order system

\[
G(s) = \frac{s^3 + 15s + 50}{s^4 + 5s^3 + 33s^2 + 79s + 50}
\]

The latter system was simulated for a unit step input with zero initial conditions and the output \(x(t)\) was collected at \(N = 80\) equidistant points in the time interval \([0,8]\). The least squares method was used with the robust simplex method as the nonlinear optimisation algorithm. The formulation derived from the output signal residual, i.e.

\[
\min_{\hat{\theta}} \left< R(\hat{\theta}, t), R(\hat{\theta}, t) \right> = \int_0^{T_f} (x(t) - \hat{x}(\hat{\theta}, t))^2 dt
\]

where \(\hat{x}(\hat{\theta}, t)\) is the output produced by the second order model when simulated with a step input and zero initial conditions for a given \(\hat{\theta} = (\theta_1, \theta_2)^T\). For every new estimate \(\theta_1\) and \(\theta_2\) produced by the simplex algorithm the second order model was simulated and the inner product was evaluated until the minimum was reached. An extremely large of computational (cpu) effort was therefore needed, especially when inadequate initial estimates were chosen thereby leading to a very slow rate of convergence.

With initial parameter estimates for \(\theta_1\) and \(\theta_2\) of 1 and 40 respectively the optimisation routine predicted a second order reduced model \(4.347/(s^2 + 5.538s + 4.347)\) while with initial estimates 0 and 50 the model \(92.804/(s^2 + 119.5s + 92.804)\) was produced. The resulting reduced order model is clearly dependent on the choice of initial parameter estimates and it may be thought that the optimisation routine has converged to two different local minima, a possibility in any
such nonlinear optimisation. Figure 4.6 shows the contours of the integral of the squared signal residual criterion and the presence of a long valley is evident. Figure 4.7 shows the value of the latter criterion as a function of $\theta_1$ in a line taken along the valley and indicates that $\theta_1=5.538$, $\theta_2=4.347$ is a unique minimum that is potentially difficult to reach and that $\theta_1=119.5$, $\theta_2=92.804$ is just a point along the valley that the optimisation has not progressed beyond.

### 4.4.2 Linear MIMO model reduction

**Example 1.**

Although not detailed in the text, all the weighted residual techniques can be applied to linear MIMO system identification and control. We shall briefly quote the results from the application of one such model reduction method to a two input, two output system with transfer function matrix

$$
G(s) = \begin{bmatrix}
\frac{5s^4 + 108.5s^3 + 2443.5s + 25515.5s + 2509.704s + 3111.225}{D'} \\
\frac{7.5s^4 + 142.1s^3 + 2862.3s + 2910.6s + 3673.917s + 4225.284}{D'} \\
\frac{4.5s^4 + 95.4s^3 + 742.14s^2 + 2509.704s + 3111.225}{D'} \\
\frac{6.75s^4 + 145.1s^3 + 1123.74s^2 + 3673.917s + 4225.284}{D'}
\end{bmatrix}
$$

where

$$
D' = (s+2)(s+4)(s+6)(s+8)(s+10) = s^5 + 30s^4 + 340s^3 + 1800s^2 + 4384s + 3840
$$

The two input signals were chosen as:

$$
r_1(t) = 20\sin(0.5t) \quad r_2(t) = 25\sin(t)
$$

and the two outputs were computed on a time interval $[0,5]$ with $N=100$, samples being taken at equidistant points in time. The transfer function matrix with second order elements was identified by least squares method as:

$$
G_a(s) = \begin{bmatrix}
7.472s^2 + 18.698 & 5.605s + 22.916 \\
8.518s^2 + 21.337 & 9.578s + 31.480
\end{bmatrix}
$$

where

$$
D = s^2 + 16.407s + 28.083
$$

In figure 4.8 a comparison is made between the true outputs and those of the reduced model. It can be seen that the reduced model is a very good approximation to the original system.
Fig. 4.6. Integral of squared signal residual.
Fig. 4.7. Integral of squared signal residual along valley.
Fig. 4. A MIMO system and reduced model simulation.
Example 2.

Here, the problem is to simulate the behaviour of a dc electric generator under the control of an automatic voltage regulator (a.v.r.). This problem arose from a study of the operation of the generator as part of a power supply system in a military vehicle by Mr Keith Gregory in the department of Electrical and Electronic Engineering at Loughborough University of Technology. The block diagram of system arrangement is shown in Fig.4.9 below.

![Block Diagram of System Arrangement](image)

Figure 4.9. System arrangement.

In particular, we are concerned with the behaviour of the a.v.r. which creates the greatest problems in the circuit analysis since its simulation is very slow and requires large amounts of computational effort. The main section of this automatic voltage regulator circuit is shown in Fig.4.10.

![Electric Circuit of an a.v.r.](image)

Figure 4.10. Electric circuit of an a.v.r.

The device maintains the output voltage of the dc generator constant under varying load conditions. The output from the circuit $V_{out}$ controls the excitation level of the generator. The
main input $V_1$ is effectively the terminal voltage of the generator, the second input $V_2$ is normally constant but can be varied to limit the generator output under overload conditions. For normal operation if $V_1$ falls $V_{out}$ rises and if $V_2$ falls $V_{out}$ falls.

The central part of the circuit is the differential operational amplifier $A$. The amplifier output is $V_{out}$ and is related to the potential difference between the two inputs + and -. The frequency response of the amplifier is effectively specified by the external components $R_7$, $R_8$, $R_9$, $C_4$, and $C_5$, with $R_5$, $R_6$, and $C_3$ causing the amplifier to give an output proportional to the voltage $V_3$ and its integral. The voltage $V_3$ is derived from $V_1$ via the dividing network $R_1$, $R_3$ and $R_4$ and is smoothed by $R_2$, $C_1$ and $C_2$.

When simulating the operation of the generator, in the time domain, under the control of this device it is necessary to determine $V_{out}$ for any known values of $V_1$ and $V_2$. Calculating a transfer function for such a circuit is tedious and error prone and is generally based on the assumption that the operational amplifier is perfect. This assumption can be avoided by using a simple equivalent circuit for $A$ such as that shown in Fig.4.11 below.

![Figure 4.11. Electric circuit of an amplifier.](image)

The application of standard network analysis techniques to the complete circuit is straightforward, for instance mesh analysis gives nine integral equations of the form

$$\dot{V} = Ri + \frac{1}{C}\int i dt$$

where $i$ is the mesh current. These equations can be solved in the time domain using standard numerical techniques, but the solution requires large amounts of computing time because the equations are stiff, integration step lengths of 100ns typically being required. Obviously it is desirable to avoid such step lengths in simulations over periods of seconds so a transfer function representation of some form is to be preferred.

If the numerical solution of the circuit mesh equations is used as a source of data, a transfer function of some order may be identified which can be solved in the time domain at step lengths of
40μs, as required by the generator equations. The use of such a transfer function would give a big reduction in the computing times for the generator simulation. The problem is therefore that of model reduction, i.e. use the input/output data of the circuit to find a transfer function of reasonable order which, if put in place of the circuit equations, would give a suitable approximation to the behaviour of the system.

To produce the output data V_{out} needed to approximate the circuit model of the a.v.r. by a fourth order transfer function, the circuit was simulated with an integration step length of 200ns and for inputs V_1 and V_2 as

\[
V_1 = \begin{cases} 
27 & \text{for } t \leq 0.0304 \\
27 - \frac{1.5(500t)}{(1.0 + 500)} & \text{for } t > 0.0304 
\end{cases}
\]

and

\[
V_2 = \begin{cases} 
6.2 & \text{for } t \leq 0.0606 \\
6.2 - \frac{0.2(500t)}{(1.0 + 500)} & \text{for } t > 0.0606 
\end{cases}
\]

The output values were collected at N = 600 equidistant points in the time interval [0, 0.12], i.e. at intervals of 0.0002 sec; Fig.4.12 shows the graphical representation of V_{out}. The delayed excitations V_1, V_2 were chosen because of the suffness of the system and the prohibitive time taken for data acquisition from both transient and steady state phases of the system response. These inputs also enable us to collect reasonable amount of data from both the transient state (0.0304 < t ≤ 0.12) and the steady state (0 ≤ t ≤ 0.0304) of the response V_{out}. The least squares point collocation method produced the following transfer function matrix:

\[
G(s) = \begin{bmatrix} f_1(s) & f_2(s) \end{bmatrix}
\]

where

\[
f_1(s) = \frac{84.132s^3 - 1616878.167s^2 - 378789795.1209s - 6411244539.342}{D}
\]

\[
f_2(s) = \frac{33367.3343s^3 + 25715173.085s^2 + 3460497056.143s + 42570034721.4}{D}
\]

and

\[
D = s^4 + 1545.927s^3 + 2818236.948s^2 + 1604387979.894s + 24989106304.39
\]

To assess the performance of such a fourth order model the characteristics of the dc electric generator (the exciter field voltage Fig.4.13; the generator current and voltage outputs Fig.4.14 and Fig.4.15; and the timing ramp Fig 4.16) under the control of both the original circuit model and its fourth order approximant when the input V_2 is kept constant at 6.2 were plotted. Although the a.v.r. output is slightly different from that of its the reduced order model as shown in Fig.4.17,
Fig. 4.12. The a.v.r. output data ($V_{ou}$) used for model reduction.
Fig. 4.13. The exciter field voltage.

- Circuit voltage
- Model voltage
Fig. 4.14. The generator current output.
Fig. 4.15. The generator voltage output.
Fig. 4.16. The timing ramp.
Fig. 4.17. The outputs of the a.v.r. and its reduced order model.
this does not result in significant differences in the generator characteristics. What is more important is that the generator can be simulated with a 42% less computational effort (cpu time) using the fourth order model.

Notice that techniques requiring zero initial conditions will not be able to be used in such circumstances, i.e. circumstances where using few data points of a previous steady state and few data points from the actual transient state of the system response for identification or model reduction purposes. Those techniques will require huge amounts of data points between the transient and the steady state of the system response, especially when if the system is very slow.

4.5. Summary

The problem of model reduction from input/output data has been reduced to that of parameter identification and the concepts of weighted residual methods, which have in the past been classically used to solve ordinary differential equations, have been extended to cover this problem. The system integral equation, from which the residual is formed, has been shown to be a useful vehicle to derive reduced order models from noisy input/output data since any residual thereby formed is primarily dependent on the system gain and slow transient modes. Thus excellent results have been obtained following application of various weighted residual methods though the least squares point collocation method is favoured by the author for it is an optimisation scheme and requires less computational effort. The applications presented include SISO model reduction with and without additive disturbances and MIMO model reduction problems. In MIMO model reduction the least squares method was applied to a practical problem involving the reduction of a high order model of an automatic voltage regulator (a.v.r.). The original model of the a.v.r. was difficult to obtain and the only way of obtaining the corresponding reduced model was by using input and output data of the system. A range of potential Galerkin approaches has been presented and it is noteworthy that use of Chebyshev polynomials and an associated weighted inner product yields models that have a relatively even signal error over the sampling interval.

When applying weighted residual methods to linear systems described by their integral equation, model parameter estimates result from the solution of a set of linear equations and hence any convergence difficulties associated with nonlinear problems generated by weighted signal residual methods are avoided. Additionally, computational effort is kept to a minimum by use of the integral equation residual. As a guide, computation times using the integral equation residual are typically two orders of magnitude smaller than when using the signal residual in problems with four or five parameters.
5. SYSTEM ORDER DETERMINATION

In this chapter we present two different methods for identifying the order of a continuous linear SISO system using input/output data. A non-square matrix is formed whose columns represent the values of successive multiple integrations of the input and output data at many equidistant points in the time interval of observation. The first method is based on the number of non-zero singular values of this matrix. The second method produces a square matrix by projecting the columns of the latter non-square matrix onto an orthogonal space spanned by shifted Legendre polynomials and deduces the order of the system from the number of non-zero eigenvalues of the square matrix thereby obtained. The effect of disturbances on the estimated model order is analysed and the techniques are illustrated by numerical examples.

5.1. Introduction and problem statement

Although much attention has been given to parameter identification of linear systems, relatively little effort has been employed in identifying the system order; it is assumed to be known in most situations. In practice this assumption is not always fulfilled and therefore the number of identifiable parameters is not predetermined. Thus the decision of the order is very important and any identification procedure without a priori knowledge of this parameter may be judged unsatisfactory for many practical applications. Moreover, of those techniques for system order determination which do exist most are designed for discrete systems as are the majority of parameter identification methods. Also such methods are often computationally demanding since they are largely based on iterative procedures.

The techniques which will be developed here are designed for continuous linear system order determination. Their main advantage over those discussed in the literature survey is that, given an initial over estimate of the model order, the true system order is determined by a precise formulation. Little computational effort is required and the methods are found to be robust in identifying the correct order even when data is subject to additive disturbances. The problem is stated as follows: given input/output data observations of a continuous linear SISO system described by its ordinary differential equation

\[ x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_n x(t) = b_1 u^{(n-1)}(t) + \ldots + b_n u(t) \]  \hspace{1cm} (5.1)
where \( x^{(i)}(0) = x_{i0}, i = 1, 2, ..., n-1 \)

Since the methods are based on multiple integrals of data it is better to replace (5.1) with its equivalent integral equation representation, i.e.

\[
\sum_{k=1}^{n} a_k X^{(k)}(i) - \sum_{k=1}^{n} b_k U^{(k)}(i) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} = -x(i) \quad (5.2)
\]

The problem is now to determine the order \( n \) of the system using only the successive integrals of input/output data, i.e. \( \{U^{(k)}(i), X^{(k)}(i); k=1,2,\ldots\} \).

To develop the analytical relation between data and its multiple integrals we devote the next two sections to the solution of the system ordinary differential equation (5.1) and its multiple integrals.

### 5.2. Solution of input/output equations

A classical way of finding the solution of equation (5.1) is to map the ordinary differential equation in the time domain into an algebraic equation in the complex frequency domain via Laplace transformation. The algebraic equation is solved and the result is mapped back into the time domain via inverse Laplace transformation to provide the solution of the original ordinary differential equation. The Laplace transform of (5.1) gives

\[
[s + a_1 s^{n-1} + \ldots + a_n]X(s) = [b_1 s^{n-1} + \ldots + b_n]U(s) + [c_1 s^{n-1} + \ldots + c_n] \quad (5.3)
\]

where \( X(s), U(s) \) are the Laplace transform of \( x(t) \) and \( u(t) \), and the coefficients \( \{c_i; i = 1,2,\ldots,n\} \) are identical to those in the integral equation (5.2). Notice that Laplace transform of (5.2) could have been used to produce the same equation (5.3). Finally equation (5.3) may be solved for the transformed output, i.e.
\[ X(s) = \frac{c_1 s^{n-1} + \ldots + c_n}{s^n + a_1 s^{n-1} + \ldots + a_n} U(s) + \frac{b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} U(s) \]  

(5.4)

Now \( X(s) \) can be expressed as a partial fraction expansion in terms of the roots of its denominator polynomial (assuming that \( U(s) \) is a rational function of \( s \)). These are clearly the system poles, i.e. the roots of the characteristic polynomial:

\[ P(s) = s^n + a_1 s^{n-1} + \ldots + a_n \]

plus any additional roots in the denominator of \( U(s) \). The characteristic polynomial \( P(s) \) can in general be factorised as:

\[ P(s) = (s + \gamma_1)^{r_1} (s + \gamma_2)^{r_2} \ldots (s + \gamma_k)^{r_k} (s + \gamma_{k+1}) \ldots (s + \gamma_{k+n-R}) \]

where \( R = r_1 + r_2 + \ldots + r_k \); \( \gamma_1, \gamma_2, \ldots, \gamma_k \) are repeated roots and \( \gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_{k+n-R} \) are distinct roots. The transformed output can then be expanded as:

\[ X(s) = \sum_{j=1}^{r_1} \frac{\beta_1}{(s+\gamma_j)} + \ldots + \sum_{j=1}^{r_k} \frac{\beta_k}{(s+\gamma_j)} + \sum_{j=k+1}^{k+n-R} \frac{B_j}{(s+\gamma_j)} + H(s) \]  

(5.5)

where \( \{\beta_1, j=1,2,\ldots,k; j=1,2,\ldots,r_k\}, \{B_j; j=1,2,\ldots,k+n-R\}\) are constants known as residues and \( H(s) \) is a rational function of \( s \) whose denominator is that of \( U(s) \). Mapping back the transformed output into the time domain using inverse Laplace transformation we obtain the general solution of (5.1) which can be written as:

\[ x(t) = \sum_{j=1}^{r_1} \beta_1 j^{l-1} \exp(-\gamma_j t) + \ldots + \sum_{j=1}^{r_k} \beta_k j^{l-1} \exp(-\gamma_j t) + \sum_{j=k+1}^{k+n-R} B_j j^{l-1} \exp(-\gamma_j t) + h(t) \]  

(5.6)

We note that the shape of the output time function \( x(t) \) is determined principally by two factors: the poles of the system and the shape of the time function \( h(t) \). The shape of \( h(t) \) is similar to that of the input \( u(t) \) except that it is phase shifted and multiplied by the system gain \( \rho \), i.e.
Notice that if the input is not a periodic signal we have $T = 0$ and $h(t) = pu(t)$. We define the fundamental solution set of (5.1) as being the union of the subsets corresponding to each of the repeated roots and the subset corresponding to distinct roots, i.e:

$$\{w_1(t), w_2(t), \ldots, w_n(t)\} = \{t^{j-1}\exp(-\chi_0); j=1,2,\ldots,\tau_1\} U \ldots U \{t^{j-1}\exp(-\chi_0); j=1,2,\ldots,\tau_k\}
\cup \{\exp(-\chi_{k+1} t), \exp(-\chi_{k+2} t), \ldots, \exp(-\chi_{n-R} t)\}$$ (5.8)

Hence the solution can be written as

$$x(t) = \sum_{i=1}^{n} \beta_i w_i(t) + h(t)$$ (5.9)

where the coefficients $\{\beta_i; i=1,2,\ldots,n\}$ can be determined.

### 5.3. Multiple Integration

In this section we shall find an analytical relationship between the signal output $x(t)$ and its successive integrals

$$X^{(q)}(t) = \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} \ldots \int_0^{\tau_{q-1}} x(\tau_1) d\tau_1 d\tau_2 \ldots d\tau_{q-1} d\tau_q \quad \text{for } q \geq 2$$

$$X^{(1)}(t) = \int_0^t x(\tau_1) d\tau_1$$ (5.10)

However before doing so we shall firstly find a general formulation obeyed by successive integrals of the general expression:

$$x^{(j)}(t) = \sum_{\chi_j} \beta_j t^{j-1} \exp(-\chi_1 t)$$
Integrating once gives:

\[
\int_0^t x_{\gamma_1}(t) \, dt = \int_0^t \sum_{j=1}^r \beta_{\gamma_1 j} t^{j-1} \exp(-\gamma_1 t) \, dt = \sum_{j=1}^r \int_0^t t^{j-1} \exp(-\gamma_1 t) \, dt
\]

\[
= \sum_{j=1}^r \beta_{\gamma_1 j} \left[ \frac{t^j}{j!} + \frac{(j-1)t^{j-2}}{\gamma_1^2} + \cdots + \frac{(j-1)(j-2)\cdots 1}{\gamma_1^j} \right]_0^t
\]

Collecting terms in \( \gamma_1, \gamma_2, \ldots, \gamma_k \) we obtain:

\[
\int_0^t x_{\gamma_1}(t) \, dt = \left[ \beta_{11}^{(1)} + \beta_{12}^{(1)} t + \cdots + \beta_{1r_1}^{(1)} t^{r_1-1} \right] \exp(-\gamma_1 t)
\]

where \( \beta_{1j}^{(1)}; j = 1, 2, \ldots r_1 \) are constants which can be determined. Thus

\[
\int_0^t x_{\gamma_1}(t) \, dt = \sum_{j=1}^{r_1} \beta_{1j}^{(1)} t^{j-1} \exp(-\gamma_1 t) - \beta_{11}^{(1)}
\]

continuing this process we see that the \( q \)-th (\( q \geq 1 \)) multiple integral of \( x_{\gamma_1}(t) \) will have the form:

\[
\begin{aligned}
X_{\gamma_1}(t) &= \sum_{j=1}^{r_1} \beta_{1j}^{(q)} t^{j-1} \exp(-\gamma_1 t) - \beta_{11}^{(q)} - \beta_{11}^{(q-1)} t - \cdots - \frac{\beta_{11}^{(q-1)} t^{q-1}}{(q-1)!} \\
&= \sum_{j=1}^{r_1} \beta_{1j}^{(q)} t^{j-1} \exp(-\gamma_1 t) + \sum_{j=1}^{r_k} \beta_{kj}^{(q)} t^{k-1} \exp(-\gamma_k t) + \sum_{j=k+1}^{r} \beta_{kj}^{(q)} t^{j-1} \exp(-\gamma_l t)
\end{aligned}
\]

and hence the \( q \)-th multiple integral of \( x(t) \), i.e:

\[
X_{\gamma_1}(t) = X_{\gamma_1}^{(q)}(t) + X_{\gamma_2}^{(q)}(t) + \cdots + X_{\gamma_k}^{(q)}(t) + X_{\gamma_{k+1}}^{(q)}(t) + \cdots + X_{\gamma_{k+n-R}}^{(q)}(t)
\]

\[
= \sum_{j=1}^{r_1} \beta_{1j}^{(q)} t^{j-1} \exp(-\gamma_1 t) + \sum_{j=1}^{r_k} \beta_{kj}^{(q)} t^{k-1} \exp(-\gamma_k t) + \sum_{j=k+1}^{r} \beta_{kj}^{(q)} t^{j-1} \exp(-\gamma_l t)
\]

\[
+ \sum_{j=1}^{q} \beta_{1j}^{(q)} t^{j-1} + \sum_{j=1}^{q} \beta_{kj}^{(q)} t^{j-1}
\]

(5.11)
where \( \{ \lambda_j; j=1,2,\ldots,q \} \) are constants which can be determined and \( H^{(q)}(t) \) is the \( q \)th integral of \( h(t) \) as defined by (5.10). \( X^{(q)}(t) \) can also be written as:

\[
X^{(q)}(t) = \sum_{j=1}^{n} a_j^{(q)} w_j(t) + \sum_{j=1}^{q} \frac{\lambda_j}{(q-j)!} + H^{(q)}(t) \quad q \geq 1
\] (5.12)

where \( \{ w_j(t); j=1,2,\ldots,n \} \) are the elements of the fundamental set defined in (5.8) and \( \{ a_j^{(q)}; j=1,2,\ldots,n \} \) are corresponding coefficients.

5.4. A non-square matrix for model order determination

We shall propose a method for determining the order of a linear SISO system. The method consists of investigating the behavior of the singular values of the following rectangular matrix

\[
M = \begin{bmatrix}
X^{(1)}(t_0) & X^{(2)}(t_0) & \cdots & X^{(m)}(t_0) & U^{(2)}(t_0) & \cdots & U^{(m)}(t_0) & t_0 & \cdots & t_0^{m-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
X^{(1)}(t_N) & X^{(2)}(t_N) & \cdots & X^{(m)}(t_N) & U^{(2)}(t_N) & \cdots & U^{(m)}(t_N) & t_N & \cdots & t_N^{m-1}
\end{bmatrix}
\] (5.13)

where \( \{ t_j; j=0,1,\ldots,N \} \) are the discrete points in time at which input/output data is observed and \( m \) is an over estimate of the model order. The matrix \( M \) is of dimension \((N+1, 3m-2)\), with \( N+1 \gg 3m-2 \) in general.

**Theorem:**

If \( n \) is the order of the system and \( m>n \) then there exists an input signal \( u(t) \) such that the matrix \( M \) has \( n_s=(m-n) \) zero singular values. The order of the system is therefore given by \( n=m-n_s \).

**Proof:**

To prove the theorem we need to show that there are \( n_s \) columns in the matrix \( M \) which can be expressed as a linear combination of the other \( n+2m-2 \) columns. We can select these columns to be...
and show that every one of them can be expressed as a linear combination of the rest of the columns of matrix \( M \). To do this we must show that for all \( 1 \leq j \leq m \), \( \mathbf{X}^{(n+j)}(t) \) and \( (\mathbf{X}^{(i)}(t) : i=1,2,\ldots,n) \), \( (\mathbf{U}^{(i)}(t) : i=2,3,\ldots,m) \) and \( (t^i : i=1,2,\ldots,m-1) \) are linearly dependent. Using the integral equation representation for linear systems (5.2) we can generate \( n \) linear combinations relating \( \{\mathbf{X}^{(n+j)}(t) : j=1,2,\ldots,n\} \) to \( \{\mathbf{X}^{(i)}(t) : i=1,2,\ldots,n\} \), \( \{\mathbf{U}^{(i)}(t) : i=2,3,\ldots,m\} \) and \( \{t^i : i=1,2,\ldots,m-1\} \). If the theorem holds for \( n_2 > n \) it automatically holds for \( n_2 < n \), therefore we only need to prove it for the case where \( n_2 > n \). Integrating the integral equation (5.2) \( n_2 \) times from 0 to \( t \) we generate the following set of algebraic equations:

\[
\begin{align*}
\mathbf{a}_n \mathbf{X}^{(n+1)}(t) &= -\mathbf{X}^{(1)}(t) - \sum_{k=1}^{n-1} \mathbf{a}_k \mathbf{X}^{(k+1)}(t) + \sum_{k=1}^{n} \mathbf{b}_k \mathbf{U}^{(k+1)}(t) - \sum_{k=1}^{n} \mathbf{c}_k \frac{t^k}{k!} \\
\mathbf{a}_n \mathbf{X}^{(n+2)}(t) &= -\mathbf{X}^{(2)}(t) - \sum_{k=1}^{n-1} \mathbf{a}_k \mathbf{X}^{(k+2)}(t) + \sum_{k=1}^{n} \mathbf{b}_k \mathbf{U}^{(k+2)}(t) - \sum_{k=1}^{n} \mathbf{c}_k \frac{t^{k+1}}{(k+1)!} \\
&\vdots \\
\mathbf{a}_n \mathbf{X}^{(2n+1)}(t) &= -\mathbf{X}^{(n+1)}(t) - \sum_{k=1}^{n-1} \mathbf{a}_k \mathbf{X}^{(k+n+1)}(t) + \sum_{k=1}^{n} \mathbf{b}_k \mathbf{U}^{(k+n+1)}(t) - \sum_{k=1}^{n} \mathbf{c}_k \frac{t^{k+n}}{(k+n)!} \\
&\vdots \\
\mathbf{a}_n \mathbf{X}^{(n+n_2)}(t) &= -\mathbf{X}^{(n_2)}(t) - \sum_{k=1}^{n-1} \mathbf{a}_k \mathbf{X}^{(k+n_2)}(t) + \sum_{k=1}^{n} \mathbf{b}_k \mathbf{U}^{(k+n_2)}(t) - \sum_{k=1}^{n} \mathbf{c}_k \frac{t^{k+n_2-1}}{(k+n_2-1)!}
\end{align*}
\]

From which we can write:
where the coefficients \( \{ \alpha_k^{(j)}, \beta_k^{(j)}, \gamma_k^{(j)} \; j=1,2,\ldots,n_3 \} \) can be determined. Using forward substitutions in (5.15) we can show that for all \( 1 \leq j \leq n \), \( X^{(n+j)}(t) \) can be expressed as a linear combination of \( \{ X^{(i)}(t); i=1,2,\ldots,n \} \), \( \{ U^{(i)}(t); i=2,3,\ldots,m \} \) and \( \{ t^i; i=1,2,\ldots,m-1 \} \), i.e:

\[
X^{(n+j)}(t) = \sum_{k=1}^{m} \alpha_k^{(n+j)} X^{(k)}(t) + \sum_{k=1}^{m} \beta_k^{(n+j)} U^{(k)}(t) + \sum_{k=1}^{m} \gamma_k^{(n+j)} \frac{t^k}{k!}
\]

for \( j=1,2,\ldots,n_5 \) (5.16)

where \( \{ \alpha_k^{(j)}, \beta_k^{(j)}, \gamma_k^{(j)} \; j=1,2,\ldots,n_3 \} \) can be determined from \( \{ \alpha_k^{(j)}, \beta_k^{(j)}, \gamma_k^{(j)} \; j=1,2,\ldots,n_3 \} \).

This means that there are \( n_5 \) columns which are linearly dependent on the other columns of matrix \( M \) and hence there are at least \( n_5 \) zero-singular values. If we show that \( \{ X^{(i)}(t); i=1,2,\ldots,n \}, \{ U^{(i)}(t); i=2,3,\ldots,m \} \) and \( \{ t^i; i=1,2,\ldots,m-1 \} \) are linearly independent we can say that the matrix \( M \) has exactly \( n_5 \) zero singular values. We can do that provided that the input \( u(t) \) is a non periodic signal and chosen so that \( \{ U^{(i)}(t); i=1,2,\ldots,m \} \) and \( \{ t^{i+1}; i=1,2,\ldots,m \} \) are linearly independent. This means that any linear combination

\[
\sum_{i=1}^{m} \alpha_i^{(i)} X^{(i)}(t) + \sum_{i=1}^{m} \beta_i^{(i)} U^{(i)}(t) + \sum_{i=1}^{m} \gamma_i^{(i)} t^{i+1} = 0
\]

unless \( \alpha_i^{(i)} = \beta_i^{(i)} = 0 \), \( \forall i > 1 \). If \( \{ X^{(i)}(t); i=1,2,\ldots,n \}, \{ U^{(i)}(t); i=2,3,\ldots,m \} \) and \( \{ t^i; i=1,2,\ldots,m-1 \} \) were linearly dependent then constants \( \{ \alpha_i^{(i)}, \beta_i^{(i)}, \gamma_i^{(i)}; i=1,2,\ldots,m \} \) and \( \{ \gamma_i^{(i)}; i=1,2,\ldots,m-1 \} \) can be found such that they are not all zero and
\[
\sum_{i=1}^{n} \alpha_i^r X^{(r)}(t) + \sum_{i=2}^{m} \beta_i^r U^{(r)}(t) + \sum_{i=1}^{m-1} \gamma_i^r t^i = 0 \tag{5.18}
\]

Now for a non-periodic signal \( u(t) \) we have \( h(t) = \rho u(t) \) and \( H^{(q)}(t) = \rho U^{(q)}(t) \). If we replace \( \{X^{(r)}(t); i=1,2,\ldots,n\} \) in (5.18) by the right-hand-side of (5.12) we obtain an equation of the form

\[
\sum_{i=1}^{n} \sigma_i^r w_i(t) + \sum_{i=1}^{m} \mu_i^r U^{(r)}(t) + \sum_{i=1}^{m-1} \nu_i t^i = 0 \tag{5.19}
\]

where \( \{\sigma_i^r; i=1,2,\ldots,n\}, \{\mu_i^r; i=1,2,\ldots,m\} \) and \( \{\nu_i; i=1,2,\ldots,m\} \) are not all zero. Notice that \( \{\sigma_i^r; i=1,2,\ldots,n\} \) cannot all be zero because (5.19) would contradict (5.17). Rearranging (5.19) we have

\[
\sum_{i=1}^{m} \mu_i^r U^{(r)}(t) = -\sum_{i=1}^{n} \sigma_i^r w_i(t) - \sum_{i=1}^{m} \nu_i t^i \tag{5.20}
\]

Equation (5.20) can only be true if the expression of the input \( u(t) \) contains a linear combination of the elements of the fundamental set of solutions \( \{w_i(t); i=1,2,\ldots,n\} \), i.e.

\[
u(t) = f(t) + \sum_{i=1}^{n} \xi_i w_i(t) \tag{5.21}
\]

where \( \{\xi_i; i=1,2,\ldots,n\} \) are not all identically zero and \( f(t) \) is the solution of the homogeneous ordinary differential equation

\[
\sum_{i=1}^{m} \mu_i t^{i(m-1)}(t) = 0 \tag{5.22}
\]

Choosing an input signal such as given by (5.21) means exciting the system at one of its natural modes. If such inputs are avoided then condition (5.17) becomes sufficient to make matrix \( M \) have exactly \( n_2 \) zero-singular values. This completes the proof of the theorem.
5.4.1 A simplified problem

The previous theorem was proved for a general situation. However the problem can be simplified by either an adequate choice of input signal or by imposing zero initial conditions when generating the input/output data.

*Standard inputs*

Standard input signals such as impulses, steps and ramps considerably simplify the form of the matrix $M$ whose singular values are to be determined. This also results in less computational effort thereby required to find the singular values. If one of these standard inputs is selected, the columns $\{U^{(i)}(t); i=2,3,...,m\}$ and $\{t^i; i=1,2,...,m-1\}$ will be linearly dependent and therefore only the maximum number of linearly independent columns are needed in such circumstances. The following matrices will be used when such standard inputs are selected to stimulate the system:

$$M_{\text{impulse}} = \begin{bmatrix} X^{(1)}(t_0) & \cdots & X^{(m)}(t_0) & t_0 & \cdots & t_0 \\ . & \ddots & . & . & . & . \\ . & . & \ddots & . & . & . \\ X^{(1)}(t_N) & \cdots & X^{(m)}(t_N) & t_N & \cdots & t_N \end{bmatrix}$$

$$M_{\text{step}} = \begin{bmatrix} X^{(1)}(t_0) & \cdots & X^{(m)}(t_0) & t_0 & \cdots & t_0 \\ . & \ddots & . & . & . & . \\ . & . & \ddots & . & . & . \\ X^{(1)}(t_N) & \cdots & X^{(m)}(t_N) & t_N & \cdots & t_N \end{bmatrix}$$

$$M_{\text{ramp}} = \begin{bmatrix} X^{(1)}(t_0) & \cdots & X^{(m)}(t_0) & t_0 & \cdots & t_0 \\ . & \ddots & . & . & . & . \\ . & . & \ddots & . & . & . \\ X^{(1)}(t_N) & \cdots & X^{(m)}(t_N) & t_N & \cdots & t_N \end{bmatrix}$$

Each of these matrices has $n_s$ zero singular values.

*Zero initial conditions*

Imposing zero initial conditions when generating the data also simplifies the computational effort by reducing the size of the matrix $M$ which then has the form
The input \( u(t) \) can also be a step, impulse or a ramp.

5.5. Use of shifted Legendre polynomials

In this section we will generate a square matrix by projecting the columns of matrix \( M \) onto an appropriate number of shifted Legendre polynomials \( \{ P_{q}(0); \, \ell = q, q+1,... \} \). The projection of a column \( X^{(i)}(t); \, t \in [0, T] \) onto a shifted Legendre polynomial \( P_{q}(0); \, t \in [0, T] \) is defined by the following inner product

\[
\langle X^{(i)}(t), P_{q}(0) \rangle = \int_{0}^{T} X^{(i)}(t)P_{q}(t) \, dt
\]

where \( T = t_{N} \). We note that

\[
\langle i, P_{q}(0) \rangle = 0 \quad \text{if } \ell \neq i
\]

(the proof for this can be found in Appendix C) and the result of projecting \( \{ X^{(i)}(t); \, i=1,2,...,m \} \), \( \{ U^{(i)}(t); \, i=1,2,...,m \} \), \( \{ i; \, i=1,2,...,m-1 \} \) onto \( (2m-1) \) shifted Legendre polynomials \( \{ P_{q}(0); \, \ell = q, q+1,..., q+2m-2 \} \), where \( q = m-1 \), produces the square matrix

\[
M_{L} = \begin{bmatrix}
\langle X^{(1)}(0), P_{q}(0) \rangle & ... & \langle X^{(m)}(0), P_{q}(0) \rangle & \langle U^{(2)}(0), P_{q}(0) \rangle & ... & \langle U^{(m)}(0), P_{q}(0) \rangle \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\langle X^{(1)}(0), P_{q}(0) \rangle & ... & \langle X^{(m)}(0), P_{q}(0) \rangle & \langle U^{(2)}(0), P_{q}(0) \rangle & ... & \langle U^{(m)}(0), P_{q}(0) \rangle
\end{bmatrix}
\] (5.24)

where \( q' = q + 2m - 2 \). We also note that the presence of initial conditions has been eliminated by this
process of projection.

**Theorem:**

If \( n \) is the order of the system and \( m > n \) then the matrix \( M_L \) has \( n_s = (m-n) \) zero-eigenvalues.

Therefore the order of the system is given by \( n = m - n_s \).

**Proof:**

Equation (5.16) gives:

\[
X^{(n+j)}(t) = \sum_{k=1}^{n} \alpha_j^{(k)} X^{(k)}(t) + \sum_{k=1}^{m} \beta_j^{(k)} U^{(k)}(t) + \sum_{k=1}^{m-1} \frac{1}{k!} \frac{d^k}{dt^k} \]

\[ j = 1, 2, ..., n_s \]

This means that \( n_s \) columns

\[
<X^{(n+j)}(t), P_j(t)> = \sum_{k=1}^{n} \alpha_j^{(k)} <X^{(k)}(t), P_j(t)> + \sum_{k=1}^{m} \beta_j^{(k)} <U^{(k)}(t), P_j(t)> \]

\[ j = 1, 2, ..., n_s \]

\[ l = q, q+1, ..., q+2m-2 \]

are linearly dependent on the other columns. Since the input signal is chosen so that \( (X^{(i)}(t); i = 1, 2, ..., n) \) and \( (U^{(i)}(t); i = 2, 3, ..., m) \) are linearly independent, the matrix \( M_L \) must have exactly \( n_s \) zero eigenvalues.

### 5.5.1 A simplified problem

As previously stated an appropriate choice of input signals simplifies the computational effort. Therefore for standard inputs (steps, impulses, ramps) the matrix to be used to compute the eigenvalues becomes

\[
M_L = \begin{bmatrix}
<X^{(1)}(t), P_q(t)> & ... & <X^{(m)}(t), P_q(t)>\\
\vdots & \ddots & \vdots \\
<X^{(1)}(t), P_q(t)> & ... & <X^{(m)}(t), P_q(t)>
\end{bmatrix}
\]
where $q' = q + m - 1$. Notice that the number of non-zero eigenvalues is equal to the system order $n$.

In the case of a ramp input signal the shifted Legendre polynomials must be chosen such that $q > m + 1$.

5.6. The effect of disturbances

In practical situations the measured input/output data may be subject to unknown additive disturbances and in this section we shall investigate the effect of these disturbances on system order determination. The relationship between the measured and the true system input/output data is then given by:

$$u(t) = r(t) + p(t)$$

$$y(t) = x(t) + q(t)$$

where $r(t)$ and $y(t)$ are the measured data, $x(t)$ and $u(t)$ being the true signals of the system. Very often the disturbances are simple offsets due to calibration of the apparatus but we shall consider the general case of polynomial disturbances.

5.6.1 Effect of polynomial disturbances

Suppose that additive disturbances on both the input and the output are polynomials with unknown coefficients, i.e.

$$u(t) = r(t) + \sum_{i=0}^{n_u} \psi_i t^i$$

$$y(t) = x(t) + \sum_{i=0}^{n_y} \varphi_i t^i$$

(5.26)

where $n_u, n_y, \{\psi_i; i=1,2,\ldots,n_u\}$ and $\{\varphi_i; i=1,2,\ldots,n_y\}$ are unknown. The output of a linear system when subject to the input $u(t)$ is the sum of two outputs, that due to $r(t)$ and that due to the corresponding additive polynomial. The total output $y(t)$ will therefore have the form:

$$x(t) = \sum_{i=1}^{n} A_i w_i(t) + h'(t) + \sum_{i=0}^{n_y} \varphi_i t^i$$

(5.27)

where $h'(t) = pr(t)$. The measured output will therefore be:
\[ x(t) = \sum_{i=1}^{n_1} A_i w_i(t) + h(t) + \sum_{j=0}^{n_2} y_j^r(t) \]  \hspace{1cm} (5.28)

where \( n_1 = \text{max}(n_0, n_y) \). By using previous analysis that led to (5.12), the \( q \)th integral of the measured output will have the form

\[ Y^{(q)}(t) = \sum_{j=1}^{n} A_j w_j(t) + \sum_{j=0}^{n+q} y_j^r(t) + \rho R^{(q)}(t) \quad q \geq 1 \]  \hspace{1cm} (5.29)

The only difference in structure between (5.29) and (5.12) is that (5.29) has \( (n_1+1) \) more terms, i.e. \( (t^q, t^{q+1}, ..., t^{q+n_1}) \), therefore the matrix \( M \) to be considered in this situation must have at least \( (n_1+1) \) more columns than previously. If \( n_1 \) is unknown then \( (n_b+1) \) columns can be added to matrix \( M \) (with \( n_b \geq n_3 \)) to give:

\[
M = \begin{bmatrix} 
Y^{(1)}(t_0) & ... & Y^{(m)}(t_0) & R^{(2)}(t_0) & ... & R^{(m)}(t_0) & t_0 & ... & t_0 & m+n_b \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
Y^{(1)}(t_N) & ... & Y^{(m)}(t_N) & R^{(2)}(t_N) & ... & R^{(m)}(t_N) & t_N & ... & t_N & m+n_b 
\end{bmatrix}
\]

Following similar analysis with shifted Legendre polynomials \( M_L \) becomes:

\[
M_L = \begin{bmatrix} 
<Y^{(1)}(t), P_q(t)> & ... & <Y^{(m)}(t), P_q(t)> & <R^{(2)}(t), P_q(t)> & ... & <R^{(m)}(t), P_q(t)> \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
<Y^{(1)}(t), P_q(t)> & ... & <Y^{(m)}(t), P_q(t)> & <R^{(2)}(t), P_q(t)> & ... & <R^{(m)}(t), P_q(t)> 
\end{bmatrix}
\]

where \( q > m+n_b \) and \( q' = q+2m-2 \). Notice that with shifted Legendre polynomials the matrix \( M_L \) has the same form as before, except that the polynomials are of higher degrees.
5.6.2 A simplified problem

We note that if the input signal is of a standard (steps, impulses, ramps) form the problem can also be simplified simply by choosing the matrix to be

\[
M_{\text{ramp}} = M_{\text{step}} = M_{\text{impulse}} = \begin{bmatrix}
Y^{(1)}(t_0) & \cdots & Y^{(m)}(t_0) & t_0 & \cdots & t_0^{m+n_b} \\
\vdots & & \vdots & \ddots & \vdots & \vdots \\
\vdots & & \vdots & \ddots & \vdots & \vdots \\
Y^{(1)}(t_N) & \cdots & Y^{(m)}(t_N) & t_N & \cdots & t_N^{m+n_b}
\end{bmatrix}
\]

Further simplifications can be made using the shifted Legendre polynomials of appropriate order since the matrix becomes

\[
M'_L = \begin{bmatrix}
<Y^{(1)}(t),P_q(t)> & \cdots & <Y^{(m)}(t),P_q(t)> \\
\vdots & & \vdots \\
<Y^{(1)}(t),P_q(t)> & \cdots & <Y^{(m)}(t),P_q(t)>
\end{bmatrix}
\]

for \( q \geq m+n_b \) and \( q' = m+q-1 \).

5.7. Numerical results

In practice we provide an initial over estimate \( m \) of the model order. The matrix \( M \) is then computed by suitable numerical multiple integration of the input/output data sets; here the trapezoidal integration scheme is used. The inner product integrals which constitute \( M_L \) are computed numerically using the Newton-Cotes seven point formula. The results are quoted for measured data in full floating point form and also for such data following twelve bit analogue to digital conversion. The routine to find singular values of \( M \) reduces \( M \) to upper triangular form by Householder transformations. It then uses Givens plane rotations to reduce the triangular form to a bidiagonal form, and finally the QR algorithm is used to obtain the singular value decomposition (Wilkinson [18]). The routine used to find the eigenvalues of \( M_L \) involves the reduction of \( M_L \) to an upper Hessenberg form and the QR algorithm (Wilkinson [17]).
Example 1.

Firstly we consider a simple first order system

\[ \dot{x}(t) + x(t) = u(t) \]

The total observation time was chosen to be \( T_f = t_N = 5 \) with \( N = 500 \). The input was a unit-step i.e. \( r(t) = 1 \ \forall \ t > 0 \). Two different estimates (\( m=2 \) and \( m=4 \)) of the order were taken and the singular values of \( M_{step} \) and eigenvalues of \( M'_L \) are as shown in tables 5.1 and 5.2. Results for the situation in which the measured output differs from the actual output by a constant offset are also documented.

<table>
<thead>
<tr>
<th>Singular values of ( M_{step} )</th>
<th>( m=2 )</th>
<th>( m=4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No offset</td>
<td>Offset = 4.0</td>
<td>No offset</td>
</tr>
<tr>
<td>167.3</td>
<td>667.03</td>
<td>338.24</td>
</tr>
<tr>
<td>17.98</td>
<td>67.8</td>
<td>46.5</td>
</tr>
<tr>
<td>1.25</td>
<td>0.492</td>
<td>8.48</td>
</tr>
<tr>
<td>0.37.10^{-14}</td>
<td>0.11.10^{-13}</td>
<td>1.46</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.12.10^{-13}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.9.10^{-14}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.37.10^{-14}</td>
</tr>
</tbody>
</table>

Table 5.1. Singular values of \( M_{step} \) for example 1.

<table>
<thead>
<tr>
<th>Eigenvalues of ( M'_L )</th>
<th>( m=2 )</th>
<th>( m=4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No offset</td>
<td>Offset = 4.0</td>
<td>No offset</td>
</tr>
<tr>
<td>-0.175</td>
<td>-0.175</td>
<td>-0.162.10^{-1}</td>
</tr>
<tr>
<td>0.99.10^{-9}</td>
<td>0.99.10^{-9}</td>
<td>-0.10.10^{-9}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.18.10^{-10}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.33.10^{-11}</td>
</tr>
</tbody>
</table>

Table 5.2. Eigenvalues of \( M'_L \) for example 1.
The singular values were calculated on a computer which has $\varepsilon = 10^{-7}$ as the value of $\varepsilon$ such that $1 + \varepsilon > 1$. Hence values less than $10^{-7}$ are considered as negligible. From the above results we can see that the offset did not change the number of non-zero eigenvalues nor the number of zero singular values and in all cases the system order is correctly identified as one.

**Example 2.**

This second example consists of a second order system:

$$\ddot{x}(t) + 4\dot{x}(t) = 4r(t)$$

with the initial conditions

$$x(0) = 0.5, \quad \dot{x}(0) = -2.0$$

The input signal $r(t)$ was again a unit-step and the time interval for measurements was $T_f = T_N = 5$ with $N = 500$. The results shown below are for both types of data, i.e. floating point and following 12 bit analogue to digital conversion. Over estimates of the system order were chosen as $m = 3, 4$ and $5$. The singular values and eigenvalues were found to be:

<table>
<thead>
<tr>
<th>Real data</th>
<th>A/D</th>
<th>Real data</th>
<th>A/D</th>
<th>Real data</th>
<th>A/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>269.99</td>
<td>269.99</td>
<td>347.83</td>
<td>347.83</td>
<td>398.03</td>
<td>398.03</td>
</tr>
<tr>
<td>33.49</td>
<td>33.49</td>
<td>47.45</td>
<td>47.45</td>
<td>59.02</td>
<td>59.02</td>
</tr>
<tr>
<td>5.48</td>
<td>5.48</td>
<td>8.68</td>
<td>8.68</td>
<td>11.73</td>
<td>11.73</td>
</tr>
<tr>
<td>0.77</td>
<td>0.77</td>
<td>1.52</td>
<td>1.52</td>
<td>2.33</td>
<td>2.33</td>
</tr>
<tr>
<td>0.28</td>
<td>0.28</td>
<td>0.30</td>
<td>0.30</td>
<td>0.38</td>
<td>0.38</td>
</tr>
<tr>
<td>0.1.10^{-13}</td>
<td>0.13.10^{-4}</td>
<td>0.117</td>
<td>0.117</td>
<td>0.19</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.64.10^{-13}</td>
<td>0.14.10^{-4}</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.16.10^{-14}</td>
<td>0.30.10^{-5}</td>
<td>0.17.10^{-13}</td>
<td>0.12.10^{-4}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.11.10^{-13}</td>
<td>0.32.10^{-5}</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.43.10^{-14}</td>
<td>0.26.10^{-5}</td>
</tr>
</tbody>
</table>

Table 5.3. Singular values of $M_{\text{step}}$ for example 2.
The noise effect induced by A/D quantisation error makes determination of zero singular values and eigenvalues more difficult and in such instances the relative sizes of adjacent singular values and eigenvalues must be noted. It is then clearly possible to distinguish negligible values from non-zero values. Again we note that the system order is correctly identified as two.

**Example 3.**

This is the same example as the previous one with the exception that the output and the input are subject to additive polynomial disturbances i.e.

\[ g(t) = 1 + t \quad \text{and} \quad h(t) = 3 + 2t + t^2 \]

In such a situation we use the square matrix \((M_L^T)\) formulation. The inner products of the multiple integrals and the shifted Legendre polynomials had \(q = 6\) for \(m=3\), \(q = 7\) for \(m=4\) and \(q = 8\) for \(m=5\). The observation interval time was \(t_N = 5\) with \(N = 250\). The numerical results show that the number of non-zero eigenvalues \(M_L'\) is two in all cases.

**Table 5.4. Eigenvalues of \(M_L'\) for example 2.**

<table>
<thead>
<tr>
<th>(m=3)</th>
<th>(m=4)</th>
<th>(m=5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real data</td>
<td>A/D</td>
<td>Real</td>
</tr>
<tr>
<td>-0.141</td>
<td>-0.141</td>
<td>0.087</td>
</tr>
<tr>
<td>0.11</td>
<td>0.11</td>
<td>-0.03</td>
</tr>
<tr>
<td>(-0.4 \times 10^{-9})</td>
<td>(-0.16 \times 10^{-5})</td>
<td>(0.29 ( \times 10^{-11})</td>
</tr>
<tr>
<td>(+j0.15 \times 10^{-10})</td>
<td>(+j0.15 \times 10^{-10})</td>
<td>(-0.27 \times 10^{-5})</td>
</tr>
<tr>
<td>(-j0.15 \times 10^{-10})</td>
<td>(-j0.15 \times 10^{-10})</td>
<td>(-0.68 \times 10^{-6})</td>
</tr>
<tr>
<td>(-j0.15 \times 10^{-10})</td>
<td>(-j0.15 \times 10^{-10})</td>
<td>(-0.10 \times 10^{-11})</td>
</tr>
</tbody>
</table>

**Table 5.5. Eigenvalues of \(M_L'\) for example 3.**

<table>
<thead>
<tr>
<th>(m=3)</th>
<th>(m=4)</th>
<th>(m=5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.29 \times 10^{-1})</td>
<td>(-0.12 \times 10^{-2})</td>
<td>(0.48 \times 10^{-2})</td>
</tr>
<tr>
<td>(-0.65 \times 10^{-3})</td>
<td>(0.103 \times 10^{-3})</td>
<td>(-0.130 \times 10^{-4})</td>
</tr>
<tr>
<td>(-0.25 \times 10^{-8})</td>
<td>(-0.912 \times 10^{-8})</td>
<td>(-0.98 \times 10^{-7})</td>
</tr>
<tr>
<td>(-0.450 \times 10^{-8})</td>
<td>(0.26 \times 10^{-8})</td>
<td>(-0.69 \times 10^{-8})</td>
</tr>
</tbody>
</table>
Example 4.

A final example consists of a third order system where the characteristic polynomial has a repeated root. The system, subject to a step input \( r(t) = 1 \), is

\[
\dot{x}(t) + 5x(t) + 8\ddot{x}(t) + 4\dot{x}(t) = 4r(t)
\]

\[
\ddot{x}(0) = -14.0, \quad \dot{x}(0) = -2.0, \quad x(0) = 0.5
\]

with a corresponding characteristic polynomial

\[ P(s) = (s+2)^2(s+1) \]

The time interval of measurement is \( T_f = t_N = 8 \) with \( N = 400 \). Only one estimate of the order was taken, i.e. \( m = 5 \), and the results are quoted in full floating point.

<table>
<thead>
<tr>
<th>m=5</th>
<th>Singular values of ( M_{\text{step}} )</th>
<th>Eigenvalues of ( M_L' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No disturbance</td>
<td>No disturbance</td>
<td>( q(t) = 3 + 2t + t^2 )</td>
</tr>
<tr>
<td>( 225.883 )</td>
<td>(-0.117 )</td>
<td>( 0.67 \times 10^{-2} )</td>
</tr>
<tr>
<td>( 198.604 )</td>
<td>( 0.32 \times 10^{-1} )</td>
<td>(-0.114 \times 10^{-2} )</td>
</tr>
<tr>
<td>( 32.377 )</td>
<td>(-0.35 \times 10^{-3} )</td>
<td>( 0.311 \times 10^{-4} )</td>
</tr>
<tr>
<td>( 7.006 )</td>
<td>( 0.117 \times 10^{-10} )</td>
<td>(-0.22 \times 10^{-7} )</td>
</tr>
<tr>
<td>( 1.447 )</td>
<td>( 0.298 \times 10^{-10} )</td>
<td>( 0.71 \times 10^{-8} )</td>
</tr>
<tr>
<td>( 0.5001 )</td>
<td>( 0.1125 )</td>
<td></td>
</tr>
<tr>
<td>( 0.000321 )</td>
<td>( 0.229 \times 10^{-10} )</td>
<td></td>
</tr>
<tr>
<td>( 0.33 \times 10^{-13} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.6. Singular values of \( M_{\text{step}} \) and eigenvalues of \( M_L' \) for example 4.

The numerical values show that a repeated root of the characteristic polynomial has no effect in identifying the order of the system under study.
5.8. Summary

Two methods have been proposed for system order identification in continuous SISO systems; these can easily be extended to include MIMO model order determination. The first method is based on the behaviour of singular values of a non-square matrix the elements of which represent multiple integrals of input/output data sets. The second method is based on inner producting the columns of this non-square matrix with shifted Legendre polynomials to produce a square matrix. The number of zero eigenvalues of this square matrix determines the system order. The major advantage of these techniques over many others in the literature is that only one test is sufficient to exactly determine the order of the system under consideration. Non-zero initial conditions and additive polynomial disturbances at both input and output have no effect on finding the correct order. Although the methods can identify the order from data generated using a wide range of nonperiodic input signals, standard inputs such as steps, impulses and ramps are to be preferred since they significantly reduce the dimension of the problem.
CHAPTER 6.
6. LUMPED PARAMETER APPROXIMATION OF DISTRIBUTED PARAMETER SYSTEMS

The problem of approximating distributed parameter systems by lumped parameter models will be reduced to an identification problem where the unknown parameters can be identified from measured input/output data using various weighted residual methods. Numerical examples include application of the least squares method to identify lumped parameter models which approximate distributed models of parabolic (heat equation) and hyperbolic (wave equation) structures. The effect of noisy observations is also considered.

6.1. Introduction

Distributed parameter systems (DPS's) are systems where the state variables vary continuously in one or more spatial directions within physical system boundaries. Unlike continuous lumped parameter systems (LPS's) which are modelled by ordinary differential equations, continuous DPS's are modelled by partial differential or integro-differential equations. DPS dynamics of parabolic and hyperbolic type can in general be represented by an operator statement, i.e.

\[ \mathcal{L}_1(x(z,t)) = \mathcal{L}_2(x(z,t)) + u(z,t) \quad z \in \Omega \]  

(6.1)

with initial and boundary conditions

\[ x(z,0) = x_0 \quad z \in \Omega \]

\[ \mathcal{F}_x(x(z,t)) = 0 \quad z \in \Gamma \]

where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are partial differential operators, \( \Omega \) is the spatial domain with boundary \( \Gamma \), \( \mathcal{F}_x \) is a vector partial differential operator whose dimension is determined by the number of boundaries and \( u(z,t) \) is a forcing term. Even though DPS's as described by (6.1) are infinite dimensional, usually they must be controlled by finite dimensional controllers. There are two methods of designing controllers for such systems. The first method is to approximate the DPS to be controlled by a finite number of lumped parameter models and use classical methods based on lumped systems to design the controller. The second technique, called the direct method, is to design an infinite
dimensional controller then reduce its model for implementation purposes. Our concern is not how
to design controllers but is how to obtain a lumped parameter approximation to a distributed
parameter system. The obvious way is to decompose the spatial domain $\Omega$ into a sufficient, finite
number of smaller domains and obtain the corresponding mathematical description. This can be
achieved by discretizing the spatial domain using either finite difference or finite element
techniques. More details about discretization methods can be found in Tzafestas [79]. In both cases
the distributed parameter system will be decomposed into a finite number of subsystems which are
coupled through interaction variables. The corresponding mathematical description of the
decomposed system will have the form

$$\hat{L}_i(x_i(t)) = g(x_i(t), u_i(t), w_i) \quad (i = 1,2,...,N_s)$$

with the initial and boundary conditions

$$x_i(0) = x_{i0} \quad (i = 1,2,...,N_s)$$

$$\hat{L}_i(x_i(0)) \geq \Omega \quad (i = 1,2,...,N_s)$$

where $\hat{L}_i$ is an ordinary differential operator, $N_s$ is the number of subsystems and $w_i$ is the
interaction function which represent the interaction with other subsystems and which can be
expressed as

$$w_i = h(x_1, x_2, ..., x_k, ..., x_{N_s}) \quad k \neq i \quad (i = 1,2,...,N_s)$$

This approach to approximating DPS's by a finite number of interconnected lumped parameter
models has attracted the attention of some researchers. Macleod and Campbell [77] used lumped
discrete models of parallel and cascade types to approximate distributed systems. The parameters of
these models were identified from input/output data. The technique is based on spatial and temporal
pulse response derived from the Green's function associated with the boundary conditions. The
authors first derive a "parallel model-discrete space" from pulse responses at each point in space,
and "parallel model-discrete time" from the pulse response at various points in time. They then
combine the space and time pulse response models in various ways to obtain the overall model of
DPS. For the cascade modelling, the system was discretized in time and space into small segments
(using finite differences) and a series of interconnected identical transfer functions is formed from
the pulse response of each segment. This approach was applied to a diffusion type system. The
authors suggest that if the system is homogeneous the cascade model is preferable since it requires fewer spatial observations than the parallel model. However, if the system is not homogeneous the cascade type cannot be used. Anderson and Parks [78] approximated an LC transmission line, an RC line (heat equation), a towed cable and a deep-sea mine hauling pipe by lumped systems. Controllability properties were examined in the frequency domain in order to delimit the region of validity of the approximation with the conclusion that approximations were valid over a limited, low pass, temporal frequency band and fast acting (high frequency) controls do not give good approximations.

In the following sections we shall determine lumped parameter models which approximate a distributed parameter system at a given point in the spatial domain from direct observations of input/output data at that point. The problem will be reduced to that of identifying the parameters of lumped models using weighted residual methods applied to the integral equation representation of the system dynamics.

6.2. Problem statement

Given a spatial coordinate \( z = z_i \) of a subsystem at which the output \( x(t) \) is observed over an interval of time \([0, T_f]\) when the system is subject to an input signal \( u(t) \), the problem is to obtain a lumped parameter model of the form

\[
\dot{x}^{(n)}(t) + a_1 \dot{x}^{(n-1)}(t) + \cdots + a_n \dot{x}(t) = b_1 u^{(n-1)}(t) + \cdots + b_n u(t)
\]  

which conveys the important dynamics of the subsystem under consideration, where \( \dot{x}(t) \) is the output predicted by the lumped parameter model when excited with the input \( u(t) \).

6.3. Reduction to identification problem

It was previously stated that when identifying the parameters of a lumped parameter linear SISO system from input/output data it is better to use the equivalent integral equation representation for it is a more robust vehicle in determining the parameters. We therefore replace (6.3) by its equivalent integral equation

\[
\dot{x}(t) + \sum_{k=1}^{n} a_k \dot{x}^{(k)}(t) - \sum_{k=1}^{n} b_k U^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} = 0
\]  

(6.6)
where the parameters \( \{c_k; k=1,2,\ldots,n\} \) represent the effect of initial conditions. The problem is now to identify the parameter vector \( \Omega = \{a_k,b_k,c_k; k=1,2,\ldots,n\}^T \) from measurements made of the system output \( y(t) \) at a given point \( z_1 \) in the spatial domain \( \Omega \) and the system input \( r(t) \), such that (6.6) conveys the important dynamics of the distributed parameter system. This means that the input/output pair of the lumped parameter model (6.6) must be very close to that of the subsystem of coordinate \( z_1 \). If the measured input/output data \( r(t) \) and \( y(t) \) where

\[
\begin{align*}
    u(t) &= r(t) + p(t) \\
    y(t) &= x(t) + q(t)
\end{align*}
\]

are substituted into (6.6) the equation becomes

\[
y(t) + \sum_{k=1}^{n} a_k Y^{(k)}(t) + \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} c_k \int_{0}^{t} \frac{k-1}{(k-1)!} p(t) + \sum_{k=1}^{n} b_k P^{(k)}(t) = \mathcal{R}(\Omega, t) (6.7)
\]

where \( p(t), q(t) \) are the additive disturbances and \( \{p^{(k)}(t); k=1,2,\ldots,n\}, \{Q^{(k)}(t); k=1,2,\ldots,n\} \) are their successive integrals respectively, the residual \( \mathcal{R}(\Omega, t) \) arising from the substitution of \( x(t) \) and \( u(t) \) by \( y(t) - q(t) \) and \( r(t) + p(t) \) in equation (6.6). The parameter vector \( \Omega = \{a_k,b_k,c_k; k=1,2,\ldots,n\}^T \) can now be determined by forcing the residual \( \mathcal{R}(\Omega, t) \) close to zero on \([0, T_f]\) using various weighted residual methods described in chapter three.

Equation (6.7) and (4.5) previously used for order reduction problems are structurally identical. Since the order reduction problem from input/output data was treated as an identification problem, the lumped parameter approximation of distributed parameter systems will therefore be treated in exactly the same manner.

In practice the distributed parameter systems are approximated by a finite number \( N_5 \) of interconnected subsystems each of which is approximated by a lumped parameter model. So far we have formulated the problem of approximating a single subsystem by a lumped parameter model. However the technique can easily be extended to cover the approximation of all subsystems in one single formulation. The \( N_5 \) interconnected subsystems can be considered as single MIMO system where \( r \) inputs give rise to \( m \) outputs. The problem can therefore be treated as a MIMO lumped parameter approximation and formulated in a similar way as that of MIMO system parameter identification.

**6.4. Numerical results**

In this section the approximation of DPS's described by various heat and wave equations are
approximated by lumped parameter models is considered. The parameters of the lumped models are determined from input/output data by least squares point collocation (3.22), the numerical implementation of which is given in §3.5.1.

The effect of noisy data on the lumped model parameters is investigated when approximating the one-dimensional heat equation. Other aspects of the deduced lumped parameter model such as its order and its frequency domain characteristics are also investigated.

6.4.1 One-dimensional heat equation

Consider the following partial differential equation which represents a mathematical model for the heat conduction in a thin bar

\[
\frac{\partial x(z,t)}{\partial t} = \alpha \frac{\partial^2 x(z,t)}{\partial z^2}, \quad \alpha > 0
\]

(6.13)

with boundary and initial conditions

\[
\begin{align*}
  x(0,t) &= \beta \\
  x(1,t) &= 0 \\
  x(z,0) &= 0 \quad z \in [0, 1]
\end{align*}
\]

(6.14)

which models a step input in temperature at one end of the bar. The solution of the partial differential equation (6.13) subject to the latter initial and boundary conditions is given by

\[
x(z,t) = \beta (1-z) - \frac{2\beta}{\pi} \sum_{k=1}^{\infty} \frac{\sin(\pi k z) \exp(-\pi^2 k^2 \alpha t)}{k}
\]

(6.14)

This analytical solution is used to generate \( N = 300 \) output data points at a fixed coordinate \( z = 0.7 \) and at equidistant intervals \( \Delta T = 0.02 \) of time, i.e. \( T_f = 6 \). The constants of the solution were chosen as \( \beta = 10 \) and \( \alpha = 1 \). The parameters of the approximating lumped models are determined for several values of the chosen order \( n \) and the results are shown in table 6.1 below in the form of transfer functions expanded into partial fractions.
The results in the above table indicate a lumped parameter model of order three should yield a reasonable approximation to the distributed. The simplifications occur through pole/zero cancellations in the global transfer function every time the identified lumped parameter model is assumed to be of order \( n \) greater than three. We note that the parameters of the resulting transfer functions for \( n > 3 \) are very close to those of the third order transfer function. We can also note that the important characteristics which are inherent in the solution (6.14) are well reflected by the transfer function of the approximating lumped model. The dominant modes of the solution (6.14), i.e. when \( k = 1 \) and 2 are close to those of the transfer function. This can be seen in the exponential expressions which constitute the transient part of (6.14) and that of the lumped parameter model which approximates it, i.e.

\[
\exp(-\pi^2 t) = \exp(-9.83t) \quad \text{and} \quad \exp(-4\pi^2 t) = \exp(-38t)
\]

The transient step response predicted by the third order lumped parameter model and that of the system at \( z = 0.7 \) are shown in Fig 6.1. To complete the comparison of the two models we assess the performance of the lumped parameter model in the frequency domain and compare it to that of the original distributed parameter system. The transfer function of the distributed parameter system was calculated analytically and found to be:

<table>
<thead>
<tr>
<th>Order ( n )</th>
<th>Transfer function</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( F(s) = \frac{1.945}{s + 6.481} )</td>
</tr>
<tr>
<td>2</td>
<td>( F(s) = \frac{5.339}{s + 9.95} - \frac{6.665}{s + 27.17} )</td>
</tr>
<tr>
<td>3</td>
<td>( F(s) = \frac{5.065}{s + 9.837} - \frac{13.196}{s + 38.47} + \frac{7.793}{s + 60.82} )</td>
</tr>
<tr>
<td>4</td>
<td>( F(s) = \frac{5.065}{s + 9.837} - \frac{13.169}{s + 38.46} + \frac{7.768}{s + 60.90} + \varepsilon(s) )</td>
</tr>
<tr>
<td>5</td>
<td>( F(s) = \frac{5.064}{s + 9.837} - \frac{13.50}{s + 38.64} + \frac{8.092}{s + 85.00} + \varepsilon(s) )</td>
</tr>
<tr>
<td>8</td>
<td>( F(s) = \frac{5.056}{s + 9.834} - \frac{12.611}{s + 38.60} + \frac{7.75}{s + 69.11} + \varepsilon(s) )</td>
</tr>
</tbody>
</table>

Table 6.1.
Fig. 6.1. The responses of the system (at z=0.7) and the third order lumped model.

(One dimensional wave equation)
\[
G(z,s) = \frac{\sinh \sqrt{\frac{s}{\alpha}} (1-z)}{\sinh \sqrt{\frac{s}{\alpha}}}
\]  

(6.15)

where \( z = 0.7 \) and the transfer function of the lumped parameter model approximation when \( n = 3 \) was

\[
F(s) = \frac{-0.184s^2 - 53.036s + 6904.132}{s^3 + 109.121s^2 + 3316.167s + 23013.776}
\]

(6.16)

The performance of the two transfer functions \( G(z,s) \) and \( F(s) \) in the frequency domain is shown by the Nyquist plot Fig. 6.2 and the Bode diagram Fig. 6.3 and Fig. 6.4. It can be seen the approximation is excellent up to a frequency of 100 rad/s.

To see the effect of noise on the parameters of the lumped model we corrupted the solution (6.14) with a Gaussian noise \( N(0, 0.06) \) and used the resulting signal in the identification scheme. The transfer function was identified as

\[
F(s) = \frac{1.906s^2 + 0.6099s + 5.594}{s^3 + 6.708s^2 + 4.793s + 18.662}
\]

and the predicted transient step response of this lumped model is shown together with the measured and the true system step responses in Fig. 6.5.

6.4.2 Two-dimensional heat equation

A thick pipe (inner radius \( r_1 = 1 \) cm, outer radius \( r_2 = 4 \) cm) is half iron and half copper. The pipe is immersed in a liquid which maintains the outer temperature at \( x(r_2, t) = 0^\circ c \) while steam at \( x(z, t) = 20^\circ c \) (\( |z| < r_1 \)) passes through the interior. It is assumed that the steam causes a convective heat flux of 0.38 \( (20 - x(z,t))^1.25 \) cal / cm / s. The initial temperature is \( x(z,0) = 0^\circ c. \)
Fig. 6.2. The frequency responses of the system (at $z=0.7$) and the third order lumped model.
Fig. 6.3. Frequency analysis of the Gains.
Fig. 6.4. Frequency analysis of the phases.
Fig. 6.5. The effect of noise on the lumped parameter model approximation.
The heat conduction equation describing the above system:

$$C_p \rho \frac{\partial x(z,t)}{\partial t} = \alpha \left[ \frac{\partial^2 x(z,t)}{\partial x_1^2} + \frac{\partial^2 x(z,t)}{\partial x_2^2} \right] + F(x,z,t) \quad (6.17)$$

where $x_1$ and $x_2$ are the coordinates of $z$, i.e. $z = (x_1, x_2)$ and $C_p$ heat capacity, $\rho$ density, $\alpha$ conductivity. It is clear that in our case $F(x,z,t) = 0$ since we do not have any heat generation or heat loss. For the numerical example the characteristics of the material were:

<table>
<thead>
<tr>
<th></th>
<th>copper</th>
<th>iron</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_p$</td>
<td>0.09</td>
<td>0.11</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.94</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho$</td>
<td>8.96</td>
<td>7.87</td>
</tr>
</tbody>
</table>

A numerical package (TWODEPEP [119]) using finite elements for solving partial differential equations in two dimensions and was employed to generate $N = 500$ output data points at equidistant intervals $\Delta T = 0.02$ of time and at a coordinate point $z = (-1.5, 1.5)$. The transfer function obtained for a selected order $n = 3$ was:

$$F(s) = \frac{-0.0023s^2 + 1.4210s + 0.7315}{s^3 + 8.9526s^2 + 10.761s + 3.0296}$$

The transient step response given by the above model is good approximation to the step system response solution as shown in Fig 6.6.
Fig. 6.6. The responses of the system and the third order lumped model.

(Two dimensional heat equation)
7.4.3 One-dimensional wave equation

A further example consists on studying small transverse vibrations of a small stretched string.

The string, initially of unit length, fixed at both ends, is subject to a constant tension and an external force $u(z,t)$ per unit length acting normally to the string.

Associated assumptions are:
- The string is perfectly flexible.
- The displacements $x$ are small compared to the length.
- The tension $T$ is constant all time and for $x \in [0,1]$.
- The mass of the string is negligible.

Under the above assumptions the partial differential equation describing this system is of the form

$$ \frac{\partial^2 x(z,t)}{\partial t^2} = \frac{T}{\rho p} \frac{\partial^2 x(z,t)}{\partial z^2} + \frac{u(z,t)}{\rho} $$

where $\rho$ is the density per unit length of the string, with boundary conditions

$$ x(0,t) = 0 \quad t > 0 $$
$$ x(1,t) = 0 $$

and initial conditions

$$ x(z,t) = g(z) \quad \frac{\partial x(z,t)}{\partial t} = 0 \quad \text{at } t = 0 $$

where $g(z)$ is chosen to be:

$$ g(z) = \begin{cases} 
  -2\alpha z & 0 \leq z \leq \frac{1}{2} \\
  -2\alpha(1-z) & \frac{1}{2} \leq z \leq 1 
\end{cases} $$

Letting $c = F / \rho$, $\lambda_k = c k \pi$ and choosing as input $u(z,t) = A \sin(\omega t)$ the analytical solution of the above problem is

$$ x(z,t) = \sum_{k=0}^{\infty} G_k(t) \sin(k \pi z) $$

(6.19)
where

\[ G_k(t) = B_k \cos(\lambda_k t) + B_k^* \sin(\lambda_k t) + 2A(1 - \cos(k\pi)) \frac{\sin(\alpha t)}{k\pi(\lambda_k^2 - \omega^2)} \]

the constant \( B_k \) and \( B_k^* \) being

\[ B_k = -\frac{8\alpha}{(k\pi)^2} \sin \frac{k\pi}{2}, \quad B_k^* = -\frac{2A\alpha(1 - \cos k\pi)}{\lambda_k k\pi(\lambda_k^2 - \omega^2)} \]

\( \lambda_k \neq \omega \)

The input chosen was \( u(z,t) = \sin(2t) \), the constant \( c = F/I = 1 \), \( \alpha = 0.0001 \) and the analytical solution (6.19) was used to generate \( N = 1000 \) output data points at \( z = 0.7 \) and at equidistant intervals of \( \Delta T = 0.01 \). The transfer function was identified as

for \( n = 2 \)

\[ F(s) = \frac{0.00005s + 1.038}{s^2 + 0.00046s + 9.892} = \frac{0.00005(s + 20760)}{(s + 0.0002 - j3.145)(s + 0.0002 + j3.145)} \]

for \( n = 3 \)

\[ F(s) = \frac{0.0018s^2 + 1.037s + 3.3608}{s^3 + 3.242s^2 + 9.886s + 32.007} = \frac{0.0018(s + 3.25)(s + 566s)}{(s + 3.24)(s + 0.0013 + j3.143)(s + 0.0013 - j3.143)} \]

\[ \frac{0.0018(s + 566s)}{(s + 0.0013 + j3.143)(s + 0.0013 - j3.143)} \]

The pole/zero cancellation when \( n = 3 \) seems to suggest that a lumped parameter which approximates the string dynamics at \( z = 0.7 \) is of order two. We also note that the dominant harmonic (the fundamental) in the solution (6.19) has been identified, i.e. \( \lambda_1 = \pi = 3.14 \). Fig.6.7 shows how well the string vibrations at \( z = 0.7 \) can be approximated by a second order lumped parameter model response to the same input \( u(t) \).
Fig. 6.7. The responses of the system and the second order lumped model.

(One dimensional wave equation)
6.4.4 Two-dimensional wave equation

An elastic square membrane of density per unit area $\rho$, is subject to a force $u(z,t)$ perpendicular to its surface. The membrane is fixed along its boundaries. The partial differential equation describing the displacements along the force axis together with the initial and boundary conditions are as follows:

$$\frac{\partial^2 x(z,t)}{\partial t^2} = c^2 \left( \frac{\partial^2 x(z,t)}{\partial x_1^2} + \frac{\partial^2 x(z,t)}{\partial x_2^2} \right) + \frac{u(z,t)}{\rho}$$

(6.20)

where $x_1$ and $x_2$ are the spatial coordinates. The initial conditions are

$$x(z,t) = g(z)$$
$$\frac{\partial x(z,t)}{\partial t} = 0$$

at $t = 0$

and the appropriate boundary conditions are

$$x(0,x_2,t) = 0$$
$$x(1,x_2,t) = 0$$
$$x(x_1,0,t) = 0$$
$$x(x_1,1,t) = 0$$

$t \geq 0$

The parameters of the partial differential equation were chosen to be $c = \rho = 1$. For a sinusoidal input $u(z,t) = A \sin(\omega t)$ and initial displacement of the form $g(z) = \alpha x_1 x_2$ the analytical solution to the above problem is given by

$$x(x_1, x_2,t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[ B_{ij} \cos \lambda_{ij} t + B_{ij}^* \sin \lambda_{ij} t + \frac{4A(1-\cos\phi)(1-\cos\omega t)\sin\alpha}{ij\pi^2(\lambda_{ij}^2 - \omega^2)} \right] \sin(i\pi x_1)\sin(j\pi x_2)$$

(6.21)

where $\lambda_{ij}^2 \neq \omega^2$ and

$$B_{ij} = -4A\alpha(1-\cos\phi) \frac{1}{ij\pi^2(\lambda_{ij}^2 - \omega^2)}$$

$$\lambda_{ij}^2 = \kappa^2(1 + j^2) \quad \text{and} \quad B_{ij} = 4\alpha \cos(\phi) \cos(\omega t) \frac{1}{ij\pi^2}$$
For an input \( u(z,t) = 10 \sin(0.5t) \), \( z = (0.7, 0.7) \), \( \alpha = 0.001 \) and \( N = 700 \) points given at equidistant intervals \( \Delta T = 0.01 \) of time the transfer function identified was:

for \( n = 2 \)

\[
F(s) = \frac{0.0006s + 1.095}{s^2 + 0.0189s + 19.979} = \frac{0.0006(s + 1781.0)}{(s + 0.0054 + j 4.470)(s + 0.0054 - j 4.470)}
\]

for \( n = 3 \)

\[
F(s) = \frac{0.98 \times 10^{-5}(s + 0.1109 \times 10^6)(s - 1.749)}{(s + 0.0020 + j 4.46)(s + 0.0020 - j 4.46)(s - 1.748)} \frac{0.98 \times 10^{-5}(s + 0.1109 \times 10^6)}{(s + 0.002 + j 4.46)(s + 0.002 - j 4.46)}
\]

Here again the pole/zero cancellation seems to suggest that a reasonable lumped parameter model is of order two. The dominant harmonic in the solution (6.21) has also been well identified, i.e \( \lambda_{11} = \pi \star 2^{1/2} = 4.44 \). The approximation by a second order model is a good approximation as it can be seen in Fig.6.8 where the predicted and the system transient responses are shown.

6.5. Summary

We have investigated the problem of approximating DPS's from input/output data by lumped parameter models. The problem was reconstituted as that of identifying the parameters of the integral equation representation of linear systems using weighted residual methods discussed in chapter three. Excellent results were obtained following the application of the least squares point collocation method to problems of approximating distributed parameter systems of both parabolic and hyperbolic type, at a desired point of in the spatial domain, by lumped parameter models. The applications include approximation of heat conduction in a thin bar, heat convection in a thick pipe and forced vibrations of an elastic string and an elastic membrane. In all applications the transient behaviour of the distributed system was well approximated by the corresponding lumped model transient behaviour. The dominant modes of the original system were also well identified. The effect of noisy observations is included in the application of approximating the heat equation. Although the applications given were for approximating DPS's at one single point in the spatial domain the whole system can be approximated by several lumped parameter models using measurements at several points and formulating the problem as that of MIMO parameter identification.
Fig. 6.8. The responses of the system and the second order lumped model.
(Two dimensional wave equation)
CHAPTER 7.
7. CONCLUSIONS AND FUTURE WORK

An alternative mathematical formulation for describing continuous system dynamics has been introduced. This integral equation formulation explicitly reflects the effect of initial conditions of the system under consideration and a technique to recover these conditions is given. This representation of system dynamics has also an important property of rejecting the noise and therefore remains a valid description of continuous systems in noisy situations. The integral equation is extended to cover the description of linear MIMO and nonlinear system dynamics under a wide range of operating conditions, i.e. with various descriptions of additive disturbances. The formulation is derived in generality for linear systems though it is clear that careful individual formulation is required for nonlinear systems.

The basic concept of weighted residual methods, classically used to produce approximate solutions to ordinary differential equations, is extended to two fundamental problems in control engineering, i.e. continuous system identification and model reduction. The two latter problems were grouped with the problem of ordinary differential equation solution and an operator statement representation of system dynamics was used as a unifying background.

The system integral equation from which a residual is formed is used as a basic vehicle for parameter identification of continuous linear SISO, MIMO and nonlinear systems using various weighted residual methods. The equation residual is formed by selecting an integral equation of correct structure but unknown parameters to describe the dynamics of the system. The parameters of this integral equation are then identified from the observed system input/output data using weighted residual methods. The effect of additive polynomial disturbances is included in the analysis and several numerical examples illustrating the application of weighted residual methods to system parameter identification problems in the presence of unknown offsets are given. The application to the nonlinear system identification problem is illustrated by a practical example where the parameters of a nonlinear ordinary differential equation describing the unforced rolling motion of a ship are correctly identified. A feature of the integral equation formulation is that parameters arise naturally from non-zero initial conditions and must be identified alongside the integral equation parameters. It is also essential to include parameters in the estimation that may not be of significant interest such as parameters of unquantified additive disturbances should any be thought to be present. However, considerable simplification can be achieved when the identification problem is solved using the Galerkin method with high order shifted Legendre or Chebyshev polynomials as weighting functions in the attendant inner products. This eliminates
the effect of initial conditions and can even eliminate the effect of polynomial disturbances from the problem. Therefore since the weighted residual methods can be extended to either identify the coefficients of any polynomial disturbance or eliminate them (using Galerkin methods with high order polynomials) from the equation residual thereby formed we conclude that the integral equation serves the system identification problem well in an engineering environment where the most common additive disturbances are offsets (due to bad calibration of apparatus) and zero mean Gaussian noise. Among the various weighted residual methods applied to parameter identification problems the least squares point collocation method is preferred by the author for it is the only optimisation scheme and requires less computational effort. However, if the system observed input/output data is subject to high order polynomial additive disturbances then Galerkin methods become preferable for their ability to eliminate the effect of such disturbances from the problem. For multi-input multi-output system parameter identification the formulation is given such that the number of unknowns to be estimated is kept to a minimum and the reduction in order of any ordinary differential equation is indicated by pole/zero cancellation in the identified transfer function.

A further feature of using integral equation for system identification purposes is that one must be careful in choosing the input signal to generate the system output data points. Standard inputs such as impulses, steps and ramps are excluded from consideration in the presence of non-zero initial conditions and/or additive disturbances of polynomial form. Exponential and sinusoidal signals also demand careful consideration as to whether or not they may be used as inputs. Such inputs and disturbances generate similar terms in the integral equation and do not permit the individual influence of the input to be identified.

Care must also be taken in selecting the step size of integration when using the trapezoidal rule as the quadrature scheme to compute the multiple integrals of the system input and output which constitute the equation residual. Systems with fast modes require smaller step size in order to pick up more information about the fast decaying modes of the system response. This leads to using larger amount of data points since information about the slow transient and steady state modes of the system response is also needed. The advantage of using the trapezoidal rule is that it has an easy nesting process for computing the successive multiple integrals of the system input/output data from the previous integrals, therefore simplifying the implementation procedure.

When applying weighted residual methods to linear parameter systems described by their integral equation, model parameter estimates result from the solution of a set of linear equations and hence any convergence difficulties associated with nonlinear problems generated by weighted signal residual methods are avoided. Additionally computational effort is kept to a minimum by
the use of the integral equation residual. As a guide, computation times using the integral equation residual are typically two orders of magnitude smaller than when using the signal residual in problems with four or five parameters.

The problem of model reduction using input/output data has been reduced to that of identifying the parameters of the reduced order model using the integral equation representation of system dynamics to form a corresponding equation residual. The equation residual is formed by choosing an integral equation of incorrect structure and unknown parameters to approximate the important dynamics of the original system. The parameters of this integral equation residual are then identified by forcing the residual close to zero on the observation time interval using weighted residual methods. The integral equation serves the model reduction problem well since it restrains the effect of the system fast modes in the equation residual via repeated integration of the data. Therefore any residual thereby formed is primarily dependent on the system gain and slow transient modes. Excellent approximations of linear SISO and MIMO systems were obtained using several weighted residual methods applied to the integral equation residual thereby formed. In particular excellent results were produced for a practical problem of approximating the dynamics of an automatic voltage regulator (a.v.r.) by a model of lower order. The a.v.r. model was characterised by the interaction of very fast and very slow modes, thus making any simulation demand a large computational effort. The reduced model approximant thereby obtained reduces the simulation time by forty percent and produces a very good performance in the control of the circuit voltage.

Two methods for continuous SISO system order determination have also been proposed. The first method deduces the system order from the number of non-zero singular values of a non-square matrix the elements of which are formed from repeated integration of the observed system input/output data. The second method uses shifted Legendre polynomials to square the latter matrix then deduces the system order from the number of non-zero eigenvalues of the square matrix thereby obtained. The advantage of both techniques is that given an initial estimate of the model order which is greater than the true order of the system, the order of the model is found by a precise formulation. The analysis includes the effect of additive disturbances and excellent results were obtained following the application of both methods to identifying the order of continuous SISO systems from noisy input/output data, in the presence of additive polynomial disturbances and non-zero initial conditions. The computational effort is considerably reduced by the use of standard input signals such as impulses, steps and ramps. The second method is preferable when the system input/output data is subject to high order polynomial additive disturbances since the method can be used to eliminate such disturbances using higher order shifted Legendre polynomials.

Approximation of distributed systems by lumped parameter models using input/output data is
also reduced to that of lumped parameter identification using an integral equation residual. The least squares method was successfully applied to approximate various distributed parameter systems of the parabolic and hyperbolic types. In the examples investigated, lumped parameter approximation accurately identified the dominant time constants of the distributed system.

**Future work**

The integral equation formulation was derived in the context of non-delay time-invariant continuous systems. Further research is required to extend the formulation to time-varying and to delayed systems and investigate the application to identification of such systems.

When solving the problem of system identification, model reduction and lumped parameter approximation of distributed parameter systems using the least squares method, the input/output data and its successive multiple integrals are required as one batch of numbers over the whole observation time interval. This entails a heavy data storage overhead especially in the case of MIMO system identification problems. Further work is needed to overcome this situation. One possibility is to reformulate the problem such that data is processed sequentially, therefore producing an on-line identification method. This can be achieved for example by using the recursive least squares given by Young [50] applied to successive rows of the integral equation residuals.

The purpose of obtaining a reduced-order model may be to design a control or a filter based on the model and thus have simpler design. Although the performance of the reduced model obtained using weighted residual methods applied to an integral equation residual was judged as satisfactory when compared to that of the original model, future investigation is needed to see whether the performance of any controller or filter whose design is based on the reduced model thereby obtained is satisfactory when applied in the control of the original system. When approximating distributed parameter systems by lumped parameter models similar investigations are needed to assess the performance of any controller or filter whose design is based on the lumped parameter model approximant.
REFERENCES
REFERENCES

1. Order determination

[1] Chan, C. W., Harris, C. J. and Wellstead, P. E.


[6] Soderstrom, T.

[7] Soderstrom, T.

[8] Young, P., Jakeman, A. and McMeurtrie, R.


[12] Soderstrom, T.


[14] Unbehauen, H. and Gohring, B.


2. LPS identification

[19] Ljung, L. and Glover, K.

[20] Ljung, L.
"On the Estimation of Transfer Functions", Automatica, 21, 6, 1985, pp. 677-696.

[21] Isermann, R., Baur, U., Bamberger, W., Kneppo, P. and Siebert, H.

[22] Saridis, G. N.
[23] Eykhoff, P.


[26] Rake, H.

[27] Astrom, K. J. and Eykhoff, P.

[28] Balakrishnan, A., V. and Peterka, V.


[30] Young, P.C.

[31] Saridis, G.N.


[33] Unbehauen, H.

[34] Strejc, V.


[36] Strejc, V.
[37] Ichikawa, K.

[38] Hwang, C. and Guo, T. Y.


[40] Janiszowski, K.

[41] Astrom, K. J.

[42] Puthenpura S. and Sinha, N. K.

[43] Peterka, V.
"Bayesian System Identification", *Automatica*, 17, 1, 1981, pp. 41–53.

[44] Godfrey, K. R.

[45] Halme, A. , Selkainaho, J. and Soininen, J.

[46] Ljung, L.

[47] Edmunds, J. M.

[48] Gawthrop, P. J.
[49] Roberts, J. B.

[50] Young, P. C.

[51] Whitfield, A. H. and Messali, N.

3. DPS identification

[52] Polis, M. P. and Goodson, R. E.

[53] Polis, M. P.

[54] Kubrusly, C. S.

[55] Courdesses, M., Polis, M. P. and Amouroux, M.

[56] Wozny, M. J. and Carpenter, W. T.

[57] Perdreauville, F. J. and Godson, R. E.

[58] Collins, P. L. and Khatri, H. C.
[59] Seinfeld, J. H.

[60] Fairman, F. W. and Shen, D. W. C.

[61] Seinfeld, J. H. and Chen, W. H.


[63] Hamza, M. H. and Sheirah, M. A.

[64] Polis, M. P., Goodson, R. E. and Wozny, M. J.

[65] Ward, E. D. and Goodson, R. E.

[66] Kobayashi, T. and Hitotsuya, S.

[67] Kobayashi, T.

[68] Sehitoglu, H.

[69] Kitamura S. and Nakagiri, S.
[70] Pierce, A.

[71] Banks, H. T. and Lamm, P. D.

[72] Dale, O. B. and Cohen, R.

[73] Carpenter, W. T., Wozny, M. J. and Goodson, R. E.

[74] Sunahara, Y., Osumi, A. and Imamura, M.

[75] Kubrusly, C. S. and Curtain, R. F.

[76] Courdesses, M.

[77] Macleod, C. J. and Campbell, D. R.

[78] Anderson, B. D. O. and Parks, P. C.

[79] Tzafestas, S.G.
4. Order reduction

[80] Hickin, J. and Sinha, N. K.

[81] Bonvin, D. and Mellichamp, D. A.

[82] Mahapatra, G.B.


[84] Kokotovic, P. C., O'Malley, R. E. and, Sannuti, P.

[85] Litz, L. and Roth, H.

[86] Lastman, G. J., Sinha, N. K. and Rozsa, P.
"On the Selection of States to be Retained in a Reduced", I.E.E. Proceedings, 131, 1, 1984, 15–22.

[87] Daly, K. C. and Colebourn, A. P.

[88] Warwick, K.

[89] Shamash, Y.

[90] Chen, C. F. and Shieh, L. S.
[91] Zakian, V.

[92] Hutton, M. and Friedland, B.

[93] Hutton, M. and Rabin, M. J.

[94] Lepschy, A. and Viaro, U.

[95] Lucas, T. N. and Davidson A. M.

[96] Farsi, M., Warwick, K. and Guilandoust, M.

[97] Wilson, D. A.

[98] Riggs, J.B. and Edgar, T. F.

[99] Wilson, D. A. and Mishra R. N.

[100] Eitelberg, E.

[101] Sinha, N. K. and Bereznai, G. T.

[102] Sinha, N. K. and Pille, W.
[103] Genesio, R. and Pomè, R.


[104] Bistritz, Y. and Langholz, G.


[105] Subbayan, R. and Vaithilingam, M. C.


[106] El-Attar, R. A. and Vidyasagar, M.


[107] Langholz, G. and Bistritz, Y.


[109] Glover, K.


[110] Hassan, M. M. M. and Cook, P. A.


[111] Paraskevopoulos, P.N.


[112] Finlayson, B. A.


[113] Whitfield, A. H. and Messali, N.

[114] Whitfield, A. H. and Williams, N. G.


[116] Shamsh, Y.

[117] Chen, T. C., Chang, C. Y. and Han, K. W.

[118] Harshavardhana, P., Jonckeere, E. A. and Silverman, L. M.

[119] Sewell, G.
Appendix A.

AN INTEGRAL EXPRESSION FOR SYSTEM IDENTIFICATION
AND
MODEL ORDER REDUCTION

We shall introduce an alternative technique for system identification and model reduction problems which eliminates the effect of non-zero initial conditions. The technique is based on an appropriate choice of a set of functions which are \( C^n \) differentiable on the observation time interval \([0, T_f]\) and which vanish together with their derivatives up to order \((n-1)\) on the boundaries of this interval. A numerical example will be given to illustrate the application of the method.

A.1. Statement of the problem

Consider a general linear single-input single-output system of order \( n \) described by the ordinary differential equation

\[
\begin{align*}
\frac{d^m}{dt^m} x(t) + \sum_{i=1}^{n} a_i x^{(i)}(t) - \sum_{i=1}^{m} b_i u^{(m-i)}(t) &= 0 \quad (m \leq n) \\
\end{align*}
\]  
(A.1)

and suppose that we wish to identify the parameters \( \{ a_i; i=1,2,\ldots,n \} \), \( \{ b_i; i=1,2,\ldots,m \} \) from observations made of the system output \( x(t) \) and the system input \( u(t) \), without having to perform any numerical differentiation of the signals, which in practice may lead to serious errors.

A.2. Problem solution

The parameter vector \( \Theta = (a_i, b_j; i=1,2,\ldots,n; j=1,2,\ldots,m)^T \) to be identified has \( N_p = (n+m) \) components. We therefore need a set of \( N_p \) linearly independent algebraic equations to uniquely determine the \( N_p \) parameters. Let \( \{ \phi_j(t); j=1,2,\ldots,N_p \} \) be a set of \( N_p \) functions which are at least \( C^n \) differentiable on the observation time interval \([0,T_f]\) and which vanish together with their derivatives up to order \((n-1)\) on the boundaries of this interval of observations, i.e.

\[
\{ \phi_j^{(k)}(t); k=0,1,\ldots,n; j=1,2,\ldots,N_p \} \text{ are continuous on } [0,T_f]
\]

and

\[
\{ \phi_j^{(k)}(T_f), \phi_j^{(k)}(0); k=0,1,\ldots,n-1; j=1,2,\ldots,N_p \} = 0
\]
The $N_p$ required equations are then generated by multiplying equation (A.1) by each of the functions \{$\phi_j(t); j=1,2,...,N_p$\} and integrating the resulting equations over the observation time interval $[0,T_f]$. The result of this operation leads to the following equations

$$
\sum_{i=1}^{n} a_i \int_{0}^{T_f} \phi_j(t)x^{(n-i)}(t)dt - \sum_{i=1}^{m} b_i \int_{0}^{T_f} \phi_j(t)u^{(m-i)}(t)dt = 0 \quad (j=1,2,...,N_p)
$$

(A.2)

Before performing the integration of equation (A.2), we need to find the integral of the general expression

$$
\int_{0}^{T_f} \phi_j(t)g(t)dt
$$

\begin{equation}
(A.3)
\end{equation}

where $g(t)$ is any function of time. Using successive integration by parts expression (A.3) becomes

$$
\begin{align*}
\int_{0}^{T_f} \phi_j(t)g^{(i)}(t)dt &= \sum_{k=1}^{i} (-1)^{k-1} \int_{0}^{T_f} \phi_j(t)g^{(k)}(t)dt + (-1)^i \int_{0}^{T_f} \phi_j(t)g^{(i)}(t)dt \\
&= (-1)^i \int_{0}^{T_f} \phi_j(t)g^{(i)}(t)dt
\end{align*}
$$

\begin{equation}
(A.4)
\end{equation}

and since \{$\phi_j^{(k)}(T_f); \phi_j^{(k)}(0); k=0,1,...,n-1; j=1,2,...,N_p$\} = 0 the final expression will become

$$
\int_{0}^{T_f} \phi_j(t)g^{(i)}(t)dt = \int_{0}^{T_f} \phi_j^{(0)}(0)g^{(i)}dt
$$

\begin{equation}
(i \leq n)
\end{equation}

and therefore, by analogy equation (A.2) becomes

$$
(-1)^{n} \int_{0}^{T_f} \phi_j^{(n)}(t)x(t)dt + \sum_{i=1}^{n} a_i (-1)^i \int_{0}^{T_f} \phi_j^{(n-i)}(t)x(t)dt - \sum_{i=1}^{m} b_i (-1)^i \int_{0}^{T_f} \phi_j^{(m-i)}(t)u(t)dt = 0 \quad (j=1,2,...,N_p)
$$

(A.5)

Having chosen the functions \{$\phi_j(t); j=1,2,...,N_p$\}, their derivatives up to order $n$ can be calculated and a set of $N_p$ equations in the system parameters with coefficients depending on $\phi_j(t)$ and their derivatives, the system input $u(t)$ and the system output $x(t)$ can be obtained.
A.3. Choice of the input

The input signal must be carefully selected for it is not difficult to generate an ill-conditioned problem. For instance if the input signal is a step then

$$\int_0^{T_f} \phi_j^{(i)}(t)u(t)dt = 0 \quad \forall i > 0 \quad \text{and} \quad \forall j = 1, 2, \ldots, N_p$$

and in the case of a simple sine wave input we have

$$\int_0^{T_f} \phi_j^{(i)}(t)u(t)dt = \alpha \int_0^{T_f} \phi_j^{(i+k)}(t)u(t)dt \quad \text{for} \ k = 1, 3, 5, \ldots$$

The impulse signal must also be excluded for the same reasons. A general criterion for choosing the input signal $u(t)$ is that it must be at least $C^m$ differentiable on the observation time interval $[0, T_f]$. Also together with its derivatives up to order $m$ must form a set of $(m+1)$ linearly independent signals on $[0, T_f]$.

A.4. The effect of an offset

Suppose that the observed input/output data are subject to some constant additive disturbances, i.e:

$$u(t) = r(t) + d_1$$
$$y(t) = x(t) + d_2$$

where $r(t)$ and $y(t)$ are the measured input/output data and, $u(t)$ and $x(t)$ are the true input/output pair of the system. Equation (A.1), when the system is subject to offsets would become

$$y^{(n)}(t) + \sum_{i=1}^{n} a_i y^{(n-i)}(t) - \sum_{i=1}^{m} b_i r^{(m-i)}(t) = a_d d_2 + b_m d_1 = cd$$ \hspace{1cm} (A.6)

If we differentiate equation (A.6) once we have:

$$y^{(n+1)}(t) + \sum_{i=1}^{n} a_i y^{(n-i+1)}(t) - \sum_{i=1}^{m} b_i u^{(m+i-1)}(t) = 0$$ \hspace{1cm} (A.7)
Equations (A.7) and (A.1) are of the same form except that (A.7) is of order \((n+1)\) rather \(n\). Therefore a choice of \(N_p\) at least \(\leq (n+1)\) differentiable functions \(\{\phi_j(t); j=1,2,\ldots,N_p\}\) whose derivatives up to order \(n\) vanish on the boundaries of \([0, T_f]\) would give a set of \(N_p\) algebraic equations of the form

\[
\begin{align*}
\sum_{i=1}^{n} a_i \phi_j^{(n+1)}(t) y(t) dt + \sum_{i=1}^{m} b_i \phi_j^{(m+1)}(t) r(t) dt = 0 & \quad (i=1,2,\ldots,N_p) \\
\end{align*}
\]

(A.8)

The technique can be extended to include additive disturbances of polynomial form.

There are two problems which arise when using this method for either identification or model reduction purposes. The first problem is that of finding functions \(\{\phi_j(t); j=1,2,\ldots,N_p\}\) which vanish together with their derivatives at both limits of the observation time interval. The second problem is that these functions may exert an undesirable time weighting on the residual.

A.5. Numerical application

Identification example

We consider the problem of identifying the parameter of a linear SISO system whose transfer function is

\[
F(s) = \frac{3s^2 + 2s + 6}{s^3 + 6s^2 + 11s + 6}
\]

The above system was simulated with an input \(u(t) = 2\sin 2t + 2\sin 3t\) after converting the ordinary differential equation into a state space observable form. The initial state vector was chosen as \(z_1(0) = 3, z_2(0) = 1, z_3(0) = 2\). One thousand (\(N = 1000\)) output data points were collected at equidistant intervals (\(\Delta T = 0.01\)) of time. The functions \(\{\phi_j(t); j=1,2,\ldots,6\}\) were selected as

\[
\phi_j(t) = (1 - \cos \frac{2\pi}{T_{f_i}})^3 
\]

where \(\{T_{f_i}; j=1,2,\ldots,6\}\) are the final the final times of observation and which were chosen as \(T_{f1} = 2, T_{f2} = 3, T_{f3} = 5, T_{f4} = 7, T_{f5} = 9, T_{f6} = 10\). This is an easy way of generating \(N_p\) function \(\phi_j(t)\) from one formula, i.e. by dividing the whole observation time interval into \(N_p\) smaller intervals and generating the \(N_p\) desired equations by multiplying the ordinary differential equation by \(\phi_j(t)\) before integrating the product on \([0, T_f]\). The derivatives of \(\{\phi_j(t); j=1,2,\ldots,6\}\) were calculated numerically using the following formulae...
With real data and no disturbances the transfer function was identified as

\[ G(s) = \frac{3s^2 + 2s + 6}{s^3 + 6s^2 + 11s + 6} \]

With an offset on the output (offset = 4) and using 12-bit A/D conversion the identified function was

\[ G(s) = \frac{3s^2 + 2.002s + 6.002}{s^3 + 5.998s^2 + 11.007s + 6.002} \]

**Model reduction example**

The example is the same as that previously considered in §4.4.1 where the original transfer function was

\[ F(s) = \frac{375000(s+0.08333)}{s^7 + 83.635s^6 + 4097s^5 + 70342s^4 + 853703s^3 + 2814271s^2 + 3310875s + 281250} \]

The system was simulated under the same conditions as before except for the input signal which was \( u(t) = t^2 \), because a step input is prohibited even with zero initial conditions. To approximate \( F(s) \) with a third order transfer function \( G(s) \) we needed six algebraic equations. The functions \( \{\phi_j(t); j=1,2,...,6\} \) were generated using the same formula as in the identification example above but this time with different \( \{T_{fj}; j=1,2,...,6\} \) which were \( T_{f1} = 5, T_{f2} = 6, T_{f3} = 7, T_{f4} = 8, T_{f5} = 9, T_{f6} = 10 \). The third transfer function was identified as

\[ G(s) = \frac{-0.094s^2 + 0.664s + 0.061}{s^3 + 4.227s^2 + 5.817s + 0.546} \]

Notice that the parameters of this transfer function are very close to those found in §4.4.1 using weighted residual methods.
Appendix B.

A COMPLEX DOMAIN POINT COLLOCATION METHOD FOR
MODEL REDUCTION

We shall introduce a new technique for system model reduction which is based on point collocation in the complex domain. This method is very simple and gives excellent results but requires further research to theoretically justify the selection of the collocation points. The suggestions given in this Appendix are based on purely experimental results.

B.1 Problem statement

Consider a system of order \( n_s \) described by the transfer function

\[
F(s) = \frac{\tilde{b}_m s^{m-1} + \ldots + \tilde{b}_n}{s + \tilde{a}_1 s^{n_s-1} + \ldots + \tilde{a}_n} = \frac{Q(s)}{P(s)} \quad m_s \leq n_s
\]  

The order reduction problem is that of finding the parameters \( \{a_k; k=1,2,\ldots,n_s\}, \) \( \{b_k; k=1,2,\ldots,m_s\} \) of a transfer function

\[
G(s) = \frac{b_1 s^{m-1} + \ldots + b_n}{s + a_1 s^{n-1} + \ldots + a_n} = \frac{B(s)}{A(s)} \quad m \leq n
\]  

where \( n \leq n_s \), such that, when subject to the same input \( u(t) \) the two transfer functions \( F(s) \) and \( G(s) \) give rise to almost same outputs.

B.2 Problem solution

Since the requirement is that the outputs given by the two transfer functions should be the same, the problem becomes that finding the parameter vector \( \Omega = (a_1,\ldots,a_n,b_1,\ldots,b_m)^T \) such that residual
\[ R(\Omega, s_i) = X(s) - \hat{X}(s) \]  

(B.3)

is forced close to zero, where \( \hat{X}(s) = G(s)U(s) \) and \( X(s) = F(s)U(s) \).

One of the standard weighted residual techniques we suggest to solve this problem is the point collocation method, i.e.

\[
\text{solve } R(\Omega, s_i) = 0 \quad i = 1, 2, \ldots \quad (B.4)
\]

Suppose that the set of collocation points \( \{s_i; i = 1, 2, \ldots\} \) contains \( n_c \) complex values and \( n_r \) real values, then for every complex value equation (B.4) yields two equations and for every real value it yields one equation only, i.e.

\[
\begin{align*}
\text{real } R(\Omega, s_i) &= 0 & i = 1, 2, \ldots, n_c \\
\text{img } R(\Omega, s_i) &= 0 & i = 1, 2, \ldots, n_c \\
R(\Omega, s_i) &= 0 & i = 1, 2, \ldots, n_r
\end{align*}
\]

(B.5)

If \( N_p = 2n_c + n_r \) is the dimension of the parameter vector \( \Omega \) then equation (B.4) can be written as

\[ R(\Omega, s_i) = 0 \quad i = 1, 2, \ldots, n_c + 1, \ldots, n_c + n_r \quad (B.6) \]

where

\[
\begin{align*}
s_i &= c_i + jd_i & i = 1, 2, \ldots, n_c \\
s_i &= w_i & i = n_c + 1, \ldots, n_c + n_r
\end{align*}
\]

If we reformulate the problem in terms of the unknown parameters, equation (B.6) becomes

\[
\begin{align*}
\frac{Q(s_i)}{P(s_i)}U(s_i) - \frac{B(s_i)}{s_i^n + A(s_i)}U(s_i) &= 0 & i = 1, 2, \ldots, N_p - n_c
\end{align*}
\]

(B.7)

which is equivalent to writing

\[
\begin{align*}
\frac{Q(s_i)}{P(s_i)} - \frac{B(s_i)}{s_i^n + A(s_i)} &= 0 & i = 1, 2, \ldots, N_p - n_c
\end{align*}
\]

(B.8)
or

\[- Q(s_1)A(s_1) + P(s_1)B(s_1) = Q(s_1)^n \quad i = 1, 2, \ldots, N_p - n_c \]  

(B.9)

The collocation points \( \{s_i; i = 1, 2, \ldots, N_p - n_c\} \) must be carefully selected and that is the main object of the following section.

B.3 Choice of the collocation points

There are four steps for choosing the collocation points \( \{s_i; i = 1, 2, \ldots, N_p - n_c\} \) each of which is explained below.

Step 1. Partial fraction expansion

In order to be able to select a set of collocation points \( \{s_i; i = 1, 2, \ldots, N_p - n_c\} \) we must first expand the system response \( X(s) = F(s)U(s) \) into partial fraction expansion, where every complex conjugate pair is combined to form a second order partial fraction, hence leaving every mode with real coefficients. To explain this procedure we suppose for simplicity reasons that all the \( n \) poles of \( F(s) \) are simple and, \( k \), of them are real, \( n - k \) are complex. The system response can must be expanded as

\[
X(s) = H(s)U(s) + \sum_{i=1}^{k} \frac{\alpha_i}{s + \chi_i} + \sum_{i=1}^{m} \frac{\lambda_{11}s + \lambda_{12}}{s^2 + \beta_{11}s + \beta_{12}} \]  

(B.10)

where \( m_c = (n - k)/2 \) and \( H(s)U(s) \) is the forced mode of the response.

Step 2. Mode dominance

Using Parseval's theorem we compute the energy corresponding to every mode of the partial fraction, for example for a first order mode

\[
\frac{\alpha_i}{s + \chi_i} \quad \text{energy} = \frac{\alpha_i^2}{2\gamma_i} 
\]

and for a second order mode
We define the dominancy of a mode as follows: The larger the energy of a mode (compared to energies of other modes) the more dominant its contribution to the system response.

**Step 3. Choice of the order**

If the partial fraction expansion (B.10) is written in a decreasing order of mode dominancy (most dominant modes first), one can decide what the best order of the reduced model can be, simply by deciding how many modes with very small contribution in energy there are. However for those who prefer paying the cost of having a less accurate model a smaller order can be selected. This order can be defined in terms of the orders of the most dominant modes in the partial fraction expansion, i.e.

\[
\begin{align*}
\frac{\lambda_1 s + \lambda_2}{s^2 + \beta_{11} s + \beta_{12}} \quad \text{energy} = \frac{\beta_{12} \lambda_1^2 + \lambda_{12}^2}{2}\beta_{11}\beta_{12} \quad \text{etc...}
\end{align*}
\]

where \(k_i; i=1,2,...,p\) are the orders of the first \(p\) most dominant modes in the expansion and \(k_u\) is the order of the forced mode \(H(s)U(s)\). Therefore the problem of choosing \(n\) depends on the number of the most dominant modes \(p\), which in turn depends on the user's choice and the accuracy required.

**Step 4 Collocation points**

The choice of the collocation points depends on the poles of the \(p\) most dominant modes (on the real part and on the imaginary part of these poles). If the poles are a pair of complex conjugate, i.e. \(p_j = \alpha_j \pm i\beta_j\ (\alpha_j < 0)\), then an adequate choice will be \(s_j = \gamma_j + i\delta_j\), with \(\gamma_j = \text{abs}(\alpha_j)\) and \(\delta_j = \beta_j\) or \(\delta_j = -\beta_j\). If the pole is a real (multiple or simple), i.e. \(p_j = \alpha_j\), then a good choice will be \(s_j = \text{abs}(\alpha_j)\). Other values of \(s\) if needed (to complete the set of equations) can be selected using arbitrary points from the neighborhood of the values already chosen (i.e. same order of magnitude). No complex conjugate pair should be selected if one is to avoid linear dependency of
The equations. If all dominant modes are real and \( F(s) \) is not a decreasing function of \( s \) on \([0,\infty)\) then one may have to include points from around both sides of the extrema in order to avoid an unstable reduced order model.

The choice of \( \{s_i; i=1,2,...,N_p-n_c\} \) is based on purely experimental results, it is by no means an optimal choice and therefore further research is required to theoretically justify the results. The technique gives excellent results even where other methods have failed.

**B.4 Numerical applications**

*Example 1.*

Here we consider the same seventh order example as before, i.e. the seventh order system with the transfer function

\[
F(s) = \frac{375000(s+0.08333)}{s^7 + 83.635s^6 + 4097s^5 + 70342s^4 + 853703s^3 + 2814271s^2 + 3310875s + 281250}
\]

The partial fraction of the system step response \( X(s) = F(s)U(s) \) written in an increasing order of mode dominancy is

\[
X(s) = \frac{1.111}{s} + \frac{0.124}{s+0.09} - \frac{0.13s+0.57}{s^2 + 4.04s + 5.02} - \frac{0.0003s - 0.056}{s^2 + 15.34s + 239.52} + \frac{0.00003s + 0.0013}{s^2 + 64.15s + 2537.351}
\]

The dominant modes are the first three modes, i.e. those corresponding to the poles \( p_1 = 0 \) (with energy = \( \infty \)), \( p_2 = 0.09 \) (with energy = 0.08), \( p_3 = 2.024 \pm j \) 0.964 (with energy = 0.01). The energy contributed by the last two modes were negligible, they were less than 5 \( \times 10^{-7} \). The collocation points needed to generate six equations were selected from the neighbourhood of the poles of the dominant modes, i.e. \( s_1 = 0, s_2 = 0.1, s_3 = 0.2, s_4 = 0.3 \) and \( s_5 = 2+j \). The transfer function of the third order model approximant was found as

\[
G(s) = -\frac{0.0412s^2 + 0.562s + 0.0466}{s^3 + 3.879s^2 + 4.960s + 0.4201}
\]

Notice how the parameters of this transfer function are close to those found when using weighted residual methods in §4.4.1.
**Example 2.**

This example has been considered by Ouyang *et al* [115], where the denominator of the reduced model is constructed from the poles of large dispersion based on the concept of power decomposition. The numerator is found by using the frequency response matching technique. The transfer function of the system under consideration was

\[
F(s) = \frac{13.2s^3 + 84s^2 + 167.2s + 96.8}{s + 10s^3 + 35s^2 + 50s + 24}
\]

and the reduced second order model found by the authors was

\[
R(s) = \frac{13.1667s + 32.2667}{s^2 + 6s + 8}
\]

The authors compared the performance of this reduced model in both the time and frequency domains to those of the reduced models previously found by Chen *et al* [117], Shamash [116] and Harshavardhana *et al* [118]. the reduced models were

\[
R_1(s) = \frac{-13.1344257s + 23.473727}{s^2 + 5.3183360s + 5.827510} \quad \text{(by Harshavardhana *et al*)}
\]

\[
R_2(s) = \frac{167.1999s + 96.799992}{34.3003s^2 + 50s + 24} \quad \text{(by Chen *et al*)}
\]

\[
R_3(s) = \frac{9.2778s + 8.06667}{s^2 + 3s + 2} \quad \text{(by Shamash)}
\]

The performance of \( R(s) \) was found to be much better than those in [117] and [116] which were obtained using Padé approximation method. The reduced model found by Harshavardhana *et al* [118] was obtained using the Hankel-norm approximation method.

The expansion into partial fraction of the system step response is

\[
X(s) = \frac{4.0333}{s} - \frac{1}{s + 2} - \frac{2.5}{s + 4} - \frac{0.2}{s + 1} - \frac{0.333}{s + 3}
\]
We need four equations to identify the four parameters of a second order model. The poles are all real, so we need four values of $s$ to generate four equations. The dominant modes are those corresponding to poles $p_1 = 0$ (with energy $= \infty$), $p_2 = 2$ (with energy $= 0.85$), $p_3 = 4$ (with energy $= 0.78$), $p_4 = 1$ (with energy $= 0.02$). The collocation points selected were $s_1 = 0$, $s_2 = 2$, $s_3 = 4$, $s_4 = 1$, and the second order transfer function was found as

$$G(s) = \frac{13.1697s + 22.7642}{s^2 + 5.2677s + 5.6440}$$

Notice how the parameter of $G(s)$ are close to those of $R_1(s)$. The performance in both time and frequency domains of these two transfer functions are identical to that of the original system as Fig.B.1 and Fig.B.2 show. The transfer function $R(s)$ found by Ouyang et al. [115] is a good approximation to $F(s)$ whereas the $R_2(s)$ and $R_3(s)$ are very poor approximations.

### B.5 Remarks

For a dominant complex mode with poles not near enough the imaginary axis (not oscillatory mode) a real value of $s$ can be used instead of the corresponding complex one.

If a correct static gain is required, then either use a response due to inputs such as steps or ramps, or select a collocation point at the origin, i.e. $s = 0$ (if real), $s = 0 + j\epsilon$ (if complex). If however the system is known to operate under particular inputs such as sine waves and impulses (no static gain required) then responses subject to those particular inputs may be used to derive the reduced model required.

If, for accuracy reasons we decide to increase the order of the reduced model, we may do so by increasing it with the order of the next dominant mode in the sequence (two if the next dominant mode is complex, one if it is a real mode). In decreasing the order, we may do so by decreasing it with the order of the less dominant mode to be dropped from the sequence already selected.

If, for some reason, we want to keep certain poles of the original model, this can be done by simply selecting this pole as one of the collocation points.

In reducing the order of a transfer function by a certain integer value, it is sometimes better to reduce the order of the numerator by the same integer value.

The dominancy of the modes depends very much on the input used.
Fig. B.1. Unit step responses of the original system and the reduced models.
Fig. B.2. The frequency responses of the original system and the reduced models.
Appendix C.

LEGENDRE AND CHEBYSHEV POLYNOMIALS

In this Appendix we recall some pertinent properties of Legendre and Chebyshev polynomials.

C.1 Legendre polynomials

Legendre polynomials \( P_n(x); n = 0, 1, \ldots \) are polynomials which are defined for \( x \in [-1, 1] \) and which satisfy the Legendre ordinary differential equation

\[
\frac{2}{(x-1)} \frac{d^2 P_n(x)}{dx^2} + 2x \frac{P_n(x)}{dx} - n(n+1)P_n(x) = 0 \quad \text{(C.1)}
\]

where \( n \) is the order of the polynomial \( P_n(x) \). They are given by the general formula

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad \text{(C.2)}
\]

These polynomials are also related to each other by the recurrence formula

\[
P_0(x) = 1 \quad \text{and} \quad P_1(x) = x,
\]

\[
P_n(x) = \frac{1}{n} \left[ (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x) \right] \quad \text{for } n \geq 2 \quad \text{(C.3)}
\]

C.1.1 Orthogonality property

If we try to calculate the integral

\[
I = \int_{-1}^{1} P_n(x)x^m \, dx \quad \text{for } m < n
\]

then by replacing the polynomial \( P_n(x) \) with its values (C.2) we obtain

\[
2^n n! \, I = \int_{-1}^{1} x^m \frac{d^n}{dx^n} (x^2 - 1)^n \, dx
\]

Integrating once by parts gives
The integrated term (i.e. the square bracket) is zero because the \((n-1)\)th derivative of \((x^2-1)^n\) contains \((x^2-1)\) as a common factor. Continuing the integration process \(m\) times we finally get

\[
\int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n dx = (-1)^m m! \int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n dx = 0.
\]

which is equal to zero if \(m\) is less than \(n\). We therefore have

\[
\int_{-1}^{1} P_n(x)x^m dx = 0 \quad \text{for } m < n
\]

We deduce that for any polynomial \(Q_m(x)\) of degree \(m\) less than \(n\) the integral

\[
\int_{-1}^{1} Q_m(x)P_n(x) dx = 0
\]

(C.4)

In particular if \(Q_m(x) = P_m(x)\) we obtain

\[
\int_{-1}^{1} P_m(x)P_n(x) dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}
\]

which implies that the Legendre polynomials are orthogonal.

### C.2 Chebyshev polynomials

Chebyshev polynomials \([T_n(x); n = 0, 1, \ldots]\) are polynomials which are defined for \(x \in [-1, 1]\) and are given by the relation

\[
T_n(x) = \cos(n \arccos x)
\]

(C.5)

where it is natural to put \(T_{-n}(x) = T_n(x)\). Well known trigonometric formulas give at once

\[
T_0(x) = 1
\]

\[
T_1(x) = x
\]

(C.6)

\[
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{for } n \geq 1
\]

It can easily be shown that these polynomials satisfy the second order ordinary differential equation
The Chebyshev polynomials are orthogonal with the weight function $1/(1-x^2)^{1/2}$, i.e.

$$
\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 
0 & m \neq n \\
\pi/2 & m = n \neq 0 \\
\pi & m = n = 0
\end{cases}
$$

(C.8)

Any power $m$ of $x^m$ can be expressed as a linear combination of Chebyshev polynomials of order $n$ less or equal to $m$. The first few such expressions are

$$
\begin{align*}
1 &= T_0(x) \\
x &= T_1(x) \\
x^2 &= (T_0(x) + T_2(x))/2 \\
x^3 &= (3T_1(x) + T_3(x))/4 \\
x^4 &= (3T_0(x) + 4T_2(x) + T_4(x))/8 \\
x^5 &= (10T_1(x) + 5T_3(x) + T_5(x))/16 \\
x^6 &= (10T_0(x) + 15T_2(x) + 6T_4(x) + T_6(x))/32
\end{align*}
$$

The general coefficient in the expansion $x^m = \sum_{k=0}^{m} c_k T_k(x)$ is given by

$$
c_k = \frac{2}{\pi} \int_{-1}^{1} x^m T_k(x) \frac{dx}{\sqrt{1-x^2}}
$$

From the above result it follows that

$$
\int_{-1}^{1} \frac{T_n(x) x^m}{\sqrt{1-x^2}} dx = \sum_{k=1}^{m} c_k \int_{-1}^{1} \frac{T_n(x) T_k(x)}{\sqrt{1-x^2}} dx = 0 \quad \text{for } m < n
$$

and therefore for any polynomial $Q_m(x)$ of order $m$ less than $n$ we have

$$
\int_{-1}^{1} \frac{T_n(x) Q_m(x)}{\sqrt{1-x^2}} dx = 0
$$

(C.9)
C.3 Shifted Legendre and Chebyshev polynomials

The domain of definition of Legendre and Chebyshev polynomials can be shifted from \([-1, 1]\) to \([0, T_f]\) by changing the variable \(x\) into \(t \in [0, T_f]\), i.e. by letting

\[ x = \frac{2t}{T_f} - 1 \]

The polynomials \(\{P_n(x), T_n(x); n=0,1,...\}\) become polynomials in the variable \(t \in [0, T_f]\), i.e. \(\{P^*_n(t), T^*_n(t); n=0,1,...\}\) which are called shifted Legendre and shifted Chebyshev polynomials respectively. The orthogonality properties of these polynomials are preserved under the transformation, i.e.

\[
\int_{0}^{T_f} P_n(t) P_m(t) \, dt = \begin{cases} 
0 & m \neq n \\
\frac{T_f}{2n+1} & m = n
\end{cases} 
\quad \text{and} \quad 
\int_{0}^{T_f} \frac{T^*_n(t) T^*_m(t)}{\sqrt{1 - (2t/T_f - 1)^2}} \, dt = \begin{cases} 
0 & m \neq n \\
\pi T_f & m = n \neq 0 \\
\pi T_f/2 & m = n = 0
\end{cases}
\]

Properties (C.4) and (C.9) also remain valid when the variable \(x\) is changed into \((2t/T_f - 1)\), i.e. for any polynomial \(Q_m(t)\) of order \(m\) less than \(n\) we still have

\[
\int_{0}^{T_f} Q_m(t) P_n(t) \, dt = 0 \quad \text{and} \quad \int_{0}^{T_f} \frac{Q_m(t) T^*_n(t)}{\sqrt{1 - (2t/T_f - 1)^2}} \, dt = 0
\]

which is also true when \(Q_m(t) = t^m\).
APPENDIX D.
(Published Work)
Appendix D1.

INT. J CONTROL, 1987, VOL. 45, NO. 4, 1431–1445

Integral-equation approach to system identification

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This paper presents an integral-equation approach to the problem of parameter identification in continuous linear SISO, MIMO and linear-in-parameters non-linear systems. Parameter estimates are deduced from an appropriate set of overdetermined equations, which are solved in a least-squares sense. The effects of deterministic disturbances at system input and output are also included in the analysis.

1. Introduction

Parameter identification of continuous-time system models is possibly less prevalent than that of discrete-time models (Åström and Eykhoff 1971, Strejc 1980), but is nonetheless an extensive area of current research (Eykhoff 1974, Young 1981). Recent developments (Chang and Wang 1982, Hwang and Guo 1984) have focused on expression of section of the system transient as an orthogonal function expansion, which is then used in an integral equation governing the system dynamics. The orthogonality of the basis functions leads to extensive simplification of the integral equation, and accurate identification of deterministic linear MIMO systems has been reported (Hwang and Guo 1984) in the absence of external disturbances. A major disadvantage of orthogonal-function expansion approaches is the requirement that all the data acquired over the observation interval is needed to find the coefficients in the expansion.

This paper presents an integral-equation approach for continuous linear SISO, linear MIMO and linear-in-parameters non-linear system identification that can be implemented in the latter batch manner or as a recursive algorithm. The effect of deterministic disturbances at system input and output is included in the analysis, with identification schemes being proposed in two situations: first when the disturbances are known exactly and secondly when such disturbances are only known in functional form but are linear in their unknown parameters. A similar integral-equation approach has been derived (Golubev and Horowitz 1982) from a frequency-domain error criterion; however, their specific integral-equation formulation is only valid for zero initial conditions, zero disturbances and zero mean noise. The approach developed in this paper is derived in the time domain, and the effect of non-zero initial conditions and disturbances occurs naturally.

2. An integral equation for linear SISO systems

Any non-delay linear SISO system can be expressed as an ordinary differential equation of the form

\[ v^n(t) + a_1 v^{n-1}(t) + \ldots + a_n v(t) = b_1 u^{n-1}(t) + \ldots + b_n u(t) \]  

Received 13 June 1986
subject to initial conditions \( x^{(i)}(0) = x_0 \) \((i = 0, 1, \ldots, n-1)\), where \( x^{(i)}(t) = d^i x(t)/dt^i \), or by the equivalent transfer function

\[
\frac{X(s)}{U(s)} = \frac{b_1 s^{-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}
\]  

in which the input to the system \( u(t) \) produces a corresponding output \( x(t) \). Equations (1) and (2) are the most common expressions for non-delay linear SISO dynamics, with (1) providing a time-domain description and (2) giving rise to a frequency-domain description. An alternative time-domain description will now be presented in the form of a multiple integral equation.

Firstly we introduce some notation that will be used to concisely express a multiple integral. We define

\[
x^{(\cdot\cdot\cdot)}(t) = \int_0^t \ldots \int_0^{t_1} x(t_1) \, dt_1 \ldots dt_{n-1} \, dt_n, \quad \text{for } n \geq 2
\]

\[
X^{(1)}(t) = \int_0^t x(t_1) \, dt_1
\]

For a linear SISO system described by (1), consider evaluation at an arbitrary time \( t_1 \) and integrate from 0 to \( t \) with respect to \( t_1 \):

\[
[x^{(\cdot\cdot\cdot)}(t) - x^{(\cdot\cdot\cdot)}(0)] + a_1 [x^{(\cdot\cdot\cdot)}(t) - x^{(\cdot\cdot\cdot)}(0)] + \ldots + a_{n-1} [x(t) - x(0)] + a_n X^{(1)}(t)
\]

\[
= b_1 [u^{(\cdot\cdot\cdot)}(t) - u^{(\cdot\cdot\cdot)}(0)] + \ldots + b_{n-1} [u(t) - u(0)] + b_n U^{(1)}(t)
\]

The latter equation holds at all points in time, and hence consideration of its evaluation at \( t_2 \) followed by integration from 0 to \( t \) with respect to \( t_2 \) gives

\[
[x^{(\cdot\cdot\cdot)}(t) - x^{(\cdot\cdot\cdot)}(0)] + a_1 [x^{(\cdot\cdot\cdot)}(t) - x^{(\cdot\cdot\cdot)}(0)] + \ldots + a_{n-1} [x^{(1)}(t) - tx(0)] + a_n X^{(1)}(t)
\]

\[
= b_1 [u^{(\cdot\cdot\cdot)}(t) - u^{(\cdot\cdot\cdot)}(0)] + \ldots + b_{n-1} [U^{(1)}(t) - tu(0)] + b_n U^{(2)}(t)
\]

Continuing this process to the point at which (1) has been integrated \( n \) times and collecting terms, we have

\[
\sum_{k=1}^{n} a_k X^{(k)}(t) - \sum_{k=1}^{n} b_k U^{(k)}(t) + \sum_{k=1}^{n} c_k x^{(k-1)}(t) = -x(t)
\]

where

\[
c_k = -v^{(k-1)}(0)
\]

\[
c_k = -v^{(k-1)}(0) - \sum_{j=1}^{k-1} a_j v^{(k-j-1)}(0) + \sum_{j=1}^{k-1} b_j v^{(k-j-1)}(0) \quad (k = 2, 3, \ldots, n)
\]

The effect of the initial conditions on input and output derivatives is contained entirely in the terms \( \sum_{k=1}^{n} c_k x^{(k-1)}(k-1)! \), and these terms can only be omitted from the integral-equation formulation of (3) if all initial conditions are zero.

It is interesting to note that an equation of similar structure to (3) can be obtained via indefinite integration. Thus if we denote

\[
\mathcal{X}^{(n)}(t) = \int_{t_{n-1}}^{t} \ldots \int_{t_1}^{t} v(t) \, dt
\]
then indefinite integration of (1) gives
\[
\sum_{k=1}^{n} a_k R^{(k)}(t) - \sum_{k=1}^{n} b_k Q^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} = -x(t)
\]
Again the fundamental presence of powers of \( t \) is obvious in this indefinite integral equation. This latter equation is of no computational use precisely because it is indefinite and the constants \( \delta_{x}(k = 1, 2, ..., n) \) are only defined once the lower limit of each integral is stated.

3. Linear SISO system identification
In certain physical situations the measured system input and output may differ from the actual system input and output by an additive deterministic disturbance, the most common such disturbance being an offset. We now turn to the problem of identifying the system parameters \( \{a_k, b_k, k = 1, 2, ..., n\} \) of the ordinary differential equation (1), of the transfer function (2) and of the integral equation (3) in the absence and in the presence of such additive disturbances. The measured system input and output, denoted \( r(t) \) and \( y(t) \) respectively, are then related to the actual system input and output, \( u(t) \) and \( x(t) \) respectively, by
\[
\begin{align*}
u(t) &= r(t) + p(t) \\
y(t) &= x(t) + q(t)
\end{align*}
\]
where \( p(t), q(t) \) are deterministic disturbances, see Fig. 1.

\[\text{Figure 1} \quad \text{SISO system with disturbances}\]

Since \( x(t) = y(t) - q(t) \) and \( u(t) = r(t) + p(t) \), the integral equation (3) governing \( x(t) \) for a given \( u(t) \) can be rewritten as an integral equation governing \( y(t) \) for a given \( r(t) \) as
\[
\sum_{k=1}^{n} a_k Y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{t^{k-1}}{(k-1)!} - q(t) - \sum_{k=1}^{n} d_k Q^{(k)}(t) = -y(t)
\]
Equation (4) holds at any point in time \( t = t_i \), and we now consider identification of the system parameters \( \{a_k, b_k, k = 1, 2, ..., n\} \) by imposing (4) at many points in time \( t_i \) \( (t = 0, 1, ... N) \) Given that the measured input and output are available at these discrete points in time, the multiple integrals \( R^{(k)}(t_i) \) and \( Y^{(k)}(t_i) \) can be evaluated by a suitable quadrature scheme. A particularly simple and computationally efficient formulation is available if the trapezoidal integration rule is used with equidistant sampling in time; this is usually the case in digital sampling. Thus if \( t_i = i \Delta T, i = 0, 1, ..., N \), then we have
\[
\begin{align*}
R^{(k)}(t_i) &= R^{(k)}(t_{i-1}) + \left[ R^{(k-1)}(t_i) + R^{(k-1)}(t_{i-1}) \right] \Delta T / 2 \\
Y^{(k)}(t_i) &= Y^{(k)}(t_{i-1}) + \left[ Y^{(k-1)}(t_i) + Y^{(k-1)}(t_{i-1}) \right] \Delta T / 2
\end{align*}
\]
where

\[ R^{01}(t_i) = r(t_i), \quad Y^{01}(t_i) = y(t_i) \]

Thus in the set of equations

\[ \sum_{k=1}^{n} a_k Y^{k1}(t_i) - \sum_{k=1}^{n} b_k R^{k1}(t_i) + \sum_{k=1}^{n} c_k \frac{t_i^{k-1}}{(k-1)!} - q(t_i) = \sum_{k=1}^{n} a_k Q^{k1}(t_i) \]

\[- \sum_{k=1}^{n} b_k P^{k1}(t_i) = -y(t_i) \quad (i = 0; 1, \ldots, N) \]  

(6)

the values of the multiple integrals \( \{Y^{k1}(t_i), R^{k1}(t_i), k = 1, 2, \ldots, n, i = 0, 1, \ldots, N\} \) are known, as are the values \( \{t_i^{k-1}/(k-1)!, k = 1, 2, \ldots, n\} \) via the sampling-time instants. Estimation of the system parameters \( \{a, b, c; k = 1, 2, \ldots, n\} \) via the set of equations (6) will now be tackled in two situations, which are distinguished by the degree to which knowledge of the disturbances is available.

3.1 Quantitative disturbance descriptions available

In this situation \( p(t) \) and \( q(t) \) are assumed to be known precisely; therefore the case when disturbances are known to be zero is covered by this section. Since \( p(t) \) and \( q(t) \) are known at each point in time \( t_i \), the actual system inputs and outputs can be recovered via

\[ u(t_i) = r(t_i) + p(t_i) \]

\[ x(t_i) = y(t_i) - q(t_i) \]

and (3), from which the equation set (6) arose, can be considered at each \( t_i \).

\[ \sum_{k=1}^{n} a_k X^{k1}(t_i) - \sum_{k=1}^{n} b_k U^{k1}(t_i) + \sum_{k=1}^{n} c_k \frac{t_i^{k-1}}{(k-1)!} = -x(t_i) \quad (i = 0, 1, \ldots, N) \]  

(7)

The multiple integrals \( \{X^{k1}(t_i), U^{k1}(t_i), k = 1, 2, \ldots, n, i = 0, 1, \ldots, N\} \) can be calculated from the recovered values \( x(t_i), u(t_i) \) using a quadrature rule such as (5) and, together with the values \( t_i^{k-1}/(k-1)! \), (7) provides a set of linear equations for the system parameters \( \{a, b, c; k = 1, 2, \ldots, n\} \) and the initial-condition parameters \( \{c; k = 1, 2, \ldots, n\} \). It is essential that the initial condition parameters are included in the estimation process when the system has non-zero initial conditions, which is the case in most physical situations. These terms do not appear in the analysis of Golubev and Horowitz (1982), where the time-domain integral equation was derived from a frequency-response error criterion. For the quadrature scheme to be accurate, \( N \) will be large, and hence the set of \( (N+1) \) linear equations (7) is generally overdetermined for the \( 3n \) unknowns \( \{a, b, c; k = 1, 2, \ldots, n\} \) and must be solved in a least-squares sense.

3.2 Qualitative disturbance descriptions available

In certain situations we may only have a qualitative feel for the deterministic disturbances that are present in a system, and quantifying such disturbances may require additional work. Typically we may be aware that a measurement transducer incorporates an offset, and a calibration exercise would be required to evaluate this offset. Such qualitative knowledge can be included in the preceding analysis when the disturbance is linear in its unknown parameters. Suppose that the disturbances have
the form
\[ p(t) = \sum_{i=1}^{n} a_i g_i(t), \quad q(t) = \sum_{i=1}^{n} \beta_i h_i(t) \]
in which the functions \( \{g_i(t), s = 1, 2, \ldots, p_s\}\) are known, although the parameters \( \{a_s, s = 1, 2, \ldots, p_s\}\) are unknown. Then (6) gives
\[ \sum_{k=1}^{n} a_k \alpha_{k}(t) - \sum_{k=1}^{n} b_k R_k^{(k)}(t) + \sum_{k=1}^{n} c_k \left[ \frac{t^{k-1}}{(k-1)!} \right] - \sum_{k=1}^{n} \beta_k h_k(t) + \sum_{k=1}^{n} a_k \alpha_{k}(t) \]
\[ \quad - \sum_{k=1}^{n} \sum_{s=1}^{p_s} b_k G_k^{(s)}(t) = -y(t) \quad (i = 0, 1, \ldots, N) \] (8)
or
\[ \sum_{k=1}^{n} a_k \alpha_{k}(t) - \sum_{k=1}^{n} b_k R_k^{(k)}(t) + \sum_{k=1}^{n} c_k \left[ \frac{t^{k-1}}{(k-1)!} \right] - \sum_{k=1}^{n} \beta_k h_k(t) \]
\[ \quad - \sum_{k=1}^{n} \sum_{s=1}^{p_s} b_k G_k^{(s)}(t) = -y(t) \quad (i = 0, 1, \ldots, N) \] (9)
where \( \beta_{ab} = \beta_a q_k \) and \( \beta_{ab} = \beta b_s \). The effect of the unquantified disturbances \( p(t) \) and \( q(t) \) has now been included in the analysis, and if the set of \( N + 1 \) linear equations (9) is solved with respect to the system parameters \( \{a_k, b_k; k = 1, 2, \ldots, n\} \), the initial-condition parameters \( \{c_k; k = 1, 2, \ldots, n\} \) and the disturbance parameters \( \{\beta_{ab}; k = 1, 2, \ldots, n; s = 1, 2, \ldots, q_s\} \) then the system parameters will be correctly identified. Again, \( N \) will be large for accurate quadrature of the multiple integrals, and the set of \( N + 1 \) linear equations (9) must be solved in a least-squares sense with respect to the total set of \( (3 + q_s + p_s)n + q_s \) unknown parameters. As before, it is essential to include parameters in the estimation that may not be of significant interest.

The effect of polynomial disturbances (e.g., offsets and ramps) can be included in the analysis without explicitly introducing the \( p_s + q_s \) unknown parameters that would generally be required. By way of an example, consider a system subject to constant, but unknown, offsets at input and output, i.e., \( u(t) = r(t) + p_0 \) and \( y(t) = x(t) + q_0 \). Then (8) becomes
\[ \sum_{k=1}^{n} a_k \alpha_{k}(t) - \sum_{k=1}^{n} b_k R_k^{(k)}(t) + \sum_{k=1}^{n} c_k \left[ \frac{t^{k-1}}{(k-1)!} \right] - q_0 - \sum_{k=1}^{n} a_k q_0 t^k \]
\[ \quad - \sum_{k=1}^{n} b_k p_0 k! = -y(t) \quad (i = 0, 1, \ldots, N) \]
or
\[ \sum_{k=1}^{n} a_k \alpha_{k}(t) - \sum_{k=1}^{n} b_k R_k^{(k)}(t) + \sum_{k=1}^{n+1} c_k \left[ \frac{t^{k-1}}{(k-1)!} \right] - q_0 \]
\[ \quad - \sum_{k=1}^{n+1} b_k p_0 k! = -y(t) \quad (i = 0, 1, \ldots, N) \] (10)
where the parameters \( \{d_k; k = 1, 2, \ldots, n + 1\} \) include the effects of both the system initial conditions and the offsets on the governing integral equation. Clearly the system parameters \( \{a_k, b_k; k = 1, 2, \ldots, n\} \) are found by solving the generally overdetermined set of \( N + 1 \) equations (10) with respect to \( \{a_k, b_k; k = 1, 2, \ldots, n\} \) and \( \{d_k; k = 1, 2, \ldots, n + 1\} \). If the system has only one offset present then the estimation...
procedure can also evaluate this offset. Thus if the system has an offset $q_0$ between the actual output $x(t)$ and the measured output $y(t)$ then $d_{s+1} = a_0 q_0$, and since $a_0$ has been estimated, $q_0$ is available from $q_0 = d_{s+1}/a_0$.

4. Linear MIMO system identification

The preceding SISO linear system-identification scheme is now extended to cover MIMO systems where $r$ measured inputs $r(t)$ give rise to $m$ measured outputs $y(t)$. Such measured signals differ from the actual system inputs $u(t)$ and outputs $x(t)$ by additive disturbances, i.e.,

$$u(t) = r(t) + p(t)$$

$$y(t) = x(t) + q(t)$$

where $p(t)$ and $q(t)$ are $r$- and $m$-dimensional vectors respectively, as shown in Fig. 2.

![Figure 2. MIMO system with disturbances](image)

For a proper MIMO linear system the $m$ ordinary differential equations relating the $r$ inputs $u(t)$ to the $m$ outputs $x(t)$ can be written as

$$y_i^{(n)}(t) + \sum_{k=1}^{n} a_k y_i^{(k)}(t) = \sum_{j=1}^{r} \sum_{k=1}^{n} b_{jk} u_j^{(k)}(t) \quad (l=1, 2, \ldots, m) \quad (11)$$

Not all $n$ modes of the system may be observable in each of the outputs, and therefore one or more of the ordinary differential equations in (11) may be of order less than $n$. However, with the system as expressed in (11), the number of parameters to be estimated is generally minimized, and the reduction in order of any differential equation will be indicated by pole-zero cancellation. Following the same multiple integration arguments used in the SISO case, the equivalent set of system integral equations is

$$\sum_{k=1}^{n} a_k x_i^{(k)}(t) = \sum_{j=1}^{r} \sum_{k=1}^{n} b_{jk} y_j^{(k)}(t) + \sum_{k=1}^{r} c_k \frac{t^{k-1}}{(k-1)!} = -y_i(t) \quad (l=1, 2, \ldots, m) \quad (12)$$

and the associated set of integral equations for measured inputs and outputs is

$$\sum_{k=1}^{n} a_k y_i^{(k)}(t) = \sum_{j=1}^{r} \sum_{k=1}^{n} b_{jk} p_j^{(k)}(t) + \sum_{k=1}^{r} c_k \frac{t^{k-1}}{(k-1)!} - q_i(t) - \sum_{k=1}^{r} a_k q_i^{(k)}(t)$$

$$-\sum_{j=1}^{r} \sum_{k=1}^{n} b_{jk} p_j^{(k)}(t) = -y_i(t) \quad (l=1, 2, \ldots, m) \quad (13)$$

As in the SISO case, this latter set of integral equations can be used to identify the system parameters $\{a_k, b_{jk}, l=1, 2, \ldots, r; k=1, 2, \ldots, n\}$, when the disturbances $p(t)$ and $q(t)$ are known precisely and when such disturbances are known only qualitatively but are linear in their unknown parameters.
When $p(t)$ and $q(t)$ are known exactly the actual system inputs and outputs can be recovered at any point in time $t_i$ via

$$u(t_i) = r(t_i) + p(t_i)$$
$$x(t_i) = y(t_i) - q(t_i)$$

and (12) can be considered at such points in time:

$$\sum_{k=1}^{r} a_k X^{(k)}(t_i) - \sum_{j=1}^{r} \sum_{i=1}^{s} b_{ij} Y^{(i)}(t_j) + \sum_{k=1}^{r} c_k \frac{t_i^{k-1}}{(k-1)!} = -x_i(t_i)$$

$$(l = 1, 2, ..., m; i = 0, 1, ..., N)$$

This is a set of $N + 1$ linear equations for the $n(mr + 1)$ system parameters $\{a_k, \{b_{ij}; l = 1, 2, ..., m, j = 1, 2, ..., r\}; k = 1, 2, ..., n\}$ and the $mn$ initial condition parameters $\{c_k, l = 1, 2, ..., m, k = 1, 2, ..., n\}$. Accurate quadrature of the multiple integrals will only be guaranteed if $N$ is large, and parameter estimation generally follows from a linear least-squares analysis. However, the size of the estimation problem for complex MIMO systems must be noted, i.e., there is a total of $mn(r + 1) + 1$ parameters to be estimated. Thus for a jet engine with 5 inputs, 33 states and 5 outputs (Edmunds 1979) there is a total of 991 unknowns, and $N$ must be greater than or equal to 991 to provide a solution.

The procedure for including only linear-in-parameters qualitative effects is identical with the SISO case and will not be detailed here. However it is once again emphasized that polynomial disturbances should be considered separately since their associated multiple integrals only give rise to powers of $t$ and only marginally increase the order of the estimation problem.

5. Non-linear system identification

The preceding integral-equation approach can also be extended to cover the identification of certain non-linear systems. Initially we shall consider a SISO system governed by an ordinary differential equation of the form

$$x^{(s)}(t) + a_1 x^{(s-1)}(t) + ... + a_n x(t) + \sum_{s=1}^{\infty} \gamma_s f_s[x(t), u(t), t]$$

$$= b_1 u^{(s-1)}(t) + ... + b_n u^{(s)}(t)$$

(15)

where the non-linearity can be expressed by the functions $\{f_s[x(t), u(t), t], s = 1, 2, ..., n\}$. The integral-equation equivalent is

$$x(t) + a_1 X^{(1)}(t) + ... + a_n X^{(n)}(t) + \sum_{s=1}^{\infty} \gamma_s F_s^{(n)}[x(t), u(t), t]$$

$$+ \sum_{k=1}^{n} c_k \frac{t_i^{k-1}}{(k-1)!} = b_1 U^{(1)}(t) + ... + b_n U^{(n)}(t)$$

(16)

where the parameters $\{c_k, k = 1, 2, ..., n\}$ cover the effect of non-zero initial conditions and

$$F_s^{(n)}[x(t), u(t), t] = \int_0^t \int_0^{\tau_1} \cdot \int_0^{\tau_{s-1}} f_s[x(t_1), u(t_1), t_1] dt_1 \cdot d\tau_{s-1} dt_s$$

When input and output disturbances are known, $u(t)$ and $x(t)$ are again recoverable.
from the measured input and output and (16) can be considered at the set of points in
time \( t_i \), \( i = 0, 1, \ldots, N \), i.e
\[
\sum_{k=1}^{n} a_k X^{(k)}(t_i) - \sum_{k=1}^{n} b_k u^{(k)}(t_i) + \sum_{k=1}^{n} \gamma_k e^{(k)}[u(t_i), u(t_i), t_i] \\
+ \sum_{k=1}^{n} c_k t_i^{(k)} = -x(t_i) \quad (i = 0, 1, \ldots, N) \tag{17}
\]
The set of \( N + 1 \) equations (17) is linear in the system parameters \( \{a_k, b_k; k = 1, 2, \ldots, n\} \) and the initial condition parameters \( \{c_k; k = 1, 2, \ldots, n\} \), and a least-squares solution of this generally overdetermined equation set should accurately identify all the system parameters.

Identification of linear-in-parameters non-linear systems may also lead to a linear least-squares analysis when the disturbances are only known qualitatively but are linear in their unknown parameters. Specific investigation of the particular non-linear system is needed to ascertain whether or not this is the case. Consider an example in which a system is governed by a non-linear ordinary differential equation
\[
\ddot{x} + a_1 x + \gamma_1 x^2 = b_1 \dot{u} + b_2 u
\]
The equivalent integral equation is
\[
a_1 X^{(1)}(t) - b_1 U^{(1)}(t) - b_2 U^{(2)}(t) + \gamma_1 \int_0^t x^2(t_1) \, dt_1 \, dt_2 + c_1 + c_2 t = -x(t)
\]
If the measured output is the actual output plus a constant offset and the measured input is equal to the actual input, i.e. \( y(t) = x(t) + q_0 \), \( r(t) = u(t) \), then the associated integral equation for \( y(t) \) and \( r(t) \) is
\[
a_1 [Y^{(1)}(t) - q_0 t] - b_1 R^{(1)}(t) - b_2 R^{(2)}(t) + \gamma_1 \int_0^t \int_0^{t_1} [y(t_1) - q_0]^2 \, dt_1 \, dt_2 \\
+ c_1 + c_2 t = -y(t) + q_0
\]
or
\[
a_1 Y^{(1)}(t) - 2\gamma_1 q_0 Y^{(2)}(t) - b_1 R^{(1)}(t) - b_2 R^{(2)}(t) + \gamma_1 \int_0^t \int_0^{t_1} y^2(t_1) \, dt_1 \, dt_2 \\
+ [c_1 - q_0] + [c_2 - a_1 q_0 t] + \gamma_1 q_0^2 \frac{t^2}{2!} = -y(t)
\]
Hence if we pose an overdetermined set of linear equations as
\[
x_1 Y^{(1)}(t_i) + x_2 Y^{(2)}(t_i) - \beta_1 R^{(1)}(t_i) - \beta_2 R^{(2)}(t_i) + \gamma_1 \int_0^{t_i} \int_0^{t_1} y^2(t_1) \, dt_1 \, dt_2 \\
+ d_1 + d_2 t_i + d_3 \frac{t_i^2}{2!} = -y(t_i) \quad (i = 0, 1, \ldots, N)
\]
and identify the parameters \( x_1, x_2, \beta_1, \beta_2, \gamma_1, d_1, d_2, d_3 \) by a least-squares analysis then the original system parameters and the unknown offset can be found from
\[
a_1 = x_1, \quad \gamma_1 = \gamma_1, \quad b_1 = \beta_1, \quad b_2 = \beta_2
\]
and
\[
-2\gamma_1 q_0 = x_2 \quad \text{or} \quad q_0 = -\frac{x_2}{2\gamma_1}
\]
The governing differential equation for a more general linear-in-parameters non-linear system may be written

\[ \sum_{s=1}^{n} \gamma_s G_s \left[ x^{(s)}(t), \ldots, x^{(1)}(t), x(t), u^{(n-1)}(t), \ldots, u^{(1)}(t), u(t), r(t) \right] = 0 \]

and the equivalent integral equation can only be written as

\[ \sum_{s=1}^{n} \gamma_s G_s \left[ x^{(s)}(t), \ldots, x^{(1)}(t), x(t), u^{(n-1)}(t), \ldots, u^{(1)}(t), u(t), r(t) \right] = 0 \]

where

\[ G_s^{(n)} = \int_0^\infty \int_0^\infty \cdots \int_0^\infty g_s(r_1 \ldots r_{n-1}) dr_1 \ldots dr_{n-1} \]

With the inclusion of input and output disturbances, the associated integral equation for \( y(t) \) and \( r(t) \) is

\[ \sum_{s=1}^{n} \gamma_s G_s \left[ x^{(s)}(t) - q^{(s)}(t), \ldots, x(t) - q(t), r^{(n-1)}(t) + \rho^{(n-1)}(t), \ldots, r(t) + \mu(t), r(t) \right] = 0 \]

Even in the situation where \( p(t) \) and \( q(t) \) are known exactly, derivatives of both measured input and output signals are required to form the coefficients of a set of equations for the unknown \( \gamma_s; s = 1, 2, \ldots, n \) In the presence of even very low levels of noise (e.g., from A/D quantization), higher derivatives will be determined inaccurately, and the identification scheme can only be expected to work when low-order derivatives are present in the non-linear functions. If the disturbances are only known qualitatively but are linear in their unknown parameters then investigation of the particular non-linearity is required to determine whether or not all unknown parameters may be estimated by a linear least-squares analysis. A numerical example of such a non-linear system is given in § 7.4

6. Choice of input signal

An interesting feature of the integral-equation approach to system identification is that the set of input signals that can be applied to the system is limited Referring to the SISO linear system and its governing integral equation (3), it can be seen that if the actual input to the plant is of the monic polynomial form \( u(t) = t^l (l = 0, 1, \ldots, n - 2) \) then the columns of (7) are linearly dependent on the columns generated by any non-zero initial conditions Hence a linear least-squares solution for the system parameters is not available In the case where the disturbances are only known qualitatively, the measured input \( r(t) \) must be such that the columns \( \{ R^{(s)}(t); s = 1, 2, \ldots, n \} \) in the set of overdetermined linear equations (9) are not linear combinations of the columns arising from non-zero initial conditions and from unknown disturbance parameters In the case of constant unknown offsets, polynomial inputs of the form \( r(t) = t^r \) \( r = 0, 1, \ldots, n - 1 \) are prohibited. It is therefore interesting to note that standard test signals such as impulses, steps and ramps are excluded from use with this identification technique when the system has non-zero initial conditions Clearly the columns \( \{ R^{(s)}(t); s = 1, 2, \ldots, n \} \) must also be linearly independent of one another. This often precludes the use of purely exponential and sinusoidal functions.

7. Results

7.1. Numerical implementation

The results quoted in this section are based on use of the trapezoidal integration scheme (5) with a Householder transformation scheme (Noble 1976, Wilkinson and
Reinsch (1971) providing the least-squares solution to all sets of overdetermined linear equations, each of which is built from observations over the whole time interval 0 to T. Such an en-bloc approach entails a heavy data-storage overhead, which could be avoided by use of the recursive least-squares solution technique (Plackett 1950, Young 1974) applied to successive rows of the integral equation.

Identification results are quoted for measured data in full floating-point form and also for such data following 12-bit analogue-to-digital conversion. In the latter situation the data is accurate to one digit in 4096, i.e. to approximately 0.025% when scaled to full range.

7.2 SISO linear systems

A first-order system with \( a_1 = b_1 = 1 \), i.e. \( \dot{x} + x = u \), was used as a vehicle to show the effectiveness of the trapezoidal integration scheme for calculating the values of the multiple integrals and the influence of integration step size \( \Delta T \) upon parameter estimation. With initial conditions and disturbances at input and output known to be zero, only the parameters \( a_1 \) and \( b_1 \) need be estimated. Figure 3 shows the variation of those estimates with step size \( \Delta T \) time units for a total observation time of \( T = 5 \) units when the input to the plant is \( r(t) = 2t/(1 + 2t) \).

For the given system, which has a unit time constant, acceptable estimation occurs for \( \Delta T \approx 0.1 \) with improved accuracy being attained by even shorter sampling intervals. The same first-order system is also used to illustrate the effect of 12-bit A/D quantization on parameter estimation. With both input and output being passed into

![Parameter estimation vs. \( \Delta T \)](image-url)
a 12-bit A/D (full scale range $-5.12$ V to $5.12$ V) without rescaling and sampling parameters $\Delta T = 0.01$, $N = 300$, $T = 3$, the results shown in Table 1 were obtained. Quantization clearly reduces the accuracy of the estimates, though by a tolerable degree.

<table>
<thead>
<tr>
<th>Identified model</th>
<th>without A/D</th>
<th>with A/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>$b_1$</td>
<td>1000</td>
<td>1000</td>
</tr>
</tbody>
</table>

Table 1

For the higher-order systems that follow, the poles $P_i$ ($i = 1, 2, ..., n$) and zeros $Z_i$ ($i = 1, 2, ..., n - 1$) of the system and any corresponding model are also quoted. A second-order system was tested with $u(t) = 2t/(1 + 2t)$, $x(0) = 2$, $\dot{x}(0) = -19$, known zero disturbances and sampling parameters $\Delta T = 0.01$, $N = 200$, $T = 2$ with the results shown in Table 2. Once again accurate identification results despite the limited quantity of sampled data. The accuracy could be further improved by decreasing $\Delta T$ and increasing $T$.

<table>
<thead>
<tr>
<th>Identified model</th>
<th>without A/D</th>
<th>with A/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$a_1$</td>
<td>1000</td>
<td>9997</td>
</tr>
<tr>
<td>$a_2$</td>
<td>2100</td>
<td>2099</td>
</tr>
<tr>
<td>$b_1$</td>
<td>1000</td>
<td>1001</td>
</tr>
<tr>
<td>$h_1$</td>
<td>1500</td>
<td>1499</td>
</tr>
<tr>
<td>$Z_1$</td>
<td>-1500</td>
<td>-1499</td>
</tr>
<tr>
<td>$P_1$</td>
<td>-300</td>
<td>-3000</td>
</tr>
<tr>
<td>$P_2$</td>
<td>-700</td>
<td>-6997</td>
</tr>
</tbody>
</table>

Table 2

The latter system was also used to illustrate the inclusion of unquantified deterministic disturbances in the analysis. The system, subject to initial conditions $\chi(0) = 2$, $\dot{\chi}(0) = -19$, was stimulated by an input $r(t) = \sin t$, and the measured output $y(t)$ was taken as $y(t) = \chi(t) + 3$. The identification scheme (10) was implemented wherein it is assumed that the system has a constant offset at the output though the value of the offset is unknown. The choice of $r(t) = \sin t$ does not cause any problems in the identification of second-order systems since only the integrals $R^{11}(t) = 1 - \cos t$ and $R^{12}(t) = t - \sin t$ are involved in the overdetermined equations columns, and together with the columns $Y^{11}(t)$, $Y^{12}(t)$, $1$, $t$, $t^2/2!$ they form a linearly independent set of functions. Such an input could not, however, be used for a third-order system identification since $R^{13}(t) = t^2/2! + \cos t - 1$ is a linear combination of $R^{11}(t)$ and $t^2 2!$. However, an input of the form $r(t) = A \sin \omega_1 t + A_2 \sin \omega_2 t$ would suffice.
Following the notation of (10) with $d_3$ as the coefficient of $t^2/2!$ and sampling with $T = 0.01$, $N = 700$, $T = 7$ we have the results shown in Table 3. Again the identification procedure yields accurate parameter estimates. The value of the unquantified offset is evaluated from $d_3 = a_2 q_0$ as $q_0 = 3.001$ with full floating-point data and $q_0 = 3.000$ with 12-bit digitized data.

<table>
<thead>
<tr>
<th>System</th>
<th>without A/D</th>
<th>with A/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$a_1$</td>
<td>10.00</td>
<td>9.996</td>
</tr>
<tr>
<td>$a_2$</td>
<td>21.00</td>
<td>20.98</td>
</tr>
<tr>
<td>$b_1$</td>
<td>1.00</td>
<td>1.000</td>
</tr>
<tr>
<td>$b_2$</td>
<td>15.00</td>
<td>14.99</td>
</tr>
<tr>
<td>$Z_1$</td>
<td>-15.00</td>
<td>-14.99</td>
</tr>
<tr>
<td>$P_1$</td>
<td>-3.00</td>
<td>-3.000</td>
</tr>
<tr>
<td>$P_2$</td>
<td>-7.00</td>
<td>-6.997</td>
</tr>
<tr>
<td>$d_3$</td>
<td>-</td>
<td>62.96</td>
</tr>
</tbody>
</table>

Table 3.

7.3 A linear MIMO system

A three-output two-input linear system with transfer function matrix

$$G(s) = \begin{bmatrix}
\frac{3}{s+4} & \frac{2.7(s+5)}{(s+4)(s+6)} \\
\frac{12}{(s+4)(s+6)} & \frac{3.5(s+10)}{(s+4)(s+6)} \\
\frac{2}{s+4} & \frac{0.5(s+10)}{(s+4)(s+6)}
\end{bmatrix}$$

was stimulated by inputs $u_1(t) = 20 \sin 0.5t$, $u_2(t) = 25 \sin t$ and subject to initial conditions $x_1(0) = 40$, $x_2(0) = -15.3$, $x_3(0) = 2$, $x_4(0) = -7.8$, $x_5(0) = 1.0$, $x_6(0) = -4.0$. Offsets of 5, 1 and 0 units were added to the actual outputs to create three measurements $y_1(t)$, $y_2(t)$, $y_3(t)$. Extension of (13) to include the effects of unknown offsets leads to a set of equations in 23 unknowns, and using $\Delta T = 0.01$, $N = 900$, $T = 9$ for each measurement the pertinent parameters lead to the following identified transfer-function matrices:

**exact data:**

$$\begin{bmatrix}
1 & 2.999(s + 0.6000) & 2.7(s + 4.999) \\
0 & 0.00015(s + 26.6600) & 1.2(s + 8.999) \\
3.4996(s + 1.0000) & 0.5001(s + 9.996)
\end{bmatrix}$$

**12-bit A/D data:**

$$\begin{bmatrix}
1 & 3.003(s + 0.0001) & 2.700(s + 5.015) \\
0 & -0.003(s - 1337.0) & 1.199(s + 0.872) \\
3.507(s + 1.001) & 0.498(s + 10.07)
\end{bmatrix}$$
Clearly the exact data yield virtually exact identification once the appropriate pole-zero cancellations and the negligible term 0.00015s are accounted for. The 12-bit A/D sampled data yields slightly less accurate identification.

7.4 A linear-in-parameters non-linear system

The unforced rolling motion of a ship may be approximately described (Roberts 1982 a, b) by the normalized equation

\[ \ddot{x} + (a_1 + n_1 t) \dot{x} + (a_2 + n_2 t^2) x = 0 \]  

(18)

Estimation of the parameters \(a_1\), \(a_2\), \(n_1\), \(n_2\) is to be considered when the roll angle \(\phi(t)\) is measured directly and in the presence of an unknown offset; Gawthrop (1984) has previously considered this problem.

Double integration of (18) gives the integral equation for \(\phi(t)\) as

\[ a_1 x_1'(t) + a_2 x_2'(t) + n_1 \int_0^t \int_0^t [\dot{x}(\tau_1) x(\tau_1) \, d\tau_1 \, d\tau_2 \]

\[ + n_2 \int_0^t \int_0^t x^2(\tau_1) \, d\tau_1 \, d\tau_2 - x(0) - a_1 t \phi(0) - a_2 t \phi(0) = -x(t) \]

In the presence of an unknown offset \(q_0\), the measured roll angle is \(y(t) = x(t) + q_0\), and the associated integral equation for \(y(t)\) is

\[ x_1 y_1'(t) + x_2 y_2'(t) + \gamma_1 \int_0^t \int_0^t [\dot{x}(\tau_1) x(\tau_1) \, d\tau_1 \, d\tau_2 + \gamma_2 \int_0^t \int_0^t x^2(\tau_1) \, d\tau_1 \, d\tau_2 \]

\[ + \gamma_3 \int_0^t \int_0^t x^3(\tau_1) \, d\tau_1 \, d\tau_2 + d_1 + d_2 t + d_3 t^2 = -y(t) \]  

(19)

where

\[ x_1 = a_1, \quad x_2 = a_2 + 3q_0 n_2 \]

\[ \gamma_1 = n_1, \quad \gamma_2 = n_2, \quad \gamma_3 = -3q_0 n_2 \]

\[ d_1 = -\phi(0) - q_0, \quad d_2 = -a_1 \phi(0) - \dot{x}(0) - a_1 q_0, \quad d_3 = -a_2 q_0 - n_2 q_0 \]

It is noticeable that the cubic non-linearity in \(x(t)\) gives rise to cubic and quadratic non-linear expressions for \(y(t)\), together with contributions to linear and initial condition coefficients. Equation (19) can be imposed at points in time \(\tau(t) = 0, 1, \ldots, N\) to form a set of \(N + 1\) equations in the unknowns \(a_1, a_2, \gamma_1, \gamma_2, \gamma_3, d_1, d_2, d_3\), and least-squares solutions then yield values of these unknowns. Parameter and initial-condition identification follows from equations such as

\[ a_1 = x_1, \quad n_1 = \gamma_1, \quad n_2 = \gamma_2 \]

\[ a_0 = -\dot{x}(0) - x(0) - a_1 q_0, \quad d_3 = -a_2 q_0 - n_2 q_0 \]

\[ \phi(0) = -d_1 - q_0, \quad \dot{x}(0) = -d_2 - a_1 x(0) - a_1 q_0 \]

though this latter equation set is not unique (e.g. \(a_2\) could be calculated from \(a_2 = (-d_3 - n_2 q_0^2)/q_0)\).

With parameter values \(a_1 = 0.02, a_2 = 1.00, n_1 = 0.001, n_2 = 0.0002\) and initial conditions \(\phi(0) = 40, \dot{x}(0) = 0.1\), the latter system produces a transient as shown in Fig 4. For identification purposes, discrete samples were taken with \(T = 0.01, N = 900, T = 9\) in situations of zero and 5 units offset. Two further data sets were
Figure 4. Unforced rolling motion of a ship

<table>
<thead>
<tr>
<th>System</th>
<th>Identification</th>
<th>with exact data</th>
<th>with A/D sampled data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Offset = 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>0.02</td>
<td>0.0199</td>
<td>0.0187</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1.00</td>
<td>0.9999</td>
<td>1.0000</td>
</tr>
<tr>
<td>$r_1$</td>
<td>0.01</td>
<td>0.0100</td>
<td>0.0101</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0.0002</td>
<td>0.0002</td>
<td>0.0002</td>
</tr>
<tr>
<td>$r_3$</td>
<td>0</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$d_1$</td>
<td>-4.00</td>
<td>-4.00</td>
<td>-4.00</td>
</tr>
<tr>
<td>$d_2$</td>
<td>-0.9</td>
<td>-0.8976</td>
<td>-0.8900</td>
</tr>
<tr>
<td>$d_3$</td>
<td>0</td>
<td>-0.0007</td>
<td>-0.0029</td>
</tr>
<tr>
<td>Offset = 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>0.02</td>
<td>0.0199</td>
<td>0.0187</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1.015</td>
<td>1.015</td>
<td>1.0</td>
</tr>
<tr>
<td>$r_1$</td>
<td>0.01</td>
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</tr>
<tr>
<td>$r_2$</td>
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<td>0.0002</td>
</tr>
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<td>-45.00</td>
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</tr>
<tr>
<td>$d_3$</td>
<td>-5.025</td>
<td>-5.025</td>
<td>-5.028</td>
</tr>
</tbody>
</table>

Table 4
Integral-equation approach to system identification

created by 12-bit full-range A/D sampling of the latter data sets. The results shown in Table 4 were obtained.

The exact data give virtually exact identification, while the 12-bit A/D sampled data are marginally worse. The principle source of error in the latter case being the numerical differentiation of \( y \) required to form \( \dot{y} \). Gawthrop (1984) has reported reasonable parameter estimation (e.g., \( x = -0.0232 \)) using exact data with \( T = 0.01 \), \( N = 5000 \) and \( T = 50 \), and concluded that inaccurate initial condition identification (e.g., \( x(0) = 43.608 \)) may be due to an identifiability problem. However, even the severely limited data set used above gives accurate parameter and initial condition identification and there is apparently no identifiability problem inherent in the system.

8. Conclusions

An integral-equation formulation has been presented for parameter identification in continuous linear SISO, MIMO and linear-in-parameters non-linear systems. Parameters that arise naturally from non-zero initial conditions must also be estimated in the procedure. Effects of disturbances at both input and output can be included in the analysis. A feature of the procedure is that care must be taken in the choice of test input signal, with certain standard inputs such as impulses, steps and ramps being excluded from consideration in the presence of non-zero initial conditions. Golubev and Horowitz (1982) derived the same procedure for linear SISO systems subject to zero initial conditions via a frequency-domain formulation and investigated parameter estimation in the presence of significant levels of noise. The time-domain approach presented in this paper must now be extended to include such noise levels.

References

ASTEROM, K. J. and EYKHOFF, P., 1971, *Automatica*, 7, 123
GAWTHROP, P. J., 1984, *IMA J. Math Control Inf.*., 1, 117
PLACKETT, R. L., 1950, *Biometrika*, 37, 149
STIFICH, V., 1984, *Automatica*, 20, 535
Continuous system order identification from plant input–output data

A. H. WHITFIELD† and N. MESSAL†

This paper presents two techniques for system order determination in continuous SISO systems. The first method is based on the behaviour of singular values of a specific non-square matrix, the elements of which are calculated from multiple integrals of input and output data. The second method uses shifted Legendre polynomials to produce a square matrix whose non-zero eigenvalues then identify the order of the system under consideration.

1. Introduction

The basis of most parameter identification techniques is that the system order and therefore the number of identifiable parameters is predetermined. The problem of system order determination has therefore received considerable attention over the last two decades and a variety of methods has been proposed for this purpose. The most common methods are those based on the minimization of Akaike's (1978) criteria (the final prediction error (FPE) and the Akaike information criterion (AIC)) (see Edmunds 1985, Soderstrom 1977, Chan, et al. 1974), and those based on the system information matrix (see Van den Boom and Van den Enden 1974, Woodside 1971, Wellstead 1978, Sagara et al. 1982, Unbehauen and Gohring 1974). Among methods based on the system information matrix are those which investigate the near singularity of the product moment matrix, the condition number, the determinant ratio and the behaviour of the determinant. Most of the latter techniques use iterative schemes to identify the order of the system. Among other techniques developed for order determination are a pole-zero cancellation method (Soderstrom 1975) and the minimization of the residual error variance (Young et al. 1980).

This paper presents two different techniques for use with continuous time systems which are based on matrices formed by multiple integrals of input and output data. The advantage of the proposed methods is that, given an initial estimate of the model order which is greater than the true order of the system, the order of the model is found by a precise formulation. First, a non-square matrix is formed using multiple integrals of the measured input–output data; the system order is then determined by the difference between the estimated order and the number of zero singular values of the non-square matrix. Shifted Legendre polynomials are then used to produce a square matrix where the number of non-zero eigenvalues determines the order of the system.

The techniques are presented in the context that the input to the system is a step, though the analysis can be extended to include other inputs.

2. Shifted Legendre polynomials

Before presenting the methods for system order determination, we briefly summarize some pertinent properties of shifted Legendre polynomials. Legendre

Received 17 November 1986
polynomials \( \{ Q_i(x) \} \), \( i = 0, 1, 2, \ldots, n \) are defined for \( x \in [-1, 1] \) and are given by the following recurrence relation.

\[
\begin{align*}
    Q_0(x) &= 1 \\
    Q_1(x) &= x \\
    Q_n(x) &= \frac{1}{n} [(2n-1)xQ_{n-1}(x) - (n-1)Q_{n-2}(x)] \\
    n &= 2, 3, 4, \ldots
\end{align*}
\]

In the time domain \( t \in [0, T_f] \) we define shifted Legendre polynomials \( P_i(t) \), \( i = 0, 1, 2, \ldots \) by substituting \( x = (2t/T_f) - 1 \) in (1) for all \( t \) with \( 0 \leq t \leq T_f \). Shifted Legendre polynomials are therefore defined by:

\[
\begin{align*}
P_0(t) &= 1 \\
P_1(t) &= \frac{2t}{T_f} - 1 \\
P_n(t) &= \frac{1}{n} [(2n-1)P_{n-1}(t) - (n-1)P_{n-2}(t)] \\
n &= 2, 3, \ldots; \quad 0 \leq t \leq T_f
\end{align*}
\]

Shifted Legendre polynomials form a complete set of orthogonal polynomials on \([0, T_f]\). Therefore any time function \( f(t) \) which is defined on \([0, T_f]\) and is square integrable can be expanded as a shifted Legendre series (Hwang and Guo 1984):

\[
f(t) = \sum_{i=0}^{\infty} f_i P_i(t)
\]

The orthogonality of the shifted Legendre polynomials is demonstrated by the inner product:

\[
\langle P_m(t), P_n(t) \rangle = \int_0^{T_f} P_m(t)P_n(t) \, dt = \begin{cases} 
    \frac{T_f}{2n+1}, & m = n \\
    0, & m \neq n
\end{cases}
\]

3. System order determination

The methods of system order determination will be presented with the assumption that the system response to a step input is available. We briefly introduce some notation for future reference.

Any non-delay SISO system subject to a step input can be described by the ordinary differential equation:

\[
x^{(n)}(t) + a_1x^{(n-1)}(t) + \ldots + a_n x(t) = c
\]

where \( n \) is the order of the system, \( x^{(i)}(t) = d^i x(t)/dt^i \), and the system is subject to initial conditions \( x^{(i)}(0) = x_{i0} \) for \( i = 0, 1, \ldots, (n-1) \). The characteristic polynomial of (5) has the following form:

\[
p(s) = s^n + a_1 s^{n-1} + \ldots + a_n
\]
which can in general be factorized as:
\[ p(s) = (s + \gamma_1)\cdots(s + \gamma_j)\cdots(s + \gamma_k - \lambda) \cdots(s + \gamma_k + R) \]
where \( R = r_1 + r_2 + \ldots + r_s \). \( \gamma_1, \gamma_2, \ldots, \gamma_k \) are repeated roots and \( \gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_{k+R} \) are distinct roots. We define the fundamental solution set of (5) as being the union of the subsets corresponding to each of the repeated roots and the subset corresponding to distinct roots, i.e.
\[ \{w_1(t), w_2(t), \ldots, w_d(t)\} \]
\[ = \{t^{j-1} \exp(-\gamma_1 t), \ j = 1, 2, \ldots, r_1\} \cup \ldots \cup \{t^{j-1} \exp(-\gamma_k t), \ j = 1, 2, \ldots, r_k\} \]
\[ \cup \{\exp(-\gamma_{k+1} t), \exp(-\gamma_{k+2} t), \ldots, \exp(-\gamma_{k+R} t)\} \]
Therefore, the general solution of (5) is of the form.
\[ x(t) = \sum_{j=1}^{k} \sum_{i=1}^{r_j} \beta_{ij} t^{i-1} \exp(-\gamma_i t) + \ldots + \sum_{j=1}^{k} \beta_{kj} t^{k_1-1} \exp(-\gamma_k t) \]
\[ + \sum_{j=1}^{k+R} B_j \exp(-\gamma_i t) + d \]
where \( \{\beta_{ij} : \ j = 1, 2, \ldots, k; \ i = 1, 2, \ldots, r_j\}, \{B_j : \ j = 1, 2, \ldots, k + n - R\} \) are constants which can be determined by the initial conditions and \( d = c/a_n \).

### 3.1. Multiple integration

The following notation for the multiple integrals considerably simplifies the subsequent analysis.
\[ X^{(i)}(t) = \int_0^t \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_n} x(t_1) \, dt_1 \ldots dt_n \, dt_n \text{ for } n \geq 2 \]
\[ X^{(n)}(t) = \int_0^t x(t) \, dt_1 \]
Before we investigate the process of multiply integrating the output \( x(t) \), we shall find a general formulation obeyed by successive integrals of the general expression
\[ x_n = \sum_{j=1}^{k} \beta_{ij} t^{i-1} \exp(-\gamma_i t) \]
integrating once gives:
\[ \int_0^t x_n \, dt = \sum_{j=1}^{k} \beta_{ij} \int_0^t t^{j-1} \exp(-\gamma_i t) \, dt = \sum_{j=1}^{k} \beta_{ij} \left[\frac{t^{j-1}}{\gamma_i} + \frac{(j-1)t^{j-2}}{\gamma_i^2} + \ldots + \frac{(j-1)(j-2)...1}{\gamma_i^{j-1}}\right] \exp(-\gamma_i t) \]
Collecting terms in \( t, t^2, \ldots, t^j \), we obtain
\[ \int_0^t x_n \, dt = \left[\beta_{i1}^1 + \beta_{i2}^2 t + \ldots + \beta_{in}^n t^{n-1}\right] \exp(-\gamma_i t) \]
where \( \beta_{ij}, \ j = 1, 2, \ldots, r_i \) are constants which can be determined. Thus
\[ \int_0^t x_n \, dt = \sum_{j=1}^{k} \beta_{ij}^n t^{j-1} \exp(-\gamma_i t) \]
Continuing this process we see that the qth integral of \( x_n \) will have the form

\[
X^{(q)}(t) = \sum_{j=1}^{\infty} \beta_j^{(q)} t^{j-1} \exp(-\gamma_j t) - \frac{\beta_j^{(q)} t^{j-1}}{(q-1)!} - \cdots - \frac{\beta_j^{(q)} t^{j-1}}{(q-1)!}
\]

and hence the qth integral of \( x(t) \): i.e.

\[
X^{(q)}(t) = X^{(q)}_0 + X^{(q)}_1 + \cdots + X^{(q)}_n + \cdots + X^{(q)}_n + \cdots + \frac{dt^q}{q!}
\]

where \( c_1, c_2, \ldots, c_{e+1} \) are constants which can be determined. \( X^{(q)}(t) \) can also be written as

\[
X^{(q)}(t) = \sum_{j=1}^{\infty} a_j^{(q)} w_j(t) + c_{e+1} + c_e t + \cdots + c_1 \frac{t^q}{q!}
\]

where \( \{w_j(t): j = 1, 2, \ldots, n\} \) are the elements of the fundamental set defined in (6) and \( \{a_j^{(q)}: j = 1, 2, \ldots, n\} \) are respective coefficients. If we let

\[
V_j(t) = w_j(t), \quad j = 1, 2, \ldots, (n-1) \quad \text{and} \quad V_n(t) = \frac{c_{e+1}}{a_1^{(q)} + w_n(t)}
\]

Then we can write

\[
X^{(q)}(t) = \sum_{j=1}^{\infty} a_j^{(q)} V_j(t) + c_{e+1} + c_e t + \cdots + c_1 \frac{t^q}{q!} \tag{8}
\]

3.2. Non-square matrix

The first technique of system order identification consists of investigating the behaviour of the singular values of a non-square matrix \( F \) defined by

\[
F = \begin{bmatrix}
X^{(1)}(t_0) & X^{(2)}(t_0) & \cdots & X^{(m)}(t_0) & U^{(1)}(t_0) & \cdots & U^{(m)}(t_0) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
X^{(1)}(t_1) & X^{(2)}(t_1) & \cdots & X^{(m)}(t_1) & U^{(1)}(t_1) & \cdots & U^{(m)}(t_1)
\end{bmatrix} \tag{9}
\]

where \( t_0, t_1, \ldots, t_q \) are the discrete points in time at which both input and output are observed, \( m \) is an over estimate of the model order and

\[
U^{(i)}(t) = \frac{t^i}{i!}, \quad i = 1, 2, \ldots, m
\]

are the \( i \)th integrals of the step input as defined for \( X^{(i)}(t) \) in (7). Thus there are \((N+1)\) input–output data measurements and \( N \geq 2m \) in general.

**Theorem**

If \( n \) is the order of the system and if \( m > n \) then the matrix \( F \) has \( n_e = (m - n) \) zero-singular values. The order of the system is therefore given by \( n = m - n_e \).
Proof

Using the general formula (8) we can generate successive multiple integrals $X^{(i)}(t)$, $i = 1, 2, \ldots, m$.

\[
\begin{align*}
X^{(1)}(t) &= \sum_{j=1}^{n} d_{j}^{(1)} V_{j}(t) + c_{0}t \\
X^{(2)}(t) &= \sum_{j=1}^{n} d_{j}^{(2)} V_{j}(t) + c_{0}t + c_{1} \frac{t^{2}}{2!} \\
&\vdots \\
X^{(m)}(t) &= \sum_{j=1}^{n} d_{j}^{(m)} V_{j}(t) + c_{0}t + c_{1} \frac{t^{2}}{2!} + \ldots + c_{m} \frac{t^{m}}{m!}
\end{align*}
\]

(10)

If we consider the first $n$ equations we have

\[
\begin{align*}
da_{1}^{(1)} V_{1}(t) + da_{2}^{(1)} V_{2}(t) + \ldots + da_{n}^{(1)} V_{n}(t) &= X^{(1)}(t) - c_{0}t \\
da_{1}^{(2)} V_{1}(t) + da_{2}^{(2)} V_{2}(t) + \ldots + da_{n}^{(2)} V_{n}(t) &= X^{(2)}(t) - c_{0}t - c_{1} \frac{t^{2}}{2!} \\
&\vdots \\
da_{1}^{(m)} V_{1}(t) + da_{2}^{(m)} V_{2}(t) + \ldots + da_{n}^{(m)} V_{n}(t) &= X^{(m)}(t) - c_{0}t - \ldots - c_{m-1} \frac{t^{m-1}}{n!}
\end{align*}
\]

(11)

Equation (11) is a set of linear algebraic equations which, if solved gives

\[
V_{j}(t) = \sum_{i=1}^{n} z_{ij} X^{(i)}(t) + \sum_{i=1}^{n} \theta_{ij} \frac{t^{i}}{i!}, \quad j = 1, 2, \ldots, n
\]

(12)

where \(\{x_{ij}, \theta_{ij}: i = 1, 2, \ldots, n; j = 1, 2, \ldots, n\}\) are constants determined by \(\{c_{i}: i = 1, 2, \ldots, n\}\) and \(\{d_{j}^{(k)}: j = 1, 2, \ldots, n; k = 1, 2, \ldots, n\}\). Equation (12) shows that any $V_{j}(t), j = 1, 2, \ldots, n$ is a linear combination of \(\{X^{(i)}(t): i = 1, 2, \ldots, n\}\) and \(\{t^{i}: i = 1, 2, \ldots, n\}\). If we replace $V_{j}(t), j = 1, 2, \ldots, n$ by the right-hand side of (12) in the last \((m-n)\) equations of (10) we have:

\[
\begin{align*}
X^{(m-1)}(t) &= \sum_{j=1}^{n} d_{j}^{(m-1)} \left[ \sum_{i=1}^{n} z_{ij} X^{(i)}(t) + \sum_{i=1}^{n} \theta_{ij} \frac{t^{i}}{i!} \right] + c_{m-1} t^{m-1} + \ldots + c_{m} \frac{t^{m}}{(m+1)!}
\end{align*}
\]

(13)

Thus defining

\[
\eta_{k} X^{(k)}(t) = \sum_{j=1}^{n} d_{j}^{(k)} z_{ij} X^{(i)}(t)
\]

\[
c_{0}t + \ldots + c_{m} \frac{t^{m}}{m!} = \sum_{j=1}^{n} \sum_{i=1}^{n} \theta_{ij} \frac{t^{i}}{i!} + c_{0}t + \ldots + c_{m} \frac{t^{m}}{m!}
\]
∀k such that \( n < k \leq m \) then
\[
\begin{align*}
X^{(n+1)}(t) &= \sum_{i=1}^{n} \eta_{i,n+1} X^{(i)}(t) + c'_1 t + \ldots + c'_1 \frac{t^{n+1}}{(n+1)!} \\
\vdots & \\
X^{(m)}(t) &= \sum_{i=1}^{n} \eta_{i,m} X^{(i)}(t) + c'_m t + \ldots + c'_m \frac{t^{m}}{m!}
\end{align*}
\]  
(14)

Equation (14) shows that any \( X^{(i)}(t) \) where \( n < i \leq m \), can be expressed as a linear combination of \( \{ X^{(i)}(t) : j = 1, 2, \ldots, n \} \) and \( \{ t^k : k = 1, 2, \ldots, i \} \) ∀ \( t \in [t_0, t_n] \). Thus the following \( n \) columns \( (n_s = m - n) \) of matrix \( F \)
\[
\begin{bmatrix}
X^{(n+1)}(t_0) & X^{(n+2)}(t_0) & \ldots & X^{(m)}(t_0) \\
\vdots & \vdots & & \vdots \\
X^{(n+1)}(t_s) & X^{(n+2)}(t_s) & \ldots & X^{(m)}(t_s)
\end{bmatrix}
\]  
(15)

are linearly dependent on the first \( n \) and the last \( m \) columns of the matrix \( F \) and therefore \( F \) must have \( n_s = (m - n) \) zero singular values.

In practice we provide an initial over-estimate \( m \) of the model order. The matrix \( F \) is then computed by suitable numerical integration of the output signal. The singular values of \( F \) are computed, the number of zero singular values \( (n_s) \) is noted and the system order is given by \( n = (m - n_s) \).

3.3 Use of shifted Legendre polynomials

We have shown that the \( i \)th integral of the system output can be written as
\[
X^{(i)}(t) = \eta_{1,i} X^{(1)}(t) + \eta_{2,i} X^{(2)}(t) + \ldots + \eta_{n,i} X^{(n)}(t) + c'_i t + \ldots + c'_i \frac{t^i}{i!}
\]  
(16)

where \( n < i \leq m \). We have also stated that any function \( f(t) \) which is defined on \([0, T_f]\) and is square integrable can be expanded as a shifted Legendre series and that
\[
\langle P_{d}(t), P_{m}(t) \rangle = \int_0^{T_f} P_{d}(t)P_{m}(t) \, dt = 0, \quad \text{for } m \neq n
\]

Therefore any polynomial of the form
\[
f(t) = q_r t^r + q_{r-1} t^{r-1} + \ldots + q_0, \quad r \geq 0
\]  
(17)

can be expanded as
\[
f_d(t) = b_r P_r(t) + b_{r-1} P_{r-1}(t) + \ldots + b_0 P_0(t)
\]
and (16) can be expressed as
\[
X^{(i)}(t) = \sum_{j=1}^{n} \eta_{j,i} X^{(j)}(t) + \sum_{j=1}^{i} \delta_{ij} f_d(t), \quad n < i \leq m
\]  
(18)

where the monomials \( \{ t^j \} : j = 1, 2, \ldots, i \) have been written as shifted Legendre function expansions \( \{ f_j(t) : j = 1, 2, \ldots, i \} \) Now
\[
\langle f_j(t), P_q(t) \rangle = 0, \quad j < q
\]
and hence (18) gives

$$\langle X^{0k}(t), P_d(t) \rangle = \sum_{j=1}^{\infty} \eta_j \langle X^{00}(t), P_d(t) \rangle$$

\(\forall t\) with \(n < i \leq m\), \(q > m\) and \(T_r = T_v\). Thus the \((m+m)\) square matrix \(F_r\) defined as

$$F_r = \begin{bmatrix}
\langle X^{11}(t), P_{e1}(t) \rangle & \cdots & \langle X^{1m}(t), P_{e1}(t) \rangle \\
\vdots & \ddots & \vdots \\
\langle X^{11}(t), P_{e+m-1}(t) \rangle & \cdots & \langle X^{1m}(t), P_{e+m-1}(t) \rangle
\end{bmatrix}$$

(19)

has only \(n\) linearly independent columns and therefore \(n\) non-zero eigenvalues when \(q > m\).

Since \(x(t)\) and the multiple integrals \(\{X^{0k}(t)\} \quad i = 1, 2, \ldots, m\) are only available at discrete points in time \(t_0, t_1, \ldots, t_m\), the inner product integrals which constitute \(F_r\) have, in practice, to be computed numerically. The second technique for system order identification therefore consists of forming the matrix \(F_r^*\); the number of non-zero eigenvalues of \(F_r^*\) then determines the system order.

4. Effect of disturbances

We have noted that the general solution of (5) was

$$x(t) = \sum_{j=1}^{\infty} \beta_{1j} t^{i-1} \exp (-\gamma_1 t) + \cdots + \sum_{j=1}^{\infty} \beta_{4j} t^{i-1} \exp (-\gamma_4 t)$$

$$+ \sum_{j=\infty+1}^{\infty} B_j \exp (-\gamma_j t) + d$$

when subject to a step input \(r(t) = 0\). In practical situations, the input and the measured output may often be subject to additive disturbances, i.e.

$$\begin{cases}
u(t) = r(t) + g(t) \\
y(t) = x(t) + h(t)
\end{cases}$$

(20)

where \(r(t)\) and \(y(t)\) are the measured input and output, \(u(t)\) and \(x(t)\) being the true system input and output. We shall now investigate the functionality of the two system order identification schemes in the presence of typical but unquantifiable disturbances.

4.1. Effect of a constant offset

Suppose that the additive disturbances on both the input and the output are constant offsets, i.e.

$$\begin{cases}
g(t) = \eta \\
h(t) = \sigma
\end{cases}$$

(21)

where \(\eta\) and \(\sigma\) are unknown constants. The input to the system \(u(t) = r(t) + \eta\) is still a step if \(r(t)\) is a step and \(\eta\) is a constant. Therefore the measured output will still be of
the form (6), i.e.

\[ y(t) = \sum_{i=1}^{p} \beta_i t^{i-1} \exp(-\gamma_i t) + \ldots + \sum_{j=q+1}^{p} \beta_j t^{j-1} \exp(-\gamma_j t) + \ldots + \sum_{j=q+1}^{\infty} B_j \exp(-\gamma_j t) + \rho \]  

Equations (22) and (6) are structurally identical. Thus by using the measured input \( r(t) \) and output \( y(t) \) and by forming the matrices

\[
F = \begin{bmatrix}
Y^{11}(t_0) & Y^{12}(t_0) & \ldots & Y^{1m}(t_0) & R^{11}(t_0) & \ldots & R^{1m}(t_0) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
Y^{n1}(t_v) & Y^{n2}(t_v) & \ldots & Y^{nm}(t_v) & R^{n1}(t_v) & \ldots & R^{nm}(t_v)
\end{bmatrix}
\]

and

\[
F_p = \begin{bmatrix}
\langle Y^{11}(t), P_p(t) \rangle & \ldots & \langle Y^{1m}(t), P_p(t) \rangle \\
\vdots & \ddots & \vdots \\
\langle Y^{n1}(t), P_{p+p-1}(t) \rangle & \ldots & \langle Y^{nm}(t), P_{p+p-1}(t) \rangle
\end{bmatrix}
\]

the analysis of the preceding sections still applies. Thus even in the presence of constant offsets, for an assumed system order \( m > n \), the actual system order is given by \( n = (m - n) \) where \( n \) is the number of zero singular values of \( F \) or alternatively by the number of non-zero eigenvalues of \( F_p \).

### 4.2. Effect of other polynomial disturbances

When the disturbance is a polynomial with unknown coefficients, a slight amendment to the square matrix \( F_p \) still permits the system order to be identified. Suppose that the disturbances have the form:

\[
g(t) = \sum_{i=0}^{n_1} \Psi_i t^i, \quad h(t) = \sum_{i=0}^{n_2} \phi_i t^i
\]

where \( n_1, n_2 \) \( \{\Psi_i; \quad i = 0, 1, \ldots, n_1\}, \{\phi_i; \quad i = 0, 1, \ldots, n_2\} \) are unknown. The output due to a polynomial input is a sum of the complementary function (linear combination of elements of the fundamental set) and a particular integral (polynomial of the same degree as the input). Therefore the measured output will have the form:

\[
y(t) = \sum_{j=1}^{s} A_j w_j(t) + \sum_{j=1}^{s} \beta_j t^j
\]

where \( s = \max(n_1, n_2) \). Thus the integral of the measured output can be written as

\[
Y^{m}(t) = \sum_{j=1}^{s} A_j^m V_j(t) + \sum_{j=1}^{s} \beta_j^m t^j
\]

Thus by the analysis that led to (18) we have

\[
Y^{n}(t) = \sum_{j=1}^{s} \mu_j Y^{n}(t) + \sum_{j=1}^{s} \delta_{ij} f_j(t)
\]

\( \forall n < s \leq m \)
Continuous system order identification

and

\[ \langle Y^{\mu}(t), P_{\lambda}(t) \rangle = \sum_{n=1}^{N} \mu_n \langle Y^{\mu}(t), P_{\lambda}(t) \rangle \]  \hfill (26)

\( \forall \) with \( n < \mu < \mu, q > (m + s) \) and \( T_r = T_n \). Hence the matrix

\[ F = \begin{bmatrix}
\langle Y^{11}(t), P_{\lambda}(t) \rangle & \cdots & \langle Y^{1s}(t), P_{\lambda}(t) \rangle \\
\vdots & \ddots & \vdots \\
\langle Y^{\mu 1}(t), P_{\mu + \alpha - 1}(t) \rangle & \cdots & \langle Y^{\mu s}(t), P_{\mu + \alpha - 1}(t) \rangle 
\end{bmatrix} \]  \hfill (27)

has only \( n \) linearly independent columns and therefore \( n \) non-zero eigenvalues when \( q > (m + s) \). Thus we form \( F \) numerically for a sufficiently large \( q \) and, as before, the number of non-zero eigenvalues of \( F \) determines the system order.

5. Numerical results

The results are quoted for measured data in full floating point form and also for such data following twelve bit analogue to digital conversion. The routine to find singular values of \( F \) reduces \( F \) to upper triangular form by Householder transformations. It then uses Givens plane rotations to reduce the triangular form to a bidiagonal form, and finally the QR algorithm is used to obtain the singular value decomposition (Wilkinson 1978). The Newton–Cotes seven point formula was used as the quadrature scheme required for the inner products involving shifted Legendre polynomials. The routine used to find the eigenvalues of \( F \) involves the reduction of \( F \) to an upper Hessenberg form and the QR algorithm (Wilkinson and Reinsch 1971).

Example 1

Firstly we consider a simple first order system

\[ x(t) + x(t) = u(t) \]

The total observation time was chosen to be \( T_n = 5 \), with \( N = 500 \). The input was a unit-step \( u(t) = 1, \forall t > 0 \) Two different estimates \((m = 2 \) and \( m = 4 \)) of the order were taken and the singular values and the eigenvalues are as shown below. Results for

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>No offset</td>
<td>Offset = 40</td>
<td></td>
</tr>
<tr>
<td>No offset</td>
<td>Offset = 40</td>
<td></td>
</tr>
<tr>
<td>( m = 2 )</td>
<td>( m = 4 )</td>
<td></td>
</tr>
<tr>
<td>167 3</td>
<td>667 03</td>
<td>338 24</td>
</tr>
<tr>
<td>17 98</td>
<td>67 8</td>
<td>46 5</td>
</tr>
<tr>
<td>1 25</td>
<td>0 492</td>
<td>8 48</td>
</tr>
<tr>
<td>0 37 ( \times 10^{-14} )</td>
<td>0 11 ( \times 10^{-13} )</td>
<td>1 46</td>
</tr>
<tr>
<td>0 18</td>
<td>0 453</td>
<td>0 12 ( \times 10^{-13} )</td>
</tr>
<tr>
<td>0 9 ( \times 10^{-14} )</td>
<td>0 11 ( \times 10^{-13} )</td>
<td>0 37 ( \times 10^{-14} )</td>
</tr>
</tbody>
</table>

Table 1 Singular values of \( F \) for Example 1.
the situation in which the measured output differs from the actual output by a constant offset are also documented.

The singular values were calculated on a computer which has $\varepsilon = 10^{-7}$ as the value of $\varepsilon$ such that $1 + \varepsilon > 1$. Hence values less than $10^{-7}$ are considered as negligible. From the above results, we can see that the offset did not change the number of non-zero eigenvalues nor the number of zero singular values, and in all cases the system order is identified as one.

**Example 2**

This second example consists of the second order system:

$$\ddot{x} + 4\dot{x} + 3x = 4r$$

with the initial conditions

$$x(0) = 0.5, \quad \dot{x}(0) = -2.0$$

The input signal $r(t)$ was again a unit-step and the time interval for measurements was $T_x = 5$ with $N = 500$. The results shown below are for both types of data, i.e. floating point and following 12 bit analogue to digital conversion. Estimates of the system order were chosen as $m = 3, 4$ and $5$. The singular values and eigenvalues were as shown in Tables 3 and 4.

<table>
<thead>
<tr>
<th>$m = 2$</th>
<th>$m = 4$</th>
<th>$m = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No offset</td>
<td>Offset = 40</td>
<td>No offset</td>
</tr>
<tr>
<td>$-0.175$</td>
<td>$-0.175$</td>
<td>$-0.162 \times 10^{-1}$</td>
</tr>
<tr>
<td>$0.99 \times 10^{-9}$</td>
<td>$0.99 \times 10^{-9}$</td>
<td>$-0.10 \times 10^{-9}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-0.18 \times 10^{-10}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.33 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

Table 2. Eigenvalues of $F_m$ for Example 1

<table>
<thead>
<tr>
<th>$m = 3$</th>
<th>$m = 4$</th>
<th>$m = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real data</td>
<td>A/D</td>
<td>Real</td>
</tr>
<tr>
<td>269.99</td>
<td>33.49</td>
<td>5.48</td>
</tr>
<tr>
<td>347.83</td>
<td>47.45</td>
<td>8.68</td>
</tr>
<tr>
<td>347.83</td>
<td>47.45</td>
<td>8.68</td>
</tr>
<tr>
<td>398.03</td>
<td>59.02</td>
<td>11.73</td>
</tr>
<tr>
<td>398.03</td>
<td>59.02</td>
<td>11.73</td>
</tr>
<tr>
<td>398.03</td>
<td>59.02</td>
<td>11.73</td>
</tr>
<tr>
<td>398.03</td>
<td>59.02</td>
<td>11.73</td>
</tr>
</tbody>
</table>

Table 3. Singular values of $F_m$ for Example 2.
Continuous system order identification

### Table 4 Eigenvalues of \( F_e \) for Example 2.

<table>
<thead>
<tr>
<th>( m = 3 )</th>
<th>( m = 4 )</th>
<th>( m = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real A/D</td>
<td>Real A/D</td>
<td>Real A/D</td>
</tr>
<tr>
<td>-0.141</td>
<td>0.087</td>
<td>-0.047</td>
</tr>
<tr>
<td>0.11</td>
<td>-0.03</td>
<td>0.0057</td>
</tr>
<tr>
<td>-0.16 × 10^{-5}</td>
<td>-0.27 × 10^{-5}</td>
<td>-0.23 × 10^{-10}</td>
</tr>
<tr>
<td>-0.16 × 10^{-5}</td>
<td>-0.68 × 10^{-6}</td>
<td>(-0.10 × 10^{-11})</td>
</tr>
<tr>
<td>(+0.15 × 10^{-10})</td>
<td>(+0.31 × 10^{-11})</td>
<td>(-0.9 × 10^{-7})</td>
</tr>
<tr>
<td>(-0.10 × 10^{-11})</td>
<td>(-0.46 × 10^{-7})</td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Eigenvalues of \( F_e \) for Example 3

### Example 3

This is the same example as the previous one with the exception that the output and the input are subject to polynomial additive disturbances, i.e.,

\[
g(t) = 1 + r, \quad h(t) = 3 + 2r + r^2
\]

In such a situation we use the square matrix \((F_e)\) formulation. The inner products of the multiple integrals and the shifted Legendre polynomials had \( q = 6 \) for \( m = 3 \), \( q = 7 \) for \( m = 4 \) and \( q = 8 \) for \( m = 5 \). The observation interval time was \( T,v = 5 \) with \( N = 250 \). The numerical results show that the number of non-zero eigenvalues \( F_e \) is two in all cases.

### Table 5. Eigenvalues of \( F_e \) for Example 3

<table>
<thead>
<tr>
<th>( m = 3 )</th>
<th>( m = 4 )</th>
<th>( m = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.29 × 10^{-1}</td>
<td>-0.12 × 10^{-2}</td>
<td>0.48 × 10^{-2}</td>
</tr>
<tr>
<td>-0.65 × 10^{-2}</td>
<td>0.103 × 10^{-3}</td>
<td>-0.130 × 10^{-4}</td>
</tr>
<tr>
<td>-0.25 × 10^{-8}</td>
<td>-0.912 × 10^{-8}</td>
<td>-0.98 × 10^{-7}</td>
</tr>
<tr>
<td>-0.450 × 10^{-8}</td>
<td>-0.26 × 10^{-8}</td>
<td>-0.69 × 10^{-6}</td>
</tr>
</tbody>
</table>

### Example 4

A final example consists of a third order system where the characteristic polynomial has a repeated root. Thus the system, subject to a step input \( r(t) = 1 \), is

\[
\ddot{v}(t) + 5\dot{v}(t) + 8v(t) + 4x = 4r(t)
\]

\[
v(0) = -140, \quad \dot{v}(0) = -20, \quad v(0) = 5
\]
Continuous system order identification

<table>
<thead>
<tr>
<th>m = 5</th>
</tr>
</thead>
</table>

Singular values of F | Eigenvalues of F
---|---

| No disturbance | No disturbance | \(q(t) = 3 + 2t + t^2\) | \(p(t) = 1 + t\)
---|---|---|---
225 883 | -0.117 | 0.67 \(\times 10^{-2}\) |
198 604 | 0.32 \(\times 10^{-1}\) | -0.114 \(\times 10^{-2}\) |
32 377 | -0.35 \(\times 10^{-3}\) | 0.311 \(\times 10^{-4}\) |
7 006 | 0.117 \(\times 10^{-10}\) | -0.22 \(\times 10^{-7}\) |
1 447 | 0.298 \(\times 10^{-10}\) | 0.71 \(\times 10^{-8}\) |
0 5001 | 0 1125 | 0 000321 |
0 229 \(\times 10^{-10}\) | 0 33 \(\times 10^{-13}\) |

Table 6. Singular values of F and eigenvalues of F, for Example 4

The corresponding characteristic polynomial is \(P(s) = (s + 2)^2(s + 1)\). The time interval of measurement is \(T_N = 8\) with \(N = 400\). Only one estimate of the order was taken, i.e. \(m = 5\), and the results are quoted in full floating point.

The numerical values show that a repeated root of the characteristic polynomial has no effect in identifying the order of the system under study

6. Conclusions

Two methods have been proposed for system order identification in continuous SISO systems. They can easily be extended to include MIMO model order determination. Non-zero initial conditions and constant offset disturbances at both input and output have no effect on finding the correct order. The major advantage of these techniques over many others in the literature is that only one test is sufficient to exactly determine the order of the system under consideration. The test input signal is restricted to standard inputs such as steps, impulses and ramps. Additionally, the use of inner products with Legendre polynomials can cope with additive polynomial disturbances at both input and output

References

AKAIKE, H., 1978, Int J Control, 27, 323
HWANG, C., and GUO, T., 1984, Int J Control, 39, 807
SAGARA, S., GOTANDA, H., and WADA, K., 1982, Int J Control, 35, 637
VAN DEN boom, A. J. W., and VAN DEN ENDEN, A. W. M., 1974, Automatica, 10, 245
WELLSTEAD, P. E., 1978, Automatica, 14, 89
WOODSIDE, C. M., 1971, Automatica, 7, 727
Appendix D3.

Weighted residual approaches to system identification
and model reduction

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and

N MESSALI *

(This work has been accepted for publication in Int. J. Syst. Sci., March 1988.)

This paper extends the weighted residual concept of ordinary differential equation solution to
the two problems of continuous linear system identification and model reduction. Although
continuous system dynamics are classically described in terms of the system ordinary differential
equation, an equivalent integral equation description exists and it is this latter equation which is
used as the basis of the various identification and model reduction techniques. Several individual
approaches within the broad class of weighted residual approaches are detailed and a number of
numerical examples are presented to illustrate the application of such schemes.

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1. Introduction

System identification and model reduction are two of the most prevalent topics in the current literature on control engineering. The former problem is characterised by the availability of input/output data from which a suitable parametric system description is sought. The latter problem is characterised by the availability of a high order parametrised model to which an appropriate low approximation is sought. A high order model can, of course, be simulated for specified initial conditions and therefore input/output data can be generated and used for the purpose of model reduction. This paper uses input/output data sets as the basis for solutions to the two problems of system identification and model reduction. In both problems we shall consider linear SISO systems though the methods presented are readily extended to linear MIMO systems and linear-in-parameters non-linear systems.

Parameter identification of continuous-time system models has a long history as shown by the surveys in Eykhoff (1974), Young (1981). Frequently identification methods are based on the system transfer function and the associated system ordinary differential equation though Whitfield and Messali (1987) have recently presented a robust method based on an equivalent integral equation formulation. This paper uses predominantly the same integral equation description but presents the latter method as only one of a number of techniques that will be presented and classified as weighted residual methods.

The problem of model reduction has also received considerable attention over the last two decades. One group of methods is based on aggregation theory and includes the dominant pole approach (Mahapatra 1971, Bonvin and Mellichamp 1982) and a perturbation method (Kokotovic and O'Malley 1976, Kokotovic et al 1980). A second approach is to consider approximations to the high order model transfer functions. Such methods include the continued fraction approach (Chen and Shieh 1968), a time moment method (Zakian 1973) and coefficient matching in the Laurent series between the reduced and original model (Shamash 1974, Daly and Colebourn 1979). A further approach is to consider minimization of an error criterion which measures the difference between the high order model and the approximant outputs for specified inputs (Wilson 1974, Riggs and Edgar 1979, Wilson and Mishra 1979). Recent attention has focused on model reduction by balanced realisation (Glover 1984). All the latter methods rely on the availability of a high order parametric description of the plant. Order reduction methods which utilise only input/output data sets are based primarily on error criteria and include time domain (Sinha and Pille 1971, Sinha and Berezinai 1971) and frequency domain (Genesio and Pome 1975, Whitfield and Williams 1987) approaches. The major advantage of utilising input/output data is that a higher order system can be approximated by a low order model without the need to explicitly identify the
parameters pertinent to the high order system. Such data will therefore be used in the various methods of model reduction presented in this paper.

The techniques to be presented derive directly from the class of weighted residual methods which have been used classically to produce approximate solutions to differential equations (Finlayson 1980). The three individual areas of equation solution, system identification and model reduction are classified as fundamental problems and the application of the weighted residual methods to each problem is developed from a general framework. Specific results from the application of such methods to various identification and model reduction problems are then presented.

2. Linear system considerations

In this section we denote the output of a linear system by \( x(t) \), this output resulting from certain initial conditions when the system is subject to a prescribed input \( u(t) \).

2.1 Time domain system representations

The most common time domain expression for the dynamics of a non-delay time invariant linear SISO system is an ordinary differential equation of the form

\[
x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_n x(t) = b_1 u^{(n-1)}(t) + \ldots + b_n u(t)
\]  

subject to initial conditions \( x^{(i)}(0) = x_{i0} \) \( (i=0,1,\ldots,n-1) \). We can write such a general ordinary differential equation as a linear differential operator equation \( L_d[x(t),u(t)] = 0 \), subject to the same initial conditions on \( x(t) \) and its derivatives.

An alternative description of this system is provided by the integral equation expression (Whitfield and Messali 1987)

\[
x(t) + \sum_{k=1}^{n} a_k x^{(k)}(t) - \sum_{k=1}^{n} b_k u^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{k-1}{(k-1)!} = 0
\]

where

\[
c_1 = -x(0)
\]

\[
c_k = -x^{(k-1)}(0) - \sum_{j=1}^{k-1} a_j x^{(k-j-1)}(0) + \sum_{j=1}^{k-1} b_j u^{(k-j-1)}(0) \quad (k=2,3,\ldots,n)
\]

and
\[ X^{(1)}(t) = \int_0^t x(t) \, dt \]
\[ X^{(k)}(t) = \int_0^t \cdots \int_0^{t_{k-2}} x(t_{k-1}) \, dt_{k-1} \, dt_{k-2} \quad \text{for } k \geq 2 \]

We can write the integral equation (2) as a linear integral operator equation \( L[x(t), u(t)] = 0 \). Thus we can express the dynamics of such a linear system by a general operator statement \( L[x(t), u(t)] = 0 \), where \( L \) includes the system initial conditions if it is an integral operator or is subject to such initial conditions if it is a differential operator.

2.2 Three fundamental problems

Having posed the dynamics of a linear system as

\[ L[x(t), u(t)] = 0 \quad (3) \]

we shall now distinguish between three basic problems.

The first problem is that of the solution of the system equation (3). Here we seek to find the function \( x(t) \) which solves equation (3) for a prescribed input \( u(t) \) and initial conditions \( x^{(i)}(0) = x_i \), \( i = 0, 1, \ldots, n-1 \) when the structure and coefficients of \( L \) are given precisely i.e. the constants \( n \), \( \{a_k, b_k; k = 1, 2, \ldots, n\} \) in (1) and (2) are given and the constants \( \{c_k; k = 1, 2, \ldots, n\} \) in (2) are calculable. Although (2) is an entirely valid description of the linear system, the ordinary differential equation (1) is the standard vehicle to calculate \( x(t) \) both analytically and numerically.

A second problem is that of system identification. Here it is generally presumed that the order of the system, \( n \), is known and that a time history of the output \( x(t) \) is available for a given input \( u(t) \). In practice the input and output may only be available at discrete points in time i.e. as time series \( \{x(t_i); i = 0, 1, \ldots, N_t - 1\}, \{u(t_i); i = 0, 1, \ldots, N_u - 1\} \). The problem then is to identify the parameters \( \{a_k, b_k; k = 1, 2, \ldots, n\} \) in the system ordinary differential equation (1). The integral equation (2) may act as a more robust vehicle for this purpose since, if either the input or output signals are corrupted by noise, forming the derivative components of (1) may yield numerically unacceptable approximations. Since the parameters \( \{c_k; k = 1, 2, \ldots, n\} \) are fundamental to the integral equation expression of system dynamics, they must also be identified in this latter approach (Whitfield and Messali 1987).

A third and final problem is that of model reduction. Here it is presumed that either a high order model has been identified or that input/output data from a high order system is available.
The problem is to identify parameters \( \{ a_k, b_k; k=1,2,\ldots,n \} \) in a reduced model which still convey the important dynamics of the higher order system with order \( n \geq n \).

3. Weighted residual methods

3.1 A general residual formulation

Weighted residual methods (Finlayson 1980) provide a class of approaches to the problem of solving an ordinary differential equation of which (1) is a typical example. Such methods could also be used to solve an integral equation of the form (2) and we shall therefore present such methods in the context of general linear operator equation (3).

All weighted residual methods calculate a set of parameter values \( \{ \alpha_i; i=1,2,\ldots,N \} \) for an approximate solution \( \hat{x}(t) \) to the operator equation (3) where:

\[
\hat{x}(t) = \sum_{i=1}^{N} \alpha_i \psi_i(t)
\]  

(4)

and \( \{ \psi_i(t); i=1,2,\ldots,N \} \) are combinations of preselected basis functions such as monomials \( \{ t^i; i=1,2,\ldots,N \} \) or orthogonal polynomials \( \{ P_{i-1}(t); i=1,2,\ldots,N \} \), chosen so that \( \hat{x}(t) \) satisfies the given initial conditions on \( x(t) \). The technique for determining the parameters \( \{ \alpha_i; i=1,2,\ldots,N \} \) is dictated by the particular method chosen. In the context of linear operator equation solution, a residual \( R(t) \) arises from the application of the known linear operator to the approximate solution \( \hat{x}(t) \). Thus

\[
R(t) = L[\hat{x}(t), u(t)]
\]  

(5)

and the weighted residual methods calculate the solution parameters \( \{ \alpha_i; i=1,2,\ldots,N \} \) by forcing \( R(t) \) to zero in various ways.

This paper seeks to investigate various system identification and model reduction techniques which also arise from the application of weighted residual principles. In the context of system identification an equation residual is formed by application of a linear operator with correct structure but potentially incorrect model parameters while in the model reduction problem an equation residual is formed by application of a linear operator with incorrect structure and consequently incorrect parameters. In both situations the residual \( R(t) \) arises from the application of an approximate operator \( \hat{L} \) to the actual system input and output i.e.

\[
R(t) = \hat{L}[x(t), u(t)]
\]  

(6)
In the case of system identification and model reduction via the ordinary differential equation (1), we seek the parameters \( \{a_k, b_k; k=1,2,\ldots,n\} \) while via the integral equation (2) we seek the parameters \( \{a_k, b_k, c_k; k=1,2,\ldots,n\} \) which suitably force the residual to zero.

A range of alternative approaches to system identification and model reduction may be generated by consideration of a signal residual. This latter residual is defined as:

\[
\mathcal{R}(t) = x(t) - \hat{x}(t)
\]

where \( x(t) \) is the measured output of the system and \( \hat{x}(t) \) is the output predicted by a model when subject to the same input and initial conditions.

We can now encompass all three basic problems in one general weighted residual formulation. We define the general problem as that of finding a parameter vector \( \Omega \) which suitably forces a residual \( \mathcal{R}(\Omega, t) \) to zero, noting that in the case of operator equation solution the parameter vector \( \Omega \) has components \( \{\alpha_i; i=1,2,\ldots,N\} \) and the residual is formed by (5) while in the case of system identification \( \Omega \) has components \( \{a_k, b_k; k=1,2,\ldots,n\} \) for the ordinary differential equation (1), \( \{a_k, b_k, c_k; k=1,2,\ldots,n\} \) for the integral equation (2) and the residual is formed by (6). In general we denote the number of parameters in the vector \( \Omega \) by \( N_p \). For \( t \in [0,T_f] \) we write an inner product of two functions \( g_1(t) \) and \( g_2(t) \) as \( \langle g_1(t), g_2(t) \rangle \); for example the inner product generally used in this paper is

\[
\langle g_1(t), g_2(t) \rangle = \int_0^{T_f} g_1(t) g_2(t) dt
\]

The standard weighted residual methods for operator equation solution will now be written in terms of the latter general formulation and therefore all such methods are equally applicable to the problems of system identification and model reduction.

**Method of moments**

\[
\text{solve } \Omega \quad \langle \mathcal{R}(\Omega, t), t^{-1} \rangle = 0 \quad i=1,2,\ldots,N_p
\]

**Galerkin**

\[
\text{solve } \Omega \quad \langle \mathcal{R}(\Omega, t), \psi_i(t) \rangle = 0 \quad i=1,2,\ldots,N_p
\]

where \( \{\psi_i(t); i=1,2,\ldots\} \) form a complete set of functions on \([0,T_f]\).
Least squares

\[
\min_{\hat{Q}} \sum_{i=0}^{N_c-1} \mathcal{R}(\hat{Q}, t_i)^2
\]

i.e. solve \[
\sum_{i=0}^{N_c-1} \mathcal{R}(\hat{Q}, t_i) \frac{\partial \mathcal{R}(\hat{Q}, t_i)}{\partial \hat{Q}} = \Omega
\]

Point collocation

\[
\text{solve } \sum_{i=0}^{N_c-1} \mathcal{R}(\hat{Q}, t_i) \delta(t_i) = 0 \quad i=0,1,\ldots,N_p-1
\]

i.e. solve \[
\mathcal{R}(\hat{Q}, t_i) = 0 \quad i=0,1,\ldots,N_p-1
\]

where \(t_i\) are values of \(t, t_i \in [0,T_f]\).

Orthogonal collocation

\[
\text{solve } \sum_{i=0}^{N_c-1} \mathcal{R}(\hat{Q}, t_i) \delta(t_i) = 0 \quad i=0,1,\ldots,N_p-1
\]

i.e. solve \[
\mathcal{R}(\hat{Q}, t_i) = 0 \quad i=0,1,\ldots,N_p-1
\]

where \(t_i\) are roots of polynomials which are orthogonal with respect to the inner product defined on \([0,T_f]\).

Least squares point collocation

\[
\min_{\hat{Q}} \sum_{i=0}^{N_c-1} \mathcal{R}(\hat{Q}, t_i)^2
\]

i.e. solve \[
\sum_{i=0}^{N_c-1} \mathcal{R}(\hat{Q}, t_i) \frac{\partial \mathcal{R}(\hat{Q}, t_i)}{\partial \hat{Q}} = \Omega
\]

where \(N_c > N_p\) and \(t_i\) are values of \(t, t_i \in [0,T_f]\).

Least squares quadrature

\[
\min_{\hat{Q}} \sum_{i=0}^{N_q-1} w_i \mathcal{R}(\hat{Q}, t_i)^2
\]

i.e. solve \[
\sum_{i=0}^{N_q-1} w_i \mathcal{R}(\hat{Q}, t_i) \frac{\partial \mathcal{R}(\hat{Q}, t_i)}{\partial \hat{Q}} = \Omega
\]

where \(N_q > N_p\) and \(t_i, w_i : i=0,1,\ldots,N_q-1\) are suitable quadrature points (on \([0,T_f]\)) and weights which approximate the least squares inner product problem defined by (10).
We briefly note a similarity between application of the trapezoidal rule in least squares quadrature and the least squares point collocation formulation. Thus when \( N_q = N_0 \) and the quadrature points \( t_i; i=0,1,...,N_q-1 \) are equally spaced with \( \Delta T = t_{i+1} - t_i \) then using the trapezoidal rule in (14) leads to an equation set

\[
\min_{\Omega} \Delta T \left[ \frac{1}{2} R(t_i) + \sum_{i=1}^{N_q-2} R(t_i) + \frac{1}{2} R(t_{N_q-1}) \right]
\]

i.e. solve

\[
\frac{1}{2} R(t_i) \frac{\partial R(t_i)}{\partial \Omega} + \sum_{i=1}^{N_q-2} R(t_i) \frac{\partial R(t_i)}{\partial \Omega} + \frac{1}{2} R(t_{N_q-1}) \frac{\partial R(t_{N_q-1})}{\partial \Omega} = 0
\]  

while the least squares point collocation scheme (13) produces a set of equations

\[
\sum_{i=0}^{N_q-1} R(t_i) \frac{\partial R(t_i)}{\partial \Omega} = 0
\]

while the least squares point collocation scheme (13) produces a set of equations

\[
\sum_{i=0}^{N_q-1} R(t_i) \frac{\partial R(t_i)}{\partial \Omega} = 0
\]

We note that (16) and (15) only differ by the factor 1/2 in the contributions from \( R(t_i) \) and \( R(t_{N_q-1}) \). Thus for \( N_q \) relatively large, which will generally be the case in the identification problem and can be forced in the operator equation solution and model reduction problems, (15) will only differ marginally from (16) and we can expect very similar estimates of \( \Omega \) from both methods.

3.2 Comparison of methods

As previously commented, application of the various weighted residual methods is well developed for the problem of operator equation solution. We shall now contrast the various methods when applied to this latter problem and when applied to the problems of system identification and model reduction.

One fundamental difference between the problems is that when considering equation solution the number of unknowns, \( N_p \), in the approximate expansion can be increased. Hence in the integral non-least squares methods (method of moments and Galerkin) the residual can be made orthogonal to an increasing number of functions while in the collocation non-least squares methods (point collocation, orthogonal collocation) the residual can be forced to zero at an increasing number of points, both strategies ultimately producing a more accurate solution. However the number of unknowns is fixed in both the identification and model reduction problems and the number of equations produced by (8), (9), (11) and (12) may not be sufficient to adequately force the residual to zero in a global sense. This is particularly true in the model reduction problem
where the number of free parameters is generally small. Hence the problems of system identification and model reduction may well be best addressed by least squares approaches.

A second difference between the three problems results from the nature of the input/output data available for system identification and model reduction in that such data is available at only discrete, usually equidistant, points in time. Thus the inner product integrals required by the method of moments, Galerkin and least squares must be evaluated numerically using the available data. This is in contrast to the equation solution problem where such integrals can often be performed analytically following application of the equation operator to the continuous approximation of the solution.

Several other distinctions will now be drawn between the three fundamental problems when addressed by the various weighted residual methods.

**Method of moments**

The inner products involving the monomial basis function that are required by this method can yield a rather ill-conditioned set of equations in each of the fundamental problems. This phenomenon is particularly apparent for larger values of $N_p$ and is therefore less significant in the system identification and model reduction problems when $N_p$ tends to be relatively small. Nonetheless use of extended precision arithmetic during the solution of (8) is recommended when applying this technique to all three fundamental problems.

**Galerkin**

For the operator equation solution problem the basis functions $\{\psi_i(t); i=1,2,...,N_p\}$ used in the inner products (9) are fixed as the functions defined in the approximate solution expansion (4) and are therefore constrained by the need to impose any initial conditions upon this expansion. However there is no such restriction in the system identification and model reduction problems and a wide choice of complete function sets $\{\psi_i(t); i=1,2,...\}$ is available. Typically we could choose the standard inner product (7) and use

$$\psi_i(t) = P_{i-1}(t)$$  \hspace{1cm} i=1,2,... \hspace{1cm} (17)

where $P_{i-1}(t)$ is the shifted Legendre polynomial of degree $(i-1)$, such polynomials forming an orthogonal basis on $[0,T_f]$ (Hwang and Guo 1984). Since such polynomials are simply linear combinations of the monomial basis functions, the set of equations (9) formed by this choice of $\{\psi_i(t); i=1,2,...,N_p\}$ should yield identical solutions to that given by the equation set (8) produced by the method of moments. However the orthogonality of the shifted Legendre polynomials on $[0,T_f]$ produces a better conditioned set of equations and solutions produced numerically from (9)
are generally more accurate than those produced numerically from (8) and are therefore to be preferred.

Another useful approach within the Galerkin classification is given by defining the inner product as

\[
< g_1(t), g_2(t) > = \frac{1}{T_f} \int_0^{T_f} \frac{g_1(t) g_2(t)}{\sqrt{(t/T_f)(1-t/T_f)}} \, dt
\]

and choosing the basis functions as \( \psi_i(t) = T_{i-1}^*(t) \), the shifted Chebyshev polynomials of degree \( (i-1) \) on \([0, T_f]\), such polynomials forming an orthogonal basis with respect to the latter inner product on \([0, T_f]\). Solution of the Galerkin equations (9) then provides a model which approximately minimizes the maximum value of the residual on \([0, T_f]\).

Alternative sets of basis functions are provided by considering the Fourier series of \( R(t) \). Thus choosing the standard inner product (7) and

\[
\psi_i(t) = \begin{cases} \cos \pi (i-1)t/T_f & i = 1, 3, 5, \ldots \\ \sin \pi t/T_f & i = 2, 4, 6, \ldots \end{cases}
\]

and solving (9) forces the first \( N_p \) coefficients of the general Fourier series expansion of \( R(t) \) with period \( T_f \) to zero. Further alternative choices within this category are

\[
\psi_i(t) = \cos \pi (i-1)t/T_f \quad i = 1, 2, \ldots
\]

which forces the first \( N_p \) coefficients of \( R(t) \), considered as an even function with period \( 2T_f \), to zero and

\[
\psi_i(t) = \sin \pi t/T_f \quad i = 1, 2, \ldots
\]

which performs the corresponding role when \( R(t) \) is considered as an odd function with period \( 2T_f \).

Least squares

Since the inner product integrals which arise in this method have to be computed by a quadrature scheme, comparison of the fundamental problem formulations will be treated under the least squares quadrature heading.

Point and orthogonal collocation

The limited number of free parameters in both the system identification and model reduction problems severely restricts the application of the point and orthogonal collocation methods since the residual is forced to zero at a minimal number of points.
Least squares point collocation and least squares quadrature

These methods are equally applicable to each of the three fundamental problems. The input/output data points are restricted in the system identification and model reduction problems and hence the values of input and output at quadrature rule collocation points will almost certainly need to be derived from interpolation of the measured data points. As suggested previously, there is a close similarity between least squares point collocation and least squares quadrature when the trapezoidal rule is used as the quadrature scheme in the latter technique. The least squares point collocation method has been previously considered in isolation (Whitfield and Messali 1987).

3.3 Comparison of residuals

At this stage we emphasise that problems of entirely different structure result from the use of equation and signal residuals when performing system identification and model reduction. The signal residual is derived from a simulation of the system subject to a prescribed input and initial conditions. Thus, even for a linear system, the signal residual will depend non-linearly upon the model parameters and all the weighted residual methods will require either the solution of a set of non-linear equations or a complex non-linear optimisation. In either case the resulting problem is generally of considerable magnitude. By contrast a linear system, or even linear-in-parameters non-linear system, yields an equation residual which is linear in the unknown model parameters and the weighted residual methods produce either a set of linear equations or a linear least-squares problem. In both cases the solution is unique and easily computed.

4. The integral equation formulation

We have already stated that accurate reconstruction of integrals in the integral equation (2) from potentially noise corrupted input/output data is the major motivation for its use as the vehicle for system identification and model reduction. We shall now consider certain other features of the integral equation formulation which are pertinent to these two fundamental problems.

4.1 Restrictions on input functions

The parameters \( c_k; \ k=1,2,\ldots,n \) in (2) arise solely from initial conditions. If all initial conditions are known to be zero then we have \( c_k=0; \ k=1,2,\ldots,n \) and we need only estimate the parameters \( a_k, b_k; \ k=1,2,\ldots,n \). In such a situation the input \( u(t) \) must be chosen such that \( [X^{(k)}(t), U^{(k)}(t); \ k=1,2,\ldots,n] \) form a set of \( 2n \) linearly independent functions of time. As a consequence, inputs such as a simple exponential function \( u(t) = \exp(-\lambda t) \) may be ruled out, since \( U^{(2)}(t) = -U^{(1)}(t)/\lambda \), as may a simple sinusoid \( u(t) = \sin(\omega t) \), since \( U^{(3)}(t) = -U^{(1)}(t)/\omega^2 \). Identification
is often required in zero internal initial condition situations and such zero initial conditions can always be imposed on a high order model simulation when creating input/output data for the model reduction problem.

In the most general identification we may not have zero initial conditions and we therefore need to identify the full parameter set \( \{a_k, b_k, c_k; k=1,2,\ldots,n\} \). In such situations \( u(t) \) must be chosen such that \( (x^{(k)}(t), U^{(k)}(t), k^{k-1}; k=1,2,\ldots,n) \) forms a set of \( 3n \) linearly independent functions of time. Thus, while a step input is useful in zero initial condition situations, it is actually eliminated from consideration as an input in the unknown initial condition situation since for \( u(t) = \gamma \) (a constant), \( U^{(k)}(t) = \gamma k^{k-1} \) and the functions \( \{x^{(k)}(t), U^{(k)}(t), k^{k-1}; k=1,2,\ldots,n\} \) do not possess the requisite linear independence.

4.2 The effect of additive disturbances

The system identification problem may encompass a situation in which the measured system input and output, denoted \( r(t) \) and \( y(t) \) respectively, differ from the actual system input and output, denoted \( u(t) \) and \( x(t) \) respectively, by unknown additive deterministic disturbances. Thus

\[
\begin{align*}
  u(t) &= r(t) + p(t) \\
  y(t) &= x(t) + q(t)
\end{align*}
\]

where \( p(t) \) and \( q(t) \) are the unknown disturbances. The integral equation (2) describing the dependence of \( x(t) \) on \( u(t) \) can then be written as the integral equation

\[
y(t) + \sum_{k=1}^{n} a_k x^{(k)}(t) - \sum_{k=1}^{n} b_k u^{(k)}(t) + \sum_{k=1}^{n} c_k x^{(k)}(t - 1) - \frac{k-1}{(n-1)!} - q(t) \sum_{k=1}^{n} a_k Q^{(k)}(t) - \sum_{k=1}^{n} b_k P^{(k)}(t) = 0
\]

(Whitfield and Messali 1987) which describes the dependence of the measured output \( y(t) \) on the measured input \( u(t) \) and the disturbances \( p(t) \) and \( q(t) \). In certain circumstances only qualitative knowledge of \( p(t) \) and \( q(t) \) can still yield accurate identification. Thus if \( p(t) \) and \( q(t) \) can be expressed as

\[
\begin{align*}
  p(t) &= \sum_{j=1}^{n_p} \alpha_j g_j(t) \\
  q(t) &= \sum_{j=1}^{n_q} \beta_j h_j(t)
\end{align*}
\]

where \( \{\alpha_j; j=1,2,\ldots,n_p\}, \{\beta_j; j=1,2,\ldots,n_q\} \) are unknown and \( \{g_j(t); j=1,2,\ldots,n_p\}, \{h_j(t); j=1,2,\ldots,n_q\} \) are known then (22) can be written (Whitfield and Messali 1987)
Simultaneous estimation of the disturbance parameters \( \{ \beta_j; k=1,2,\ldots,n \}; j=1,2,\ldots,q \}, \{ c_j; k=1,2,\ldots,n \}; \) and the system parameters \( \{ \alpha_j; k=1,2,\ldots,n \}; \) should then permit correct identification of the system when both input and output signals are contaminated with unquantified disturbances. Once again care must be taken to ensure the linear independence of the time varying functions in (22). Thus if the disturbances are constant unknown offsets, \( p_0 \) at input \( q_0 \) at output, then (22) reduces to

\[
y(t) + \sum_{k=1}^{n} a_k Y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{k-1}{2\pi!} \sum_{j=1}^{n} \sum_{k=1}^{n} \beta_j Y_j^{(k)}(t) - \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j G_j^{(k)}(t) = 0
\]

(23)

Thus the multiple integrals \( \{ P^{(K)}(t), Q^{(K)}(t); k=1,2,\ldots,n-1 \} \) have generated terms proportional to \( t^k; k=1,2,\ldots,n-1 \) which are linearly dependent on the corresponding terms generated by non-zero initial conditions. The only independent term generated by such offsets is proportional to \( t^n \) and hence estimation of the system parameters \( \{ a_k, b_k; k=1,2,\ldots,n \} \) is achieved by estimation of the parameters \( \{ a_k, b_k; k=1,2,\ldots,n+1 \} \) in the equation

\[
y(t) + \sum_{k=1}^{n} a_k Y^{(k)}(t) - \sum_{k=1}^{n} b_k R^{(k)}(t) + \sum_{k=1}^{n} d_k \frac{k-1}{2\pi!} = 0
\]

5. Results

We now document the results of applying several of the previously defined weighted residual techniques to various problems of system identification and model reduction. Unless otherwise stated the residual used in each method arises from the system integral equation, the formulation following that given in the previous section. The input/output data required to produce low order models is in the form of discrete time series \( \{ u(t_i), x(t_i); i=0,1,\ldots,N_i-1 \} \) where the points \( t_i \) are equidistant in time and span the interval \( [0,T_f] \). Thus the inner product integrals required by the method of moments and the Galerkin technique must be evaluated by a suitable quadrature rule; here the seventh order Newton-Cotes scheme is used.

Results quoted as 'least squares' have come from implementation of the least squares point collocation equations (13); such results differ only negligibly from those produced by the least squares quadrature technique. In documenting the Galerkin technique, the results denoted 'Legendre'
have been produced by choosing the shifted Legendre polynomials on \([0,T_f]\), (17), as basis functions while those denoted 'general Fourier', 'even Fourier' and 'odd Fourier' have been produced by basis functions choices (19), (20) and (21) respectively. The inner product used for all the latter basis functions is the unweighted inner product (7). Results denoted 'Chebyshev' have been produced by the Galerkin technique using shifted Chebyshev polynomials on \([0,T_f]\) as basis functions in association with the weighted inner product (18).

Where analogue to digital (A/D) conversion is quoted, the signal minima and maxima have been scaled to full range on \([0,4095)\) with intermediate values being taken from the nearest integer of the scaled signal within that range. Such 12-bit A/D conversion incurs a measurement error of approximately 0.025%. When an offset value is quoted such an offset was imposed on the true system output but was presumed unknown as far as the identification was concerned. No offset indicates a knowledge of zero offset and therefore identification of any offset value was unnecessary.

**Example 1.**

Here a system with transfer function

\[
\frac{X(s)}{U(s)} = \frac{s+15}{s^2+10s+21}
\]

was subject to initial conditions \(x(0)=2\), \(y(0)=-19\) and an input \(u(t)=2t/(1+2t)\). Discrete data was collected on the time interval \([0,7]\) with \(N_s=700\) sample points. With no noise contamination the methods of least squares, moments and each Galerkin approach (i.e. Legendre, Chebyshev, general, even and odd Fourier) all identified the system transfer function as

\[
\frac{1.0000s+14.993}{s^2+9.997s+20.99}
\]

The small error in the estimated parameters occurs entirely as a result of sampling the continuous input/output signals at discrete points in time.

When both input and output signals were subject to 12-bit A/D conversion the results shown in table 1 were obtained. Clearly the presence of quantisation error and unknown observation offset have little effect on parameter estimation for the least squares approach and the polynomial based approaches of the method of moments and Galerkin (Legendre and Chebyshev) though the Fourier approaches are somewhat less robust. This will often be the case since Legendre and Chebyshev polynomial expansions of the integral equation residual will generally converge quicker than a
Fourier series expansion.

<table>
<thead>
<tr>
<th>Identification method</th>
<th>No offset</th>
<th>Offset = 3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least squares</td>
<td>$0.995s + 15.000$</td>
<td>$1.07s + 14.952$</td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{s^2 + 9.996s + 21.000}$</td>
<td>$\frac{2}{s^2 + 9.979s + 20.893}$</td>
</tr>
<tr>
<td>Method of moments</td>
<td>$0.988s + 15.005$</td>
<td>$0.991s + 15.006$</td>
</tr>
<tr>
<td>and Galerkin (Legendre)</td>
<td>$\frac{2}{s^2 + 9.995s + 21.007}$</td>
<td>$\frac{2}{s^2 + 9.993s + 21.004}$</td>
</tr>
<tr>
<td>Galerkin (Chebyshev)</td>
<td>$1.008s + 14.982$</td>
<td>$0.933s + 15.041$</td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{s^2 + 9.994s + 20.975}$</td>
<td>$\frac{2}{s^2 + 10.012s + 21.084}$</td>
</tr>
<tr>
<td>Galerkin (General Fourier)</td>
<td>$0.8517s + 15.14$</td>
<td>$0.878s + 14.957$</td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{s^2 + 10.000s + 21.19}$</td>
<td>$\frac{2}{s^2 + 10.015s + 21.080}$</td>
</tr>
<tr>
<td>Galerkin (Even Fourier)</td>
<td>$0.989s + 15.008$</td>
<td>$0.88s + 15.05$</td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{s^2 + 9.998s + 21.011}$</td>
<td>$\frac{2}{s^2 + 9.994s + 21.10}$</td>
</tr>
<tr>
<td>Galerkin (Odd Fourier)</td>
<td>$0.988s + 15.04$</td>
<td>$0.83s + 14.973$</td>
</tr>
<tr>
<td></td>
<td>$\frac{2}{s^2 + 10.01s + 21.066}$</td>
<td>$\frac{2}{s^2 + 9.940s + 21.019}$</td>
</tr>
</tbody>
</table>

Table 1.

Example 2.

The linear system with transfer function

$$G(s) = \frac{375000(s+0.08333)}{s^7 + 83.635s^6 + 4097s^5 + 70342s^4 + 853703s^3 + 2814271s^2 + 3310875s + 281250}$$

has been used previously (Genesio and Pome 1975, Sinha and Bereznai 1971, Sinha and Pille 1971) to illustrate model reduction techniques. The response of this system was simulated for a unit step input, zero initial conditions and data was collected at $N_t = 100$ discrete equidistant points.
in time on the interval \([0,10]\). Application of the various weighted residual methods produced the second and third order reduced models shown in table 2.

<table>
<thead>
<tr>
<th>Identification method</th>
<th>Second order</th>
<th>Third order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least squares</td>
<td>(0.029s + 0.296)</td>
<td>(-0.046s^2 + 0.589s + 0.039)</td>
</tr>
<tr>
<td></td>
<td>(s^2 + 2.201s + 2.512)</td>
<td>(s^3 + 4.013s^2 + 5.140s + 0.364)</td>
</tr>
<tr>
<td>Method of moments and</td>
<td>(0.023s + 0.282)</td>
<td>(-0.042s^2 + 0.576s + 0.034)</td>
</tr>
<tr>
<td>Galerkin (Legendre)</td>
<td>(s^2 + 2.093s + 2.399)</td>
<td>(s^3 + 3.924s^2 + 4.991s + 0.313)</td>
</tr>
<tr>
<td>Galerkin (Chebyshev)</td>
<td>(0.0084s + 0.3229)</td>
<td>(-0.038s^2 + 0.558s + 0.018)</td>
</tr>
<tr>
<td></td>
<td>(s^2 + 2.2420s + 2.743)</td>
<td>(s^3 + 3.778s^2 + 4.737s + 0.176)</td>
</tr>
<tr>
<td>Galerkin (General Fourier)</td>
<td>(-0.031s + 0.592)</td>
<td>(-0.048s^2 + 0.592s + 0.077)</td>
</tr>
<tr>
<td></td>
<td>(s^2 + 3.919s + 5.001)</td>
<td>(s^3 + 4.131s^2 + 5.408s + 0.680)</td>
</tr>
<tr>
<td>Galerkin (Even Fourier)</td>
<td>(0.034s + 0.250)</td>
<td>(-0.045s^2 + 0.588s + 0.045)</td>
</tr>
<tr>
<td></td>
<td>(s^2 + 1.969s + 2.126)</td>
<td>(s^3 + 4.026s^2 + 5.168s + 0.407)</td>
</tr>
<tr>
<td>Galerkin (Odd Fourier)</td>
<td>(0.057s + 0.195)</td>
<td>(-0.048s^2 + 0.599s + 0.049)</td>
</tr>
<tr>
<td></td>
<td>(s^2 + 1.799s + 1.663)</td>
<td>(s^3 + 4.093s^2 + 5.284s + 0.442)</td>
</tr>
</tbody>
</table>

Table 2.

The outputs produced by applying a unit step to the actual system and the second and third order reduced models produced by the least squares technique are shown in figure 1. Clearly a third order model adequately describes the system dynamics though it is arguable whether or not the second order model does so. Figure 2 shows similar outputs for second order and third order models produced by other authors. The second order model was identified by Sinha and Pille (1971) as \(G_a(s) = 0.3401/(s^2 + 2.2412s + 2.9316)\) while the third order model was given by Sinha and Bereznai (1971) as \(G_a(s) = (-0.1142s^2 + 0.854s + 0.4)/(s^3 + 6.6677s^2 + 9.6050s + 3.4836)\). Clearly the least squares weighted residual approach improves upon the transient approximation of the latter two models. The step responses from the other weighted residual techniques differ only marginally from those in figure 1 produced by the least squares point collocation approach. Figure 3 shows the step responses of the second and third order models derived by using shifted Chebyshev polynomials and
Fig. 1. Least squares reduced model responses.
Fig. 2. Previous second and third order model responses.
Fig. 3. Step responses from Galerkin (Chebyshev) models.
the weighted inner product (18) within the Galerkin technique. As expected, the second order model is inferior to the third order model though its transient clearly illustrates an almost even distribution of error over the sampling period.

The above seventh order model output was also corrupted by additive Gaussian noise \( N(0,0.006) \) to simulate the effect of collecting noisy data from a high order system. The third order models identified by the various techniques are shown in table 3. At first sight it appears that the least squares method produces a significantly different input/output model following the visible pole/zero cancellation. In fact the third order models derived from noiseless data, shown in shown in table 2, differ only negligibly in their response since they identify one very fast mode together with a second order mode whose real and imaginary parts are very similar to that produced by the least squares method when applied to the noisy data set.

<table>
<thead>
<tr>
<th>Identification method</th>
<th>Third order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Least squares</td>
<td>( \frac{0.0255s^2 + 0.331s - 0.000418}{s^3 + 2.333s^2 + 2.8082s - 0.0036} = \frac{(0.0255s + 0.332)(s - 0.0012)}{(s^2 + 2.334s + 2.811)(s - 0.0012)} )</td>
</tr>
<tr>
<td>Method of moments and Galerkin (Legendre)</td>
<td>( -\frac{0.0036s^2 + 0.5488s + 1.0339}{s^3 + 6.2133s^2 + 11.188s + 8.766} )</td>
</tr>
<tr>
<td>Galerkin (Chebyshev)</td>
<td>( \frac{0.0418s^2 + 0.2913s + 0.3217}{s^3 + 2.978s^2 + 4.540s + 2.720} )</td>
</tr>
<tr>
<td>Galerkin (General Fourier)</td>
<td>( -\frac{0.0201s^2 + 0.9667s + 1.331}{s^3 + 9.363s^2 + 16.870s + 11.224} )</td>
</tr>
<tr>
<td>Galerkin (Even Fourier)</td>
<td>( -\frac{0.0425s^2 + 0.7058s + 0.8267}{s^3 + 6.510s^2 + 11.131s + 7.0229} )</td>
</tr>
<tr>
<td>Galerkin (Odd Fourier)</td>
<td>( -\frac{0.1185s^2 + 0.9582s + 0.5023}{s^3 + 6.840s^2 + 11.113s + 4.2942} )</td>
</tr>
</tbody>
</table>

Table 3.

Figure 4 shows a visual comparison of the output of the third order reduced model identified via the Galerkin (odd Fourier) technique. Clearly the noise level is significant and has a consequent,
Fig. 4. The effect of observation noise upon reduced model identification.
though not disastrous, effect upon parameter identification in this latter scheme which is one of the weakest implementations of the Galerkin technique.

To illustrate the effectiveness of the techniques in presence of unquantified polynomial disturbances, the actual input/output data of the original seventh order model was subject to the following additive disturbances when the measured input was $r(t)=2\sin(0.5t)$ and the initial value $x(0)=0.2$:

$$p(t)=1+t \quad q(t)=3+2t+t^2$$

The second order model transfer function identified using the Galerkin technique (with Legendre polynomials $P_{i-1}(t)$ for $i\geq 5$) was

$$G_a(s)=\frac{1.058}{s^2+7.043s+8.530}$$

Figure 5 shows the actual output $x(t)$ of the seventh order model together with the actual output of the latter second order approximation when both models are subject to the same input $r(t)=2\sin(0.5t)$.

**Example 3.**

Although not detailed in the text, all the weighted residual techniques can be applied to linear MIMO system identification and control. We shall briefly quote the results from the application of one such model reduction method to a two input, two output system with transfer function matrix

$$G(s)=\begin{bmatrix}
    \frac{5s^4+108.5s^3+811.5s^2+2443.5s+2551.5}{D'} & \frac{4.5s^4+95.4s^3+742.14s^2+2509.704s+3111.225}{D'} \\
    \frac{7.5s^4+142.1s^3+994s^2+2862.3s+29106}{D'} & \frac{6.75s^4+145.125s^3+1123.74s^2+3673.917s+4225.284}{D'}
\end{bmatrix}$$

where

$$D'=(s+2)(s+4)(s+6)(s+8)(s+10)=s^5+30s^4+340s^3+1800s^2+4384s+3840$$

The two input signals were chosen as:

$$r_1(t)=20\sin(0.5t) \quad r_2(t)=25\sin(t)$$

and the two outputs were computed on a time interval $[0,5]$ with $N_t=100$, samples being taken at equidistant points in time. The transfer function matrix with second order elements was identified by least squares method as:
Fig. 5. Model simulation following identification in presence of unknown polynomial disturbances.
\[
G_a(s) = \begin{bmatrix}
7.472s + 18.698 & 5.605s + 22.916 \\
8.518s + 21.337 & 9.578s + 31.480 \\
\end{bmatrix}
\]

where

\[D = s^2 + 16.407s + 28.083\]

In figure 6 a comparison is made between the true outputs and those of the reduced model. It can be seen that the reduced model is a very good approximation of the original system.

**Example 4.**

The major emphasis in this section has thus far been placed on the residual formed from the system integral equation where application of the various weighted residual methods leads to either a set of linear equations or a linear least squares problem. In this example we illustrate some of the potential difficulties that can arise from use of the signal residual and the subsequent non-linear optimisation. The specific problem is to find a second order approximation of the form

\[
\frac{\theta_2}{s^2 + \theta_1 s + \theta_2}
\]

to the fourth order system

\[
G(s) = \frac{s^2 + 15s + 50}{s^4 + 5s^3 + 33s^2 + 79s + 50}
\]

The latter system was simulated for a unit step input with zero initial conditions and data was collected at \(N_t=80\) equidistant points in the time interval \([0,8]\). The least squares method was used with the robust simplex method (Nelder and Mead 1965) as the non-linear optimisation algorithm. With initial parameter estimates for \(\theta_1\) and \(\theta_2\) of 1 and 40 respectively the optimisation routine predicted a second order reduced model \(4.347/(s^2+5.538s+4.347)\) while with initial estimates 0 and 50 the model \(92.804/(s^2+119.5s+92.804)\) was produced. The resulting reduced order model is clearly dependent on the choice of initial parameter estimates and it may be thought that the optimisation routine has converged to two different local minima, a possibility in any such non-linear optimisation. Figure 7 shows the contours of the integral of the squared signal residual criterion and the presence of a long valley is evident. Figure 8 shows the value of the latter criterion as a function of \(\theta_1\) in a line taken along the valley and indicates that \(\theta_1=5.538, \theta_2=4.347\) is a unique minimum that is potentially difficult to reach and that \(\theta_1=119.5, \theta_2=92.804\) is just a point along the valley that the optimisation has not progressed beyond.
Fig. 6. A MIMO system and reduced model simulation.
Fig. 7. Integral of squared signal residual.
Fig. 8. Integral of squared signal residual along valley.
6. Conclusions

The class of weighted residual techniques provides a set of well established methods for the solution of ordinary differential equations. We have extended this basic concept to cover two fundamental problems in control engineering: continuous system identification and model reduction. The weighted residual methods have been presented in a general operator framework though the discrete nature of input/output data available for model reduction implies that integrals involved in many of the formulations must be computed numerically.

The system integral equation, from which a residual is formed, is used as the basic vehicle since it has useful noise rejection properties and any residual thereby formed is primarily dependent on the system gain and slow transient modes. Thus excellent results have been obtained to both problems of identification and model reduction following application of various weighted residual methods though the least squares point collocation method is favoured by the authors. A range of potential Galerkin approaches has been presented and it is noteworthy that use of Chebyshev polynomials and an associated weighted inner product yields models that have a relatively even signal error over the sampling interval.

When applying weighted residual methods to linear systems described by their integral equation, model parameter estimates result from the solution of a set of linear equations and hence any convergence difficulties associated with non-linear problems generated by weighted signal residual methods are avoided. Additionally, computational effort is kept to a minimum by use of the integral equation residual. As a guide, computation times using the integral equation residual are typically two orders of magnitude smaller than when using the signal residual in problems with four or five parameters.

References
