Complex networks in nature and society

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By

Wu Zhang

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The thesis marks the consummation of my learning and study in the field of physics. The work would not have been possible without the scholarly support and mentor from Professor Feo Kusmartsev, my first supervisor from the Department of Physics. The enormous help that he has offered to me throughout the years is beyond verbal descriptions. Dr. Anna Kusmartseva, my second supervisor from the same department, has also contributed significantly to the completion of my degree.

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Abstract

The first chapter of this thesis provides an introduction to fundamental concepts concerning econophysics, Ising model, and opinion networks. After a glance in a field of econophysics, Chapter 2 illustrates the economic behaviour via the implementation of two methods. The statistical analysis of real economic data will be briefly stated and followed by the agent-based dynamic model describing the commercial activities. Agent-based dynamic model investigates the intrinsic dynamics of trading behaviour and individual income by modelling transaction processes among agents as a network in the economic system.

To take a further look into the network, we introduce a mathematical model of ferromagnetism in statistical mechanics which is called Ising model. Every element in the network can be treated as a two-state \( \{+1,-1\} \) or sometimes \( \{+1,0\} \) node. The similar methodology is used in the three-or-more-state situation. This kind of modelling method is widely applied in networks of neurosciences, economics, and social sciences.

Chapter 3 implements and modifies Ising model of a random neuron network with two types of neurons: inhibitory and excitatory. We numerically studied two mutually coupled networks through mean-field interactions. After 3-step alternation, the model provides some fascinating insights into the neuronal behavior via simulation. In particular, it determinates factors that lead to emergent phenomena in dynamics of neural networks.

On the other hand, it also plays a vital role in building up the opinion network. We first show the development of Ising model to opinion network. Then the coupled opinion network model and some of the analytical results are carefully given in Chapter 4. Two opinion networks are interfering each other in the system. This model can describe the opinion network more precisely and give more accurate predictions of the final state.

At last, a case of U.S. presidential campaign in 2016 is studied. To investigate a complex system which is associated with a multi-party election campaign, we have focused on the situation when we have two competing parties. We compare the prediction of the theory with data describing the dynamics of the average opinion of the U.S. population collected on a daily basis by various media sources during the last 500 days before the final Trump-Clinton election. The qualitative outcome is in reasonable agreement with the prediction of our theory. In fact, the analyses of these data made within the paradigm of our theory indicate that even
in this campaign there were chaotic elements where the public opinion migrated in an unpredictable chaotic way. The existence of such a phase of social chaos reflects the main feature of the human beings associated with some doubts and uncertainty and especially associated with contrarians which undoubtedly exist in any society. Besides, a modern tool, Twitter, with rapid information spreading speed affects the whole procedure substantially. We also take a closer look at the influence of the usage of Twitter on competitors, Trump and Clinton. Once the first sign from Trump began stirring on Twitter, it quickly began to ferment. Using Twitter not only brings strength to Trump as he wished, but also sending potentially backward to Clinton in this nationwide competition.
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Chapter 1 Introduction

1.1 Introduction of networks

Our everyday life is filled with physics. The knowledge of physics is widely applied in the calculation, designing, experiment, and production of the household electronic appliances as well as dwelling houses. It can be said that the theories in physics have laid part of the foundation of modern civilisation.

We live in a world of various networks. The notion of a network is central to our everyday life. For example, the Internet has brought enormous changes to our lifestyles. Our physical existence relies on the biological networks. And we have been participating in the comprehensive network of economic and social activities.

The physicists’ exploration of a network, be it natural or human-made, started in the 1990s. They have recognised that network systems are highly complex, and are likely to evolve and develop. The advancement of the exploration marks the advent of a new field in science with new notions and concepts. Further, it indicates a new trend in our philosophical reflections. In mathematics, physics, biology, computer sciences, etc., the classical theories may be transformed presented in new forms.\[1\]

There are many examples of the real life. The simplest example is a Network of Roads. This network consists of normal roads. Highways and cities are connected by the roads. As we may see, there are many nodes in the network. Each node stands for a city or a place in the road network with different functionalities. So the multi-functionality of the network depends on the constant communication and interaction of the nodes in it. The human cities or countries cannot be in a complete isolated state. Even for countries such as North Korea, it can never be completely separated from the network of the international communities formed by many countries and states.
Here I draw on a real-life example to illustrate the idea of a social network. One of the most obvious examples of social network in our life is probably the network of friends and acquaintances. I once asked one of colleagues (pseudonym Thomas) to outline systematically his friends network for me, which is presented in Figure 1.1. In the figure, his friends are presented in names in the form of nodes which are connected to others with lines. In Figure 1.1 we can easily distinguish Thomas from others. And we may notice that some people are connected with solid lines and some are in dashes. I probed the question and he recognised that he tried to distinguish between genuine friends and the acquaintances with different types of lines. This is a very fine visualisation of the strong and weak ties in an individual’s network of friends.

The renowned American social psychologist Stanley Milgram once conducted a series of experiments (the small-world experiments) which have been developed into his theories of ‘six degrees of separation’, also known as the famous ‘small-world problem’. It is an idea that people living in the real world are connected via a chain of friends. Everyone is able to connect to another ideally in six or fewer steps.

To put the degrees of separation theory in a specific network, assuming that all people in the network are randomly connected. Assume that I have a friend from Canada (pseudonym Anna). Anna has 50 “friends”. Let us make a random connection with Anna with 50 other
people. If each of her friends has another 50 friends chosen randomly, then two such people will be connected to $50 \times 50 = 2500$ friends. Then if each of the new friends has another 50 friends then three of them will connect $50 \times 50 \times 50 = 125000$ people. Four of them will connect $50 \times 50 \times 50 \times 50 = 6,250,000$. And finally, if we have six step of separation we will connect $50 \times 50 \times 50 \times 50 \times 50 = 15,625,000,000$ people. So, if we connect people randomly then in 6 steps we will connect all people on the earth and the phenomenon of the small world is solved. Is it really so that people are connected randomly? Let us consider the example of the small social network: the network of the local community which may exist in any small town either in Europe or in another part of the world. Such communities are very similar to those where Anna lives, it is a small town situated in a 100 km from Vancouver. She has close friends and regular visiting church. She has no random connections to the people lives around the globe, but still, she has some links which are spread across the small community located in the small town around the church. These links are mostly very strong. But Anna has both strong and weak links and they are spread around the globe as well. These two types of links are another aspect of the social networks related to a strength of the personal relationship.

Figure 1.2. There we have noticed that there were two types of links: first, the strong one, presented by a solid line and the other, weak ones, presented by a dashed line.

![Diagram of social networks]

**Fig.1.2 Weak and strong links in social networks**

Importance of weak Links has been discovered by social psychologist, Granovetter, in 1983. He found that such links are most important for keeping the society strong. Indeed, he found the Paradox and shown that weak links are social glue of human society. These weak links are very important for many aspects of our life. They are extremely important spreading
of information. He conducted statistical analysis of many cases how people found their jobs and in practically all cases they have been using the weak links. So, it is most important to use the weak links to make your search most efficiently. The paradox is that the strong links are unimportant. The matter is that when the links are strong, the cycle of friends is very small and all information are shared among friends and no new information are available.
1.2 Econophysics in network

The interaction and mutual influence between physics and economics have lasted for at least three hundred years. Since the 19th century, there have been examples of the application of the knowledge of physics and mathematics in the development of economic theories. For example, the influence of Josiah Willard Gibbs, the physicist, can be found in Paul Samuelson’s renowned work ‘Foundation of Economic Analysis’. In the 20th century, a number of regional and worldwide economic crises, including the 2007-2008’s global subprime crisis, have led to the continuous reflection and revision of the then and currently prevalent economic theories. About three decades after the birth of financial economics in the 1960s, the field of Econophysics emerged, aiming to introduce theories and methodologies from physics to aid the reinvention theoretical models in finance and economics.

The contemporary development in Econophysics can be attributed to the rapid advancement in computer and internet technologies since the 1990s, which makes the processing of significant amount of economic data possible. In the context of economic globalisation, there is also larger demand for new theories and methodologies explain new economic trends and phenomena. As an emerging ‘interdisciplinary science’, Econophysics aims to apply ‘theories and methods, formerly formulated by physicists, such as statistical mechanics, power laws, stochastic process and nonlinear dynamics’ to tackle new issues and problems in the global markets and economies. According to Romanian socioeconomic physicists Gheorghe Sävoiu and Ion Siman, the ideal Econophysics is ‘under the hypothesis that economic world behaves like a collection of electrons or a group of water molecules that interact with each other’ and the perplexing economic systems are reduced to ‘a few elegant general principles with the help of some serious mathematics borrowed from the study of disordered materials’. Nevertheless, scholars such as Victor Yakovenko points out that ‘Econophysics does not literally apply laws of physics’ directly to humans. Rather, it uses mathematical methods developed in ‘statistic physics to study statistical properties of complex economic systems consisting of a large number of humans.

Two major trends can be recognised in contemporary Econophysics, the experimental and the theoretical, which include two major tasks of the discipline: the examination of data series from the financial markets, and the creation of models to match empirically determined values. Presently, a number of theories in physics, such as kinetic theory of gas, chaos theory, statistical mechanics, etc., have been widely applied in developing new economic
theoretical models. Econophysics can also help explaining empirical data that mainstream economics has failed to explain. Since Econophysics has adopted a phenomenological approach, which has developed models on the basis of the ‘analogies between economic processes and laws, and phenomena and laws from fundamental or applied physics (electronics) especially of Electricity, Condensed Matter, Thermodynamics or Engineering (Econo-engineering models)\[^{24}\].

Nevertheless, critiques can be made regarding some basic assumptions of Econophysics. Since the physicists work in the field of physical matters in which many factors, elements, and constants are regular and stable. Whereas the factors and elements involved in human economic activities cannot always be as perfect as natural laws of physics or atoms and electrons. The tendency to idealise the highly complex human economic world, in which the unpredictable human behaviours need to be taken into account as a variable in the social sciences such as the study of economics, will likely increase the deficiency of communication between econophysicists and economists.

Thus, in the future, the development of both new economic theories and the emerging field of Econophysics will require a closer communication between the scholars in the two fields. While the conventional economists may need to engage the disciplines of the natural sciences, the physicists will need to recognise the efforts made by the social scientists in explaining complicated human world.

There are two main research methods in econophysics. The first is statistical analysis of real economic data. It collects data from real market activities. Through statistics, it may gain the comprehensive understanding of the characteristics of economic behaviours, such as correlation, fluctuation, etc. The second is called agent-based dynamic model. Through the building of agent-based dynamic model based on empirical data, the research can identify patterns to understand trends of development. In chapter 2 we are going to show some examples by using these methods.

As for the statistical analysis of real economic data, the Econophysics focus on the empirical study of different phenomena, through which regularities and particularities can be identified. To be specific, it is through the analysis of economic data, the econophysicists expect to identify the relationship between economic variables and mathematical laws, which may resemble experiments in the study of physics. For example, in the studies of stock market
prices, foreign exchange rates, and commodity prices, a number of stylised facts have been found in the stock market that challenges conventional research outcomes. It can further the research in the management of risks. In the studies of company development, changes in GDP, and the growth in individual income, the research can provide reliable data and information for companies’ development strategies and the governments’ policies of macro-control. Chapter two’s focus is on the research into the individual income growth.

The agent based dynamics modelling is the network analysis of economic phenomena. It aims to explain the reasons for economic phenomena through the construction of models. The method relies on the aids of computer simulation of the development and transformation within a given framework of rules in which economic agents (the decision makers in economic activities such as investors, banks, governments, etc.) are participating. The simulation can not only analyse linear, static and stable processes, but also can deal with nonlinear, dynamic, and even turbulent processes, which the conventional economic models cannot handle. [5] [70]
1.3 Ising Model in network

Named after the physicist Ernst Ising, the Ising model is well-known in physics. It is a mathematical model of ferromagnetism in statistical mechanics to describe the collection of magnetic moments associated with the spins of atoms. It considers the spin states of each atom, which change over time with respect to the spin states of its neighbors. Spins can be treated as located fixed points in one of two states (+1 or −1) arranged in a graph, usually a lattice, allowing each spin to interact with its neighbors.

It can be derived from quantum mechanical considerations through several educated guesses and rough simplifications. It is an extremely interesting model despite its simplicity. There are several reasons for the great attention that it has received from both physicists and mathematicians. First of all, it is the simplest model of statistical mechanics where phase transitions can be rigorously established. Second, ferromagnetic phase transitions are “universal”, in the sense that critical exponents appear to be identical in several different situations. Thus, properties of other models may be inferred by studying one model like the Ising Model. Additionally, the Ising model can be interpreted in a language of probability theory. The magnetization describing the collection of spinning atoms can be viewed as a sum of Bernoulli random variables that are identically distributed, but not independent. Here, the law of large numbers and the central limit theorem could be understood well using this physical intuition.

To begin with, a lattice is needed. For example, we could take $\mathbb{Z}^d$, the set of points in $\mathbb{R}^d$, all of whose coordinates are integers. In one dimension, it is often referred to as a chain. In two dimensions this is usually called the square lattice, and the cubic lattice in three dimensions case. Consider a set of lattice sites $\Lambda$, each with a set of adjacent sites (e.g. a graph) forming a d-dimensional lattice. For each lattice site $k \in \Lambda$ there is a discrete variable $\sigma_k$ such that $\sigma_k \in \{+1, -1\}$, representing the site's spin. A spin configuration, $\sigma = (\sigma_k)_{k \in \Lambda}$ is an assignment of spin value to each lattice site.

For any two adjacent sites $i, j \in \Lambda$ one has an interaction $J_{ij}$. Also, a site $j \in \Lambda$ has an external magnetic field $h_j$ interacting with it. The energy of a configuration $\sigma$ is given by the Hamiltonian function
\[ H(\sigma) = - \sum_{<i,j>} J_{ij} \sigma_i \sigma_j - \mu \sum_j h_j \sigma_j \] 

where the first sum is over pairs of adjacent spins (every pair is counted once). The notation \( <i,j> \) indicates that sites \( i \) and \( j \) are nearest neighbors. The magnetic moment is given by \( \mu \).

Note that the sign in the second term of the Hamiltonian above should actually be positive because the electron's magnetic moment is antiparallel to its spin, but the negative term is used conventionally. The configuration probability is given by the Boltzmann distribution with inverse temperature \( \beta \geq 0 \),

\[ P_\beta(\sigma) = \frac{e^{-\beta H(\sigma)}}{Z_\beta} \]  

where \( \beta = (k_B T)^{-1} \), and the normalization constant

\[ Z_\beta = \sum_\sigma e^{-\beta H(\sigma)} \]  

is the partition function. For a function \( f \) of the spins, one denotes

\[ < f >_\beta = \sum_\sigma f(\sigma) P_\beta(\sigma) \]  

the mean value of \( f \). The configuration probabilities \( P_\beta(\sigma) \) represent the probability of being in a state with configuration \( \sigma \) in equilibrium.

The Ising model is easy to define, but its behavior is wonderfully rich. It is often used to analyze and predict dual opinion distributions in a model population of people. Further applications and modifications will be stated in Chapter 4.
1.4 Opinion network and voting

The human society is constituted of numerous human agents constantly interacting with each other. Human interactions occur under various circumstances which often involve highly complex and unpredictable behaviours. Human interactions necessarily lead to the formation of human communities and institutions in the society. These communities and institutions are often ordered in hierarchical manners with diversified structures. Opinions of individual and group human agents play an important part in the forming and sustaining the communities and institutions. The formation of people’s opinions can be a very complicated process, in which a multitude of influencing factors, such as ideologies, legal codes, morality, and ethics etc., are involved.

At the end of 19th century and early 20th century, French social psychologist Gabriel Tarde pointed out the importance of interpersonal interactions in the formation of group and public opinions[6]. Today, in light of the emergence of new communication technologies marked by social media, the interactions between individual human agents have transcended geological boundaries. Some overarching themes and concepts such as social interaction and network, have been picked up and studied carefully by the social scientists.[7-9] Some studies have employed theoretical models and concepts in physics.[8,10,11] Some have engaged the methods in statistical physics.[12,13] The particular approach adopting theories and methods in physics to study social interactions and networks is termed ‘sociophysics’.

In sociophysics, theoretical models proposed by scholars such as Granovetter (the threshold model)[8] and Bass (the diffusion of opinions)[16] introduced statistical physics into the study of the formation of dynamic opinions. Their models are useful in understanding how disputable opinions in the public arena can reach a state of consensus over time. In this process, individuals do not necessarily shift opinions through the interactions within their social networks, namely, their peers or neighbours; as described by the class of voter models.[17,18] Nevertheless, these models have somewhat failed to capture the more chaotic dynamics of the many unpredictable changes and shifts between competing opinions before a consensus is achieved.

Thus, there is a need to introduce the Ising spin models in physics to help better understand the unpredictable individual human behaviours in opinion shifting.[19-21] Using the methods of statistical physics, in the Ising spin models, individual’s opinions can be represented as the
individual spin state. And the state of opinion consensus can be regarded as the ferromagnetic ordering of spins in the Ising model, which is associated with the ground state.\cite{19,22} Since in real life scenarios, the shift of opinions often occurs in ‘chaotic’ manners, and the development of the process can occur unconsciously for human individuals.\cite{23} In the Ising models, the shift of opinions can be understood as spin-flip process. Further, this chaotic and dynamic process of the shift of opinions can be represented with statistics.

As mentioned above, social communities, institutions, of which the various social networks are constituted, involves a multitude of individual human agents. These agents are constantly interacting with each other. This is very similar to the many-body systems in physics. Thus, we may view that it is the interactions occurred at the micro level that constitutes the entire or large part of the system at the macro level. This view is also analogous to the many situations in classical statistical mechanics and physics. Nevertheless, the driving forces of social interactions can hardly be idealised as equivalent to the natural forces in physical and material systems. This is due to the highly unpredictable nature of individual human behaviours. Moreover, there have been few complete mathematical descriptions of such forces that are supposedly driving the social dynamics.
Chapter 2

The mathematical methods and models that are used in the description of the existence of atoms and their statistical properties have been applied in the analysis of economic and financial problems. To be specific, this chapter is to reveal some of the laws of economic patterns, such as the causes and effects of crisis. This chapter is organised in three parts. The first part focuses on the use of real economic data statistical analysis to obtain the distribution of money, wealth and income in economy systems. The second and third parts of this chapter are less descriptive. They are more concerned with the agent-based dynamic model in Econophysics. These two parts focus on how the agent-based dynamic model can be applied in trading networks. The model emphasises the accuracy and confidence level, comparing with other techniques that have already been used in the industry.

2.1 The statistical analysis of real economic data

The cumulative methodology tries to conform to the laws of quantum statistical mechanics. We will use this method to show that both Boltzmann and Bose-Einstein distribution are able to provide very good fits to the low-income tiers of the UK’s income data. We also see the evolution of the UK market over a number of years (1999-2007). The quantum statistical techniques applied below work well in describing the unpredictable and chaotic human behaviours in economic activities.

The distribution and composition of individual earned income can always gain an insight into the make-up of the UK economy as it is today. We use the data from The HM Revenue and Customs service (HMRC) which collects yearly-earned income statistics on the amount of income per a set bin width (k£) in UK and present the mean income for the UK as contributed by the number of individuals in 1999-2000 in the form of a pie chart shown in Figure 2.1
These statistics are based on a population of 27,045,000 people who were eligible as UK taxpayers and declared their individual incomes to the HMRC regularly. It is clearly seen that a small quantity of people (251,000 individuals) make up nearly half of the UK’s total earned income (48%). This is contrast to many, who earn an average income of £23,100, which only make up 6% of the UK’s total earned income. This observation shows that there are not many people earning very large amounts of money, while a large proportion of people fall into the lower income tier, to which we will pay more attention.
Figure 2.2: Cumulative probability distribution of yearly individual income in the UK as calculated from raw data in log-log

To obtain the distribution in Figure 2.2, careful calculations of the cumulative frequency are required. This cumulative frequency is computed from ‘number of individuals’ column as a percentage of the ‘total number of individuals’. The output is then summed up from the bottom upwards to give a correct cumulative frequency. The adjusted earned income is calculated by an alternative method due to the inherent way the data has been published. Here, the separate columns of ‘total amount of money’, per bin width, are divided by the ‘total number of individuals’ (also known as the number of returns) in the same bin width. As the value along the x-axis increases (more money earned), the number of people earning that amount of money decreases from observation of figure 2.2. A consequence of treating the statistics in this cumulative way is the normalisation condition, which means that we have smoothed off the data, taking the average point between each bin width.

Furthermore, the income distribution of the UK can be grouped into two parts, a Boltzmann (exponential) curve at low income and a Pareto (power law) tail at high income.\textsuperscript{[24]} According to the work by Victor M.Yakovenko,\textsuperscript{[30]} the Boltzmann distribution is fitted well to the UK
data. Firstly, the probability of the existence of a physical system or subsystem in a state with energy is given by the exponential function

\[
P(\epsilon) = A \, e^{\frac{\epsilon}{T}}
\]

(2.1)

where \(A\) is the normalising constant and \(T\) is the temperature in its original model.

Then if we divide the system into two (normally unequal) parts, the total energy is the sum of the energy from both parts, \(\epsilon = \epsilon_1 + \epsilon_2\), where as the probability is the product of probabilities, \(P(\epsilon) = P(\epsilon_1)P(\epsilon_2)\). The unique solution of the set of equations is in the form of exponential functions such as (2.1). For Boltzmann distribution, this derivation can be used in the statistical character of system and the conservation of energy.

\[
\text{Figure 2.3: UK income data (99-07) Boltzmann and BE distribution fit}
\]

Apart from physic description, it can also be thought of finding the statistical equilibrium state of the market where the total average amount of money is fixed. In Figure 2.3, the red
line shows this distribution aiming to fit the UK data. It should be noted that they consist with each other quite well before the Pareto tail appears.

Using Bose statistics with the understanding of 'economic units which have the same amount of money are indistinguishable'\[^{31-33}\] is like the novel state of matter in which many atoms occupy the same lowest-energy (ground) state.\[^{34}\] This usually happens with dilute gases. When gases are cooled down to temperatures near absolute zero, they will form a Bose-Einstein condensate. Analogously, the Bose Einstein distribution will emerge when the average amount of money and number of economic agents remain constant during some periods. The length of period must be long enough for a single transaction to take place. We take one year in Figure 2.3.

The probability of an economic unit to have an amount of income or money, m, is described by the quantity known as the occupation number \( n(m) \),

\[
n(m) = \frac{h}{e^{\frac{x - \mu}{T}} - 1}
\]  

(2.2)

with the normalisation condition

\[
\sum_m n(m) = N
\]  

(2.3)

where \( h \) describes a degeneracy of state or a layer of agents having money. The equation above is normally applied to gases in a quantum state and describes the number of particles at each energy level. The chemical potential, \( \mu \), controls the density. It is also assumed that these particles have a temperature \( T \).

This distribution has been fitted to the UK data in Figure 2.3 depicted by the blue line. The fit is good everywhere except the highest money tier, described by a Pareto tail. It is more likely to fit the power law distribution of the order \( Cx^\alpha \).

From the fit of the distribution to the relative data set, a comparison of the apparent money temperature can be undertaken. The money temperature, which is equal to the average amount of money per agent

\[
T = \langle m \rangle = \frac{\text{Total money}(M)}{\text{Number of individuals}(N)}
\]  

(2.4)
can be calculated from the empirical data published online by the HMRC.[35] Figure 2.4 illustrates the comparison between the actual figures and those obtained from the distribution fits. Both the BE distribution and the Boltzmann distribution follow the same trend as the actual data, if not slightly below these actual values. Because the values obtained for $T$ from the curve fitting may include points that verge on the Pareto tail distribution as it is not a clear transition from one income tier to the other, some overlaps exist and the actual values are uniformly above both fitting distributions.

Figure 2.4: Comparison of empirical data and distribution fitting of the temperature of money

On the other hand, the actual temperature figures are calculated from the ‘total money’/ ‘total number of individuals’ and this includes the high-income tiers, so it results in a higher temperature than the expected figures from the distribution describing lower-income values. However, due to the nature of the publication, there is no clearly determined cut off and transition point for each year from the low tier to the higher tier.
Treating the data cumulatively has its drawbacks. We did not deal with equidistant bins widths due to the inherent way the data is published by the HMRC. Bin widths go up in a variety of ranges depending on the size that suits the statistics from the particular year. For example, the widths vary by year and sample size. They do not steadily increase from £1,000-2,000 each time for every year. The widths change as the amount of money grows, jumping from £40,000 to £60,000. This, in turn, means the bins covering a broad range of income variation can introduce some inaccuracy as we are comparing bin width averages when the data is taken cumulatively.
2.2 Agent-based dynamic model—Wealth Distribution

Agent-based models are applied in the investigation of the intrinsic dynamics of the economic behaviours. The model can to some extent explain the results obtained in data statistical method and consider the influence of the fluctuations.

To investigate the people’s individual income distribution, we assume that every person only has two economic activities. One is personal investment (e.g. Gambling, Stock, Bank interest etc.), the other is daily consumption (e.g. Living cost, Food, Clothing etc.). We assume that the investment rate is related to a random distribution while the daily consumption keeps the same value which is a constant. We also assume that each agent owns same amount of initial wealth. Salary is an important part of the wealth distribution. We assume that the salary at each time step stays constant.

We assume that there are 1000 individuals and each of them have the same amount of money which is 1000 pounds. Each person uses their money for 1000 time steps running as daily investment and consumption, plus a constant amount of salary with each economic activity. Then the time evolution of wealth can be obtained as below.

1\textsuperscript{st} step:

\[ W_1 = W_0 (1 + P_0 - C) + S \]  \hspace{1cm} (2.10)

2\textsuperscript{nd} step:

\[ W_2 = W_1 (1 + P_1 - C) + S \]  \hspace{1cm} (2.11)
\[ = [W_0 (1 + P_0 - C) + S](1 + P_1 - C) + S \]  \hspace{1cm} (2.12)

\[ n \textsuperscript{th} step: \]

\[ W_n = W_{n-1} (1 + P_{n-1} - C) + S \]  \hspace{1cm} (2.13)

where \( W_t \) is the wealth, \( P_t \) is personal investment at time step \( t \), \( C \) stands for a constant daily consumption and \( S \) is the salary.

Firstly, the salary \( S \) is set as 6, and the consumption \( C \) is set as 0.3 in every iteration. We set the value of \( P_t \) as a normal distribution with expect \( \mu \) of 0.1 and variance \( \sigma^2 \) of 0.01. The probability density function of normal distribution is

\[ f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \]  \hspace{1cm} (2.14)
Figure 2.12: The figure shows the values of $P_t$ corresponding to the right order number. There are 1,000 numbers. The X-axis direction represented the order number, the Y-axis on behalf of the value of $P_t$ which is indicating how many money people will investment with their wealth. The value of $P_t$ followed by normal distribution, which the expect of the distribution is 0.1 and the variance is 0.01.

As a natural phenomenon, all investments are potentially risky. This means that every investor will not necessarily make profit from all their investment activities. The negative values of $P_t$ indicate that people lose their money in the investment activity.
Figure 2.13: The figure above shows the final individual income distribution. The X-axis direction indicated the range of income (in Pounds), the Y axis direction is represented the number of population. The amount of deposit owned by every person initially is 1000. The distribution results from 1000 steps of comprises of money with a constant consumption value 0.3.

From the Figure 2.13 we can see that most people’s income ranges between £26 000 and £32 000. And the peak appears about £29 000. So, it is similar to the result in Figure 2.14. This shows that the model is able to roughly describe the income distribution.
Figure 2.14: The figure above shows the distribution of United Kingdom taxpayers’ income in the period 2009-2010. The X-axis direction indicated the range of income (lower limit), the Y-axis direction is represented the number of population. Red dot line is the distribution of model simulation, the same distribution from figure 2.13. Green dot line is the distribution of UK taxpayers’ income in the period 2009-2010. The mean income is £28 400. \[38\]

<table>
<thead>
<tr>
<th>Mean income (Pounds)</th>
<th>Number of population (thousand)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6750</td>
<td>514</td>
</tr>
<tr>
<td>7500</td>
<td>1050</td>
</tr>
<tr>
<td>9060</td>
<td>2140</td>
</tr>
<tr>
<td>11000</td>
<td>2700</td>
</tr>
<tr>
<td>13500</td>
<td>3850</td>
</tr>
<tr>
<td>17400</td>
<td>5450</td>
</tr>
<tr>
<td>24500</td>
<td>6800</td>
</tr>
<tr>
<td>37900</td>
<td>5490</td>
</tr>
<tr>
<td>57800</td>
<td>1340</td>
</tr>
<tr>
<td>82200</td>
<td>621</td>
</tr>
</tbody>
</table>

The table above is the United Kingdom taxpayer income, 2009-2010.
This part of research is to investigate how the investment rates affect the income distribution. As demonstrated in the former process, the values of investment rate $P_i$ are set as a normal distribution. The variance $\sigma^2$ and expect $\mu$ are fixed as 0.01 and 0.1, respectively. In the simulation, we are going to use different expect $\mu$ values for the normal distribution.

The Figure 2.15 above shows the normal distribution about different expect value, Yellow is 0.05. Red is 0.1. Green is 0.15.
The Figure 2.16 above shows the distribution of income. The X-axis direction indicated the range of income (in Pounds), the Y axis direction is represented the number of population. As the expect getting large correspond to the distribution from right to left shown in the figure. Yellow is 0.05. Red is 0.1. Blue is 0.15. Green is the distribution of UK taxpayers’ income in the period 2009-2010.

From Figure 2.16, we can see that as the investment rate getting larger, the personal income increases. There will be a question why people get more wealth after using large amount of money in investment because the risk of investment. Here is the model of the process

\[ W_n = W_{n-1}(1 + P_{t_{n-1}} - C) + S \]  \hspace{1cm} (2.15)

Even there are some values of \( P_t \) are negative, but the expected value is positive. Most of the values of \( P_t \) will be positive as shown in Figure 2.12. That is why as the investment rate getting higher, the higher personal income would be. We can also see from the Figure 2.16 that when the investment value \( P_t \) is getting higher, the distribution becomes smoother. This indicates the risk of investment. This clearly shows that when people put more money in the investment, more people will become poor. Only a small percentage of people will own a majority of the wealth. From the real data of income distribution, the total income comprises earned income and investment income, the proportion of investment income is 9% of total
income. So, we assume that the expected value of investment rate is positive when designing the model.\[39\]

In the following processes, we are going to set the investment rate $P_t$ generated from another distribution—Levy distribution whose probability density function is

$$f(\mu, \sigma, x) = \frac{e^{-\frac{\sigma}{2(x-\mu)}} \sqrt{\sigma}}{\sqrt{2\pi(x-\mu)^2}}$$

with location parameter $\mu$ and dispersion parameter $\sigma$. The expectation is

$$E(x) = \int_0^\infty x f(x) \, dx = \infty$$

The Truncated Levy Distribution is a generic description for a Levy distribution that has some cut-off far in the power law tails. Such a cut-off will ensure that the variance of the distribution is finite. One possible cut-off is the straight dotted line shown in Figure 2.17 below.\[40\]

Figure 2.17, show the levy distribution with dispersion parameter $\sigma$ equal to 0.1 and location parameter $\mu$ equal to 0.05(yellow), 0.1(Red), 0.15(Green). The cut-off line is indicated by a straight dotted line. This means we only pick up stochastic variables from section $\{0, 0.3\}$.  

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The Figure 2.18 above shows the distribution of income. The X-axis direction indicated the range of income (in Pounds), the Y axis direction is represented the number of population. As the expect getting large correspond to the distribution from right to left shown in the figure. Yellow is 0.05. Red is 0.1. Blue is 0.15. Green is the distribution of UK taxpayers’ income in the period 2009-2010.

Figure 2.18 shows similar trend as Figure 2.16. This is a description of the wealth distribution after a period of process. A small group of people own large amount of wealth while the rest of total wealth is shared by the majority of people in the society. Additionally, it indicates that as the investment parameter increases, both the risk and wealth increase.
Figure 2.19 shows the distribution of income. The X-axis direction indicated the range of income (in Pounds), the Y axis direction is represented the number of population. The red line with fluctuation is the simulation result, and all the other colour lines are represented the distribution of UK taxpayers’ income in the period 2004-2008 and 2009-2010.

From the distribution of UK taxpayers’ income in the period 2004-2008 and 2009-2010, there are two characteristics changing over time. One characteristic is the low mean income ranges between £0 to £15 000 as assumed. The number of population becomes smaller as the time changes from the 2004 to 2010. This indicates that fewer people are in this low mean income range as time goes on. The other characteristic is the middle mean income range between £15 000-£50 000. In this range, the number of population increase during the year of 2004 to 2010. This means that more people will be in this middle range. Above all, we could say that as time goes on, the mean income of taxpayer increases.

The red line is the simulation result by using the model (2.13). Where consumption C is set as 0.3, salary S is set as 0.3. The investment parameter $P_t$ is generated by Levy distribution.

$$f(\mu, \sigma, x) = \frac{e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma \sqrt{2\pi}}$$

(2.19)
We use Levy distribution with dispersion parameter $\sigma$ equal to 0.1 and location parameter $\mu$ equal to 0.12 and the cut-off value as 0.8. The red line in Figure 2.19 estimates the future income distribution approximately and broadly in line with the two characteristics appearing in real data. In general, the red line shows some fluctuations which indicate the limitation of the model. Model improvement is needed in the future research.
2.3 Global Economy Model

In this part, we are going to find an expression for the choice of consumption made by a representative household which will optimise that household’s dynamic utility. A household’s utility is the ‘satisfaction’ that a household gets from either choosing to spend or save their income, subject to their budget constraints. This structure of economic and consumer behaviour was constructed by Ramsey. \[29\] It has been refined and modified over the years.

Secondly, we intend to adapt these models to allow for varying degrees of generality and structured networks which represent realistic groups of economies displayed empirically. By submitting different values for parameters and variables such as utility functions and consumption, we intend to show how these variances affect interest rates and monetary policy in the real world.

Our initial analysis will be a global economy model consisting of two countries which interact by lending and borrowing their output each other. By constructing an equation for the calculation of one country’s current wealth we can use dynamic programming and Lagrange multiplication to derive the optimal choice of time path for consumption for any household in each country. Once this has been established we can extend the model to include three separate economies (or countries) which converge to form a more generalised model for the real world.

2.3.1 Model assumption

The model used in this derivation is built on the Ramsey’s growth model which was constructed in 1928 and is based on a large number of identical households within an economy, or country, which seek to optimise their own utility through their choices of consumption. \[41\] There are two countries in the model. Within each country, each household receives wages for the provision of labour, l, which produces capital, k. Each household also receives interest income on their assets, w, and purchase goods for consumption, c. Finally, a country has a rate of time preference, \( \rho \), which represents the diminishing value of a households’ utility or assets over time, i.e. a household will prefer to consume now rather than far in the future if they have a higher rate of time preference. A country’s time
preference therefore is an indication of how ‘patient’ that country is and how willing households are to save, with the lower the rate meaning the more patient a country is.

There is a world interest rate, \( r \), which applies to both countries but otherwise each country’s households adhere to common parameters specific for their country of residence. Normalising the ‘mass’ of households in each country to unity, and using a bar over parameters to distinguish the more patient country from the less patient (more patient country with time preference \( \bar{\rho} \) (with bar)), the objective for a representative household in each country is to maximise the following utility functions:

\[
\int_{\tau=\tau}^{\infty} \exp(-\bar{\rho} (\tau - t)) u(c) \, d\tau \quad (2.20)
\]

\[
\int_{\tau=\bar{\tau}}^{\infty} \exp(-\bar{\rho} \bar{(} \tau - t)) \bar{u}(\bar{c}) \, d\tau \quad (2.21)
\]

The output for the two countries are denoted buy \( a \) and \( \bar{a} \). This is calculated by the production function deriving from the Cobb-Douglas function\(^{[41]}\);

\[
a = Ak^\alpha t^{\alpha - 1} \quad (2.22)
\]

Where \( A>1 \) is the level of technology and can depend on time and \( \alpha \) is a constant with \( 0<\alpha<1 \), for now we will assume that \( A \) is also constant and hence, so is \( \alpha \).

There is a global resource constraint which states that total output is always equal total consumption, therefore as our model consists of only two countries it can be written as;

\[
c + \bar{c} = a + \bar{a} \quad (2.23)
\]

Following this, assets can be described as debt claims which represent lending from one country to the other and therefore must net to zero between the two with interest paid/received at the rate, \( r \). The change in claims or assets over time is therefore given by;

\[
dw = (a + r(t)w - c)dt \quad (2.24)
\]

hence, we have
\[ \dot{v} = a + rw - c \]  

(2.25) can be written as (2.26) by combine with the global resource constraint;

\[ a + r(t)w - c = -\bar{a} + r(t)w + \bar{c} \]  

(2.26)

In order to maximise the utility, we use Lagrange multiplication with the above equation of motion as the constraint. The Hamiltonian for this maximum principle is as follows;

\[ J = u(c)e^{-\rho t} + v(t)(a + rw - c) \]  

(2.27)

Where \((t)\) is the Lagrange multiplier. The household’s optimal choice for consumption will be obtained by maximising the Hamiltonian at each point in time. To do this it is helpful to analogue the Hamiltonian, \(J\{w,v\}\), with the Hamiltonian, \(H\), for the motion of a particle;

\[ H\{x, p\} \]  

(2.28)

with \(x\) being position and \(p\) momentum. Knowing this it can then be written that;

\[ \text{force, } \dot{p} = -\frac{\partial H}{\partial x} \]  

(2.29)

\[ \text{velocity, } \dot{x} = \frac{\partial H}{\partial p} \]  

(2.30)

Therefore, using the same principle on our Hamiltonian \(J\{w, v\}\), for wealth, we can write the following;

\[ \dot{v}(t) = -\frac{\partial J}{\partial W} = -v(t)r \]  

(2.31)

Based on this we can find the dynamic constraints for our Hamiltonian;

First, as this optimisation is a maximisation of the Hamiltonian we start by using the maximum principle for the control variable, \(c\), that is to differentiate \(J\) by \(c\) and equate to 0:

\[ \frac{\partial J}{\partial c} = e^{-\rho t}u'(c) - v(t) = 0 \]

\[ \Rightarrow v(t) = e^{-\rho t}u'(c) \]  

(2.32)
Differentiating this with respect to time using the product rule gives;

\[ \dot{v}(t) = -\rho e^{-\rho t} u'(c) + e^{-\rho t} u''(c) \dot{c} \]

\[ \Rightarrow -\rho v(t) + e^{-\rho t} u''(c) \dot{c} \]

By equating equations (2.31) and (2.33) we derive an expression for the time path of \( c \):

\[ -\rho v(t) + e^{-\rho t} u''(c) \dot{c} = -v(t)r \]

\[ \Rightarrow e^{-\rho t} u''(c) \dot{c} = v(t)(\rho - r) \]

\[ \Rightarrow e^{-\rho t} u''(c) \dot{c} = e^{-\rho t} u'(c)(\rho - r) \]

\[ \dot{c} = \frac{u'(c)}{u''(c)} (\rho - r) = \left( -\frac{u''(c)}{u'(c)} \right)^{-1} (r - \rho) \]

(2.35)

This equation describing the time path of the evolution of consumptions is known as the Euler equation and from this the equivalent Euler equation for the more patient country is a trivial substitution;

\[ \dot{c} = \frac{\pi'(c)}{\pi''(c)} (\rho - r) = \left( -\frac{\pi''(c)}{\pi'(c)} \right)^{-1} (r - \rho) \]

(2.36)

The final constraint is on the form of the tranversality condition which states that the value of the household’s and hence a country’s assets, \( w \), multiplied by the Lagrange multiplier, \( \nu(t) \) must approach zero as time approaches infinity i.e. if we think of infinity as the planning horizon, the utilizing household does not want to have any assets left at the end of time. Therefore, the transversality condition is:

\[ \lim_{t\to\infty} e^{-\rho t} \nu(t)w(t) = \lim_{t\to\infty} e^{-\rho t} u'(c)w(t) = 0 \]

(2.37)

As the interest rate is a dynamic variable dependant on other inputs we now need to find an expression for \( r \) to allow us to form an equation for the time path of \( c \) which does not contain interest rate. If we take the first derivative of the resource restraint with respect to time we can see that;

\[ \dot{c} + \ddot{c} = 0 \]

(2.38)

Therefore;
\[
\left(-\frac{u''(c)}{u'(c)}\right)^{-1}(r - \rho) + \left(-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right)^{-1}(r - \bar{\rho}) = 0 \quad (2.39)
\]

Rearranging we get;

\[
\left(-\frac{u''(c)}{u'(c)}\right)(r - \bar{\rho}) = \left(-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right)(r - \rho)
\]

\[
r = \frac{\left(-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right)\rho + \left(-\frac{u''(c)}{u'(c)}\right)\bar{\rho}}{\left(-\frac{u''(c)}{u'(c)}\right) + \left(-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right)} \quad (2.40)
\]

This result shows that the interest rate is a weighted average of the time preference of the two households in the two countries with the weights depending on \(-\frac{u''(c)}{u'(c)}\). This expression is known as the absolute risk aversion and describes the level of a household’s (or country’s) aversion to risks in unknown future returns on their wealth. Depending on the specific utility function \(u(c)\), it can be constant or relative i.e. have a dependence on consumption.

By substituting the result for \(r\) back into equation (2.35) we can write an equation for the time path of \(c\) which is a function of time preference and utility only;

\[
c = \frac{\bar{\rho} - \rho}{\left(-\frac{u''(c)}{u'(c)}\right) + \left(-\frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right)} = -\dot{c} \quad (2.41)
\]

2.3.2 Solution with different utility functions

First we must consider another constraint on each country’s borrowing. This is \(w^* \leq w \leq \bar{w}^*\) where \(w^* < 0\) represents the maximum borrowing by the first country and \(\bar{w}^* > 0\) represents the maximum borrowing by the second country. We also have \(w^* < w < \bar{w}^*\) showing that there are only two countries who lend to each other. As \(\rho > \bar{\rho}\) the time path for \(c\) described by equation (2.40) implies that the consumption for the patient country (\(\bar{c}\)) will start low and increase over time whereas the consumption for the less patient country will start high and decrease over time, this will continue until \(w = w^* < 0\) at which point \(c = a + r(w^*)w^*\) and \(\bar{c} = a - r(w^*)w^*\) with \(r(w^*)\) being the interest rate corresponding to that level of consumption, which can be solved by substituting these expressions for \(c\) and \(\bar{c}\) into equation (2.40).
We should also consider the case where there are no constraints on borrowing. To do this we must find an initial level of consumption, \( c_0 \), which is consistent with the transversality condition. For this we can use the utility function:

\[
u(c) = -\exp(-\varepsilon c) \quad \text{and} \quad \bar{u}(\bar{c}) = -\exp(-\bar{\varepsilon} \bar{c}) \quad (2.42)
\]

Here we have introduced \( \varepsilon \), the intertemporal elasticity of substitution. This is a measure of responsiveness of the growth rate of consumption to the real interest rate. Here it is also worth noting that the utility function expresses the preferences of the household with respect to perceived risk and expected return. The three utility functions we will be using are shown in figure 2.20. From the utility function we have

\[
\left(\frac{-u''(c)}{u'(c)}\right) = \varepsilon \quad \text{and} \quad \left(\frac{-\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})}\right) = \varepsilon
\quad (2.43)
\]

Therefore, the interest rate is

\[
r = \frac{\varepsilon \rho + \varepsilon \bar{\rho}}{\varepsilon + \bar{\varepsilon}}
\quad (2.44)
\]

\( r \) is a constant. By integrating equation (2.41), solving for the level of consumption at time, \( t \), the equation then becomes

\[
c = c_0 + \frac{\bar{\rho} - \rho}{\varepsilon + \bar{\varepsilon}} t
\quad (2.45)
\]

Substituting this expression for \( c \) into our utility function, we get

\[
u(c) = -\exp\left(-\varepsilon \left(c_0 + \frac{\bar{\rho} - \rho}{\varepsilon + \bar{\varepsilon}} t\right)\right)
\quad (2.46)
\]

\[
u'(c) = \varepsilon \exp(-c_0 \varepsilon) \exp\left(-\frac{\varepsilon(\bar{\rho} - \rho)}{\varepsilon + \bar{\varepsilon}} t\right)
\quad (2.47)
\]

However, with this conclusion, the consumption of the impatient country (with time preference \( \rho \)) would become negative as their entire output would be paid in interest to the patient country. This is clearly unrealistic. Therefore another non-exponential utility function must be considered
\[ u(c) = \frac{1}{1 - \varepsilon} c^{1-\varepsilon} \quad \text{and} \quad \bar{u}(\bar{c}) = \frac{1}{1 - \varepsilon} \bar{c}^{1-\varepsilon} \] (2.48)

This is known as the isoelastic utility function.

In this case we have

\[ \left( - \frac{u''(c)}{u'(c)} \right) = \frac{\varepsilon}{c} \quad \text{and} \quad \left( - \frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right) = \frac{\varepsilon}{\bar{c}} \] (2.49)

Then substituting these into equation (2.41), we have

\[ \frac{dc}{dt} = \bar{\rho} - \rho \frac{\varepsilon}{c + \bar{c}} \] (2.50)

By using the \( \varepsilon = \bar{\varepsilon} \), and \( c + \bar{c} = a + \bar{a} \) we obtain:

\[ c = \frac{a + \bar{a}}{1 + e^{(\rho - \bar{\rho})t} + c_0} \] (2.51)

If we now stipulate that \( \varepsilon \geq 1 \), there will no longer be the unrealistic outcome of negative consumption. That is because this solution, without borrowing constraints, will cause \( c \) to tend to the limit of either 0 (the impatient country) or \( a + \bar{a} \) (the patient country) over time. So in effect, there is a maximum level of borrowing which is given by \( w^* = a/\bar{\rho} \).

Finally we can consider a quadratic utility function;

\[ u(c) = \alpha c - \frac{1}{2} \beta c^2 \quad \text{and} \quad \bar{u}(\bar{c}) = \alpha \bar{c} - \frac{1}{2} \bar{\beta} \bar{c}^2 \] (2.52)

In this case we have;

\[ \left( - \frac{u''(c)}{u'(c)} \right) = \frac{\beta}{a - \beta c} \quad \text{and} \quad \left( - \frac{\bar{u}''(\bar{c})}{\bar{u}'(\bar{c})} \right) = \frac{\bar{\beta}}{\bar{a} - \bar{\beta} \bar{c}} \] (2.53)

By using the \( c + \bar{c} = a + \bar{a} \) obtained and substituting to equation (2.41) we have;

\[ c = \frac{a + \bar{a} - \frac{\alpha}{\beta} + c_0 \exp((\bar{\rho} - \rho)t) \frac{\alpha}{\beta}}{1 + c_0 \exp((\bar{\rho} - \rho)t)} \] (2.54)
This leaves a closed form solution for $c$ where, as the time path of consumption reaches equilibrium, the patient country tends to a level of $\frac{\bar{a}}{\bar{b}}$ and the impatient country retains the remainder of the combined output, $\alpha + \bar{a} - \frac{\pi}{\bar{b}}$.

Figure 2.20: Graph showing the three utility functions. The blue line shows exponential utility, which represents a risk averse household. The green line shows relative (isoelastic) utility function which represents a household which is more risk averse than the exponential function. The red line shows the quadratic utility function, this represents a household, which is risk-seeking. [42]
2.3.3. Extension of the model to multiple countries.

The next step in making the model closer to a real world global economy, or system of economies, is to extend the model with the addition of a third country. However, maintaining the assumption that each country can only be a borrower or lender to another, not both (as is the case in the two-country model) as there is a single global good traded, as shown in Figure 2.21

![Network of three countries in an economy.](image)

The same assumptions will be made as with the two-country model, with the subscript $X_1$, $X_2$, $X_3$ used to define each country’s parameters, rather than the bar as previously used. In addition, the system of countries now has non-zero net global assets (or wealth). Whereas before the only assets were debt claims and therefore the net of the two assets equated to zero. There is now a constant global wealth, $W_0$, which takes into account any capital a country owns, and can be set to any value. As it does not affect the net borrowing or lending, which will still net to zero. Therefore, $W_0$ is related only to the sum of each country’s capital $k_i$;

\[
\text{country's total assets} = k_i + w_i
\]

\[
\sum_1^3 w_i = 0 \quad \sum_1^3 k_i = W_0
\]

(2.55)

The global resource constraint still holds and now becomes;

\[
\sum_1^3 a_i = \sum_1^3 c_i
\]

(2.56)
and therefore by differentiating with respect to time we see that;

\[ \sum_i^3 c_i = 0 \]

*we have* \( c_1 + c_2 + c_3 \)  (2.57)

The equation of motion for the time path of a country’s net borrowing/lending is still as before;

\[ w_i = a_i + rw_i - c_i \]  (2.58)

Therefore using the same optimisation tools as before, we can still assume that;

\[ \dot{c}_i = \frac{u'_i(c_i)}{u''(c_i)} (\rho_i - r) = \left( -\frac{u''(c_i)}{u'_i(c_i)} \right)^{-1} (r - \rho_i) \]  (2.59)

Following this we can write;

\[ \left( -\frac{u''(c_1)}{u'_1(c_1)} \right)^{-1} (r - \rho_1) + \left( -\frac{u''(c_2)}{u'_2(c_2)} \right)^{-1} (r - \rho_2) + \left( -\frac{u''(c_3)}{u'_3(c_3)} \right)^{-1} (r - \rho_3) = 0 \]  (2.60)

And similarly, as shown for the two country model, this can be rearranged to give a function for the world interest rate, \( r \);

\[ r = \left( -\frac{u''(c_1)}{u'_1(c_1)} \right)^{-1} \rho_1 + \left( -\frac{u''(c_2)}{u'_2(c_2)} \right)^{-1} \rho_2 + \left( -\frac{u''(c_3)}{u'_3(c_3)} \right)^{-1} \rho_3 \]

\[ \left( -\frac{u''(c_1)}{u'_1(c_1)} \right)^{-1} + \left( -\frac{u''(c_2)}{u'_2(c_2)} \right)^{-1} + \left( -\frac{u''(c_3)}{u'_3(c_3)} \right)^{-1} \]  (2.61)

As with the two-country model, the equilibrium interest rate is then a weighted average of the rate of time preference of different households with the weights depending upon the utility function. Therefore, on the intertemporal elasticity of substitution and consumption for the other two countries.

The next stage is to insert the function for \( r \) into the equation for the time path of consumption and solve for \( c \). The first utility function that is considered is the exponential utility function \( u_i(c_i) = -\exp(-\epsilon_i c_i) \) where;

\[ \left( -\frac{u''(c_j)}{u'_1(c_1)} \right)^{-1} = \frac{1}{\epsilon_i} \]  (2.62)
Substituting directly into our value for $r$, we get;

$$r = \frac{\rho_1 \varepsilon_2 \varepsilon_3 + \rho_2 \varepsilon_1 \varepsilon_3 + \rho_3 \varepsilon_1 \varepsilon_2}{\varepsilon_1 \varepsilon_2 + \varepsilon_1 \varepsilon_3 + \varepsilon_2 \varepsilon_3} \quad (2.63)$$

Therefore the global interest rate is constant for the exponential utility function and following this, the time path for consumption can be given by substituting this value for $r$ into equation (2.59);

$$c_i = c_{i0} + (\frac{1}{\varepsilon_i}) \left( \frac{\rho_1 \varepsilon_2 \varepsilon_3 + \rho_2 \varepsilon_1 \varepsilon_3 + \rho_3 \varepsilon_1 \varepsilon_2}{\varepsilon_1 \varepsilon_2 + \varepsilon_1 \varepsilon_3 + \varepsilon_2 \varepsilon_3} - \rho_i \right) t \quad (2.64)$$

and by integration we can then say that;

$$c_i = c_{i0} + (\frac{1}{\varepsilon_i}) \left( \frac{\rho_1 \varepsilon_2 \varepsilon_3 + \rho_2 \varepsilon_1 \varepsilon_3 + \rho_3 \varepsilon_1 \varepsilon_2}{\varepsilon_1 \varepsilon_2 + \varepsilon_1 \varepsilon_3 + \varepsilon_2 \varepsilon_3} - \rho_i \right) t \quad (2.65)$$

This shows that the consumption grows at constant rate equal to $r - \rho_i$ assuming that $r > \rho_i$. If $r < \rho_i$ then we again have the situation where the impatient country’s consumption decreases at a constant rate and would eventually become negative.

For the isoelastic utility function $u(c_i) = c_i^{1-\varepsilon_i}$, solving for the time path of consumption is more complicated. The global interest rate is related to each of the county’s consumption;

$$r = \frac{\rho_1 c_1 \varepsilon_2 \varepsilon_3 + \rho_2 c_2 \varepsilon_1 \varepsilon_3 + \rho_3 c_3 \varepsilon_1 \varepsilon_2}{c_3 \varepsilon_1 \varepsilon_2 + c_2 \varepsilon_1 \varepsilon_3 + c_1 \varepsilon_2 \varepsilon_3} \quad (2.66)$$

Assuming that $1 > \rho_1 > \rho_2 > \rho_3 > 0$ and that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$, this equation shows that an increase in consumption will cause the global interest rate to fall, with the least patient country’s consumption having the most influential effect. This is consistent with what one would expect to see in real economies, as households would react to reduced returns on savings due to lower rates by saving less and consuming more. However, the question of which variable is affecting the other is raised, as here interest rate is described as being influenced by consumption whereas one could also say that consumption is more strongly influenced by the interest rate set. The solution for the time path of an individual country or household’s consumption will be an exponential function depending on both the other countries’ own consumptions at that time.
The model can be expanded further to include a fourth country, where there are now several complications, as it is now possible for each country to trade with more than two other countries and thus the flow of wealth is no longer predetermined by the construction of the network. Figure 2.22 shows the potential different constructions for a network of four counties based on the specified limitations of trade. If any countries were permitted to trade freely with all other countries in the network, the addition of trade between countries four and two and one and three would introduce an additional path for the flow of wealth and hence complicate the possible equation of motion for wealth of each country.

Figure 2.22a (left), 2.22b (right) Network of four country economies with trade in one direction. (a) Shows countries trading with only two other countries (b) shows countries trading with all other countries in the network.

If we continue to expand the global system of economies cyclically, with the assumption that each country only has two trading partners (a buyer and seller, or borrower and lender), as we include more countries we see the pattern in the expression for interest rate follows the trend set between equation (2.40) for two countries into equation (2.66) for three continues, namely that the interest rate is determined by the weighted average of each country’s time preference, dependant on the absolute risk aversion. Therefore (with the relative risk utility function) the greater the number of countries incorporated in the network of economies, the less impact a change in consumption for each individual country will have. This indicates that there is a kind of ‘safety in numbers’ for groups of economies looking to stabilise interest rates and consumption growth and justifies the inclusion of trade agreements in large multi-national economies such as the EU.

So far, we have described systems of economies which have cyclical flows of wealth between countries; this is unrepresentative of what one may expect to find in real economies. Therefore, we should now try to introduce a concept which seeks to describe networks of economies where, rather than a group of countries all trading in relatively equal proportions
with one another, we now have one country becoming dominant within the network and exporting to all other countries. We will also look at the case where one country becomes a sole importer, or borrower from all other countries within the network. This situation can be analogised with events in the recent ‘Eurozone crisis’ where nations such as Ireland and Greece have been bailed out by other countries within the EU.

The use of this type of topology, such as spanning trees and networks to determine a description of countries, influences on market prices and rates has been proven to be a robust method with respect to time horizon and is also stable during market crises, such as that in the Eurozone. In the case of foreign exchange rates this type of topology, appropriately used, gives a useful guide to determining the underlying economic or regional causal relationships for individual currencies and to understanding the dynamics of exchange rate price determination as part of a complex network [44]. We now intend to see if our method of determining the time path of consumption and hence the global interest rate can also produce empirically supported expressions to describe the dynamics of wealth and interest rates within a more complex network of economies.

First, we will consider the case where one country becomes effectively a sole exporter (or lender) for all other countries within the network, we call this country the ‘Hub’. For this analysis we assume that this country only exports to all other countries, does not import anything i.e. trade is only in one direction and all other countries’ non- Hub trade is negligible compared to their trading with the Hub. This is shows in Figure 2.23.
For this system of economies, we can now write two equations of motion for the change in wealth, one for the Hub, which has wealth $W$ and one for any of the individual economies it supplies, with wealth $w_n$. Assuming there are $n$ countries which the Hub supplies to;

\[
\frac{dW}{dt} = a - c + rd \sum_n w_n \tag{2.67}
\]

\[
\frac{dw_n}{dt} = a_n - c_n - rw_n \tag{2.68}
\]

We set that $w_1 = w_2 = \cdots = w_{n-1} = w_n$ all countries being supplied by the hub have the same wealth, then equation (2.67) becomes;

\[
\frac{dW}{dt} = a - c + nw_n \tag{2.69}
\]

By using the same method as before, we have the Hamiltonian for the hub country and differentiate with respect to $w_n$ as follows:

\[
J = e^{-\rho t} u(c) + v(t)(a - c + rnw_n) \tag{2.70}
\]

\[- \frac{\partial J}{\partial w_n} = -nrv(t) = \dot{v}(t) \tag{2.71} \]

By using the maximum principle, we find the optimal consumption;

\[
\frac{\partial J}{\partial c} = e^{-\rho t} u'(c) - v(t)
\]

\[
\Rightarrow v(t) = e^{-\rho t} u'(c) \tag{2.72}
\]

Differentiating this expression with respect to time, then we have

\[
\dot{v}(t) = -\rho e^{-\rho t} u' + e^{-\rho t} u''(c) \dot{c} \tag{2.73}
\]

Equating equation (2.71) and (2.73) and substituting equation (2.72) we have;

\[-\rho u'(c) + u''(c) \dot{c} = -nr u'(c) \]
\[ \Rightarrow \dot{c} = \left( -\frac{u''(c)}{u'(c)} \right)^{-1} (nr - \rho) \] (2.74)

This is the expression for the time path of the hub country’s consumption, by following the same steps, but starting with the equation (2.68) results with the expression for the time path of consumption for one of the countries being supplied by the hub;

\[ \dot{c}_n = \left( -\frac{u''(c_n)}{u'(c_n)} \right)^{-1} (r - \rho_n) \] (2.75)

the global resource constraint now takes the form;

\[ a + \sum a_n = c + \sum c_n \] (2.76)

which we can have;

\[ \dot{c} + \sum \dot{c}_n = 0 \]

\[ \Rightarrow \left( -\frac{u''(c)}{u'(c)} \right)^{-1} (nr - \rho) + \sum_n \left( \left( -\frac{u''(c_n)}{u'(c_n)} \right)^{-1} (r - \rho_n) \right) = 0 \] (2.77)

Then we have the expression for \( r \),

\[ r = \frac{\left( -\frac{u''(c)}{u'(c)} \right)^{-1} \rho + \sum_n \left( -\frac{u''(c_n)}{u'(c_n)} \right)^{-1} \rho_n}{n \left( -\frac{u''(c)}{u'(c)} \right)^{-1} + \sum_n \left( -\frac{u''(c_n)}{u'(c_n)} \right)^{-1}} \] (2.78)

This result shows that the expression for the global interest rate is similar to a model with a cyclical network of economies; however, with the presence of \( n \) attached to the absolute risk aversion for the hub country. Upon analysis this is a sensible and fairly intuitive result as it shows that the global rate is calculated as previously, by a weighted average of time preferences but with the hub’s absolute risk aversion and hence it’s elasticity of substitution and consumption behaviours having a greater influence than any other country, with this influence proportional to the number of countries that the hub supplies.
By applying the relative absolute risk aversion we obtain;

$$r = \frac{c}{\epsilon} \rho + \sum_n \frac{c_n \rho_n}{\epsilon_n}$$

(2.79)

now let $\epsilon = \epsilon_1 = \cdots = \epsilon_n$ we see that $r$ now becomes;

$$r = \frac{n c \rho + \sum_n c_n \rho_n}{n^2 c + \sum_n c_n}$$

(2.80)

From this we see that an increase in consumption for the hub country would result in a proportionate decrease in the global interest rate. This is consistent with the result from a cyclical network of countries in an economy. Although this model is crudely generalised, this system of economies containing a hub country can be seen to exist in the real world. The most apparent example of this is China. China exports to many countries around the world which do not subsequently export any significant goods or lending to China. As a result of this and changes in the Chinese economy there will be ripple effects on economies all over the world and consequently the Chinese economy is observed and monitored intently by economists and analysts from around the world.$^{[45]}$ Similarly, on a smaller scale within the EU we see that as a result of surviving the financial crisis relatively well, Germany has become the central hub for European nations seeking to borrow.

The suitable progression of this analysis is to now look at the opposite situation, where one country becomes dependant on a group of other nations who all lend or export to that one country, as shown in the Figure 2.24
This type of situation has become a reality in recent years within the Eurozone, with nations such as Ireland and Greece requiring EU bailouts. This requires leading European nations to all contribute to lending one nation the amount needed to service their debt. Once again this is a crude analogy, but the principles are consistent and this provides a good starting point for our analysis.

The equations of motion for the wealth of the sink country, $W$ and of any ‘lending’ company, $w_n$ are as follows;

$\frac{dW}{dt} = a - c - r \sum_n w_n \tag{2.81}$

$\frac{dw_n}{dt} = a_n - c_n + rw_n \tag{2.82}$

If we make the same assumption as before, of having all non-sink countries with equal wealth then by observation we can see that following the same optimising steps as previously with the hub model, the resulting expression for global interest rate will be;

$$r = \frac{\left( -\frac{u''(c)}{u'(c)} \right)^{-1}}{n \left( -\frac{u''(c)}{u'(c)} \right)^{-1}} + \sum_n \left( -\frac{u''(c_n)}{u'(c_n)} \right)^{-1} \rho_n \tag{2.83}$$

This result is less intuitive than that of the hub model. We can see again that the dominant country, this time being the sink country, has a greater influence on the global rate than any of the individual lending nations. However, this time the influence is reversed. We can now see that if the sink nation were to increase consumption, the global interest rate would also rise, contrary to what we have seen with both cyclical and hub models of economies. One could argue that in this situation the rise in interest would therefore discourage the sink country from continuing to borrow and begin to balance the wealth dynamic, however the opposite is also true. That being that increased returns would encourage the other nations to continue lending despite the sink’s increasingly negative wealth. Another way of looking at it would be that in order for the sink country to maintain its level of consumption despite negative wealth, interest rates have to rise from necessity. This inconsistency is a result of an
unrealistic global interest rate. This situation is reflected with the Eurozone bailouts for countries such as Greece, Cyprus and Ireland to relieve government debt after the financial crisis.
2.4 Conclusion

In summary, we have employed both the statistical model and the network model to simulate people’s individual income distribution. These models attempt to include a wide variety of people who may vary in their income and provide good fit to real data. Within the agent-based model, it is built up on an open network for people with the economic activities and describing their daily monetary activities including investment, daily consumption and salary-gaining. The more the people invest, the more the income they will receive. This is what people have regarded as a common sense in daily life. Nevertheless, most investment activities are risky. If people use most of their wealth for investment, the result will not be shown the same as in the simulation results.

Then agent-based model has been applied to construct model of global economy. Initially, it was established that in an open economy with a global interest rate and only one world good, the interest rate is determined by a weighted average of the countries time preference. However, this only holds with the provision that each country only trades with two others. In other words, one buyer/borrower and one seller/lender. It was also shown that as more countries are added to a cyclical network of linked economies, each country’s absolute risk aversion, and hence their consumption behaviours, have proportionally less influence on the global rate. Therefore, an economy containing more separate countries can be considered more stable and resilient to individual economies defaulting and experiencing deflating consumption. In a cyclical network of economies, the global interest rate has shown to be a weighted average of the time preferences of the countries in that network, with the weight depending on the countries absolute risk aversion, which form their consumption behaviours. In particular, it has been shown that a rise in consumption will result in a fall in the global interest rate. Empirically, it cannot be said that this observation is supported, as the causality of which variable influences which needs to be looked at more closely.

Following this, it was observed that, in the situation in which one country becomes dominant in a group of economies and effectively acts as a hub (lending/exporting to all others) that dominant country’s consumption behaviours will have a greater influence on the global interest rate proportional to the number of countries in the group. The relationship between consumption and interest rate remained as it was for cyclical networks. A sink country (borrowing/importing from all others) will also have a greater influence on the global rate
than other countries. However, crucially the influence of that country’s absolute risk aversion is now reversed. This means that a rise in consumption for the sink will result in a rise in the global interest rate. Qualitatively these results are more difficult to justify. However, the networks of countries being supplied by a hub are easily analogised with countries such as China and Germany, which in recent history have performed strongly in economic environments and subsequently become sole lenders to many countries. Similarly, the state where a sink nation develops is also very similar to recent Eurozone bailouts of governments such as those in Ireland and Greece after the financial crisis.
Chapter 3 Ising model- An analysis

3.1 Introduction

Here we have developed a mathematical model of a random neuron network with two types of neurons: inhibitory and excitatory. Every neuron was modeled as a functional cell with three states, parallel to hyperpolarized, neutral and depolarized states in vivo. These either induce a signal or not into their postsynaptic partners. First a system including only one network was simulated numerically using the software developed in Python. Our simulations show that under physiological initial conditions, the neurons in the network all switch off, irrespective of the initial distribution of states. However, with increased inhibitory connections beyond 85%, spontaneous oscillations arise in the system. This raises the question whether there exist pathologies where the increased amount of inhibitory connections leads to uncontrolled neural activity. There has been preliminary evidence elsewhere that this may be the case in autism and Down syndrome.\[46-49\]

At the next stage we numerically studied two mutually coupled networks through mean field interactions. We find that via a small range of coupling constants between the networks, pulses of activity in one network are transferred to the other. However, for high enough coupling there appears a very sudden change in behavior. This leads to both networks oscillating independent of the pulses applied. These uncontrolled oscillations may also be applied to neural pathologies, where unconnected neuronal systems in the brain may interact via their electromagnetic fields. Any mutations or diseases that increase how brain regions interact can induce this pathological activity resonance.

Our simulations provided some interesting insight into neuronal behavior, in particular factors that lead to emergent phenomena in dynamics of neural networks. This can be tied to pathologies, such as autism, down's syndrome, the synchronization seen in Parkinson’s and the desynchronization seen in epilepsy. The model is very general and also can be applied to describe social network and social pathologies.

We want to have developed a mathematical model of a random neuron network with two types of neurons: inhibitory and excitatory. Every neuron was modeled as a functional cell with three states, parallel to hyperpolarized, neutral and depolarized states in vivo. These either induce a signal or not into their postsynaptic partners. Here we systematically alter the
Ising model to move it closer to a neuronal model, and look at the effect of each change. The 3 steps we look at are: randomizing the neighbors, introducing a threshold and changing the polarity of the connection.
3.2 Alteration Method

The original Ising Model has been introduced in Chapter 1. Now we will modify the model to fit in our situation better. The first step is randomizing the neighbors. However, in this part, we move this part straightly to the procedure of simulation by introducing a random number generator with a uniform distribution to select a neighbor from anywhere in the network.

Then, we will begin with the Ising model with 3 connections for each element in the network, creating a regular lattice. We sum the states of the neighbours to determine the new state of the element using the signum function. The states, defined as the same as the classical model, can be either +1 or -1. 20-time-step iterations were applied, since the final state was always seen before the 15th time step. Any changes that we applied will be described in each subsection.

The Hamiltonian of the classical Ising model is shown in equation (3.1):

$$H_{\text{Ising}} = \sum J_{ij} \sigma_i \sigma_j - \sum H_i \sigma_i,$$

(3.1)

where $J_{ij}$ is the coupling constant between spins at sites $i$ and $j$, $\sigma_i$ is the spin state at sites $i$, which takes the value of $\pm 1$ and $H_i$ is the magnetic field acting on the spin located at site $i$. This model normally describes the magnetic states in condensed matter.

We now will alter the traditional Ising model, to bring it closer to neuronal style behaviour. For this we added a threshold ($\mu$) into the signum function, as well as the state $\sigma$ having 3 possible states. These states can be: +1 for active, -1 for hyperpolarised, 0 for inactive. Therefore the state $\sigma$ for one neuron $\alpha$ can be written as follows:

$$\sigma_\alpha(\text{sig}) = \begin{cases} 
+1 & \text{if } \text{sig} > \mu \\
0 & \text{if } 0 \leq \text{sig} \leq \mu \\
-1 & \text{if } \text{sig} < 0
\end{cases}$$

(3.2)

Sig here is the signal received from neuron $\alpha$'s inputs. The signal is the sum of the weighted states of its inputs. For example the signal applied to neuron $\alpha$ is shown in equation (3.3)

$$\text{sig}_\alpha = \sum_{j=\alpha}^{\alpha_{\text{in}}} C_{\alpha j} \sigma_j,$$

(3.3)
where \( C_{\alpha j} \) is the coupling constant between the input \( j \) and the neuron \( \alpha \). This signal then affects the new state of the neuron following equation (3.2). This can be succinctly written as equation (3.4), by the average of 2 signum functions (for the case where all 3 states are correctly implemented

\[
\sigma_\alpha(sig) = \frac{1}{2} [ \text{Sgn}(sig_\alpha) + \text{Sgn}(sig_\alpha - \mu) ]
\]  

(3.4)

These equations can be combined into equation (3.5), which is similar to a discretised Hopfield model.

\[
\sigma_\alpha(t + 1) = \text{Sgn} \left( \sum_{j=1}^{N} C_{ij} H(\sigma_j(t) - \mu) \right)
\]  

(3.5)

Where \( H(x) \) is the Heaviside step function, and \( \mu \) is the action potential threshold. The next step in progression of this model would be to correctly implement a functional hyperpolarised state. This will lead to the threshold being dependent on the prior state of a neuron (equation (3.6)). A hyperpolarised cell is harder to activate than a cell at resting potential.

\[
\mu(\sigma(t)) = \begin{cases} 
0.136 & \text{if } \sigma(t) \geq 0 \\
0.227 & \text{if } \sigma(t) < 0 
\end{cases}
\]  

(3.6)

These values are calculated based on the resting potential of a neuron being -70mV, the threshold being -55mV, hyperpolarised potential being -80mV and the fully depolarised state being +55mV.

For each experiment, a network of 1000 neuronal objects was built. Each neuron is randomly assigned 3 inputs from the 1000 objects, and each connection is given a polarity. A positive polarity reflects an excitatory glutamatergic synapse, and a negative connection is an inhibitory cholinergic synapse. The initial states are randomly picked from a uniform distribution, according to a fixed ratio of ±1. The same is done with the polarity. The inter-neuronal coupling constant is set at 0.1, and the threshold is set at 0.136. This threshold is equivalent to a neuron-reaching threshold in vivo. A zero state in our model corresponds to -70mV, threshold -55mV, and +1 state being fully depolarised at +30mV in vivo. The states are then iterated with accordance to equations (3.2-3.4) above. This simulates the evolution of the neuronal states after we set their initial conditions. There are two possible outcomes for
the system. Either the neurons reach some constant final distribution of states, or oscillations
arise. A constant final arrangement is where there is no information transfer in the system;
whereas oscillations indicate that there are dynamic signals at play. We predict that varying
the initial conditions can control the final behavior of the system. We will test which initial
variables affect the final evolution most strongly.

Following a pilot simulation, it was seen that the system reaches a final behaviour, whether
stationary or oscillatory, after 25 time step iterations. Therefore that was the maximum
number of time steps we used in future runs. 50 experiments were created for each set of
initial ratios, and the results collected on the same graphs to visualise clearer averages.
3.3 Simulation results

Following the alteration process --Randomising the neighbors, adding the threshold, and Adding connections with negative polarity, we will illustrate simulation results step by step.

3.3.1 Randomising the neighbors

Here, instead of using 3 neighbours, established in the last section, that are fixed with respect to each element, we used a random number generator with a uniform distribution to select randomly a point from anywhere as its neighbor in the network. We found that having random connections increases the magnitude of divergence in the system. Where the original Ising model only spreads out slightly dependent on the initial conditions, our version diverges greatly even when we initialise with a 50:50 state split as shown in Figure 3.1. In the symmetric case, both 1 and -1 can win out, seen by a mean of 50 (out of 100 elements) across 100 experiments. The final states do not oscillate, tested by running 200 iterations on each experiment. Increasing the connectivity has no visible effect on the divergence.

![Progression of neuronal states](image)

**Figure 3.1: Progression of Neuronal States of modified model**

Following the states of the neurons with each iteration step from an initial 50:50 state
distribution, we produce Figure 3.1. Observing above picture, all experiments diverge fully. To see this more clearly, we draw two histograms of distribution of neural states for both the original model and our modified model. Those are shown below as Figure 3.2a and Figure 3.2b.

![Distribution of neural states](image_url)

**Figure 3.2a:** A histogram derived from the modified model with the final states of 1000 experiments.
From comparison between Figure 3.2a and Figure 3.2b, the magenta and cyan peaks are the initial state distributions at approximate 500 in both figures. However, in Figure 3.2a, the final states either all become 1 or all become -1. In contrast, Figure 3.2b shows the same results but for the original Ising model, where the final states have barely diverged.

3.3.2. Adding a threshold

Adding just one threshold whilst keeping the fixed connections of the original Ising model causes an even faster full divergence (Figure 3.3b), where the -1 state always takes over the whole system. This makes sense, since a threshold adds in an asymmetry, and so an initial 1:1 distribution is off center from the threshold point. For the 1000 experiments, the means for the -1 state was 1000 out of 1000 elements, with a variance of 0 (Figure 3.3a).
Figure 3.3a: A histogram of the state distributions after adding a threshold.

Figure 3.3b: Progression of neuronal states after adding a threshold.

Observed from the above two figures, the initial 1:1 state split, the -1 state takes over all 1000 elements has been seen by the very fine lines at 1000 and 0 respective to the -1 and 1 states. The variance for both final peaks is 0. Figure 3.3b clearly shows that the rate of divergence is
a lot faster and stronger than with just the random connections whose progression of neuronal states has shown in Figure 3.1.

Diverging a little more rapidly compared to the state progression with only a threshold, a 1:1 distribution that we initially have could be described by drawing the figure of its neuronal state progression (Figure 3.4)

![Progression of neuronal states](image)

*Figure 3.4: The progression of states in the system with respect to time.*

Adding both the threshold and random connections gives results very similar to just the threshold Ising results, as seen in Figure 3.4. In this way, we are managed to have the negative states win out. Therefore, we can assume the threshold effect is stronger and dominates the effect of the random connections. However, we will still test both random and fixed neighbours in later tests for completion.

**3.3.3. Adding connections with negative polarity**

We are now inducing a connection polarity at each input, acting multiplicatively upon the
inputs to an element. Looking at the histograms, there wasn't much information seen. However, some interesting phenomena arose when looking at their progression with respect to time. First we looked at only adding the polarity to the Ising model. Here, as the amount of negative connections increases, a phase-like phenomenon appeared, flipping the states of the system with every round of iteration (Figure 3.5).
Figure 3.5a, 3.5b, 3.5c going from top (previous page) to bottom. 5a is the Ising model with 30% inhibitory connections. The state frequencies remain more or less constant. 5b here the model has 70% negative connections, and an oscillatory phenomenon can now be seen. This is further emphasized in figure 3.5c, where all the connections are inhibitory.

Now we add in having random neighbours into the model. Having 30% negative connections suppresses the divergences seen in prior runs, shown in Figure 3.6a. However, with 100% negative connections, divergence is seen alongside the oscillatory behaviour (Figure 3.6b).
Figure 3.6a (above) and b (below): a, The progression of states of 50 experiments with random neighbours and 30% negative connections. The negative connections seem to suppress the divergent activity seen from Figure 3.1 and 3.2a. B, The progression of states with random neighbours, but now with 100% negative connections. Hence the divergence is seen partnered together with the state-flipping phenomenon.

Adding in the threshold so that all 3 factors are in play, we seen a return of the strong divergence of state frequencies. With 30% negative connections, the -1 state still fully wins out (Figure 3.7a). But as the negative connections increase up to 70%, this separation is reduced. At 100% negative connections, the oscillatory behaviour wins out again.
Figure 3.7a, 3.7b and 3.7c top to bottom: Progression of states of 50 experiments, all with random neighbours, having a threshold and varying the amount of negative connections of 30%
(a), 70% (b) and 100% (c). At 30% the -1 state wins out. At 70%, the negative connections seem to suppress this winning state. At 100%, the oscillations return that were seen in Figure 3.6.

These oscillations found here occur at unphysiological amounts of inhibitory connections, when 85% of coupling constants C are negative (equation 5). However, this raises the question if there exists a pathology that could increase the amount of inhibitory synapses in a particular brain region. This would lead to any signal causing uncontrolled oscillatory activity spike, such as those we see in simulations. There has been evidence that increased inhibitory activity can disrupt the excitatory-inhibitory balance in the brain, leading to the neurological diseases of autism and down's syndrome.\textsuperscript{[47,49]}

From these tests, it appears under normal physiological inhibitory ratios, any initial signal decays rapidly till the system is at rest. Therefore a good next step is to apply pulses of activity to our model network, and follow their evolution. There are 4 parameters (randomizing neighbors, thresholds, negative connections and pulses) that can be tested when pulses are applied to the networks: pulse rate, inhibitory ratio, intra-network coupling, and finally how pulses can be transferred via mean field coupling to the other network. We'll start with looking at a single network.
Figure 3.8a (top left), 3.8b (top right), and 3.8c (bottom): Progression of mean fields.

Following a network's evolution with a pulse applied every 5-time steps, the blue network has the pulse applied and the black network is the control without pulse. From top to bottom, the networks have 20% (Figure 3.8a), 15% (Figure 3.8b) and 10% (Figure3. 8c) inhibitory connections. The lower the amount of inhibitory connections, the slower the activity decays in the system. This is similar to the saturation seen in real neurons under tetanic stimulation.
After that, simulations ran with increased pulse rate of every 2-time step. With 20% inhibitory connections, we have Figure 3.9a. As you can see there is not enough time for the signal to decay fully, leading to activity similar to unfused tetanus. In Figure 3.9a, there is the same experiment as in Figure 3.9a, but the 2 networks are allowed to interact with mean field with strength 0.1. Therefore, the second network seems to increase the recovery rate of
neurons in network 1, almost via a dampening effect. This will be investigated later in the following subsections. When we slow down the pulse rate to only every 5 time steps, and take just 15% as the inhibitory ratio, like in Figure 3.8b, we can get Figure 3.9c by the same code. The difference is that the 2 networks are allowed to interact via mean field. This leads to tetanus in both networks. This tetanus is lost when the inhibitory ratio is 20%, as in Figure 9b, even with increased pulse rate.

Before investigating the mean field interaction between two networks further, we will test the coupling constant with in a single network first. There are 3 possible useful coupling constants C when we use 3 input neurons. We apply 1, 2 and 3 inputs to be excitatory to fully activate the neuron respectively. For example, they are $C > \mu$ (10c), $C > \mu/2$ (10b), $C > \mu/3$ (10a). The results here are unusual however. You'd expect that when $\mu/2 > C > \mu/3$, the decay rate would be faster than for the other 2 cases. However, there appears to be a sweet spot when $C = 0.1$, that leads to the fastest signal decay, similar to when $C < \mu/3$ (10d)
Simulations are conducted with changes in the intra-network coupling constant $C$. Pulses are applied every 10 time steps, inhibitory ratio in 20%. The black lines are the control, non-pulsating networks, and pulses are applied to the blue network. The constants tested were 0.05 (a), 0.1 (b), 0.2 (c) and 0.01, control (d) with a threshold of 0.136. The fastest decay rates are observed for b and d. These results seem odd.

Figure 3.10a-d (left to right): progression of mean fields.
Inhibitory ratio was 20%, pulses every 10 time steps into network A (blue). The constants tested here were 0.01 (a), 0.05 (b), 0.995 (c) and 0.100 (d). a: The coupling is too weak, so only a small reaction is seen in network B. As the coupling constant increases through to 0.99, the reaction signal in network B gets stronger, as well as both activation peaks getting narrower. The narrowing seen is not intuitive, and facilitates a faster recovery rate. Between c and d, the very small increase in coupling constant leads to a much larger increase in Network B activity. Therefore the coupling of 0.1 appears to be a threshold for effective cross network signalling.
Figure 3.12 (above): Plotting under simulations of 3 state populations (positive state, negative state and neutral state)

Figure 3.12 is a plot following the 3 state populations in the simulation from Figure 3.11d. The positive state mirrors the mean field seen in previous figures. The neutral “0” state appears to have a quicker reaction speed, and acts as the intermediate state moving neurons from state +1 to state -1.
In Figure 3.13a, the coupling constant here was 0.145. The reaction signal in Network B is only slightly higher than that shown in Figure 3.11d. This further suggests that a constant of 0.1 is all that is required for clean signalling. However, a slight increase in the coupling of 0.1475 between networks A and B in Figure 3.13b shows uncontrolled resonance between both networks, masking the pulse activity completely. This suggests another route for pathological action.
3.4. Conclusion

Here we studied Ising like model applying to a broad range of system. We found some understanding the emergent phenomena in interacting neural networks that, we believe, would open up new possibilities to characterize various neural conditions. Recent research on social networks shows similar phenomena such as for example that our mood is far more strongly influenced by those around us than we tend to think. Not only that, we are also beholden to the moods of friends, or friend of friends, and of friends of friends of friends - that is, people three degrees of separation away from us. The disposition of people around us can pass through our social network like a virus and we influence each other at least on a distance of three degree of separation.

A whole range of phenomena such as happiness and depression, obesity, drinking and smoking habits, ill-health, the inclination to turn out and vote in elections, a taste for certain music or food, a preference for online privacy, even the tendency to attempt or think about suicide are transmitted through networks of friends. The ways how this Is transmitted has psychological routes and are not entirely understood. One thing is clear that the information about mood, habits and other stuff we or our friends or friends of friends have propagate through the network like electricity through a power network. Although we have studied here the neuron excitations the model also shows how the social viruses are propagating through the social network and became dominant.
Chapter 4 opinion networks in voting

In this Chapter, the McCulloch-Pitts model built on an artificial neuron is first introduced briefly. Then the binary network model and its link to Ising model will be articulated, followed by a modified model--the coupled network model to describe social opinion network in period of the presidential election. To illustrate the new model, its formalism and analytical results on fixed points will be stated step by step. Then, we investigate the dependence on the ratio of the initial conditions so that we could find out more on relationship between current information and preference on final results. Finally, U.S. election campaign in 2016 will be examined comprehensively including support rates, possible preference, time series analysis, and period analysis. Besides mathematical research, we also take real-life activities into consideration. For example, Trump used Twitter to help his view spreading and take advantage of the underlying uncertainty to some extent.

4.1 Introduction

The early study of complex system was raised from field of cognitive science. An artificial neuron model was introduced by Warren McCulloch, a neuroscientist, and Walter Pitts, a logician, in 1943.[50] It is called McCulloch-Pitts neural model, also known as linear threshold gate.[51] The model describes the procedure of a neuron accepting signals and then spreading them. It tried to understand how the brain could produce highly complex patterns by using many basic neuro tangled together. For one artificial neuron, the transfer equation is

$$\sigma_k = \theta \left( \sum_{i=0}^{m} \omega_{ki} x_i \right),$$

(4.1)

which is the weighted sum of inputs according to $\omega_{ki}$ after transferring by function $\sigma$. For two interacting nodes along discrete time steps, we can deduce similar evolution equations from Eq.(4.1),

$$\sigma_k^A (t + 1) = \left( 1 - \sigma_i^B (t) \right) \theta \left( \sum_{i=0}^{m} \omega_{ki} \sigma_i^A (t) \right)$$

(4.2)

and
where we suppose, there are \( m \) inputs towards each node and only one input between two nodes A and B.

\[
\sigma_k^B(t + 1) = (1 - \sigma_k^A(t)) \theta \left( \sum_{i=0}^{m} \omega_{ki} \sigma_i^B(t) \right)
\]  

\[(4.3)\]

Figure 4.1. Illustration on Evolution of two interacting nodes A and B at discrete time step \( t \) to time step \( t+1 \). There are two nodes A and B. At time \( t \) for each node, \( m \) input signals from \( m \) different randomly chosen nodes apart from nodes A and B influence the node at time \( t \). Meanwhile, those two nodes affect each other. At time step \( t+1 \), each node outputs its own signal, which is ready to influence others at the following time.

During investigating the multi-agent social chaotic networks, the interacting agents shows an interesting phenomenon that in a whole system, parts of the system might suggest macroscopic collective properties similar to the local microscopic interactions. These have some similarity to statistical mechanics’ situation. In order to complete this property more precisely in terms of mathematical description, a division of the whole system is needed. For example, each community could be regard as a set of social infrastructures and such networks could consist of its subnetworks interfering each other\cite{52}.

we have focused on the situation when we have two competing parties, A and B. In coupled opinion networks, each party is regarded as a network of \( N \) agents who hold their own opinion on choosing A or B. During the transient part of the election each agent may arbitrary change his preferences. The network structures studied were based on a directed random graph.
### 4.2 Formalism

The coupled network is a system consisting of two mutually exclusive networks A and B, each of which includes n interacting agents denoted by $x^A_i(t)$ and $x^B_i(t)$, respectively. In each subnetwork, A or B, the randomly inter-influencing agents have fixed in-degree of size K in their own network. And each agent in network A could also have influence on agents in network B by random selection. It is provided that the connection between agent $i$ in network A and its corresponding agent $I$ in network B is bi-directional. So, network A and network B are not much different.

According to the majority principle\[^{[54]}\], we set the influence factor $c_{ij}$ between agent $I$ and agent $j$ in same network equal to 1, the maximum possible value. Here, for fixed $I$, $j$ runs through all connected nodes with node $i$ from 1 to $N$ such that all incoming stimuli to node $I$ are all ferro-magnetic in the initial Ising model, i.e. $c_{ij} = 1$. We also assume that the inter-network links are all anti-ferromagnetic, i.e. $c_{ij} = 1$ since it is unrealistic that an agent votes for both parties A and B at same time interval. Then, we define $x^A_i(t)$ and $x^B_i(t)$ to depict the acceptance of two competing choices A and B at the discrete time $t$ from agent $I$ in network A and network B, respectively. +1 means YES and 0 means NO. To make things simple, an agent in network A is assumed to be influenced by only one randomly chosen agent in network B while this kind of inter-coupling connections are also bi-directional as connections inside of one network.

Firstly, the net-internal stimulus for agent $I$ at time $t$ are denoted by $h^A_i(t)$ and $h^B_i(t)$ for network A and B, respectively. They are defined by

$$h^A_i(t) = \sum_{\{j\}} c_{ij}^A x^A_j(t), \quad \text{and} \quad h^B_i(t) = \sum_{\{j\}} c_{ij}^B x^B_j(t).$$

(4.13)

Here, $j_i$ indicates all agent $j$ who link with the agent $I$ in network I with total number K, i.e. K randomly chosen neighbours of agent $i$. It takes account to all its effectives fields. For the theta function, we define

$$
\begin{align*}
\Theta(x) &= 1 \quad \text{if} \ x > 0 \\
\Theta(x) &= 0 \quad \text{if} \ x \leq 0.
\end{align*}
$$
Then, once the initial values $x_i^A(0)$ and $x_i^B(0)$ are known, the discrete time evolutions of $x_i^A(t)$ and $x_i^B(t)$ including inter-coupling interferences are given by

$$x_i^A(t + 1) = (1 - x_i^B(t)) \Theta(h_i^A(t) - h^A),$$  \hspace{1cm} (4.14)

and

$$x_i^B(t + 1) = (1 - x_i^A(t)) \Theta(h_i^B(t) - h^B),$$  \hspace{1cm} (4.15)

where $h^A$ and $h^B$ are suitable thresholds in network A and B. The first term in each equation means the inter coupling between network A and B, while the second term means the intra coupling in its own network deriving from the theta function of overall stimulus of K randomly connected agents towards agent i. However, under our assumptions that a single vote could totally stimulate the target agent and that two networks are identical, $h^A = h^B = : h = 0$. For K=3, the amount of interaction rules is 254 according to Wolfram’s cellular automata rule.

At last, the magnetizations of both networks are defined by

$$m^A(t) = \frac{1}{N} \sum_{j=1}^{N} x_j^A(t),$$  \hspace{1cm} (4.16)

and

$$m^B(t) = \frac{1}{N} \sum_{j=1}^{N} x_j^B(t).$$  \hspace{1cm} (4.17)

Same as previous definition of magnetization, $m^A(t)$ and $m^B(t)$ measure the acceptance of opinion A or opinion B at time spot t. The changes of their values will not stop until the final consensus could be reached.
4.3 Analytical results

In the case of uncoupled network, the mean field magnetization is introduced by $m(t)$ at discrete time $t$. The time evolution of $m(t)$ in a system with in-degree size 3 is defined by

$$m(t + 1) = 1 - (1 - m(t))^3,$$  \hspace{1cm} (4.18)

where it means that the magnetization at next step is a linear superposition of 3 contributions to the YES state at time $t$. Eq. (6) has only one fixed point $m^* = 1$. This stable fixed point shows the completely polarised state in magnetic field. It also means the final point of consensus in an election campaign. However, the opposite state, which is the complementary polarized state $m^* = 0$, is a repellor without any probability to reach. Additionally, the larger the initial value $m(0)$, the faster the system goes to its fixed point. The system dominated by Eq. (6) could also describe the spreading process of infectious disease in certain amount of people where each person is influenced by three randomly chosen people.

Now, we move to a different situation of the two coupled networks. With the assumption that each agent $I$ in network A only affects its corresponding agent $I$ in network B, we can predict any certain result due to the randomness in choosing inter links which leads to the existence of correlations in the inter-network connectivity. If we discard the assumption above, the links between two networks are randomly constructed and the evolutions of magnetizations of network A and network B along time steps are given by

$$m_A(t + 1) = (1 - m_B(t)) (1 - (1 - m_A(t))^3)$$  \hspace{1cm} (4.19)

and

$$m_B(t + 1) = (1 - m_A(t)) (1 - (1 - m_B(t))^3),$$  \hspace{1cm} (4.20)

respectively. There are three fixed points getting from calculations, $(m^*_A, m^*_B) = (0,0)$, $(m^*_A, m^*_B) = (1,0)$ and $(m^*_A, m^*_B) = (0,1)$. Trying to understand their practical meaning, we can see that $(0,0)$ indicates a stable attractor into complete polarization in both networks, while the other two, describing the debating situation, are unstable.
4.4 Dependence on the ratio of the initial votes

Noticing that there is no control parameter in the model, we could deduce that the initial states dominate the asymptotic behaviour of the system.\[61\] In order to examine it, let us introduce the ratio $r$ which is the ratio between magnetizations in two networks $m_A(0)$ and $m_B(0)$ at time 0,

$$r = \frac{m_B(0)}{m_A(0)}, \quad (4.21)$$

where $m_A(0)$ cannot be zero. If $0 \leq r \leq 1$, it means $m_A(0) \geq m_B(0)$. According to Eq.(4) and Eq.(5),

$$r = \frac{m_B(0)}{m_A(0)} = \frac{1}{N} \sum_{j=1}^{N} x_j^B (0) \cdot \frac{\sum_{j=1}^{N} x_j^B (0)}{\sum_{j=1}^{N} x_j^A (0)}. \quad (4.22)$$

Here, the quantities $x_j^A (0)$ and $x_j^B (0)$, for $j=1,2..., N$, are sampled from a uniform distribution to make $m_A(0) = \frac{1}{N} \sum_{j=1}^{N} x_j^A (0)$ and $m_B(0) = \frac{1}{N} \sum_{j=1}^{N} x_j^B (0)$. Once the value of $m_A(0)$ is fixed in an interval of $[0,1]$, $m_B(0)$ could be valued in interval of $[0, rm_A(0)]$. Hence, we have $m_A(0) \geq m_B(0)$. 

![Graphs showing the distribution of $x_j^A (0)$ and $x_j^B (0)$](image-url)
**Figure 4.2.** Distribution of the asymptotic magnetizations associated with the different final states of the coupled networks for ratios $r = 0.25, 0.50, 0.75, 0.90, 0.95$ and $r = 1$. associated with initial fraction of the votes.

The distribution of votes for the final results is presented in Fig. 4.2 with 6 different values of the ratio $r$. First, we look at the extreme case $r=0$, which implies $m_B(0)=0$ such that there is no coupled links between two networks according to Eq. (4.22) and the system degrades to an uncoupled network. It will become the totally polarized states going to fixed points $(0,1)$ or $(1,0)$. Then, the concentrate moves to $r>0$, for small values of $r$, nearly full polarization could be observed for $r=0.1$ in Fig. 4.2 while there is not much change for $r=0.25$. When $r=0.5$, we can see clearly in Fig.4.2 the complete polarization phenomenon vanishes. With an increase in the ratio $r$ and growing width of spectrum bands, the probability of supporting part A becomes larger. For instance, when $r=0.7$ in Fig.4.2 the gap between part A winning and part B winning diminishes. At last, when the value of $r$ is equal to 1, where $m_A(0) = m_B(0)$, the votes peak at the centre with number of 5000 (half of all agents in network), which implies a tie situation or a balance in votes.

In summary, the ratio $r$ plays a decisive role in the potential variation of the whole system with chaotic behaviour where there exist not only intra-links in each one of two networks, also the inter-links between them. To highlight, for the values of $r$ near 1.0, the acceptance rates for both networks vary around 0.5, which means there is no evident preference towards part A or part B.
4.5 Case study

The coupled opinion network model stated above shows an overview prospective. Now we shall look at a particular case, 2016 U.S. presidential election (Trump vs. Hillary), which is one outcome of all the possible phases generating from the model. In 2016, as we all know that Mr Donald Trump has been elected as the 45th president of United States, we still remember the intense and fierce campaign competition between Donald Trump and Hillary Clinton. In the next step, we are going to find out the interaction between Donald Trump and Hillary Clinton during the campaign, based on the real polling data analysis. Finally, we focus on the polling data to find out how they are interact according to the practical situation we combine analysis results and real life environment, to prove what we assumed.

First stage, during the voting progress, we assume that all the US citizenship only have two option to choose, support Trump or Clinton. The opinion polls data on a day-to-day basis has been collected from different states, where after that for each day it was averaged over all states. The polls data is presented in Figure 4.3.

![Figure 4.3](image_url)

*Figure 4.3 shows the Opinion poll data from the general election of presidential campaign in the U.S.A. The x-axis represents number of days accumulate from July 2015 to November 2016. The y-axis indicates the polling support rates in percentage for Trump(Blue) and Clinton(Red). Data collected from web sited of realclearpolitics.com* [58]
From the first glance of curves in Figure 4.3 which illustrates the opinion polls data from U.S. president election campaign in 2016, random fluctuations are obvious. In Figure 4.3, The red upper curve is for Hillary Clinton, while the blue lower curve is for Donald Trump. It is hard to find out the chaotic behaviour and related underlying information just from this figure, we will investigate more later in this section.

![Image of Figure 4.3 showing opinion polls data]

Figure 4.3. Ratio $r$ of the two magnetizations $m^{\text{Trump}}$ and $m^{\text{Hillary}}$, where we regard each time spots as the initial value of both magnetizations in the systems. We then check how the ratio $r$ (i.e. the initial status) affects ultimate result at any time when the coupled system begins evolving.

While we look at the poll data collected, the election result is expected to predict in advance. Based on the analysis in section 4 in this article, we compute the ratio $r=m^{\text{Trump}}/m^{\text{Hillary}}$ and draw Figure 4.4. Then we could notice that the ratio $r$ is almost inside the range of 0.8 to 1.0. According to the paper, the acceptance rates in this circumstance for both parties should be around 50%, which implies a tie situation. In fact, the result of 2016 USA election, Clinton gained 48.5% votes while Trump had 46.4% votes. There is only a difference of 2.1%.

To make a comparison with our theory by meanings of time series analysis on data of supportive rates, Figure 4.3 and Figure 4.4 indicating the autocorrelation are drawn for Trump and Hillary series, respectively. The Autocorrelation function is given by

$$A(k) = \frac{\sum_{t=1}^{N-k}(m(t) - \bar{m})(m(t+k) - \bar{m})}{\sum_{t=1}^{N}(m(t) - \bar{m})^2}$$  \hspace{1cm} (4.23)
Figure 4.5. Autocorrelation Functions of Support rates of Trump(Left) and Hillary(Right). \( K \) is the lag number. In the left graph, the autocorrelation is negative until \( k \) increases to around 118. While in the right graph, the curve goes up and down of the \( x \)-axis in a roughly period.

From the distribution of campaign results and autocorrelation figure we can see that some fluctuations, which represent many uncertainties. This is why so hard to predict the results of campaign from phenomenon. But where are the uncertainties come from? As we know that all the residents of United States have authority to vote, this is the popular vote. In general, some states have traditional personal political preference; it is hard to change their mind to support different parties. So, the key point to win the campaign is to gain the support from the states without traditional personal political preference, this is why there are uncertainties during the campaign. In other way, the uncertainties are kind of good opportunities for presidential candidates, if it is used properly, the candidate will receive “positive” feedback, vice versa obtain “negative” outcome.

Figure 4.6 Cross-correlation diagram between support rates of Trump and Hillary. At the lag number \( k=0 \), the cross correlation reaches its peak. According to observation, we would like
to find out its period using method of Fourier transformation to see more closely of its property and interesting conclusion if lucky.

Using similar method as the front, we can calculate the cross-correlation coefficients between the two samples and draw the result in Figure 4.5. Series X and Y represent support rates of Trump and Clinton, respectively. Once we have function of covariance function

\[ \gamma_{XY}(\tau) = E[(X_t - \mu_X)(Y_{t+\tau} - \mu_Y)] \] (4.24)

we could calculate cross-correlation given by

\[ \rho_{XY}(\tau) = \frac{1}{\sigma_X \sigma_Y} E[(X_t - \mu_X)(Y_{t+\tau} - \mu_Y)] = \frac{1}{\sigma_X \sigma_Y} \gamma_{XY}(\tau) \] (4.25)

where we suppose the series X and Y represent support rates of Trump and Hillary, respectively, and \( \sigma_X, \sigma_Y \) are standard derivations for series X and Y. Additionally, due to the uniform distribution of data of each day’s support rate, \( E[\cdot] \) means the average, where \( E[X] = \frac{1}{N} \sum_{t=1}^{N} X_t \) for series X.

Intuitively, it is a time series with a certain regularity and periodicity, which means there is a certain correlation between the data of Trump and Clinton. Actually, it describes the degree of correlation between the random signal of Trump and Clinton at any two different moments \( t1, t2 \).

Time series can be regarded as mixture by sine waves and cosine waves of different frequencies\(^{[54]}\). With the help of Fourier transform, we can convert a function to a series of periodic functions, which transforms time series data into frequency sequence data. The periodogram is originally used to detect and estimate the amplitude of the sinusoidal component hidden in the noise whose frequency is known. Here, we will consider the possibility that the unknown periodic component is implicit in the sequence. For the time series, the Fourier series can be expanded as:

\[ x_t = \alpha_0 + \sum_{j=1}^{k} (\alpha_j c_{jt} + \beta_j s_{jt}) + e_t \] (1) (4.26)

Where \( c_{jt} = \cos(2\pi f_j t), s_{jt} = \sin(2\pi f_j t) \), \( N \) represents the number of observations, \( k \) represents the number of periodic components, \( f_j \) represents frequency and \( e_t \) the error term.
Since $\sum_{t=1}^{N} c_{it}^2 = \sum_{t=1}^{N} s_{it}^2 = \frac{N}{2}$ and all terms in (1) are mutually orthogonal at $t=1,...,N$, the LSE of $\alpha_0$ and $(\alpha_i, \beta_i)$ are

$$
\alpha_0 = \bar{x}
$$
$$
\alpha_i = \frac{2}{N} \sum_{t=1}^{N} x_t c_{it}
$$
$$
\beta_i = \frac{2}{N} \sum_{t=1}^{N} x_t s_{it}
$$

Where $i=1, 2...k$. Therefore, the periodogram consists of $k=(N-1)/2$ values:

$$
I(f_i) = \frac{N}{2} (a_i^2 + b_i^2), i = 1, 2, ... k
$$

(4.27)

Where $I(f_i)$ represents the intensity at frequency $f_i$. In this way, given a stationary sequence with $N$ observations and frequency $f$, if we can get the repeated implementation of them from the random process, a series of $a_f, b_f$ and the population $I(f)$ could be constructed. Thus, we could calculate the mean value of $I(f)$

$$
E[I(f)] = 2[E[c_0] + 2 \sum_{k=1}^{N-1} E[c_k] \cos(2\pi fk)]
$$

(4.28)

Where $c_k$ represents the estimate of covariance function and it can be proved that

$$
I(f) = 2 \left[ c_0 + 2 \sum_{k=1}^{N-1} c_k \cos(2\pi fk) \right], 0 \leq f \leq \frac{1}{2}
$$

(4.29)

Where $I(f)$ is called sample spectrum.

When $N$ is very large, it is proved that the estimate of the coefficient of the auto-covariance tends to the theoretical covariance $\gamma_k$

$$
\lim_{N \to \infty} E[c_k] = \gamma_k
$$

When $N$ tends to infinity, take the limit of (14), the power spectrum $p(f)$ is defined as follows

$$
p(f) = 2 \left[ \gamma_0 + 2 \sum_{k=1}^{N-1} \gamma_k \cos(2\pi fk) \right]
$$

(4.30)
By varying the frequency, we can get the image of power spectral and find the cycle of the time series.

First, with the help of Eviews 8, we can prove that the cross-correlation sequence is smooth.

Null Hypothesis: CORRELATION has a unit root
Exogenous: Constant
Lag Length: 2 (Automatic - based on SIC, maxlag=18)

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<th>t-Statistic</th>
<th>Prob.*</th>
</tr>
</thead>
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</tr>
<tr>
<td>Test critical values: 1% level</td>
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</tr>
<tr>
<td>10% level</td>
<td>-2.569733</td>
</tr>
</tbody>
</table>


Figure 4.7. Main figures in Periodic analysis

Then, after using software SPSS to conduct the spectral analysis, we can get the spectrum analysis chart as follows

*Figure 4.8. cross correlation frequency cycle diagram. The diagram has two significant local maximum points. Then it almost goes to zero. In order to determine the period, what we need is the peak point \( f=0.008 \).

As can be seen from the graph, the spectrum achieves its peak value at \( f=0.008 \). Hence, the true frequency of the sequence is \( f*N=0.008*513=4.104 \) and the cycle equals to \( 1/f=0.243 \).
Finally, we can calculate the cycle of the cross-correlation is $0.243 \times 513 = 124$. Since the quarter of one period is around one month, the candidates may take significant actions once a month.
4.6 Use of Twitter

Uncertainty does not suggest backwards if one could know well of its underlying information. The winning of Trump took advantage of this unpredictable property. In our modified model, the counterpart of the stochastic element is the influential strength of which Trump used Twitter. For example, he posted his views and potential acts through Twitter to gain more support among voters who are active in social network, especially youngsters. As a social method, Twitter plays an increasingly important role for candidates to approach supporters and be against competitors in U.S. president election campaign, especially for the republican Trump the businessman in real estate. If there’s anything that goes some way to explaining Trump’s popularity in the midst of his quasi-fascistic views that reached a nadir with his call to ban all Muslims from entering the United States, it is his social media prowess. Trump has more than 5.5 million Twitter followers and 4.5 million Facebook fans. It means a lot to Trump. He is also a skilled live-tweeter. He knows that live-tweeting a popular event is an opportunity to engage with a wide audience in real time. Dan Pfeiffer, Obama’s highly-regarded former digital and social media guru, has said Trump is “way better at the internet than anyone else in the GOP which is partly why he is winning.”

From the basic model, threshold $h^A$ and $h^B$ may not stay constant as we supposed in former section. It will be more interesting if we regard the threshold as a combination of a random element and a constant that are all from external world. They also variate according to time. Let the threshold for node i at time t in network A and B be

$$ h^A(t, i) = h_0 + \sum_{\alpha=1}^{N(t)} h_i^\alpha(t), $$

(4.31)

and

$$ h^B(t, i) = h_0 - \sum_{\alpha=1}^{N(t)} h_i^\alpha(t), $$

(4.32)

respectively, where N(t) is the number of tweets that Trump posted, $h_i^\alpha(t)$ randomly shows the strength of slice $\alpha$th of Twitter towards the supporting opinion of node i to network A which has opposite effect to network B and $h_0$ is the deterministic and identified constant threshold. No one can say for sure that how one tweets attract or disappoint people and then influence the original-holding opinion. So, the model becomes
\[ x^A_t(t + 1) = \left( 1 - x^B_t(t) \right) \Theta(h^A_t(t) - h^A(t, i)), \quad (4.33) \]

and

\[ x^B_t(t + 1) = \left( 1 - x^A_t(t) \right) \Theta(h^B_t(t) - h^B(t, i)), \quad (4.34) \]

By this way, the effect of the use of Twitter would emerge both in model and in reality. We will then take deeper sight of collected data trying to find out some potential relationships in two ways: observation and time series analysis.

First, direct from the observation, from 28th Oct. 2015 to 12th May 2016, Trump has posted 3200 Twitter messages, on average 16.16 per day. He Twittered 33 posts on 16th Mar. which is the most from Feb. 2016 for celebrating and appreciating of Missouri and other six states in the first selection of Trump even under the five cities on 15th Mar.

Observing Figure 4.1 and Figure 4.7 at same time, we could notice that, on Nov. 2015, the support rate of Trump was low at 33% compared to 53% from Hillary. However, that figure kept increasing rapidly till Nov. and the twitter frequency was quite high even peaked at 58 per day. Then the distribution began fluctuating within a narrow range while the post frequency also fluctuated at about 18 per day. This fluctuation lasted about 5 months till May 2016.
Figure 4.10. The figure above shows the number of tweets Trump posted from the time period from July 2016 to July 2017. The x-axis shows the time periods, and the y-axis represent the number of tweets. The data collected from Twitonomy.\textsuperscript{[59]}

Then on Sep. to Nov. 2016, which is the several months just before the U.S. election date, to save the disadvantaged position, Trump tweets far more than ever, from about 20 jumping to 60 and more, even 87 tweets on 20th Oct. 2016.

We focus on a certain special time period which is from August 2016 to November 2016, just before the general election ending. In other way to say this period is the fiercest and strongest competition during the whole campaign.
Figure 4.11 The figure above shows the number of tweets Trump had posted from 19th August 2016 to 8th November 2016. Data indicating tweets frequency are abstracted from original data poll from end of August till the election day in 2016.

Figure 4.12 numbers of tweets and the fluctuation form Trump’s support rates of the general election. Figure 4.12 above shows the relationship between numbers of tweets and the fluctuation of the general election. The orange line indicates the fluctuation of general election, along with the curve in figure 4.11, we have the fluctuation in numbers, then magnify all the numbers by certain quantity 50 to adjust figures in a proper range.

We now first look inside more for Trump. The redline represent fluctuation of supporting rate. Here, we take all absolute value to focus on the changes rather than changing directions. Above all, we can see that the effect of tweets sometimes negative, sometimes positive, especially the trend of fluctuation during the time in October, the supporting is getting down, meantime by post large number of tweets, the supporting increase in end of October. This is indicating that the twitter, the social network has significant influence during the campaign; even though not always positive results, but it is breaking the traditional campaign machine.

Intuitively, we will also take sight to the potential effect on Trump’s competitor. Figure 4.13 gives the corresponding information from Clinton’s data pool. It is important to compare Twitter’s, an innovative tool, impacts on both candidates. In later this section, same analysis method will be used on two situations, which making the comparison more direct.
Figure 4.13 numbers of tweets and the fluctuation form Clinton’s support rates of the general election. Figure 4.13 above shows the relationship between numbers of tweets and the fluctuation of the general election. The grey line indicates the fluctuation of general election, along with the curve in figure 4.11, we have the fluctuation in numbers, then magnify all the numbers by certain quantity 50 to adjust figures in a proper range.

Using method of time series analysis and Fourier spectral period analysis, we obtain the cross-correlation and cross correlation frequency cycle diagrams (Figure 4.14, Figure 4.15, respectively).
Figure 4.14 Cross-Correlations between Difference and tweets daily for Trump(upper) and Clinton(Lower), respectively. Both graph show pieces of mess. There may be not much good conclusions to have till now. Surprising findings will be showed in the following step.

Figure 4.15 cross correlation frequency cycle diagram daily for Trump(Left) and Clinton(Right). Frequency cycle diagrams is getting from spectral analysis to find out the period through cross correlation and Fourier function transformation. In the left graph, there is only one peak point that much taller than others. While in the right graph, there are two. However, in our analysis, only peak matters. These two curves gave us one same peak point $f=0.2$.

As can be seen from the graphs, the spectrum achieves its peak value at $f=0.2$ in each graph. Hence, the periods for two would be $1/f=5$. Things change every one and half days, which indicates a quite rapid reaction to twitter posting. However, those two graphs do not show any significant difference in terms of period, since the periods we obtain are the same. As the periods are the same, in other way to say, the fluctuation change in the same time. This indicate both of them are effected during Trump post the tweets.
Figure 4.16. Distribution of support rate differences along corresponding to the number of tweets per day. (Trump: upper; Clinton: lower). First, points (x=tweets per day, y=support rate) from 19th August 2016 to 8th November 2016 for Trump and Clinton are plotted in coordination respectively. Then linear regression is done on both with regression function $y = 0.573x + 7.7953$ and $y = 0.809x + 4.7306$. If comparing the multipliers of x, we could find whether tweets influenced Trump more or Clinton more.

Linear regression analysis may tell something. Obviously, 0.573 is less than 0.809. We could say that there are more effects on Clinton than Trump through Trump’s twitter. In other words, using twitter is not only a boost of Trump’s view and opinion, but also an impair to Clinton’s.
We should say that Donald Trump try a new way to use the opportunity of uncertainty. It is Twitter, a social networking and news service platform. Every Twitter register user can post message, so do Donald Trump, he post many messages during the campaign. These messages represent what Donald Trump is real would like to share, and also he become more and more popular during the campaign.

People always pay more attention to new things, which with loads of passion, enthusiasm and actively. Donald Trump did well in this point; share more about him in various ways. Putting it differently, because of his high exposure rate, most people of unconcern about political know Donald Trump is in the campaign. When this group of people had been asked to vote, Donald Trump is the first candidate pop-up in their mind. Stand to reason that Donald Trump received more support from the group people of unconcern in politics.
4.7 Social media and political campaigning

How social medias can be weaponised for political gain? Recently, an incident between social media Facebook and political consulting firm Cambridge Analytica appeared in public view. It is an example shows how the social medias influence our opinions.

Facebook is a well-known, most popular social media in our daily lives, it provides a kind of platform for you to share yourself and communicate with your friends, friends of your friend and even with the perfect stranger. When you sign up a Facebook account on web or downloaded the Facebook application on your devices (e.g. mobile phone, computer etc.), you can upload your own details, for examples photo, video, personal status and even your contact list. Also, you may see others’ details. These activities create a social network, you can share whatever you like, meanwhile you have the opportunity to leave comment to compliment and judge others. By collecting these behaviours within the social networks, can create personality profiles. Next, through psychological research based on the personality profiles, can infer many personal details. For example; age, ethnicity, gender, parental separation, sexual orientation, religious views and political preference.

Cambridge Analytica currently are accused of using techniques like personal details as their research basis. The company harvested information from millions of US Facebook users. The company offer kinds of personality prediction applications, once you downloaded to your devices and use it, the application will access their Facebook profile data. After collected the data and analysed, the voters could be separate to several groups with different opinion, the swing voters will be identified and well look after, then the special precise and targeted political messages will be delivered to the swing group of people.
4.8 Conclusion

In this chapter, the coupled opinion network has been developed from traditional uncoupled network, which pervasively known as McCulloch-Pitts neural model or linear threshold gate. The main analysis, which based on the model with linear threshold, shows the mechanism of opinion spread and the dependence on the ratio between the initial figures of two networks. To illustrate this point in case of 2016 U.S. president election, Figure 4.4 shows that the ration always lies between 0.8 to 1.0 from which we could give no solid prediction on result of the election. Supporting rate distribution in figure 4.3 clearly describes little difference between votes to Trump and to Clinton at the very end of the general election.

For insider look at the rates’ distribution, time series analysis plays an important role. Autocorrelation and cross-correlation imply the possible underlying period. Using Fourier transformation function spectral analysis on the cross-correlation, the period is four-month length in which things seem to change every month.

Various factors impact the public’s opinion every moment in which they expose themselves to abundant information world. Nowadays, the fast and the most efficient meanings to spread views and communicate with others are via the internet. The U.S. president candidate Donald Trump took advantage of Twitter, one of the most popular online news and social networking service where users post and interact with messages, called "tweets." As The New York Times said, Election Day was a reminder of Twitter’s influence in media and the distribution of information. In America, the immediacy and speed of Twitter is unmatched by any other network such as Facebook who reaped the benefit of news breaking on Twitter.
Figure 4.17 Direct (A) and indirect (B) reciprocity between two competitors roughly shows above. Situation A describes a traditional way that Trump and Clinton (or two networks) interact. In situation B, with the appearance of Twitter, Trump posted enough tweets so that he can obtain benefit from such behaviour. At meantime, the increasing of Trump’s support rate somehow implies decreasing support towards Clinton to some extent.

Figure 4.18 shows a recent Tweet Trump had posted, the background is The UK Prime Minister Theresa May has condemned President Donald Trump’s decision to retweet videos posted by Jayda Fransen, leader of far-right group Britain First. Then Trump tells Theresa May to focus on terrorism, not him.[62]

Figure 4.17 uses simple graphical representation of direct and indirect mutual promotion or interference. Figure 18 shows a typical example of how Trump use Twitter in politics. To see how strong Twitter is as the main tool for Trump’s success in the election through mathematical language, we first modified the model by adding the stochastic term, indicating the effect on opinion change and decision making, into the external constant threshold. Put differently from previous comparison method, we now compare the Twitter’s potential
influence on Trump and Clinton in that only examining effect on Trump’s support is not sufficient to show the extent. To find out whether Twitter influence more on Trump or on Clinton, we do some data analysis work. After correlation analysis, the correlation coefficient between Trump’s support absolute fluctuation and tweets number and coefficient between Clinton’s support absolute fluctuation and tweets number are surprisingly exact the same. Still surprisingly, after Fourier transformation spectral analysis, both periods are 5 days which is a considerably small period compared with 4 months in former case. It again emphasizes how quick Twitter can be. The difference emerges in linear regression analysis where the number of tweets affects more on Clinton than on Trump. Trump’s first appearance on Twitter quickly began fermenting online. The use of Twitter has not only bringing Trump what is beyond his wishes, but also draw Clinton backward in the presidential election.
Conclusion

A whole range of phenomena such as happiness and depression, obesity, drinking and smoking habits, ill-health, the inclination to turn out and vote in elections, a taste for certain music or food, a preference for online privacy, even the tendency to attempt or think about suicide are transmitted through networks of friends. The ways how this is transmitted has psychological routes and are not entirely understood. One thing is clear that the information about mood, habits and other stuff we or our friends or friends of friends have propagated through the network like electricity through a power network.

There have been many theoretical models of social networks. These networks contain interactive units or agents that are often associated with different social behaviours. These behaviours may vary in different degrees, of which the distribution needs to be considered. Each of these units works as a variable in the Ising model. And there are only two possible states regarding the value of the variable, which are the Yes or No option for an opinion. There has been much scholarly attention on the network dynamics, which are the result of the interaction of the many units in the many-body system.

At the beginning of the thesis, I have presented the effectiveness of the cumulative method in the description of the unpredictable chaotic behaviours of human individuals. Nevertheless, some drawbacks emerge with the method as well.

Then, the agent-based model is built up on an open network of people and describing their daily monetary activities including investment, daily consumption and salary-gaining. The more the people invest, the more the income they will receive. Nevertheless, no investment is riskless. If people use most of their wealth for investment, the result will not be shown the same as in the simulation results. The final state would be that most people keep small part of wealth and a minority of people own a large amount of wealth.

Then agent-based model has also been applied to construct model of global economy in the investigation of the modest revenue and consumption model with leverage. We consider the deterministic two-country situation, we saw a loom of the influence to the interest rates and consumptions from the extent of the debt. Then the model has been developed by adding a stochastic term. It also expanded from two countries to multiple countries. In the situation in which one country becomes dominant in a group of economies and effectively acts as a hub
(lending/exporting to all others) or sink (borrowing/importing from all others) have a greater influence on the global interest rate proportional to the number of countries in the group. Qualitatively, these results are more difficult to justify.

Chapter 3 gave an in-depth mathematical analysis of the coupled network and illustrated neuron behavior modeled by this kind of network based on Ising model. By such method, one can characterize various neural conditions or even other chaotic complex systems. Although we have studied the neuron excitations in this Chapter, this model can be used to show how the social viruses are propagating through the social network and became dominant.

At last, the opinion network in voting has also been modelled as the coupled networks. Here, it focused more on information spreading and influence on opinion changing. As a case study, we took a closer sight into the Trump-Hillary president election in 2016. Unlike the traditional election, Trump is crazy about twittering. He twitted to clarify his points of view and comment on others’ posts. On the one hand, Trump has figured out a relatively efficient way to let himself famous among the public. Of course, he would like to gain more support in the general election. To some extent, taking advantage of Twitter put an obstacle in the way of Hillary’s success. However, on the other hand, this behaviour may enhance the extant of uncertainty in such a chaotic system. No one would have any confidence to see any predictions.

Nowadays, increasing amount of people can afford the internet fees and TV license and get access to digital mass communications. Persuasive communication is prevalent in many different contexts such as governments, companies, and political parties. The use of Twitter or other mass communication tools in the presidential election is not purely spreading the information or views. This behaviour is somehow persuasive. The goal of this kind of communication is to encourage the audience to believe and act on the communicators’ point of view. A recent finding suggests that an effective approach called psychological targeting makes it possible to actively affect the behaviours of large groups of people by psychological assessment of their digital footprint.\[^{68}\] If the candidate put the psychological targeting method in practice through policy interventions in the election campaign, he/she is likely to win. The technique of persuasion, if better understood by the audience, may contrariwise help the audience better assess the media contents that they are consuming. This will in some ways affect the election result. It seems to add some certainty to the uncertainty.
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Appendix


ISING MODEL - AN ANALYSIS, FROM OPINIONS TO NEURONAL STATES;  Author(s): Kusmartsev, Vassili F.; Zhang Wu; Kusmartseva, Anna; Balanov, A.; Janson, N.; Kusmartsev, F. V.


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