Seismic response analysis of linear and nonlinear secondary structures

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Seismic Response Analysis of Linear and Nonlinear Secondary Structures

by

Stavros Kasinos

A Doctoral Thesis
Submitted in partial fulfilment of the requirements for the award of
Doctor of Philosophy of Loughborough University

March 2018
To my Parents
Declaration

This thesis presents part of the research conducted at the School of Architecture, Building and Civil Engineering of Loughborough University between October 2013 to June 2015 and January 2016 to March 2018. It is the result of my own work and it does not include any outcome of work that has been the result of collaboration, unless stated otherwise. The thesis approximately contains 68,000 words, 94 figures and 140 references.

S. Kasinos
March, 2018
Understanding the complex dynamics that underpin the response of structures in the occurrence of earthquakes is of paramount importance in ensuring community resilience. The operational continuity of structures is influenced by the performance of nonstructural components, also known as secondary structures. Inherent vulnerability characteristics, nonlinearities and uncertainties in their properties or in the excitation pose challenges that render their response determination as a non-straightforward task. This dissertation settles in the context of mathematical modelling and response quantification of seismically driven secondary systems.

The case of bilinear hysteretic, rigid-plastic and free-standing rocking oscillators is first considered, as a representative class of secondary systems of distinct behaviour excited at a single point in the primary structure. The equations governing their full dynamic interaction with linear primary oscillators are derived with the purpose of assessing the appropriateness of simplified analysis methods where the secondary-primary feedback action is not accounted for. Analyses carried out in presence of pulse-type excitation have shown that the cascade approximation can be considered satisfactory for bilinear systems provided the secondary-primary mass ratio is adequately low and the system does not approach resonance. For the case of sliding and rocking systems, much lighter secondary systems need to be considered if the cascade analysis is to be adopted, with the validity of the approximation dictated by the selection of the input parameters.

Based on the premise that decoupling is permitted, new analytical solutions are derived for the pulse driven nonlinear oscillators considered, conveniently expressing the seismic response.
as a function of the input parameters and the relative effects are quantified. An efficient numerical scheme for a general-type of excitation is also presented and is used in conjunction with an existing nonstationary stochastic far-field ground motion model to determine the seismic response spectra for the secondary oscillators at given site and earthquake characteristics.

Prompted by the presence of uncertainty in the primary structure, and in line with the classical modal analysis, a novel approach for directly characterising uncertainty in the modal shapes, frequencies and damping ratios of the primary structure is proposed. A procedure is then presented for the identification of the model parameters and demonstrated with an application to linear steel frames with uncertain semi-rigid connections. It is shown that the proposed approach reduces the number of the uncertain input parameters and the size of the dynamic problem, and is thus particularly appealing for the stochastic assessment of existing structural systems, where partial modal information is available e.g. through operational modal analysis testing. Through a numerical example, the relative effect of stochasticity in a bi-directional seismic input is found to have a more prominent role on the nonlinear response of secondary oscillators when compared to the uncertainty in the primary structure.

Further extending the analyses to the case of multi-attached linear secondary systems driven by deterministic seismic excitation, a convenient variant of the component-mode synthesis method is presented, whereby the primary-secondary dynamic interaction is accounted for through the modes of vibration of the two components. The problem of selecting the vibrational modes to be retained in analysis is then addressed for the case of secondary structures, which may possess numerous low frequency modes with negligible mass, and a modal correction method is adopted in view of the application for seismic analysis. The influence of various approaches to build the viscous damping matrix of the primary-secondary assembly is also investigated, and a novel technique based on modal damping superposition is proposed. Numerical applications are demonstrated through a piping secondary system multi-connected on a primary frame exhibiting various irregularities in plan and elevation, as well as a multi-connected flexible secondary system.

Overall, this PhD thesis delivers new insights into the determination and understanding of the response of seismically driven secondary structures. The research is deemed to be of academic and professional engineering interest spanning several areas including seismic engineering, extreme events, structural health monitoring, risk mitigation and reliability analysis.
Abstract

Keywords: Analytical dynamics; bilinear hysteretic systems; earthquake engineering; nonlinear oscillators; nonstationary random process; nonstructural components; pulse; rigid-plastic; rocking block; secondary structures; seismic response; sliding; stochastic models.
I would like to express my gratitude and sincere appreciation to my academic supervisors Dr. Alessandro Palmeri and Dr. Mariateresa Lombardo for their advice, technical guidance and mentoring throughout the period of my research work at Loughborough. I am grateful for attending the French-German summer school on Modelling and Numerical Methods for Uncertainty Quantification in Porquerolles in 2014, the 12th International Conference on Applications of Statistics and Probability in Civil Engineering in 2015 and the International Workshop on Seismic Loss, Rehabilitation and Post-earthquake Crisis Management of Critical Infrastructure in 2017. I am also thankful for the intellectual challenges offered and experiences in contributing to six undergraduate final year projects, supervising a visiting undergraduate student, working as a Research Assistant and delivering the lectures of the Finite Element Structural Analysis module.

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Contents

Declaration i

Abstract iii

Acknowledgments vii

Contents ix

List of Figures xv

List of Tables xxv

Nomenclature xxvii

1 Introduction 1
  1.1 Aim and Objectives . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
  1.2 Outline of the Thesis . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5

2 Literature Review 9
  2.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
  2.2 Seismic Hazard Models . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10
  2.3 Analysis of Secondary Structures . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
    2.3.1 Modelling . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
## Contents

2.3.2 Analysis Methods .......................................................... 11
  2.3.2.1 Combined P-S System ............................................... 11
  2.3.2.2 Cascade Approximation ........................................... 12
  2.3.2.3 Component-Mode Synthesis Method .............................. 13

2.3.3 Nonlinear SDoF Oscillators .............................................. 13

2.4 Modal Correction Methods .................................................. 15

2.5 Damping Characterisation ................................................... 17

2.6 Irregularities ........................................................................ 17

2.7 System Uncertainty of the Primary Structure ......................... 19
  2.7.1 Uncertainty in the Partial Rigidity of Connections in P ....... 19
  2.7.2 Effect of Uncertainty in P Quantified on S ....................... 19

2.8 Summary ............................................................................. 20

3 Combined Vibration and Decoupling of Nonlinear Secondary Oscillators 23
  3.1 Introduction ...................................................................... 23
  3.2 Combined Vibration Response of 2DoF Linear-Nonlinear System .............................................. 24
    3.2.1 Bilinear S - Linear P .................................................. 24
    3.2.2 Sliding S - Linear P .................................................. 27
    3.2.3 Rocking S - Linear P ................................................ 28
    3.2.4 Implementation details .............................................. 31
  3.3 Nonlinear Secondary Oscillators in Cascade ......................... 32
  3.4 Decoupling Criteria ............................................................ 33
    3.4.1 Sinusoidal Pulse Ground Acceleration ......................... 33
    3.4.2 Dimensional Analysis ............................................... 34
    3.4.3 Numerical Application ............................................. 36
      3.4.3.1 Bilinear Secondary Oscillator ............................ 37
      3.4.3.2 Sliding Secondary Oscillator ............................ 39
      3.4.3.3 Rocking Secondary Block ................................. 42
  3.5 Summary ............................................................................. 46

4 Pulse-driven Nonlinear Secondary Oscillators in Cascade 51
  4.1 Introduction ...................................................................... 51
  4.2 Combined Vibration Response of 2DoF Linear System ............. 52
    4.2.1 Complex Modal Analysis .......................................... 54
    4.2.2 Closed-Form Solution ............................................. 55
5 Stochastically Excited Secondary Oscillators in Cascade 91

5.1 Introduction .......................... 91

5.2 Stochastic Ground Motion Model ......................... 92
  5.2.1 Model Formulation .......................... 92
  5.2.2 Predictive Equations ......................... 95
  5.2.3 Ground Motion Simulation ..................... 97
  5.2.4 Response of Secondary Oscillators ............. 99

5.3 Numerical Application .......................... 99
  5.3.1 Simulated Motions .......................... 100
  5.3.2 Primary System .......................... 100
  5.3.3 Bilinear Secondary Oscillator ................. 104
  5.3.4 Sliding Secondary Block ..................... 108
<table>
<thead>
<tr>
<th>5.3.5 Rocking Secondary Block</th>
<th>114</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4 Summary</td>
<td>120</td>
</tr>
</tbody>
</table>

### 6 Uncertainty Characterisation in the Properties of Linear MDoF Systems 125

6.1 Introduction 125

6.2 Governing Equations 126

6.2.1 Principal Axes 126

6.2.2 Linear Primary System 127

6.2.3 Stochastic Ground Motion Model 128

6.3 Uncertainty in the Modal Space 129

6.3.1 Effect on the Geometric Space 132

6.3.2 Identification of Model Parameters 133

6.3.2.1 Parametric Analysis 135

6.3.2.2 Stochastic Analysis 135

6.4 Response of Secondary Oscillators 137

6.4.1 Model and Engineering Demand Parameters (EDP) 139

6.4.2 Numerical Analyses 140

6.4.2.1 Simulated Motions 142

6.4.2.2 Primary System 142

6.4.2.3 Secondary Oscillators 144

6.5 Summary 149

### 7 Component-Mode Synthesis for Linear MDoF Secondary Structures 153

7.1 Introduction 153

7.2 Combined Vibration via Component-Mode Synthesis 154

7.2.1 Undamped Vibration 154

7.2.2 Modal Coordinate Transformation 155

7.3 Criteria on the Number of Vibrational Modes 157

7.4 Modal Correction Methods 157

7.5 Construction of the Viscous Damping Matrix 158

7.5.1 Proportional Damping 159

7.5.2 Generalisation of Proportional Damping 160

7.5.3 Modal Damping 161

7.6 Numerical Examples 163

7.6.1 Example 1: Piping System in Irregular Building 163
3.1 Bilinear (a), sliding (d) and free-standing rocking (g) secondary oscillators on linear SDoF primary oscillator; corresponding free-body diagrams (b, e, h); force-displacement (c, f) and moment-rotation relationships. ................. 25

3.2 Kinetic energy ratio variation with slenderness $\alpha$ and mass ratio $\gamma$. .................. 31

3.3 Ground acceleration histories of a single-cycle pulse; $\phi = 0$ (a) and $\phi = \pi/2$ (b). 34

3.4 Acceleration, velocity and displacement histories of the 1994 Northridge earthquake recorded at the Rinaldi station and single-cycle pulse approximations; $a_g = \frac{7}{8}\omega_g, \omega_g = \frac{5}{2}\pi, \phi = 0$ (a, c, e), and $a_g = 1.3\omega_g, \omega_g = \frac{20}{13}\pi, \phi = \frac{\pi}{2}$ (b, d, f). 35

3.5 Effect of the S-P parameter $\gamma$ on the absolute acceleration response of P (left), and the displacement response of the bilinear S (right): $\omega_g^s = 0.14, \omega_s^s = 0.3$ (a, b); $\omega_g^s = 1, \omega_s^s = 0.3$ (c, d); $\omega_g^s = 0.14, \omega_s^s = 1$ (e, f) and $\omega_g^s = 1, \omega_s^s = 1$ (g, h). .................................................. 40

3.6 Effect of the S-P mass ratio parameter $\gamma$, the dimensionless circular frequency $\omega_s^s$, the pulse frequency $\omega_g^s$, the specific strength $a_s^s$, and the post-yield to pre-yield stiffness ratio $\psi_s$ on the absolute acceleration response of P (a, c, e, g) and the displacement response of the bilinear S (b, d, f, h), respectively, in the form of region plots corresponding to 5%, 10%, 15% and 20% cumulative error. Reference values of $\omega_g^s = 0.14, \omega_s^s = 0.3, a_s^s = 1, \psi_s = 0, \zeta_s = 0.02$ and $\zeta_p = 0.05$ are assumed. ......................... 41
3.7 Effect of the S-P mass ratio parameter $\gamma$, and the damping ratios $\zeta_s$ and $\zeta_p$ on the absolute acceleration response of P (a, c) and the displacement response of the bilinear S (b, d), respectively, in the form of region plots corresponding to 5%, 10%, 15% and 20% cumulative error. Reference values of $\omega^*_g = 0.14$, $\omega_s^* = 0.3$, $a_s^* = 1$, $\psi_s = 0$, $\zeta_s = 0.02$ and $\zeta_p = 0.05$ are assumed. .................. 42

3.8 Effect of the S-P parameter $\gamma$ on the absolute acceleration response of P (left), and the displacement response of the sliding S (right): $\omega^*_g = 0.14$ (a, b) and $\omega^*_g = 1$ (c, d). .................. 43

3.9 Effect of the S-P mass ratio parameter $\gamma$, the dimensionless pulse frequency $\omega^*_g$, the specific strength $a^*_s$ and the damping ratio $\zeta_p$ on the absolute acceleration response of P (a, c, e) and the displacement response of the sliding S (b, d, f), respectively, in the form of region plots corresponding to 5%, 10%, 15% and 20% cumulative error. Reference values of $\omega^*_g = 0.14$, $a^*_s = 0.5$ and $\zeta_p = 0.05$ are assumed. .................. 44

3.10 Effect of the S-P parameter $\gamma$ on the absolute acceleration response of P (left), and the displacement response of the rocking S (right): $\omega^*_g = 0.14$ (a, b) and $\omega^*_g = 1$ (c, d). .................. 46

3.11 Effect of the S-P mass ratio parameter $\gamma$, the slenderness $\alpha$ and the dynamic parameter $p^*$ on the absolute acceleration response of P (a, c) and the rotation response of the rocking S (b, d), respectively, in the form of region plots corresponding to 5%, 10%, 15% and 20% cumulative error. Reference values of $\omega^*_g = 0.14$, $\zeta_p = 0.05$, $\alpha = 0.25$, $p^* = 0.136$ and $a^*_g = 0.28$ are assumed. .................. 47

3.12 Effect of the S-P mass ratio parameter $\gamma$, the slenderness $\alpha$ and the dynamic parameter $p^*$ on the absolute acceleration response of P (a, c) and the rotation response of the rocking S (b, d), respectively, in the form of region plots corresponding to 5%, 10%, 15% and 20% cumulative error. Reference values of $\omega^*_g = 0.14$, $\zeta_p = 0.05$, $\alpha = 0.25$, $p^* = 0.136$ and $a^*_g = 0.28$ are assumed. .................. 48

4.1 Normalised modal shape. .................. 53

4.2 Bilinear (a), sliding (c) and free-standing rocking (e) secondary oscillators in cascade and corresponding force-displacement (b, d) and moment-rotation relationship (f). .................. 58

4.3 Error due to linearisation of the rocking equations. .................. 59

4.4 Combined primary-secondary linear structure. .................. 67
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
<td>Comparison of the geometric and modal solution for the combined system. Absolute acceleration response quantified on the primary system (a, c, e) and displacement response quantified on the linear secondary oscillator (b, d, f).</td>
</tr>
<tr>
<td>4.6</td>
<td>Cumulative errors in the response of P (a) and S (b) due to the modal solution. Reference values of $a = 1$, $\varphi = 1$, $\zeta_p = 0.05$, $\zeta_a = 0.02$, $\phi = 0$ and $\beta = 1.3$ are assumed.</td>
</tr>
<tr>
<td>4.7</td>
<td>Comparison of the combined and cascade solution. Absolute acceleration response quantified on the primary system (a, c, e) and displacement response quantified on the linear secondary oscillator (b, d, f).</td>
</tr>
<tr>
<td>4.8</td>
<td>Cumulative errors in the response of P and S due to the cascade approximation; $a = 0.1$ (a, b) and $\omega^*_s = 1$ (c, d). Reference values of $\varphi = 1$, $\zeta_p = 0.05$, $\zeta_a = 0.02$, $\phi = 0$ and $\beta = 1.3$ are assumed.</td>
</tr>
<tr>
<td>4.9</td>
<td>Analytical and numerical evaluation of the displacement and absolute acceleration response for the primary structure due to a sine (a, c) and cosine (b, d) pulse, respectively.</td>
</tr>
<tr>
<td>4.10</td>
<td>Analytical and numerical evaluation of the displacement and velocity response for the bilinear secondary oscillator due to a sine (a, c) and cosine (b, d) pulse, respectively.</td>
</tr>
<tr>
<td>4.11</td>
<td>Analytical and numerical evaluation of the displacement and velocity response for the sliding secondary oscillator due to a sine (a, c) and cosine (b, d) pulse, respectively.</td>
</tr>
<tr>
<td>4.12</td>
<td>Analytical and numerical evaluation of the rotation and angular velocity response for the rocking secondary oscillator and corresponding energy plots due to a sine (a, c, e) and cosine (b, d, f) pulse, respectively.</td>
</tr>
<tr>
<td>4.13</td>
<td>Absolute acceleration time histories and effect of the dimensionless modal coordinate $\varphi$ and damping ratio $\zeta_p$ on the peak response of the primary structure, due to a sine (a, c, e) and cosine (b, d, f) pulse, respectively.</td>
</tr>
<tr>
<td>4.14</td>
<td>Effect of the dimensionless frequency $\omega_s^<em>$ and specific strength $a_s^</em>$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c) and cosine (b, d) pulse, respectively.</td>
</tr>
<tr>
<td>4.15</td>
<td>Effect of the dimensionless frequency $\omega_s^*$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively.</td>
</tr>
</tbody>
</table>
4.16 Effect of the damping ratio $\zeta_s$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively. ............................................. 81

4.17 Effect of the post-yield to pre-yield stiffness ratio $\psi_s$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e) and cosine (b, d, f) pulse, respectively. ................................. 82

4.18 Effect of the dimensionless modal coordinate $\varphi$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively. ......................... 84

4.19 Effect of the damping ratio $\zeta_p$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively. .......................... 85

4.20 Effect of the specific strength $a^*_{gs}$, the dimensionless modal coordinate $\varphi$ and the damping ratio $\zeta_p$ on the peak displacement response of the sliding secondary oscillator, due to a sine (a, c, e) and cosine (b, d, f) pulse, respectively. ......................... 86

4.21 Effect of the parameter $a^*_{gs}$, the slenderness $\alpha$, the dynamic parameter $p$ and the coefficient of restitution $\varepsilon$ on the peak rotation response of the rocking secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively. .................................................. 88

4.22 Effect of the dimensionless modal coordinate $\varphi$ and the damping ratio $\zeta_p$ on the peak rotation response of the rocking secondary oscillator, due to a sine (a, c) and cosine (b, d) pulse, respectively. ................................. 89

5.1 Procedure for the generation of a ground motion realisation (adapted from [132]). 93

5.2 Filter response (a) and gamma modulating function (b) for selected parameters. 94

5.3 Comparison of 5% damped elastic response spectra of 50 synthetic motions and the 1994 Northridge earthquake ($F = 1, M = 6.69, R = 20.3 \text{ km}, V_s = 1223 \text{ m/s}$) recorded at the LA Wonderland Ave station (a), and the 1999 Chi-Chi, Taiwan earthquake ($F = 1, M = 7.62, R = 42.5 \text{ km}, V_s = 643 \text{ m/s}$) recorded at the HW A038 station (b). .................................................. 98
5.4 Comparison of logarithmic median ± one standard deviation of 5% damped elastic response spectra of 500 synthetic motions with corresponding spectra from the NGA prediction models of Campbell-Bozorgnia (CB), Abrahamson-Silva (AS) Chiou-Youngs (CY) and Boore-Atkinson (BA). Strike-slip faulting ($F = 0$) and $V_s = 760$ m/s are assumed. $M = 7.0$, $R = 40$ km (a), $M = 8.0$, $R = 20$ km (b). .......................................................... 99

5.5 Comparison of logarithmic median ± one standard deviation of 5%-damping elastic response spectra of 1000 synthetic motions with corresponding spectra from the NGA prediction models of Campbell-Bozorgnia (CB), Abrahamson-Silva (AS) Chiou-Youngs (CY) and Boore-Atkinson (BA). Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M$ : {6, 6.5, 7, 7.5, 8} and $R$ : {10, 30, 50} km are assumed. .......................................................... 101

5.6 Effect of the damping ratio $\zeta_p$ (a) and the modal coordinate $\varphi$ (b) on the coefficient of variation of the absolute acceleration response of P. ..................... 102

5.7 Effect of the damping ratio $\zeta_p$ on the empirical CDF of the peak absolute acceleration spectra of the linear P due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M$ : {6, 6.5, 7, 7.5, 8} and $R$ : {10, 30, 50} km are assumed. .......................................................... 103

5.8 Effect of the modal coordinate $\varphi$ on the empirical CDF of the peak absolute acceleration spectra of the linear P due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M$ : {6, 6.5, 7, 7.5, 8} and $R$ : {10, 30, 50} km are assumed. .......................................................... 105

5.9 Effect of the ductility ratio $\mu_d$ on the empirical CDF of the peak displacement spectra of the bilinear S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M$ : {6, 6.5, 7, 7.5, 8} and $R$ : {10, 30, 50} km are assumed. .......................................................... 106

5.10 Effect of the circular frequency $\omega^*_s$ on the empirical CDF of the peak displacement spectra of the bilinear S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M$ : {6, 6.5, 7, 7.5, 8} and $R$ : {10, 30, 50} km are assumed. .......................................................... 107

5.11 Effect of the post-yield to pre-yield stiffness ratio $\psi_b$ on the empirical CDF of the peak displacement spectra of the bilinear S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M$ : {6, 6.5, 7, 7.5, 8} and $R$ : {10, 30, 50} km are assumed. .......................................................... 109
5.12 Effect of the damping ratio $\zeta_s$ on the empirical CDF of the peak displacement spectra of the bilinear $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\} \text{ km}$ are assumed. ................................. 110

5.13 Effect of the damping ratio $\zeta_p$ on the empirical CDF of the peak displacement spectra of the bilinear $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\} \text{ km}$ are assumed. ................................. 111

5.14 Effect of the modal coordinate $\varphi$ on the empirical CDF of the peak displacement spectra of the bilinear $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\} \text{ km}$ are assumed. ................................. 112

5.15 Effect of the specific strength $a_s$ on the empirical CDF of the peak displacement spectra of the sliding $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\} \text{ km}$ are assumed. ................................. 113

5.16 Effect of the damping ratio $\zeta_p$ on the empirical CDF of the peak displacement spectra of the sliding $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\} \text{ km}$ are assumed. ................................. 115

5.17 Effect of the modal coordinate $\varphi$ on the empirical CDF of the peak displacement spectra of the sliding $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\} \text{ km}$ are assumed. ................................. 116

5.18 Effect of the slenderness $\alpha$ on the empirical CDF of the peak rotation spectra of the rocking $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\} \text{ km}$ are assumed. ................................. 117

5.19 Effect of the dynamic parameter $p$ on the empirical CDF of the peak rotation spectra of the rocking $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\} \text{ km}$ are assumed. ................................. 118
5.20 Effect of the restitution coefficient $\varepsilon$ on the empirical CDF of the peak rotation spectra of the rocking $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed. ............................................. 119

5.21 Effect of the damping ratio $\zeta_p$ on the empirical CDF of the peak rotation spectra of the rocking $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed. ............................................. 121

5.22 Effect of the modal coordinate $\varphi$ on the empirical CDF of the peak rotation spectra of the rocking $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed. ............................................. 122

6.1 Rotation of orthogonal horizontal components. ..................................................... 126

6.2 Uncertainty in the first mode. ................................................................. 129

6.3 Beam element with rotational springs (adapted from [113]) (a) and structural frame model (b). ................................................................. 134

6.4 Acceleration time history of the Imperial Valley 1940 earthquake (a) and Fourier Amplitude spectrum (b). ................................................................. 135

6.5 Influence of connection flexibility on the circular frequencies (a), mode shapes (b) and lateral dynamic response envelope (c). ........................................ 136

6.6 Influence of connection flexibility on the statistics of $\alpha$ for correlated, uncorrelated cases (top, bottom) and input $\text{CoV} = 0.2, 0.3$ (left, middle), respectively; distribution of $\alpha_{21}$ at $\text{CoV} = 0.2$ (right). ........................................ 137

6.7 Influence of connection flexibility on the statistics of $\beta$ for correlated, uncorrelated cases (top, bottom) and input $\text{CoV} = 0.2, 0.3$ (left, middle), respectively; distribution of $\beta_{11}$ at $\text{CoV} = 0.2$ (right). ........................................ 138

6.8 Influence of connection flexibility on the dynamic envelope for correlated, uncorrelated cases (top, bottom) and input $\text{CoV} = 0.2, 0.3$ (left, middle), respectively; response history at $v = 0.5$ (right). ........................................ 139

6.9 Structural frame model (a) and correlation coefficient of orthogonal components (b). ................................................................. 140

6.10 Acceleration ground motion components of the Northridge 1994 earthquake recorded at Burbank–Howard Rd. station (a, c) and corresponding Fourier Amplitude spectra (b, d). ........................................ 141
6.11 Comparison between 5% damped elastic response spectra of recorded and simulated major (a) and intermediate (b) orthogonal components for Northridge 1994 earthquake. .......................... 142

6.12 Randomised first modal shape (a) and spectral matrix (b); absolute acceleration response histories of the primary structure at the attachment point due to uncertainty in the modal properties, excluding (c) and including (d) the effect of uncertainty in the ground acceleration. .......................... 143

6.13 Comparison of empirical CDF for the displacement (a) and absolute acceleration (b) EDPs for $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = 0$ (solid), $\sigma_\alpha = 0.04, \sigma_\beta = 0.07, \sigma_\gamma = 0.1$ (dotted) and $\sigma_\alpha = 0.08, \sigma_\beta = 0.14, \sigma_\gamma = 0.2$ (dashed). .......................... 144

6.14 Spectra of linear secondary system due to uncertainty in the ground and $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = 0$ (left), $\sigma_\alpha = 0.04, \sigma_\beta = 0.07, \sigma_\gamma = 0.1$ (right): influence of $\zeta_s$ and $T_s$ on the expected EDP (a, b); $\zeta_s = 0.02$ (c, d) and $\zeta_s = 0.05$ (e, f). .......................... 145

6.15 Spectra of bilinear secondary system due to uncertainty in the ground, and $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = 0$ (left), $\sigma_\alpha = 0.04, \sigma_\beta = 0.07, \sigma_\gamma = 0.1$ (right): $\psi_s = 0$, $\mu_d = 5$ (a, b), $\psi_s = 0$, $\mu_d = 8$ (c, d) and $\psi_s = 0.2$, $\mu_d = 5$ (e, f). .......................... 146

6.16 Spectra of sliding secondary system due to uncertainty in the ground, and $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = 0$ (a), $\sigma_\alpha = 0.04, \sigma_\beta = 0.07, \sigma_\gamma = 0.1$ (b). .......................... 147

6.17 Spectra of rocking secondary system due to uncertainty in the ground, and $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = 0$ (left), $\sigma_\alpha = 0.04, \sigma_\beta = 0.07, \sigma_\gamma = 0.1$ (right): $p = 1$ and $\varepsilon = 0.3$ (a, b) and $\varepsilon = 0.5$ (c, d). .......................... 148

6.18 Comparison of empirical CDF for the EDPs of the linear (a), bilinear (b), sliding (c) and rocking (d) secondary systems, respectively, for $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = 0$ (solid), $\sigma_\alpha = 0.04, \sigma_\beta = 0.07, \sigma_\gamma = 0.1$ (dotted) and $\sigma_\alpha = 0.08, \sigma_\beta = 0.14, \sigma_\gamma = 0.2$ (dashed). .......................... 150

7.1 Primary-secondary case study: specification (a) and base-fixed system configuration (b). .......................... 164

7.2 Ground acceleration time histories of the Imperial Valley 1940 (a), Erzican 1992 (c) and Irpinia 1980 (e) earthquakes and corresponding Fourier Amplitude spectra (b, d, f). .......................... 167

7.3 FRF for cascade and CMS on a light (a) and heavy (b) S system. .......................... 168

7.4 FRF for vertical mass irregularity on second (a, b) and third storey (c, d), and in-plan stiffness irregularity (e, f) quantified on P (left) and S (right), respectively. 169
List of Figures

7.5 Displacement EDPs due to vertical mass irregularity at various storeys for El Centro (a, b), Erzincan (c, d) and Irpinia (e, f) earthquakes, quantified on P (left) and S (right), respectively, in the x direction. ............... 171

7.6 Displacement EDPs due to stiffness irregularity in-plan, for El Centro (a, b), Erzincan (c, d) and Irpinia (e, f) earthquakes, quantified on P (left) and S (right), respectively, in the x and y directions. ......................... 172

7.7 Acceleration EDPs of P and S systems, for El Centro (a, b), Erzincan (c, d) and Irpinia (e, f) earthquakes, with respect to vertical mass (left) and in-plan stiffness (right) irregularities, respectively. ......................... 174

7.8 Primary-secondary case-study model. ............................. 175

7.9 Exact FRF for various damping models (a), and corresponding cumulative differences (b) quantified on S. ........................................ 176

7.10 FRF for cascade and CMS with various modal correction methods for single (a) and paired Rayleigh (b), modal (c) and Caughey (d) damping models. ..... 178

7.11 Cumulative inaccuracies of various modal correction methods for single (a), and paired Rayleigh (b), Modal (c) and Caughey (d) damping models. ......... 179

7.12 Discrepancies in the phase plane for various modal correction methods for Erzincan (a) and Irpinia (b) earthquakes, respectively. ......................... 180

7.13 Cumulative inaccuracies in the displacement (a, b) and acceleration (c, d) time histories, for Erzincan (left) and Irpinia (right) earthquakes, respectively. ......... 181

7.14 Displacement (left) and acceleration (right) vibration envelopes for the Exact, MDM, MAM and DyMAM cases, from (left to right), respectively, as well as, single and paired Rayleigh (R1, R2), modal (MD) and Caughey (CH) damping models, for Imperial Valley (top), Erzincan (middle) and Irpinia (bottom) earthquakes, respectively. ............... 183
# List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Chapter overview</td>
<td>6</td>
</tr>
<tr>
<td>5.1</td>
<td>Fitted distributions assigned to the R-DK model [132]</td>
<td>96</td>
</tr>
<tr>
<td>5.2</td>
<td>Regression coefficients and errors of the R-DK model [132]</td>
<td>97</td>
</tr>
<tr>
<td>5.3</td>
<td>Correlations between error terms of the R-DK model [132]</td>
<td>97</td>
</tr>
<tr>
<td>6.1</td>
<td>Comparison of $EDP_{s,50}$ and $EDP_{s,90}$</td>
<td>149</td>
</tr>
<tr>
<td>7.1</td>
<td>Ground motion records</td>
<td>165</td>
</tr>
</tbody>
</table>
### Chapter 3

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>B and H</td>
<td>Width and height of the rocking S</td>
</tr>
<tr>
<td>$c_s$ and $c_p$</td>
<td>Viscous damping coefficients for S and P</td>
</tr>
<tr>
<td>$f_b$, $f_s$ and $f_r$</td>
<td>Bilinear, sliding and rocking restoring force of S</td>
</tr>
<tr>
<td>$g$</td>
<td>Acceleration due to gravity</td>
</tr>
<tr>
<td>$I_o$</td>
<td>Rocking S moment of inertia about $O$ or $O'$</td>
</tr>
<tr>
<td>$k_s$ and $k_p$</td>
<td>Stiffness of the S and P oscillator</td>
</tr>
<tr>
<td>$M$</td>
<td>Applied moment</td>
</tr>
<tr>
<td>$m_s$ and $m_p$</td>
<td>Mass of the S and P oscillator</td>
</tr>
<tr>
<td>$O$ and $O'$</td>
<td>Centres of rotation of the rocking S</td>
</tr>
<tr>
<td>$p$</td>
<td>Geometrical parameter of the rocking S</td>
</tr>
<tr>
<td>$R$</td>
<td>Half diameter of the rocking S</td>
</tr>
<tr>
<td>$r$</td>
<td>Ratio of kinetic energy after and prior to impact</td>
</tr>
<tr>
<td>$T(t)$, $U(t)$ and $E(t)$</td>
<td>Kinetic, potential and total energy</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
<tr>
<td>$t_f$</td>
<td>Duration of the time history</td>
</tr>
<tr>
<td>$\ddot{u}_a(t)$</td>
<td>Absolute acceleration response of P</td>
</tr>
</tbody>
</table>
\( \ddot{u}_g(t) \) \nomenclature Horizontal ground acceleration array
\( \dot{u}_p^- \) and \( \dot{u}_p^+ \) \nomenclature Pre and post-impact velocity of P
\( u_s(t) \) and \( u_p(t) \) \nomenclature Displacement of the S and P oscillator relative to the ground
\( u_s^p(t) \) \nomenclature Displacement of S relative to the motion of P
\( u_y \) \nomenclature Yield displacement of the bilinear S
\( v_{II}, r_{II} \) and \( n_{II} \) \nomenclature Number of variables, reference dimensions and dimensionless products
\( y(t) \) \nomenclature State vector
\( z(t) \) \nomenclature Auxiliary state variable of the bilinear S
\( \alpha \) \nomenclature Slenderness angle of the rocking S
\( a_g, \omega_g \) and \( \phi \) \nomenclature Amplitude, frequency and phase angle of the pulse
\( a_s \) \nomenclature Specific strength of the bilinear S
\( \gamma \) \nomenclature S-P mass ratio
\( \epsilon \) \nomenclature Percentage cumulative error
\( \varepsilon \) \nomenclature Coefficient of restitution
\( \varepsilon_{\text{max}} \) \nomenclature Maximum allowable value of the coefficient of restitution
\( \zeta_s \) and \( \zeta_p \) \nomenclature Equivalent viscous damping ratios for S and P
\( \dot{\theta}_s(t) \) \nomenclature Rotation response of the rocking S
\( \dot{\theta}_s^- \) and \( \dot{\theta}_s^+ \) \nomenclature Pre and post-impact angular velocity
\( \mu_s \) \nomenclature Coefficient of sliding friction
\( \Pi_i \) \nomenclature \( i \)-th independent dimensionless product
\( \psi_s \) \nomenclature Post-yield to pre-yield stiffness ratio
\( \omega_s \) and \( \omega_p \) \nomenclature Circular frequencies of S and P

**Chapter 4**

\( 0_n \) \nomenclature Zero vector of dimensions \((n \times 1)\)
\( C_p \) \nomenclature Modal damping matrix of P
\( C_{sp} \) and \( K_{sp} \) \nomenclature Coupling damping and stiffness arrays of P
Nomenclature

\( c_s \)  Damping constant of S
\( D \)  Dynamic matrix
\( g \)  Partitioned influence vector of seismic input
\( I_{n_p} \)  Identity matrix of size \( n_p \)
\( M, C \) and \( K \)  Full partitioned mass, damping and stiffness matrices
\( M_p \) and \( K_p \)  Mass and stiffness base-fixed matrices for P
\( m, c \) and \( k \)  Reduced matrices of inertia, damping and stiffness
\( m_p \)  Modal mass of P
\( m_s \) and \( k_s \)  Mass and stiffness coefficients for S
\( O_{r \times s} \)  Zero matrix of order \((r \times s)\)
\( q(t) \)  Array listing the generalised coordinates of S and P
\( q_s(t) \) and \( q_p(t) \)  Generalised coordinates of S and P
\( u(t) \)  Partitioned array of displacements relative to the ground
\( \ddot{u}_a(t) \)  Absolute acceleration of P at the position of attachment
\( u_p(t) \)  DoF of P at the position of attachment
\( u_s(t) \)  DoF of S relative to P
\( u_s^g(t) \) and \( u_p(t) \)  DoF of S and DoFs or P relative to the ground
\( v \)  Vector whose non-zero entry marks the attachment position
\( \bar{x}(t) \)  Matrix listing the complex-valued modal coordinates
\( y(t) \)  State vector array
\( z(t) \)  Partitioned array of state variables
\( \alpha \)  Diagonal matrix of complex eigenvalues
\( a \)  Dimensionless modal mass ratio
\( \beta \)  Dimensionless participation mass
\( \Gamma \)  Modal transformation matrix
\( \bar{\Gamma} \)  Complex modal matrix
\( \Delta C_p \) and \( \Delta K_p \)  Residual damping and stiffness matrices of P
\( \Delta t \)  Time step
\( \Theta \)  Transition matrix
Nomenclature

$\mu_d$  
Ductility ratio

$\mu_s$  
Sliding friction coefficient

$\boldsymbol{\tau}$  
Partitioned seismic incidence vector

$\boldsymbol{\tau}_p$  
Seismic incidence of P

$\Phi_p$  
Normalised modal vector

$\Phi_p$  
Full modal matrix of P

$\varphi$  
Dimensionless modal coordinate at the attachment position

$\Omega_p$  
Full diagonal spectral matrix of P

$\omega_{p,i}$  
i-th circular frequency of P

$\bar{\omega}_s$ and $\bar{\omega}_p$  
Natural frequencies of damped vibration for S and P

Chapter 5

$F$  
Faulting mechanism

$h [t - \tau, \lambda (\tau)]$  
Impulse response function

$I_a, D_{5-95}$ and $t_{\text{mid}}$  
Arias intensity, effective duration and the time to reach 45% of Arias intensity for the target ground motion

$M$  
Earthquake magnitude

$Q (t, \kappa)$  
Time-modulating function

$R$  
Source-to-site distance

$s_i (t, \lambda (t_i))$  
Basis functions

$t_p$  
p-th percentile variate of the associate distribution

$V_s$  
Shear-wave velocity

$\nu_i$  
Standard normal random variables

$w (\tau)$  
Gaussian white-noise process

$x(t)$  
Nonstationary acceleration process

$y(t)$  
Corrected response

$\beta$  
Dimensionless regression coefficient

$\Delta t$  
Discretisation step of the process

$\theta_g$  
Set of ground motion model parameters
### Nomenclature

\( \kappa = \{ \kappa_1, \kappa_2, \kappa_3 \} \)  
Parameter set defining the intensity and shape of the modulating function

\( \lambda(\tau) = \{ \omega_f(\tau), \zeta_f(\tau) \} \)  
Parameter set listing the time-varying frequency and damping ratio

\( \nu = \{ \nu_1, \ldots, \nu_6 \} \)  
Set of jointly normal random variables

\( \sigma_h(t) \)  
Standard deviation of the process

\( \tau \)  
Dummy time variable

\( \omega_c \)  
Corner frequency of the high-pass filter

\( \omega_{\text{mid}} \) and \( \omega' \)  
Value and the slope of the frequency of the filter at \( t = t_{\text{mid}} \)

### Chapter 6

\( A_b \) and \( A_{ci} \)  
Cross sectional area of beams and the \( i \) – th storey column

\( a_1(t) \) and \( a_2(t) \)  
As-recorded acceleration time series

\( a_1^p(t) \) and \( a_2^p(t) \)  
Principal components

\( E \)  
Young’s modulus

\( h \)  
Finite element length

\( I \)  
Moment of inertia

\( I_b \) and \( I_{ci} \)  
Moment of inertia of beams and the \( i \) – th storey column

\( k \)  
Rotational stiffness

\( l \)  
Length of the beam

\( M, C \) and \( K \)  
Full mass, damping and stiffness matrices of P

\( \tilde{M}, \tilde{C} \) and \( \tilde{K} \)  
Mass, stiffness and damping deterministic matrices

\( \tilde{M}(\alpha), \tilde{K}(\alpha, \beta), \tilde{C}(\alpha, \beta, \gamma) \)  
Stochastic mass stiffness and damping matrices

\( M_l \) and \( M \)  
Lumped masses for the top storey and elsewhere

\( m \)  
Number of modes to be retained in the analysis

\( m_S \) and \( M_P \)  
Mass of S and P

\( n \)  
Total number of degrees of freedom

\( \mathbf{p}(\alpha) \)  
Seismic incidence vector

\( \mathbf{q}(t) \)  
Array listing the modal coordinates of P
\( \hat{q}(t) \)  & Stochastic modal coordinate \\
\( t_0 \) and \( t_n \)  & Initial and final time instants of the time series \\
\( u(t) \)  & Displacement vector of \( P \) \\
\( \hat{u}(t) \)  & Stochastic response \\
\( \hat{u}_g(t) \)  & Vector listing the as-recorded acceleration time series \\
\( \hat{\hat{u}}_g(t) \)  & Vector of principal components \\
\( \hat{\hat{u}}_g \)  & Stochastic horizontal ground acceleration array \\
v  & Fixity factor \\
\( \nu_\alpha, \nu_\beta \) and \( \nu_\gamma \)  & Variances of the random variables in the modal subspace \\
\( \alpha, \beta \) and \( \gamma \)  & Zero-mean random matrices \\
\( \delta M(\alpha), \delta K(\alpha, \beta), \delta C(\alpha, \beta, \gamma) \)  & Mass, stiffness and damping fluctuations \\
\( \delta \)  & Rotation angle \\
\( \delta_0 \)  & Rotation angle associated with the principal components \\
\( \zeta \)  & Equivalent viscous damping ratio of \( P \) \\
\( \eta \) and \( \xi \)  & Unique set of principal axes of the horizontal components \\
\( \mu \)  & Mean \\
\( \rho_{a_1(t) a_2(t)} \)  & Correlation coefficient \\
\( \sigma \)  & Standard deviation \\
\( \tau \)  & \((n \times 2)\) matrix of seismic incidence \\
\( \varphi_i \)  & The \( i \)-th column of the matrix \( \Phi^\top \) \\
\( \Phi \)  & Normalised modal matrix of \( P \) \\
\( \hat{\Phi}(\alpha), \hat{\Omega}(\beta) \) and \( \hat{\zeta}(\gamma) \)  & Stochastic modal, spectral and damping matrices \\
\( \hat{\Phi}(\alpha) \)  & Normalised stochastic modal matrix \\
\( \Omega \)  & Diagonal spectral matral matrix of \( P \) \\

### Chapter 7

\( c, c_S, c_P, c_{SP} \) and \( c_{PP} \)  & Reduced damping matrices in the modal subspace \\
e  & Eccentricity \\
g  & Reduced influence vector of seismic incidence
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I_r )</td>
<td>( r )-dimensional identity matrix</td>
</tr>
<tr>
<td>( K, K_P, K_S, K_{SP} ) and ( K_{PP} )</td>
<td>Full stiffness matrices</td>
</tr>
<tr>
<td>( k ) and ( k_{PP} )</td>
<td>Reduced stiffness matrices</td>
</tr>
<tr>
<td>( M, M_P ) and ( M_S )</td>
<td>Full mass matrices</td>
</tr>
<tr>
<td>( m, m_P ) and ( m_{SP} )</td>
<td>Reduced mass matrices</td>
</tr>
<tr>
<td>( m_P ) and ( m_S )</td>
<td>Number of modal coordinates</td>
</tr>
<tr>
<td>( N_{SP} )</td>
<td>Primary-secondary pseudo-static influence matrix</td>
</tr>
<tr>
<td>( n, n_P ) and ( n_S )</td>
<td>Total number of DoFs</td>
</tr>
<tr>
<td>( O_{r \times s} )</td>
<td>Zero matrix of size ( (r \times s) )</td>
</tr>
<tr>
<td>( p_P, p_S ) and ( p_{PP} )</td>
<td>Arrays of modal participation factors</td>
</tr>
<tr>
<td>( q(t), q_P(t) ) and ( q_S(t) )</td>
<td>Arrays collecting modal coordinates</td>
</tr>
<tr>
<td>( T_P ) and ( T_S )</td>
<td>Periods of vibration</td>
</tr>
<tr>
<td>( u(t), u_P(t) ) and ( u_S(t) )</td>
<td>Arrays collecting DoFs</td>
</tr>
<tr>
<td>( \ddot{u}_g(t) )</td>
<td>Ground acceleration</td>
</tr>
<tr>
<td>( \alpha_M ) and ( \alpha_K )</td>
<td>Proportionality damping coefficients for mass and stiffness</td>
</tr>
<tr>
<td>( \Gamma )</td>
<td>Modal transformation matrices</td>
</tr>
<tr>
<td>( \Delta b )</td>
<td>Static correction vector</td>
</tr>
<tr>
<td>( \Delta u )</td>
<td>Modal correction term</td>
</tr>
<tr>
<td>( \zeta_P ) and ( \zeta_S )</td>
<td>Viscous damping ratios</td>
</tr>
<tr>
<td>( \theta(t) )</td>
<td>Response of the auxiliary DyMAM oscillator</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>Eccentricity ratio</td>
</tr>
<tr>
<td>( \mu, \mu_i )</td>
<td>Mass ratios</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Radius of gyration of the floor plan</td>
</tr>
<tr>
<td>( \tau, \tau_P ) and ( \tau_S )</td>
<td>Vectors of seismic incidence</td>
</tr>
<tr>
<td>( \Phi, \Phi_P ) and ( \Phi_S )</td>
<td>Modal matrices</td>
</tr>
<tr>
<td>( \Psi_{SP} )</td>
<td>Primary-secondary modal coupling matrix</td>
</tr>
<tr>
<td>( \Omega, \Omega_P ) and ( \Omega_S )</td>
<td>Diagonal spectral matrices</td>
</tr>
<tr>
<td>( \omega_I ) and ( \omega_{II} )</td>
<td>Circular frequencies assumed for the average viscous damping ratios in the Rayleigh’s damping</td>
</tr>
</tbody>
</table>
$\omega_F$ and $\zeta_F$ Filter frequency and damping ratio

**Notations and Symbols**

- $\text{diag}$ Diagonal matrix
- $\exp [\cdot]$ Matrix exponential function of $(\cdot)$
- $\mathbb{E} (\cdot)$ Expectation of $(\cdot)$
- $F_{\theta_g,i}(\cdot)$ Marginal CDF fitted to the $i$-th element of $\theta_g$
- $f (\cdot)$ and $g (\cdot)$ Operators mapping the generic elements $i, j$ of an $(m \times m)$ matrix onto a sparse matrix of dimensions $(m^2 \times m^2)$
- $H(\cdot)$ Heaviside unit step function of $(\cdot)$
- $\text{sgn}(\cdot)$ Signum function of $(\cdot)$
- $\text{Var} [\cdot]$ Operator giving the variance for each element of $[\cdot]$
- $\Gamma(\cdot)$ Gamma function
- $\Phi [\cdot]$ Standard normal cumulative distribution function
- $\in$ Belongs to
- $\otimes$ Kronecker product
- $|\cdot|$ Absolute value of $(\cdot)$
- $|\cdot|_{\max}$ Peak absolute value of $(\cdot)$
- $\cdot^*$ Dimensionless parameter $\cdot$
- $(\cdot)^*$ Complex conjugate of $(\cdot)$
- $(\cdot)'$ Derivative of $(\cdot)$ with respect to $t$
- $(\cdot)^{-1}$ Matrix inverse of $(\cdot)$
- $(\cdot)^\top$ Matrix transpose of $(\cdot)$

**Abbreviations**

- **2D** Two dimensional
- **2DoF** Two-degree-of-freedom
- **3D** Three dimensional
- **AbsTol and RelTol** Absolute and relative tolerances
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDF</td>
<td>Cumulative distribution function</td>
</tr>
<tr>
<td>CMS</td>
<td>Component-mode synthesis method</td>
</tr>
<tr>
<td>CoV</td>
<td>Coefficient of variation</td>
</tr>
<tr>
<td>DoF</td>
<td>Degrees of freedom</td>
</tr>
<tr>
<td>DyMAM</td>
<td>Dynamic Modal Acceleration Method</td>
</tr>
<tr>
<td>EDP</td>
<td>Engineering demand parameter</td>
</tr>
<tr>
<td>FRF</td>
<td>Frequency response function</td>
</tr>
<tr>
<td>HPF</td>
<td>High-pass filter</td>
</tr>
<tr>
<td>IRF</td>
<td>Impulse-response function</td>
</tr>
<tr>
<td>MAM</td>
<td>Modal Acceleration Method</td>
</tr>
<tr>
<td>MDoF</td>
<td>Multi-degree-of-freedom</td>
</tr>
<tr>
<td>OAPI</td>
<td>Open application programming interface</td>
</tr>
<tr>
<td>P</td>
<td>Primary structure</td>
</tr>
<tr>
<td>PDF</td>
<td>Probability density function</td>
</tr>
<tr>
<td>Refine</td>
<td>Interpolation output</td>
</tr>
<tr>
<td>S</td>
<td>Secondary system</td>
</tr>
<tr>
<td>SDoF</td>
<td>Single-degree-of-freedom</td>
</tr>
</tbody>
</table>
CHAPTER 1

Introduction

Secondary structures, often referred to in the technical literature as nonstructural elements or building attachments (or appendages), are the auxiliary components or contents of a building which do not form part of the primary load-bearing structure [1]. They are classified into architectural components, comprising of partitions, cladding, suspended ceilings, fences and chimneys; mechanical, plumbing and electrical components, consisting of fans, antennas, piping systems, transformers and pumps; and building equipment and contents, such as furniture, shelving and storage units.

Their dynamic analysis and design is a topic of broad engineering interest and plays a critical role in community resilience and the seismic assessment of building structures. Damage to nonstructural components during seismic events can cause injuries or deaths, damage to the property as well as functional loss through interruption of services, which can lead to further human and economic losses. Orizaba (1973), Idaho (1983), Whittier Narrows (1987), Northridge (1994) and Chile (2010) are all examples of earthquakes that resulted in significant nonstructural damage [1, 2, 3, 4, 5]. As a matter of fact, a comprehensive review undertaken by Taghavi and Miranda [6] concluded that the highest economic loss is attributed to the damage of nonstructural components, which were identified to account for up to 80% of the construction costs.

Earthquakes cause nonstructural damage in four main ways [7]:

1.
1. *Inertial forces* cause partially restrained or unrestrained items to slide, rock or overturn. These forces depend on the ground acceleration, mass and location of the nonstructural component, e.g. the higher the location from the base, the higher tends to be the excitation amplification;

2. *Structural deformations* cause nonstructural components to displace. Brittle components, for instance, such as secondary panels may crack and, once subjected to the out-of-plane inertial forces, dislodge;

3. *Building separations*, which involve damage of components crossing expansion joints of buildings due to differential displacements, e.g. piping;

4. *Nonstructural interaction* causes damage due to the differential movement of adjacent secondary structures of varying dynamic characteristics. Accordingly, the extent of damage is not only influenced by their dynamic properties, but also by the characteristics of the ground motion and the primary structure, their location, proximity and interaction to the primary or other secondary components, the type of anchorage and restraint location, as well as the distribution and location of the load.

Numerous characteristics render nonstructural components vulnerable to earthquake events. For instance, their function is different to seismic resistance, and are often made of brittle materials which are highly sensitive to vibration. They are subjected to amplified motions and their mass and stiffness are usually lower then the supporting structure, which can result in tuned natural frequencies and resonant effects. They also tend to have lower damping ratios compared to their supporting structure, and differential movement can arise if they are connected at multiple points [1].

Further to the vulnerability characteristics, there are various challenges associated with their analysis. Inherent nonlinearities and uncertainties in the specification of the ground shaking (i.e. nonstationarities in amplitude and frequency) as well as the properties of the primary or secondary structure (i.e. strength and stiffness of members and connections) induce variation in their seismic performance and need to be accounted for. Moreover, the primary-secondary dynamic interaction often needs to be considered, and therefore the combined structure needs to be analysed. Since the dynamic system usually possesses a large number of degrees-of-freedom and non-classical vibration modes, the analysis can be cumbersome. Furthermore, due to the natural frequencies of the secondary structure being potentially close to the primary, a higher number of modes may be required to evaluate the seismic response with sufficient accuracy.
The primary focus in earthquake engineering has historically been on structural resistance, providing guidance to ensure life safety. This has mainly been addressed through prescriptive requirements, e.g. limiting the internal forces, and implicit consideration of uncertainties, i.e. through partial safety factors [8]. Aiming for predictable performance, as well as allowing different design targets to be achieved, performance-based engineering philosophy has emerged as a broad spectrum of design solutions underpinned by well-defined performance objectives, allowing the estimation of consequences arising with the chosen design and quantification of uncertainties in a rigorous probabilistic manner [9]. Nevertheless, the extension of performance-based concepts to secondary structures has lagged behind that of primary systems, while past earthquakes have evidenced that current methods for their seismic analysis lack the necessary robustness, resulting in expensive and often unreliable solutions.

Motivated by the shortcomings of existing techniques, as well as the pronounced vulnerability characteristics of secondary structures, the significant damage consequences and the inherent uncertainties in the ground and structural properties, it is the purpose of this thesis to provide insights into the seismic response analysis of secondary systems.

1.1 Aim and Objectives

The aim of this thesis is the development of efficient computational methods for the response analysis of secondary structures subjected to seismic excitations. In order to achieve this aim, the following objectives are pursued:

1. **Review the existing methods for the seismic analysis of secondary structures.**
   An extensive literature review will be carried out to highlight the current state of practice and state of the art in the context of secondary structures subjected to earthquakes, identifying areas for development. Modelling and analysis methods will be considered, as well as methods for representation of uncertainties and characterisation of the seismic hazard.

2. **Characterise the combined vibration response of nonlinear secondary oscillators and identify the conditions under which simplified analysis methods can be adopted.**
   The equations governing the combined vibration of various nonlinear secondary oscillators connected to linear primary oscillators will be formulated. The response will be evaluated for each system in presence of full-cycle pulses, which resemble the pulses observed in near-source earthquakes, and it will be compared to the respective cascade
solution, where the dynamic interaction is neglected. Decoupling criteria in the form of cumulative errors in the response time histories will be presented to identify the conditions whereby the cascade approximation is permissible for engineering applications.

3. **Develop analytical and numerical solutions for the nonlinear response of cascaded secondary oscillators.**
   Based on the assumption that the secondary-primary dynamic interaction can be regarded as negligible, closed-form solutions will be derived for the response of nonlinear secondary oscillators in cascade, to full-cycle pulses. Numerical expressions will be also presented for a general-type excitation and will be validated against the analytical ones. Response spectra will be presented for the secondary oscillators connected to linear primary structures providing insights into understanding their seismic behaviour.

4. **Examine the effect of uncertainty in the seismic input on the nonlinear response of cascaded secondary oscillators.**
   The response analysis of secondary oscillators to full-cycle pulses will be extended to the case of far-field strong ground motions. In doing that, an existing stochastic ground motion model will be adopted and the effects of uncertainty in the seismic input will be quantified on the response spectra of the secondary oscillators.

5. **Develop an efficient method for characterising uncertainty in the modal properties of the primary structure and quantify its effects on secondary systems.**
   Random vibration tools will be exploited and a novel efficient technique will be presented, whereby uncertainty is characterised in the modal properties of a primary multi-degree-of-freedom (MDoF) system. The propagation of uncertainty from the primary structure to the secondary oscillators will be assessed, and the relative effect in comparison to uncertainty in the ground motion will be quantified.

6. **Study the combined vibration of linear multi-degree-of-freedom primary-secondary systems**
   A variant of the component-mode synthesis method will be adopted to study the combined vibration response of flexible multi-connected linear secondary systems. The method allows for a practical alternative to the conventional combined analysis for systems where the dynamic interaction needs to be accounted for. Numerical investigations will be performed on a piping system as well as a flexible secondary system.
7. **Extend the improved modal correction method for the analysis of secondary structures.**

   The Dynamic Modal Acceleration Method (DyMAM) for linear systems will be extended to the case of MDoF secondary structures to improve the accuracy of the dynamic response with reduced modal information.

8. **Develop a novel technique for constructing the viscous damping matrix and quantify the effects of viscoelastic damping.**

   The effect of alternative techniques for constructing the damping matrix will be examined on the primary-secondary interface, and a novel method based on modal damping superposition will be presented.

9. **Examine the effect of irregularity and quantify its effects on secondary structures.**

   The effect of mass and stiffness irregularities of the primary supporting system will be quantified on the response of a MDoF multi-attached piping secondary structure.

### 1.2 Outline of the Thesis

The thesis is divided into eight chapters, the List of Publications and the List of References. Table 1.1 maps each of the main chapters against the modelling and analysis methods adopted while an overview is provided below.

_Chapter 2_ reviews the technical literature, with emphasis on existing methods for modelling and analysing secondary structures subjected to earthquakes. Modal correction methods for linear systems, stochastic models for synthetic ground motion generation as well as methods for characterising uncertainty in the structural properties are reviewed, along with a discussion of their limitations.

_Chapter 3_ deals with the simplest case of secondary structure, namely, that of a single-degree-of-freedom (SDoF) oscillator. In particular, bilinear, sliding and free-standing rocking secondary oscillators are considered, chosen as representative candidates of a wider spectrum of components, and the equations governing their combined vibration (i.e. full dynamic interaction) with a linear primary oscillator are formulated. The systems’ response is then determined for the case of mathematically convenient full-cycle trigonometric pulses that resemble near-source ground motions. The results are compared to the corresponding cascade solution, and decoupling criteria are put forward for a range of input parameter combinations.
In Chapter 4, the combined vibration of a pulse-driven two-degree-of-freedom (2DoF) linear system is first examined, and novel closed-form solutions are derived. Based on the findings of Chapter 3 and considering the case of negligible dynamic interaction, the cascade approximation is adopted. Within the limits of linear approximation, new analytical solutions are derived for the pulse excited oscillators considered in Chapter 3, and an efficient numerical scheme based on the interpolation of a generic excitation is presented. Nonlinear spectra are presented and the response of the secondary systems is examined.

In Chapter 5, an existing stochastic ground motion model accounting for both amplitude and frequency nonstationarities is adopted, and its predictive relationships are used to generate a suite of far-field synthetic ground motions for given earthquake (i.e. type of fault, rupture distance and moment magnitude) and site characteristics (i.e. shear wave velocity). The ensemble is used to extend the work of Chapter 4, and the stochastic nonlinear spectra for cascaded secondary oscillators are presented.

Chapter 6 deals with the cascade analysis of secondary oscillators vibrating on linear multi-degree-of-freedom (MDoF) primary systems. First, a novel method is proposed, whereby uncertainty in the properties of a primary structure is conveniently characterised in the reduced modal space with modal shapes, frequencies and damping ratios comprising the random quantities. Contrary to the conventional characterisation of uncertainty in the full geometric space,
the proposed method allows a significant reduction in the number of uncertain parameters and the size of the dynamic problem. An identification procedure is then presented, where the model parameters are calibrated over various levels of connection flexibility in a steel frame with aleatory semi-rigid connections, allowing the direct evaluation of the random dynamic response without resorting to the full geometrical model. Furthermore, the stochastic ground motion model adopted in Chapter 5 is used to generate synthetic principal horizontal motions and, in conjunction with the proposed uncertainty representation model, allows investigating the relative effects of uncertainties on the seismic response of secondary structures.

Chapter 7 is devoted to the combined analysis of linear MDoF secondary systems. A convenient variant of the component-mode synthesis method is first presented, whereby the primary-secondary dynamic interaction is accounted for through the modes of vibration of the two components. The problem of selecting the vibrational modes to be retained in analysis is then addressed for the case of secondary structures, which may possess numerous low-frequency modes with negligible mass, and the dynamic mode acceleration method (DyMAM) is adopted in view of the application for seismic analysis. The influence of various approaches to build the viscous damping matrix of the primary-secondary assembly is also investigated, and a novel technique based on modal damping superposition is proposed. Numerical applications are demonstrated through a piping secondary system multi-connected on a primary frame exhibiting various irregularities in plan and elevation, as well as a multi-connected flexible system.

Finally, Chapter 8 presents the emerging conclusions and contributions of this thesis and provides recommendations for further research.
CHAPTER 2

Literature Review

2.1 Introduction

The purpose of this chapter is to highlight the current state of the art in the context of seismically driven secondary (S) structures, reviewing the main shortcomings and limitations of existing studies as well as identifying research areas that will contribute to the development of this PhD project.

Section 2.2 deals with the characterisation of the seismic input. The main models commonly utilised in the representation of the seismic hazard are identified. The use of previously recorded accelerograms is discussed along with existing stochastic ground motion models for the generation of synthetic motions.

Section 2.3 critically reviews existing modelling and analysis methods in view of the seismic response evaluation of S structures. A discussion on single-degree-of-freedom nonlinear S oscillators is also included.

Section 2.4 is devoted on the review of modal correction methods, that comprise an effective tool for improving the solution accuracy in the response evaluation of multi-degree-of-freedom linear systems.

The central theme of Section 2.5 is the characterisation of dissipative forces, an active research area in linear structural dynamics.
An intrinsic aspect related to the seismic response of S structures is the seismic behaviour of the primary (P) system which can be influenced by irregularities arising either in plan or elevation and is the topic of Section 2.6.

Section 2.7 finally discusses the effects of uncertainty in the structural, nonstructural elements, as well as in the earthquake input on the response of S systems. A review of existing works on the uncertainty in the partial rigidity of connections of steel frames is also included.

2.2 Seismic Hazard Models

In the context of performance-based earthquake engineering (PBEE), all sources of uncertainty need to be consistently accounted for [9]. Hazard models capable of capturing the intrinsic randomness of the seismic signals are used to characterise the variation in the seismic response of P and S systems.

An important class of models comprises of those based on single or multiple scalar intensity measures. In the former case, intensity measures (e.g. the spectral acceleration at the fundamental frequency for a structure), are related to magnitude and distance to rupture sources via attenuation models. Hazard curves are then extracted by exploiting probabilistic seismic hazard analysis, that also takes into account the occurrence of earthquakes. In the latter case, improvements may be sought if multiple intensity measures are used (e.g. spectral acceleration as a function of the circular frequency) [9].

The utilisation of recorded accelerograms is accompanied with a main shortcoming, i.e. they represent only a single realisation of the probabilistic hazard. To circumvent this, the use of scaled recorded motions has been proposed, but concerns have been raised, as they may render unphysical characteristics [10, 9]. More recently, several techniques have been put forward for the generation of artificial ground motions, whose characteristics aim to match the ones of target accelerograms, including various wavelet-based methods (e.g. [11, 12, 13]), analytical-based ones for near-fault ground motions [14], as well as the ones evolved from filtered white noise processes [15, 16].

As an example of the latter, a model proposed by Rezaeian and Der Kiureghian [17, 18] is particularly appealing, encompassing completely separable temporal and spectral non-stationarities. The model has been extensively used in the literature [19, 20] and has been further extended to multi-component simulations [21] based on the work of Penzien and Watabe [22], who identified a set of principal axes along which the orthogonal components of a ground motion can be considered as statistically uncorrelated. Synthetic motions generated through
the model for a given design scenario can be readily validated through comparisons with the Next Generation of Ground-Motion Attenuation models (NGA) [23], namely, Abrahamson and Silva, 2008 [24], Boore and Atkinson, 2008 [25], Campbell and Bozorgnia, 2008 [26], and Chiou and Youngs, 2008 [27].

2.3 Analysis of Secondary Structures

In recent decades, research efforts have been devoted to identify efficient modelling and analysis methods for evaluating the dynamic response of S structures. In what follows, the main ones are presented.

2.3.1 Modelling

Owing to the non-exhaustive list of S structures, no universally-accepted requirements are currently in place for modelling their behaviour. Villaverde [1] classified nonstructural components in three categories, namely, rigid, flexible and hanging from above. Rigid components include the ones anchored to floors (e.g. engines), that can be modelled either as linear or nonlinear single-degree-of-freedom (SDoF) oscillators and depend on the properties of their anchors (e.g. ductility and stiffness). Flexible components, which can also be multiply attached (i.e. pipelines), need to be modelled as multi-degree-of-freedom (MDoF) oscillators, with all attachment points being considered. The last category comprises of elements typically hanged from ceilings (e.g. lighting systems), rarely damaged by earthquakes and are thus frequently neglected from the seismic analysis. In the case these impact other elements they are modelled as single-mass pendulums.

2.3.2 Analysis Methods

2.3.2.1 Combined P-S System

Conventional techniques available in the current literature deal with the analysis of S structures in conjunction with their supporting P structures. Although this method implicitly accounts for the dynamic interaction of the two subsystems, it has been associated with numerous limitations. Firstly, it is deemed impractical for engineering practice, as for every change introduced, the composite system has to be resolved; furthermore, the design has to be carried out by the same team. Secondly, it is computationally expensive due to the presence of excessive number of degrees of freedom. Finally, the combined system will be difficult to model if nonlinearities
are to be considered and it will typically possess complex-valued eigenproperties (dissimilarities in the inertia and stiffness terms), while the solution may be cumbersome [28, 1].

2.3.2.2 Cascade Approximation

To circumvent the aforementioned shortcomings, a decoupled approach can often be employed. In the case of a time history analysis, for instance, the two subsystems are sequentially analysed. The response time history of the P structure is first determined at the attachment points and is then used as input to the S system. It has long been recognised, however, that this approach neglects the dynamic interaction between the two subsystems and the analysis can also be computationally expensive [28].

Recognising the importance of the conventional response spectrum method (RSM), an alternative approach can often be adopted, where the S structure is separately analysed using a response spectrum at each attachment point (so-called floor response spectrum (FRS)). The response time history of the P structure at the attachment level is first determined, which then allows successively computing the FRS (for the particular case of multiply supported structures the spectra for all the attachments are either enveloped, or the individual responses are obtained and then combined). The response of the S structure is then obtained through response spectrum analysis. In doing so, a range of ground acceleration histories are used and the resulting spectra are enveloped. To further reduce the computational effort, it is also common to utilise artificial ground acceleration histories compatible with the design spectrum. From a general viewpoint, care must be taken as different time histories enveloping a particular spectrum may result in different, and often contrasting outcomes. In this case, methods that make use of the response spectrum method can be utilised to obtain FRS from a design spectrum avoiding the use of time-history analysis [29, 30].

Although the FRS permits the decoupling of the two subsystems the P-S dynamic interaction, tuning and nonclassical damping effects are not accounted for and as a result it may lead to overconservatism. Furthermore, it can only be used for singly attached S systems within their linear-elastic regime and can therefore result in wrong predictions of the structural response [1, 31, 32]. In an effort to ameliorate its shortcomings, variants of the FRS have been proposed to account for the dynamic interaction. These methods were focused on multiply supported MDOF S systems including the works of Asfura and Der Kiureghian [33] as well as Gupta and Jaw [34].
2.3 Analysis of Secondary Structures

2.3.2.3 Component-Mode Synthesis Method

S systems can be highly sensitive to accelerations and inter-storey drifts, and their seismic performance is influenced by the P-S dynamic interaction, which in many situations needs to be accounted for [35]. Aiming at overcoming the drawbacks of the above delineated approaches, the component-mode synthesis (CMS) method [36] and its variants [37, 38, 39, 40], comprise an efficient computational strategy to handle the dynamic interaction between P and S linear systems under dynamic loads. The strength of the method rests in the equations of motion, which are projected on the reduced modal space, which is conveniently defined by the relevant modes of vibration of the P structure and S attachment. Consequently, the response of the two components can simultaneously be evaluated with an acceptable computational time, without resorting to the combined structural model.

For analysis using RSM, the method comprises of the definition of the ground response spectrum, determination of dynamic properties of the P-S system (i.e. natural frequency, damping ratios), evaluation of the maximum modal response of the S system, and combination of these responses using modal combination rules. It is critical that the combined P and S system will typically not possess classical modes of vibration, having complex valued natural frequencies and mode shapes. The response is therefore highly sensitive on the combination rule used for such systems [1]. Procedures proposed for determination of the dynamic properties of the P-S system include those of Muscolino [41], while modal combination rules for non-classically damped structures have also been formulated by Falsone and Muscolino [42, 43] which do not require the complex solution of an eigenvalue problem. Later extended by Muscolino and Palmeri [44], the procedure is applicable to the case of linear light MDoF S structures attached to linear MDoF systems. Although these methods have been previously criticised as unrealistic, and only possible for the linear-elastic range [30], efforts made for an approximate nonlinear representation remain limited.

2.3.3 Nonlinear SDoF Oscillators

Owing to the non-exhaustive list of S structures available, numerous failure modes may characterise their structural behaviour. High accelerations may cause internal damage in anchored S, and sliding or rocking (leading to overturning) may result in the breakage of restraints/service lines in weakly anchored or unanchored S [7]; the latter being highly nonlinear and difficult to model. In view of the plethora of nonlinear S only the case of bilinear, sliding and rocking oscillators will be addressed in this section. The bilinear idealisation is chosen as the simplest
approximation of nonlinear behaviour, whereby sliding and rocking account for the behaviour characterising common failure modes [45].

Several studies have been devoted to the nonlinear analyses of rigid single-degree-of-freedom (SDoF) blocks. The dynamics of bilinear oscillators have been investigated in presence of sinusoidal excitation by Caughey [46] who identified that the system exhibits soft-type resonance, while unbounded resonance occurs beyond a critical value of the excitation. Makris and Black [45] carried out dimensional analysis in bilinear blocks driven by idealised pulse-type excitations, showing that for a given dimensionless yield displacement and strength the response is self-similar regardless of the pulse duration and intensity. More recently, Voyagaki et al. [47] proposed a transformation method for their yielding response.

The behaviour of sliding systems has been examined by Pratt and Williams [48], who proposed a combined analytical-numerical periodic solution for the steady state response. Further studies include those of Westermo and Udwadia [49], who derived analytical and numerical periodic solutions. More recently, Makris and Constantinou [50] proposed an exact method for the harmonic analysis of the transient and steady-state solution of a constant Coulomb oscillator, further extending the analysis to the case of a linear/Coulomb oscillator. Hong and Liu refined the model formulation and derived further exact solutions to harmonic loading [51]. Finally, Voyagaki et al. [52] proposed a shift approach for the response evaluation and interpretation of the oscillators. The authors have also presented further analytical [53] and numerical [54] solutions for the response to idealised near-fault acceleration pulses.

The dynamic behaviour of rocking blocks has been extensively examined in the technical literature. Yim et al. [55] studied the response of free-standing blocks to horizontal and vertical ground motions, uncovering high sensitivity to fluctuations in the slenderness ratio, the size, and the ground motion characteristics. Spanos and Koh [56], later examined the response of harmonically excited rocking blocks resting on rigid foundation identifying safe and unsafe regions. Zhang and Makris [57, 58] investigated the transient response of free-standing rocking blocks driven by trigonometric pulses showing that in this case the block can overturn in two distinct modes. The authors extended their analyses to the transient response of anchored blocks with elastic-brittle as well as elastic-plastic restrainers [59, 60]. Palmeri and Makris [61, 62] examined the response of rigid structures rocking on viscoelastic foundation. More recently, building on the work of Zhang and Makris, Dimitrakopoulos and DeJong [63] derived new closed-form solutions and similarity laws for the free-standing block. Voyagaki et al. [64] have revisited the problem showing that the nonlinear equations result in more stable response when compared to their linearised counterparts. In a subsequent paper, the authors
2.4 Modal Correction Methods

have derived closed-form solutions and overturning criteria for a two-dimensional rigid block driven by idealised acceleration pulses [65].

All the above-referenced studies deal with blocks driven by the external excitation rather than the response of the P structure, as is the case for S. Most of the remaining literature deals with the cascade analysis of S (i.e. neglecting the P-S dynamic interaction), based on the premise that the S is sufficiently ‘light’ and is not in-tune with the P structure. In these conditions, and if the equipment is assumed linear, decoupling criteria are employed for deciding whether this approach is allowable [66]. In presence of heavy S or when the equipment vibrates close to, or is tuned with the P structure, this analysis approach can lead to unrealistic results [30]. No decoupling criteria exist to aid the designer for nonlinear equipment. In such cases, therefore, one cannot evaluate the response without resorting to the combined S-P assembly. Nonetheless, most of the studies dealing with the combined analysis consider only linear S [67, 68].

2.4 Modal Correction Methods

In the determination of the dynamic structural response of linear MDoF individual systems, mode superposition principles are exploited. As such, the mode displacement method (MDM) is typically employed, where the high-frequency modes are truncated, based on the assumption that their contribution is negligible beyond a certain threshold (e.g. when the cumulated participating modal mass exceeds 90% [69]). This may lead to large inaccuracies in the evaluation of displacements and their derivatives, increasing complexity and computational demand. To alleviate this, various modal correction techniques have been proposed in the literature.

In the well known mode acceleration method (MAM) [70] a pseudo-static adjustment is appended to the MDM solution to account for the contribution of the higher modes, under the assumption that inertial effects of the high-frequency modes are negligible [71]. Several variants of MAM have been presented, including the works of Maddox [72], Hansteen and Bell [73] and later Cornwell et al. [74]. Despite the apparent value, a critical analysis undertaken by Soriano and Filho [75] showed that all the above methods are equivalent. MAM has also been extended to the case of non-stationary stochastic loading using Karhunen-Loéve expansion [76]. It has long been recognised that if the truncated modal frequencies are close to the input excitation frequencies, MAM can result in considerable errors.

Aiming in overcoming these drawbacks various force derivative methods (FDM), which are based on the time-derivatives of the excitation function, have been proposed, including the
works of Leung [77]. The latter, has further been extended to the case of damped systems by Camarda et al. [78] and has been used by Akgun [79]. As pointed out, however, these methods can result to large inaccuracies in non-classically damped systems [80].

A dynamic correction method (DCM) has been put forward by Borino and Muscolino [80], where the dynamic correction term obtained with a low number of modes, is appended to the pseudostatic response, and is also applicable to non-classically damped systems. If the excitation frequency is a unit-step, this approach simplifies to the MAM.

As pointed out by D’Aveni and Muscolino [81], the aforementioned methods require an analytical expression of the forcing function. Wilson et al. [82] suggested the superposition of Ritz vectors as an alternative, which also includes the static correction, resulting in a coupled set of equations which have to be numerically solved. Nour-Omid and Clough [83] proposed the use of Lanczos vectors to transform the equations of motion in a tri-diagonal form which can efficiently be solved numerically. Other variants of those methods have also been reported [84, 85]. Despite the apparent value, these methods have been criticised as practitioners can gain a better understanding of the dynamic behaviour by employing modal analysis [86].

Aiming at resolving these drawbacks, an improved DCM (IDCM) method formulated by D’Aveni and Muscolino [81] has been proposed, based on the use of modal analysis and Ritz vectors. The method enjoys the advantage of applicability to sampled excitation functions (e.g. accelerograms). The method, can be regarded as a generalisation of DCM, FDM and MAM.

Di Paola and Failla [87] highlighted that the correction terms in the aforementioned methods, are expressed in the full geometric rather than the reduced modal space, and proposed an analytical procedure to improve the response for classically damped systems. Palmeri and Lombardo [88] pointed out that the procedure is limited by convergence criteria and the benefits can be negligible.

In the context of the stochastic response evaluation of structural systems, correlation approaches include those of Maldonado and Singh [89] who exploited FDM to evaluate the random response of structural systems as well as Der Kiureghian and Nakamura [90] who proposed a modal combination rule to estimate cross-correlation coefficients. Coupled with the aforementioned limitations, these methods have been heavily criticised due to their underlying requirement of bounded statistical moments for the input, thus, failing for the case of a white noise excitation [86]. Benfratello and Muscolino [86], proposed a method based on the IDCM for the stochastic structural response evaluation. Although the authors demonstrated its application on a P-S system (without the CMS), it has been criticised as computationally inefficient [91]. Cacciola et al. [91], formulated a new stochastic mode acceleration method.
for Gaussian excitation systems, yet it offers increased computational demand and its application can be deemed difficult for continuous systems. Palmeri and Lombardo [88] formulated a Dynamic Modal Acceleration Method (DyMAM), which requires an auxiliary oscillator for each dynamic excitation with the number being significantly lower than the system’s DOFs, being computationally efficient and applicable to random and deterministic excitations as well as continuous and discrete systems.

To this end, no single procedure is widely acceptable with the design codes prescribing participation threshold criteria and the truncation errors remaining uncorrected. In the design and analysis domain of P-S systems, the need for these approaches is driven by the inherent degree of complexity and analysis particularisation. Notwithstanding such techniques have only been pursued for the case of individual subsystems, the CMS method motivates their applicability to the case of composite systems.

### 2.5 Damping Characterisation

An intrinsic aspect related to the seismic response of S systems is the characterisation of dissipative forces, which has been an active research area in the field of linear structural dynamics. In the time-domain analysis, damping is typically idealised as viscous, due to the associated modelling simplifications and the difficulties in representing the actual mechanisms of energy dissipation. Two procedures are readily available for constructing a consistent damping matrix of individual systems based on estimation of modal damping ratios. If the system possesses classical normal modes (i.e. if and only if the Caughey and O’Kelly condition is met [92]), a particular case of viscous damping, known as ‘Rayleigh damping’, can be assumed, expressing it as a linear combination of mass and stiffness. As a matter of fact, a more general form is available via a series expression the ‘Caughey damping’, in which Rayleigh is viewed as a special case. Lastly, a viable alternative is the superposition of the significant modal damping matrices [93].

### 2.6 Irregularities

The dynamic response of building structures to earthquakes is affected by irregularities in their arrangement, which typically arise from architectural, functional or accidental requirements (e.g. usage variations, inconsistencies in the construction process, damage, etcetera), either in plan (i.e. asymmetric distributions of mass, stiffness and strength) or in elevation (e.g. due to
discontinuities in structural elements or variations in the occupancy). In fact, irregular buildings tend to exhibit complicated modes of vibration [94, 95], e.g. with concentrated deformations in soft storeys or large torsional effects. As a result, they often suffer higher levels of damage when compared to regular structures, which prompted modern building codes to impose restrictions on various aspects of seismic design, with implications on structural modelling, allowed methods of analysis and behaviour factor.

With regards to vertical mass distribution, Eurocode 8 [69] specifies for regular buildings a criterion of no abrupt variations in the mass of individual storeys, without explicitly quantifying what would be an abrupt change. Conversely, other codes dictate that a vertical mass irregularity exists when the mass of a storey exceeds 150% [96, 97, 98] or 200% [99] of the mass of an adjacent storey (however a roof significantly lighter than the floor below would not be considered as an irregularity). Furthermore, with respect to the lateral stiffness, EC8 requires for a regular structure to be approximately symmetrical in plan in two orthogonal axes, with prescriptive limits given on the structural eccentricity orthogonal to the direction of the analysis.

Alongside the development of the above code requirements, several studies have been carried out to examine the seismic response of building structures with irregularities. Valmundsson and Nau [100] highlighted some inconsistencies in the mass, stiffness and strength criteria of regularity set out by the UBC. Das and Nau [101] studied the effects of vertical irregularity for both mass and stiffness, suggesting that the UBC restrictions might be too conservative. Choi [102] evaluated the seismic response of multi-storey frames, showing that the most severe cause of irregularity is when the change in the mass happens at the uppermost floors. Aydin [103] suggested that the results of an equivalent lateral force (ELF) procedure tend to overestimate those of time history analyses, independently of the degree of irregularity. More recently, Varadharajan et al. [104] proposed a single parameter to quantify the irregularity in terms of both magnitude and location, while design code quantification classifies irregularity on the basis of magnitude only.

Tezcan and Alhan [105] investigated torsionally irregular multi-storey structures by varying the location of shear walls and comparing ELF with dynamic analyses. Ozmen [106] examined the conditions that cause large torsional effects, while Kumar et al. [107] quantified the performance of symmetric and asymmetric buildings via pushover analyses. Lavan and De Stefano [108] and Gokdemir et al. [109] have recently studied the torsional effects induced by the non-coincidence of the centres of mass and stiffness on the seismic performance of frame structures.
All the above-referenced studies are focussed on the response analysis of irregular P structures, without addressing the effects on any S system. Nevertheless, given their key role to ensure the serviceability of buildings as well as the current shift towards performance-based earthquake engineering [9], an accurate estimation of the consequences that structural irregularities may have on S components appears of key importance.

### 2.7 System Uncertainty of the Primary Structure

#### 2.7.1 Uncertainty in the Partial Rigidity of Connections in P

It has long been recognised that the vast majority of connections used in the construction of steel frames function as semi-rigid joints [110]. The level of partial rigidity determines the amount of bending moment at the beam ends and the relative rotation between columns and beams (i.e. stiffer connections result in larger end moments in the beams and smaller deformations). Nevertheless, conventional methods used for the analysis of steel structures assume idealised beam-to-column connections, either fully rigid (i.e. complete rotational continuity) or pinned (i.e. no moment transfer), which may result in considerable inaccuracies in the bending moment diagrams, the prediction of the static and dynamic response and the sizing of members during the design [111].

Numerous contributions are available in the literature dealing with the dynamic analysis of structures with flexible connections (among others, Kawashima and Fujimoto [112]; Xu and Zhang [113], Sekulovic et al. [114] and references presented therein). All these studies are limited to deterministic models which can only be justifiable for fixed connections (fully restrained frames) and pinned ones (simple frames). Inherent uncertainties in the realisation of semi-rigid connections call for the application of probabilistic models for their stiffness and strength. This is particularly important in the case of wind and earthquake loads, as any uncertainty in such parameters directly affects the modal frequencies and modal shapes and in turn the dynamic performance of semi-rigid steel frames [114]. At present, limited efforts have been devoted in this area, including the ones investigating the effects of semi-rigid connections on the reliability of steel frames [115, 116].

#### 2.7.2 Effect of Uncertainty in P Quantified on S

The response of S structures, further to their own dynamic properties and the ground shaking specification, depends also on the characteristics of the supporting P system. Accordingly,
uncertainties in the structural properties may arise when modelling the material, geometry and boundary conditions of members, inducing variations in the dynamic response. Relevant contributions to the technical literature involve methods based on series expansions for linear systems, including the study of Katafygiotis and Papadimitriou [117], as well as Jensen and Iwan [118] and Muscolino et al. [119], who concluded that uncertainty in the stiffness strongly affects the S system’s response and reliability. More recently, investigations on coupled linear systems have shown that uncertainty on modal frequencies and damping ratios can indeed lead to significant variation on the response of S systems when these are tuned or nearly tuned with P structures [120].

Despite the apparent need to account for all uncertainty sources, the notion exists that uncertainty in the earthquake loading is more significant than the one associated with the structural behaviour [9]. To the authors’ best knowledge, however, no studies have been reported in the dedicated literature where the relative contribution of uncertainty in the P structure is quantified and compared with the effects of the inherent randomness in the seismic action.

2.8 Summary

This chapter has reviewed the existing contributions in the technical literature for the seismic response analysis of S structures, identifying the main shortcomings and areas for development. Based on the discussions carried out in the previous sections the following main points are concluded:

- Stochastic ground motion models represent an appealing alternative for characterising the seismic hazard;
- The current state of development for the analysis of S structures lags behind that of the primary ones. Further contributions are needed to model nonlinear S systems as well as the development of mathematical models to quantify their response;
- The component-mode synthesis (CMS) is a convenient method for the analysis of linear multi-degree-of-freedom systems accounting for the dynamic interaction and reducing numerical complexities;
- The cascade approximation is a simplified analysis method where the dynamic interaction is neglected. No decoupling criteria currently exist for the analysis of nonlinear systems. Further work is required to generate these, identifying the conditions where the cascade analysis is permitted;
2.8 Summary

• In line with the CMS, the Dynamic Modal Acceleration Method can be an appealing alternative to increase the response accuracy of linear S systems. Further studies are required to extend the method to composite P-S systems;

• The effects of irregularity on the response of S systems needs to be quantified;

• The relative contribution of uncertainty in the primary structure in comparison with the seismic input needs to be quantified on the S system;

The above key points identified, will be the subject of this dissertation.
CHAPTER 3

Combined Vibration and Decoupling of Nonlinear Secondary Oscillators

3.1 Introduction

The central theme of this chapter is to investigate the dynamic interaction effects of two-degree-of-freedom (2DoF) primary-secondary systems.

The equations governing the combined response (i.e. full dynamic interaction) of nonlinear secondary oscillators coupled with linear primary ones are first formulated in Section 3.2. Accordingly, bilinear, sliding and free-standing rocking blocks are considered for this purpose as representatives of a wider spectrum of secondary structures of engineering interest. The bilinear idealisation is chosen as the simplest example of nonlinear behaviour [45], while sliding and rocking account for the behaviour characterising common failure modes [121].

In Section 3.3, the cascade analysis of the nonlinear oscillators under investigation, is addressed whereby the feedback action of the secondary system on the primary one is neglected.

Finally, based on the resulting coupled equations of Section 3.2 a set of decoupling criteria are generated and are presented in Section 3.4, in the form of accumulated errors in the response history of each system, for various input parameters. The analysis is carried out for the case
of a pulse-type base excitation, chosen herein as an analytical approximation of near-source ground motions.

The results presented throughout this Chapter allow highlighting the conditions under which simplified analysis methods can be used and thus form the basis for the cascade response analysis, that is the topic of Chapter 4.

3.2 Combined Vibration Response of 2DoF Linear-Nonlinear System

The equations of motion governing the combined vibration response of a linear SDoF primary oscillator with a SDoF bilinear, sliding and free-standing secondary block are derived and are presented in the following.

3.2.1 Bilinear S - Linear P

The case of a bilinear-linear 2DoF S-P system is first considered, as depicted in Figure 3.1(a). The system is subjected to the horizontal ground acceleration \( \ddot{u}_g(t) \), where the overdot denotes differentiation with respect to time and \( u_s(t) \), \( u_p(t) \) are the unidirectional displacements of S and P, relative to the ground.

Figure 3.1(b) shows the forces acting on S and P, where \( m_s \) and \( m_p \) are the associated masses; \( k_p u_p \) represents the elastic restoring force of P, whereby \( k_p \) is the stiffness of a linear spring; \( c_s \dot{u}_s \) and \( c_p \dot{u}_p \) are the associated damping forces, where \( c_s = 2 \zeta_s \omega_s m_s \) and \( c_p = 2 \zeta_p \omega_p m_p \) are viscous damping coefficients, in which \( \{ \zeta_s, \zeta_p \} \) comprise the respective equivalent viscous damping ratios while \( \omega_s = \sqrt{k_s/m_s} \) and \( \omega_p = \sqrt{k_p/m_p} \) are the natural circular frequencies. In the above, \( u_s(t) = u_s^a(t) - u_p(t) \) conveniently expresses the response of S relative to the motion of P.

Furthermore, \( f_b \) represents the bilinear restoring force of S, expressed as:

\[
f_b(u_s(t), \dot{u}_s(t)) = \psi_s \omega_s^2 m_s u_s(t) + a_s m_s (1 - \psi_s) z(t), \tag{3.1}
\]

in which \( \psi_s \), in the range \( 0 \leq \psi_s < 1 \), is the post-yield to pre-yield stiffness ratio (Figure 3.1(c)); \( a_s \) represents the specific strength of the system, i.e. the level of the absolute response acceleration (i.e. \( \ddot{u}_p(t) + \ddot{u}_g(t) \)) required for the secondary oscillator to yield; \( z(t) \) is an auxiliary state variable satisfying \( |z(t)| \leq 1 \), ruled by [122]:

...
3.2 Combined Vibration Response of 2DoF Linear-Nonlinear System

Figure 3.1: Bilinear (a), sliding (d) and free-standing rocking (g) secondary oscillators on linear SDoF primary oscillator; corresponding free-body diagrams (b, e, h); force-displacement (c, f) and moment-rotation relationships.

\[
\dot{z}(t) = \frac{\ddot{u}_s(t) \omega_s^2}{a_s} \left[ 1 - H(\ddot{u}_s(t)) H(z(t) - 1) - H(-\ddot{u}_s(t)) H(-z(t) - 1) \right], \quad (3.2)
\]
where \( H(x) \) denotes the Heaviside unit step function:

\[
H(x) = \begin{cases} 
1, & x \geq 0 \\
0, & x < 0 
\end{cases}.
\]  (3.3)

Setting \( \psi_s = 0 \) in Eq. (3.1) corresponds to an elastic-perfectly-plastic case; alternatively, setting \( \psi_s = 1 \) results in the linear regime of motion with \( f_b(t) = \omega_s^2 m_s u_s(t) \).

The dynamic equilibrium of the mass \( m_s \) in the horizontal direction gives:

\[
\ddot{u}_s(t) = -2 \zeta_s \omega_s \dot{u}_s(t) - \frac{f_b(u_s(t), \dot{u}_s(t))}{m_s} - \ddot{u}_p(t) - \ddot{u}_g(t);  \quad (3.4)
\]

with assumed initial conditions \( u_s(0) = 0 \) and \( \dot{u}_s(0) = 0 \).

The equation of motion of \( P \) is derived in a similar manner as:

\[
\ddot{u}_p(t) = -\gamma \ddot{u}_s(t) - 2 \zeta_p \omega_p \dot{u}_p(t) - \omega_p^2 u_p(t) / (1 + \gamma) - \ddot{u}_g(t);  \quad (3.5)
\]

with initial conditions \( u_p(0) = 0 \) and \( \dot{u}_p(0) = 0 \), where \( \gamma = m_s/m_p \) is the secondary-primary mass ratio.

Equations (3.4) and (3.5) can be cast in a state space form (i.e. explicit expressions of the state variables) and be solved together. In this case, the state vector is:

\[
y(t) = \begin{bmatrix} u_s(t) \dot{u}_s(t) z(t) u_p(t) \dot{u}_p(t) \end{bmatrix}^T, \quad (3.6)
\]

derived as:

\[
\dot{y}(t) = \begin{bmatrix} \ddot{u}_s(t) \
-(1 + \gamma) \begin{bmatrix} 2 \zeta_s \omega_s \dot{u}_s(t) + \psi_s \omega_s^2 u_s(t) + a_s (1 - \psi_s) z(t) \end{bmatrix} + 2 \zeta_p \omega_p \dot{u}_p(t) + \omega_p^2 u_p(t) \\
\dot{u}_p(t) \\
\gamma \begin{bmatrix} 2 \zeta_s \omega_s \dot{u}_s(t) + \psi_s \omega_s^2 u_s(t) + a_s (1 - \psi_s) z(t) \end{bmatrix} - 2 \zeta_p \omega_p \dot{u}_p(t) - \omega_p^2 u_p(t) - \ddot{u}_g(t) \end{bmatrix}, \quad (3.7)
\]
3.2 Combined Vibration Response of 2DoF Linear-Nonlinear System

3.2.2 Sliding S - Linear P

On the basis of Figure 3.1(d), a rigid-perfectly plastic SDoF secondary system is considered as the limiting case of the above restoring force by setting $\psi \rightarrow 0$, $\zeta \rightarrow 0$ and $\omega_\alpha^2 \rightarrow +\infty$. Accordingly, the system exhibits infinite pre-yielding stiffness and infinite ductility.

The restoring force, thus, takes the form:

$$f_s = \begin{cases} 
\mu_s g m_s, & \dot{u}_s(t) > 0 \\
\in [-\mu_s g m_s, \mu_s g m_s], & \dot{u}_s(t) = 0 \\
-\mu_s g m_s, & \dot{u}_s(t) < 0 
\end{cases}$$

(3.8)

in which $\mu_s = a_s/g$ is the coefficient of sliding friction assuming horizontal contact surface and $g$ is the acceleration due to gravity. Contrary to the typical two-value representation of the contact friction, the formalism given by Eq. (3.8) also contains information about sticking (i.e. $\dot{u}_s(t) = 0$), where $f_s$ may take any value between $-\mu_s g m_s$ and $\mu_s g m_s$ [51].

As a result, equilibrium of the forces (Figure 3.1(e)) gives the equation of motion for the sliding regime of motion:

$$\ddot{u}_s(t) = -\mu_s g sgn(\dot{u}_s(t)) - \ddot{u}_p(t) - \ddot{u}_g(t),$$

(3.9)

and $sgn(\cdot)$ is the signum function (i.e. $sgn(x) = +1$ if $x > 0$, $sgn(x) = -1$ if $x < 0$, and $sgn(x) = 0$ if $x = 0$). The above equation is valid only within the intervals of slipping (i.e. $\dot{u}_s(t) \neq 0$), while no relative motion is exhibited (i.e. $u_s = \dot{u}_s = 0$), when in the sticking phase.

In the case of P, the equation of motion is identical to the bilinear case, namely Eq. (3.5), but the term $u_s(t)$ corresponds herein to the sliding response given by Eq. (3.9). Furthermore, during the sticking phase, $\ddot{u}_s(t) = 0$ and thus the first term in Eq. (3.5) has to be neglected.

In this case, the equation can be visualised as the same of a SDoF P oscillator whose mass is augmented by $m_s$.

The initiation condition for the sliding regime of motion is therefore set to $|\ddot{u}_p(t) + \ddot{u}_g(t)| = \mu_s g$ (Figure 3.1(f)), $\ddot{u}_p(t)$ being a solution of Eq. (3.5). Following initiation, an instantaneous stop or a full stop can occur in the system once the velocity drops to zero ($\dot{u}_s(t) = 0$). In the former case, the motion will reverse or it will continue in the same direction, while in the latter case the system will remain at rest until the initiation condition is exceeded again.

In solving Equations (3.5) and (3.9) the state vector takes the form:
\[
\mathbf{y}(t) = \begin{bmatrix} \dot{u}_s(t); \ddot{u}_s(t); \dot{u}_p(t); \ddot{u}_p(t) \end{bmatrix}^\top, \tag{3.10}
\]

and the time derivative during sliding motion is:

\[
\dot{y}(t) = \begin{cases} 
\dot{u}_s(t) \\
-(1 + \gamma) a_s \text{sgn}(\dot{u}_s(t)) + 2 \zeta_p \omega_p \dot{u}_p(t) + \omega_p^2 u_p(t) \\
\gamma a_s \text{sgn}(\dot{u}_s(t)) - 2 \zeta_p \omega_p \ddot{u}_p(t) - \omega_p^2 u_p(t) - \ddot{u}_g(t) 
\end{cases}, \tag{3.11}
\]

### 3.2.3 Rocking S - Linear P

Let us consider the case of a rectangular secondary block standing free on a primary oscillator. If the coefficient of sliding friction is sufficiently large (\(\mu_s \to +\infty\)), the block can experience pure rocking motion, oscillating about its centres of rotation (i.e. pivot points) \(O\) and \(O'\), as illustrated in Figure 3.1(g). Based on the assumption of zero vertical ground acceleration, equilibrium of moments about the centres of rotation gives the equation governing its response during the rocking regime of motion, that can be written in compact form as [57]:

\[
\ddot{\theta}_s(t) = -p^2 \left\{ \sin(\theta_s(t)) + \frac{\ddot{u}_p(t) + \ddot{u}_g(t)}{g} \cos(\theta_s(t)) \right\}; \tag{3.12}
\]

where \(A(t) = \alpha \text{sgn}(\theta_s(t)) - \theta_s(t)\) and \(\theta_s(t)\) is the rotation response; \(\alpha = \tan^{-1}(B/H)\) is the slenderness angle, being a function of the width (B) and height (H) of the block; \(p = \sqrt{3g/(4R)}\) is a geometrical parameter (e.g. \(p \approx 2\) rad/s for an electrical transformer), where \(R\) is half the block’s diagonal. Contrary to [57], the input excitation in Eq. (3.12) (i.e. the term \(\ddot{u}_p(t) + \ddot{u}_g(t)\)) is further enriched with the response of P, \(\ddot{u}_p(t)\).

Furthermore, horizontal force equilibrium for P gives:

\[
\dddot{u}_p(t) = -2 \zeta_p \omega_p \dot{u}_p(t) - \omega_p^2 u_p(t) + \frac{f_r(t)}{m_p}; \quad \dddot{u}_p(t) = 0; \quad u_p(0) = 0; \quad \dot{u}_p(0) = 0, \tag{3.13}
\]

where \(f_r(t)\) can be interpreted as a restoring force, which accounts for the coupling of S and P and can be evaluated according to Figure 3.1(h) as:
in which \( u_a^a(t) \) is the translational horizontal displacement of the block, relative to the ground and is readily determined from Figure 3.1(g) as:

\[
u_a^a(t) = \text{sgn}(\theta_s(t))R \sin(\alpha) - R \sin(A(t)) + u_p(t) .
\]

Differentiating twice Eq. (3.15) with respect to time and substituting the result in Eq. (3.14) gives:

\[
f_r(t) = -\gamma m_p \left\{ R(\dot{\theta}_s(t))^2 \sin(A(t)) + R \ddot{\theta}_s(t) \cos(A(t)) + \ddot{u}_p(t) + \ddot{u}_g(t) \right\}.
\]

Figure 3.1(i) depicts the moment rotation relationship of the system, which will initially possess infinite stiffness until the applied moment reaches a value of \( M = m_u g R \sin(\alpha) \), and a softening branch (negative stiffness) will follow thereafter. Interestingly, this behaviour reminiscences the sliding secondary block (Figure 3.1(f)), except that the behaviour of the latter is hysteretic and the stiffness is non-negative.

The initiation condition for Eq. (3.12) is \( |\ddot{u}_a(t)| = g \tan(\alpha) \). Similar to the sliding case, \( \ddot{u}_p(t) \) is obtained by directly solving Eq. (3.5) after setting \( \ddot{u}_a(t) = 0 \). Following initiation, the block will rotate about the pivot point \( O \) and \( \theta_s \) will increase in magnitude until the kinetic energy \( T(t) \) stored in the system reaches a zero value and the total energy of the system \( E(t) \), is all in the form of potential energy \( U(t) \). The rotation \( \theta_s \) will then decrease until the block eventually reaches its initial position (\( \theta_s = 0 \)) during impact, at which point the total energy is all in the form of kinetic energy, where:

\[
T(t) = \frac{1}{2} I_o (\dot{\theta}_s(t))^2 ,
\]

and

\[
U(t) = m_u g R \left[ \cos \{ H(|\theta_s(t)|) \alpha - |\theta_s(t)| \} - \cos(\alpha) \right].
\]

Assuming that the block does not bounce back, the regime of motion will then switch, and the block will continue rotating smoothly about the pivot point \( O' \). Based on [123], conservation of linear momentum gives:
(3.19) where the superscripted plus and minus signs denote here the associated quantity after and prior to impact, respectively.

Conservation of angular momentum about \( O' \) further results in:

\[
I_o \dot{\theta}_s^- - m_s R B \sin(\alpha) \dot{\theta}_s^- + \frac{1}{2} m_s H \dot{u}_p^- = I_o \dot{\theta}_s^+ + \frac{1}{2} m_s H \dot{u}_p^+ ,
\]

(3.20)

where \( I_o \) is the block’s moment of inertia about \( O \) or \( O' \) (assuming that the rocking block has a vertical axis of symmetry). For a rectangular block, for instance, \( I_o = (4/3) m_s R^2 \) and therefore Eq. (3.20) reduces to:

\[
8 R^2 \dot{\theta}_s^- - 6 R B \sin(\alpha) \dot{\theta}_s^- + 3 H \dot{u}_p^- = 8 R^2 \dot{\theta}_s^+ + 3 H \dot{u}_p^+ .
\]

(3.21)

Upon substitution of Eq. (3.21) in Eq. (3.19), one obtains:

\[
\dot{\theta}_s^+ = \frac{-2 + \gamma - 3(2 + \gamma) \cos(2\alpha)}{-8 - 5\gamma + 3\gamma \cos(2\alpha)} \dot{\theta}_s^- ,
\]

(3.22)

and

\[
\dot{u}_p^+ = \dot{u}_p^- + \frac{6 H \gamma \sin^2(\alpha)}{8 + 5\gamma - 3\gamma \cos(2\alpha)} \dot{\theta}_s^- .
\]

(3.23)

From Eq. (3.22), the ratio of kinetic energy after and prior to impact is:

\[
r = \left( \frac{\dot{\theta}_s^+}{\dot{\theta}_s^-} \right)^2 = \left[ \frac{-2 + \gamma - 3(2 + \gamma) \cos(2\alpha)}{-8 - 5\gamma + 3\gamma \cos(2\alpha)} \right]^2 .
\]

(3.24)

The coefficient of restitution \( \varepsilon \) accounts for the attenuation of the response due to impact by reducing the post-impact angular velocity \( \dot{\theta}_s^+ \) [55, 61], i.e. \( \dot{\theta}_s^- = \varepsilon \dot{\theta}_s^- \). Based on the above expression, its maximum permissible value is \( \varepsilon_{\text{max}} = \sqrt{r} \) in the range \( 0 < \varepsilon \leq \varepsilon_{\text{max}} < 1 \). Accordingly, the higher \( \varepsilon \), the smaller the energy loss due to impact.

Notably, the above equations represent only the case of the maximum coefficient of restitution as implementing a smaller value would require altering also Eq. (3.23).

Figure 3.2 plots \( r \) as a function of the slenderness angle \( \alpha \), and it is shown that it can take values between one (for slender blocks, when \( \alpha \to 0 \)) and zero. Evidently, when \( \gamma > 0 \) additional energy is lost during impact owing to the translational velocity of \( P \), while as \( \gamma \to 0 \),
the behaviour approaches the case of a block rocking on a rigid foundation (i.e. \( \varepsilon_{\text{max}} \to 1 - 3 \sin^2(\alpha)/2 \) and \( \dot{u}_p = \ddot{u}_p \)).

The expressions derived herein, are analogous to the ones presented in [123] for base isolated rigid rocking blocks. Equations (3.12) and Eq. (3.13) are finally expressed in state-space form and are solved together, while \( \varepsilon_{\text{max}} = \sqrt{r} \) (based on Eq. (3.24)) is used to attenuate the angular velocity at every impact. Accordingly, the associated state vector is:

\[
y(t) = \begin{pmatrix} \theta_s(t) \\ \dot{\theta}_s(t) \\ u_p(t) \\ \dot{u}_p(t) \end{pmatrix}^T, \tag{3.25}
\]

and the time derivative vector reads:

\[
\dot{y}(t) = \begin{pmatrix} -p^2 \left\{ \sin(A(t)) + \cos(A(t)) \left[ \frac{-2 \zeta \omega_p \dot{u}_p(t) - \omega_p^2 u_p(t) - \gamma R \dot{\theta}_s(t)^2 \sin(A(t)) + \gamma R p^2 \sin(A(t)) \cos(A(t))}{g(1+\gamma) - \gamma R p^2 \cos^2(A(t))} \right] \right\} \\ \dot{\theta}_s(t) \\ -2 \zeta \omega_p \dot{u}_p(t) - \omega_p^2 u_p(t) - \gamma R \dot{\theta}_s(t)^2 \sin(A(t)) + \gamma R p^2 \sin(A(t)) \cos(A(t)) \right\} \right\} \\ \dot{u}_p(t) \\ \dot{u}_g(t) \end{pmatrix} \tag{3.26}
\]

3.2.4 Implementation details

Numerical integration of the equations derived in the previous Section is carried out with build-in MATLAB [124] Ordinary Differential Equation (ODE) solvers. Specifically, ODE45 is used, which is based on an explicit 4th and 5th order Runge-Kutta formulation, namely, the Dormand-Prince pair [125, 126].
The implementation has been performed with consistent initial conditions and by setting MATLAB’s odeset parameter values \( \text{AbsTol} = \text{RelTol} = 10^{-8} \) and \( \text{Refine} = 4 \), which refer to relative and absolute solution tolerances as well as interpolation output, respectively.

Furthermore, while for the bilinear case the presence of the state variable allows direct integration of Eq. (3.7), the option ‘Events’ has been invoked for the sliding and rocking cases. In particular, this allowed for approximately identifying state events (i.e. transition points of piecewise solutions such as the initiation and change in the regime of motion) and breaking down the solution in parts which have been later pieced together.

When the system exhibits no sliding or rocking motion (i.e. initiation), or following overturning of the rocking block, it was found critical, for computational efficiency and solution accuracy, to seek a solution out of Eq. (3.5) after setting \( \ddot{u}_s(t) = 0 \) rather than solving Eq. (3.11) and Eq. (3.26), respectively. This is due to the presence of zeroed state variables in the full expressions, which inevitably introduces numerical errors, requiring lower solution tolerances and thus a much slower convergence.

Specifically for the case of the rocking block, as the oscillations tend to reduce in amplitude once the strong motion phase of the seismic event is finished, the system will continuously transition between the two regimes of motion with an increasing frequency. As a result, the integration scheme has to be interrupted several times to identify state events, on occasions where the response of the block can be regarded as negligible. Hence, rather than continuing integrating the equations until the end of the duration of motion (which is theoretically possible), it is important that a stopping criterion is used. Herein, when the total energy of the system becomes sufficiently small, the rocking block is brought to rest. The ratio of the system’s total energy is readily determined by evaluating Eq. (3.17) at the time of impact, and Eq. (3.18) at the verge of overturning (i.e. when \( \theta_s = \alpha \)), leading to the stopping condition:

\[
\frac{2}{3} \frac{R (\dot{\theta}_s(t))^2}{g (1 - \cos(\alpha))} < 10^{-5} .
\]  

Integration proceeds thereafter, solely for the primary system based on Eq. (3.5), using the initial conditions from the last step, until a new initiation is identified.

### 3.3 Nonlinear Secondary Oscillators in Cascade

In the preceding section, the combined response (i.e. full dynamic interaction) of a class of nonlinear SDoF secondary oscillators vibrating on linear SDoF oscillators was derived. Notably, the parameter \( \gamma = m_s/m_p \) in Equations (3.7), (3.11) and (3.26) accounts for the relative
significance of the feedback action on P, due to the presence of S. In practical applications, however, particularly in the case of nonlinear secondary systems in multi-degree-of-freedom structures, the resulting equations of motion that govern the full dynamic system can be cumbersome to consider and therefore the feedback action on P is usually not accounted for, based on the assumption that its contribution can be regarded negligible, leading to the so-called ‘cascade approximation’. With reference to the preceding section, the latter can be realised when the secondary-primary mass ratio is sufficiently low \(m_s \ll m_p\) and \(\gamma \to 0\). Setting \(\gamma = 0\) in the above, the equations decouple and can sequentially be considered, i.e. P motion is calculated first, then S. In this case, the equations are identical to the ones derived for a cascade system where the dynamic interaction is neglected. It is noted that the case \(\gamma = 0\) does not imply a massless S system but merely the absence of the dynamic interaction.

### 3.4 Decoupling Criteria

In this section, the conditions under which simplified analysis methods can be adopted for the seismic response evaluation of nonlinear secondary structures are investigated. Specifically, the three nonlinear cases, namely, the bilinear, sliding and rocking S, vibrating in conjunction with a linear SDoF P, are considered. Under the assumption of a pulse-type ground acceleration, and based on the equations derived in § 3.2, the results of a parametric study are presented in the following, and decoupling criteria are proposed.

#### 3.4.1 Sinusoidal Pulse Ground Acceleration

The simplest case of a ground acceleration comprising of a full-cycle pulse is considered:

\[
\ddot{u}_g(t) = \begin{cases} 
    a_g \sin (\omega_g t + \phi), & 0 \leq t \leq 2\pi/\omega_g, \\
    0, & \text{otherwise}
\end{cases}
\]

where \(a_g\), \(\omega_g\) and \(\phi = \{0, \pi/2\}\) are the amplitude, frequency and phase angle of the pulse, respectively, the latter corresponding to a sine and cosine pulse, and \(t_f = 2\pi/\omega_g\) is the duration of the time history, as depicted in Figure 3.3.

Although the external excitation considered herein cannot be regarded as physically realisable, it is a mathematically convenient approximation of near-fault ground motions, allowing the derivation of closed-form expressions for shedding light in the response of dynamical sys-
tems to ground excitations. This type of excitation has extensively considered in the literature for this purpose [57].

Figure 3.4, adapted from [58], plots the ground acceleration, velocity and displacement histories of the 1994 Northridge earthquake record at the Rinaldi station and those of the two pulses considered. Interestingly, it is shown, that both pulses can approximate the record; that is, the component lies between a forward and a forward-and-back pulse. Furthermore, the pulses possess continuous and differentiable displacement histories that gradually build up, and zero velocity after the end of the excitation [57]. Such characteristics are not present in half-cycle pulses which are thus not considered [53].

3.4.2 Dimensional Analysis

In order to investigate the response of the systems, it is useful to recast the response quantities of interest in dimensionless form, based on the principle of dimensional homogeneity, such that the number of terms required to fully characterise the response is smaller than the number of engineering design variables defining the problem.

For the linear-bilinear system in § 3.2.1 the response \( u(\gamma, \omega_g, \omega_p, \omega_s, \zeta_p, \zeta_s, a_g, \phi, \psi_s, a_s) \) is a function of ten variables \( (v_{\Pi} - 1 = 10) \) with two reference dimensions \( (r_{\Pi} = 2) \), namely length \([L]\) and time \([T]\). Based on Buckingham’s \( \Pi \) - theorem [127], there exist \( n_{\Pi} = v_{\Pi} - r_{\Pi} = 9 \) independent dimensionless products \( (\Pi_1, \Pi_2 \ldots \Pi_{n_{\Pi}}) \) giving rise to a reduced set of \( n_{\Pi} - 1 = 8 \) variables fully characterising the response. Herein, \( a_g \) and \( \omega_p \) are chosen as repeating variables, leading to the dimensionless displacement \( u^* (\gamma, \omega_g^*, \omega_p^*, \zeta_p, \zeta_s, \phi, \psi_s, a_s^*) = \frac{u \omega_p^2}{a_g} \) that is also a function of the dimensionless time \( t^* = \omega_p t \). Accordingly, \( \omega_g^* = \omega_g / \omega_p \) and \( \omega_s^* = \omega_s / \omega_p \) are the frequencies of the pulse and the secondary system, respectively, relative to
3.4 Decoupling Criteria

that of the primary structure, namely, $\omega_p$, while $a^*_s = a_s/a_g$ is the dimensionless strength of the system (e.g. yielding occurs when $|a^*_s| < 1$). Alternatively, the ductility ratio $\mu_d$ can be used in place of $a^*_s$, if the response is to be determined in terms of ductility spectra. The displacement
response of the linear-bilinear system is thus normalised with \( a_g/\omega_p^2 \), a representative measure of the persistence of the excitation [45].

In a similar manner, the dimensionless displacement of the linear-sliding system reads \( u^* (\gamma, \omega^*_g, \zeta_p, \phi, a^*_s) = u \omega_p^2/a_g \). Finally, the response of the linear-rocking system is expressed as \( \theta^*_s (\gamma, \omega^*_g, \zeta_p, \phi, \alpha, p^*, \varepsilon, a^*_s) = \theta_s/\alpha \) in which \( p^* = p/\omega_p \) is the dimensionless dynamic parameter of the rocking block and \( a^*_s = a_g/g \) the dimensionless pulse amplitude. It should be noted that, while introducing the ratio \( \theta_s/\alpha \) was not strictly required by the Buckingham’s \( \Pi \)-theorem, as both \( \theta_s \) and \( \alpha \) are dimensionless, in this way \( \theta^*_s = 1 \) identifies the condition where overturning is incipient.

### 3.4.3 Numerical Application

Equations (3.7), (3.11) and (3.26) are numerically implemented as detailed in § 3.2.4 with the aim of identifying the conditions in which the cascade approximation is acceptable for engineering purposes. In doing so, the response for each S-P assembly (i.e. considering the full-dynamic interaction with \( \gamma \neq 0 \)) is compared to the cascade approximation (i.e. setting \( \gamma = 0 \)) and the percentage errors are evaluated over various parameter combinations through:

\[
\epsilon = \frac{\int_{t_0}^{t_f} |x_{\text{ref}}(t) - x(t)| \, dt}{\int_{t_0}^{t_f} |x_{\text{ref}}(t)| \, dt} \times 100, \tag{3.29}
\]

where \( x \) represents the response quantity of interest with respect to its reference value \( x_{\text{ref}} \) and \( |\cdot|_{\text{max}} \) denotes the peak absolute value of \( (\cdot) \). Evidently, Eq. (3.29) accumulates the error over the duration \( t_f \) of the response time history.

The results are reported for the case of a single-cycle sine pulse, i.e. \( \phi = 0 \), and the range \( 0 \leq \gamma \leq 1.5 \) is considered. Furthermore, each of the remaining parameters is assigned a prescribed range, while all the other parameters are kept constant, so that the influence of a single parameter is investigated each time. Specifically, the dimensionless pulse frequency range is chosen as \( 0.07 \leq \omega^*_p \leq 1.2 \) (i.e. \( 0.35 \leq t_f \leq 6 \) [58]), with a reference value \( \omega^*_p = 0.14 \) (i.e. \( t_f = 3s, \omega_p = 14.76 \text{ rad/s} \)). Likewise, the range of damping ratios of \( P \) is \( 0 \leq \zeta_p \leq 0.1 \), with a reference value \( \zeta_p = 0.05 \). For the bilinear system, the frequency range is \( 0.01 \leq \omega^*_s \leq 2 \), the strength is \( 0.1 \leq a^*_s \leq 2.5 \), chosen to cover the elastic regime (i.e. \( |a^*_s| > 1 \)) and the inelastic regime (i.e. \( |a^*_s| < 1 \)), the post-yield to pre-yield stiffness ratio is \( 0 \leq \psi^*_s \leq 0.5 \) and the damping ratio is \( 0 \leq \zeta_s \leq 0.05 \), with the associated reference values being \( \omega^*_s = 0.3, a^*_s = 1, \psi_s = 0 \) and \( \zeta_s = 0.02 \), respectively. For the sliding system, \( 0.1 \leq a^*_s \leq 1.2 \) is assumed, similar to the bilinear case, to investigate the sticking and sliding regimes of motion,
with reference value $a_s^* = 0.5$. Finally, for the rocking system, the slenderness and dynamic parameter are chosen as $0.05 \leq \alpha \leq 0.6$ and $0.014 \leq p^* \leq 0.34$ (i.e. $0.2 \leq p \leq 5$ normalised by $\omega_p = 14.76$), with the associated reference values of $\alpha = 0.25$ and $p^* = 0.136$ (i.e. $p = 2$). Given that rocking initiates when $a_s^* > 0.2$ and for $a_s^* > 0.32$ overturning is reached at an earlier instant in time without oscillations, the dimensionless pulse amplitude range is chosen as $0.2 \leq a_s^* \leq 0.5$ with a reference value $a_s^* = 0.28$.

### 3.4.3.1 Bilinear Secondary Oscillator

Figure 3.5 shows the effect of $\gamma$ on the response of the bilinear-linear S-P system at different combinations of the frequency parameters $\omega_g^*$ and $\omega_s^*$, with all other parameters assuming the reference values reported above. The case of $\omega_g^* = 0.14$ and $\omega_s^* = 0.3$ is first examined in terms of absolute acceleration (Figure 3.5(a)) and displacement response (Figure 3.5(b)) time histories of S and P, respectively, where the thick solid line shows the response due to $\gamma = 0$ (i.e. the cascade approximation), and the thin solid one corresponds to $\gamma = 1.5$ (i.e. the extreme case where S is 50% heavier than P). It is noted that although the focus herein is for $\gamma < 1$ the case of $\gamma > 1$ is reported to identify trends (e.g. any discontinuities in the response when the mass ratio reverses).

Two intermediate values are also depicted, namely $\gamma = 0.5$ (dashed line) and $\gamma = 1$ (dotted line), while the grey lines show other 46 values equally distributed in the range $0 \leq \gamma \leq 1.5$.

As expected, the response of both P and S increases with $\gamma$ as the contribution of the feedback action of S on P becomes more prominent. The results shown correspond to cumulative errors of 11% and 4% for $\gamma = 0.5$, 21% and 9% for $\gamma = 1$, 30% and 12% for $\gamma = 1.5$, for P and S, respectively, with the effect more profound on P and eventually influencing the response of S.

Setting $\omega_g^* = 1$, higher discrepancies are shown on the response of both P (Figure 3.5(c)) and S (Figure 3.5(d)). In particular, the cumulative errors are 45% and 29% for $\gamma = 0.5$, 84% and 59% for $\gamma = 1$, 109% and 85% for $\gamma = 1.5$, for P and S, respectively.

When $\omega_g^* = 0.14$ and $\omega_s^* = 1$, the system is in resonance and, contrary to the previous cases, the discrepancies are more profound on the response of S. Specifically, the cumulative errors amount to 15% and 85% for $\gamma = 0.5$, 26% and 35% for $\gamma = 1$, 35% and 55% for $\gamma = 1.5$, for P (Figure 3.5(e)) and S (Figure 3.5(f)), respectively. Interestingly, the results in this case are not ordered as $\gamma = 0.5$ results in highest error on S compared to $\gamma = 1$ and $\gamma = 1.5$, suggesting that it may be difficult to derive decoupling criteria when the system is in resonance.
Finally, when $\omega^*_g = \omega^*_s = 1$ the associated errors of P (Figure 3.5(g)) and S (Figure 3.5(h)) further increase to 168% and 40% for $\gamma = 0.5$, 180% and 75% for $\gamma = 1$, 169% and 146% for $\gamma = 1.5$.

Figures 3.6 and 3.7 present regions of 5%, 10%, 15% and 20% cumulative errors of the system under consideration, which allow assessing the appropriateness of the cascade approximation for a given parameter combination. Figures 3.6(a) and 3.6(b) show the effect of $\gamma$ and $\omega^*_s$ on P and S, respectively, when the remaining parameters assume reference values. Evidently, the influence of $\omega^*_s$ is more prominent on the response of S. Provided $\omega^*_g$ does not approach unity and as $\gamma \to 0$, the cascade approximation can be used. Specifically, provided that $\gamma < 0.02$ and $\gamma < 0.004$ for P and S, respectively, the cascade approximation results in errors less than 5% for all $\omega^*_s$ considered. Similarly, this is also the case when $\omega^*_s < 0.05$ and $\omega^*_s < 0.21$ for P and S, respectively, for all values of $\gamma$.

The influence of $\omega^*_g$ is examined in Figures 3.6(c) and 3.6(d). As shown, $\omega^*_g$ appears to be less dominant on S than the case of $\omega^*_s$; that is, higher values of $\gamma$ can be considered for the same level of error, and provided that $\omega^*_g > 0.7$, the error regions appear constant as demonstrated by pseudo-vertical lines in these figures.

The case of $a^*_s$ is considered in Figures 3.6(e) and 3.6(f). Overall, the choice of $a^*_s$ appears to be less critical on the evolution of the errors when compared to the two previous cases. Evidently, provided that $\gamma < 0.1$, the cascade approximation is permissible for both P and S, resulting in cumulative errors lower than 5%. Furthermore, when $a^*_s > 1.8$ the error regions are constant. While for P the response depends both on the selection of $\gamma$ and $a^*_s$, provided that $a^*_s < 0.7$, regardless of the choice of $\gamma$, the cascade approximation will result in errors less than 5% for S.

The effect of $\psi_s$ is examined in Figures 3.6(g) and 3.6(h). Overall, the curves are ordered and, provided that $\gamma < 0.14$, the cascade approximation is permissible, resulting in errors less than 5%, regardless of the choice of $\psi_s$.

Figures 3.7(a), 3.7(b), 3.7(c) and 3.7(d) show the region plots with respect to changes in $\zeta_s$ and $\zeta_p$ with higher cumulative errors for P. As shown, the cascade approximation is satisfactory for both these cases, provided that $\gamma < 0.18$, with errors lower than 5%. Evidently, the results appear less sensitive to changes in the damping ratios $\zeta_s$ and $\zeta_p$ when compared to the previous cases (relatively lower error regions).

Overall, the results indicate that the feedback action of S on P can influence the response of both systems. The permissibility of the cascade approximation for engineering applications is primarily dictated by the selection of the two frequency parameters $\omega^*_g$, $\omega^*_s$ and $a^*_s$, while varia-
tions on $\psi_s$, $\zeta_s$ and $\zeta_p$ were found to have lower effects on the evolution of the errors. Provided that the system does not approach resonance, the cascade approximation can be considered satisfactory when $\gamma < 0.1$ and will improve as $\gamma \to 0$.

### 3.4.3.2 Sliding Secondary Oscillator

The effect of $\gamma$ is investigated on the response history of the sliding-linear S-P system (Figure 3.8). The case of $\omega_g^s = 0.14$ is first considered (Figures 3.8(a) and 3.8(b)). As shown, the presence of the secondary oscillator alters the absolute acceleration response of P, even when there is no sliding motion (i.e. $t^* < 4$), simply because there is a change in the mass. The discontinuities shown in the absolute acceleration history of P (e.g. at $t^* = 53$) occur at the instants where the regime of motion switches and the velocity of S approaches zero, until the ODE solver eventually identifies the time instant within the prescribed tolerance (see: § 3.2.4). Similar to the bilinear case, the variations in the response increase with $\gamma$, resulting in cumulative errors of 22% and 9% for $\gamma = 0.5$, 39% and 9% for $\gamma = 1.5$, 51% and 15% for $\gamma = 1.5$, for P and S, respectively, suggesting that the cascade approximation in this case underestimates the sliding distance experienced by S.

Figures 3.8(c) and 3.8(d) illustrate the response histories when $\omega_g^s = 1$. Evidently, significant variations are shown, both in magnitude and frequency leading to overestimation of the sliding response by the cascade approximation. Specifically, the results shown, amount to cumulative errors of 112% and 40% for $\gamma = 0.5$, 146% and 89% for $\gamma = 1$, 181% and 120% for $\gamma = 1.5$, for P and S, respectively.

Similarly to the analyses results presented for the bilinear S, Figure 3.9 illustrates regions of the 5%, 10%, 15% and 20% errors, for the parameters governing the S system now under consideration. The case of $\omega_g^s$ is examined in Figures 3.9(a) and 3.9(b). The results indicate that the choice of $\omega_g^s$ highly influences the evolution of the errors. In the case of S, these are large inaccuracies occurring for $\omega_g^s = 0.55$. Provided that $\gamma < 0.01$ (i.e. when S is about 100 times lighter than P), the cascade approximation will result in errors lower than 5% for all values of $\omega_g^s$ considered.

The effect of the dimensionless specific strength $a^s_g$ is shown in Figures 3.9(c) and 3.9(d), for P and S, respectively. The results reported herein are limited to the case where $a^s_g \leq 1.2$ and the system exhibits sliding motion. As shown, the selection of $a^s_g$ can play a critical role in deciding whether the cascade approximation is permitted. Interestingly, when $a^s_g < 0.23$ the approximation results in errors lower than 5% for S, for all values of $\gamma$, suggesting that decoupling is particularly effective when higher displacements are exhibited. Alternatively, the
Figure 3.5: Effect of the S-P parameter $\gamma$ on the absolute acceleration response of P (left), and the displacement response of the bilinear S (right): $\omega^*_g = 0.14, \omega^*_s = 0.3$ (a, b); $\omega^*_g = 1, \omega^*_s = 0.3$ (c, d); $\omega^*_g = 0.14, \omega^*_s = 1$ (e, f) and $\omega^*_g = 1, \omega^*_s = 1$ (g, h).
Figure 3.6: Effect of the S-P mass ratio parameter $\gamma$, the dimensionless circular frequency $\omega_s^*$, the pulse frequency $\omega_g^*$, the specific strength $a_s^*$ and the post-yield to pre-yield stiffness ratio $\psi_s$ on the absolute acceleration response of P (a, c, e, g) and the displacement response of the bilinear S (b, d, f, h), respectively, in the form of region plots corresponding to 5%, 10%, 15% and 20% cumulative error. Reference values of $\omega_g^* = 0.14$, $\omega_s^* = 0.3$, $a_s^* = 1$, $\psi_s = 0$, $\zeta_s = 0.02$ and $\zeta_p = 0.05$ are assumed.
approximation can be considered satisfactory for all values of $a_s^*$, provided that $\gamma < 0.01$ and $\gamma < 0.006$ for P and S, respectively.

The influence of $\zeta_p$ is illustrated on the evolution of the errors in Figures 3.9(e) and 3.9(f). Contrary to the two previous cases, the choice of $\zeta_p$ appears to be less dominant on deciding whether the cascade approximation is permissible. Specifically, smooth regions are illustrated and, as long as $\gamma < 0.052$ and $\gamma < 0.15$, the total errors will not exceed 5%, for P and S, respectively.

### 3.4.3.3 Rocking Secondary Block

The results of the rocking-linear S-P system are investigated next, assuming in all the simulations that the coefficient of restitution takes its maximum allowable value. Figure 3.10
3.4 Decoupling Criteria

Figure 3.8: Effect of the S-P parameter $\gamma$ on the absolute acceleration response of P (left), and the displacement response of the sliding S (right): $\omega_g^* = 0.14$ (a, b) and $\omega_g^* = 1$ (c, d).

illustrates the influence of $\gamma$ on the seismic responses. For the reference case under consideration, i.e $\omega_g^* = 0.14$, the presence of the secondary mass causes variations on the response of P (Figure 3.10(a)) even prior to the initiation of rocking motion ($t^* < 5.9$). Similar to the sliding case, the presence of discontinuities in the acceleration history of P (e.g. $t^* = 26$) is attributed to the change in the regime of motion. As shown, the response is highly sensitive to $\gamma$, suggesting that the contribution of the feedback action of S needs to be accounted for. This can be justified as when $\gamma \neq 0$, the velocity of P experiences a finite jump during impact which cannot be captured by the cascade approximation. As shown, when $\gamma \geq 1$ the cascade approximation fails to predict the true overturning behaviour of the block (Figure 3.10(b)). The associated cumulative errors are 50% and 98% for $\gamma = 0.5$, 57% and 97% for $\gamma = 1$, 59% and 98% for $\gamma = 1.5$, for P and S, respectively.

When $\omega_g^* = 1$, the cumulative errors in Figures 3.10(c) and 3.10(d) further increase, amounting to 111% and 160% for $\gamma = 0.5$, 126% and 398% for $\gamma = 1$, 146% and 670% for $\gamma = 1.5$, for P and S, respectively.
Figure 3.9: Effect of the S-P mass ratio parameter $\gamma$, the dimensionless pulse frequency $\omega^*_g$, the specific strength $a^*_s$ and the damping ratio $\zeta_p$ on the absolute acceleration response of P (a, c, e) and the displacement response of the sliding S (b, d, f), respectively, in the form of region plots corresponding to 5%, 10%, 15% and 20% cumulative error. Reference values of $\omega^*_g = 0.14$, $a^*_s = 0.5$ and $\zeta_p = 0.05$ are assumed.
Figures 3.11 and 3.12 portray the associated error regions for the acceleration and rotation response of P and S, respectively. The effect of $\omega_g^*$ is first considered in Figures 3.11(a) and 3.11(b). As shown, the response is highly sensitive to the chosen value of $\omega_g^*$. While the curves appear to be ordered for the case of P, with $\gamma < 0.01$ being satisfactory in maintaining the error to 5%, this is not the case for S. Specifically, the results do not allow clear patterns to emerge. As shown, there exists a region, i.e. $0.23 \leq \omega_g^* \leq 0.32$ whereby the errors retain minimum values (less than 5%) as both the cascade solution and the combined one correctly predict overturning. Alternatively, when $\omega_g^* < 0.23$ the cascade solution fails to predict the true rocking behaviour of the block, resulting in higher errors. Finally, when $\omega_g^* > 0.32$ no overturning occurs and the results appear more ordered. This behaviour is attributed to the highly nonlinear nature of the system, as well as the pronounced sensitivity of the rocking response to the angular velocity when the rocking motion begins. Consequently, lower values of $\gamma$ have to be considered (i.e. $\gamma < 0.001$) if the cascade approximation needs to be adopted or, alternatively, the approximation needs to be adaptively implemented by carrying out sensitivity analysis prior to a given set of input parameters.

Figures 3.11(c) and 3.12(d) illustrate the effect of the normalised pulse amplitude $a_g^*$ on the error regions. Similar to the previous case, setting $\gamma \leq 0.02$ can limit the errors to less than 5% for P, while for the case of S low errors are only identified when both the cascade approximation and the full dynamic interaction predict overturning (i.e. $a_g^* > 0.35$) and provided that rocking motion is initiated ($a_g^* \geq 0.25$).

The case of $\zeta_p$ is examined in Figures 3.11(e) and 3.11(f). Provided that $\gamma < 0.01$ and $\gamma < 0.0015$ the errors are maintained at 5% for P and S, respectively.

Figures 3.12(a) and 3.12(b) depict the associated errors with respect to changes in the slenderness parameter $\alpha$. Evidently, setting $\gamma < 0.02$ limits the errors with respect to P regardless of the choice of $\alpha$. In the case of S, the effectiveness of the cascade approximation increases with lower slenderness (i.e. $\alpha < 0.18$) provided that overturning is predicted in both solutions. When the slenderness is in the range $0.18 \leq \alpha \leq 0.32$ the approximation depends on the choice of $\gamma$ and when $\alpha > 0.32$ no rocking motion occurs.

Figures 3.12(c) and 3.12(d) illustrate no clear trends in the case of the dimensionless parameter $p^*$ with respect to the effectiveness of the cascade approximation. Similar to the previous case the response is governed by the combination of $p^*$ and $\gamma$. It is thus indicated to either limit $\gamma$ (i.e. $\gamma < 0.001$) or adaptively estimate the errors prior to the analysis.
3. Combined vibration and decoupling of SDoF S

Figure 3.10: Effect of the S-P parameter $\gamma$ on the absolute acceleration response of P (left), and the displacement response of the rocking S (right): $\omega_g^* = 0.14$ (a, b) and $\omega_g^* = 1$ (c, d).

3.5 Summary

In this chapter, the simplest case of a primary-secondary (P-S) structure, namely that of a 2DoF system is considered. The equations governing the full dynamic interaction of bilinear, sliding and free-standing rocking nonlinear secondary (S) oscillators coupled with linear primary (P) ones have been derived. Aiming at identifying the conditions whereby decoupling is permissible, dynamic analyses have then been carried out in presence of full-cycle pulse excitation, representative of near-field earthquake loading. Regions of validity have been estimated for the cascade analysis of the systems considered as well as the associated approximation errors.

The cascade approximation, largely used in the current engineering practice, was found satisfactory for the analysis of bilinear-linear 2DoF systems, provided that the dynamic system does not approach resonance and the S-P mass ratio falls below 0.1. The validity of the cascade analysis is dictated mainly by the frequency parameters of the ground motion and the secondary system, as well as the S system’s strength. The post-yield to pre-yield stiffness ratio and the
Figure 3.11: Effect of the S-P mass ratio parameter $\gamma$, the dimensionless pulse frequency $\omega_\ast^g$, the pulse amplitude $a_\ast^g$ and the damping ratio $\zeta_p$ on the absolute acceleration response of P (a, c, e) and the rotation response of the rocking S (b, d, f), respectively, in the form of region plots corresponding to 5%, 10%, 15% and 20% cumulative error. Reference values of $\omega_\ast^g = 0.14$, $\zeta_p = 0.05$, $\alpha = 0.25$, $p^\ast = 0.136$ and $a_\ast^g = 0.28$ are assumed.
damping ratios were shown to have minor contributions on the evolution of the errors and, as such, might be disregarded when deciding whether the cascade approximation is acceptable.

The selection of the ground frequency parameter and the strength parameter highly influence the evolution of the errors in sliding-linear 2DoF systems. Cascade analysis is permitted, provided that the S-P mass ratio is lower than 0.01. It was found that decoupling is particularly effective at low strength when larger displacements are exhibited. Similarly to the previous case, damping ratios were found to have minor contributions on the errors.

Dynamic analyses on rocking-linear 2DoF systems have indicated high sensitivity to the input parameters. The cascade analysis is only permissible provided that the S-P mass ratio is lower than 0.001. Because of the way in which the cumulative error has been defined (see Eq. (3.29)), the effectiveness of the approximation was found to increase when less oscillations are exhibited; that is, when overturning is reached.
3.5 Summary

In cases where the S-P mass ratio is not sufficiently small, the derived expressions comprise an a priori measure of adaptively assessing the approximation error for a given set of input parameters.
4.1 Introduction

In the preceding chapter, the conditions dictating the use of the cascade approximation have been discussed for a class of pulse-driven 2DoF secondary-primary systems. Purpose of this chapter is the development of analytical and numerical solutions for the cascade response analysis of nonlinear secondary oscillators that are excited by the response of pulse-driven linear multi-degree-of-freedom primary structures.

The dynamics of a two-degree-of-freedom (2DoF) primary-secondary linear system are first examined in Section 4.2 and some closed-form solutions are presented.

In Section 4.3, analytical as well as numerical solutions are derived for the cascade analysis of bilinear, sliding and free-standing rocking secondary oscillators by solving the associated linearised equations of each system. The analytical solutions allow the direct evaluation of the response as a function of the input parameters and are used to confirm the validity of the numerical solutions. The latter allows the determination of the dynamic response under a general-type of seismic excitation, that is the topic of subsequent chapters.
4. Pulse-driven SDoF S in cascade

The proposed formulations are then further illustrated in Section 4.5 through numerical applications. The influence and dependencies between the input parameters of the ground motion, the primary structure and the secondary oscillators are quantified on the response of the secondary systems under consideration through deterministic response spectra.

4.2 Combined Vibration Response of 2DoF Linear System

Let us consider the case of a single degree-of-freedom (DoF) secondary (S) oscillator vibrating on an \( n_p \) DoF classically damped primary (P) structure. The equation of motion of the combined system within its linear-elastic range, assumed at rest for \( t \leq 0 \), reads:

\[
M \ddot{u}(t) + C \dot{u}(t) + K \cdot u(t) = -M \cdot \tau \dddot{u}_g(t) ; \quad u(0) = 0_n ; \quad \dot{u}(0) = 0_n ,
\]

(4.1)

where \( u(t) = \{ u_s(t)^\top; u_p(t)^\top \}^\top \) comprises the partitioned array listing the \( n = n_p + 1 \) DoFs of the combined system relative to the ground, in which \( u_s(t) \) denotes the DoF of S and \( u_p(t) = \{ u_{p,1}(t), \ldots, u_{p,n_p}(t) \}^\top \) collects the DoFs of P; the superscripted \( ^\top \) is the transpose operator. In the above, the DoF of S relative to P is \( u_s(t) = u_s(t) - u_p(t) \) where \( u_p(t) \) is the response of P at the position of attachment. Furthermore, \( \tau = \{ 1; \tau_p^\top \}^\top \) is the partitioned array of seismic incidence; \( \dddot{u}_g(t) \) is the horizontal ground acceleration; \( u(0) \) and \( \dot{u}(0) \) are the vectors of the initial conditions; \( 0_n \) is a zero vector of dimensions \( (n \times 1) \). Additionally, \( M, K \) and \( C \) denote the mass, stiffness and damping matrices of the combined system in the geometric space, assembled as [128]:

\[
M = \begin{bmatrix}
    m_s & 0_{1 \times n_p} \\
    0_{n_p \times 1} & M_p
\end{bmatrix} ; \quad K = \begin{bmatrix}
    k_s & K_{sp} \\
    K_{sp}^\top & K_p + \Delta K_p
\end{bmatrix} ; \quad C = \begin{bmatrix}
    c_s & C_{sp} \\
    C_{sp}^\top & C_p + \Delta C_p
\end{bmatrix},
\]

(4.2)

where \( \{ m_s, k_s \} \) are the mass and stiffness coefficients for S and \( \{ M_p, K_p \} \) are the corresponding base-fixed matrices for P, respectively; \( O_{r \times s} \) is a zero matrix of order \( (r \times s) \); \( K_{sp} = k_s v^\top \) and \( \Delta K_p = k_s v \cdot v^\top \) comprise the out-of-diagonal coupling stiffness array and the supplementary stiffness matrix of P due to the presence of S, respectively, where \( v^\top = \{ 0, \ldots, 0, -1, 0, \ldots, 0 \} \) is a vector, whose non-zero entry at the \( j \)-th degree of freedom denotes the position of attachment. Similarly, \( c_s = 2 \zeta_s \omega_s m_s \) is the damping constant and \( C_p = 2 \zeta_p M_p \cdot \Phi_p \cdot \Omega_p \cdot \Phi_p^\top \cdot M_p \) is the damping matrix for S and P, in which \( \{ \zeta_s, \zeta_p \} \)
comprise the associated equivalent viscous damping ratios, $\omega_p$ denotes the circular frequency of $S$ and $\bar{\Omega}_p = \text{diag} \left\{ \omega_{p,1}, \ldots, \omega_{p,n_p} \right\}$ is the full diagonal spectral matrix, listing all the modal circular frequencies of $P$; $C_{sp} = c_s \mathbf{v}^\top$ is the coupling damping array; $\Delta C_p = c_p \mathbf{v} \cdot \mathbf{v}^\top$ is the additional damping matrix of $P$. Furthermore, $\Phi_p$ is the $(n_p \times n_p)$ full modal matrix, which is obtained by solving the real-valued eigenproblem:

$$M_p \cdot \Phi_p \cdot \bar{\Omega}_p^2 = K_p \cdot \Phi_p,$$

(4.3)

with the ortho-normal condition $\Phi_p^\top \cdot M_p \cdot \Phi_p = I_{n_p}$, where $I_{n_p}$ is the identity matrix of size $n_p$.

The number of DoFs in the dynamic analysis can significantly be reduced by projecting the differential equations of motion onto the modal subspace. If only the fundamental vibration mode of $P$ is retained, the following coordinate transformation can be adopted:

$$\mathbf{u}(t) \approx \Gamma \cdot \mathbf{q}(t); \quad \Gamma = \begin{bmatrix} 1 & \varphi \end{bmatrix} \begin{bmatrix} \Phi_p \end{bmatrix},$$

(4.4)

where $\mathbf{q}(t) = \begin{bmatrix} q_s(t) \mid q_p(t) \end{bmatrix}^\top$ is a partitioned array listing the $m = 2$ generalised coordinates for $S$ and $P$, respectively; $\Phi_p$ is the $(n_p \times 1)$ normalised modal vector (i.e. $\max \{ \Phi_p \} = 1$) satisfying $\Phi_p^\top \cdot M_p \cdot \Phi_p = m_p$, where $m_p$ is the modal mass of $P$; $\varphi$, in the range $0 \leq \varphi \leq 1$, represents the dimensionless modal coordinate at the position of attachment (i.e. an element of $\Phi_p$), as shown in Figure 4.1.

Substituting Eq. (4.4) in Eq. (4.1) and premultiplying the result by $\Gamma^\top$, the dynamic response of the resulting 2DoF system, is governed by:
\[ m \ddot{q}(t) + c \dot{q}(t) + k q(t) = g \ddot{u}_g(t); \quad q(0) = \Gamma^\top \cdot M \cdot u(0); \quad \dot{q}(0) = \Gamma^\top \cdot M \cdot \dot{u}(0), \quad (4.5) \]

where \( m, k \) and \( c \) are partitioned matrices of inertia, stiffness and damping in the reduced modal subspace, respectively, and \( g \) is the influence vector of the seismic input:

\[ m = \begin{bmatrix} \frac{1}{a} & \frac{\varphi}{a} \end{bmatrix}; \quad k = \begin{bmatrix} \omega_s^2 & 0 \\ 0 & \omega_p^2 \end{bmatrix}; \quad c = \begin{bmatrix} 2 \zeta_s \omega_s & 0 \\ 0 & 2 \zeta_p \omega_p \end{bmatrix}; \quad g = - \begin{bmatrix} 1 + a \beta \end{bmatrix}, \quad (4.6) \]

in which \( a = m_s \varphi/m_p \) and \( \beta = m_p^{-1} \Phi_p^\top \cdot M_p \cdot \tau_p \) are two dimensionless parameters representing the modal mass ratio and the normalised participation mass, respectively. It is worth noting here that, in addition of the two pairs of parameters \( \{\omega_s, \zeta_s\} \) and \( \{\omega_p, \zeta_p\} \), the three dimensionless parameters \( a, \beta \) and \( \varphi \) fully describe the P-S dynamic interaction within the linear domain.

Eq. (4.5) can be recast in a state-space form of 4 first-order differential equations as:

\[ \dot{z}(t) = D \cdot z(t) + V \ddot{u}_g(t), \quad (4.7) \]

in which \( z(t) = \begin{bmatrix} q^\top(t) & \dot{q}^\top(t) \end{bmatrix}^\top \) is the partitioned array collecting the state variables, while:

\[ D = -A^{-1} \cdot B = \begin{bmatrix} O_{m \times m} & I_m \\ -m^{-1} \cdot k & -m^{-1} \cdot c \end{bmatrix}; \quad V = \begin{bmatrix} O_{m \times 1} \\ m^{-1} \cdot g \end{bmatrix}, \quad (4.8) \]

and:

\[ A = \begin{bmatrix} c \cdot m \\ m \cdot O_{m \times m} \end{bmatrix}; \quad B = \begin{bmatrix} k \cdot O_{m \times m} \\ O_{m \times m} \cdot -m \end{bmatrix}. \quad (4.9) \]

### 4.2.1 Complex Modal Analysis

For non-classically damped systems (if and only if \( K \cdot M^{-1} \cdot C \neq C \cdot M^{-1} \cdot K \)), it is not possible to decouple the equations of motion using real-valued eigenvalues and eigenvectors [92]. Decoupling of Eq. (4.5) is made possible via the complex modal analysis [129, 130]. Accordingly, the following transformation of coordinates can be adopted:
4.2 Combined Vibration Response of 2DoF Linear System

\[ z(t) = \bar{\Gamma} \cdot \bar{x}(t), \]  

(4.10)

where \( \bar{x}(t) \) lists the complex-valued modal coordinates and \( \bar{\Gamma} \) is the complex modal matrix, which is a solution of the complex-valued eigenvalue problem:

\[ D \cdot \bar{\Gamma} = \bar{\Gamma} \cdot \bar{\alpha}, \]  

(4.11)

\( \bar{\alpha} \) being a diagonal matrix listing the 4 complex eigenvalues. The eigenvalues and eigenvectors are ordered in complex conjugate pairs as:

\[ \bar{\alpha} = \text{Diag} [\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_1^*, \bar{\alpha}_2^*] \; ; \; \bar{\Gamma} = [\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_1^*, \bar{\gamma}_2^*]. \]  

(4.12)

Substitution of Eq. (4.10) into Eq. (4.7) and premultiplying the result by \( p^{-1} \cdot \bar{\Gamma}^\top \cdot A \), we obtain a set of 4 decoupled differential equations:

\[ \dot{\bar{x}}(t) = \bar{\alpha} \cdot \bar{x}(t) + \bar{\Gamma}^{-1} \cdot V \ddot{u}_g(t) ; \; \bar{x}_0 = \bar{\Gamma}^\top \cdot A \cdot z_0, \]  

(4.13)

where \( p = \bar{\Gamma}^\top \cdot A \cdot \bar{\Gamma} \) and \( z(0) = \{ q(0)^\top ; \dot{q}(0)^\top \} \)\( ^\top \). The solution of Eq. (4.13) is obtained as:

\[ \bar{x}(t) = \exp[\bar{\alpha} t] \cdot \bar{x}_0 + \int_0^t \exp[\bar{\alpha}(t - \tau)] \cdot \bar{\Gamma}^{-1} \cdot V \ddot{u}_g(\tau) \; d\tau, \]  

(4.14)

where \( \exp[\bar{\alpha} t] \) denotes the matrix exponential function. The response in the geometric space \( y(t) = \{ u^\top(t) ; \dot{u}^\top(t) \} \)\( ^\top \) finally reads:

\[ y(t) = \Pi \cdot z(t) = \Pi \cdot \bar{\Gamma} \cdot \bar{x}(t) ; \; \Pi = \begin{bmatrix} \bar{\Gamma} & O_{n \times 2} \\ O_{n \times 2} & \bar{\Gamma}^\top \end{bmatrix}. \]  

(4.15)

4.2.2 Closed-Form Solution

Let us consider the case of a pulse-type ground acceleration, given by Eq. (3.28). It can be mathematically proved that in this case Eq. (4.14) reduces to:

\[ \bar{x}(t) = \exp[\bar{\alpha} t] \cdot \bar{x}(0) + H(t) \cdot \bar{\Gamma}^{-1} \cdot V \; ; \; \bar{x}(0) = \bar{\Gamma}^\top \cdot A \cdot z(0), \]  

(4.16)

where the diagonal matrix \( H(t) \) is evaluated as:
\[
H(t) = a_g G \cdot \left\{ \exp \left[ \tilde{\alpha} t \right] \cdot \left( \tilde{\alpha} \sin(\phi) + \omega_g \cos(\phi) I_{2m} \right) \\
- \tilde{\alpha} \sin(\omega_g t + \phi) - \omega_g \cos(\omega_g t + \phi) I_{2m} \right\},
\]
(4.17)
and
\[
G = [\tilde{\alpha}^2 + \omega_g^2 I_{2m}]^{-1}.
\]
(4.18)

4.2.3 Numerical Solution

A numerical approach can be adopted for solving the eigenvalue problem of Eq. (4.11), for a linear piecewise input e.g. a ground shaking which is known through a tabulated record of ground acceleration, with a certain sampling frequency \(\Delta t\). Denoting \(t_i, i = 0, 1, \ldots d\) as distinct time instants with constant time step \(\Delta t\) for \(0 < t \leq t_d\), and assuming a piecewise linear forcing function, the discretised solution of Eq. (4.7) reads [129]:

\[
z(t_{i+1}) = \Theta(\Delta t) \cdot z(t_i) + \Gamma_0(\Delta t) \cdot V \ddot{u}_g(t_i) + \Gamma_1(\Delta t) \cdot V \ddot{u}_g(t_{i+1}),
\]
(4.19)

where,

\[
\Gamma_0(\Delta t) = [\Theta(\Delta t) - L(\Delta t)] \cdot D^{-1}; \quad \Gamma_1(\Delta t) = [L(\Delta t) - I_{2m}] \cdot D^{-1},
\]
(4.20)
and

\[
L(\Delta t) = \frac{1}{\Delta t} [\Theta(\Delta t) - I_{2m}] \cdot D^{-1}; \quad D^{-1} = \begin{bmatrix}
-k^{-1} \cdot e_{1m} & -k^{-1} \cdot m_{1m} \\
0 & I_{m \times m}
\end{bmatrix}.
\]
(4.21)

In the above, the transition matrix \(\Theta(\Delta t) = \exp [D \cdot \Delta t]\) only depends on \(\Delta t\), which is tacitly assumed sufficiently small, so that the interpolation of the force is satisfactory.
4.3 Nonlinear Secondary Oscillators in Cascade

In many situations, \( m_s \ll m_p/\phi^2 \) and a cascade approach is admissible. In this section, the response of nonlinear SDoF S vibrating in cascade with a linear SDoF P driven by a pulse-type ground acceleration is considered. Suppose that the response of S relative to P as well as the response of P at the position of attachment are of interest. Neglecting the feedback action of S in Eq. (4.5) by setting \( a = 0 \), the equations decouple and can sequentially be solved:

\[
\ddot{u}_p(t) + 2\zeta_p\omega_p \dot{u}_p(t) + \omega_p^2 u_p(t) = -\phi \beta \ddot{u}_g(t)
\]
\[
\ddot{u}_s(t) + 2\zeta_s\omega_s \dot{u}_s(t) + \omega_s^2 u_s(t) = -\ddot{u}_a(t),
\]

where \( \ddot{u}_a(t) = \ddot{u}_p(t) + \ddot{u}_g(t) \) is the absolute acceleration response of P at the position of attachment that is used to excite S.

4.3.1 Equations of Motion

The cascade approximation allows conveniently considering various types of nonlinear S components for a given P. Notably, the response of the linear P is readily available from Eq. (4.22a), while the governing equations of motion of the three selected nonlinear secondary oscillators are identical to the ones presented in § 3.2. More specifically, Eq. (3.4), Eq. (3.9) and Eq. (3.12), are used for the bilinear (Figure 4.2(a)), sliding (Figure 4.2(c)) and rocking (Figure 4.2(e)) S, respectively.

In the latter case, if the block is slender, angle \( \theta_s(t) \) is sufficiently small and Eq. (3.12) can be linearised by assuming \( \sin(\pm\alpha - \theta_s(t)) \approx \pm\alpha - \theta_s(t) \) and \( \cos(\pm\alpha - \theta_s(t)) \approx 1 \). Figure 4.3 confirms that the errors in the trigonometric functions reduce with lower \( \alpha \).

The equation of the linearised system then reads:

\[
\dot{\theta}_s(t) = -p^2 \left\{ \alpha \text{ sgn}(\theta_s(t)) - \theta_s(t) + \frac{\ddot{u}_a(t)}{g} \right\};
\]
\[
\theta_s(0) = 0; \quad \dot{\theta}_s(0) = 0.
\]

and can be used in place of Eq. (3.12).
Figure 4.2: Bilinear (a), sliding (c) and free-standing rocking (e) secondary oscillators in cascade and corresponding force-displacement (b, d) and moment-rotation relationship (f).

4.3.2 Closed-Form Solutions

The pulse-type excitation (see Eq. (3.28)) and the piecewise linear secondary systems presented in § 4.3.1 and § 3.2, enable the derivation of analytical solutions, which are presented in what follows. Due to the piecewise nature of the governing equations of motion, the solutions are derived separately for each regime of motion and are then pieced together to construct the response time history. The purpose of deriving closed-form solutions is twofold. First, they al-
low directly expressing the response as a function of the input parameters of the ground motion and the primary structure, conveniently examining the relative dependencies and contributions. Second, they will be used to validate the numerical solutions that will be presented in § 4.3.3 for general-type excitations.

### 4.3.2.1 Bilinear Secondary Oscillator

The case of a bilinear $S$ is considered first. Accordingly, integrating Eq. (3.4) twice with respect to time and enforcing the initial conditions, the solution can be expressed as:

$$u_s(t) = e^{-\zeta_s\omega_s t} \left[ C_1^b \sin(\bar{\omega}_s^{\psi} t) + C_2^b \cos(\bar{\omega}_s^{\psi} t) \right] + C_3^p \sin(\omega_g t + \phi) + C_4^p \cos(\omega_g t + \phi) + e^{-\zeta_p\omega_p t} \left[ C_5^b \sin(\bar{\omega}_p t) + C_6^b \cos(\bar{\omega}_p t) \right] + C_7^b(t),$$

(4.24)

where $\bar{\omega}_s^{\psi} = \omega_s \sqrt{\psi_s - \zeta_s^2}$ and $\bar{\omega}_p = \omega_p \sqrt{1 - \zeta_p^2}$ comprise the natural frequencies of damped vibration for $S$ and $P$, respectively, and $A = \zeta_p^2 \omega_p^2 + \bar{\omega}_p^2$. The response depends on a set of coefficients defined as:
In which $C = \zeta_p^2 \omega_p^2 - 2 \zeta_p \zeta_s \omega_p \omega_s$. Similarly,

$$C^b_7(t) = \begin{cases} 
\frac{t[w_k(h(2u_{s0} - a_s z)+2 \bar{\omega}_p C_6 + 2 \bar{\omega}_p \zeta \omega_p C_6) + 2A(C_3 \cos(\phi) - C_4 \sin(\phi))]}{2 \bar{\omega}_p}, & \psi_s = 0, \zeta_s = 0 \\
\frac{a_s z(1 - e^{-2 \zeta_s \omega_s t} - 2 \zeta_s \omega_s t)}{4 \zeta_s^2 \omega_s^2}, & \psi_s = 0, \zeta_s \neq 0 \\
C^b_8, & \text{otherwise}
\end{cases}$$

$$C^b_8 = \begin{cases} 
0, & \text{if } \psi_s = 0, \\
\frac{z(\psi_s - 1) a_s}{\psi_s \omega_s^2}, & \text{otherwise}
\end{cases}$$

The above coefficients are also functions of the parameters of P and the ground acceleration through:
4.3.2.2 Sliding Secondary Block

Notably, setting respectively.

\[ C_1 = \frac{1}{\omega_p} [(C_8 \omega_g - C_7 \zeta_p \omega_p) \sin(\phi) - (C_7 \omega_g + C_8 \zeta_p \omega_p) \cos(\phi) + \zeta_p \omega_p u_{p,0} + \dot{u}_{p,0}] ; \]  
(4.27a)

\[ C_2 = u_{p,0} - C_7 \sin(\phi) - C_8 \cos(\phi) ; \]  
(4.27b)

\[ C_3 = C_7 \omega_g^2 - a_g ; \]  
(4.27c)

\[ C_4 = C_8 \omega_g^2 ; \]  
(4.27d)

\[ C_5 = C_1 (\omega_p - \zeta_p \omega_p) (\zeta_p \omega_p + \omega_p) - 2 \zeta_p \omega_p \omega_p C_2 ; \]  
(4.27e)

\[ C_6 = 2 C_1 \zeta_p \omega_p + C_2 (\omega_p - \zeta_p \omega_p) (\omega_p + \zeta_p \omega_p) ; \]  
(4.27f)

\[ C_7 = \frac{a_g \beta (\omega_g - \omega_p) (\omega_g + \omega_p)}{2 (2 \zeta_p^2 - 1) \omega_p^2 + \omega_g^2 + \omega_f^2} ; \]  
(4.27g)

\[ C_8 = \frac{2 a_g \beta \zeta_p \omega_p}{2 (2 \zeta_p^2 - 1) \omega_p^2 + \omega_g^2 + \omega_f^2} . \]  
(4.27h)

where \( C_1^b, C_2^b \) and \( C_1, C_2 \) depend on the initial conditions \( u_{s,0}, \dot{u}_{s,0} \) and \( u_{p,0}, \dot{u}_{p,0} \) for S and P, respectively.

The velocity time history is finally determined by differentiating Eq. (4.24):

\[ \dot{u}_s (t) = e^{-\zeta \omega_s t} [(C_1^b \bar{\omega}_s^\psi - \zeta_s \omega_s C_2^b) \cos(\bar{\omega}_s^\psi t) - (C_1^b \zeta_s \omega_s + \bar{\omega}_s^\psi C_2^b) \sin(\bar{\omega}_s^\psi t)] + e^{-\zeta \omega_s t} [(C_1^b \bar{\omega}_s^\psi - \zeta_p \omega_p C_5^b) \cos(\bar{\omega}_s^\psi t) - (\zeta_p \omega_p C_5^b + \bar{\omega}_s^\psi C_6^b) \sin(\bar{\omega}_s^\psi t)] + \omega_p [C_3^b \cos(\omega_p t + \phi) - C_4^b \sin(\omega_p t + \phi)] + C_7^f (t) . \]  
(4.28)

Notably, setting \( \psi_s = 1, C_3 = -a_g, C_4 = C_5 = C_6 = 0 \) in the above, and making the substitutions \( \bar{\omega}_s^\psi = \bar{\omega}_p, \omega_p = \omega_p, \zeta_s = \zeta_p, u_{s,0} = u_{p,0} \) and \( \dot{u}_{s,0} = \dot{u}_{p,0} \), one can obtain the response \( u_p (t), \dot{u}_p (t) \) of the linear primary SDoF oscillator.

4.3.2.2 Sliding Secondary Block

The pertinent solution to Eq. (3.9) for a sliding S reads:

\[ u_s (t) = -\frac{1}{2} u_s g t^2 \text{sgn}(\dot{u}_s (t)) + C_1^s + C_2^s t + C_3^s \sin(\omega_g t + \phi) + C_4^s \cos(\omega_g t + \phi) + e^{-\zeta \omega_p t} [C_5^s \sin(\bar{\omega}_p t) + C_6^s \cos(\bar{\omega}_p t)] , \]  
(4.29)
where the coefficients $C_1^s - C_6^s$ take the form:

\[
C_1^s = u_{s,0} - C_3^s \sin(\phi) - C_4^s \cos(\phi) - C_6^s; \tag{4.30a}
\]

\[
C_2^s = \dot{u}_{s,0} - C_3^s \omega_g \cos(\phi) + C_4^s \omega_g \sin(\phi) - C_5^s \bar{\omega}_p + C_6^s \zeta_p \omega_p; \tag{4.30b}
\]

\[
C_3^s = -\frac{C_3}{\omega_g^2}; \tag{4.30c}
\]

\[
C_4^s = -\frac{C_4}{\omega_g^2}; \tag{4.30d}
\]

\[
C_5^s = \frac{C_5 (\bar{\omega}_p^2 - \omega_p^2)}{\left(\bar{\omega}_p^2 + \zeta_p^2 \omega_p^2\right)^2} - 2 \zeta_p \bar{\omega}_p \omega_p C_6; \tag{4.30e}
\]

\[
C_6^s = \frac{C_6 (\bar{\omega}_p^2 - \omega_p^2) + 2 \zeta_p \bar{\omega}_p \omega_p C_5}{\left(\bar{\omega}_p^2 + \zeta_p^2 \omega_p^2\right)^2}. \tag{4.30f}
\]

where $C_1^s$, $C_2^s$ are functions of the initial conditions $u_{s,0}$ and $\dot{u}_{s,0}$ and $C_1 - C_8$ are the same coefficients defined in Eq. (4.27).

The velocity response history is obtained from Eq. (4.29) as:

\[
\dot{u}_s(t) = -\mu_s g t \text{sgn}(\dot{u}_s(t)) + C_3^s + C_4^s \omega_g \cos(\phi) - C_5^s \omega_p \sin(\phi) + e^{-\zeta_p \omega_p t} \left[ C_5^s \sin(\bar{\omega}_p t) - C_6^s \cos(\bar{\omega}_p t) \right] \tag{4.31}
\]

Upon determination of an analytical expression for $\dot{u}_p(t)$ as delineated in § 4.3.2.1, differentiation with respect to time and substitution of the resulting expression in $|\ddot{u}_p(t) + \ddot{g}(t)| = \mu_s g$, gives:

\[
-\mu_s g \sin(\omega_g t) - C_4 \cos(\omega_g t) - e^{-\zeta_p \omega_p t} \left[ C_5 \sin(\bar{\omega}_p t) + C_6 \cos(\bar{\omega}_p t) \right] = \mu_s g, \tag{4.32}
\]

for $\phi = 0$, where $u_{p,0} = \dot{u}_{p,0} = 0$. Determination of the initiation time instant for sliding motion requires a solution to the above transcendental equation. This falls beyond the scope of this study, and therefore a numerical scheme (i.e. the bisection method [131]) is adopted.

### 4.3.2.3 Rocking Secondary Block

A solution to Eq. (4.23) gives the rotation response of the rocking S block:

\[
\dot{\theta}_s(t) = -\frac{C_3^s \sin(\omega_g t) - C_4 \cos(\omega_g t) - e^{-\zeta_p \omega_p t} \left[ C_5 \sin(\bar{\omega}_p t) + C_6 \cos(\bar{\omega}_p t) \right]}{\mu_s g}, \tag{4.32}
\]
\[\theta_s(t) = C_1^r \sinh(pt) + C_2^r \cosh(pt) + C_3^r \sin(\omega_p t + \phi) + C_4^r \cos(\omega_p t + \phi) + e^{-\zeta_p \omega_p t} [C_5^r \sin(\omega_p t) + C_6^r \cos(\omega_p t)] + \alpha \text{sgn} [\theta_s(t)],\]  
(4.33)

where the coefficients \(C_1^r - C_8^r\) are:

\[
\begin{align*}
C_1^r &= \frac{1}{p} \left[ \dot{\theta}_{s,0} - C_5^r \omega_p \cos(\phi) + C_4^r \omega_p \sin(\phi) - C_5^r \omega_p + C_6^r \zeta_p \omega_p \right]; \\
C_2^r &= \theta_{s,0} - C_4^r \sin(\phi) - C_5^r \cos(\phi) - C_6^r - \alpha_r \text{sgn} [\theta_s(t)]; \\
C_3^r &= -\frac{p^2 C_4}{g(p^2 + \omega_p^2)}; \\
C_4^r &= -\frac{p^2 C_4}{g(p^2 + \omega_p^2)}; \\
C_5^r &= -\frac{p^2 [p^2 C_5 + C_5 (\bar{\omega}_p - \zeta_p \omega_p) (\bar{\omega}_p + \zeta_p \omega_p) + 2 \zeta_p \bar{\omega}_p \omega_p C_6]}{g \left[ p^4 + 2 p^2 (\bar{\omega}_p - \zeta_p \omega_p) (\bar{\omega}_p + \zeta_p \omega_p) + (\bar{\omega}_p^2 + \zeta_p^2 \omega_p^2)^2 \right]}; \\
C_6^r &= -\frac{p^2 \left[ 2 \zeta_p \bar{\omega}_p \omega_p C_5 + C_6 \left( \zeta_p^2 \omega_p^2 - p^2 - \bar{\omega}_p^2 \right) \right]}{g \left[ p^4 + 2 p^2 (\bar{\omega}_p - \zeta_p \omega_p) (\bar{\omega}_p + \zeta_p \omega_p) + (\bar{\omega}_p^2 + \zeta_p^2 \omega_p^2)^2 \right]}.
\end{align*}
\]

in which \(C_1^r, C_2^r\) depend on the initial conditions \(\theta_{s,0}, \dot{\theta}_{s,0}\), and \(C_1 - C_8\) are given in Eq. (4.27).

The angular velocity is evaluated from Eq. (4.33) as:

\[\dot{\theta}_s(t) = C_1^p \cos(pt) + C_2^p \sin(pt) + C_3^p \omega_p \cos(\omega_p t + \phi) + C_4^p \omega_p \sin(\omega_p t + \phi) + e^{-\zeta_p \omega_p t} \left[ (C_5^p \bar{\omega}_p - C_6^p \zeta_p \omega_p) \cos(\bar{\omega}_p t) - (C_5^p \zeta_p \omega_p + C_6^p \bar{\omega}_p) \sin(\bar{\omega}_p t) \right],\]  
(4.35)

Notably, setting \(C_3 = -a_p\) and \(C_4^r = C_5^r = C_6^r = 0\) in the above one can obtain the response of a free-standing block as reported in [57]. Similar to the sliding S, a numerical scheme is utilised in identifying the initiation time instant and the time of impact.

### 4.3.3 Numerical Solutions

In the preceding subsection, new analytical solutions have been derived and have been presented for the case of pulse-type excitation. Although such solutions allow preliminary investigations to be carried out on the systems considered, it is desirable to accurately evaluate
the response under a general-type excitation. Notably, MATLAB’s build-in ODE solvers (see § 3.2.4) are computationally demanding for accurate analysis. To this end, the numerical procedure presented in § 4.2.3 for linear systems, is conveniently extended in the following to the nonlinear cases considered; the procedure is made possible owing to the piecewise linear form of the S oscillators.

Accordingly, the response for each of the components takes the form:

\[ y(t_{i+1}) = \Theta(\Delta t) \cdot y(t_i) + \Gamma_0(\Delta t) \cdot V \cdot \eta(t_i) + \Gamma_1(\Delta t) \cdot V \cdot \eta(t_{i+1}), \]  

(4.36)

where the response vector \( y(t) \), the dynamic matrix \( D \), the transition matrix \( \Theta \) as well as the products \( \Gamma_0(\Delta t) \cdot V \) and \( \Gamma_1(\Delta t) \cdot V \) are provided for each case below.

### 4.3.3.1 Bilinear Secondary Oscillator

For the bilinear case (Eq. (3.4)), \( y(t) = y_s(t) = \{u_s(t), \dot{u}_s(t)\}^\top \) and \( \eta(t) = \ddot{u}_a(t) + a_s z (1 - \psi_s) \). Additionally:

\[ D = \begin{bmatrix} 0 & 1 \\ -\psi_s \omega_s^2 & -2 \zeta_s \omega_s \end{bmatrix}; \]

\[ \Theta(\Delta t) = \begin{bmatrix} A & B \\ A' & B' \end{bmatrix} = \begin{bmatrix} e^{-\zeta_s \omega_s \Delta t} (\zeta_s r_1 + r_2) & \frac{r_1 e^{-\zeta_s \omega_s \Delta t}}{\omega_s} \\ -\psi_s \omega_s e^{-\zeta_s \omega_s \Delta t} & e^{-\zeta_s \omega_s \Delta t} (r_2 - \zeta_s r_1) \end{bmatrix}, \]

(4.37)

\[ r_1 = \begin{cases} \omega_s \Delta t, & \text{if } \psi_s = 0, \; \zeta_s = 0 \\ \omega_s \sin \left( \omega_s^\psi \Delta t \right) / \omega_s^\psi, & \text{otherwise} \end{cases}; \quad r_2 = \cos \left( \omega_s^\psi \Delta t \right), \]

(4.38)

and
\[ \Gamma_0(\Delta t) \cdot V = \begin{cases} 
-\frac{\Delta t^2}{3}, & \text{if } \psi_s = 0, \zeta_s = 0 \\
-\frac{\Delta t^2}{2} & \\
\frac{2 B \zeta_0 \Delta t + B - \Delta t(\zeta_0 \Delta t + 1)}{4 \zeta_0^2 \Delta t} & \\
\frac{\Delta t - B}{2 \zeta_0 \Delta t} - B & \\
\frac{2 \psi_s \omega_s (A \Delta t - 2(A - 1) \zeta_s)}{\psi_s^2 \omega_s^2 \Delta t} & \\
\frac{2 \zeta_s \omega_s^2 \Delta t}{\psi_s^2 \omega_s^2 \Delta t} & \\
\frac{2 B' \zeta_0 \Delta t (\zeta_0 \Delta t - 1)}{2 \zeta_0^2 \omega_s^2 \Delta t} & \\
\frac{B' - 2 \zeta_0 \omega_s \Delta t (\zeta_0 \Delta t - 1) - 2(A - 1) \zeta_s}{4 \zeta_0^2 \omega_s^2 \Delta t} & \\
\frac{B' - 2 \zeta_0 \omega_s \Delta t - 2(A - 1) \zeta_s}{\psi_s^2 \omega_s^2 \Delta t} & \\
\frac{B' - 2 \zeta_0 \omega_s \Delta t - 2(A - 1) \zeta_s}{\psi_s^2 \omega_s^2 \Delta t} & \end{cases} \] (4.39)

\[ \Gamma_1(\Delta t) \cdot V = \begin{cases} 
-\frac{\Delta t^2}{3}, & \text{if } \psi_s = 0, \zeta_s = 0 \\
-\frac{\Delta t^2}{2} & \\
\frac{2 B \zeta_0 \Delta t + B - \Delta t(\zeta_0 \Delta t + 1)}{4 \zeta_0^2 \Delta t} & \\
\frac{\Delta t - B}{2 \zeta_0 \Delta t} - B & \\
\frac{2 \psi_s \omega_s (A \Delta t - 2(A - 1) \zeta_s)}{\psi_s^2 \omega_s^2 \Delta t} & \\
\frac{2 \zeta_s \omega_s^2 \Delta t}{\psi_s^2 \omega_s^2 \Delta t} & \\
\frac{2 B' \zeta_0 \Delta t (\zeta_0 \Delta t - 1)}{2 \zeta_0^2 \omega_s^2 \Delta t} & \\
\frac{B' - 2 \zeta_0 \omega_s \Delta t (\zeta_0 \Delta t - 1) - 2(A - 1) \zeta_s}{4 \zeta_0^2 \omega_s^2 \Delta t} & \\
\frac{B' - 2 \zeta_0 \omega_s \Delta t - 2(A - 1) \zeta_s}{\psi_s^2 \omega_s^2 \Delta t} & \\
\frac{B' - 2 \zeta_0 \omega_s \Delta t - 2(A - 1) \zeta_s}{\psi_s^2 \omega_s^2 \Delta t} & \end{cases} \] (4.40)

The response of P is readily available by setting \( y(t) = \mathbf{y}_p(t) = \{u_p(t), \dot{u}_p(t)\}^\top \), \( \eta(t) = \varphi \beta \ddot{u}_g(t) \) and \( \psi_s = 1 \), as well as substituting the parameter set \( \{\omega_s, \zeta_s, a_s, \bar{\omega}_s^\psi\} \) with \( \{\omega_p, \zeta_p, a_p, \bar{\omega}_p^\psi\} \).

### 4.3.3.2 Sliding Secondary Block

In the case of a sliding secondary oscillator (Eq. (3.9)), \( y(t) = \mathbf{y}_s(t) = \{u_s(t), \dot{u}_s(t)\}^\top \) and \( \eta(t) = \ddot{u}_a(t) + \mu_s g \text{sgn}(\dot{u}_a(t)) \). Furthermore:

\[ D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \Theta(\Delta t) = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}, \] (4.40)

and

\[ \Gamma_0(\Delta t) \cdot V = \begin{bmatrix} -\frac{\Delta t^2}{3} \\ -\frac{\Delta t^2}{2} \end{bmatrix}; \quad \Gamma_1(\Delta t) \cdot V = \begin{bmatrix} -\frac{\Delta t^2}{3} \\ -\frac{\Delta t^2}{2} \end{bmatrix}. \] (4.41)
4.3.3.3  Rocking Secondary Block

For the rocking case (Eq. (4.23)), the response vector is \( y(t) = y_s(t) = \{\dot{\theta}_s(t), \ddot{\theta}_s(t)\}^\top \) and \( \eta(t) = p^2 \left( \alpha \text{sgn}(\dot{\theta}_s(t)) + \frac{\ddot{\theta}_s(t)}{g} \right) \). Additionally:

\[
D = \begin{bmatrix} 0 & 1 \\ p^2 & 0 \end{bmatrix}; \quad \Theta(\Delta t) = \begin{bmatrix} \cosh(p \Delta t) & \frac{\sinh(p \Delta t)}{p} \\ p \sinh(p \Delta t) & \cosh(p \Delta t) \end{bmatrix}, \quad (4.42)
\]

and

\[
\Gamma_0(\Delta t) \cdot V = \begin{bmatrix} \frac{\sinh(p \Delta t) - p \Delta t \cosh(p \Delta t)}{p^2 \Delta t} \\ -p \Delta t \sinh(p \Delta t) + \cosh(p \Delta t) - 1 \end{bmatrix}; \quad \Gamma_1(\Delta t) \cdot V = \begin{bmatrix} \frac{p - \sinh(p \Delta t)}{p^2} \\ \frac{1 - \cosh(p \Delta t)}{p^2 \Delta t} \end{bmatrix}. \quad (4.43)
\]

4.3.4  Numerical Implementation

The solutions presented in § 4.3.2 and § 4.3.3 have been implemented in MATLAB [124]. They are separately evaluated for each regime of motion and are pieced together to construct the time history of the response. In doing this, the roots of the associated transcendental equations are numerically evaluated. Each time the regime of motion changes, the time step is divided into two parts and an iterative procedure based on the bisection method [131] is used to determine the exact time the change is introduced in the system.

4.4  Dimensional Analysis

Similar to § 3.4.2, the response quantities are recast in dimensionless form. Accordingly, for the combined linear system in § 4.2, the dimensionless response reads \( u^* (\alpha, \beta, \varphi, \omega_\beta^*, \omega_s^*, \zeta_p, \zeta_s, \phi) = u \omega_p^2 / a_g \). Similarly, under the cascade approximation \((a = 0)\) in § 4.3, the dimensionless displacement for P reads \( u_p^* (\omega_s^*, \zeta_p, \phi, \varphi, \beta) = u_p \omega_p^2 / a_g \). Additionally, the associated response quantities for S are \( u_s^* (u_p^*, \omega_s^*, \zeta_s, a_s^*, \psi_s) = u_s \omega_p^2 / a_g, u_s^* (u_p^*, a_s^*) = u_s \omega_p^2 / a_g \) and \( \theta_s^* (u_p^*, \alpha, p^*, \varepsilon, a_s^*) = \theta_s / \alpha, \) for the bilinear, sliding and rocking S, respectively, where the dimensionless parameters are defined in the same way as in § 3.4.2.
4.5 Numerical Applications

A set of numerical applications are presented in the following, with the aim of assessing the validity of the expressions derived in the preceding sections and examining the seismic response of secondary structures.

Figure 4.4 shows a MDoF primary system comprising of a planar 5-storey single-bay moment-resisting frame. The structure is subjected to the unidirectional action of a horizontal pulse-type ground acceleration (see Eq. (3.28)) and a secondary system is attached at its generic floor, modelled as (i) linear (ii) bilinear, (iii) sliding and (iv) free-standing rocking SDoF oscillator. Although only the combined linear system is depicted in Figure 4.4, the same configuration is assumed hereafter for each of the nonlinear cases.

Floors are assumed rigid in their own plane, while the self-weight and super-dead load are the two sources of mass for the structure and are lumped at the floor level. The total number of DoFs is \( n_p = 75 \) (i.e. 15 DoFs per storey, 2 finite elements used for each frame element) and the total number of modes is 30. The fundamental period of vibration in the direction of interest \( x \) is \( T_{p,1} = 0.426 \text{s} \) \( (\omega_p = 14.76 \text{rad/s}) \) corresponding to a participation of 84% of the modal mass and \( \beta = 1.3 \).

4.5.1 Combined Linear System

The vibration response of the combined linear system (§ 4.2) is examined first. Figure 4.5 compares the full geometric response obtained by solving Eq. (4.1) with the modal analytical
(Eq. (4.16)) and numerical (Eq. (4.19)) solutions, when only the fundamental mode is retained in the analysis. The results quantified in terms of the absolute acceleration of P and the displacement of S are reported for the full dynamic interaction (i.e. \( a = 1 \), \( \varphi = 1 \), \( \zeta_p = 0.05 \), \( \zeta_s = 0.02 \) and \( \phi = 0 \).

As shown, the analytical solution precisely matches the numerical one for all cases considered. Furthermore, the three solutions coincide when \( \omega_s^* = 0.3 \), even though the primary structure is in resonance (i.e. \( \omega_g^* = 1 \)), suggesting that in this case the first mode of vibration is sufficient to capture the dynamics of the combined P-S system (Figures 4.5(a), 4.5(b)). When the secondary system is in tune with the primary structure (i.e. \( \omega_s^* = 1 \)), errors develop in the evolution of the response of both systems. As demonstrated in Figures 4.5(c) - 4.5(f), these appear to be comparable for \( \omega_g^* = 2\pi/(3\omega_p) = 0.43 \) (i.e. a pulse duration of \( t_f = 1 \) s) and \( \omega_s^* = 1 \).

Figures 4.6(a), 4.6(b) plot the percentage cumulative errors for P and S, respectively, evaluated through Eq. (3.29). The results are reported for \( 0.1 \leq \omega^*_s \leq 4 \) and \( \pi/(3\omega_p) \leq \omega^*_g \leq 1.2 \) (i.e. \( 0.35 \leq t_f \leq 6 \) [58]).

As shown, provided that \( \omega_g^* > 0.35 \), errors increase with \( \omega_g^* \) and as \( \omega_s^* \to 1 \); furthermore, they are similar for both P and S. Interestingly, the modal approximation becomes particularly effective when \( \omega_s^* < 0.7 \), i.e. for relatively flexible S oscillator, while for \( \omega_s^* > 2 \) the error surfaces reach a plateau. Indeed, although \( \epsilon \) increases as \( \omega_s^* \to 1 \) and \( \omega_s^* \to 1 \), the effects of the latter are higher. Overall, when \( \beta = 1.3 \), provided that \( \omega_s^* \leq 0.75 \), \( \omega_s^* \geq 1.25 \) and \( \omega_g^* \) does not approach unity, the modal approximation is satisfactory i.e. \( \epsilon \leq 30\% \) and is thus adopted in the subsequent analyses.

Figure 4.7 compares the modal solutions for the combined system when \( a = 0.1 \) with the cascade approximation. As shown, when \( \omega_s^* = 0.3 \) and \( \omega_g^* = 1 \), the P-S interaction is minimal and the cascade solution provides a good approximation of the response for both P and S (Figure 4.7(a) and 4.7(b)). On the contrary, large discrepancies develop in the response when \( \omega_s^* = 1 \), regardless of the value of \( \omega_g^* \) (Figure 4.7(c) - 4.7(f)); these discrepancies are more profound on S.

Regions of validity for the cascade analysis are portrayed in Figure 4.8 in the form of cumulative errors, evaluated using Eq. (3.29). When \( a = 0.1 \), similar trends are shown in Figures 4.8(a) and 4.8(b) to those identified in Figure 4.6, except that higher errors are predicted herein. Evidently, errors increase with \( \omega_g^* \) provided that \( \omega_g^* > 0.5 \), while the results appear less ordered below this threshold. The results signify that dynamic interaction becomes critical when \( \omega_s^* \to 1 \), and therefore in such condition the cascade approximation should not be
Figure 4.5: Comparison of the geometric and modal solution for the combined system. Absolute acceleration response quantified on the primary system (a, c, e) and displacement response quantified on the linear secondary oscillator (b, d, f).
used. Furthermore, the resulting errors for S are significantly higher than P at \( \omega^*_s = 1 \), while comparable elsewhere; a cut-off limit has been applied in the plots at \( \epsilon = 160\% \).

Figures 4.8(c) and 4.8(d) present the error surfaces at \( \omega^*_g = 1 \) for P and S, respectively. Overall, the results are in good agreement with those presented in [66] for frequency domain analysis. Provided that \( \omega^*_s \leq 0.75 \) or \( \omega^*_s \geq 1.25 \) and \( a \leq 0.1 \), the cascade approximation is satisfactory. While the latter condition is usually satisfied with the majority of S being much lighter than P, the analyst must only resort to simplified methods if the frequency ratio \( \omega^*_s \) is not close to unity.

### 4.5.2 Analytical and Numerical Solution Validation

The analysis proceeds with the validation of the analytical (§ 4.3.2) and numerical (§ 4.3.3) solutions for the nonlinear oscillators in cascade. The results presented herein correspond to

- \( \omega^*_g = 2\pi/(3\omega_p) = 0.14 \) (i.e. \( t_f = 3 \text{s} \)) and \( \phi : \{0, \pi/2\} \) for the ground acceleration; \( \omega^*_s = 0.3 \), \( \psi_s = 0 \) and \( a^*_s = 0.1 \) for the bilinear and sliding S; \( \alpha = 0.25 \), \( p^* = 2/\omega_p \), \( \varepsilon = 0.7 \) and \( a^*_g = 0.28 \) for the rocking S; furthermore, the overturning condition \( \theta^*_s = 1 \) is assumed for the latter. The parameters \( \zeta_p, \zeta_s, \varphi, \beta \) retain the values defined in the preceding sections.

Figure 4.9 shows the displacement and absolute acceleration response histories of P for the two pulse-type excitations being considered. Overall, an excellent agreement is observed between the analytical and numerical solutions, confirming the validity of the derived expressions. Notably, the case of \( \phi = \pi/2 \) results in higher amplitudes in the response; as expected, these occur within the forced vibration regime for both cases, while the amplitude of vibration reduces during the free vibration, i.e. \( t^* > 44 \). Evidently, the frequency of oscillation
Figure 4.7: Comparison of the combined and cascade solution. Absolute acceleration response quantified on the primary system (a, c, e) and displacement response quantified on the linear secondary oscillator (b, d, f).
of the displacement response increases when $\phi = \pi/2$ (Figures 4.9(a) and 4.9(b)), albeit the pronounced similarities with $\phi = 0$ that have been reported in Figure 3.4. Figures 4.9(c) and 4.9(d) show the absolute acceleration response histories that are used as input to the secondary oscillators in the cascaded analysis. The horizontal dotted lines depicted at $a_s^* = 0.1$, represent the yielding threshold of the bilinear secondary system as well as the initiation instant for sliding motion; similarly, the dashed ones at $g^* \tan(\alpha) = 0.91$ mark the initiation of rocking motion. Interestingly, when $\phi = \pi/2$, $\ddot{u}_p^* = -0.305$ at $t^* = 0$.

The validity of the derived expressions is confirmed on the displacement and velocity response of the bilinear S, as shown in Figure 4.10. When $\phi = 0$ (Figure 4.10(a)), the deformation is initially small as the system vibrates within its elastic regime of motion (Figure 4.9(c)). After yielding, a peak displacement of $u_s^* = -209$ is reached at $t^* = 33.9$ and the system is finally brought to rest at $u_s^* = -103$, a position that is evidently different than its initial equilibrium position. When $\phi = \pi/2$ (Figure 4.10(b)), the system initiates motion whilst being on
the plastic branch of the force-deformation relationship. A peak displacement of $u_s^* = -74$ occurs at $t^* = 19.2$ and the system finally comes to rest with $u_s^* = 39$.

The sliding response histories in Figure 4.11 retain similar trends to the bilinear S, further confirming the validity of the expressions. This is due to the parameter values chosen, namely, $\omega_s = 0.3$, $\zeta_s = 0.02$ and $\psi_s = 0$, resulting in the bilinear solution approaching the rigid-plastic limit. As demonstrated through Figure 4.9(c), when $\phi = 0$, sliding motion is initiated at $t^* = 1.84$ and deformation increases thereafter (Figure 4.11(a)) until a peak displacement of $u_s^* = -242$ is reached at $t^* = 36.9$, at which point $\dot{u}_s^* = 0$. For $\phi = \pi/2$, sliding motion is initiated from $t^* = 0$ and the peak displacement is $u_s^* = -82$ at $t^* = 20.1$.

The rotation and angular velocity response of the rocking S is examined in Figure 4.12. As shown, in all cases the analytical and numerical expressions derived for the linearised system in Eq. (4.23) are in excellent agreement. Figures 4.9(c) and 4.9(d) also show that the system enters rocking motion at $t^* = 8.9$ and $t^* = 1.5$ for $\phi = 0$ and $\phi = \pi/2$, respectively. The response progressively increases thereafter and maximum values are seen for both cases within
the forced vibration regime of motion. Notably, when comparing the response to the nonlinear system of Eq. (3.12), the linearised formulation incorrectly predicts overturning when $\phi = 0$ as $\theta_s^* = 1$ at $t^* = 44.3$ (Figure 4.12(a)). Although the linear approximation appears satisfactory in predicting the time of overturning $t^* = 27$ with respect to its reference value $t^* = 27.7$, when $\phi = \pi/2$ (Figure 4.12(b)), the results indicate that lower values of $\alpha$ may need to be considered. The effect of the coefficient of restitution is shown in Figures 4.12(c) and 4.12(d) through the sharp attenuations on the peaks values of $\dot{\theta}_s^*$ and the energy dissipation in Figures 4.12(e) and 4.12(f).

### 4.5.3 Floor Response Spectra

A series of floor response spectra have been evaluated based on the cascade approximation and are presented herein with the aim of understanding the influence of the input parameters on the response. The parameters are $\zeta_p : \{0.01, 0.03, 0.05, 0.07, 0.09\}$ and $\phi : \{0.2, 0.4, 0.6, 0.8, 1\}$. 

Figure 4.10: Analytical and numerical evaluation of the displacement and velocity response for the bilinear secondary oscillator due to a sine (a, c) and cosine (b, d) pulse, respectively.
for P; $\omega_g^* : \{0.3, 0.5, 0.7, 1.3, 1.5\}$ (i.e. $\omega_g^* \leq 0.75$ and $\omega_g^* \geq 1.25$; see: § 4.5.1), $\zeta_s : \{0.01, 0.02, 0.03, 0.04, 0.05\}$, $\mu_d : \{1, 3, 5, 7\}$ and $\psi_s : \{0, 0.1, 0.2, 0.3, 0.4\}$ for the bilinear secondary system; $a_g^* : \{1, 1.3, 1.6, 1.9, 2.2, 2.5\}$ for the sliding secondary system; $a_g^* : \{0.24, 0.27, 0.31, 0.36, 0.44, 0.57, 0.8, 1.33, 4\}$, $\alpha : \{0.15, 0.2, 0.25, 0.3\}$, $p : \{1, 1.5, 2, 2.5, 3\}$ and $\varepsilon : \{0.2, 0.3, 0.5, 0.7\}$ for the rocking system, where the numbers in bold font represent reference values. These are kept constant so that the influence of a single parameter is investigated each time. The results are reported for $0 \leq \omega_g^* \leq 1.2$ and $\phi : \{0, \pi/2\}$ and are quantified in terms of the peak absolute value of the response.

### 4.5.3.1 Primary System

The response of the primary structure is examined first. Figures 4.13(a) and 4.13(b) show the influence of $\omega_g^*$ on the absolute acceleration response history of P for $\phi = 0$ and $\phi = \pi/2$, respectively, when $\zeta_p = 0.05$ and $\varphi = 1$. The grey lines show the response for values of $\omega_g^*$ in
Figure 4.12: Analytical and numerical evaluation of the rotation and angular velocity response for the rocking secondary oscillator and corresponding energy plots due to a sine (a, c, e) and cosine (b, d, f) pulse, respectively.
the range \(0 \leq \omega_g^* \leq 1.2\). The peak response occurs at \(\omega_g^* = 0.92\) and \(\omega_g^* = 0.76\) for the two cases, respectively.

Figures 4.13(c) and 4.13(d) illustrate the influence of \(\varphi\) on the response spectra (\(\zeta_p = 0.05\)). Overall, the response is a smooth function of \(\omega_g^*\) comprising of three distinct phases. For \(\phi = 0\) these are \(0 \leq \omega_g^* \leq 0.2\), \(0.2 \leq \omega_g^* \leq 0.5\) and \(0.5 \leq \omega_g^* \leq 1.2\). As shown, the response increases with \(\omega_g^*\) and the rate of increase is higher for each phase, while a reduction is shown beyond the peak value (e.g. \(\omega_g^* > 0.92\) for \(\varphi = 1\)). As expected, increasing \(\varphi\) increases the response and causes the peak value to be reached at a higher value of \(\omega_g^*\). Furthermore, the effect of \(\varphi\) is proportional to \(\omega_g^*\) (i.e. negligible at low \(\omega_g^*\)). Similarly, for \(\varphi = \pi/2\) the phases are \(0 \leq \omega_g^* \leq 0.35\), \(0.35 \leq \omega_g^* \leq 0.62\) and \(0.62 \leq \omega_g^* \leq 1.2\). Contrary to the previous case, the response decreases with \(\omega_g^*\) over the first phase. Moreover, the effect of \(\varphi\) is considerable also in the low frequency range. Lastly, higher amplitudes are predicted in the curves than \(\phi = 0\).

The influence of \(\zeta_p\) is depicted in Figures 4.13(e) and 4.13(f) (\(\varphi = 1\)). As expected, increasing \(\zeta_p\) results in a reduction in the response spectra for both cases, and the effect is more significant for higher frequencies.

### 4.5.3.2 Bilinear Secondary Oscillator

The analysis proceeds with the response spectra for the bilinear S. The case of an elastoplastic undamped system (\(\psi_s = \zeta_s = 0\)) is considered first. As shown in Figures 4.14(a) and 4.14(b) \((a_s^* = 0.2)\), the response is a decreasing function of \(\omega_g^*\) and as \(\omega_g^* \rightarrow \infty\), the response approaches the rigid-plastic (RP) limit. Furthermore, higher values are seen when \(\phi = 0\). When \(\omega_g^* = 0.3\), Figures 4.14(c) and 4.14(d) indicate that the response is highly sensitive to \(a_s^*\). As \(a_s^* \rightarrow \infty\), the response approaches the linear (LN) limit, where there is no plastic deformation. Increasing \(a_s^*\) from 0.65 to 1.10 has the highest influence on the response, with the curves changing shape and having a single peak value at \(\omega_g^* \approx \omega_g^* = 0.3\).

Figures 4.15 to 4.19 plot the response in terms of constant ductility inelastic spectra where \(\mu_d = u_{s,ult}/u_{s,yld}\) denotes the ductility ratio, \(u_{s,ult}\) and \(u_{s,yld}\) being the ultimate and yielding displacement of the system, respectively. The effect of each input parameter is separately investigated while the remaining ones assume the reference values. The effect of \(\omega_g^*\) is investigated first in Figure 4.15. Overall similar trends are shown for both \(\phi = 0\) and \(\phi = \pi/2\). The response decreases with \(\omega_g^*\) for \(\omega_g^* < 0.4\), while the results appear less ordered in the high frequency range. As expected, an amplification is shown in the response when \(\omega_g^* \approx \omega_g^*\).
Figure 4.13: Absolute acceleration time histories and effect of the dimensionless modal coordinate $\varphi$ and damping ratio $\zeta_p$ on the peak response of the primary structure, due to a sine (a, c, e) and cosine (b, d, f) pulse, respectively.
Increasing $\mu_d$ increases the amplitude of the response due to the inelastic deformations and causes a shift in the amplification frequency towards the lower range.

Figure 4.16 shows the effect of $\zeta_s$ on the spectra. For all cases considered, increasing $\zeta_s$ results in a reduction in the displacement response. As shown, the effect is more profound near the resonant frequency of the curves (e.g. $\omega^*_g \approx \omega^*_s$ for $\mu_d = 1$, Figure 4.16(a)) and negligible for $\omega^*_g > 0.5$. The effect is also negligible when $\phi = 0$ and $\mu_d = 5, 7$ (Figures 4.16(e) and 4.16(g)).

Figure 4.17 shows the influence of $\psi_s$ on the response spectra. Increasing $\psi_s$ results in a residual stiffness that causes a reduction on the response, with the effect being more evident near the resonant frequency and for higher values of $\mu_d$ (i.e. Figures 4.17(e) and 4.17(f)). Similar to the previous case, the effects are negligible for $\omega^*_g > 0.5$. Furthermore, $\psi_s$ causes a shift in the peak displacement of the curves towards the high frequency range when $\phi = 0$ (Figures 4.17(a), 4.17(c) and 4.17(e)).
Figure 4.15: Effect of the dimensionless frequency $\omega_s^*$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively.
Figure 4.16: Effect of the damping ratio $\zeta_s$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively.
Figure 4.17: Effect of the post-yield to pre-yield stiffness ratio $\psi_s$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e) and cosine (b, d, f) pulse, respectively.
Figures 4.18 and 4.19 illustrate the influence of the parameters $\varphi$ and $\zeta_p$ of the primary structure on the response of the bilinear secondary oscillator, respectively. As expected, the results appear ordered with the response increasing with $\varphi$ (Figure 4.18) and reducing with $\zeta_p$ (Figures 4.19). Contrary to the primary structure (Figure 4.13), the relative contribution of both these parameters appears to have minor effects on the response of the bilinear secondary system. An exception to this is the case of $\zeta_p \leq 0.03$ for $\phi = \pi/2$ and $\omega_g^* > 0.6$ (Figures 4.19(d), 4.19(f) and 4.19(h)).

### 4.5.3.3 Sliding Secondary Block

The response spectra of the sliding secondary oscillator are presented in Figure 4.20.

Figures 4.20(a) and 4.20(b) confirm that variation in $\alpha_s^*$ can significantly influence the response. Overall, similar trends are shown for both $\phi = 0$ and $\phi = \pi/2$, with the peak displacement spectra reducing with $\alpha_s^*$. Interestingly, the curves comprise of three distinct phases that arise due to the response of the primary structure (see Figure 4.13). The curves appear less ordered for $\omega_g^* > 0.5$ and $\alpha_s^* \leq 1.3$.

The influence of $\varphi$ is demonstrated in Figures 4.20(c) and 4.20(d). For all cases (i.e. $\phi = 0, \pi/2$), the response spectra increase with $\varphi$ and the results appear ordered. Contrary to the bilinear system (§ 4.5.3.2) the influence of $\varphi$ cannot be considered negligible. Evidently, the rate of increase reduces at higher values of $\varphi$ (the distance between the curves progressively reduces).

Figures 4.20(e) and 4.20(f) illustrate that the response reduces with increasing values of $\zeta_p$ and the results are ordered. Similar to the case of $\varphi$, the variations are appreciable on the response.

### 4.5.3.4 Rocking Secondary Block

The case of the rocking secondary block is examined in Figures 4.21 and 4.22.

Overall, $\alpha_g^*$ increases the overturning tendency of the block, up to a point that no overturning occurs for $\alpha_g^* < 0.31$ and $\alpha_g^* < 0.27$ for $\phi = 0$ and $\phi = \pi/2$, respectively. Interestingly minor effects are seen for $\alpha_g^* < 0.57$, $\omega_g^* > 0.6$ for $\phi = 0$ and the results appear more ordered for $\phi = \pi/2$ (Figure 4.21(b)). Discontinuities are seen for $\alpha_g^* = 0.8$ at $\omega_g^* = 0.6$ (Figure 4.21(a)) and $\alpha_g^* = 0.24$ at $\omega_g^* = 0.15$ (Figure 4.21(b)), that signify the importance of modelling the rocking behaviour of secondary structures.

Increasing the slenderness $\alpha$ of the block reduces its overturning likelihood; indeed, no overturning is predicted for $\alpha > 0.2$ and $\alpha > 0.25$ for $\phi = 0$ and $\phi = \pi/2$, respectively,
Figure 4.18: Effect of the dimensionless modal coordinate $\varphi$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively.
Figure 4.19: Effect of the damping ratio $\zeta_p$ and ductility ratio $\mu_d$ on the peak displacement response of the bilinear secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively.
Figure 4.20: Effect of the specific strength $a^*_s$, the dimensionless modal coordinate $\phi$ and the damping ratio $\zeta_p$ on the peak displacement response of the sliding secondary oscillator, due to a sine (a, c, e) and cosine (b, d, f) pulse, respectively.
(Figures 4.21(c) and 4.21(d)). Minor effects are shown for $\alpha > 0.1$, $\omega^*_g > 0.6$ for $\phi = 0$. Evidently, the results are not ordered ($\alpha = 0.15, 0.20, \phi = 0$), suggesting that the response can only be predicted by modelling the rocking behaviour, and performing some sensitivity analyses.

Figures 4.21(e) and 4.21(f) show that the response increases with the dynamic parameter $p$. For both $\phi = 0$ and $\phi = \pi/2$, overturning occurs for $p \geq 2$. Contrary to the previous cases, the results are ordered, suggesting that a scaling factor can be adopted for estimating the response of various rocking blocks.

Figures 4.21(g) and 4.21(h) illustrate the effect of the coefficient of restitution $\varepsilon$ on the attenuation of the response. Similar to the case of $p$, the spectra are ordered with the exception of a discontinuity for $\varepsilon = 0.5$ at $\omega^*_g = 0.13$. As shown, the effects of $\varepsilon$ reduce with $\omega^*_g$ for $\phi = \pi/2$.

The influence of the parameters of the primary structure have also been investigated. As demonstrated, the overturning potential increases with $\phi$. In particular, reducing $\phi$ from 1 to 0.8 results in a 37% reduction of the peak value at $\omega^*_g = 0.3$ for $\phi = 0$ (Figure 4.22(a)). Although similar trends are observed for $\phi = \pi/2$ (Figure 4.22(b)), there are discontinuities in the spectra (e.g. $\phi = 0.8$ at $\omega^*_g = 0.1$).

Figure 4.22(c) illustrates that increasing $\zeta_p$ from 0.03 to 0.05 causes a 11% reduction in the spectra at the resonant frequency ($\omega^*_g = 0.3$) for $\phi = 0$. Interestingly, there are discontinuities associated with low damping values at $\omega^*_g = 1$ and $\omega^*_g = 0.7$ for $\phi = 0$ (Figure 4.22(c)) and $\phi = \pi/2$ (Figure 4.22(d)), respectively.

### 4.6 Summary

In this chapter, the dynamic response of pulse-driven secondary (S) oscillators attached to a primary (P) structure has been examined. Closed-form solutions have been derived for the combined vibration response of a 2DoF linear system, and decoupling criteria have been presented for the analysis of linear SDoF secondary structures. Analytical and numerical solutions have also been derived for the cascade analysis of nonlinear bilinear, sliding and rocking S oscillators. These solutions have been used to investigate the sensitivity of the S oscillators for various input parameters in the form of floor response spectra. The latter is indeed a powerful tool, often used in the engineering practice to assess the seismic performance of light S attachments. It is particularly appealing because it allows circumventing the numerical difficulties that are present in the analysis of combined systems, leading to substantial savings in com-
Figure 4.21: Effect of the parameter $a^*\gamma$, the slenderness $\alpha$, the dynamic parameter $p$ and the coefficient of restitution $\varepsilon$ on the peak rotation response of the rocking secondary oscillator, due to a sine (a, c, e, g) and cosine (b, d, f, h) pulse, respectively.
4.6 Summary

Computational cost. However, the results of the parametric investigations reported in this chapter demonstrate that the cascade approximation is accurate enough for engineering analysis of linear systems provided that the S-P frequency ratio is \( \omega_s^* \leq 0.75 \) or \( \omega_s^* \geq 1.25 \) and the modal mass ratio is \( a \leq 0.1 \).

The analytical solutions derived directly express the response time history of the secondary oscillators considered as a function of the ground parameters. These were used to examine the influence of various parameters on the response and formed the basis for validating the proposed numerical solutions. The latter can be used for a general-type ground excitation (see: Chapter 5).

Floor response spectra have been presented for each of the nonlinear secondary oscillators. The constant ductility inelastic spectra for the bilinear oscillator highlighted that variation in \( \mu_d \) and \( \psi_h \) results in a shift in the resonant frequency \( (\omega_{g*} \approx \omega_{s*}) \) towards the lower and higher range, respectively. Increasing \( \zeta_s \) was found to have profound effects in the vicinity of the

Figure 4.22: Effect of the dimensionless modal coordinate \( \varphi \) and the damping ratio \( \zeta_p \) on the peak rotation response of the rocking secondary oscillator, due to a sine (a, c) and cosine (b, d) pulse, respectively.
resonant frequency. The relative contribution of $\varphi$ and $\zeta_p$ was shown to have minor effects on the response.

Increasing $a_s^*$ was shown to reduce the peak spectra of the sliding oscillator and the results were shown to be less ordered for $\omega_n^* > 0.5$ and $a_s^* \leq 1.3$. Contrary to the bilinear oscillator the effect of $\varphi$ and $\zeta_p$ cannot be considered negligible; furthermore, the resulting spectra were shown to be ordered.

The overturning potential of the rocking oscillator was found to increase with $a_g^*$ and $\alpha$. Although these trends are rather intuitive, i.e. the stronger the ground shaking and the more slender the rocking block, the higher the likelihood of overturning, some discontinuities appearing in the rocking spectra are attributed to the strong nonlinearities of this S system suggest that its seismic response can only be predicted by accurately modelling the rocking behaviour. The overturning potential was also shown to increase with $p$, $\varphi$ and reduce with $\varepsilon$, $\zeta_p$ and the results were found ordered, suggesting that a scaling factor can be adopted to envelope the response for these parameters.

Overall, the response analysis presented in this chapter demonstrates how, even in the simple case of a pulse-like input, the seismic response of S oscillators is highly sensitive to the mechanical parameters of both the P structure and S oscillator. Great attention should then be paid when quantifying these parameters, and sensitivity analyses should always be conducted to assess the impact of any variation.
Stochastically Excited Secondary Oscillators in Cascade

5.1 Introduction

In the last chapter, numerical solutions were developed for the analysis of piecewise linear secondary oscillators, whose response was examined under pulse-type excitation. This excitation was chosen as a simplified approximation of near-field ground motions, permitting preliminary investigations. Clearly, this type of excitation is purely mathematical in nature. Appropriate characterisation of the seismic hazard requires the use of recorded ground motions or stochastic ground motion models (either record-based or point-source) that also take into consideration uncertainties or nonstationarities, which can significantly influence the response of inelastic systems. Arguably, the use of recorded signals suffers from scarcity of fit-for-purpose recorded events for specified earthquake characteristics, rendering the use of stochastic models as an appealing alternative.

In this chapter, an existing record-based stochastic model is used to generate a suite of synthetic far-field strong ground motions for specific earthquake and site characteristics, and the response of nonlinear secondary oscillators is examined using Monte Carlo simulations. Specifically, Section 5.2 overviews the stochastic ground motion model and its validity is con-
firmed through comparisons with recorded ground motions and the Next Generation of Ground-Motion Attenuation (NGA) prediction models. Through a numerical example, the results of a parametric investigation are then presented in Section 5.3 for the main seismicity characteristics (i.e. earthquake magnitude and rupture distance) in the form of stochastic response spectra of bilinear, sliding and free-standing rocking secondary oscillators. In doing this, the numerical procedure proposed in Chapter 3 is adopted. Purpose of this chapter is to quantify and facilitate understanding of the response of nonlinear secondary oscillators solely due to stochastic ground motions. Consideration of uncertainties in their properties or those of the primary structure falls beyond the scope of this chapter, as the latter will be explicitly addressed in the following chapters.

5.2 Stochastic Ground Motion Model

Among the stochastic models available in the technical literature, the recently-proposed record-based ground motion model by Rezaeian and Der Kiureghian (R-DK) [132] is deemed to adequately account for all uncertainties contributing to the variability of the ground motion for the scope of this study, and is thus chosen herein for characterising the seismic hazard [9].

5.2.1 Model Formulation

The procedure for simulating a single horizontal ground acceleration component using the R-DK model is schematically represented in Figure 5.1. Adopting the formulation proposed in [17], the continuous form of a unidirectional Gaussian ground acceleration process is defined through:

$$x(t) = \frac{Q(t, \kappa)}{\sigma_h(t)} \int_{-\infty}^{t} h[t - \tau, \lambda(\tau)] w(\tau) d\tau,$$

in which $x(t)$ denotes the nonstationary acceleration process; $Q(t, \kappa)$ is a deterministic time-modulating function, depending on a set of parameters $\kappa$ defining its intensity and shape; $w(\tau)$ is a Gaussian white-noise process; $h[t - \tau, \lambda(\tau)]$ is the impulse-response function (IRF) of a filter with time-dependent parameters collectively denoted as $\lambda(\tau)$, accounting for the spectral nonstationarity; $\sigma_h^2(t) = \int_{-\infty}^{t} h^2[t - \tau, \lambda(\tau)] d\tau$ is the variance of the process. The above expression therefore represents a filtered white-noise process of unit variance modulated in time through $Q(t, \kappa)$, that is equal to the standard deviation of the process, fully defining the temporal nonstationarity.
5.2 Stochastic Ground Motion Model

The filter IRF is chosen as:

$$h[t - \tau, \lambda(\tau)] = \begin{cases} \frac{\omega_f(\tau)}{\sqrt{1 - \zeta_f^2(\tau)}} \exp \left[ -\zeta_f(\tau)\omega_f(\tau)(t - \tau) \right] \times \sin \left[ \omega_f(\tau)\sqrt{1 - \zeta_f^2(\tau)}(t - \tau) \right], & \text{if } \tau \leq t, \\ 0, & \text{otherwise} \end{cases}$$

representing the pseudo-acceleration response of a linear SDoF oscillator (see Figure 5.2(a)), where the spectral parameter set $\lambda(\tau) = \{\omega_f(\tau), \zeta_f(\tau)\}$ collects the time-varying frequency and damping ratio.

Herein, based on [18], a linear function is adopted for the frequency and a constant value for the damping:

$$\omega_f(\tau) = \omega_{mid} + \omega'(\tau - t_{mid}); \quad \zeta_f(\tau) = \zeta_f,$$

where $\omega_{mid}$ and $\omega'$ are the filter frequency and its derivative at $t = t_{mid}$.

A Gamma function is adopted for the modulating function:

$$Q(t, \kappa) = \kappa_1 t^{\kappa_2 - 1} \exp (-\kappa_3 t),$$

in which the temporal parameter set $\kappa = \{\kappa_1, \kappa_2, \kappa_3\}$ is to be identified, satisfying $\kappa_1, \kappa_3 > 0$ and $\kappa_2 > 1$. Specifically, $\kappa_1$ accounts for the intensity of the ground motion random process, while $\kappa_2$ and $\kappa_3$ control the shape of the modulating function and the duration of the strong
motion phase, respectively. Figure 5.2(b) illustrates the modulating function for a given set of parameters.

The parameters \( \{\kappa_1, \kappa_2, \kappa_3\} \) are related to the physical ones \( \{\bar{I}_a, D_{5-95}, t_{\text{mid}}\} \), namely the Arias intensity, the effective duration and the time to reach 45% of Arias intensity for the target ground motion under consideration, respectively. Accordingly, the gamma probability density function (PDF) with shape and scale parameters, \( 2\kappa_2 - 1 \) and \( 1/(2\kappa_3) \), respectively, is proportional to the variance function \( Q^2(t, \kappa) \) [18]. It follows that \( t_p \), the \( p\% \) percentile variate of the associated cumulative distribution function (CDF), can be expressed in terms of the probability \( p\% \) and the parameters \( \kappa_2 \) and \( \kappa_3 \). Hence, \( \kappa_2 \) and \( \kappa_3 \) are chosen for a given value of \( D_{5-95} \) and \( t_{\text{mid}} \) in order to satisfy:

\[
D_{5-95} = t_{95} - t_5; \quad t_{\text{mid}} = t_{45}.
\]

Once \( \kappa_2 \) and \( \kappa_3 \) are estimated, \( \kappa_1 \) can be computed for a given value of \( \bar{I}_a \) as:

\[
\kappa_1 = \sqrt{\frac{2g}{\pi} \bar{I}_a \frac{(2\kappa_3)^{2\kappa_2 - 1}}{\Gamma(2\kappa_2 - 1)}},
\]

where \( \Gamma(\cdot) \) represents the gamma function.
5.2 Stochastic Ground Motion Model

Denoting $t_i, i = 0, 1, \ldots, d$ as distinct time instants with a constant time step $\Delta t$, and letting $k = \text{int}(t/\Delta t)$ for $0 < t \leq t_d$, the discretised form of the stochastic model in Eq. (5.1) reads:

$$x(t) = Q(t, \kappa) \sum_{i=1}^{k} s_i(t, \lambda(t_i)) v_i; \quad t_k \leq t < t_{k+1},$$

(5.7)

in which $v_i$ are standard normal random variables used to define the white-noise process, and $s_i(t, \lambda(t_i))$ are deterministic basis functions, where:

$$s_i(t, \lambda(t_i)) = \frac{h[t - t_i, \lambda(t_i)]}{\sqrt{\sum_{j=1}^{k} h^2[t - t_j, \lambda(t_j)]}}; \quad t_k \leq t < t_{k+1}; \quad i = 1, \ldots, k. \quad (5.8)$$

Once the model parameters are identified, the random variables $v_i$ and the basis functions $s_i(t, \lambda(t_i))$ are used to generate the desired number of realisations of the process in Eq. (5.7).

Given the simulated process, a critically damped high-pass filter (HPF) is adopted to adjust the low frequency content of the model, ensuring zero residual displacement and velocity. The corrected simulated acceleration $\ddot{y}(t)$ is then obtained as the response of:

$$\ddot{y}(t) + 2 \omega_c \dot{y}(t) + \omega_c^2 y(t) = x(t); \quad y(0) = 0; \quad \dot{y}(0) = 0,$$

(5.9)

where $\omega_c$ is the HPF frequency ($\omega_c \approx 0.2 - 0.4 \pi \text{rad/s}$).

5.2.2 Predictive Equations

The parameter set $\theta_g = \{ I_a, D_{5-95}, t_{\text{mid}}, \omega_{\text{mid}}/2\pi, \omega'/2\pi, \zeta_f \}$ in conjunction with $\omega_c$ and the standard normal random vector $v$ of size $k$, fully define the R-DK model and can be regarded as a single realisation that could arise from ground motions of similar sites and characteristics. Empirical predictive relationships were constructed by fitting the model to a subset of ground motions from the Next Generation Attenuation (NGA) strong motion database [132]. Thus, given the faulting mechanism $F$ ($F = 0$ for strike slip and $F = 1$ for reverse fault), the shear-wave velocity $V_s$, the moment magnitude of the earthquake $M$ and the source-to-site distance $R$, the set $\theta_g$ can be obtained without the need of a previously recorded motion. In order to do so, each of the parameters in $\theta_g$ is first assigned a probability distribution according to Table 5.1 as detailed in [132]. In the case of $\omega'/2\pi$ a truncated exponential PDF is assumed.
Table 5.1: Fitted distributions assigned to the R-DK model [132].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Fitted distribution</th>
<th>Distribution bounds</th>
<th>Mean</th>
<th>Std. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_a$ (s g)</td>
<td>Lognormal</td>
<td>(0, $\infty$)</td>
<td>0.0468</td>
<td>0.164</td>
</tr>
<tr>
<td>$D_{5-95}$ (s)</td>
<td>Beta</td>
<td>[5, 45]</td>
<td>17.3</td>
<td>9.31</td>
</tr>
<tr>
<td>$t_{\text{mid}}$ (s)</td>
<td>Beta</td>
<td>[0.5, 40]</td>
<td>12.4</td>
<td>7.44</td>
</tr>
<tr>
<td>$\omega_{\text{mid}}/2\pi$ (Hz)</td>
<td>Gamma</td>
<td>(0, $\infty$)</td>
<td>5.87</td>
<td>3.11</td>
</tr>
<tr>
<td>$\omega'/2\pi$ (Hz/s)</td>
<td>Eq. (5.10)</td>
<td>$[-2, 0.5]$</td>
<td>-0.0892</td>
<td>0.185</td>
</tr>
<tr>
<td>$\zeta_f$</td>
<td>Gamma</td>
<td>[0.02, 1]</td>
<td>0.213</td>
<td>0.143</td>
</tr>
</tbody>
</table>

The parameters are then transformed to the standard normal space through the following transformation:

$$f_{\omega'/2\pi}(\omega'/2\pi) = \begin{cases} 
4.85 \exp(6.77 \omega'/2\pi), & -2 < \omega'/2\pi < 0 \\
4.85 \exp(-17.10 \omega'/2\pi), & 0 < \omega'/2\pi < 0.5 \\
0, & \text{otherwise}
\end{cases} \quad (5.10)$$

The parameters are then transformed to the standard normal space through the following transformation:

$$\nu_i = \Phi^{-1} \left[ F_{\theta_{g,i}}(\theta_{g,i}) \right]; \quad i = 1, \ldots, 6, \quad (5.11)$$

leading to the vector $\nu$ with elements $\nu_i$, $i = 1, \ldots, 6$. In the above, $\Phi [\cdot]$ denotes the standard normal cumulative distribution function (CDF) and $F_{\theta_{g,i}}(\cdot)$ the marginal CDF fitted to the $i$-th element of $\theta_g$. The resulting predictive equations are then:

$$\begin{align*}
\nu_1 &= \beta_{1,0} + \beta_{1,1}F + \beta_{1,2} \left( \frac{M}{7} \right) + \beta_{1,3} \ln \left( \frac{R}{25 \text{ km}} \right) + \beta_{1,4} \ln \left( \frac{V_s}{750 \text{ m/s}} \right) + \eta_1 + \varepsilon_1;
\nu_i &= \beta_{i,0} + \beta_{i,1}F + \beta_{i,2} \left( \frac{M}{7} \right) + \beta_{i,3} \left( \frac{R}{25 \text{ km}} \right) + \beta_{i,4} \left( \frac{V_s}{750 \text{ m/s}} \right) + \eta_i + \varepsilon_i;
\end{align*} \quad i = 2, \ldots, 6, \quad (5.12)$$

in which $\beta_{i,j}$ are the associated dimensionless regression coefficients, given in Table 5.2. Furthermore, $\eta_i + \varepsilon_i$ is the total regression error for each element corresponding to correlated normal random variables with standard deviation $\sqrt{\tau_{i}^2 + \sigma_{i}^2}$ and correlation coefficients as.
Table 5.2: Regression coefficients and errors of the R-DK model [132].

<table>
<thead>
<tr>
<th>i</th>
<th>$\beta_{i,0}$</th>
<th>$\beta_{i,1}$</th>
<th>$\beta_{i,2}$</th>
<th>$\beta_{i,3}$</th>
<th>$\beta_{i,4}$</th>
<th>$\tau_i$</th>
<th>$\sigma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.844</td>
<td>-0.071</td>
<td>2.944</td>
<td>-1.356</td>
<td>-0.265</td>
<td>0.274</td>
<td>0.594</td>
</tr>
<tr>
<td>2</td>
<td>-6.195</td>
<td>-0.703</td>
<td>6.792</td>
<td>0.219</td>
<td>-0.523</td>
<td>0.457</td>
<td>0.569</td>
</tr>
<tr>
<td>3</td>
<td>-5.011</td>
<td>-0.345</td>
<td>4.638</td>
<td>0.348</td>
<td>-0.185</td>
<td>0.511</td>
<td>0.414</td>
</tr>
<tr>
<td>4</td>
<td>2.253</td>
<td>-0.081</td>
<td>-1.810</td>
<td>-0.211</td>
<td>0.012</td>
<td>0.692</td>
<td>0.723</td>
</tr>
<tr>
<td>5</td>
<td>-2.489</td>
<td>0.044</td>
<td>2.408</td>
<td>0.065</td>
<td>-0.081</td>
<td>0.129</td>
<td>0.953</td>
</tr>
<tr>
<td>6</td>
<td>-0.258</td>
<td>-0.477</td>
<td>0.905</td>
<td>-0.289</td>
<td>0.316</td>
<td>0.682</td>
<td>0.760</td>
</tr>
</tbody>
</table>

Table 5.3: Correlations between error terms of the R-DK model [132].

<table>
<thead>
<tr>
<th>$\nu_1$</th>
<th>$\nu_2$</th>
<th>$\nu_3$</th>
<th>$\nu_4$</th>
<th>$\nu_5$</th>
<th>$\nu_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_1$</td>
<td>1</td>
<td>-0.36</td>
<td>0.01</td>
<td>-0.15</td>
<td>0.13</td>
</tr>
<tr>
<td>$\nu_2$</td>
<td>1</td>
<td>0.67</td>
<td>-0.13</td>
<td>-0.16</td>
<td>-0.20</td>
</tr>
<tr>
<td>$\nu_3$</td>
<td>1</td>
<td>-0.28</td>
<td>-0.20</td>
<td>-0.22</td>
<td></td>
</tr>
<tr>
<td>$\nu_4$</td>
<td>1</td>
<td>-0.20</td>
<td>0.28</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu_5$</td>
<td>Sym.</td>
<td>1</td>
<td>-0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu_6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

provided in Table 5.3. The resulting elements of $\mathbf{\nu}$ are thus jointly normal with mean values according to Eq. (5.12) and variances $\tau_i^2 + \sigma_i^2$ while uncertainty is accounted through the error.

5.2.3 Ground Motion Simulation

Given a set of earthquake and site characteristics ($F$, $V_s$, $M$, $R$) corresponding to a design scenario, an ensemble of ground motions is simulated without the need for a recorded motion. For each realisation, the set $\mathbf{\nu} = \{\nu_1, \ldots, \nu_6\}$ of correlated Gaussian random variables is first generated using Eq. (5.12). Using the marginal distributions given in Table 5.1 each parameter is projected back to the physical space by inverting Eq. (5.11), i.e. $\mathbf{\theta}_{g,i} = F_{\mathbf{\theta}_{g,i}}^{-1} \left[ \Phi \left( \mathbf{\nu}_i \right) \right]$ leading to $\mathbf{\theta}_g$. The first three parameters are converted to the parameters $\mathbf{\kappa} = \{\kappa_1, \kappa_2, \kappa_3\}$ through Eqs. (5.5) and (5.6), leading to the set $\{\kappa_1, \kappa_2, \kappa_3, \omega_{\text{mid}}, \omega'_{\text{mid}}, \zeta_f\}$. The synthetic accelerogram $\ddot{y}(t)$ is finally generated using Eq. (5.7) and the high pass filter in Eq. (5.9).

Notably, the independent standard normal random variables $v_i$, $i = 1, \ldots, n$ in Eq. (5.7) and the correlated ones $\nu_i$, $i = 1, \ldots, n$ in Eq. (5.12) thus comprise the source of randomness.
in the seismic excitation. It must be emphasised here that the model was calibrated for ‘strong’ shaking, considering earthquakes with magnitudes $M \geq 6$ and rupture distances $10 \text{ km} \leq R \leq 100 \text{ km}$ and can thus only be used within this range. Obviously, depending on the specific needs of the design team, different stochastic models could be used.

Figure 5.3 compares the 5% damped elastic response spectra of a suite of $n_{\text{sym}} = 50$ synthetic motions with the two horizontal components of target accelerograms (thick solid lines). Specifically, Figure 5.3(a) shows the case of $F = 1$, $M = 6.69$, $R = 20.3 \text{ km}$ and $V_s = 1223 \text{ m/s}$, corresponding to the 1994 Northridge earthquake (LA Wonderland Ave station), while Figure 5.3(b) illustrates the case of $F = 1$, $M = 7.62$, $R = 42.5 \text{ km}$ and $V_s = 643 \text{ m/s}$, corresponding to the 1999 Chi-Chi, Taiwan earthquake (HWA038 station). Both these cases assume $\omega_c = 0.2 \pi \text{ rad/s}$. Evidently, the spectra confirm that the target accelerogram components (black lines) can indeed be regarded as single realisations of the suite of synthetic motions (grey lines) which lie within the variability range over the period range considered (i.e. up to 5 s).

In order to assess the validity of the model, comparisons have been made with existing prediction equations used in the engineering practice. Figure 5.4 compares the median and median $\pm$ one standard deviation of the 5% damped elastic response spectra of a suite of $n_{\text{sym}} = 500$ simulated motions with the spectra obtained from the NGA prediction equations (i.e. Campbell and Bozorgnia [26], Abrahamson and Silva [24], Chiou and Youngs [27], Boore and Atkinson [25]). The case of $M = 7.0$, $R = 40 \text{ km}$ (Figure 5.4(a)) and $M = 8.0$, $R = 20 \text{ km}$ (Figure 5.4(b)) are considered, where $F = 0$ and $V_s = 760 \text{ m/s}$ are assumed.
5.3 Numerical Application

parameters chosen herein permit a direct comparison with the results reported in [9], thus verifying the implementation of the model. As shown, the median and dispersion curves of the R-DK model closely assemble the NGA ones, suggesting that the former provides a good description of the ground motion variability. Furthermore, the results are in excellent agreement with the ones reported in [9].

![Comparison of logarithmic median ± one standard deviation of 5% damped elastic response spectra of 500 synthetic motions with corresponding spectra from the NGA prediction models of Campbell-Bozorgnia (CB), Abrahamson-Silva (AS), Chiou-Youngs (CY) and Boore-Atkinson (BA). Strike-slip faulting ($F = 0$) and $V_s = 760\text{ m/s}$ are assumed. $M = 7.0$, $R = 40\text{ km}$ (a), $M = 8.0$, $R = 20\text{ km}$ (b).](image)

**Figure 5.4:** Comparison of logarithmic median ± one standard deviation of 5% damped elastic response spectra of 500 synthetic motions with corresponding spectra from the NGA prediction models of Campbell-Bozorgnia (CB), Abrahamson-Silva (AS), Chiou-Youngs (CY) and Boore-Atkinson (BA). Strike-slip faulting ($F = 0$) and $V_s = 760\text{ m/s}$ are assumed. $M = 7.0$, $R = 40\text{ km}$ (a), $M = 8.0$, $R = 20\text{ km}$ (b).

### 5.2.4 Response of Secondary Oscillators

Following the generation of synthetic motions, the horizontal ground acceleration array in Eq. (4.22a) becomes $\ddot{u}_g(t) = \ddot{y}(t)$ where $\ddot{y}(t)$ satisfies Eq. (5.9). Obviously, the general-type of the above stochastic excitation does not permit the use of the closed-form solutions proposed in § 4.3.2. Consequently, the numerical solutions presented in § 4.3.3 are adopted for the determination of the primary and secondary system response.

### 5.3 Numerical Application

Aimed at quantifying the effects of the aleatory randomness characterising the ground shaking, the response of bilinear, sliding and free-standing rocking secondary oscillators in cascade is investigated under the effect of the far-field strong ground motion model summarised in the previous sections. In doing this, the same numerical application as the one considered in
§ 4.5 is adopted and the parameters $T_{p,1} = 0.426 \text{s}$ and $\beta = 1.3$ are assumed for the primary structure.

The RD-K model is initially used to generate a suite of $n_{\text{sym}} = 1000$ samples for a given design scenario. Specifically, generic rock site conditions (strike-slip fault $F = 0$, shear wave velocity $V_s = 620 \text{m/s}$) and the range of values $M : \{6, 6.5, 7, 7.5, 8\}$ as well as $R : \{10, 30, 50\}$ km are assumed for which the model has been calibrated. Furthermore, the corner frequency is taken as $\omega_c = 0.2 \pi \text{rad/s}$ [20].

The validity of the generated signals is confirmed through comparisons with NGA models, and the resulting ground motions are used as an input on the primary structure considered. The response of the secondary oscillators is then quantified in the form of stochastic response spectra. The parameter sets assumed are $\zeta_\text{p} : \{0.01, 0.03, 0.05, 0.07, 0.09\}$ and $\varphi : \{0.2, 0.4, 0.6, 0.8, 1\}$ for the primary system; $\mu_\text{d} : \{1, 3, 5, 7\}$, $\omega_s^* : \{0.3, 0.5, 0.7, 1.3, 1.5\}$, $\psi_s : \{0, 0.1, 0.2, 0.3, 0.4\}$ and $\zeta_s : \{0.01, 0.02, 0.03, 0.04, 0.05\}$ for the bilinear S; $\alpha_s : \{0.4, 1, 1.6, 2.2, 2.8\}$ for the sliding S; $\alpha : \{0.15, 0.2, 0.25, 0.3\}$, $p : \{1, 1.5, 2, 2.5, 3\}$ and $\varepsilon : \{0.2, 0.3, 0.5, 0.7\}$ for the rocking S, where the numbers in bold, similar to § 4.5.3, represent nominal values.

5.3.1 Simulated Motions

Figure 5.5 compares the statistics of the 5%-damping elastic response spectra for the generated ground motions with those of the NGA models for the $M, R$ pairs considered. As expected, the spectra increase with $M$ and reduce with $R$ and, overall, the performance of the model is satisfactory in characterising the ground motion variability. Evidently, when the model is close to the boundaries of its validity range and fewer data were used for calibration, the curves either under- (i.e. $M = 6$) or over-estimate (i.e. $M = 8$, $R = 50$ km) the ones predicted by the NGA models, respectively. This observation is also in-line with [9].

5.3.2 Primary System

The analysis proceeds with the response of the primary system. Figure 5.6 presents the coefficient of variation (CoV) for the response absolute acceleration of P due to changes in the damping ratio $\zeta_\text{p}$ (for constant $\varphi = 1$, Figure 5.6(a)) and the modal coordinate $\varphi$ (constant $\zeta_\text{p} = 0.05$, Figure 5.6(b)) for various combinations of $M$ and $R$.

As shown, in both cases the response surfaces are ordered. In the former case, the CoV decreases with higher values of $\zeta_\text{p}$. Specifically, when $M = 7.5$ and $R = 10$ the CoV values are 0.93, 0.91, 0.88, 0.85 and 0.84 for $\zeta_\text{p}$ values of 0.01, 0.03, 0.05, 0.07 and 0.09, respectively.
Figure 5.5: Comparison of logarithmic median ± one standard deviation of 5%-damping elastic response spectra of 1000 synthetic motions with corresponding spectra from the NGA prediction models of Campbell-Bozorgnia (CB), Abrahamson-Silva (AS) Chiou-Youngs (CY) and Boore-Atkinson (BA). Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
In the latter case, it is demonstrated that uncertainty in the seismic input can considerably influence the response variability at higher elevations. For the same input characteristics (i.e. $M = 7.5$ and $R = 10$), a CoV of $0.71, 0.83, 0.89, 0.90$ and $0.88$ is identified for $\varphi$ values of $0.2, 0.4, 0.6, 0.8$ and $1$, respectively.

In order to examine the response in more detail the empirical CDFs of the response due to changes in $\zeta_p$ and $\varphi$ are determined.

The effect of $\zeta_p$ is first examined in Figure 5.7 for the $M$ and $R$ parameter combinations considered, where the thick solid lines represent reference values i.e. $\zeta_p = 0.05$. Notably, the curves are smooth, suggesting that the number of synthetic motions used is sufficient. In all cases, increasing $M$ reduces the slope of the curves, as higher expected peak response values are experienced in the system. Evidently, the same effect appears to be more dominant when lower values of $R$ are used. In particular, the median reference response of $2.38$ ms$^{-2}$ at $M = 7, R = 30$ increases by $41\%$ (i.e. $3.36$ ms$^{-2}$) when $M = 8$ (at $R = 30$) and by $277\%$ (i.e. $8.98$ ms$^{-2}$) when $R = 10$ (at $M = 7$).

Interestingly, the curves indicate that increasing $\zeta_p$ from $0.01$ to $0.03$ has the highest influence on the response. Specifically, for $M = 7$ and $R = 30$ the median values on the curves are $4.08$ ms$^{-2}$, $2.87$ ms$^{-2}$, $2.38$ ms$^{-2}$, $2.10$ ms$^{-2}$ and $1.90$ ms$^{-2}$ for $\zeta_p$ values of $0.01, 0.03, 0.05, 0.07$ and $0.09$, respectively.

Figure 5.8 illustrates the effect of $\varphi$. Similar to the previous case, the results confirm that higher values of $M$ and lower values of $R$ result in higher expected peak responses. Furthermore, variations in $\varphi$ result in curves that are distributed between the two extreme cases (i.e. $\varphi = 0.2$ and $\varphi = 1$). In particular, for $M = 7$ and $R = 30$ the median values on the curves are
Figure 5.7: Effect of the damping ratio $\zeta_p$ on the empirical CDF of the peak absolute acceleration spectra of the linear P due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
5. Stochastically excited SDoF S in cascade

The results reported herein confirm the expected behaviour for P before analysing S. Furthermore, they indicate that uncertainty in the ground motion can considerably influence the response of the primary structure and therefore needs to be accounted for. As far as the analysis of $\zeta_p$ and $\varphi$ is concerned, owing to the ordered response, it is argued that identification of several curves over given design scenarios could allow directly relating the curves to the ground motion parameters.

### 5.3.3 Bilinear Secondary Oscillator

The response of the bilinear S is investigated next in Figures 5.9 - 5.14.

Similar to the case of P, the smooth CDF curves indicate that the ground motions considered are satisfactory in capturing the stochastic spectra of S. Furthermore, the slopes reduce as $M$ increases and as $R$ decreases. The median reference response is 0.052 m at $M = 7$ and $R = 30$, and increases to 0.08 m (i.e. 54%) when $M = 8$ (at $R = 30$) and to 0.17 m (i.e. 227%) when $R = 10$ (at $M = 7$).

The effect of the ductility ratio $\mu_d$ is examined on the empirical CDF curves of the peak displacement in Figure 5.9 when all other parameters assume their reference values. As shown, variation in $\mu_d$ does not result in considerable change in the shape of the CDF, suggesting that a linear approximation could be considered satisfactory for assessment. Specifically, when $M = 7$ and $R = 30$, median values are 0.051 m, 0.047 m, 0.052 m, and 0.055 m, for $\mu_d$ of 1, 3, 5 and 7, respectively.

Figure 5.10 shows the effect of $\omega_s^*$ on the CDF curves. Overall, similar trends are observed, as in the previous cases, with respect to variations in $M$ and $R$ for all values of $\omega_s^*$ considered. As shown, the curves are ordered, and the expected displacement response is highly influenced by the choice of $\omega_s^*$. The lower $\omega_s^*$ is, the higher the response. In particular, when $M = 7$ and $R = 30$ the median values of the curves are 0.052 m, 0.044 m, 0.036 m, 0.024 m and 0.021 m, for $\omega_s^*$ values of 0.3, 0.5, 0.7, 1.3 and 1.5, respectively.

On comparing Figures 5.9 and 5.10, higher variations in the CDF curves indicate that more attention must be paid when selecting the parameter $\omega_s^*$ as compared to $\mu_d$, via sensitivity analysis.

The influence of $\psi_s$ is examined on the response in Figure 5.11. The higher $\psi_s$, the lower the expected peak response and interestingly, provided $\psi_s \neq 0$, the choice has no considerable
Figure 5.8: Effect of the modal coordinate $\phi$ on the empirical CDF of the peak absolute acceleration spectra of the linear P due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
Figure 5.9: Effect of the ductility ratio $\mu_d$ on the empirical CDF of the peak displacement spectra of the bilinear S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
Figure 5.10: Effect of the circular frequency $\omega_s^*$ on the empirical CDF of the peak displacement spectra of the bilinear S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
effects on the response. Notably, this indicates that one needs to only understand whether or not there is post-yielding stiffness, rather than its value.

Specifically, for the case that $M = 7$ and $R = 30$ the median values are 0.052 m, 0.041 m, 0.039 m, 0.039 m and 0.039 m, for $\psi_s$ values of 0, 0.1, 0.2, 0.3 and 0.4, respectively.

Figure 5.12 illustrates the effect of $\zeta_p$ which is shown to only account for minor variation in the response CDF curves. In particular, when $M = 7$ and $R = 30$ the median values are 0.054 m, 0.052 m, 0.051 m, 0.049 m and 0.048 m, for $\zeta_p$ values of 0.01, 0.02, 0.03, 0.04 and 0.05, respectively. The results therefore indicate that the selection of the parameter $\zeta_p$ may be less significant and does not need to be quantified in an accurate way.

Following the discussion on Figures 5.7 and 5.8, Figures 5.13 and 5.14 finally illustrate the effect of $\zeta_p$ and $\varphi$ on the CDF curves, when all parameters of S are assigned their reference values. In the former case, when $M = 7$ and $R = 30$ the median values are 0.063 m, 0.056 m, 0.052 m, 0.051 m, and 0.050 m corresponding to $\zeta_p$ values of 0.01, 0.03, 0.05, 0.07 and 0.09, respectively. In the latter case, the associated median values are 0.043 m, 0.045 m, 0.047 m, 0.048 m, and 0.052 m for $\varphi$ values of 0.2, 0.4, 0.6, 0.8 and 1, respectively. Similar behaviour has been confirmed for other values of $M$ and $R$, for both these cases.

### 5.3.4 Sliding Secondary Block

The analysis proceeds with the empirical CDF curves for the peak sliding displacement in Figures 5.15 - 5.17.

Overall, similar trends are observed, as in the previous cases, with the CDF curves being smooth and the expected response consistently increasing with $M$ and as $R$ decreases. Specifically, the median reference response (thick solid line) at $M = 7$ and $R = 30$ is 0.016 m and increases by 125% (i.e. 0.036 m) when $M = 8$ (at $R = 30$) and by 819% (i.e. 0.147 m) when $R = 10$ (at $M = 7$).

The influence of the specific strength $a_s$ is first examined in Figure 5.15. As shown, the results are ordered between the two extreme values of $a_s$, and the higher $a_s$, the lower the expected response (steeper CDF curves) as a higher acceleration threshold is needed to initiate sliding motion. Evidently, the contribution of $a_s$ is significant on the expected response. In particular, when $M = 7$ and $R = 30$, the corresponding median values are 0.039 m, 0.016 m, 0.004 m, 0 m, and 0 m for $a_s$ values of 0.4, 1, 1.6, 2.2 and 2.8, respectively. Similar behaviour has been confirmed for other values of $M$ and $R$.

Figures 5.16 and 5.17, examine the effect of $\zeta_p$ and $\varphi$ on the expected response. In the first case, when $M = 7$ and $R = 30$ the median values are 0.036 m, 0.022 m, 0.016 m, 0.011 m,
Figure 5.11: Effect of the post-yield to pre-yield stiffness ratio $\psi_s$ on the empirical CDF of the peak displacement spectra of the bilinear S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
Figure 5.12: Effect of the damping ratio $\zeta_s$ on the empirical CDF of the peak displacement spectra of the bilinear S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M = \{6, 6.5, 7, 7.5, 8\}$ and $R = \{10, 30, 50\}$ km are assumed.
Figure 5.13: Effect of the damping ratio $\zeta_p$ on the empirical CDF of the peak displacement spectra of the bilinear $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\} \text{ km}$ are assumed.
Figure 5.14: Effect of the modal coordinate $\varphi$ on the empirical CDF of the peak displacement spectra of the bilinear S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
Figure 5.15: Effect of the specific strength $a_s$ on the empirical CDF of the peak displacement spectra of the sliding S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
and 0.008 m corresponding to $\zeta_p$ values of 0.01, 0.03, 0.05, 0.07 and 0.09, respectively. In
the latter case, the associated median values are 0 m, 0 m, 0.002 m, 0.008 m, and 0.016 m
for $\varphi$ values of 0.2, 0.4, 0.6, 0.8 and 1, respectively. Furthermore, similar behaviour has been
confirmed for other $M$ and $R$ configurations. It therefore appears, that the selection of these
parameters for $P$ can considerably influence the behaviour of sliding $S$.

### 5.3.5 Rocking Secondary Block

The empirical CDF curves of the peak normalised rotation for the rocking $S$ are examined in
Figures 5.18 - 5.22.

Similar to the sliding case, vertical shifts in the CDFs (e.g. Figure 5.18, $M = 6, R = 50$)
are attributed to the fact that rocking motion has not initiated. Furthermore, the presence of
discontinuities in the curves (e.g. Figure 5.18, $M = 8, R = 10$) is due to overturning being
reached. The results confirm, that the overturning potential increases with $M$ and as $R$ reduces.
In particular, the median reference response (thick solid line) at $M = 7$ and $R = 30$ is 0 (i.e.
no initiation) and increases to 0.055 when $M = 8$ (at $R = 30$) and to 0.1286 when $R = 10$
(at $M = 7$).

The effect of the slenderness parameter $\alpha$ is examined in Figure 5.18. In all cases, the
curves are ordered and the higher $\alpha$ the less the rotation (steeper curves). For $M = 7$ and
$R = 30$ the median values are 0.068, 0.003, 0 and 0, for $\alpha$ values of 0.15, 0.2, 0.25 and 0.3,
respectively. Furthermore, when $R = 50$ the median values are zero, for all $\alpha$, regardless of
the choice of $M$.

Figure 5.19 illustrates the effect of the dynamic parameter $p$ on the CDF curves. As shown,
the results are ordered and considerable variations can be seen on the curves with the expected
response increasing with $p$. Evidently, the median values when $M = 7$ and $R = 30$ are all
zero. While this is also the case for $R = 50$, when $M = 7$ and $R = 10$ the median values are
0.032, 0.072, 0.129, 0.204 and 0.297 for $p$ values of 1, 1.5, 2, 2.5 and 3, respectively.

The CDF curves in Figure 5.20 indicate that increasing the restitution coefficient $\varepsilon$ can
exacerbate the response. Similar to the case of $\alpha$ and $p$, the effect depends on the choice of
$M$ and $R$. Specifically, while for $M = 7$ and $R = 30$ the median values are all zero, when
$M = 7$ and $R = 10$ these are 0.082, 0.086, 0.103 and 0.129 for $\varepsilon$ values of 0.2, 0.3, 0.5 and
0.7, respectively.

Figures 5.21 and 5.22 finally examine the effect of $\zeta_p$ and $\varphi$ on the expected response. In
both these cases the curves are ordered. In the former case, the expected response reduces with
$\zeta_p$ (steeper curves). When $M = 7$ and $R = 10$ the median values are 0.163, 0.138, 0.129, 0.12
Figure 5.16: Effect of the damping ratio $\zeta_p$ on the empirical CDF of the peak displacement spectra of the sliding $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M: \{6, 6.5, 7, 7.5, 8\}$ and $R: \{10, 30, 50\}$ km are assumed.
Figure 5.17: Effect of the modal coordinate $\phi$ on the empirical CDF of the peak displacement spectra of the sliding $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
Figure 5.18: Effect of the slenderness $\alpha$ on the empirical CDF of the peak rotation spectra of the rocking S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
Figure 5.19: Effect of the dynamic parameter $p$ on the empirical CDF of the peak rotation spectra of the rocking S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
Figure 5.20: Effect of the restitution coefficient $\varepsilon$ on the empirical CDF of the peak rotation spectra of the rocking S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
and 0.114 corresponding to $\zeta_p$ values of 0.01, 0.03, 0.05, 0.07 and 0.09. In the latter case, the overturning potential increases with $\varphi$. Specifically, the median values associated with $M = 7$ and $R = 10$ are 0, 0.061, 0.095, 0.113 and 0.129, corresponding to $\varphi$ values of 0.2, 0.4, 0.6, 0.8 and 1.

It therefore appears that the choice of input parameters can have a considerable influence on the probabilistic response of the rocking S whose behaviour has to be modelled to allow assessment to be made.

### 5.4 Summary

In this chapter, an existing fully nonstationary stochastic model for far-field strong ground motions was adopted in view of the seismic analysis and response quantification of nonlinear secondary (S) oscillators with deterministic system parameters.

The stochastic model was first overviewed and its predictive equations were used to generate a suite of synthetic accelerograms for given design scenarios, without the need for previously recorded motions. Specifically, generic rock site conditions were assumed and a set of moment magnitude and source-to-site distance combinations were used, representative of the main seismicity characteristics. The statistics of the resulting generated motions were then validated through comparisons with NGA prediction models and were used as an input to a linear primary (P) structure. Bilinear, sliding and free-standing rocking S oscillators connected at a single point to the P structure were then analysed in cascade and the empirical cumulative distribution functions (CDF) of the expected peak responses were quantified over different input parameter combinations.

It was found that uncertainty in the seismic input can considerably influence the expected response of both P and S and therefore needs to be accounted for. The effect of the ductility ratio $\mu_d$ and the damping ratio $\zeta_s$ of a bilinear S was found to have minimal effects on the CDF curves, indicating that a linear approximation may be satisfactory when modelling the S component. On the contrary, more attention needs to be paid on the choice of $\omega_s^s$. Furthermore, it was found that one needs to only address whether the post-yielding stiffness $\psi_s$ needs to be accounted for within the analysis, and not its value.

For the sliding S, the choice of the specific strength $a_s$ of S as well as the damping ratio $\zeta_p$ and the modal coordinate $\varphi$ of P can considerably affect the CDF curves. Due to the intrinsic strong nonlinearity evidenced through discontinuities in the CDF curves, the response of a rocking S can only be predicted by modelling its behaviour.
Figure 5.21: Effect of the damping ratio $\zeta_p$ on the empirical CDF of the peak rotation spectra of the rocking $S$ due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620 \text{ m/s}$) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
Figure 5.22: Effect of the modal coordinate $\phi$ on the empirical CDF of the peak rotation spectra of the rocking S due to 1000 synthetic motions. Generic rock site conditions ($F = 0$ and $V_s = 620$ m/s) and $M : \{6, 6.5, 7, 7.5, 8\}$ and $R : \{10, 30, 50\}$ km are assumed.
The ordered CDF functions evidenced through the majority of the cases, confirm that the procedure presented herein only for a single circular frequency parameter of the P structure, may be further extended to directly determine correlations between the seismic input parameters and the response CDF functions of S, over a finite set of P structures (with a range of characteristics e.g. circular frequency). A new set of appropriate predictive equations can then be developed that can be used to generate realisations of the expected performance of nonlinear S systems conditioned on specified values of the input parameters (i.e. both ground motion and P and S parameters). These can be used in practice for seismic risk characterisation and analysis of secondary structures.
6.1 Introduction

In Chapter 5 the effect of uncertainties in the ground motions was examined on the response of secondary oscillators. Prompted by the presence of uncertainties in both the seismic action and the dynamic behaviour of the primary, load-bearing structure, this chapter addresses their relative importance on the response of secondary structures.

Section 6.2 describes the procedure of generating independent synthetic principal components for a target accelerogram and presents the equations governing the motion of a linear primary structure.

In Section 6.3 a novel method for characterising the uncertainty in the primary structure is proposed, in which the modal shapes, frequencies and damping ratios constitute the random quantities. The latter are directly introduced in the reduced modal subspace rather than in the full geometrical space, thus decreasing the number of parameters and the size of the dynamic problem. A procedure is then presented in which the model parameters are identified over various configurations with indicative application on linear steel frames with uncertain semi-rigid connections subjected to deterministic seismic excitation.
6.2 Governing Equations

6.2.1 Principal Axes

For the sake of generality, let us consider the case of a ground motion record characterised through a pair of orthogonal horizontal components; specifically, $\mathbf{u}_g(t) = \{a_1(t), a_2(t)\}^\top$ is the vector listing the ‘as-recorded’ acceleration time series, and the correlation coefficient over the total duration of motion can be expressed as:

$$\rho_{a_1(t) a_2(t)} = \frac{\int_{t_0}^{t_n} a_1(t) a_2(t) \, dt}{\sqrt{\int_{t_0}^{t_n} a_1(t)^2 \, dt \int_{t_0}^{t_n} a_2(t)^2 \, dt}},$$

(6.1)

where $t_0$, $t_n$ are the initial and final time instants, respectively [21]. Following the work of Penzien and Watabe [22], a unique set of principal axes $\eta$ and $\xi$ exists (see Figure 6.1) along which the components can be regarded as statistically uncorrelated. Accordingly, introducing the rotation angle $\delta$ over the range $0^\circ$ to $90^\circ$ and adopting the following orthogonal transformation:

$$\mathbf{u}_{gr}(\delta, t) = \begin{bmatrix} \cos(\delta) & \sin(\delta) \\ -\sin(\delta) & \cos(\delta) \end{bmatrix} \cdot \mathbf{u}_g(t),$$

(6.2)
6.2 Governing Equations

One can derive the vector of principal components \( \ddot{\mathbf{u}}_{gp}(t) = \{a_1^p(t), a_2^p(t)\}^\top = \ddot{\mathbf{u}}_{gr}(\delta_0, t) \), in which \( \delta_0 \) represents the angle satisfying the condition \( \rho_{a_1,r}(\delta_0,t), \rho_{a_2,r}(\delta_0,t) = 0 \), where \( \ddot{\mathbf{u}}_{gr}(\delta, t) = \{a_1^r(\delta, t), a_2^r(\delta, t)\}^\top \) is the vector listing the ‘rotated’ acceleration time series.

6.2.2 Linear Primary System

Considering now the case of a deterministic primary-secondary dynamic system, if the secondary system is assumed to be ‘light’ [44], i.e. its mass \( m_S \) is much less than the mass of the primary \( M_P \) (\( m_S \ll M_P \)), a cascade-type approach is admissible. Accordingly, the two systems are decoupled and can be sequentially analysed. Initially, the seismic response of the primary system is determined neglecting the feedback of the secondary, with the response of the latter successively being evaluated at the points of attachment (i.e. no primary-secondary interaction is taken into account).

The differential equation governing the motion of a multi-degree-of-freedom (MDoF) primary system within the linear-elastic range, assumed at rest at time \( t = 0 \), takes a similar form as Eq. (4.1) and is given by:

\[
M \cdot \ddot{u}(t) + C \cdot \dot{u}(t) + K \cdot u(t) = -M \cdot \tau \cdot \ddot{u}_{gp}(t) ; \quad u(0) = 0_n; \quad \dot{u}(0) = 0_n,
\]  

(6.3)

where \( M, C \) and \( K \) are the \((n \times n)\) matrices of mass, equivalent viscous damping and elastic stiffness, respectively; \( u(t) = \{u_1(t) \ldots u_n(t)\}^\top \) is the array collecting the \( n \) degrees of freedom (DoFs) of the structure; \( \tau \) is a \((n \times 2)\) matrix of seismic incidence; \( \ddot{u}_{gp}(t) \) is the \((2 \times 1)\) vector of the horizontal principal components of the ground acceleration; \( 0_n \) is a zero vector of dimensions \((n \times 1)\).

The equations of motion can be projected onto the modal subspace, reducing the size of the dynamic problem from \( n \) (system’s DoFs) to \( m \leq n \) (the number of modes retained in the dynamic analysis). This requires solving the real-valued eigenproblem:

\[
M \cdot \Phi \cdot \Omega^2 = K \cdot \Phi,
\]  

(6.4)

where \( \Phi = [\phi_1 \ldots \phi_m] \) is the normalised modal matrix (i.e. \( \Phi^\top \cdot M \cdot \Phi = I_m \)), \( I_m \) being the identity matrix of size \( m \); and \( \Omega = \text{diag} \{\omega_1 \ldots \omega_m\} \) the diagonal spectral matrix. Accordingly, the dynamic response can be expressed as the sum of modal contributions:

\[
u(t) = \Phi \cdot q(t),
\]  

(6.5)
where \( \mathbf{q}(t) = \{ q_1(t), \ldots, q_m(t) \}^\top \) is the array collecting the modal coordinates, ruled by the equation of motion in the modal subspace:

\[
\ddot{\mathbf{q}}(t) + 2 \zeta \Omega \cdot \dot{\mathbf{q}}(t) + \Omega^2 \cdot \mathbf{q}(t) = \mathbf{p} \cdot \ddot{\mathbf{u}}_{gp}(t) ; \quad \mathbf{q}(0) = \mathbf{0}_m ; \quad \dot{\mathbf{q}}(0) = \mathbf{0}_m ,
\]  

(6.6)

in which \( \zeta \) is the equivalent viscous damping ratio assumed for the primary structure, and \( \mathbf{p} = -\Phi^\top \cdot \mathbf{M} \cdot \tau \).

### 6.2.3 Stochastic Ground Motion Model

Given the statistically independent principal components \( \ddot{\mathbf{u}}_{gp}(t) \) of a target accelerogram, a stochastic ground motion model can then be used to simulate bi-directional time series with temporal and spectral nonstationarities. Accordingly, the R-DK model [17, 21] whose unidirectional case has been discussed in § 5.2 can be adopted for this purpose, simulating each of the two components in turn.

In this chapter, a simplified procedure for generating synthetic accelerograms is used which varies from the one delineated in § 5.2 in that the predictive relationships are not used. In other words, the resulting synthetic motions are generated with deterministic parameters associated with specific recorded ground motions (see: § 6.4.1) and the randomness is therefore only related to the white-noise. Notwithstanding that the resulting variability in this case is smaller than the one associated with the model in § 5.2 and the synthetic motions have different trajectories, this simplified procedure has extensively been used in the literature [19].

The acceleration process of the \( r \) th component denoted by \( x_r(t) \) is defined through Eq. (5.1), where \( r = 1, 2 \). Similar to the unidirectional case, the deterministic time-modulating Gamma function \( Q(t, \kappa) \) in Eq. (5.4) depends on the parameter set \( \kappa_r = \{ \kappa_{1,r}, \kappa_{2,r}, \kappa_{3,r} \} \) that relate to the intensity and shape of each component; the Gaussian white-noise process \( w_r(\tau) \) is realised through a set of standard normal random variables \( v_{i,r} \) and the IRF \( h[t - \tau, \lambda_r(\tau)] \) in Eq. (5.2) depends on the parameter set \( \lambda_r(\tau) = \{ \omega_{f,r}(\tau), \zeta_{f,r}(\tau) \} \), that assume a linear function and a constant, respectively as in Eq. (5.3).

Given the model parameters, realisations of the processes are generated based on the random variables \( v_{i,r} \), and the basis functions \( s_i(t, \lambda_r(t_i)) \) (Eq. (5.8)) based on Eq. (5.7). Each simulated process is then high-pass filtered and the corrected acceleration record \( \ddot{y}_r(t) \) is finally obtained through Eq. (5.9). The stochastic horizontal ground acceleration array finally becomes \( \ddot{\mathbf{u}}_{gp}(t) = \{ \ddot{y}_1(t), \ddot{y}_2(t) \}^\top \), i.e. \( \ddot{a}_{x}^E(t) = \ddot{y}_r(t), \) for \( r = 1, 2. \)
6.3 Uncertainty in the Modal Space

In the preceding section, the procedure to calculate the seismic response of the primary structure with deterministic mechanical properties was summarised, in which the equations of motion were conveniently projected onto the reduced modal subspace. Modal analyses are also utilised to model in an efficient way the dynamic interaction between primary-secondary systems [38]. In accordance with the performance-based earthquake engineering (PBEE) philosophy, the seismic response must be characterised in a probabilistic sense, and thus considerations in terms of uncertainty quantification and propagation need to be explicit [9]. Contrary to the existing methods, where the sources of structural uncertainty are treated in the full geometrical space, in the present study uncertainty is directly characterised in the reduced modal subspace, significantly reducing the number of the uncertain parameters and the resulting computational burden. This can be achieved by considering some random fluctuations in the modal shapes (see Figure 6.2), modal circular frequencies and modal damping ratios, leading to the following definition of the stochastic matrices $\tilde{\Phi}(\alpha)$, $\tilde{\Omega}(\beta)$ and $\tilde{\zeta}(\gamma)$:

\begin{align}
\tilde{\Phi}(\alpha) &= \Phi \cdot [I_m + \alpha] ; \\
\tilde{\Omega}(\beta) &= \Omega \cdot [I_m + \beta] ; \\
\tilde{\zeta}(\gamma) &= \zeta \cdot [I_m + \gamma] ,
\end{align}

Figure 6.2: Uncertainty in the first mode.
in which the upper hat denotes stochastic quantities, while $\alpha$, $\beta$ and $\gamma$ are zero-mean random matrices so defined:

$$\alpha = \begin{bmatrix} 0 & \alpha_{1,2} & \cdots & \alpha_{1,m} \\ \alpha_{2,1} & 0 & \cdots & \alpha_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m,1} & \alpha_{m,2} & \cdots & 0 \end{bmatrix}; \quad \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}; \quad \gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{bmatrix}. \quad (6.8)$$

In the above, a total of $m^2 + m$ statistically independent random variables fully characterise the uncertainty in the structural system, and it is assumed that: $|\alpha_{j,i}|, |\beta_i|, |\gamma_i| \ll 1$; $\mathbb{E}[\alpha_{j,i}] = \mathbb{E}[\beta_i] = \mathbb{E}[\gamma_i] = 0$; $\mathbb{E}[\alpha_j \alpha_i] = \mathbb{E}[\beta_j \beta_i] = \mathbb{E}[\gamma_j \gamma_i] = 0$ for $i \neq j$, where $i, j = 1, \ldots, m$ and $\mathbb{E} [\cdot]$ is the expectation operator. The variances of the random variables in the modal subspace are conveniently collected in the matrices $\mathbf{v}_\alpha$, $\mathbf{v}_\beta$ and $\mathbf{v}_\gamma$, respectively. It is worth noting here that, according to the proposed structural uncertainty model, the $i$-th modal shape, frequency and damping ratio are only influenced by the set of coefficients $\{\alpha_{1,i}, \alpha_{2,i}, \ldots, \alpha_{m,i}\}, \beta_i$ and $\gamma_i$, respectively.

Projecting back onto the full geometric space the effects of the modal uncertainties, the stochastic mass $\mathbf{\hat{M}}(\alpha)$, stiffness $\mathbf{\hat{K}}(\alpha, \beta)$ and damping $\mathbf{\hat{C}}(\alpha, \beta, \gamma)$ matrices take the expressions:

$$\mathbf{\hat{M}}(\alpha) = \mathbf{M} + \delta \mathbf{M}(\alpha); \quad (6.9a)$$
$$\mathbf{\hat{K}}(\alpha, \beta) = \mathbf{K} + \delta \mathbf{K}(\alpha, \beta); \quad (6.9b)$$
$$\mathbf{\hat{C}}(\alpha, \beta, \gamma) = \mathbf{C} + \delta \mathbf{C}(\alpha, \beta, \gamma), \quad (6.9c)$$
where $\delta M(\alpha)$, $\delta K(\alpha, \beta)$ and $\delta C(\alpha, \beta, \gamma)$ are fluctuations about their nominal values, given by:

\[
\delta M(\alpha) = M \cdot \hat{\Phi}(\alpha) \cdot \hat{\Phi}^\top(\alpha) \cdot M - \tilde{M} = M \cdot \Phi \cdot \left[\alpha + \alpha^\top + \alpha \cdot \alpha^\top\right] \cdot \Phi^\top \cdot M;
\]

\[
(6.10a)
\]

\[
\delta K(\alpha, \beta) = M \cdot \hat{\Phi}(\alpha) \cdot \hat{\Omega}^2(\beta) \cdot \hat{\Phi}(\alpha)^\top \cdot M - \tilde{K} = M \cdot \Phi \cdot \left[(I_m + \alpha) \cdot \Omega^2 \cdot (I_m + \beta^\top) \cdot (I_m + \alpha^\top) - \Omega^2\right] \cdot \Phi^\top \cdot M;
\]

\[
(6.10b)
\]

\[
\delta C(\alpha, \beta, \gamma) = 2M \cdot \hat{\Phi}(\alpha) \cdot \hat{\zeta}(\gamma) \cdot \hat{\Omega}(\beta) \cdot \hat{\Phi}(\alpha)^\top \cdot M - \tilde{C} = 2M \cdot \Phi \cdot \left[(I_m + \alpha) \cdot \zeta \cdot \Omega \cdot ((\beta + \gamma + \beta^\top) \cdot (I_m + \alpha^\top) + \alpha \cdot \zeta \cdot \Omega)\right] \cdot \Phi^\top \cdot M;
\]

\[
(6.10c)
\]

and $\tilde{M}$, $\tilde{K}$ and $\tilde{C}$ are the deterministic matrices:

\[
\tilde{M} = M \cdot \Phi \cdot \Phi^\top \cdot M;
\]

\[
(6.11a)
\]

\[
\tilde{K} = M \cdot \Phi \cdot \Omega^2 \cdot \Phi^\top \cdot M;
\]

\[
(6.11b)
\]

\[
\tilde{C} = 2M \cdot \Phi \cdot \zeta \cdot \Omega \cdot \Phi^\top \cdot M.
\]

\[
(6.11c)
\]

Accordingly, the stochastic equivalent of the deterministic eigenproblem of Eq. (6.4) reads:

\[
\hat{\Phi}(\alpha) \cdot \hat{\Omega}^2(\beta) = \hat{\Phi}(\alpha) \cdot \hat{\Phi}(\alpha),
\]

\[
(6.12)
\]

where $\hat{\Phi}(\alpha)$ is the stochastic modal matrix, normalised with respect to the mass matrix:

\[
\hat{\Phi}(\alpha) = \hat{\Phi}(\alpha) \cdot \left[\hat{\Phi}(\alpha) \cdot \hat{\Phi}(\alpha)^\top \cdot M(\alpha) \cdot \hat{\Phi}(\alpha)\right]^{-\frac{1}{2}} = \hat{\Phi}(\alpha) \cdot \left[(I_m + \alpha)^\top \cdot (I_m + \alpha + \alpha^\top + \alpha \cdot \alpha^\top) \cdot (I_m + \alpha)^{-\frac{1}{2}}.
\]

\[
(6.13)
\]

Adopting the transformation of coordinates given by:

\[
\hat{u}(t) = \hat{\Phi}(\alpha) \cdot \hat{q}(t),
\]

\[
(6.14)
\]
and modifying the deterministic Eq. (6.6) in light of Eq. (6.7), the solution of the system with uncertainties in the modal properties is governed by:

\[
\ddot{\mathbf{q}}(t) + 2\zeta(\gamma)\Omega(\beta)\dot{\mathbf{q}}(t) + \Omega^2(\beta)\mathbf{q}(t) = \mathbf{p}(\alpha)\cdot \mathbf{u}_{gp}(t) ; \quad \mathbf{q}(0) = 0_m ; \quad \dot{\mathbf{q}}(0) = 0_m , \quad (6.15)
\]

where the response \(\mathbf{q}(t) = \{\hat{q}_1(t) \ldots \hat{q}_m(t)\}^\top\) is a function of the uncertainty sources considered for the structure, and \(\mathbf{p}(\alpha) = -\dot{\Phi}^\top(\alpha)\cdot \mathbf{M}(\alpha)\cdot \mathbf{r}\) is the seismic incidence vector. Notably, in the above formulation the stochastic equations of motion (Eq. (6.15)) are decoupled.

It is noted, that the assumption of zero correlation between the elements of the random matrices \(\alpha, \beta\) and \(\gamma\) is mathematically convenient but not necessarily realistic for scenarios such as uncertainty in the Young’s modulus or in the material properties. For this reason, the proposed model is adopted herein only for the case of uncertainty in the partial rigidity of connections while for other types of uncertainty the model needs to be further extended and validated to account for nonzero correlation in the associated coefficients.

### 6.3.1 Effect on the Geometric Space

Closed-form expressions for the statistics of the random quantities in the full geometrical space can be derived in terms of the input random variables in the reduced modal subspace. Accordingly, neglecting higher order terms in Eq. (6.10) and assuming for simplicity sake equal damping ratios in all modes, the mean values are:

\[
E[\mathbf{M}(\alpha)] \approx \mathbf{M} ; \quad E[\mathbf{K}(\alpha,\beta)] \approx \mathbf{K} ; \quad E[\mathbf{C}(\alpha,\beta,\gamma)] \approx \mathbf{C} , \quad (6.16)
\]

and the variances are:

\[
\text{Var}[\mathbf{M}(\alpha)] \approx \mathbf{w} \cdot \left[ f\left( v_\alpha + v_\alpha^\top \right) + g\left( v_\alpha + v_\alpha^\top \right) \right] \cdot \mathbf{w}^\top ; \quad (6.17a)
\]

\[
\text{Var}[\mathbf{K}(\alpha,\beta)] \approx \mathbf{w} \cdot \left[ f\left( (v_\alpha + 4v_\beta) \cdot \Omega^4 + \Omega^4 \cdot v_\alpha^\top \right) + g\left( v_\alpha \cdot \Omega^4 + \Omega^4 \cdot v_\alpha^\top \right) \right] \cdot \mathbf{w}^\top ; \quad (6.17b)
\]

\[
\text{Var}[\mathbf{C}(\alpha,\beta,\gamma)] \approx 4\zeta^2 \mathbf{w} \cdot \left[ f\left( (v_\alpha + v_\beta + v_\gamma) \cdot \Omega^2 + \Omega^2 \cdot v_\alpha^\top \right) + g\left( v_\alpha \cdot \Omega^2 + \Omega^2 \cdot v_\alpha^\top \right) \right] \cdot \mathbf{w}^\top , \quad (6.17c)
\]

in which \(\mathbf{w} = \mathbf{M} \cdot [\varphi_1^2 \ldots \varphi_n^2]^\top\), where \(\varphi_i^2 = \varphi_i \otimes \varphi_i\), the symbol \(\otimes\) denotes the so-called Kronecker product \([133]\) and \(\varphi_i\) is the \(i\)-th column of the matrix \(\Phi^\top\); additionally, \(\text{Var}[\cdot]\) is an
operator giving the variance of each element of the matrix within square brackets, while \( f(\cdot) \)
and \( g(\cdot) \) are operators mapping the generic elements \( i, j \) of the input \((m \times m)\) matrix onto a
sparse matrix of dimensions \((m^2 \times m^2)\):

\[
f: (i, j) \rightarrow m(i - 1) + i, m(j - 1) + j; \quad g: (i, j) \rightarrow m(i - 1) + j, m(j - 1) + i.
\] (6.18)

Notably, the proposed model of structural uncertainty representation in the reduced modal sub-
space can be effectively used for non-probabilistic models of the uncertainty, e.g. interval
[134, 135] or fuzzy [136, 137] models.

### 6.3.2 Identification of Model Parameters

Fluctuations in the mass and/or stiffness matrices lead to variations in the modal shapes and
circular frequencies. Consequently, the model parameters listed in \( v_\alpha, v_\beta \) can be identified by
rearranging the expressions in Eq. (6.7):

\[
\alpha = \Phi^\top \cdot M \cdot \hat{\Phi} - I_m; \quad \beta = \Omega^{-1} \cdot \hat{\Omega} - I_m.
\] (6.19a)

A numerical procedure is employed herein with the purpose of calibrating the probabilistic
definition of the proposed model, with indicative application to the case of semi-rigid con-
nections. Considering the unidirectional case with a deterministic seismic load (i.e. \( \ddot{u}_{gp} = \ddot{u}_g \) in Eq. (6.6)), linear rotational springs (an acceptable approximation under serviceability
limit states [138]) are used to model the connection stiffness of Euler-Bernouli beams (Fig-
ure 6.3(a)). It follows that axial and shear forces \((N, Q)\) and the associated displacements
\((u, w)\) at the generic node equate the internal ones. Accordingly, the mass and stiffness matri-
ces for the beam element with rotational springs are derived (see Appendix A in [113]) and are
functions of the rotational stiffness \( k \) defined as:

\[
k(v) = \frac{3EI}{l} \cdot \frac{v}{1 - v},
\] (6.20)

where \( E, I, l \) are the Young’s modulus, moment of inertia and length of the beam, respectively,
and \( v \) is the fixity factor at the generic end node, within the range \( \{0, 1\} \). The two limiting
cases, \( \lim_{v \to 0} k(v) = 0 \) and \( \lim_{v \to 1} k(v) = \infty \) represent a pinned connection (permitting free
rotation) or a rigid one (restraining rotation), respectively, while in actual construction the fixity factor takes intermediate values.

Figure 6.3: Beam element with rotational springs (adapted from [113]) (a) and structural frame model (b).

Figure 6.3(b) shows the case study model, taken from [114], consisting of a 10-storey single-bay frame. The Young’s modulus is \( E = 210 \text{ GPa} \) and the geometrical parameters are \( A_b = 306 \cdot 10^{-3} \text{ m}^2 \), \( I_b = 2569 \cdot 10^{-6} \text{ m}^4 \) for the beams while three values have been considered for the columns, namely: \( A_{c1} = 27 \cdot 10^{-3} \text{ m}^2 \), \( I_{c1} = 1710 \cdot 10^{-6} \text{ m}^4 \) for the first four storeys, \( A_{c2} = 21.8 \cdot 10^{-3} \text{ m}^2 \), \( I_{c2} = 798.9 \cdot 10^{-6} \text{ m}^4 \) for the middle ones; \( A_{c3} = 14.9 \cdot 10^{-3} \text{ m}^2 \), \( I_{c3} = 251.7 \cdot 10^{-6} \text{ m}^4 \) for the top three ones. The finite element length is \( h = 4 \text{ m} \) and masses of \( M_t = 3 \text{ Mg} \) and \( M = 4 \text{ Mg} \) are lumped at the nodes of each beam element for the top storey and elsewhere, respectively. The fundamental period of vibration is \( T_1 = 0.993 \text{ s} \) (73\% of modal mass participation) for the reference case (rigid connections); the total number of DoFs is \( n = 90; m = 3 \) modes were retained in the analysis, so that 93\% of the modal mass participates in the seismic motion in the direction of interest \( x \). The structure is subjected to the ground excitation of Imperial Valley 1940 earthquake whose time history and frequency content are shown in Figure 6.4(a) and 6.4(b), respectively. The commercial software SAP2000 [139] has been used to construct the relevant mass and stiffness matrices and the numerical software MATLAB [124] to carry out the linear dynamic analysis.
6.3 Uncertainty in the Modal Space

6.3.2.1 Parametric Analysis

The influence of connection flexibility on the dynamic response is investigated through parametric (deterministic) analyses by varying the rotational spring stiffness at all beam ends by the same amount through Eq. (6.20). Figure 6.5(a) reports the influence of $v$ on the circular frequencies. As shown, increasing $v$ results in a corresponding increase in the natural frequencies with the effects being more profound on the lower modes, which typically dominate the seismic structural response. It is also worth mentioning here that the results are in good agreement with [114]. Figure 6.5(b) shows the effect on the first three modal shapes, normalised with the displacement at roof level and figure 6.5(c) shows the effects on the envelope of the dynamic response. Accordingly, the dashed and solid black lines correspond to the two extreme cases (i.e. $v = 0$, $v = 1$) while each grey line shows intermediate values. Interestingly, the results are not ordered, suggesting that various intermediate stiffness values may cause higher effects.

6.3.2.2 Stochastic Analysis

This section presents a selection of the results of a stochastic analysis carried out on the case study under consideration. For each level of partial rigidity of the connections the model parameters $v_\alpha$, $v_\beta$ are identified. In doing this, a set of nominal values of the partial fixity $v$ in the range \{0, 1\} is used and $N = 500$ samples of each associated rotational stiffness are generated via Monte Carlo (MC) simulations. Each set of the generated stiffnesses is assumed to be lognormally distributed and two levels of the input coefficient of variation (CoV = 0.2, 0.3) are used. Furthermore, both correlated (i.e. same stiffness at all connections) and uncorrelated
(i.e. different stiffness with zero correlation) cases are considered.

Figure 6.6 shows the effect of the connection flexibility on the model parameters of the elements of $\alpha$. As shown, relatively low levels of stiffness ($v \approx 0.1$) cause higher variation in the model parameters. Furthermore, the correlated case (Figure 6.6(a), 6.6(b)), tends to underestimate the parameters, which appear to be smooth functions in the uncorrelated case (Figure 6.6(d), 6.6(e)). Increasing the input CoV (Figure 6.6(b), 6.6(e)) amplifies the parameters without any noticeable impact on the shape. The effects of the connection flexibility on the model parameters in $v_\beta$ are summarised in Figure 6.7. Contrary to the previous case, the correlated case (Figure 6.7(a), 6.7(b)) overestimates the parameters. Higher variation is observed around $v \approx 0.1$ and the effects are more significant on the lower frequencies. Increasing the input CoV (Figure 6.7(b), 6.7(e)) results in higher effects and the shape is preserved.

Having identified the model parameters for each configuration, the dynamic analysis is carried out using the calibrated uncertainty model. Figure 6.8 compares the peak response at roof level of the reference MC simulations with the ones predicted by the proposed model. Overall, good agreement is observed and high CoV are predicted at $v = 0.3$. Similar trends to the previous cases are shown by the input CoV and the correlated, uncorrelated cases. It follows that given a level of connection flexibility $v$ the random dynamic response can directly be simulated without resorting to the full model.
6.4 Response of Secondary Oscillators

In this section, the cascaded response of bilinear, sliding and rocking nonlinear S oscillators with deterministic mechanical parameters is considered in presence of uncertainties in the ground motion and the properties of the primary structure. The equations governing the motion of the three selected S are given in § 3.2. In particular, Eq. (3.4), Eq. (3.9) and Eq. (3.12), are used for the bilinear (Figure 4.2(a)), sliding (Figure 4.2(c)) and rocking (Figure 4.2(e)) S, respectively, while \( \ddot{u}_a(t) = \ddot{u}(t) + \ddot{u}_{gp}(t) \) comprises herein the input unidirectional absolute acceleration response of the primary structure at the position of attachment due to a single realisation.

A numerical application is presented in the following, with the aim of assessing the seismic response of a case-study building structure to the presence of both random ground shaking and

Figure 6.6: Influence of connection flexibility on the statistics of \( \alpha \) for correlated, uncorrelated cases (top, bottom) and input CoV = 0.2, 0.3 (left, middle), respectively; distribution of \( \alpha_{21} \) at CoV = 0.2 (right).
uncertain modal parameters. Figure 6.9(a) shows a MDoF primary system comprising of a 5-storey single-bay moment-resisting frame, irregular in both plan and elevation. The system is subjected to the simultaneous action of orthogonal horizontal components and position $S$ denotes the attachment point of a light secondary SDoF system at the top storey, modelled as (i) linear, (ii) bilinear, (iii) sliding and (iv) rocking oscillator. Floors are rigid in plane, while the self-weight and super-dead load constitute the mass source of the structure. The fundamental period of vibration in the direction of interest $x$ and the viscous damping ratio are $T_{px} = 0.382$ s, $\zeta_p = 0.05$, while the number of modal coordinates retained in the analysis is $m = 5$, chosen such that at least 90% of the modal mass participates in the seismic motion in the $x$ direction, a criterion set by current codes of practice (e.g. the Eurocode 8 [69]) and widely accepted by researchers and practitioners.
6.4 Response of Secondary Oscillators

The influence of various mechanical parameters is considered for the secondary oscillators, namely the period of oscillation $T_s = \frac{2 \pi}{\omega_s}$, the viscous damping ratio $\zeta_s$ and the ductility ratio $\mu_d = \frac{u_s,\text{ult}}{u_s,\text{yld}}$ for the linear and bilinear cases, $u_s,\text{ult}$ and $u_s,\text{yld}$ being the ultimate and yielding displacement of the system, respectively; the friction coefficient $\mu_s$ for the sliding oscillators; the slenderness angle $\alpha$ and the dynamic parameter $p$ for the rocking oscillator, with an ‘incipient’ overturning condition set as $|\theta_s| = \alpha$.

6.4.1 Model and Engineering Demand Parameters (EDP)

A single recorded accelerogram has been considered for defining the nonstationary characteristics of the seismic input, namely the Northridge 1994 record at Burbank–Howard Rd. station. Figure 6.8 shows the ground acceleration time histories of the two horizontal components.
and the associated Fourier Amplitude spectra (i.e. the distribution of the amplitude of the ground motion with respect to the circular frequency). The principal components have been identified by examining the correlation coefficient of its as-recorded horizontal components (Eq. (6.1, 6.2)). The parameters of the ground motion model have been obtained based on the procedure described in [18]. The uncertainty in the modal properties of the primary structural system was represented assuming uncorrelated Gaussian random variables with standard deviation $\sigma_\alpha = 0.04$, $\sigma_\beta = 0.07$ and $\sigma_\gamma = 0.1$ for the modal shapes, modal frequencies and modal viscous damping ratios, respectively, (§ 6.3.2), which are deemed as representative of the actual level of uncertainty which could be expected in the modal subspace for an existing structure with a limited level of knowledge in terms of mass, stiffness and damping.

For the subsequent analyses, the relevant engineering demand parameters (EDPs) were considered for the various structural systems, namely: the maximum displacement $|\hat{u}(t)|_{\text{max}}$ and absolute acceleration $|\ddot{u}_a(t)|_{\text{max}}$ for the linear primary structure; the maximum relative displacement $|\hat{u}_s(t)|_{\text{max}}$ for the linear, bilinear and sliding secondary oscillators; and the maximum normalised rotation $|\hat{\theta}_s(t)/\alpha|_{\text{max}}$ for the rocking block.

### 6.4.2 Numerical Analyses

MC simulations comprising of a series of linear dynamic analyses have been carried out, with $n_{\text{sym}} = 500$ realisations, using the computational software MATLAB [124]. The primary
structure has been excited by assigning the ground motion components in the \( x \) and \( y \) directions as shown in Figure 6.9(a).

In a first stage, the uncertainty model is verified and the effects are studied on the primary structure. In a second stage, the response of secondary oscillators is quantified, considering both the randomness in the seismic input and the uncertainty in the modal properties of the primary structure. Their impact is exemplified with the use of linear and nonlinear response spectra, as well as through the empirical cumulative distribution function (CDF) for the selected EDPs.
6.4.2.1 Simulated Motions

Figure 6.9(b) confirms that the correlation coefficient calculated over the pair of components is a smooth function of the angle of attack $\delta$. For the earthquake record under consideration, $\rho = 0$ when $\delta = 84^\circ$, and hence the associated components can be considered as principal.

Figure 6.11(a) and 6.11(b) compare the statistics of the linear-elastic response spectra with 5% of viscous damping for the resulting simulated orthogonal components (mean ‘$\mu$’, thin solid lines, ± two standard deviations ‘$\sigma$’, thin dashed lines) with the recorded ones (thick solid lines). Overall, a satisfactory match is observed, and the simulated earthquake signals are thus adopted for the subsequent stages of the numerical study.

![Graphs showing comparison between recorded and simulated response spectra](image)

Figure 6.11: Comparison between 5% damped elastic response spectra of recorded and simulated major (a) and intermediate (b) orthogonal components for Northridge 1994 earthquake.

6.4.2.2 Primary System

Figures 6.12(a) and 6.12(b) show the randomised first modal shape and the first five modal circular frequencies, respectively, while Figures 6.12(c) compares the initial acceleration response of the primary system due the target ground motion input (black line) with the 500 realisations obtained with the proposed randomisation of the modal properties (grey). Indeed, the randomisation seems to be satisfactory with the oscillations showing fluctuations around the deterministic ones, with the expected spread of the random realisations and the uncertainty propagating in the time history. As evident by Figure 6.12(d), accounting also for randomness in the seismic input greatly increases the overall statistical dispersion of the results. In all the above graphs, light grey lines/dots are used for the individual realisations, while the black ones denotes the mean values.
6.4 Response of Secondary Oscillators

Figure 6.12: Randomised first modal shape (a) and spectral matrix (b); absolute acceleration response histories of the primary structure at the attachment point due to uncertainty in the modal properties, excluding (c) and including (d) the effect of uncertainty in the ground acceleration.

Figures 6.13(a) and 6.13(b) compare the empirical cumulative distribution function (CDF) of the displacement and absolute acceleration EDPs, with $n_{sym} = 2,000$ realisations, respectively. Three cases are considered, namely, $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = 0$ (Case A; solid line, i.e. deterministic primary structure), $\sigma_\alpha = 0.04, \sigma_\beta = 0.07, \sigma_\gamma = 0.1$ (Case B; dotted line, representing the reference case) and $\sigma_\alpha = 0.08, \sigma_\beta = 0.14, \sigma_\gamma = 0.2$ (Case C; dashed line, with twice the level of uncertainty in the primary structure). The 50th and 90th percentile values, $EDP_{p,50}$ and $EDP_{p,90}$, corresponding to probability of non-exceedance of 50% (median) and 90%, respectively, on the empirical CDFs are used to compare the findings for these three cases.

For the displacement EDP (Figure 6.13(a)), the inclusion of uncertainty in the primary structure causes 1.7% and 4.6% increase in the median for the two levels of modal uncertainty, respectively; 2.6% and 11.4% increase in the 90th percentile. For the acceleration EDP (Figure 6.13(b)), the increase is of 1.3% and 4.5% in terms of median; 3.5% and 13.5% in terms
Figure 6.13: Comparison of empirical CDF for the displacement (a) and absolute acceleration (b) EDPs for $\sigma_\alpha = \sigma_\beta = \sigma_\gamma = 0$ (solid), $\sigma_\alpha = 0.04, \sigma_\beta = 0.07, \sigma_\gamma = 0.1$ (dotted) and $\sigma_\alpha = 0.08, \sigma_\beta = 0.14, \sigma_\gamma = 0.2$ (dashed).

of 90th percentile. The effects then tend to be more significant for higher percentile values, meaning that uncertainties in the structural parameters tend to affect more severely the structural demand in the upper tail of its statistical distribution, which is the most significant area for any structural reliability consideration.

### 6.4.2.3 Secondary Oscillators

Following the generation of the simulated ground motions and the evaluation of the seismic response for the primary structure at the point of attachment, our analyses proceed with the cascade dynamic analysis of the four secondary oscillators under consideration. In what follows, the effects of various parameters governing the response of the oscillators are presented in terms of the associated EDPs for (i) randomness in the ground motion only and (ii) combined with the uncertainty in the modal parameters of the primary structure. The rationale is that, while the aleatory randomness in the seismic input is unavoidable in practice, the uncertainty in the structural parameters tends to be more epistemic and thus can often be reduced.

Figure 6.14 shows the influence of $T_s$ and $\zeta_s$ on the seismic response of the linear oscillator. Overall, similar effects are seen on the EDPs (Figures 6.14(a), 6.14(b)) for both cases under consideration. As evident (Figures 6.14(c), 6.14(d), 6.14(e), 6.14(f)), the mean response and variance increase with $T_s$ and an amplification is seen near the resonant period of the primary structure (i.e. $T_s = T_{px}$), when the value of $\zeta_s$ becomes particularly important.

Figure 6.15 shows the constant ductility bilinear spectra for the oscillators considering two values of stiffness ratio $\psi_s$ and ductility ratio $\mu_{d1}$, while the viscous damping ratio is set as
6.4 Response of Secondary Oscillators

\[ \zeta_s = 0.02. \] For all cases, uncertainty in the structure does not cause further increase in the response variance of the system, and therefore it could be safely neglected from the analysis.

As expected, the dynamic amplification seen in the linear system is significantly reduced at \( T_s = T_{px} \) due to the energy being dissipated in the hysteretic, elasto-plastic cycles. Increasing \( \mu_d \) from 5 (Figures 6.15(a), 6.15(b)) to 8 (Figures 6.15(c), 6.15(d)) causes minor effects on the
nonlinear spectra for the elastic-perfectly-plastic case ($\psi_s = 0$). Furthermore, increasing $\psi_s$ shows a reduction in the seismic response (Figures 6.15(e), 6.15(f)) when $\mu_d = 5$.
The sliding spectra are presented in Figure 6.16. Smooth curves are observed, with progressively smaller values of EDP and lower response variance when the friction coefficient increases. Similar to the previous cases, inclusion of uncertainty in the primary structure does not cause a noticeable increase in the secondary response variance, meaning that the latter is not sensitive to little-to-moderate variations in the primary seismic response.

Figure 6.16: Spectra of sliding secondary system due to uncertainty in the ground, and \( \sigma_\alpha = \sigma_\beta = \sigma_\gamma = 0 \) (a), \( \sigma_\alpha = 0.04, \sigma_\beta = 0.07, \sigma_\gamma = 0.1 \) (b).

Figure 6.17 shows the nonlinear rocking spectra for the dynamic parameter \( p = 1 \) and the coefficient of restitution \( \varepsilon \). As shown, uncertainty in the primary structure does not affect the response variance in the spectra for the secondary system. The smaller the slenderness angle \( \alpha \) and the higher the coefficient of restitution, the more likely the subsystem is to experience relatively large rotations, which are potentially incompatible with the functionality of the system, and the higher the coefficient of variation.

Finally, Figure 6.18 compares the empirical CDFs for the EDPs of the secondary oscillators for selected values, namely, \( T_s = 0.4 \) s and \( \zeta_s = 0.02 \) for the linear case (Figure 6.18(a)); \( T_s = 0.8 \) s, \( \psi_s = 0 \) and \( \mu_1 = 5 \) for the bilinear one (Figure 6.18(b)); \( \mu_s = 0.2 \) for the sliding block (Figure 6.18(c)); \( p = 1, \varepsilon = 0.5 \) and \( \alpha = 0.25 \) for the rocking block (Figure 6.18(d)). The additional dotted and dashed curves reported in the last two graphs represent the cases where the response acceleration of the primary structure is increased by a factor of 2 and 4, respectively, so to investigate the effects of a higher degree of nonlinearity, when relevant.

Table 6.1 compares the variations in the median value (50th percentile) and 90th percentile of the EDPs with respect to the case where randomness is only considered in the ground shaking. As shown, negative values are obtained only for the linear case and are attributed to resonance effects. The largest reduction (−21.5%) is observed for the median of the linear
oscillator (case C); as discussed previously, this effect can be attributed to the increased likelihood that the secondary oscillator becomes detuned from the motion of the primary structure. On the contrary, the largest increase (+22.2\%) is seen for the 90th percentile of the sliding block (case C; × 1 input scaling).

Importantly, in all the cases that have been analysed, the uncertainty in the modal parameters of the primary structure propagates onto the seismic response of light, linear and nonlinear secondary oscillators without experiencing any significant inflation: that is, the maximum variations observed are of the same order of magnitude as the maximum coefficient of variation assumed for the random modal parameters. Furthermore, the effects of the randomness in the ground shaking are always more significant. These observations could be helpful, for instance, when applying the PBEE methodology, as under these circumstances the effects of structural uncertainty can be neglected. Obviously, larger effects can be potentially experienced by a
6.5 Summary

In this chapter the relative effects of uncertainty in the primary, load-bearing structure have been examined on the dynamic response of light, secondary systems when compared to the intrinsic randomness of the seismic action. A method for characterising the structural uncertainty was presented, where the random quantities (modal shapes, frequencies and damping ratios) are directly defined in the reduced modal subspace rather than the full geometrical space, which in turn allows reducing the number of uncertain parameters to $m^2 + m$ statistically independent coefficients, $m$ being the number of modes retained in analysis (and typically $m$ is much less than the number $n$ of the degrees of freedom in primary structure).

### Table 6.1: Comparison of EDP$_{s,50}$ and EDP$_{s,90}$

<table>
<thead>
<tr>
<th>Oscillator Input Scaling</th>
<th>EDP$_{s,50}$</th>
<th>EDP$_{s,90}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_\alpha = 0$</td>
<td>$\sigma_\alpha = 0.04$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_\beta = 0$</td>
<td>$\sigma_\beta = 0.07$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_\gamma = 0$</td>
<td>$\sigma_\gamma = 0.1$</td>
</tr>
<tr>
<td>Linear</td>
<td>–</td>
<td>0.110</td>
</tr>
<tr>
<td>Bilinear</td>
<td>–</td>
<td>0.042</td>
</tr>
<tr>
<td></td>
<td>× 1</td>
<td>0.015</td>
</tr>
<tr>
<td></td>
<td>× 2</td>
<td>0.061</td>
</tr>
<tr>
<td></td>
<td>× 4</td>
<td>0.167</td>
</tr>
<tr>
<td>Rocking</td>
<td>× 1</td>
<td>0.009</td>
</tr>
<tr>
<td></td>
<td>× 2</td>
<td>0.018</td>
</tr>
<tr>
<td></td>
<td>× 4</td>
<td>0.036</td>
</tr>
</tbody>
</table>

combined primary-secondary system in which there is a significant level of dynamic interaction (e.g. similar to the effects of detuning a tuned mass damper), but this is beyond the scope of this paper.

### 6.5 Summary

In this chapter the relative effects of uncertainty in the primary, load-bearing structure have been examined on the dynamic response of light, secondary systems when compared to the intrinsic randomness of the seismic action. A method for characterising the structural uncertainty was presented, where the random quantities (modal shapes, frequencies and damping ratios) are directly defined in the reduced modal subspace rather than the full geometrical space, which in turn allows reducing the number of uncertain parameters to $m^2 + m$ statistically independent coefficients, $m$ being the number of modes retained in analysis (and typically $m$ is much less than the number $n$ of the degrees of freedom in primary structure).
An identification procedure has been presented for the case of steel frames with uncertain semi-rigid connections under deterministic seismic excitation, in which the model parameters have been calibrated over various levels of connection flexibility and different configurations, allowing the direct evaluation of the random dynamic response without resorting to the full geometrical model. The numerical results showed that the level of connection flexibility can considerably affect the dynamic response of semi-rigid steel frames.

Used in conjunction with a convenient stochastic model for the ground shaking, the proposed approach was used to quantify the seismic performance of linear and nonlinear (i.e. bilinear, sliding, rocking) single-degree-of-freedom (SDoF) secondary oscillators vibrating in cascade with a multi-degree-of-freedom (MDoF) primary structure. In this way, a broad spectrum of different nonlinear behaviours and secondary components has been investigated.

As demonstrated with Monte Carlo simulations, the proposed structural uncertainty model renders a realistic representation of the random dynamic response of primary structures, and
is particularly appealing for existing structures, where the level of uncertainty in the modal subspace could be estimated either using the tools of the operational modal analysis or be based on the actual level of confidence on the computational model.

The performance of the secondary oscillators under consideration was quantified through linear and non-linear response spectra and the cumulative distribution functions of the relevant engineering demand parameters (EDPs). For the chosen stochastic model of ground acceleration, the randomness in the seismic input was found to be of higher importance when compared to uncertainty in the modal parameters of the primary structure, with the latter having either positive or negative effects on probability of failure of the secondary structures. That is, the random process representing the dynamic excitation experienced by the secondary SDoF oscillator is fully characterised by the stochastic ground shaking and the uncertain filter provided by the primary structure, with the randomness of the latter playing a much less significant role. Interestingly, a 3.9% reduction in the 90th percentile of the peak displacement suggests that uncertainty in the primary structure tends to reduce the likelihood of a resonant response for a linear secondary oscillators. Oppositely, increases of 3.2%, 3.4% and 1.6% in the 90th percentile of the EDP for the bilinear, sliding and rocking oscillators indicate that the effects may reverse and thus they might become significant when assessing the seismic safety of secondary systems. Furthermore, increasing the level of modal uncertainty or the level of seismic input was found to exacerbate these effects, meaning that, within a performance-based framework, considering higher levels of intensity measures may warrant a full probabilistic seismic analysis, including the effects of the uncertainty in the primary structure.
Chapter 7

Component-Mode Synthesis for Linear MDoF Secondary Structures

7.1 Introduction

In Chapter 6, the seismic response of nonlinear secondary oscillators has been examined in presence of uncertainties in the ground motion and the properties of the supporting primary system. This chapter deals with the deterministic dynamic analysis of multi-degree-of-freedom primary-secondary combined linear systems.

First, a convenient variant of the component-mode synthesis (CMS) method [38] is introduced in Section 7.2. The proposed approach is more accurate than the cascade approximation, often used in the design practice, as the primary-secondary dynamic interaction is considered through the modes of vibration of the two components.

Second, the problem of selecting the vibrational modes to be retained in the analyses is addressed in Section 7.3. While it is still doable to cumulate the mass of the first modes for the primary structure, until a certain threshold is reached, the same criterion can hardly be applied for secondary attachments, as they may possess numerous low-frequency modes with negligible mass. To overcome this problem, a convenient application of the dynamic MAM
(DyMAM) [88] is proposed in Section 7.4, to account for the contribution of the truncated modes of the secondary system.

Finally, the influence of various approaches to construct the damping matrix of the primary-secondary assembly is investigated in Section 7.5, and a novel technique based on the modal damping superposition is proposed for modelling the dissipative forces in composite systems.

The CMS method is demonstrated in Section 7.6 through numerical applications, namely, i) a piping system multi-connected to a three dimensional multi-storey moment resisting frame, with irregularities in terms of both mass distribution in elevation and lateral stiffness in plan and ii) a flexible secondary system multi-connected to a two-dimensional stiff frame.

### 7.2 Combined Vibration via Component-Mode Synthesis

#### 7.2.1 Undamped Vibration

Let us consider the case of a S structure with \(n_S\) degrees of freedom (DoFs) multiply attached to a P system with \(n_P\) DoFs. Within the linear-elastic range, the undamped seismic motion is governed by:

\[
\mathbf{M} \cdot \ddot{\mathbf{u}}(t) + \mathbf{K} \cdot \mathbf{u}(t) = -\mathbf{M} \cdot \mathbf{\tau} \cdot \ddot{\mathbf{u}}_g(t), \tag{7.1}
\]

where, following the CMS formulation in [38]: \(\mathbf{u}(t) = \left\{ \mathbf{u}_S^\top(t) ; \mathbf{u}_P^\top(t) \right\}^\top\) is the partitioned array collecting the \(n\) DoFs \((n = n_S + n_P)\) of the combined dynamic system, in which \(\mathbf{u}_S(t) = \{u_{S,1}(t), \ldots, u_{S,n_S}(t)\}^\top\) and \(\mathbf{u}_P(t) = \{u_{P,1}(t), \ldots, u_{P,n_P}(t)\}^\top\) are arrays listing the DoFs of the S and P components, respectively, and the superscripted \(\top\) is the transpose operator; \(\mathbf{\tau} = \left\{ \mathbf{\tau}_S^\top ; \mathbf{\tau}_P^\top \right\}^\top\) is the partitioned array of seismic incidence; \(\ddot{\mathbf{u}}_g(t)\) is the ground acceleration; \(\mathbf{M}\) and \(\mathbf{K}\) are the matrices of mass and elastic stiffness, respectively, which can be partitioned as:

\[
\mathbf{M} = \begin{bmatrix}
\mathbf{M}_S & \mathbf{0}_{n_S \times n_P} \\
\mathbf{O}_{n_P \times n_S} & \mathbf{M}_P
\end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix}
\mathbf{K}_S & \mathbf{K}_{SP} \\
\mathbf{K}_{SP}^\top & \mathbf{K}_P + \mathbf{K}_{PP}
\end{bmatrix}, \tag{7.2}
\]

where \(\{\mathbf{M}_S, \mathbf{K}_S\}\) and \(\{\mathbf{M}_P, \mathbf{K}_P\}\) are the two pairs of mass and stiffness matrices of the S and P systems, individually considered, in which the P structure is assumed to be fixed to the ground, while the S system is also fixed to the support points on P; and \(\mathbf{O}_{r \times s}\) denotes a zero matrix with \(r\) rows and \(s\) columns. Furthermore, \(\mathbf{K}_{SP}\) is the stiffness matrix coupling P and S; \(\mathbf{K}_{PP}\) represents the additional stiffness in the P structure due to the presence of S. The elements
of both $K_{SP}$ and $K_{PP}$ only depend on the stiffness of the links used to connect the P and S components.

### 7.2.2 Modal Coordinate Transformation

The number of DoFs in the dynamic analysis can significantly be reduced by projecting the differential equations of motion onto the modal subspaces. This requires the following $n \times m$ transformation of coordinates [38]:

$$\tilde{\mathbf{u}}(t) = \mathbf{Γ} \cdot \mathbf{q}(t),$$  \hspace{1cm} (7.3)

in which $\mathbf{q}(t) = \left\{ \begin{array}{c} \mathbf{q}_S^\top(t) \\
\mathbf{q}_P^\top(t) \end{array} \right\}^\top$ is the $m$-dimensional array $(m = m_S + m_P)$ collecting the modal coordinates of the P-S system, where those of the S component, listed in the array $\mathbf{q}_S(t) = \{q_{S,1}(t), \ldots, q_{S,m_S}(t)\}^\top$ precede those of the P structure, listed in $\mathbf{q}_P(t) = \{q_{P,1}(t), \ldots, q_{P,m_P}(t)\}^\top$; and $\mathbf{Γ}$ is a transformation matrix, conveniently assembled as:

$$\mathbf{Γ} = \begin{bmatrix} \Phi_S^\top & \Psi_{SP}^\top \\
O_{np \times ms} & \Phi_P^\top \end{bmatrix},$$  \hspace{1cm} (7.4)

where $\Phi_S = [\phi_{S,1} \ldots \phi_{S,m_S}]$ and $\Phi_P = [\phi_{P,1} \ldots \phi_{P,m_P}]$ are the $n_S \times m_S$ and $n_P \times m_P$ modal matrices for the S and P systems, respectively; and $\Psi_{SP} = [\psi_{SP,1} \ldots \psi_{SP,m_S}]$ is the $n_S \times m_P$ coupling matrix.

The two modal matrices can be obtained by solving two independent real-valued eigenproblems, which neglect the interaction effects between the two systems [44]:

$$\mathbf{M}_S \cdot \Phi_S \cdot \Omega_S^2 = \mathbf{K}_S \cdot \Phi_S; \quad \mathbf{M}_P \cdot \Phi_P \cdot \Omega_P^2 = \mathbf{K}_P \cdot \Phi_P,$$  \hspace{1cm} (7.5)

with the ortho-normal condition $\Phi_S^\top \cdot \mathbf{M}_S \cdot \Phi_S = \mathbf{I}_{m_S}$ and $\Phi_P^\top \cdot \mathbf{M}_P \cdot \Phi_P = \mathbf{I}_{m_P}$. In Eqs. (7.5), $\Omega_S$ and $\Omega_P$ are the diagonal spectral matrices, listing the modal circular frequencies of S and P, respectively; and $\mathbf{I}_r$ stands for the identity matrix of size $r$.

The coupling matrix can be obtained as:

$$\Psi_{SP} = \mathbf{N}_{SP} \cdot \Phi_P,$$  \hspace{1cm} (7.6a)

where $\mathbf{N}_{SP}$ is the matrix of pseudo-static influence of P on S, which can in turn be determined by solving the matrix equation:
\[ K_S \cdot N_{SP} = -K_{SP}. \]  
(7.6b)

Substituting Eq. (7.3) into Eq. (7.1), and premultiplying the result by \( \Gamma \top \), the equation of motion in the modal subspaces is ruled by:

\[ m \cdot \ddot{q}(t) + k \cdot q(t) = g \ddot{u}_g(t), \]  
(7.7)

where \( m \) and \( k \) are the matrices of mass and stiffness, while \( g \) is the influence vector of seismic incidence in the reduced modal subspace:

\[ m = \Gamma \top \cdot M \cdot \Gamma = \begin{bmatrix} I_{m_S} & \mathbf{m}_{SP} \\ m_{SP} & I_{mp} + m_{PP} \end{bmatrix}; \]  
(7.8a)

\[ k = \Gamma \top \cdot K \cdot \Gamma = \begin{bmatrix} \Omega_S^2 S_{O^2 m_S \times m_p} \\ O_{mp \times m_S} \Omega_P^2 P_{pp} + k_{pp} \end{bmatrix}; \]  
(7.8b)

\[ g = -\Gamma \top \cdot M \cdot \tau = \begin{bmatrix} \mathbf{p}_S \\ \mathbf{p}_P + p_{PP} \end{bmatrix}, \]  
(7.8c)

in which \( \mathbf{p}_S = \Phi_S \top \cdot M_S \cdot \tau_S \) and \( \mathbf{p}_P = \Phi_P \top \cdot M_P \cdot \tau_P \) are the two arrays collecting the modal participation factors for \( S \) and \( P \), respectively. The presence of the \( S \) system affects the mass, stiffness and participation factors of the \( P \) structure, through the additional blocks:

\[ m_{PP} = \Psi_{SP} \top \cdot M_S \cdot \Psi_{SP}; \]  
(7.9a)

\[ k_{PP} = \Phi_P \top \cdot [K_{PP} \cdot \Phi_P + K_{SP} \cdot \Psi_{SP}]; \]  
(7.9b)

\[ p_{PP} = \Psi_{SP} \top \cdot M_S \cdot \tau_S. \]  
(7.9c)

Furthermore, the P-S coupling is established in the reduced modal space by the out-of-diagonal block \( m_{SP} \), given by:

\[ m_{SP} = \Phi_S \top \cdot M_S \cdot \Psi_{SP}. \]  
(7.10)

Notably:

- Modal frequencies and modal shapes of the coupled (undamped) P-S dynamic system are the solution of the real-valued eigenproblem:
\[ m \cdot \Phi \cdot \Omega^2 = k \cdot \Phi. \]  

(7.11)

- The blocks of Eqs. (7.9) and (7.10) account for the dynamic feedback between the two components, and neglecting their contribution leads to the cascaded approximation.

### 7.3 Criteria on the Number of Vibrational Modes

In practical applications, a limited number of modes are retained in the dynamic analysis, typically the ones significantly contributing to the seismic motion. This leads to the MDM, in which the truncated modes result in an approximated structural response and may introduce considerable inaccuracies in the high-frequency range. Current codes of practice (e.g. Eurocode [69]) set out truncation thresholds for conventional structures via a set of criteria in which: (i) all modes with effective modal masses greater than 5% of the total mass need to be considered; and (ii) the sum of the effective modal masses for the retained modes, amounts to at least 90% of the total mass of the structure.

Following the work of Muscolino and Palmeri [44], analogous conditions can be expressed for the two systems in turn as:

\[ \max \{ p_{tP} \} < \sqrt{0.05M_P}; \quad \max \{ p_{tS} \} < \sqrt{0.05M_S}, \]  

(7.12)

where \( p_{tP} \) and \( p_{tS} \) comprise the arrays listing the modal participation factors for the truncated modes of P and S, respectively. Similarly:

\[ \sum_{i=1}^{m_P} \{ p_P \}_{i}^2 \geqslant 0.9M_P; \quad \sum_{j=1}^{m_S} \{ p_S \}_{j}^2 \geqslant 0.9M_S. \]  

(7.13)

### 7.4 Modal Correction Methods

It has been noted [88] that such criteria may fail in terms of stresses and strains, leading to significant errors in the design values of various checks. Moreover, these criteria cannot easily be adopted for secondary systems, which may possess numerous low-frequency modes with negligible mass. Accordingly, it is possible to improve the accuracy via a correction term appended to the approximate response (Eq. (7.3)) such that:

\[ u(t) = \tilde{u}(t) + \Delta u(t). \]  

(7.14a)
Two alternative formulations can be adopted for the modal correction term ($\Delta u$) to account for the contribution of the neglected modes:

$$\Delta u_{\text{MAM}} (t) = \Delta b \ddot{u}_g (t) ; \quad \Delta u_{\text{DyMAM}} (t) = \Delta b \omega_F^2 \theta (t) , \quad (7.14b)$$

corresponding to the MAM and DyMAM, respectively, where $\Delta b$ is the static correction vector:

$$\Delta b = b_G - \Gamma \cdot b_M , \quad (7.14c)$$

in which $b_G = -K^{-1} \cdot M \cdot \tau = \left\{ (b_S + N_{SP} \cdot b_P)^\top ; b_P^\top \right\}^\top$ and $b_M = k^{-1} \cdot g$ are the static response of the whole structure, and the response due to the modes of vibration retained in analysis (neglecting the inertial effects in Eqs. (7.1) and (7.7)), respectively, while $b_S$ and $b_P$ are solutions of:

$$K_S \cdot b_S = -M_S \cdot \tau_S ; \quad \left[ K_P + K_{PP} + K_{SP}^\top \cdot N_{SP} \right] b_P = -M_P \cdot \tau_P - K_{SP}^\top \cdot b_S , \quad (7.14d)$$

and $\theta(t)$ is the response of a single-degree-of-freedom (SDoF) oscillator satisfying:

$$\ddot{\theta} (t) + 2 \zeta_F \omega_F \dot{\theta} (t) + \omega_F^2 \theta (t) = \ddot{u}_g (t) , \quad (7.15a)$$

in which $\omega_F$ and $\zeta_F$ are chosen as:

$$\omega_F = 2 \min \{ \Omega_P \} ; \quad \zeta_F = \frac{1}{\sqrt{2}} . \quad (7.15b)$$

### 7.5 Construction of the Viscous Damping Matrix

Assuming viscously damped linear systems, it is possible to assemble the equivalent viscous damping matrix as:

$$C = \begin{bmatrix} C_S & C_{SP} \\ C_{SP}^\top \cdot C_P + C_{PP} \end{bmatrix} ; \quad c = \Gamma^\top \cdot C \cdot \Gamma = \begin{bmatrix} c_S & c_{SP} \\ c_{SP}^\top \cdot c_P + c_{PP} \end{bmatrix} , \quad (7.16)$$

where $\{ C_S, C_P \}$ and $\{ c_S, c_P \}$ represent the corresponding damping matrices on S and P in the geometrical and modal domain, respectively; $C_{SP}$ is the damping matrix coupling the two systems; $C_{PP}$ represents the residual damping in the P structure due to the presence of the S
system. Three alternative formulations can be adopted for constructing the individual blocks, and these are described in the following subsections. Specifically, the first two are taken from the literature [35, 93] while the third one is a new contribution. Once the associated matrices are defined, the combined response of the P-S system will then be governed by:

\[ m \ddot{q}(t) + c \dot{q}(t) + k q(t) = g \ddot{u}_g(t). \]  

(7.17)

### 7.5.1 Proportional Damping

The Rayleigh damping model is adopted for the two systems so the matrices \( C_S \) and \( C_P \) take the form:

\[ C_S = \zeta_S [\alpha_M M_S + \alpha_K K_S]; \quad C_P = \zeta_P [\alpha_M M_P + \alpha_K K_P], \]  

(7.18a)

in which \( \zeta_S \) and \( \zeta_P \) are the viscous damping ratios for S and P, respectively, while \( \alpha_M \) and \( \alpha_K \) are the coefficients of proportionality for mass and stiffness, evaluated as:

\[ \alpha_M = \frac{2 \omega_I \omega_{II}}{\omega_I + \omega_{II}} b; \quad \alpha_K = \frac{2 \omega_I \omega_{II}}{\omega_I + \omega_{II}} b; \quad b = \frac{2 (\omega_{II}^2 - \omega_I^2)}{\omega_{II}^2 - \omega_I^2 + 2 \omega_I \omega_{II} \ln (\omega_{II}/\omega_I)}, \]  

(7.18b)

where \( \omega_I \) and \( \omega_{II} \) are chosen circular frequencies of \( \omega_{I,S}, \omega_{I,P} \) and \( \omega_{II,S}, \omega_{II,P} \) for S and P, respectively, such that average values of \( \zeta_S \) and \( \zeta_P \) are achieved in the corresponding intervals \([\omega_{I,S}, \omega_{II,S}], [\omega_{I,P}, \omega_{II,P}] \). A single interval, \([\min\{\omega_{I,S}, \omega_{I,P}\}, \min\{\omega_{II,S}, \omega_{II,P}\}] \) can alternatively be assumed for the circular frequencies of both components. Additionally, the coupling matrix takes the form of \( C_{SP} = \zeta_S \alpha_{K,S} K_{SP} \), while the residual damping in the P structure is \( C_{PP} = \zeta_S \alpha_{K,S} K_{PP} \).

In the modal subdomain \( c_S \) and \( c_P \) are given (similar to Eqs. (7.18a)) by:

\[ c_S = \zeta_S [\alpha_{M,S} I_{m_S} + \alpha_{K,S} \Omega_S^2]; \quad c_P = \zeta_P [\alpha_{M,P} I_{m_P} + \alpha_{K,P} \Omega_P^2]. \]  

(7.19a)

Furthermore, the presence of the S system affects the damping of the P structure, through the additional block:

\[ c_{PP} = \zeta_S [\alpha_{M,S} m_{PP} + \alpha_{K,S} k_{PP}], \]  

(7.19b)
while, the P-S coupling is established by the out-of-diagonal block $c_{SP}$, given by:

$$c_{SP} = \zeta_S \alpha_{M,SP} \mathbf{m}_{SP}.$$

(7.19c)

### 7.5.2 Generalisation of Proportional Damping

It is possible to define damping ratios for a higher number of modes. Retaining the first four terms of the Caughey series, one can deduce:

$$C_S = \alpha_{S,0} \mathbf{M}_S + \alpha_{S,1} \mathbf{K}_S + \mathbf{M}_S \sum_{l=2}^{3} a_{S,l} \left[ \Phi_S \cdot \Phi_S^\top \cdot \mathbf{K}_S \right]^l,$$

(7.20a)

$$C_P = \alpha_{P,0} \mathbf{M}_P + \alpha_{P,1} \mathbf{K}_P + \mathbf{M}_P \sum_{l=2}^{3} a_{P,l} \left[ \Phi_P \cdot \Phi_P^\top \cdot \mathbf{K}_P \right]^l,$$

(7.20b)

for the S and P systems, respectively, while the coupling and residual matrices take the form:

$$C_{SP} = \alpha_{S,1} \mathbf{K}_{SP}; \quad C_{PP} = \alpha_{S,1} \mathbf{K}_{PP},$$

(7.20c)

where the coefficients $\alpha_S$ and $\alpha_P$ satisfy the succeeding algebraic equations:

$$\zeta_{S,i} = \frac{1}{2} \sum_{l=0}^{3} a_{S,l} \omega_{S,i}^{2l-1}; \quad \zeta_{P,i} = \frac{1}{2} \sum_{l=0}^{3} a_{P,l} \omega_{P,i}^{2l-1}; \quad i = \{i_1, \ldots, i_4\},$$

(7.20d)

with $\zeta_{S,i}$, $\zeta_{P,i}$ being the $i^{th}$ modal damping ratios corresponding to chosen frequencies $\omega_{S,i}$, $\omega_{P,i}$, for S and P systems, respectively. Once projected on to the modal subdomain, the corresponding blocks read:

$$c_S = \alpha_{S,0} \mathbf{I}_{ms} + \alpha_{S,1} \Phi_S^2 + \Phi_S^\top \cdot \mathbf{R}_S \cdot \Phi_S; \quad c_P = \alpha_{P,0} \mathbf{I}_{mp} + \alpha_{P,1} \Omega_P^2 + \Phi_P^\top \cdot \mathbf{R}_P \cdot \Phi_P;$$

(7.21a)

$$c_{SP} = \alpha_{S,0} \mathbf{m}_{SP} + \Phi_S^\top \cdot \mathbf{R}_S \cdot \Psi_{SP}; \quad c_{PP} = \alpha_{S,0} \mathbf{m}_{PP} + \alpha_{S,1} \mathbf{k}_{PP} + \Psi_{SP}^\top \cdot \mathbf{R}_S \cdot \Psi_{SP};$$

(7.21b)
\[ R_S = M_S \sum_{l=2}^{3} \alpha_S,l \left[ \Phi_S \cdot \Phi_S^\top \cdot K_S \right]^l; \quad R_P = M_P \sum_{l=2}^{3} \alpha_P,l \left[ \Phi_P \cdot \Phi_P^\top \cdot K_P \right]^l. \quad (7.21c) \]

Evidently, when the first two terms are only considered the damping model reduces to the case of Rayleigh. Additionally, the selection of four modes is driven by the requirement to maintain positive \( \zeta \) outside the chosen interval, while at the same time avoid ill-conditioning associated with higher mode number [93].

### 7.5.3 Modal Damping

An alternative formulation based on modal damping superposition is developed herein for constructing the viscous damping matrix of secondary structures. In the proposed method, higher modal contribution is considered, for secondary structures that typically possess numerous low-frequency modes with negligible mass.

Considering constant damping on the vibrational modes retained, \( c_S \) and \( c_P \) can be expressed as:

\[ c_S = 2 \zeta_S \Omega_S; \quad c_P = 2 \zeta_P \Omega_P, \quad (7.22a) \]

while the coupling and residual matrices can be expressed as:

\[ c_{SP} = O_{m_S \times m_P}; \quad c_{PP} = 2 \kappa_S k_{PP}. \quad (7.22b) \]

The corresponding blocks associated with the individual systems can then be assembled in the geometric space as:

\[ C_S = \tilde{C}_S + \Delta C_S; \quad C_P = \tilde{C}_P + \Delta C_P, \quad (7.23a) \]

and similarly:

\[ C_{SP} = \tilde{C}_{SP} + \Delta C_{SP}; \quad C_{PP} = \tilde{C}_{PP} + \Delta C_{PP}, \quad (7.23b) \]

where blocks with the overtilde are those associated with the modes retained in the modal analysis (\( m_S \) for the S system and \( m_P \) for the P structure), and the ones denoted by \( \Delta \) account for the higher modes (whose contribution is neglected in the conventional modal damping approach). Based on the preceding expressions, one can derive:
\[ \tilde{C}_S = 2 \zeta_S \Phi_S \cdot \Omega_S \cdot \Phi_S^\top \cdot M_S ; \quad \tilde{C}_P = 2 \zeta_P \Phi_P \cdot \Omega_P \cdot \Phi_P^\top \cdot M_P ; \quad (7.24a) \]

\[ \tilde{C}_{SP} = -\tilde{C}_S \cdot \Psi_{SP} \cdot \Phi_P^\top \cdot M_P ; \quad (7.24b) \]

\[ \tilde{C}_{PP} = \Phi_P \left[ \Psi_{SP} \cdot \tilde{C}_S \cdot \Psi_{SP} + 2 \kappa_S \kappa_{PP} \right] \Phi_P^\top \cdot M_P . \quad (7.24c) \]

Additionally, it is possible to derive the expressions for the higher mode contribution by adopting the Rayleigh damping model as:

\[ \Delta C_S = \mu_S \Delta M_S + \kappa_S \Delta K_S ; \quad \Delta C_P = \mu_P \Delta M_P + \kappa_P \Delta K_P ; \quad (7.25a) \]

\[ \Delta C_{SP} = \kappa_S \Delta K_{SP} ; \quad \Delta C_{PP} = \kappa_S \Delta K_{PP} - \mu_S \tilde{M}_{PP} , \quad (7.25b) \]

where \( \{ \mu_S, \kappa_S \} \) and \( \{ \mu_P, \kappa_P \} \) are the pairs of coefficients for the damping model applied to the S and P components, in turn, such that:

\[ \mu_S = \zeta_S \omega_{S,max} ; \quad \kappa_S = \zeta_S / \omega_{S,max} ; \quad \omega_{S,max} = \max \left[ \Omega_S \right] = \omega_{S,m_S} ; \quad (7.26a) \]

\[ \mu_P = \zeta_P \omega_{P,max} ; \quad \kappa_P = \zeta_P / \omega_{P,max} ; \quad \omega_{P,max} = \max \left[ \Omega_P \right] = \omega_{P,m_p} , \quad (7.26b) \]

while the residual modal contributions can be posed in the form:

\[ \Delta M_S = M_S - \tilde{M}_S ; \quad \Delta M_P = M_P - \tilde{M}_P ; \quad (7.27a) \]

\[ \Delta K_S = K_S - \tilde{K}_S ; \quad \Delta K_P = K_P - \tilde{K}_P ; \quad (7.27b) \]

\[ \Delta K_{SP} = K_{SP} - \tilde{K}_{SP} ; \quad \Delta K_{PP} = K_{PP} - \tilde{K}_{PP} , \quad (7.27c) \]

in which the blocks with the overtilde are evaluated as:
\[ \tilde{M}_S = M_S \cdot \Phi_S \cdot \Phi_S^\top \cdot M_S; \quad \tilde{M}_P = M_P \cdot \Phi_P \cdot \Phi_P^\top \cdot M_P; \quad (7.28a) \]

\[ \tilde{K}_S = M_S \cdot \Phi_S \cdot \Omega_S^2 \cdot \Phi_S^\top \cdot M_S; \quad \tilde{K}_P = M_P \cdot \Phi_P \cdot \Omega_P^2 \cdot \Phi_P^\top \cdot M_P; \quad (7.28b) \]

\[ \tilde{K}_{SP} = -\tilde{K}_S \cdot \Psi_{SP} \cdot \Phi_P^\top \cdot M_P; \quad (7.28c) \]

\[ \tilde{K}_{PP} = M_P \cdot \Phi_P \left[ \Psi_{SP}^\top \cdot \tilde{K}_S \cdot \Psi_{SP} + k_{PP} \right] \Phi_P^\top \cdot M_P; \quad (7.28d) \]

\[ \tilde{M}_{PP} = M_P \cdot \Phi_P \cdot \Psi_{SP}^\top \left[ M_S - \tilde{M}_S \right] \Psi_{SP} \cdot \Phi_P^\top \cdot M_P. \quad (7.28e) \]

Notably, Eqs. (7.25) to (7.28), constitute a novel characterisation of the truncated vibrational mode contribution, that is deemed necessary in formulating a consistent viscous damping matrix for the primary-secondary assembly.

### 7.6 Numerical Examples

Aimed at assessing the validity of the formulation presented in the previous section, the seismic response of coupled P-S systems has numerically been investigated via two representative case studies, each examining different aspects, and the results are reported and discussed in this section. In the first case study, the CMS method has been used to quantify the seismic response of an S piping system multi-connected to a three-dimensional P multi-storey moment-resisting frame with irregularities. In the second case study the application of the CMS has been demonstrated on a flexible secondary system and the use of modal correction methods as well as the various approaches to construct the damping matrix have been investigated.

#### 7.6.1 Example 1: Piping System in Irregular Building

Figure 7.1(a) shows the P-S combined dynamic system under consideration, which consists of a 3-dimensional 5-storey moment-resisting frame (P) multiply connected with flexible links to a MDoF piping system (S). The floors of the P frame are assumed to be rigid in their own plane, so to simulate the presence of slabs. Self-weight and super-dead load are the two sources of mass, which is concentrated at the floor level for the P structure and uniformly distributed for the S system. The total masses are \( M_P = 97.9 \text{ Mg} \) and \( M_S = 0.3 \text{ Mg} \) and the resulting S-P
mass ratio is $\mu = M_S/M_P = 0.003$, while the axial stiffness of the links is set to 277 MN/m. The fundamental periods of vibration are $T_{P,1} = 0.421$ s for the P structure and $T_{S,1} = 0.484$ s for the S piping (the latter being fixed to the ground as well as to the points of connection to P).

In its reference configuration, the P frame is doubly symmetrical in plan and has equal storey masses, fully meeting the regularity criteria in plan and elevation, while S has an unsymmetrical geometry as depicted in Figure 7.1(b).

![Figure 7.1: Primary-secondary case study: specification (a) and base-fixed system configuration (b).](image)

The number of DoFs is $n_p = 120$ for P (24 per storey), and $n_s = 336$ for S (where a finer discretisation is required), leading to a total of $n = n_p + n_s = 456$ DoFs. Only $m = m_p + m_s = 6 + 121 = 127$ modes (28%) were retained in the analysis, so that at least 90% of the modal mass for each sub-model participates in the seismic motion in the direction of interest ($x$ for all our analyses).

The reference values of the viscous damping ratios are $\zeta_P = 0.05$ and $\zeta_S = 0.02$, respectively, while the circular frequencies for the Rayleigh’s damping model are $\omega_I = 1$ rad/s and $\omega_{II} = 100$ rad/s, chosen as representative bounds of the energy content of the seismic input and kept constant throughout the analyses.

In order to trigger the P-S dynamic interaction for an accelerogram applied along $x$, the 7th mode of the S piping, with a large participation mass in the direction of interest, has been tuned to the 2nd mode of the P frame, which accounts for about 85% of $M_P$ in the $x$ direction, so that $T_{P,2} = T_{S,7} = 0.385$ s.
Table 7.1: Ground motion records

<table>
<thead>
<tr>
<th>Earthquake</th>
<th>Site / Component</th>
<th>$\Delta t$ [s]</th>
<th>PGA [$g$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Imperial Valley 1940</td>
<td>El Centro / 180</td>
<td>0.0100</td>
<td>0.258</td>
</tr>
<tr>
<td>Erzican 1992</td>
<td>Erzican / N-S</td>
<td>0.0050</td>
<td>0.489</td>
</tr>
<tr>
<td>Irpinia 1980</td>
<td>Calitri / 270</td>
<td>0.0024</td>
<td>0.152</td>
</tr>
</tbody>
</table>

7.6.1.1 Parametric Study

A series of linear dynamic analyses were carried out using the commercial software SAP2000 [139] to assemble the relevant mass and stiffness matrices and the numerical software MATLAB [124] to implement the CMS variant described in § 7.2. The OAPI (open application programming interface) of SAP2000 was used to allow the bidirectional exchange of data with MATLAB, including the model updating of the mass and stiffness matrices.

Three recorded accelerograms were used, namely El Centro 1940, Erzican 1992 and Irpinia 1980 (see Table 7.1) whose acceleration time histories and associated Fourier Amplitude spectra are shown in Figure 7.2. These records have been chosen because of their distinct characteristics, which allow exploring the performance of the combined P-S system under different loading scenarios and can be used to identify some general trends in the results. Specifically, the El Centro 1940 earthquake has been chosen as its elastic response spectrum closely resembles the elastic design spectrum of Eurocode 8 [69]. Furthermore, the Erzican 1992 accelerogram has been considered as an example of a near-fault record. Lastly, the Irpinia 1980 earthquake has been chosen due to its peculiar nature comprising of a double intense phase, due to a two-stage fault rupture [11].

The validity of the CMS has been initially confirmed for the reference frame, with a fully regular configuration (Section 7.6.1.2). The effects of irregularities in the vertical distribution of the mass have then been investigated by increasing in turn the mass of the 2nd, 3rd and 4th storey. Finally, the variations in the dynamic response due to stiffness irregularities in plan have been studied by varying the stiffness of the corner column denoted with the letter $A$ in Figure 7.1(a) (Section 7.6.1.3).

The amount of irregularity applied to the P frame has been quantified with two dimensionless ratios, namely the mass ratio between two consecutive storeys:

$$\mu_i = \frac{M_{P,i}}{M_{P,i-1}}, \quad i = 2, \ldots, 5;$$

(7.29a)
and the eccentricity ratio:

\[ \kappa = \frac{e}{\rho}, \tag{7.29b} \]

where \( e \) is the distance between centre of mass and centre of rigidity (\( c_m \) and \( c_r \), respectively, in Figure 7.1(a)); and \( \rho \) is the radius of gyration of the floor plan.

In order to allow for a fair assessment as well as maintain the P-S interaction effects, \( M_P \) and \( T_{P2} \) were kept constant and the tuning condition \( T_{P2} = T_{S7} \) was maintained, irrespectively of the level of mass and stiffness irregularity.

According to the principles of performance-based earthquake engineering [9], different S components can be sensitive to different engineering demand parameters (EDPs). In the present study, they have been selected as: the maximum absolute displacements in the P frame, \( u_P \); the maximum absolute displacements in the S piping, relative to the P frame, \( u_S \); and maximum absolute accelerations in both P and S sub-models, \( \ddot{u}_P \) and \( \ddot{u}_S \). Points \( R_P \) and \( R_S \) in Figure 7.1(a) identify the positions on the P and S sub-models where the EDPs have been calculated.

### 7.6.1.2 Validation of the CMS

In order to enable a fair comparison between the CMS (with \( m = 127 \) modes of vibration) and the full dynamic P-S system (with \( n = 456 \) DoFs), the same value of viscous damping ratio \( \zeta_P = \zeta_S = 0.05 \) has been initially assumed for the two sub-models, as this allows using the Rayleigh’s damping model for both the full system and the CMS (Eqs. (7.19a), (7.19b) and (7.19c)).

Figure 7.3 compares the frequency response function (FRF) of a representative DoF in the S piping, i.e. the \( x \) displacement of point \( R_S \) (see Figure 7.1(a)), as evaluated for three levels of approximation, namely: the full combined system (thick solid line), which can be regarded as the reference solution; the CMS (dotted line); and the cascade approximation, where the P-S dynamic interaction is neglected (dashed line). Two cases of light (\( \mu = 0.003 \), Figure 7.3(a)) and heavy (\( \mu = 0.10 \), Figure 7.3(b)) S attachment are considered.

It is evident that for the light S piping, both the CMS and cascade approximation closely match the exact response. Conversely, when the mass of the S system increases, the CMS still gives accurate predictions, with only minor inconsistencies observed in the high-frequency range (\( \omega > 65 \text{ rad/s} \)), while the response predicted by the cascade approximation also introduces a significant inaccuracy in the low-frequency range, and the fundamental frequency of vibration is overestimated.
The accuracy and computational efficiency of the CMS has also been confirmed with the time domain analyses, which have demonstrated an average reduction of 43% in the execution time compared to the full combined dynamic system in the geometrical space.
Given the good level of fidelity and efficiency exhibited by the CMS, this model has been used to investigate how irregularities in the P frame affect the coupled dynamic response of the P-S system, as detailed in the following sub-section.

![Figure 7.3: FRF for cascade and CMS on a light (a) and heavy (b) S system.](image)

### 7.6.1.3 Effects of Irregularities in the P Structure

#### 7.6.1.3.1 Frequency Response

Figure 7.4 shows the FRFs for the two scenarios under investigation, i.e. mass irregularity in elevation and stiffness irregularity in plan, in which the thick solid curves denote the reference case, without irregularities, and the curves for the higher level of irregularity are denoted with the thick dashed line, namely $\mu = 6$ for the mass irregularity (top graphs) and $\kappa = 1.2$ for the stiffness irregularity (bottom graphs).

Overall, the effect of irregularity is more evident on the high frequency range, in which the different combinations of higher modes of vibration cause large fluctuations in both P and S systems, although the variations in the S attachment appear to be less ordered, requiring a P-S coupled dynamic analysis to quantify them.

#### 7.6.1.3.2 Displacement EDPs for Mass Irregularities

Figure 7.5 summarises the results obtained for the combined P-S system under investigation when the dynamic analysis is carried out in the time domain and the mass irregularity is varied at different locations. Each row presents the results for a given accelerogram (El Centro 1940 in the top row; Erzincan 1992 in the middle row; and Irpinia 1980 in the bottom row), all applied in the $x$ direction, while the left column shows the maximum responses in the P frame (i.e. the maximum $x$ displacements of point $R_P$) and the right column the maximum responses in the S piping (i.e. the maximum
Figure 7.4: FRF for vertical mass irregularity on second (a, b) and third storey (c, d), and in-plan stiffness irregularity (e, f) quantified on P (left) and S (right), respectively.

$x$ displacements of point $R_S$, relative to the P frame); in both cases, the results are normalised with respect to the corresponding maxima observed for the regular P frame ($\mu_i = 1$).
As shown, the mass irregularity tends to cause an overall increase of $u_P$ at lower storeys ($i = 2$) with the effect reversed at higher elevations ($i = 4$), also evident in all the three accelerograms, and consistent with the FRF (7.4(a)-7.4(d)). Similar trends are also observed for $u_S$. While the details of the various curves inevitably depend on the time-frequency distribution of the energy content for each accelerogram, it is interesting to note that the EDP of the S piping appears to show similar sensitivities to the P frame to the presence of mass irregularity, suggesting that any regularity criterion assumed for the P structure could also be used for the S systems. In this respect, however, the 150% and 200% thresholds set for $\mu_i$ by various codes of practice appears to be quite arbitrary and not necessarily associated to a significant change in the seismic response of the structure.

Overall, it appears that the relative displacements in the P structure could be used for assessing the expected performance of light drift-sensitive S systems, without resorting to sophisticated methods of analysis, such as the CMS used in this paper. Effectively, the variation in amplitude of the motion due to irregularities in the P frame results in a similar variation in the S piping.

7.6.1.3.3 Displacement EDPs for Stiffness Irregularities The effects of stiffness irregularities in plan are presented in Figure 7.6, for both P (left column) and S (right column) components, using the EDPs $u_P$ and $u_S$ in the two orthogonal directions $x$ (solid lines, parallel to the direction of the earthquake) and $y$ (dashed lines, orthogonal to it). To allow for an easier comparison, both responses along $x$ and $y$ for a given accelerogram are normalised with respect to the response of each sub-model in the $x$ direction when the P frame is regular (i.e. for $\kappa = 0$).

Also in this circumstance the P and S components appear to have similar level of sensitivity to the structural irregularity. As expected, the induced torsional vibration in the P frame means that the dynamic response orthogonal to the direction of the earthquake increases with the level of stiffness irregularity, with a maximum value of $u_{Py} = 0.35$ for the Irpinia 1980 record (Figure 7.6(e)), i.e. 35% of the corresponding response of the regular frame in the $x$ direction.

Given the unsymmetrical geometry of the S piping (see Figure 7.1(b)), the point $R_S$ used to evaluate their EDPs always experiences both $x$ and $y$ vibrations even for $\kappa = 0$ (regular P frame), in which case $u_{Sy}$ is about half of $u_{Sx}$ (Figures 7.6(b), 7.6(d), 7.6(f)).

Interestingly, for El Centro 1940, the maximum response of the S piping in the $y$ direction can be as large as $u_{Sy} = 0.53$, becoming comparable to the maximum response in the direction of the ground motion, highlighting the importance that irregularities in plan can have not only on the P load-bearing structural elements, but also on S systems. This also suggests that, when
drift-sensitive non-structural attachments are required for the serviceability of buildings with irregular plans, any torsional movement should be minimised, if possible, and the effects on the S components should be quantified at the design stage.
Figure 7.6: Displacement EDPs due to stiffness irregularity in-plan, for El Centro (a, b), Erzincan (c, d) and Irpinia (e, f) earthquakes, quantified on P (left) and S (right), respectively, in the x and y directions.

7.6.1.3.4 Acceleration EDPs for Irregular Frames Figure 7.7 presents the maximum absolute accelerations of both P and S components due to the presence of mass irregularity at the third floor ($i = 3$, left column) and stiffness asymmetry (right column).
While in the case of Erzincan 1992 and Irpinia 1980 earthquakes, the curves for the two systems are very close for mass and stiffness irregularities, respectively (Figures 7.7(d), 7.7(e)), this is not always true, and indeed very different trends are observed in some cases for the P frame and the S attachment.

Additionally, the sensitivity to structural irregularities in terms of absolute accelerations increases for both P and S elements in comparison to the relative displacements, mainly because the first modes of vibration of the P structure, which are those primarily affected by irregularities in plan and elevation, contribute more to the absolute accelerations than to the relative displacements.

Although the limited number of earthquake records and the specific features of the case study do not allow drawing general conclusions, it appears that the enhanced sensitivity and irregular seismic response would require the use of the CMS to assess the expected performance of acceleration-sensitive secondary systems.

### 7.6.2 Example 2: Flexible Secondary System with DyMAM

Figure 7.8 shows the composite system under investigation, which consists of a two-dimensional stiff frame (P) multiply connected with flexible links to a MDoF flexible system (S). Self-weight and super-dead load are the two sources of mass, concentrated at the floor level for the P structure and uniformly distributed for the S system, while mass at the P-S interface is assumed to act on the P structure. The total masses are $M_P = 99.8 \text{ Mg}$ and $M_S = 47.0 \text{ Mg}$ and the resulting S-P mass ratio is $\mu = M_S / M_P = 0.47$. The fundamental periods of vibration are $T_{P,1} = 0.215 \text{ s}$ for the P structure and $T_{S,1} = 0.312 \text{ s}$ for the S sub-model (the latter being fixed to the ground as well as to the points of connection to P).

The total number of DoFs is $n = n_p + n_s = 30 + 78 = 108$, and $m = m_p + m_s = 4 + 6 = 10$ modes (9%) were retained in the analysis, so that 98% and 87% of the modal mass for each sub-model, respectively, participates in the seismic motion in the direction of interest ($x$ for our analyses). In order to trigger the P-S dynamic interaction for an accelerogram applied along $x$, the 5th mode of the S system, with a large participation mass in the direction of interest, has been tuned to the fundamental mode of the P frame, which accounts for about 84% of $M_P$ in the $x$ direction, so that $T_{P,1} = T_{S,5}$.

### 7.6.2.1 Damping Characterisation

Four variants of the viscous damping matrix are studied, namely: Rayleigh; defined via (i) single and (ii) paired intervals for P and S; (iii) Caughey; and (iv) Modal. To enable a fair
Figure 7.7: Acceleration EDPs of P and S systems, for El Centro (a, b), Erzincan (c, d) and Irpinia (e, f) earthquakes, with respect to vertical mass (left) and in-plan stiffness (right) irregularities, respectively.
assessment of the CMS method to each variant, exact-reference damping models are defined, in which all modes are retained for the case of modal damping (i.e. $m_S = n_S, m_P = n_P$); the circular frequencies of the associated Rayleigh intervals are taken as $\omega_{I,S} = \omega_{S,4}, \omega_{II,S} = \omega_{S,10}$ and $\omega_{I,P} = \omega_{P,1}, \omega_{II,P} = \omega_{P,2}$ for S and P, respectively; the modes for Caughey are $i_S = \{4, 5, 6, 8\}$ and $i_P = \{1, 2, 3, 4\}$. Once the MDM is considered, $\omega_{II,S} = \omega_{S,6}$ and $i_S = \{3, 4, 5, 6\}$. The reference values of the viscous damping ratios are $\zeta_P = \zeta_S = 0.05$, thus allowing the construction of the damping models for the full dynamic P-S system in accordance with the proposed formulation.

### 7.6.2.2 Numerical Analyses

Linear dynamic analyses were carried out using the three recorded accelerograms (details are listed in Table 7.1) and the validity of the CMS has initially been confirmed in the frequency domain, with the exact responses of the various damping models applied on individual systems (modal damping also applied on the full dynamic P-S system), being compared to that of a hysteretic model [140] (whose evaluation is only permitted in the frequency domain), treated here as a reference one, as is believed to be in better accordance with experimental data. The corresponding truncated (MDM) responses were then evaluated for each damping model and were compared with the analogous corrected ones (MAM, DyMAM) (§ 7.6.2.3). Finally, the effects on the dynamic response were quantified in the time domain (§ 7.6.2.4). In the present study, the EDPs have been selected as: the maximum relative displacements, $u_S$; and the maximum
absolute accelerations, \( \ddot{u}_S \) in the S sub-model. Point \( R_S \) in Figure 7.8 identifies the position where the EDPs have been calculated.

### 7.6.2.3 Frequency-Domain Response

A selection of results is presented in this section for the case study under consideration. The frequency response function (FRF) has been evaluated for a representative DoF in the secondary system, i.e. the \( x \) displacement of point \( R_S \) (see Fig. 7.8). Fig. 7.9(a) shows the exact FRF in the geometric space, for the various damping models studied, whose cumulative relative differences are then reported (Fig. 7.9(b)) with respect to the hysteretic model, where the overbar sign denotes normalisation by the peak value of the Modal P, S model. It is evident that there are variations in the magnitude of the fundamental frequency (\( \omega \approx 22 \) rad/s) which is overestimated by the modal, Rayleigh and Caughey models applied on individual systems, while the paired Rayleigh and full combined system appear to be in better agreement with the hysteretic.

![Graph showing frequency response function (FRF) for various damping models.](image)

**Figure 7.9:** Exact FRF for various damping models (a), and corresponding cumulative differences (b) quantified on S.

Figure 7.10 compares the FRF as evaluated for each of the models, for five levels of approximation, namely: the full CMS, which can be regarded as the exact solution; the cascaded approximation (light dashed line), where the P-S dynamic interaction is neglected; the MDM where no correction is applied (light solid line); the MAM (dotted line) and the proposed DyMAM (dark dotted line), introducing a static and dynamic correction, respectively. It is observed that the response predicted by the cascaded approximation always introduces
7.6 Numerical Examples

a significant inaccuracy in the low-frequency range, leading to an overestimation of the fundamental frequency. This is due to the high S-P mass ratio ($\mu = 0.47$), suggesting that the dynamic interaction must indeed be accounted for. Consequently, the cascaded approximation is not considered in any of the subsequent stages of the analyses.

Figure 7.11 quantifies the cumulative inaccuracies of the remaining three approximations, normalised with the maximum value of the MAM, for each model. As expected, the truncation is shown to induce an error, as predicted by the MDM for all models. Notably, while MAM is shown to improve the dynamic response in the low-frequency range, a large discrepancy is introduced in the high-frequency range (clearly seen in Fig. 7.10, at $\omega \approx 145$ rad/s). Interestingly, the proposed DyMAM is capable of sustaining the correction in the low-frequency range and concurrently ameliorating the error in the high-frequency domain. A discrepancy of the DyMAM shown at $\omega > 170$ rad/s in Fig. 7.10(a) can indeed be regarded as negligible (see Fig. 7.10(b)).

7.6.2.4 Time-Domain Response

The dynamic analysis was carried out in the time domain, for the system under investigation and the three input accelerograms. The displacement and acceleration response histories have been computed for the MDM, MAM and DyMAM approximations. Figure 7.12 compares the corresponding discrepancies for the case of modal damping evolving with time, while Fig. 7.13 reports the cumulative values normalised with the associated maxima (i.e. the peak MDM for Fig. 7.13(a) and 7.13(b) as well as the peak MAM for Fig. 7.13(c) and 7.13(d)), at the end of the time interval. One can observe that when displacements are under consideration, the highest accumulated error is given by the MDM, while conversely larger discrepancies are predicted for the case of acceleration EDPs. Interestingly, the DyMAM consistently diminishes errors, a result that appears of practical importance as it highlights its appropriateness to the various engineering demand parameters chosen for the analysis of a given system. It is also worth emphasising that, notwithstanding its implementation in this paper, the application of the MAM to the case of accelerations is currently hindered to the practitioner, due to its requirement for availability of the ground acceleration time derivatives.

Figure 7.14 summarises the results of the EDPs under consideration, quantified through each damping model and the three input accelerograms. Although the limited number of earthquake records and the specific features of the case study do not allow drawing general conclusions, the effects of the aforementioned modal correction methods hold true for the remaining damping models and accelerograms. Additionally, it appears that depending on the damp-
Figure 7.10: FRF for cascade and CMS with various modal correction methods for single (a) and paired Rayleigh (b), modal (c) and Caughey (d) damping models.

It is noted that, while mutual comparison of the various models is permitted throughout the analysis, implementation of a hysteretic damping model in the time domain analysis is currently unattainable for the purpose of numerical validation [141]. Current uncertainties in the characterisation of damping need therefore to be quantified to assess and fully understand the predicted response of vibrating secondary systems.
7.7 Summary

In this chapter, the seismic response of coupled dynamic linear systems has been addressed. A convenient variant of the component-mode synthesis method (CMS) has first been introduced for the analysis of secondary systems in primary structures. The selection of vibrational modes to be retained in analysis has been discussed and a modal correction method has been proposed, to account for the dynamic contribution of the truncated modes of a secondary system. Finally, the influence of various approaches to construct the viscous damping matrix of the primary-secondary assembly has been investigated and a novel technique based on modal damping superposition, has been proposed.

Figure 7.11: Cumulative inaccuracies of various modal correction methods for single (a), and paired Rayleigh (b), Modal (c) and Caughey (d) damping models.
Numerical investigations carried out on a secondary piping system multi-connected to a primary (P) multi-storey moment resisting frame exhibiting various degrees of irregularities have shown that:

- The CMS is an efficient alternative to the cascade approximation, as the dynamic interaction between the two components is accounted for, therefore improving the accuracy in the evaluation of the seismic response. Moreover, since the modes of vibration of the P structure and S system are used, the computational effort is less than the full combined model in the geometrical space. A further practical advantage of the CMS is that P and S components can be designed independently by different teams, which only need to exchange the relevant modal information to check the effects of the dynamic interaction.

- For the chosen case study, mass irregularities at lower elevations tend to increase the displacements of both the P frame and S piping, while opposite effects are noted at higher storeys. The two systems were found to exhibit similar levels of sensitivity to irregularity and regularity thresholds set by various codes of practice for the mass in elevation (i.e. 150% or 200% between adjacent storeys) are not associated to any significant change, qualitative or quantitative, in the seismic response of the structure, meaning that further research is needed to establish more representative regularity conditions.

- Stiffness irregularity in plan, which induces torsional effects, does not always increase the relative displacements in the P frame, with a comparable level of sensitivity found for the S piping.
• The absolute accelerations in P and S components reveal increased sensitivity to structural irregularities in comparison with the relative displacements.

The above observations suggest that the relative displacements evaluated for the P structure can often provide a good basis to assess the performance of light drift-sensitive S attachments, without requiring a coupled dynamic analysis, while higher sensitivity and more irregular seismic responses would benefit from the use of the CMS for S acceleration-sensitive systems.

Furthermore, linear dynamic analyses carried out on a flexible secondary system multi-connected to a two-dimensional multi-storey stiff frame, lend themselves the following conclusions:
• The proposed dynamic MAM (DyMAM) is capable of improving the truncation error due to the MDM while it concurrently ameliorates the discrepancy induced in the high-frequency range by the MAM. It is of paramount importance for secondary systems possessing numerous low-frequency modes with negligible mass, where truncation threshold criteria can hardly be applied. Conversely to MAM, it has been demonstrated to consistently be applicable on various EDPs, being in accordance with performance-based earthquake engineering principles. Provided that a good proportion of the mass participating with vibration is considered, its effectiveness will increase with reduced modal information.

• The proposed technique for assembling the damping matrix is shown to be a convenient alternative for modelling the dissipative forces in composite vibrating systems. The predicted vibration envelope was shown to successively reduce in size for modal, Caughey, paired and single Rayleigh damping models, respectively, regardless of the EDP under consideration. An implementation of a hysteretic damping model in the time domain analysis is deemed necessary for the purpose of numerical validation.
7.7 Summary

Figure 7.14: Displacement (left) and acceleration (right) vibration envelopes for the Exact, MDM, MAM and DyMAM cases, from (left to right), respectively, as well as, single and paired Rayleigh (R1, R2), modal (MD) and Caughey (CH) damping models, for Imperial Valley (top), Erzican (middle) and Irpinia (bottom) earthquakes, respectively.
Conclusions

The research conducted throughout this PhD project aimed to contribute towards the determination and understanding of the response of seismically excited secondary systems. Discussions and summary sections have been included at the end of each chapter. It is the purpose of this chapter to recapitulate and unify the main contributions, as well as outline future research directions.

8.1 Summary of Achievements and Dissemination

The achievements of this study are summarised in relation to the objectives of Section 1.1 as follows:

1. Review the existing methods for the seismic analysis of secondary structures.
   A comprehensive literature review has been carried out in Chapter 2 identifying the main shortcomings of existing modelling and analysis methods for secondary (S) structures, and highlighting the need to further study their seismic behaviour.

2. Characterise the combined vibration response of nonlinear secondary oscillators and identify the conditions under which simplified analysis methods can be adopted.
The response of bilinear, sliding and free-standing rocking S oscillators coupled with primary (P) oscillators was modelled in Chapter 3 and the effectiveness of the cascade approximation was examined for full-cycle pulses.

3. Develop analytical and numerical solutions for the nonlinear response of cascaded secondary oscillators.

In Chapter 4, new analytical and numerical expressions have been proposed for the cascade response of bilinear, sliding and free-standing rocking S oscillators, and floor response spectra were presented. A subset of the work has been presented at the 2nd International Workshop on Seismic Loss, Rehabilitation, and Post-Earthquake Crisis Management of Critical Infrastructure. The findings of Chapter 3 and 4 will be included in a paper for possible publication in the Journal of Engineering Mechanics.

4. Examine the effect of uncertainty in the seismic input on the nonlinear response of cascaded secondary oscillators.

In Chapter 5, an existing stochastic model for far-field synthetic ground motions was adopted and the response spectra of nonlinear S oscillators in cascade were quantified. The findings were presented at the 16th World Conference on Earthquake Engineering, and a journal paper will be submitted for possible publication in Probabilistic Engineering Mechanics.

5. Develop an efficient method for characterising uncertainty in the modal properties of the primary structure and quantify its effects on secondary systems.

A novel uncertainty characterisation approach was presented in Chapter 6 in conjunction with an identification procedure for calibration of the associated model parameters. The main findings were presented at the 12th International Conference on Applications of Statistics and Probability in Civil Engineering and at the 12th International Conference on Structural Safety and Reliability, respectively. A journal paper will be submitted for possible publication in Earthquake Engineering & Structural Dynamics.

6. Study the combined vibration of linear multi-degree-of-freedom primary-secondary systems.

In Chapter 7, the component-mode synthesis (CMS) method was adopted for the dynamic analysis of linear coupled P-S systems. Two numerical applications have been studied, namely a piping S system and a straicase S system, multi-connected to a P structure.
7. **Extend the improved modal correction method for the analysis of secondary structures.**

The Dynamic Modal Acceleration Method (DyMAM) was proposed for the analysis of MDoF linear S systems in *Chapter 7*.

8. **Develop a novel technique for constructing the viscous damping matrix and quantify the effects of viscoelastic damping.**

In *Chapter 7*, a novel approach based on modal damping superposition was put forward for constructing the viscous damping matrix of the P-S system and its influence has been examined on the frequency response of an S system. The findings were presented at the *15th International Conference on Civil, Structural and Environmental Engineering Computing*.

9. **Examine the effect of irregularity and quantify its effects on secondary structures.**

The effect of mass and stiffness irregularity in a P structure has been examined on the response of a MDoF S system in *Chapter 7*. The outcomes of this objective, in conjunction with objective 6, were published in the *Proceedings of the Institution of Civil Engineers - Structures and Buildings*.

### 8.2 Novelties and Findings

#### 8.2.1 Novelties

The main contributions to technical knowledge provided in this PhD thesis are listed below:

- Dimensionless equations describing the motion of bilinear, sliding and free-standing rocking secondary oscillators coupled with linear primary ones have been derived and can be used as an a priori measure to adaptively assess the error associated with the cascade analysis approximation for a prescribed set of input parameters;

- Regions of validity for the cascade analysis of pulse-driven bilinear, sliding and free-standing rocking secondary oscillators have been presented, identifying trends of engineering interest;

- Closed-form analytical solutions have been derived for pulse-driven non-classically damped 2DoF linear systems. Closed-form piecewise linear analytical solutions have also been
derived for the cascade analysis of pulse-driven bilinear, sliding and free-standing rocking oscillators;

- A numerical scheme for the cascade analysis of piecewise linearised bilinear, sliding and free-standing rocking secondary oscillators has been proposed and numerically validated;

- The influence of pulse-like excitation parameters and of the input primary-secondary system parameters has been examined on the floor response spectra of bilinear, sliding and free-standing rocking secondary oscillators, identifying key trends;

- The stochastic response of bilinear, sliding and free-standing rocking secondary oscillators has been quantified in presence of far-field synthetic ground motions for specified earthquake and site characteristics through the cumulative distribution response functions;

- A novel approach for characterising system uncertainty in the modal properties of linear primary structures has been proposed, leading to a reduced set of input model parameters, and thus reduced computational effort. Importantly, this approach can be used for a broad variety of structural dynamics applications, not only for the seismic performance of secondary systems;

- A procedure for calibrating the parameters of the uncertainty model has been demonstrated for the case of steel frames with uncertain semi-rigid connections, and the effects of connection flexibility were quantified on the dynamic response;

- The relative influence of uncertainties in the seismic input and the primary structure has been examined on the response of nonlinear secondary oscillators, demonstrating that the former tends to have a significant impact on the seismic response of S systems;

- A CMS variant has been presented for the coupled dynamic analysis of multiply-attached multi-degree-of-freedom (MDoF) linear secondary systems, with illustrative applications to the case of a piping and a flexible secondary system;

- The DyMAM correction method has been proposed for the analysis of MDoF linear secondary systems, improving the response accuracy with reduced modal information, particularly suited to systems possessing numerous low frequency modes with negligible mass;
8.2 Novelties and Findings

- A novel approach for constructing the viscous damping matrix of the primary-secondary assembly has been proposed, and the influence of various techniques to build the damping matrix has been examined;

- The influence of mass and stiffness irregularity of a primary structure has been examined on the response of a piping secondary system;

8.2.2 Key Findings

The main findings of this dissertation are outlined below:

- Analyses of coupled nonlinear-linear pulse-driven 2DoF S-P systems have evidenced that the cascade analysis can be used for bilinear secondary systems provided their mass is less than 10% of the primary structure’s mass and the frequency ratio does not approach unity. On the contrary, the sliding and rocking secondary systems have shown pronounced sensitivity in the combination of the input parameters, suggesting that lighter systems (i.e. 1% and 0.1%, respectively) or an a priori assessment of the approximation error will need to be considered;

- The formulated coupled equations of motion for the 2DoF systems have been confirmed through comparisons with the cascade approximation in the limit when the mass ratio $\gamma \to 0$. Furthermore, the validity of the proposed numerical solutions for the cascade response has been confirmed through comparisons with the derived analytical ones, which were found to be exact matches;

- Cascade analyses of nonlinear S oscillators have shown that, even for the relatively simple case of a full-cycle pulse-type input, the response can be highly sensitive to the mechanical parameters of the S-P assembly, highlighting the need for sensitivity analyses to assess the impact of any variation. Analyses on bilinear S highlighted the importance of the damping ratio $\zeta_s$ in the vicinity of the resonance frequency, while the contribution of the modal coordinate $\varphi$ and the damping ratio $\zeta_p$ was found minor on the response. The latter cannot be regarded negligible for the analysis of sliding S oscillators. For rocking S, the response spectra were found to be ordered for the dynamic parameter $p$, the restitution coefficient $\varepsilon$, $\varphi$ and $\zeta_p$, while the normalised amplitude $a^*_k$ and the slenderness $\alpha$ resulted in discontinuities, suggesting that the response can only be predicted by accurately modelling the rocking behaviour;
• Investigations on cascaded nonlinear S oscillators driven by far-field ground motions have shown that the ductility ratio $\mu_1$ and the damping ratio $\zeta_s$ of bilinear S has minimal effects on the response CDF curves, indicating that a linear approximation may be satisfactory when modelling the S component. On the contrary, attention needs to be paid on the choice of $\omega_s^*$. One must only address whether the post-yielding stiffness $\psi_s$ needs to be accounted for within the analysis, and not its value;

• The relative effect of uncertainty in the primary structure was found to be of lower importance when compared to the randomness of the seismic action;

• The CMS has been found to be an accurate method for analysing linear MDoF systems accounting for the S-P dynamic interaction. Furthermore, it utilises the modal information of the associated base-fixed subsystems, reducing the computational burden required by the full combined model in the geometrical space;

• The DyMAM reduces the discrepancy induced in the high-frequency range by the modal acceleration method (MAM). Contrary to the latter, it was shown to be consistently applicable on various engineering demand parameters (EDP), being in line with performance-based earthquake engineering principles;

### 8.2.3 Further Findings

Additional findings are listed below:

• Numerical investigations carried out on steel frames with uncertain semi-rigid connections have revealed that the level of connection flexibility can considerably affect their dynamic response;

• Analyses carried out on a secondary piping system coupled with a primary structure exhibiting various degrees of irregularities highlighted that regularity thresholds set by codes of practice for the mass in elevation are not associated to any significant change in the seismic response of the structure;

• The predicted vibration response envelope was shown to successively reduce in size for modal, Caughey, paired and single Rayleigh damping models, respectively, regardless of the EDP considered;
8.3 Recommendations for Further Work

Further studies can complement and advance the work in the following main ways:

- Near-field and far-field stochastic analyses;
  
  In the present study, response spectra of nonlinear SDoF S were quantified in presence of simple pulse-type excitations as well as for a far-field stochastic ground motion model. In the former case, uncertainty effects were not accounted for while in the latter, only a single circular frequency was assumed for the primary structure and aftershock effects were excluded.

  The CDF functions quantified through the analysis, indicate that the procedure, presented here only for a single circular frequency parameter of the P structure, may be further extended to identify correlations between the seismic input parameters and the response CDF functions, over a finite set of P structures. A new set of predictive equations can then be obtained that can be used to generate realisations of the expected performance of S systems. It can then be further extended to the case of near-field motions by adopting the model proposed in [14].

- Identification of the uncertainty model parameters;
  
  In this work, the parameters of the proposed structural uncertainty model were calibrated over stiffness variations associated with the semi-rigid connections of steel frames for a single case-study scenario.

  Further investigations need to be conducted to identify the coefficients for different scenarios e.g. uncertainties in the mass and for various numerical case-study models. This will allow building distributions for the input parameters as well as a comprehensive relationship with uncertainty in the geometrical space.

- Investigate the effect of system uncertainty;
  
  In this study, the effect of uncertainty due to the seismic input and the properties of the P structure was considered. The S systems were assumed deterministic.

- The implementation of the nonlinear solutions for the S oscillators has been found to be much more efficient when compared to MATLAB’s build-in ODE solvers with a good trade-off between accuracy and computational cost.
Further studies need to model uncertainty in the mechanical parameters of S and the relative effects need to be quantified.

- **Model the nonlinear dynamic behaviour of high-dimensional S systems;**

  The work conducted throughout this thesis was limited to MDoF linear as well as bilinear, sliding and free-standing idealised rocking SDoF S, where in the latter case, pure rocking motion was considered. Given the non-exhaustive list of S, modelling other types of systems can provide further insights into the dynamic behaviour of a plethora of equipment.

  Further extending the work, the combined rocking and sliding regime of motion can be modelled, as well as the rocking response of anchored systems considering various type of nonlinearity for the restraints. Moreover, higher-order nonlinear systems can also be considered. The resulting models could be included in a computational platform, resulting in a library of systems readily accessible for assessment given a design scenario.

- **Identify the optimal solution for design variables of secondary systems;**

  A combinatorial optimization based procedure can be devised to provide recommendations on the optimality of the mechanical parameters of the S systems for given design scenarios.

- **Establish regularity conditions;**

  Analyses carried out on irregular primary frames have highlighted that further research is required to establish representative regularity conditions, which can be reliable and broadly applicable.
List of Publications

All of the papers published were written over the period of the doctoral research. Papers J1, C1-C4, C6, C7 and the future papers J3-J6 are in relation to the PhD research. Papers J2 and C5 are in relation to other projects.

Refereed Journal Articles


Refereed International Conference Papers


**National Conference Papers**


**Future publications**


Bibliography


