On some polynomials and continued fractions arising in the theory of integrable systems

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Metadata Record: https://dspace.lboro.ac.uk/2134/33775

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On some polynomials and continued fractions arising in the theory of integrable systems

BY

Marie-Pierre Grosset

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy in Mathematics of Loughborough University

June 2007

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Acknowledgements

I would like to express my gratitude to my supervisor Professor Alexander Veselov who provided me with excellent and enthusiastic supervision. I could not have imagined having a better advisor and mentor for my PhD. I would like to thank him for all the inspiration, guidance and support during this work.

I am grateful also to the academic and support staff at the Mathematical Sciences Department at Loughborough University for their kind and helpful assistance in many different ways.

Finally I would like to thank my sister Jeanne-Claire for her endless encouragement and my children for their patience and understanding.
Abstract

This thesis consists of two parts. In the first part an elliptic generalisation of the Bernoulli polynomials is introduced and investigated. We first consider the Faulhaber polynomials which are simply related to the even Bernoulli polynomials and generalise them in relation with the classical Lamé equation using the integrals of the Korteweg-de-Vries equation. An elliptic version of the odd Bernoulli polynomials is defined in relation to the quantum Euler top. These polynomials are applied to compute the Lamé spectral polynomials and the densities of states of the Lamé operators.

In the second part we consider a special class of periodic continued fractions that we call $\alpha$-fractions

$$
\phi(\lambda) = b_0 + \frac{\lambda - \alpha_1}{b_1 + \frac{\lambda - \alpha_2}{b_2 + \ddots}} \quad \text{where } \alpha = (\alpha_1, \ldots, \alpha_N) \text{ is a given sequence.}
$$

This type of continued fractions appears in the theory of the dressing chain. We investigate when an algebraic function of the form $\phi(\lambda) = \frac{-B(\lambda) + \sqrt{R(\lambda)}}{A(\lambda)}$, where $R(x)$ is a polynomial of odd degree $N = 2g + 1$ with coefficients in $\mathbb{C}$, can be written as a periodic $\alpha$-fraction. We show that this problem has a natural answer given by the classical theory of hyperelliptic curves and their Jacoby varieties.
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Introduction

Johann Faulhaber, "Rechenmeister der Stadt Ulm", in his book [20] published in 1631 announced a remarkable fact that the sum of the odd powers of the first $n$ natural numbers can be expressed as a polynomial of their sum. Namely, if we denote $\lambda = 1 + 2 + \ldots + n = \frac{n^2 + n}{2}$ then the sum

$$ S_{2m-1} = 1^{2m-1} + 2^{2m-1} + \ldots + n^{2m-1} $$

is a polynomial in $\lambda$ denoted $F_m(\lambda)$. These polynomials $F_m$ are called the *Faulhaber polynomials*. The first Faulhaber polynomials are

$$ F_1(\lambda) = \lambda, \quad F_2(\lambda) = \lambda^2, \quad F_3(\lambda) = \frac{1}{3} \lambda^2(4\lambda - 1), \quad F_4(\lambda) = \frac{1}{3} \lambda^2(6\lambda^2 - 4\lambda + 1), $$

$$ F_5(\lambda) = \frac{1}{5} \lambda^2(16\lambda^3 - 20\lambda^2 + 12\lambda - 3), \quad F_6(\lambda) = \frac{1}{3} \lambda^2(16\lambda^4 - 32\lambda^2 + 34\lambda^2 - 20\lambda + 5). $$

Faulhaber published the sums up to $S_{17}$ and stated that similar polynomials in $\lambda$, with alternating signs and a leading coefficient of the form $2^{m-1}/m$, would exist for all $m$.

In full generality, this claim was first proved by Jacob in 1834 [35]. For a nice discussion of this story and the effective ways to compute the Faulhaber polynomials we refer to a very interesting paper [40] by Donald Knuth.

For the general $k$, the sums of powers $S_{k-1}(n) = 1^{k-1} + 2^{k-1} + \ldots + n^{k-1}$ can be expressed through the classical *Bernoulli polynomials* named after Jacob Bernoulli

$$ S_{k-1}(n) = (B_k(n+1) - B_k)/k, $$

where $B_k = B_k(0)$ are the *Bernoulli numbers*. All odd Bernoulli numbers except $B_1 = -\frac{1}{2}$ are known to be zero, so the odd Bernoulli polynomials (up to multiplication by $k$) can be thought of as an "analytic continuation" of the sums of powers from the natural argument $n$ to the real (or complex) $x$.

Bernoulli polynomials can be defined through the generating function

$$ \frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k $$
\[ B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2}, \]

(see e.g. [2, 16]) and have the symmetry property \( B_k(1 - x) = (-1)^k B_k(x) \). Faulhaber's claim is a simple corollary of this symmetry. In fact, the Faulhaber polynomials are related to the even Bernoulli polynomials in a simple way:

\[ B_{2m}(x + 1) = 2m F_m \left( \frac{x^2 + x}{2} \right) + B_{2m}. \]

Faulhaber did not discover the Bernoulli numbers but his work was known to Jacob Bernoulli and cited in his famous treatise in *Ars Conjectandi* in 1713.

In the first part of the thesis we introduce a generalisation of the Bernoulli polynomials in relation to the classical Lamé equation. We define two new classes of polynomials: the *elliptic Faulhaber polynomials* and the *elliptic Bernoulli polynomials*, which are generalisations of the (slightly modified) *even* Bernoulli polynomials and the *odd* Bernoulli polynomials respectively.

The *Lamé equation* has the form [16], [68]

\[-\frac{d^2 y}{dz^2} + n(n + 1) \wp(z)y = Ey,\]

where \( \wp(z) \equiv \wp(z|\omega, \omega') \) is the Weierstrass elliptic function with fundamental periods \( 2\omega \) and \( 2\omega' \). This function satisfies the differential equation

\[ \wp(z)^{\prime 2} = 4\wp(z)^3 - g_2 \wp(z) - g_3, \]

where \( g_2, g_3 \) are the corresponding elliptic invariants.

The Lamé equation was first obtained by Gabriel Lamé by applying the method of separation of variables to Laplace's equation \( \Delta \phi = 0 \) in an ellipsoidal coordinate system. It was later extensively studied by Hermite, Brioschi, Halphen, Stieltjes and others.

Since the Lamé equation has doubly-periodic coefficients, a natural question to ask is when its solution is also doubly-periodic. The main result, going back to Hermite and Halphen, is that for such a solution to exist, \( n \) must be an integer or a half-integer. We will restrict ourselves to the special case when \( n \) is an integer which we assume non-negative since replacing \( n \) by \(-n - 1\) does not change the corresponding
equation. The eigenvalue $E$ must then belong to a set of $2n+1$ characteristic values $E_n^m, m = 0, 1, \ldots, 2n$. We call the polynomials $R_{2n+1}(E) = \prod_{m=0}^{2n}(E - E_n^m)$ the Lamé spectral polynomials.

In the case when the period $2\omega$ is real and the period $2\omega'$ is purely imaginary, one can consider a real smooth periodic version of the Lamé operator

$$L_n = -\frac{d^2}{dx^2} + n(n+1)\varphi(x + \omega'),$$

on the real line shifted by the imaginary half-period. It turns out (Ince [34]) that this operator has a remarkable spectral property: its spectrum has exactly $n$ gaps. The ends of the gaps $E_n = E_n(n)$ are precisely the $2n+1$ characteristic values $E_n^m$ of the Lamé equation, which are real and distinct in this case. We would like to mention that all finite-gap potentials are described by the finite-gap theory developed in the 1970s in the work of Novikov and Dubrovin, Its and Matveev, McKean and van Moerbeke, et al [14, 15, 46, 53, 54]. The Lamé operator is a very special case of the finite-gap operators since it admits some important multi-dimensional and matrix generalisations known as Calogero-Moser systems (see [9, 10, 24]).

In chapter 2 we define the elliptic Faulhaber polynomials. Our motivation comes from the recent discovery by D. Fairlie and A. P. Veselov [19] of an interesting relation between the Faulhaber polynomials and the soliton theory of the Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0$$

It is known since 1967 due to Gardner, Kruskal, Miura and Zabusky [48, 49], that this equation has infinitely many conservation laws of the form

$$I_m[u] = \int T_m(u, u_x, u_{xx}, \ldots, u_{m-2}) dx,$$

where $T_m$ are some polynomials of the function $u$ and of its $x$-derivatives up to order $m - 2$:

$$T_1 = u, \quad T_2 = u^2, \quad T_3 = u_1^2 + 2u^3, \quad T_4 = u_2^2 + 10uu_1^2 + 5u^4.$$  

They are uniquely defined by some homogeneity property up to an addition of a total derivative or a multiplication by a constant. This constant can be fixed by
demanding that $T_m(u, u_x, u_{xx}, \ldots, u_{m-2}) = u_{m-2}^2$ plus a function of derivatives of order less than $m - 2$.

The KdV equation has famous solutions known as solitons, the simplest of which is a one-soliton solution

$$u = -2 \text{sech}^2(x - 4t),$$

corresponding to the initial profile $u(x, 0) = -2 \text{sech}^2 x$. The main result of [19] is the following formula, relating the Faulhaber polynomials $F_m$ with the integrals of the KdV equation

$$I_m[-2 \lambda \text{sech}^2 x] = (-1)^m \frac{2^{2m}}{2m - 1} F_m(\lambda).$$

The elliptic Faulhaber polynomials are related to the periodic traveling waves (known also as cnoidal waves) of the KdV equation

$$u(x, t) = 2 \varphi_{*}(x - ct),$$

where $\varphi_{*}(x)$ satisfies the differential equation

$$(\varphi_{*}')^2 = 4 \varphi_{*}^3 - g_1 \varphi_{*}^2 - g_2 \varphi_{*} - g_3$$

with $g_1 = -c$. The function $\varphi_{*}(x)$ differs from the classical Weierstrass elliptic $\wp$-function by adding a constant.

The elliptic Faulhaber polynomials are defined as

$$\mathcal{F}_m(\lambda | \Gamma, \gamma) = \oint_{\gamma} T_m[2 \lambda \varphi_{*}(z)]dz,$$

where the integral is taken over a cycle $\gamma$ on the corresponding elliptic curve $\Gamma$ given by the equation $Y^2 = 4X^3 - g_1X^2 - g_2X - g_3$ with $X = \varphi_{*}(x), Y = \varphi_{*}'(x)$. Note that since the integrand has all the residues zero, one can consider $\gamma$ here as an element of the first homology group $H_1(\Gamma, \mathbb{Z})$ and the integral as a linear function on it. Equivalently, one can say that the integrand is a differential of the second kind and thus, by de Rham's theorem, determines a special element of the first cohomology group $H^1(\Gamma, \mathbb{C})$, polynomially dependent on the parameter $\lambda$. 
Let
\[ \omega = \frac{1}{2} \int_{\gamma} dz = \frac{1}{2} \int_{\gamma} \frac{dX}{Y}, \quad \xi = -\frac{1}{2} \int_{\gamma} \varphi_*(z)dz = -\frac{1}{2} \int_{\gamma} \frac{XdX}{Y} \]
be the corresponding periods of the basic second-kind differentials on this curve. The variables \( g_1, g_2, g_3, \omega, \xi \) can be considered as the coordinates on the space \( \mathcal{U} = \{(\Gamma, \gamma)\} \), so \( \mathcal{F}_m = \mathcal{F}_m(\lambda, g_1, g_2, g_3, \omega, \xi) \). Define the weight of the variables \( \lambda, g_1, g_2, g_3, \omega, \xi \) as \( 0, 2, 4, 6, -1, +1 \) respectively.

**Theorem 1** (Chapter 2, Th. 2.4). The elliptic Faulhaber polynomial \( \mathcal{F}_m \) is polynomial in all variables \( \lambda, g_1, g_2, g_3, \omega, \xi \) with rational coefficients, homogeneous with weight \( 2m - 1 \), and linear with respect to \( \omega \) and \( \xi \). When \( g_2 = g_3 = 0 \) it reduces (up to a factor) to the classical Faulhaber polynomial
\[ \mathcal{F}_m(\lambda, 0, 0, \omega, \xi) = -\frac{4}{2m - 1} g_1^{m-1} \xi F_m(\lambda) \]

If we consider the Weierstrass reduction \( g_1 = 0 \) we come to the reduced version of these polynomials \( \mathcal{F}_m^W(\lambda; g_2, g_3, \omega, \eta) = \mathcal{F}_m(\lambda, 0, g_2, g_3, \omega, \eta) \), where \( \varphi_* = \varphi, \xi \) reduces to \( \eta = -\frac{1}{2} \int \varphi(z)dz = \zeta(\omega, g_2, g_3) \), \( \varphi \) and \( \zeta \) are the standard Weierstrass elliptic functions \([68]\). The explicit form of the first 8 of them can be found in appendix A.

These reduced elliptic Faulhaber polynomials have a double zero at zero (for \( m > 1 \)) with the second derivative at zero proportional to
\[ B_{2m} := \frac{1}{2} \int_{\gamma} (\frac{d^{m-1}}{dz^{m-1}} \varphi(z))^2 dz, \]
which we call elliptic Bernoulli numbers. They are described in section 2.4. These numbers \( B_{2m} = B_{2m}(g_2, g_3, \omega, \eta) \) depend on the elliptic invariants \( g_2, g_3 \) and also on \( \omega \) and \( \eta \)
\[ B_2 = \frac{1}{12} g_2 \omega, \quad B_4 = -\frac{3}{5} g_3 \omega + \frac{2}{5} g_2 \eta, \quad B_6 = \frac{2}{7} g_2^2 \omega - \frac{36}{7} g_3 \eta. \]
On the discriminant (when two of the roots of the equation \( 4X^3 - g_2X - g_3 = 0 \) collide) they reduce to the usual Bernoulli numbers \( B_{2m} \) up to a simple factor. More precisely, for \( m > 1 \) we have
\[ B_{2m}(\frac{1}{12} h^2, \frac{1}{216} h^3, \omega, \xi + \frac{1}{12} h \omega) = -B_{2m} h^m \xi. \]
INTRODUCTION

In chapter 3 we introduce and investigate the elliptic Bernoulli polynomials. They are related to the quantum top and to the Lamé spectral polynomials.

The following remarkable relation between the Lamé equation and the quantum Euler top, going back to Kramers and Ittmann [41], is crucial for us. Consider the quantum mechanical Hamiltonian of the Euler top (see e.g. [43])

\[ \hat{H} = a_1 \hat{M}_1^2 + a_2 \hat{M}_2^2 + a_3 \hat{M}_3^2, \]

where the components \( \hat{M}_j \) of the angular momentum operator satisfy the standard commutation relations \([\hat{M}_1, \hat{M}_2] = i \hat{M}_3, [\hat{M}_2, \hat{M}_3] = i \hat{M}_1, [\hat{M}_3, \hat{M}_1] = i \hat{M}_2\).

The operator \( \hat{H} \) naturally acts in any representation of the Lie algebra \( \text{so}(3) \). In particular, it acts in the representation space with spin \( s = n \) of dimension \( 2n + 1 \) as a finite-dimensional operator \( \hat{H}_n \). The claim is that if the parameters \( a_i = e_i \) are the roots \( e_1, e_2, e_3 \) of the equation \( 4z^3 - g_2z - g_3 = 0 \), then the characteristic polynomial of the operator \( \hat{H}_n \) coincides with the Lamé spectral polynomial

\[ \det(\lambda I - \hat{H}_n) = R_{2n+1}(\lambda) \]  

(1)

The Weierstrass condition \( e_1 + e_2 + e_3 = 0 \) is unnatural from this point of view (and moreover contradicts the "physical" condition of positivity of \( a_i \)), so we consider the case when the parameters \( a_i \) are arbitrary and define the elliptic Bernoulli polynomials \( B_{2k+1} \) as the coefficients in the expansion of the trace of the resolvent of \( \hat{H}_n \) at infinity:

\[ tr(\lambda I - \hat{H}_n)^{-1} = \sum_{k=0}^{\infty} \frac{B_{2k+1}(n)}{\lambda^{k+1}} \]  

(2)

or, equivalently by the relation

\[ B_{2k+1}(n) = tr\hat{H}_n^k. \]

They depend also on the parameters \( a_1, a_2, a_3 \) or more precisely on their symmetric functions of \( g_1, g_2, g_3 \) defined by the relation

\[ 4(z - a_1)(z - a_2)(z - a_3) = 4z^3 - g_1z^2 - g_2z - g_3 \]
INTRODUCTION

Thus, strictly speaking, we should write $B_{2k+1}(n; g_1, g_2, g_3)$ rather than $B_{2k+1}(n)$. We will use both notations depending on the context.

**Theorem 2** (Chapter 3, Th. 3.2). The trace of the $k$-th power of the quantum Euler top Hamiltonian in the representation with spin $s = n$ is a polynomial in $n$ of degree $2k + 1$, anti-symmetric with respect to $n = -\frac{1}{2}$, and whose coefficients are polynomials in $g_1, g_2, g_3$ with rational coefficients. When $g_2 = g_3 = 0$ these polynomials reduce to the classical odd Bernoulli polynomials up to a constant factor

$$B_{2k+1}(n; g_1, 0, 0) = \frac{g_1^k}{(2k + 1)2^{2k-1}} B_{2k+1}(n + 1) \quad (3)$$

When $g_1 = 0$ we have the reduced elliptic Bernoulli polynomials denoted as $B_{2k+1}^W(n, g_2, g_3)$ (W is for Weierstrass) $B_1^W = 2n + 1, \quad B_3^W = 0$,

$$B_5^W = \frac{1}{60} g_2 n(n + 1)(2n - 1)(2n + 1)(2n + 3),$$

$$B_7^W = \frac{1}{280} g_3 n(n + 1)(2n - 3)(2n - 1)(2n + 1)(2n + 3)(2n + 5),$$

$$B_9^W = \frac{1}{1680} g_2^2 n(n + 1)(2n - 1)(2n + 1)(2n + 3)(4n^4 + 8n^3 - 11n^2 - 15n + 21).$$

The coefficients of $B_{2k+1}$ are homogeneous polynomials in $g_2, g_3$ of weight $2k$ if we assume as usual that the weights of $g_2$ and $g_3$ are 4 and 6 respectively (in other words, they are modular forms of weight $2k$, see e.g. [44]).

In section 3.3 we discuss some effective ways to compute the elliptic Bernoulli polynomials, investigate their properties, and then apply them to the calculation of the coefficients of the Lamé spectral polynomials. Indeed they enable us to express the coefficients $b_k$ of the Lamé spectral polynomials $R_{2n+1}(E) = E^{2n+1} + b_1 E^{2n} + b_2 E^{2n-1} + \cdots + b_{2n}$ as functions of $n$ (and thus for all values of parameter $n$).

**Theorem 3** (Chapter 3, Th. 3.6). The coefficient $b_k = b_k(n)$ of the Lamé spectral polynomial $R_{2n+1}(E) = \prod_{m=0}^{2n}(E - E_m(n)) = E^{2n+1} + b_1 E^{2n} + b_2 E^{2n-1} + \cdots + b_{2n+1}$ is a polynomial in $n, g_2, g_3$ with rational coefficients. It can be computed using the reduced elliptic Bernoulli polynomials and the following recurrence relation with $b_0 = 1$

$$b_k = -\frac{1}{k} \sum_{j=1}^{k} B_{2j+1}^W(n) b_{k-j}.$$
The first coefficients are $b_1 = 0$, $b_2 = -\frac{9}{120}n(n+1)(2n-1)(2n+1)(2n+3)$, 
\[ b_3 = -\frac{9}{840}n(n+1)(2n-3)(2n-1)(2n+1)(2n+3)(2n+5), \]
\[ b_4 = \frac{9^2}{201600}n(n-1)(n+1)(2n-1)(2n+1)(2n+3)(56n^4 + 76n^3 - 94n^2 + 201n + 630) \]

In chapter 4 we show how both elliptic Faulhaber polynomials and elliptic Bernoulli polynomials can be used in the calculation of the Lamé densities of states. The density of states $\rho_n(E)$ is one of the most important spectral characteristics of the Schrödinger operator with periodic potential (see e.g. [56]). From the finite-gap theory (see [14, 15, 46, 53, 54]) it is known that for integer $n$ the density of states for the non-singular real periodic version of the Lamé operator $L_n = -\frac{d^2}{dx^2} + n(n+1)\varphi(x+\omega)$ has the form

\[ \rho_n(E) = \frac{1}{2\pi} \frac{P_n(E)}{\sqrt{R_{2n+1}(E)}} \]  

for a certain polynomial $P_n$ and the Lamé spectral polynomial $R_{2n+1}$. On the other hand, the high-energy asymptotics for $\rho$ can be given in terms of the KdV integrals (see [54]), which in this case are our (reduced) elliptic Faulhaber polynomials

\[ \rho_n(E) = \frac{1}{2\pi\sqrt{E}} \left[ 1 + \frac{1}{T} \sum_{k=1}^{\infty} \frac{2k-1}{2^{2k-1}} \frac{F_k^W(\lambda)}{E^k} \right], \quad \lambda = \frac{n(n+1)}{2}. \]

Let $\bar{\varphi} = -\frac{1}{\omega}$ be the average of the Weierstrass $\varphi$-function over the real period $T = 2\omega$. We will use also the standard notation $[x]$ for the integer part of a real number $x$.

**Theorem 4** (Chapter 4, Th 4.1). The coefficient $a_k = a_k(n)$ in the numerator $P_n(E) = E^n + a_1E^{n-1} + a_2E^{n-2} + \cdots + a_n$ of the Lamé density of states (4) is a polynomial in $n$ of degree $\left\lfloor \frac{5k}{2} \right\rfloor$. Its coefficients are polynomials with rational coefficients of $g_2, g_3$ and $\bar{\varphi}$, homogeneous of weight $2k$, where the weights of $\bar{\varphi}, g_2, g_3$ are 2, 4, 6 respectively.

The explicit form of the first coefficients $a_k(n)$ is

\[ a_1 = \frac{n(n+1)}{2} \bar{\varphi}, \]
\[
a_2 = -\frac{g_2}{480}(n-1)n(n+1)(6 + 25n + 16n^2),
\]
\[
a_3 = (n-2)(n-1)n(n+1)(-\frac{g_3}{3360}(45+243n+247n^2+64n^3) - \frac{g_2\delta}{960}n(n+1)(27+16n))
\]

This provides us with an alternative method (to the classical one recently elaborated further by Belokolos - Enolski and Takemura [8, 58, 59]) to find the Lamé densities of states, giving us some information for all values of \( n \).

In the second part of the thesis we consider the following continued fractions which we call \( \alpha \)-fractions

\[
\phi = b_0 + \frac{\lambda - \alpha_1}{b_1 + \frac{\lambda - \alpha_2}{b_2 + \cdots}} = [b_0, b_1, \ldots]_\alpha
\]

(5)

Here \( \alpha = (\alpha_i), \alpha_i \in \mathbb{C} \) is a given sequence, \( b_i \) are arbitrary complex numbers, and \( \lambda \) is a formal parameter. We will examine a special case of \( N \)-periodic \( \alpha \)-fractions

\[
\phi = [b_0, b_1, b_2, \ldots, b_N]_\alpha,
\]

(6)

when the sequences \( \alpha_i \) and \( b_i \) are periodic with period \( N \)

\[\alpha_{i+N} = \alpha_i, \ b_{i+N} = b_i \text{ for all } i \geq 1.\]

Such fractions appear in the theory of integrable systems, in particular in the theory of the periodic dressing chain [65].

In the particular case when \( b_N = b_0 \), we have \( \phi = [b_0, b_1, \ldots, b_{N-1}]_\alpha \), which will be called a purely \( N \)-periodic \( \alpha \)-fraction.

Because of periodicity we can write formally (6) as

\[
\phi = b_0 + \frac{\lambda - \alpha_1}{b_1 + \frac{\lambda - \alpha_2}{b_2 + \cdots} + \frac{\lambda - \alpha_{N}}{b_{N-1} + \cdots + b_N}}
\]

which implies a quadratic relation

\[
A(\lambda)\phi^2 + 2B(\lambda)\phi + C(\lambda) = 0,
\]

(7)
where $A, B, C$ are certain polynomials in $\lambda$ with coefficients polynomially depending on $b$. Thus to any periodic $\alpha$-fraction (6), there corresponds an algebraic function

$$\phi(\lambda) = \frac{-B(\lambda) + \sqrt{R(\lambda)}}{A(\lambda)}, \quad (8)$$

where

$$R(\lambda) = B(\lambda)^2 - A(\lambda)C(\lambda) \quad (9)$$

is the discriminant of (7). In that case we will say that (6) is a *periodic $\alpha$-fraction expansion* of the algebraic function (8) from the hyperelliptic extension $\mathbb{C}(\lambda, \sqrt{R(\lambda)})$ of the field of rational functions $\mathbb{C}(\lambda)$

We will discuss the following three main questions

**Question 1.** Which algebraic functions (8) admit $N$-periodic $\alpha$-fraction expansions?

**Question 2.** How many such expansions exist for a given algebraic function (8) and how can we find them?

**Question 3.** What is the geometry of the set of functions (8) from the given hyperelliptic extension $\mathbb{C}(\lambda, \sqrt{R(\lambda)})$ (i.e. with fixed $R$) which admit periodic $\alpha$-fraction expansions?

The answers depend on the parity of $N$. In the thesis we will restrict our study to the case of odd period $N = 2g + 1$. We will also assume that all the parameters $\alpha_i$ are distinct.

We should note that when $N$, which is also the degree of $R(\lambda)$, is even, one can consider the usual continued fraction expansions

$$\varphi(\lambda) = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}}$$

where $a_i$ are some polynomials of $\lambda$. Abel and Chebyshev discovered a remarkable relation between the periodicity of the expansion of $\sqrt{R(\lambda)}$, where $R(\lambda)$ is a polynomial of even degree, and the evaluation in elementary functions of "pseudo-elliptic" integrals of the form

$$I = \int \frac{f(x)dx}{\sqrt{R(x)}}$$
INTRODUCTION

(see [1],[11], [12], and a nicely written paper by van der Poorten and Tran [62] for more details) These expansions are more natural, but can not be used in the odd degree case.

Let us introduce the polynomial

$$\mathfrak{A}(\lambda) = \prod_{i=1}^{N} (\lambda - \alpha_i)$$

and call a polynomial $R$ of degree $N$ $\alpha$-admissible if

$$R(\lambda) = S^2(\lambda) + \mathfrak{A}(\lambda)$$

for some polynomial $S(\lambda)$ of degree $g$ or less, where as before $N = 2g + 1$.

We call a polynomial monic if its highest coefficient is 1 and anti-monic if it is equal to $-1$. Note that the $\alpha$-admissible polynomials $R$ are automatically monic.

The answers to the questions 1 and 2 are summarised by the following

**Theorem 5** (Chapter 5, Th 5.1). The algebraic functions $\phi(\lambda)$ admitting an $N$-periodic $\alpha$-fraction expansion have the form (7, 8) with the polynomials $A, B, C$ satisfying the following conditions.

1. $\deg B \leq g$, $A(\lambda)$ and $C(\lambda)$ are monic and anti-monic polynomials of degree $g$ and $g + 1$ respectively

2. the discriminant $R(\lambda) = B^2 - AC$ is $\alpha$-admissible.

Conversely, for an open dense subset of such triples $(A, B, C)$ the corresponding function (8) has exactly two $N$-periodic $\alpha$-fraction expansions. The corresponding $b_i$ are rational functions of both the coefficients of $A, B, C$ and the parameters $\alpha_i$, and can be found by an effective matrix factorisation procedure.

In chapter 6 we consider the purely $N$-periodic $\alpha$-fractions and show that in addition to the two conditions of Theorem 5, we have the requirement

$$C(\alpha_N) = 0,$$
under which the purely periodic \( \alpha \)-fraction expansion is generically unique.

We will call \((A, B, C)\) satisfying the conditions (1), (2) of the above theorem the \( \alpha \)-truples and note that these conditions are invariant under any permutation of the parameters \( \alpha \).

In chapter 7, we show that there is in fact a natural birational action of the direct product \( G = \mathbb{Z}_2 \times S_N \) on the set of periodic continued \( \alpha \)-fraction expansions, where the generator \( t \) of \( \mathbb{Z}_2 \) is acting simply by swapping the two different \( \alpha \)-fraction expansions given by Theorem 5. Our next result describes this action explicitly.

Let us consider the following permutation \( \pi \in S_N \) which reverses the order \( \alpha_1, \alpha_2, \ldots, \alpha_N \) to \( \alpha_N, \ldots, \alpha_2, \alpha_1 \); and the involutions \( \sigma_k \) which swap \( \alpha_k \) and \( \alpha_{k+1} \) where \( k = 1, \ldots, N - 1 \).

Define the action of \( \sigma_k \) on \( b = (b_i), i = 0, \ldots, N \) with \( b_k \neq 0 \) as follows

\[
\tilde{b}_{k-1} = b_{k-1} + \frac{\alpha_{k+1} - \alpha_k}{b_k}, \quad \tilde{b}_{k+1} = b_{k+1} - \frac{\alpha_{k+1} - \alpha_k}{b_k},
\]

(12)

the rest of \( b \) remain the same (cf Adler’s formulae [3] for the dressing chain). This determines the action of the symmetric group \( S_N \) since the \( \sigma_k \) generate it. To describe the action of \( \mathbb{Z}_2 \) it is enough to describe the action of the involution \( t \pi \in G \), which turns out to be quite simple

\[
\tilde{b}_j = -b_{N-j}, \quad j = 1, \ldots, N - 1, \quad \tilde{b}_0 = b_0 - b_N, \quad \tilde{b}_N = -b_N.
\]

(13)

**Theorem 6** (Chapter 7, Th 7.1). The formulae (12) and (13) define a birational action of the group \( G = \mathbb{Z}_2 \times S_N \) on the set of \( N \)-periodic \( \alpha \)-fractions. Its orbits consist of all \( 2N! \) possible periodic \( \alpha \)-fraction expansions for a given \( \alpha \)-triple \((A, B, C)\) and any permutation of the parameters \( \alpha \).

In the purely periodic case the symmetric group is broken down to \( S_{N-1} \) generated by \( \sigma_k \) with \( k = 1, \ldots, N - 2 \) given by (12).

In chapter 8 we discuss the geometry of the set of elements from the hyperelliptic extension field \( \mathbb{C}(\lambda, \sqrt{R(\lambda)}) \) which have a \( N \)-periodic \( \alpha \)-fraction expansion. The \( \alpha \)-admissible polynomial \( R(\lambda) \) is now fixed with distinct roots.

Let us consider the hyperelliptic curve \( \Gamma_R \) given by the equation

\[
\mu^2 = R(\lambda).
\]

(14)
The curve $\Gamma_R$ consists of the affine part $\Gamma_R^{aff}$, corresponding to the "finite" solutions of (14), and the "infinity" point, which we will denote as $P_\infty$. Since the roots of $R$ are distinct, the curve $\Gamma_R$ is non-singular and has genus $g$. Consider $g$ points $P_1, \ldots, P_g$ of $\Gamma_R^{aff}$ and call the corresponding divisor $D = P_1 + \cdots + P_g$ non-special if

$$P_i \neq \tau(P_j)$$

for any $i \neq j$, where $\tau$ is the hyperelliptic involution $\tau(\mu, \lambda) = (-\mu, \lambda)$.

Non-special divisors have the property that the linear space $L(D)$ of all meromorphic functions on $\Gamma_R$, having their poles at $D$ of order less than or equal to 1, has dimension 1, which means that it consists only of constant functions. The corresponding linear space $L(D + P_\infty)$ has dimension 2, so there exists a non-constant function $f \in L(D + P_\infty)$ with an additional pole at infinity. These functions are in a way "least singular" among the "generic" meromorphic functions on $\Gamma_R$ in the sense that any such function can not have less than $g + 1$ poles (see [27]).

We define the affine Jacobi variety $J(\Gamma_R)^{aff}$ as the set of positive non-special divisors $D = P_1 + \cdots + P_g$, $P_1, \ldots, P_g \in \Gamma_R^{aff}$.

Let $M_\alpha^g$ be the affine variety of $\alpha$-triples of polynomials $(A, B, C)$ with given discriminant $R(\lambda)$, and $\tilde{M}_\alpha^g$ be its subvariety given by the additional condition (11).

**Theorem 7** (Chapter 8, Th 8.1). There exists a bijection between the set $M_\alpha^g$ and the extended affine Jacobi variety $J(\Gamma_R)^{aff} \times \mathbb{C}$. The corresponding algebraic functions (8) can be characterised as meromorphic functions $\phi \in L(D + P_\infty)$ on $\Gamma_R$ with the non-special pole divisors $D + P_\infty$ and with $\phi \sim \sqrt{\lambda}$ at infinity.

In the purely periodic case under the assumption that $R(\alpha_N) \neq 0$, there exists a natural $2:1$ map from the set $\tilde{M}_\alpha^g$ to $J(\Gamma_R)^{aff}$. The corresponding $\phi$ from $L(D + P_\infty)$ are fixed by the condition that one of the two values of $\phi(\alpha_N)$ is zero.

The proof is based on the classical description of the Jacobi variety due to Jacobi himself [36] (see also Mumford [52]).

In the last chapter we give a short summary of the results of the thesis, and discuss some open problems.

The main results of this thesis are published in [28, 29, 30, 31].
Part I

Elliptic Faulhaber and Bernoulli polynomials
Chapter 1

Lamé equation

We start with a brief account of the theory of the elliptic functions following Weierstrass' approach.

1.1 Elliptic functions

The elliptic functions as developed by Jacobi, Weierstrass, Eisenstein and others are one of the crowning achievements of 19th century mathematics and are widely applied in physics and engineering. The classical way to understand them is to consider the inverse of an elliptic integral

\[ \int R[x, p(x)^{\frac{1}{3}}] \, dx, \]

where \( p(x) \) is a polynomial of degree 3 or 4 and \( R(x, y) \) is a rational function of 2 variables.

The simultaneous development of the properties of analytic functions of a complex variable by Gauss and Cauchy, in particular their characterisation by their zeros and poles, resulted in a new way of looking at elliptic functions. Elliptic functions have since been defined as analytic functions with double periodicity and with no singularities other than their poles. There are two classical sets of basic elliptic functions due to Jacobi and Weierstrass. We give below a short summary of the properties of the Weierstrass functions used in the thesis, following [5].
First it can be proved that the series

\[ \sum_{m,n} \frac{1}{(z - 2mw - 2nw')^3}, \tag{1.1} \]

where \( \omega \) and \( \omega' \) are any two complex numbers whose ratio is not real, and \( \sum_{m,n} \) denotes the summation over all pairs of integer values \( m \) and \( n \), converges absolutely and uniformly in each bounded region of the \( z \)-plane if the finite number of terms that become infinite there are removed. Therefore the sum of the series (1.1) is a meromorphic function whose only poles (which have order three) are the points \( 2mw + 2nw' \).

Let

\[ Q(z) = -2 \sum_{m,n} \frac{1}{(z - 2mw - 2nw')^3} \]

We can easily show that the function \( Q(z) \) has periods \( 2\omega \) and \( 2\omega' \), and that it is an odd function.

Now we introduce the Weierstrass \( \wp \) function

\[ \wp(z) = \frac{1}{z^2} + \int_0^z \left[ Q(u) + \frac{2}{u^3} \right] du, \]

assuming that the path of integration does not go through vertices in the period network different from \( u = 0 \). Thus,

\[ \wp'(z) = Q(z). \tag{1.2} \]

Termwise integration gives us

\[ \wp(z) = \frac{1}{z^2} + \sum_{m,n} \left[ \frac{1}{(z - 2mw - 2nw')^2} - \frac{1}{(2mw + 2nw')^2} \right] \tag{1.3} \]

where \( \sum_{m,n} \) denotes the summation over all pairs of integer values \( m \) and \( n \) except \( m = n = 0 \).

Since \( Q(z) \) is an odd function, it follows that \( \wp(z) \) is an even function. Moreover, since \( Q(z) \) has period \( 2\omega \), we have from (1.2)

\[ \wp'(z + 2\omega) = \wp'(z), \]
and hence 

$$\varphi(z + 2\omega) = \varphi(z) + c,$$

where $c$ is a constant. Since from (1.3) the only poles of $\varphi(z)$ are the points $2m\omega + 2n\omega'$, $\varphi(z)$ is finite at the point $\omega$. For $z = -\omega$ we have

$$\varphi(\omega) = \varphi(-\omega) + c,$$

and since $\varphi(z)$ is even, $c = 0$. Similarly

$$\varphi(z + 2\omega') = \varphi(z)$$

We see that the Weierstrass function $\varphi(z)$ is an elliptic function (of second order, since it has a total of one pole of order 2 in each period parallelogram.)

The function $\varphi(z)$ satisfies the differential equation

$$\varphi(z)^2 = 4\varphi(z)^3 - g_2\varphi(z) - g_3,$$  \hspace{1cm} (1.4)

where $g_2 = 60 \sum'_{m,n} \Omega_{m,n}^{-4}$, $g_3 = 140 \sum'_{m,n} \Omega_{m,n}^{-6}$ with $\Omega_{m,n} = 2m\omega + 2n\omega'$ and the summation being over all pairs $m, n$ except $m = n = 0$. The coefficients $g_2$ and $g_3$ are called the elliptic invariants of $\varphi(z)$.

To specify $\varphi(z)$ completely, its half-periods $\omega$ and $\omega'$ or elliptic invariants $g_2$ and $g_3$ must be specified. These two cases are denoted $\varphi(z|\omega, \omega')$ and $\varphi(z, g_2, g_3)$, respectively. But for brevity we shall use $\varphi(z)$.

Let $\omega_1 = \omega$, $\omega_3 = \omega'$ and $\omega_2 = \omega + \omega'$ be the half periods then $\varphi(\omega_1), \varphi(\omega_2)$ and $\varphi(\omega_3)$ are the roots of the cubic equation $4\varphi(z)^3 - g_2\varphi(z) - g_3 = 0$. Indeed, since $\varphi'(z)$ is odd,

$$\varphi'(\omega_1) = -\varphi'(-\omega_1)$$

On the other hand,

$$-\varphi'(-\omega_1) = -\varphi'(2\omega_1 - \omega_1) = -\varphi'(\omega_1)$$

since $\varphi'(z)$ has period $2\omega$ and so

$$\varphi'(\omega_1) = 0.$$
Let \( \varphi(\omega_1) = e_1, \varphi(\omega_2) = e_2 \) and \( \varphi(\omega_3) = e_3 \), then the differential equation (1.4) can be written

\[
\varphi(z)^2 = 4(\varphi(z) - e_1)(\varphi(z) - e_2)(\varphi(z) - e_3).
\] (1.5)

Using the formulae connecting the roots of equations with their coefficients, we obtain the following relations

\[
e_1 + e_2 + e_3 = 0,
\]

\[
e_1e_2 + e_1e_3 + e_2e_3 = -\frac{1}{4}g_2,
\]

\[
e_1e_2e_3 = \frac{1}{4}g_3.
\]

When \( g_2 \) and \( g_3 \) are real and the discriminant \( \Delta = g_2^2 - 27g_3^2 \) is positive, then \( e_1, e_2 \) and \( e_3 \) are real. Choosing them such that \( e_1 > e_2 > e_3 \) corresponds to choosing a real period \( 2\omega_1 \) and a purely imaginary period \( 2\omega_3 \).

The function \( \varphi(z) \) tends to the trigonometric and the hyperbolic functions in the limit cases when the discriminant \( \Delta = 0 \). Namely,

\[
\varphi(z, g_2, g_3) \rightarrow e_3 + \frac{(e_1 - e_3)}{\sin^2(\sqrt{e_1 - e_3})}
\]

when \( e_2 \rightarrow e_3 \) and

\[
\varphi(z, g_2, g_3) \rightarrow e_1 + \frac{e_1 - e_3}{\sinh^2(\sqrt{e_1 - e_3})}
\]

when \( e_2 \rightarrow e_1 \).

The Weierstrass zeta function \( \zeta(z, g_2, g_3) \) (or \( \zeta(z) \)) is defined by

\[
\zeta(z) = \frac{1}{z} - \int_0^z [\varphi(u) - \frac{1}{u^2}]du,
\]

so that

\[
\frac{d}{dz}\zeta(z; g_2, g_3) = -\varphi(z, g_2, g_3)
\]

Here the path of integration must not go through vertices of the period network other than the point \( u = 0 \). It follows from the above definition that \( \zeta(z) \) is an odd function of \( z \).
We get for $\zeta(z)$ the representation

$$
\zeta(z) = \frac{1}{z} + \sum_{m,n} \left[ \frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{\Omega_{m,n}^2} \right],
$$

which shows that the function $\zeta(z)$ is analytic throughout the whole plane except at its poles $\Omega_{m,n}$. Integrating the equation $\varphi(z + 2\omega) = \varphi(z)$ gives

$$
\zeta(z + 2\omega) = \zeta(z) + 2\eta,
$$

where $2\eta$ is a constant of integration. Putting $z = -\omega$ and taking into account that $\zeta(z)$ is an odd function of $z$, we have

$$
\eta = \zeta(\omega)
$$

Similarly, $\zeta(z + 2\omega') = \zeta(z) + 2\eta'$ and $\eta' = \zeta(\omega')$.

1.2 Lamé functions and Lamé spectrum

The Lamé equation

$$
-\frac{d^2y}{dz^2} + n(n+1)\varphi(z)y = Ey
$$

was first obtained by Gabriel Lamé who applied the method of separation of variables to Laplace's equation $\Delta \varphi = 0$ in the ellipsoidal coordinate system (see e.g. Whittaker and Watson [68]).

Since the coefficients of the Lamé equation are doubly periodic functions, the question naturally arises about the doubly-periodic solutions.

The main result, going back to Hermite and Halphen (see [6], [16], [68]), is that for such doubly periodic solutions to exist, $n$ must be either an integer or a half-integer. We restrict ourselves to $n$ being an integer, which we assume to be non-negative. The corresponding $E$ must take one of a set of $2n + 1$ characteristic values denoted $E_n^m$, $m = 0, \ldots, 2n$. The associated doubly-periodic solutions are called Lamé functions of degree $n$, and denoted $E_n^m(z)$. In fact, their squares are polynomials in $\varphi(z)$ of degree $n$.

The Lamé functions $E_n^m(z)$, $m = 0, \ldots, 2n$, take one of the following four forms (see [32]).
1. When \( n \) is even, \( \varepsilon_n^m(z) = Y_0(\varphi(z)) \) where \( Y_0(\varphi(z)) \) is a polynomial in \( \varphi(z) \) of degree \( \frac{n}{2} \) satisfying the differential equation

\[
(4\varphi^3 - g_2\varphi - g_3) \frac{d^2Y_0}{d\varphi^2} + (6\varphi^2 - \frac{1}{2}g_2) \frac{dY_0}{d\varphi} = [n(n + 1)\varphi - E]Y_0, \tag{1.8}
\]

or \( \varepsilon_n^m(z) = Y_1(\varphi(z)) \sqrt{(\varphi(z) - \epsilon_\beta)(\varphi(z) - \epsilon_\gamma)} \), where \( Y_1(\varphi(z)) \) is a polynomial in \( \varphi(z) \) of degree \( \frac{n-2}{2} \) satisfying

\[
(4\varphi^3 - g_2\varphi - g_3) \frac{d^2Y_1}{d\varphi^2} + (14\varphi^2 - 4\epsilon_\alpha\varphi - 4\epsilon_\alpha^2 - \frac{1}{2}g_2) \frac{dY_1}{d\varphi} = [(n-2)(n+3)\varphi - E + 3\epsilon_\alpha]Y_1, \tag{1.9}
\]

2. When \( n \) is odd, \( \varepsilon_n^m(z) = Y_2(\varphi(z)) \sqrt{(\varphi(z) - \epsilon_1)(\varphi(z) - \epsilon_2)(\varphi(z) - \epsilon_3)} \), where \( Y_2(\varphi(z)) \) is a polynomial in \( \varphi(z) \) of degree \( \frac{n-3}{2} \) \( (n \geq 3) \) satisfying

\[
(4\varphi^3 - g_2\varphi - g_3) \frac{d^2Y_2}{d\varphi^2} + 3(6\varphi^2 - \frac{1}{2}g_2) \frac{dY_2}{d\varphi} = [(n-3)(n+4)\varphi - E]Y_2,
\]

or \( \varepsilon_n^m(z) = Y_3(\varphi(z)) \sqrt{\varphi(z) - \epsilon_\alpha} \), where \( Y_3(\varphi(z)) \) is a polynomial in \( \varphi(z) \) of degree \( \frac{n-1}{2} \) satisfying

\[
(4\varphi^3 - g_2\varphi - g_3) \frac{d^2Y_3}{d\varphi^2} + (10\varphi^2 + 4\epsilon_\alpha\varphi + 4\epsilon_\alpha^2 - \frac{3}{2}g_2) \frac{dY_3}{d\varphi} = [(n-1)(n+2)\varphi - E - \epsilon_\alpha]Y_3.
\]

In the above, \( \epsilon_\alpha, \epsilon_\beta, \epsilon_\gamma \) are the roots of the cubic equation \( 4z^3 - g_2z - g_3 = 0 \)

The classical approach for finding the Lamé functions \( \varepsilon_n^m(z) \) is the following. Let us take for simplicity the case of an even \( n \) and the solution of the form \( \varepsilon_n^m(z) = Y_0(\varphi(z)) \). We substitute \( Y_0 \) into equation (1.8) The highest degree term \( \varphi^{3+1} \) disappears. Equating like-powers of \( \varphi \) leads to solving a system of \( \frac{n}{2} + 1 \) simultaneous equations which are linear with respect to the \( \frac{n}{2} \) unknown coefficients of \( Y_0 \). To find the values \( \varepsilon_n^m \), we have to solve an algebraic equation in \( E \) obtained by eliminating the unknown coefficients of \( Y_0 \).
For example, when \( n = 2 \) we substitute \( Y_0 = \varphi + \alpha \) in (1.8) to obtain a system of two simultaneous equations

\[
\begin{align*}
E - 6\alpha &= 0 \\
\alpha E &= \frac{1}{2} g_2
\end{align*}
\]

We therefore have two values for \( E \) determined by \( E^2 = 3g_2 \) and two corresponding Lamé functions

\[
E_2^1(z) = \varphi(z) + \frac{\sqrt{3g_2}}{6}
\]

and

\[
E_2^2(z) = \varphi(z) - \frac{\sqrt{3g_2}}{6}.
\]

Solutions of the form \( Y_1(\varphi(z))\sqrt{(\varphi(z) - e_1)(\varphi(z) - e_2)} \) are simply

\[
E_2^3(z) = \sqrt{(\varphi(z) - e_2)(\varphi(z) - e_3)},
\]

\[
E_2^4(z) = \sqrt{(\varphi(z) - e_1)(\varphi(z) - e_3)},
\]

and

\[
E_2^5(z) = \sqrt{(\varphi(z) - e_1)(\varphi(z) - e_2)},
\]

since the degree of \( Y_1(\varphi) \) in this case is zero. Substituting \( Y_1(\varphi(z)) = 1 \) in (1.9) gives

\[
E - 3e_\alpha = 0.
\]

The corresponding characteristic values are thus \( E_2^3 = 3e_1, E_2^4 = 3e_2 \) and \( E_2^5 = 3e_3 \) respectively, so the Lamé spectral polynomial in this case can be written as

\[
R_5(E) = (E^2 - 3g_2)(E - 3e_1)(E - 3e_2)(E - 3e_3) = (E^2 - 3g_2)(E^3 - \frac{9}{4}g_2E - \frac{27}{4}g_3)
\]

We would like to mention that there is a beautiful electrostatic interpretation of the zeroes of the Lamé functions discovered by Stieltjes (see [68]).

The problem is that, as \( n \) becomes bigger, it becomes increasingly more and more difficult to compute the eigenvalues \( E_n^m \) and the corresponding Lamé functions.

The discovery of quantum mechanics in the 20th century brought a new prospect to the old theory of the Lamé equation. Assume that \( \omega \) is real and \( \omega' \) is purely
imaginary then the potential \( s(s + 1)\varphi(x + \omega') \) is a smooth, real-valued, periodic function of real \( x \), so that the Lamé equation

\[
-\frac{d^2\psi}{dx^2} + s(s + 1)\varphi(x + \omega')\psi = E\psi
\]

is an example of a Schrödinger equation with a periodic potential, also known as Hill's equation.

Hill's equations are usually studied by applying Floquet theory. Consider the one-dimensional Schrödinger operator

\[
L\psi = -\psi'' + u(x)\psi = E\psi,
\]

where \( u(x) \) is a periodic potential \( u(x + T) = u(x) \). The eigenfunctions satisfying the condition

\[
\psi(z + T) = \mu\psi(z)
\]

are called Bloch (or Floquet) eigenfunctions. Note that any two eigenfunctions \( \psi_1 \) and \( \psi_2 \) with the same value of \( E \) satisfy the relation

\[
\begin{pmatrix}
\psi_1(z + T) \\
\psi_2(z + T)
\end{pmatrix} = M(E) \begin{pmatrix}
\psi_1(z) \\
\psi_2(z)
\end{pmatrix}
\]

for some matrix \( M(E) \) called the monodromy matrix. The eigenvalues \( \mu \) are called the Floquet multipliers of the monodromy matrix \( M \). They satisfy the characteristic equation

\[
\mu^2 - t(E)\mu + 1 = 0,
\]

where \( t(E) = tr M(E) \) is the trace of the matrix \( M \). Thus

\[
\mu = \frac{t(E) \pm \sqrt{t(E)^2 - 4}}{2}.
\]

The stability set (spectrum of the operator \( L \)) is characterised as the set of all \( E \) such that the modulus of the Floquet multipliers \( \mu(E) \) is one. This happens when
$t(E) < 2$ so $\mu = \exp(\pm \gamma p(E) T)$ The corresponding quantity $p(E)$ is called the *quasimomentum*. For a general smooth periodic potential the spectrum of $L$ consists of an infinite set of intervals (or bands) $[E_{2m}, E_{2m+1}], m = 0, 1, 2 \ldots$ separated by the forbidden zones (gaps).

The Lamé operator

$$L_n = -\frac{d^2}{dx^2} + n(n+1)p(x + \omega)$$

for integer $n$ has exactly $n$ gaps in its spectrum [34]. Namely, the spectrum of the operator $L_n$ is $[E_0, E_1] \cup [E_2, E_3] \cup \cdots \cup [E_{2n-2}, E_{2n-1}] \cup [E_{2n}, \infty)$, where the $2n+1$ band edges $E_m = E_m(n)$ are exactly the $2n+1$ characteristic values $E_n^m$ of the Lamé equation, and the corresponding periodic and anti-periodic Bloch solutions are the Lamé functions.

We have mentioned already the classical approach of how to find the Lamé spectral polynomials, due to Hermite and Halphen. News ideas came from the finite-gap theory created in the 1970s by Novikov and Dubrovin, Its and Matveev, McKean and van Moerbeke [14, 15, 46, 53, 54]. This remarkable theory combines the methods of spectral theory with algebraic and symplectic geometry. In the recent papers of Belokolos and Enolski [7] and Takemura [57, 58], this theory was applied to the investigation of the Lamé equation. We are going to develop a different approach based on the elliptic Bernoulli polynomials.
Chapter 2

Elliptic Faulhaber polynomials

Before introducing the elliptic Faulhaber polynomials we review the properties of the KdV equation and its conserved densities as these will play an essential role in our construction.

2.1 KdV equation and its integrals

Investigations of solitary waves began with their discovery by John Scott Russell in 1834 at the Union Canal at Hermiston, Edinburgh, Scotland. In 1871 Boussinesq published the first mathematical theory to support Scott Russell's experimental observation. In 1895 the Dutch physicist Diederik Johannes Korteweg and his student Gustav de Vries derived a shallow water wave equation (known nowadays as the KdV equation) and showed that it had a solitary wave solution similar to the one observed by Scott Russell. The KdV equation has the form

\[ u_t - 6uu_x + u_{xxx} = 0 \]  \hspace{1cm} (2.1)

The variable \( t \) denotes time, the variable \( x \) is the space coordinate along the canal, while the unknown function \( u = u(x,t) \) represents the elevation of the fluid above the bottom of the canal.

To find explicitly the simplest solitary wave solutions of the KdV, one looks for traveling wave solutions \( u(x,t) \) of the form

\[ u(x,t) = f(x - ct) \]
The KdV equation then reduces to the ordinary differential equation

\[ f''' = 6ff' + cf' \]

Imposing the condition of fast decay at \( \pm \infty \) one finds

\[ f(y) = \frac{1}{2}c \text{sech}^2 \left( \frac{1}{2} \sqrt{c} y \right) \]

for any positive parameter \( c \)

The corresponding family of solitary wave solutions of the KdV is given by

\[ u(x, t) = \frac{1}{2}c \text{sech}^2 \left( \frac{1}{2} \sqrt{c} (x - ct) + x_0 \right), \]

where the parameter \( x_0 \) (called phase) is arbitrary.

In 1965 Zabusky and Kruskal found numerically solitary wave solutions of the KdV equation and investigated their interactions. They named these solitary waves solitons. The most surprising phenomenon they observed was that, after interaction, the solitons reappear with identical profile and speed as if no collision had taken place. The only parameter that is changed is the phase.

Two years later, Miura, Gardner and Kruskal [48] showed that the peculiarity of the KdV equation lies in the existence of an infinite number of conservation laws. Consider an evolution equation

\[ u_t = K(u) \]

for a function \( u = u(x, t) \), where \( K \) is a function of \( u, u_x, u_{xx} \). A functional

\[ I[u] = \int T(u, u_x, \ldots) dx \]

is called an integral (or conserved quantity) of this equation if

\[ \frac{dI}{dt} \equiv 0 \]

for any solution \( u(x, t) \). The function \( T \) is then called the conserved density.

The KdV integrals

\[ I_k[u] = \int T_k(u_0, u_1, \ldots, u_{k-2}) dx \]  \hspace{1cm} (2.2)
ELLiptic Faulhaber Polynomials

play a fundamental role in the theory of the one-dimensional Schrödinger operator $L = -\frac{d^2}{dx^2} + u(x)$. They appear in the asymptotic expansion of the trace of the resolvent $R = (\mu I - L)^{-1}$ (see Gelfand-Dikii [21]) or, in the periodic case, of the density of states or quasi-momentum (see Novikov et al [54] and chapter 4).

In this section we will discuss the algebraic properties of the KdV conserved densities mainly following the classical paper by Miura, Gardner, Kruskal and Zabusky [49]. We mention that our $u$ differs from the variable in that paper by a constant factor, so the arithmetic of the coefficients is slightly different.

The KdV conserved densities $T_k = T_k(u_0, u_1, \ldots, u_{k-2})$ are polynomials in $u = u_0$ and its derivatives $u_x = u_1, u_{xx} = u_2, \ldots$ up to order $k - 2$ In general they are defined up to a total derivative, which does not affect the integrals. To eliminate this freedom, let us introduce the notion of canonical form [49]. Consider the following grading. For the monomial $u_0^{m_0} u_1^{m_1} u_k^{m_k}$ we define its rank as

$$r = \sum_{j=0}^{j=k} (2 + j) m_j$$

A form $T(u_0, u_1, \ldots, u_r)$ is called canonical if it is rank-homogeneous and irreducible in the sense that there are no monomials $u_0^{10} u_1^{11} u_k^{1k}$ in $T$ ($k < r$) with the highest derivative $u_k$ appearing linearly (i.e., $u_k = 1$).

It is clear that if the highest derivative $u_k$ appears linearly one can always eliminate such a term by doing integration by parts (which is equivalent to adding a total derivative).

**Theorem 2.1.** (Miura, Gardner and Kruskal [49]) For any $k = 1, 2, \ldots$ the KdV equation has a non-trivial conserved density $T_k = T_k(u_0, u_1, \ldots, u_{k-2})$, which is a rank-homogeneous polynomial of rank $2k$. The canonical form of the corresponding density $T_k$ is defined uniquely up to a constant factor.

The highest derivative term $u_{k-2}$ of the corresponding $T_k$ is $u_{k-2}^2$. We normalize these densities by choosing the coefficient at this term to be 1. It turns out that this makes all the other coefficients integers, which will be important for us.
Theorem 2.2. All the coefficients of the KdV density $T_k$ in the normalized canonical form are integers

Although this fact was probably known to the experts in KdV theory, it did not play any role before. We have not found the proof in the literature although it is not difficult.

Proof. One can define the KdV densities by the following recurrence formula

$$
\sigma_{m+1} = -\sigma'_m - \sum_{k=1}^{k=m-1} \sigma_k \sigma_{m-k}
$$

(2.3)

with $\sigma_1 = u$ (see e.g. [54]). These densities are not in the canonical form but are rank-homogeneous and obviously have integer coefficients. The $\sigma_k$ for even $k$ are known to be total derivatives [54], so only the $\sigma_k$ for odd $k$ lead to non-trivial integrals. The claim is that

$$
T_k = (-1)^{k-1} \sigma^*_{2k-1},
$$

(2.4)

where the polynomial $\sigma^*_{2k-1}$ is the irreducible equivalent of $\sigma_{2k-1}$. We should only show that the coefficient at $u^2_{k-2}$ in $T_k$ is equal to 1.

From (2.3) we have $\sigma_{2k-1} = -\sigma'_{2k-2} - \sum_{i=1}^{i=2k-3} \sigma_i \sigma_{2k-2-i}$. The total derivative term $\sigma'_{2k-2}$ vanishes when we take the irreducible form $\sigma^*_{2k-1}$ and so does not contribute to the term $u^2_{k-2}$. The only contributions to $u^2_{k-2}$ come therefore from $-\sum_{i=1}^{i=2k-3} \sigma_i \sigma_{2k-2-i}$, in particular from the product of terms of degree 1 such as $u_0 u_{2k-4}, u_1 u_{2k-5}, \ldots, u_{k-2} u_{k-2}$. Since the only term of degree 1 in $\sigma_i$ is $(-1)^{i-1} u_i$ the contribution to $u^2_{k-2}$ comes from $-\sum_{i=1}^{i=2k-3} (-1)^{i-1} u_{i-1} (-1)^{2k-3-i} u_{2k-3-i} = -u^2_{k-2} - 2 \sum_{i=0}^{i=k-3} u_i u_{2k-4-i}$. After integration by parts, $u_i u_{2k-4-i}$ is reduced to $(-1)^{k-1-2} u^2_{k-2}$, so the quadratic term $u^2_{k-2}$ in $\sigma^*_{2k-1}$ has the coefficient $(-1 - 2(-1)^k \sum_{i=0}^{i=k-3} (-1)^i)$ i.e. $(-1)^{k-1}$. Indeed, if $k$ is odd, $k - 3$ is even and $\sum_{i=0}^{i=k-3} (-1)^i = +1$; the coefficient of $u^2_{k-2}$ in $\sigma^*_{2k-1}$ becomes $-1 - 2(-1)(+1) = +1 = (-1)^{k-1}$. If $k$ is even, $k - 3$ is odd and $\sum_{i=0}^{i=k-3} (-1)^i = 0$, the coefficient of $u^2_{k-2}$ in $\sigma^*_{2k-1}$ becomes $-1 - 2(+1)(0) = -1 = (-1)^{k-1}$. From (2.4), the coefficient of $u^2_{k-2}$ in $T_k$ is thus $(-1)^{k-1}(-1)^{k-1} = +1$. Theorem 2.2 is proved.
Below are the first KdV densities (in the canonical form).

\[ T_1 = u \]
\[ T_2 = u^2 \]
\[ T_3 = u_1^2 + 2u^3 \]
\[ T_4 = u_2^2 + 10uu_1^2 + 5u^4 \]
\[ T_5 = u_3^2 + 14uu_2^2 + 70u^2u_1^2 + 14u^5 \]
\[ T_6 = u_4^2 - 20u_2^3 + 18uu_3^2 - 35u_1^4 + 126u^2u_2^2 + 420u^3u_1^2 + 42u^6 \]

The following proposition gives an explicit formula for the last coefficient in these polynomials

**Proposition 2.3.** The coefficient of \( u_0^k \) in the canonical form \( T_k \) is \( \frac{2^{(2k-3)!}}{k(k-2)!} \) for \( k > 1 \) and 1 for \( k = 1 \) (or equivalently \( \frac{(2k)!}{2(2k-1)(k)!} \)).

**Proof.** The term \( u_0^k \) appears with exactly the same coefficient (up to a sign) in \( T_k \) as in \( \sigma_{2k-1} \) since the integration by parts does not affect it. Consider again the relation \( \sigma_{2k-1} = -\sigma_{2k-2} - \sum_{i=1}^{i=2k-3} \sigma_i\sigma_{2k-2-i} \). The total derivative term \( \sigma_{2k-2} \) does not contain any term \( u_0^k \). The only contributions to \( u_0^k \) come therefore from \( -\sum_{i=1}^{i=2k-3} \sigma_i\sigma_{2k-2-i} \) and result from the product of powers of \( u_0 \).

Let us write \( \sigma_1 = a_1u_0 + ... \), \( \sigma_3 = a_2u_0^2 + ... \), \( \sigma_{2k-1} = a_ku_0^k + ... \). Recall that for even \( \tau \), \( \sigma_1 \) contains no powers of \( u_0 \) since it is a total derivative. It results that \( a_k \) satisfies the relation: \( a_k = -\sum_{i=1}^{i=2k-1} a_ia_{k-i} \) with \( a_1 = 1 \) since the initial condition \( \sigma_1 = u \).

It can be easily shown by induction that the sign of \( a_k \) is \( (-1)^{k-1} \); therefore \( a_k = (-1)^k \sum_{i=1}^{i=2k-1} |a_i||a_{k-i}| \). From (2.4), we obtain that \( T_k = (\sum_{i=1}^{i=2k-1} |a_i||a_{k-i}|)u_0^k + ... \). I.e. \( T_k = b_ku_0^k + ... \) where \( b_k = \sum_{i=1}^{i=2k-1} b_i \), with \( b_1 = 1 \). Using the generating function \( G(t) = \sum_{r=1}^\infty b_rt^r \) with \( b_1 = 1 \) and computing \( G(t)^2 \), we obtain that \( G(t)^2 = G(t) + t = 0 \) i.e \( G(t) = \frac{1+\sqrt{1-4t}}{2} \). Since \( G(0) = 0 \) we deduce that \( G(t) = \frac{1+\sqrt{1-4t}}{2} \) is the only possible choice. Expanding \( G(t) \) as a power series (using \( \sqrt{1+x} = \sum_{n=0}^\infty (-1)^n(2n)!x^n \) for \( |x| < 1 \)) and equating the coefficients, we obtain the required expression for \( b_k \).
2.2 Elliptic Faulhaber polynomials

Consider now the special potential of the form \( u = 2\lambda\varphi_\ast \), where \( \varphi_\ast = \varphi_\ast(z, g_1, g_2, g_3) \) is the elliptic function satisfying the differential equation

\[
(\varphi_\ast^\prime)^2 = 4\varphi_\ast^2 - g_1\varphi_\ast^2 - g_2\varphi_\ast - g_3.
\]  

(2.5)

This function differs from the classical Weierstrass function \( \varphi(z; \hat{g}_2, \hat{g}_3) \) with the equation

\[
(\varphi^\prime)^2 = 4\varphi^3 - \hat{g}_2\varphi - \hat{g}_3,
\]

where \( \hat{g}_2 = g_2 + \frac{1}{12}g_1^2 \) and \( \hat{g}_3 = g_3 + \frac{1}{12}g_1g_2 + \frac{1}{216}g_1^3 \), by adding a constant

\[
\varphi_\ast(z; g_1, g_2, g_3) = \varphi(z; \hat{g}_2, \hat{g}_3) + \frac{g_1}{12}.
\]

However it will be convenient for us to keep the additional parameter \( g_1 \) and consider the elliptic curve \( \Gamma \) in the non-reduced form

\[
Y^2 = 4X^3 - g_1X^2 - g_2X - g_3.
\]

Choose any cycle \( \gamma \) on \( \Gamma \), which does not pass through the poles of the function \( \varphi_\ast \) and define the elliptic Faulhaber polynomials as the integrals

\[
\mathcal{F}_m(\lambda|\Gamma, \gamma) = \oint_\gamma T_m[2\lambda\varphi_\ast(z)]dz.
\]  

(2.6)

Since the integrand has all the residues zero, this integral can be considered as a linear function on the first homology group of \( \Gamma \) or equivalently as an element of the first cohomology group \( H^1(\Gamma, \mathbb{C}) \). Thus, the elliptic Faulhaber polynomials can be considered as \( \lambda \)-dependent sections of the canonical cohomology bundle over the space of the elliptic curves \( \mathcal{E} = \{\Gamma\} \).

We choose the following basis in the first cohomology of \( \Gamma \)

\[
\omega = \frac{1}{2} \oint dz = \frac{1}{2} \oint \frac{dX}{Y}, \quad \xi = -\frac{1}{2} \oint \varphi_\ast(z)dz = -\frac{1}{2} \oint \frac{XdX}{Y}
\]  

(2.7)
**Theorem 2.4.** The elliptic Faulhaber polynomial $F_m$ is polynomial in all variables $\lambda, g_1, g_2, g_3, \omega, \xi$ with rational coefficients, homogeneous with weight $2m - 1$, and linear with respect to $\omega$ and $\xi$. When $g_2 = g_3 = 0$ it reduces (up to a factor) to the classical Faulhaber polynomial

$$F_m(\lambda; g_1, 0, 0, \omega, \xi) = -\frac{4}{2m - 1} g_1^{m-1} \xi F_m(\lambda) \tag{2.8}$$

Here are the first 4 elliptic Faulhaber polynomials

$$F_1 = -4\xi \lambda, \quad F_2 = (-\frac{4}{3} g_1 \xi + \frac{2}{3} g_2 \omega) \lambda^2,$$

$$F_3 = (-\frac{4}{15} g_2^2 \xi + \frac{2}{15} g_1 g_2 \omega) \lambda^2 (4\lambda - 1) - \frac{8}{5} g_2 \xi \lambda^2 (3\lambda - 2) + \frac{8}{5} g_3 \omega \lambda^2 (2\lambda - 3),$$

$$F_4 = (-\frac{4}{21} g_3^2 \xi + \frac{2}{21} g_1 g_3 \omega) \lambda^2 (6\lambda^2 - 4\lambda + 1) - \frac{8}{21} g_1 g_2 \xi \lambda^2 (26\lambda^2 - 29\lambda + 9) + \frac{8}{7} g_1 g_3 \omega \lambda^2 (3\lambda^2 - 2\lambda - 3) + \frac{2}{21} g_2^2 \omega \lambda^2 (25\lambda^2 - 40\lambda + 24)] - \frac{32}{7} g_3 \xi \lambda^2 (5\lambda^2 - 15\lambda + 9)$$

To prove Theorem 2.4 we need the following lemma about the derivatives of the function $\varphi_*$, which can be easily proved by induction using the differential equation (2.5) for the function $\varphi_*$ and its corollary

$$\varphi_*'' = 6\varphi_*^2 - g_1 \varphi_* - \frac{1}{2} g_2 \tag{2.9}$$

**Lemma 2.5.** The derivatives of the elliptic function $\varphi_*$ have the form

$$\varphi_*^{(2k)} = A_k^*(\varphi_*, g_1, g_2, g_3), \quad \varphi_*^{(2k+1)} = A_k''(\varphi_*, g_1, g_2, g_3) \varphi_*' \tag{2.10}$$

for some polynomials $A_k^*$ with rational coefficients. The polynomials $A_k^*$ satisfy the following recurrence relation

$$A_{k+1}^* = (4\varphi_*^3 - g_1 \varphi_*^2 - g_2 \varphi_* - g_3) A_k'' + (6\varphi_*^2 - g_1 \varphi_* - \frac{1}{2} g_2) A_k'', \tag{2.11}$$

where the derivative of $A_k^*$ is taken with respect to $\varphi_*$. 
We know that the KdV density $T_m(u_0, ..., u_{m-2})$ is a sum of monomials $u_0^{m_0} u_1^{m_1} ... u_k^{m_k}$ with rank

$$r = \sum_{j=0}^{j=k} (2 + j)m_j = 2m.$$  

From Lemma 2.5 it follows that if we substitute in such a monomial $u = 2\lambda \varphi_*(x)$ we will have an expression of the form $Q(\varphi_*, g_1, g_2, g_3)\lambda^M$, where $M = \sum_{j=0}^{j=k} m_j$ and $Q$ is a polynomial in $\varphi_*, g_1, g_2, g_3$ with rational coefficients. Indeed, the fact that the rank of $T_k$ is even means that the first derivative of $\varphi_*$ appears only in even powers and thus can be eliminated using the equation (2.5). It is easy to see also that if we define the grading of $\varphi_*, g_1, g_2, g_3$ to be 2, 2, 4 and 6 respectively then $Q$ is homogeneous of degree $2m$.

Let us consider now the integrals

$$K_n^* = \int \varphi_*^n(z)dz.$$  

Using Proposition 2.3, we have the immediate

**Corollary 2.6.** $F_m(\lambda|\Gamma, \gamma)$ has degree $m$ in $\lambda$ with leading coefficient $\frac{(2m+1)(2m-3)!}{m!(m-2)!} K^*_m$.

**Proposition 2.7.** The integrals $K_n^*$ satisfy the following recurrence relation of third order

$$(8n-4)K_n^* = (2n-2)g_1K_{n-1}^* + (2n-3)g_2K_{n-2}^* + (2n-4)g_3K_{n-3}^*$$  

with initial terms $K_0^* = 2\omega, K_1^* = -2\xi, K_2^* = \frac{1}{6}g_2\omega - \frac{1}{3}g_1\xi$.

Indeed, using (2.9) we have

$$K_n^* = \int \varphi_*^{n-2}(z)\varphi_*^2(z)dz = \int \varphi_*^{n-2}(z)(\frac{1}{6}\varphi_*''(z) + \frac{g_1}{6}\varphi_* + \frac{g_2}{12})dz$$  

This can be written

$$K_n^* = \frac{g_1}{6} \int \varphi_*^{n-1}(z)dz + \frac{g_2}{12} \int \varphi_*^{n-2}(z)dz + \frac{1}{6} \int \varphi_*^{n-2}(z)\varphi_*''(z)dz$$  

Integrating by parts the third integral, we obtain

$$\int \varphi_*^{n-2}(z)\varphi_*''(z)dz = -(n-2) \int \varphi_*^{n-3}(z)(\varphi_*'(z))^2dz.$$
Using (2.5), we obtain

\[ \int \varphi_n^{-2}(z)\varphi''_n(z)dz = -(n-2) \int \varphi_n^{-3}(z)(4\varphi_n^3(z) - g_1\varphi_n^2(z) - g_2\varphi_n(z) - g_3)dz. \]

Thus

\[ \int \varphi_n^{-2}(z)\varphi''_n(z)dz = -(n-2)(4K_n^* - g_1K_{n-1}^* - g_2K_{n-2}^* - g_3K_{n-3}^*). \]

Therefore \( K_n^* = \frac{g_1}{6}K_{n-1}^* + \frac{g_2}{12}K_{n-2}^* - \frac{n-2}{6}(4K_n^* - g_1K_{n-1}^* - g_2K_{n-2}^* - g_3K_{n-3}^*) = -\frac{2n-4}{3}K_n^* + \frac{n-1}{6}g_1K_{n-1}^* + \frac{2n-3}{12}g_2K_{n-2}^* + \frac{n-2}{6}g_3K_{n-3}^* \), which leads to the required recurrence relation. The expression for \( K_0^*, K_1^* \) are obvious, the form of \( K_2^* \) follows from the relation (2.9).

**Corollary 2.8.** The integrals \( K_n^* \) have the form

\[ K_n^* = A_n^{(n)}(g_1, g_2, g_3)\omega - B_n^{(n)}(g_1, g_2, g_3)\xi, \]

where \( A_n^{(n)}(g_1, g_2, g_3), B_n^{(n)}(g_1, g_2, g_3) \) are polynomials with positive rational coefficients.

Here are the next few integrals

\( K_3^* = \frac{1}{30}(g_1g_2 + 6g_3)\omega - \frac{1}{30}(2g_1^2 + 9g_2)\xi, \)

\( K_4^* = \frac{1}{840}(6g_1^2g_2 + 25g_2^2 + 36g_1g_3)\omega - \frac{1}{210}(3g_1^3 + 26g_1g_2 + 60g_3)\xi, \)

\( K_5^* = \frac{1}{2520}(4g_1^3g_2 + 33g_1g_2^2 + 24g_1^2g_3 + 168g_2g_3)\omega - \frac{1}{2520}(8g_1^4 + 102g_1^2g_2 + 147g_2^2 + 300g_1g_3)\xi \)

Combining all this we have the proof of the first part of Theorem 2.4. To prove the second part let us consider the case \( g_2 = g_3 = 0 \). The equation (2.5) becomes simply

\[ (\varphi'_n)^2 = 4\varphi_n^2 - g_1\varphi_n^2 \]

with the solution \( \varphi_n(z) = \frac{2\alpha}{\sinh^2(\alpha z)} \), where \( \alpha \) is related to \( g_1 \) by \( g_1 = -4\alpha^2 \).

For the vanishing cycle the integral \( \xi = -\frac{1}{2} \int \varphi_n(z)dz \) is identically zero while the integral \( \xi = -\frac{1}{2} \int_{-\infty}^{\infty} \varphi_n(z)dz \), taken over the real line and corresponding to the...
diverging period $\omega$, has the finite value $\alpha$. In fact, it is convenient to shift the real line in the complex plane by $\frac{\pi}{2a}$, then the function becomes $p_\ast(x + \frac{\pi}{2a}) = -\alpha^2 \text{sech}^2 \alpha x$, which (up to a coefficient $\frac{1}{2}$) is the soliton profile.

Now we can apply the result by Fairlie and Veselov [19], which claims that

$$I_m[-2\lambda \text{sech}^2 x] = (-1)^m \frac{2^{2m}}{2m - 1} F_m(\lambda). \quad (2.12)$$

Let us outline here the idea of the proof. Consider the solution of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0$$

with initial data $u(x, 0) = -n(n + 1) \text{sech}^2 x$. It is well-known that for integer $n$

$$u(x, t) \sim -2 \sum_{k=1}^n k^2 \text{sech}^2 k(x - 4k^2 t - x_k)$$

as $t \to \infty$ for some constants $x_k$ (see e.g. [13], page 79). Now one can use the fact that the integrals $I_m$ are actually the conserved quantities, so $I_m[u(x, t)] = I_m[u(x, 0)]$. Because these integrals have the property $I_m[a^2 u(ax)] = a^{2m-1} I_m[u(x)]$ this immediately gives us the equality

$$I_m[-n(n + 1) \text{sech}^2 x] = I_m[-2 \text{sech}^2 x] \sum_{k=1}^n k^{2m-1}.$$

Now to derive the formula (2.12) one needs only to show that $I_m[-2 \text{sech}^2 x] = (-1)^m \frac{2^{2m}}{2m - 1}$, which can be done in various ways (see [19]).

All this leads to the formula (2.8), which completes the proof of Theorem 2.4.

**Remark.** The elliptic Faulhaber polynomial has the form

$$F_m(\lambda) = -\frac{4}{2m - 1} \left(g_1^{m-1} \xi - \frac{1}{2} g_1^{m-2} g_2 \omega\right) F_m(\lambda) + .$$

where $F_m(\lambda)$ is the usual Faulhaber polynomial and the dots mean the terms of lower order in $g_1$.

There are two more interesting special cases: *lemniscatic* when $g_1 = g_2 = 0$ and *equianharmonic* when $g_1 = g_2 = 0$. Both of them are specialisations of the reduced version of the elliptic Faulhaber polynomials, which we are going to look at in more detail in the next section.
2.3 Reduced elliptic Faulhaber polynomials.

The reduced form of the elliptic Faulhaber polynomials corresponds to $g_1 = 0$:

$$F^W_m(\lambda; g_2, g_3, \omega, \eta) = F_m(\lambda; 0, g_2, g_3, \omega, \eta) \quad (2.13)$$

In that case the function $\varphi$ becomes simply the standard Weierstrass elliptic function $\varphi(z, g_2, g_3)$. Thus by definition the reduced elliptic Faulhaber polynomials are

$$F^W_m(\lambda|\Gamma, \gamma) = \oint_{\gamma} T_m[2\lambda\varphi(z)]dz, \quad (2.14)$$

where $\gamma$ is any cycle on the elliptic curve $\Gamma$ given by the algebraic relation

$$Y^2 = 4X^3 - g_2X - g_3,$$

which does not pass through the poles of the $\varphi$-function.

**Remark.** One can consider also another reduction replacing the Weierstrass function $\varphi$ by the Jacobi elliptic function $k^2 sn^2$, appearing in the Jacobi form of the Lamé equation [68]. The corresponding *Jacobi reduced version of the elliptic Faulhaber polynomials* can be defined by setting $g_3 = 0$.

$$F^{J}_m(\lambda, g_1, g_2, \omega, \xi) = F_m(\lambda, g_1, g_2, 0, \omega, \xi).$$

It corresponds to the elliptic curves $\Gamma$ with a chosen half-period and has the advantage that one can easily see the hyperbolic limit ($g_2 = 0$). However we prefer the Weierstrass version as it is more canonical and standard in the mathematical literature.

**Theorem 2.9.** The elliptic Faulhaber polynomials $F^W_m$ have the form

$$F^W_m = A_m(\lambda, g_2, g_3)\omega + B_m(\lambda, g_2, g_3)\eta,$$

$$A_m = \sum A^{(m)}_{k,l}(\lambda)g_2^k g_3^l, \quad B_m = \sum B^{(m)}_{k,l}(\lambda)g_2^k g_3^l,$$

where the sum is taken over all non-negative integers $k, l$ satisfying $2k + 3l = m$ and $2k + 3l = m - 1$ respectively, and $A^{(m)}_{k,l}(\lambda), B^{(m)}_{k,l}(\lambda)$ are some polynomials in $\lambda$ of degree $m$ with rational coefficients and double zero at $\lambda = 0$. 
The proof follows from Theorem 2.4, Corollary 2.8 and the weight consideration. The explicit form of the first 8 polynomials can be found in the Appendix A.

We are going to look in more detail at the highest and the lowest degree coefficients in \( \lambda \) of the polynomials \( F_m^W \). The highest coefficient of \( F_m^W \) is proportional to \( K_n = \oint \varphi(z)^n dz \). Setting \( g_1 = 0 \) in Proposition 2.7 we have

**Proposition 2.10.** The integrals \( K_n \) satisfy the following 2 term recurrence relation of third order

\[
(8n - 4)K_n = (2n - 3)g_2 K_{n-2} + (2n - 4)g_3 K_{n-3}
\]

with the initial data \( K_0 = 2\omega, K_1 = -2\eta, K_2 = \frac{1}{6}g_2 \omega \). They have the form

\[
K_n = A^{(n)}(g_2, g_3)\omega - B^{(n)}(g_2, g_3)\eta,
\]

where \( A^{(n)}(g_2, g_3), B^{(n)}(g_2, g_3) \) are polynomials with positive rational coefficients.

There is no explicit way to solve this recurrence (except for the lemniscatic and equianharmonic cases, see below), so we would like to discuss here another way to compute \( K_n \), going back to the classical work by Halphen [32] (see chapter VII page 203).

It is based on the following relation between the powers of \( \varphi \) and its even derivatives

\[
\varphi^n = B_n^{(n)} + \sum_{r=0}^{n-1} \frac{B_r^{(n)}}{(2n - 2r - 1)!} \varphi^{(2n - 2 - 2r)},
\]

with

\[
B_r^{(n)} = \frac{(2n - 2r - 2)(2n - 2r - 1)(2n - 2r)}{(2n - 2)(2n - 1)} B_r^{(n-1)} + \frac{2n - 3}{4(2n - 1)} B_{r-2}^{(n-2)} g_2 + \frac{n - 2}{2(2n - 1)} B_{r-3}^{(n-3)} g_3
\]

with \( n > 0, r = 0, \ldots, n \) and with \( B_r^{(n)} = 0 \) for \( r < 0 \) or \( r > n \). \( B_0^{(n)} = 1 \) and \( B_1^{(n)} = 0 \), for any \( n \). By construction \( B_r^{(n)} \) are polynomials in \( g_2, g_3 \) with rational positive coefficients. We will call them *Halphen coefficients*; see the first of them in Table 2.1.
This leads to the following expression for \( K_n \) in terms of Halphen coefficients

\[
K_n = \int \varphi^n(z)dz = 2B_n^{(n)}\omega - 2B_{n-1}^{(n)}\eta. \tag{2.17}
\]

Indeed, since the complete integrals of the total derivatives are zero, the only non-zero contribution comes from the constant term and the \( \varphi \) term in the right-hand side of the relation (2.15). This is of course related to the general de Rham’s theorem, which implies that on the elliptic curve \( \Gamma \) any differential \( \varphi^n(z)dz \) is cohomological to a linear combination of \( dz \) and \( \varphi(z)dz \).

Note that these particular Halphen coefficients \( B_n^{(n)} \) and \( B_{n-1}^{(n)} \) satisfy the same recurrence relation (2.10) as \( K_n \):

\[
\alpha_n = \frac{2n - 3}{4(2n - 1)}g_2\alpha_{n-2} + \frac{n - 2}{2(2n - 1)}g_3\alpha_{n-3}
\]

but with initial conditions \( \alpha_{-1} = 0, \alpha_0 = 1, \alpha_1 = 0 \) for \( \alpha_n = B_n^{(n)} \) and \( \alpha_{-1} = 0, \alpha_0 = 0, \alpha_1 = 1 \) for \( \alpha_n = B_{n-1}^{(n)} \).

In the lemniscatic case when \( g_3 = 0 \) this recurrence equation can be easily solved and leads to the following formula for \( K_n \):

\[
K_n = -4 \eta \frac{n!}{(2n)!} \prod_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (4k-1)^2g_2^{\lfloor \frac{n}{2} \rfloor} \eta \tag{2.18}
\]
for odd $n$ and

$$K_n = 2^{-n!} \frac{\left(\frac{n-1}{2}\right)}{(2n)!} \prod_{k=1}^{\left[\frac{n}{2}\right]} (4k + 1)^2 g_2^{\left[\frac{n}{2}\right]} \omega$$

(2.19)

for even $n$.

Similarly in the equianharmonic case when $g_2 = 0$, we have three cases depending on the congruence of $n$ modulo 3

$$K_n = \frac{1}{2(n-1)} \prod_{k=1}^{\left[\frac{n}{2}\right]} \frac{n - 3k + 1}{2n - 6k + 5} g_3^{\left[\frac{n}{2}\right]} \omega \quad \text{for } n \equiv 0(\text{mod } 3)$$

$$K_n = -\frac{1}{2(n-1)} \prod_{k=1}^{\left[\frac{n}{2}\right]} \frac{n - 3k + 1}{2n - 6k + 5} g_3^{\left[\frac{n}{2}\right]} \eta \quad \text{for } n \equiv 1(\text{mod } 3)$$

$$K_n = 0 \quad \text{for } n \equiv 2(\text{mod } 3)$$

In the next section we analyse the coefficients of the elliptic Faulhaber polynomials at the lowest degree $\lambda^2$, which can be considered as an elliptic generalisation of the Bernoulli numbers.

### 2.4 Elliptic Bernoulli numbers

We define the *elliptic Bernoulli numbers* $B_{2n} = B_{2n}(g_2, g_3, \omega, \eta)$ as the following integrals

$$B_{2n} = \frac{1}{2} \int \left( \frac{d^{n-1}}{dz^{n-1}} \psi(z, g_2, g_3) \right)^2 dz$$

(2.20)

They are simply related to the coefficient of the lowest degree term $\lambda^2$ in the reduced elliptic Faulhaber polynomial $\mathcal{F}_m^W(\lambda)$, which is equal to $8B_{2m-2}$. Note that the coefficient of the lowest degree term $\lambda^2$ in $\mathcal{F}_m(\lambda)$ for $m > 1$ can be obtained by replacing the coefficients $g_2, g_3, \eta$ in $B_{2m-2}$ by $g_2 + \frac{1}{12} g_1^2$ and $g_3 + \frac{1}{12} g_1 g_2 + \frac{1}{216} g_1^3, \xi + \frac{2}{12} \omega$ respectively.
The terminology is justified by the fact that on the discriminant where two roots of the polynomial $4X^3 - g_2X - g_3$ collide, these numbers reduce to the classical (even) Bernoulli numbers.

Indeed, consider the relation

$$B_{2m}(g_2 + \frac{1}{12}g_1^2, g_3 + \frac{1}{12}g_1g_2 + \frac{1}{216}g_1^3, \omega, \xi + \frac{g_1}{12}\omega) = \frac{1}{16}F''_{m+1}(0, g_1, g_2, g_3, \omega, \xi)$$

Take $g_2 = g_3 = 0$. From (2.8) in Theorem 2.4 we obtain

$$B_{2m}(\frac{1}{12}g_1^2, \frac{1}{216}g_1^3, \omega, \xi + \frac{g_1}{12}\omega) = -\frac{1}{4(2m + 1)}g_1^m\xi F''_{m+1}(0)$$

Now from the relation (2.26) between the classical Bernoulli numbers and the Faulhaber polynomials (see subsection 2.4.1)

$$B_{2m} = \frac{1}{4(2m + 1)}F''_{m+1}(0),$$

we have the following

**Proposition 2.11.** For $n > 1$ the specialisation of the elliptic Bernoulli numbers on the discriminant is

$$B_{2n}(\frac{1}{12}h^2, \frac{1}{216}h^3, \omega, \xi + \frac{1}{12}h\omega) = -B_{2n}h^n\xi,$$  \hspace{1cm} (2.21)

where $B_{2n}$ are the usual Bernoulli numbers

One can show this limit using an integral formula for the classical Bernoulli numbers which we think is new and which we prove in the next subsection.

### 2.4.1 Soliton limit: usual Bernoulli numbers

The Bernoulli numbers play an important role in analysis, number theory, algebraic topology and many other areas of mathematics. They have the following generating function

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!}z^k$$

All odd Bernoulli numbers except $B_1 = -\frac{1}{2}$ are zero and the first even Bernoulli numbers are

$$B_0 = 1, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}.$$
Theorem 2.12. The Bernoulli numbers $B_{2m}$ with $m \geq 1$ have the following integral representation

$$B_{2m} = \frac{(-1)^{m-1}}{2^{2m+1}} \int_{-\infty}^{+\infty} ((\text{sech}^2 x)^{(m-1)})^2 dx, \quad m \geq 1,$$

(2.22)

where $(\text{sech}^2 x)^{(m-1)}$ denotes the $(m - 1)$th derivative of $\text{sech}^2 x$.

This formula is motivated by the results of Fairlie and Veselov [19], who discovered a relation between the Bernoulli polynomials and the soliton theory.

The main result of [19] is the following formula relating the Faulhaber polynomials $F_m$ to the integrals of the KdV equation:

$$I_m[-2\lambda \text{sech}^2 x] = (-1)^m \frac{2^{2m}}{2m - 1} F_m(\lambda)$$

(2.23)

Recall that the Faulhaber polynomials are directly related to the Bernoulli polynomials through the formula

$$B_{2m} (x + 1) = 2mF_m \left( \frac{x^2 + x}{2} \right) + B_{2m}$$

(2.24)

where $B_k(x)$ and $B_k$ are Bernoulli polynomials and Bernoulli numbers respectively (see [40, 2]).

Let us first present the proof based on the results of [19]. Recall that the Faulhaber polynomial $F_{m+1}(\lambda)$ with $m \geq 1$ has the form

$$F_{m+1}(\lambda) = \alpha_2^{m+1} \lambda^2 + \alpha_3^{m+1} \lambda^3 + \ldots + \alpha_{m+1}^{m+1} \lambda^{m+1}$$

with some rational coefficients $\alpha_2^{m+1}, \alpha_3^{m+1}, \ldots, \alpha_{m+1}^{m+1}$ (see [40]). Since the only quadratic term in the density of the KdV integral $I_{m+1}$ is $u_{m-1}^2 dx$, we have from the relation (2.23) that

$$\int_{-\infty}^{+\infty} ((\text{sech}^2 x)^{(m-1)})^2 dx = (-1)^{m+1} \frac{2^{2m}}{2m + 1} \alpha_2^{m+1}$$

(2.25)

Differentiating the formula (2.24) twice with respect to $x$ gives

$$\left( \frac{2x + 1}{2} \right)^2 F'_m \left( \frac{x(x + 1)}{2} \right) + F_m \left( \frac{x(x + 1)}{2} \right) = \frac{1}{2m} B''_{2m}(1 + x) = (2m - 1) B_{2m-2}(1 + x)$$
because \( B_{2m}(x)^{''} = (2m)(2m - 1)B_{2m-2}(x) \). Since \( F'_m(0) = 0 \) this gives for \( x = 0 \)
\[
\frac{1}{4} F'_m(0) = (2m - 1)B_{2m-2}(1),
\]
and after a change of indices
\[
\frac{1}{4} F'_{m+1}(0) = (2m + 1)B_{2m}(1).
\]
Now using the well-known symmetry \( B_k(1 - x) = (-1)^kB_k(x) \), we have \( B_{2m}(1) = B_{2m}(0) = B_{2m} \) and thus
\[
F''_{m+1}(0) = 4(2m + 1)B_{2m}
\]
1.e.
\[
\alpha_{m+1}^n = \frac{1}{2} F''_{m+1}(0) = 2(2m + 1)B_{2m}
\]
Substituting this into (2.25) we come to the formula (2.22).

We are now going to prove the formula (2.22) directly without reference to soliton theory. We borrow the main idea from the book [26] by Graham, Knuth and Patashnik, where it is attributed to Logan.

Consider the integral
\[
J_m = \frac{(-1)^{m-1}}{2^{2m+1}} \int_{-\infty}^{+\infty} ((\text{sech}^2 x)^{(m-1)})^2 dx,
\]
for \( m \geq 1 \). Integrating \( J_m \) by parts \( m - 1 \) times gives
\[
J_m = \frac{1}{2^{2m+1}} \int_{-\infty}^{+\infty} (\text{sech}^2 x)^{(2m-2)} \text{sech}^2 x dx = \frac{1}{2^{2m+1}} \int_{-\infty}^{+\infty} \tanh x^{(2m-1)} \tanh x^{(1)} dx
\]
Let \( y = \tanh x \) then
\[
J_m = \frac{1}{2^{2m+1}} \int_{-1}^{+1} T_{2m-1}(y) dy,
\]
where the polynomial \( T_k(y) \) is the \( k \)-th derivative of \( y = \tanh x \) rewritten in terms of \( y \).
\[
T_1 = y^{(1)} = 1 - y^2
\]
These polynomials can be determined by the recurrence formula

\[ T_n(x) = (1 - x^2)T_{n-1}(x)' \]  

(2.29)

with \( T_0(x) = x \). ¹ They have integer coefficients with the highest one being equal to \((-1)^n n!\) and have the symmetry \( T_n(-x) = (-1)^n T_n(x) \). Note that the non-zero coefficients have the alternating signs and their total sum is zero. The integers \(|T_{2m-1}(0)|\) are called tangent numbers (see e.g. [26]).

Let us consider now the generating function

\[ T(x, z) = \sum_{n \geq 0} T_n(x) \frac{z^n}{n!} \]  

(2.30)

**Lemma 2.13.** The generating function for the polynomials \( T_n(x) \) is

\[ T(x, z) = \frac{\sinh z + x \cosh z}{\cosh z + x \sinh z} \]  

(2.31)

Indeed it is easy to see that when \( x = \tanh w \) the function \( T(x, z) \) becomes \( \tanh(z + w) \). Now the claim follows from the Taylor formula and the definition of the polynomials \( T_n(x) \).

From this, one can derive an interesting relation between the Bernoulli and the tangent numbers (see [26], formula (6.93) and discussion after that). We will need however a slightly different result

**Lemma 2.14.** The Bernoulli numbers can be written as

\[ B_m = \frac{1}{2^{m+1}} \int_{-1}^{1} T_{m-1}(x)dx \]  

(2.32)

for all \( m > 1 \). ¹

¹These polynomials should not be confused with the classical Chebyshev (Tchebycheff) polynomials, which are also denoted as \( T_n(x) \). We follow the notations of the book [26], where the trigonometric version of these polynomials was considered.
Indeed, this formula is obvious for odd \( m \) bigger than 1 since \( T_{m-1}(x) \) is an odd function and both sides of (2.32) are then equal to 0. In order to prove the formula for even \( m \), let us consider first the left hand side of (2.31). The function \( T(x, z) \) can be rewritten as

\[
T(x, z) = \frac{(x + \sinh z) \cosh z}{(x + \cosh z) \sinh z} = \coth z - \frac{1}{(x + \coth z) \sinh^2 z}.
\]

Integrating with respect to \( x \) gives

\[
\int T(x, z) \, dx = x \coth z - \frac{1}{\sinh^2 z} \ln |x + \coth z|,
\]

which leads to

\[
\int_{-1}^{1} T(x, z) \, dx = 2 \coth z - \frac{2z}{\sinh^2 z} = 2 \coth z + 2z \coth' z \quad (2.33)
\]

The expansion of the function \( \coth z \) can be written in terms of the Bernoulli numbers (see [2]):

\[
\coth z = \frac{1}{z} + \sum_{n=1}^{\infty} a_{2n-1} z^{2n-1}, \quad a_{2n-1} = \frac{2^{2n} B_{2n}}{(2n)!}
\]

(2.34)

This easily follows from the identity

\[
\frac{z}{2} \coth \frac{z}{2} = \frac{z}{2} + \frac{z}{e^z - 1}.
\]

From (2.33) and (2.34) it follows that

\[
\int_{-1}^{1} T(x, z) \, dx = 4 \sum_{n=1}^{\infty} n a_{2n-1} z^{2n-1} \quad (2.35)
\]

On the other hand from (2.30) we have

\[
\int_{-1}^{1} T(x, z) \, dx = \sum_{n \geq 1} \frac{z^{2n-1}}{(2n-1)!} \int_{-1}^{1} T_{2n-1}(x) \, dx \quad (2.36)
\]

since \( T_{2n}(x) \) are odd polynomials.
Comparing (2.35) and (2.36) and replacing $a_{2n-1}$ by $\frac{2nB_{2n}}{2n+1}$ we obtain

$$B_{2n} = \frac{1}{2^{2n+1}} \int_{-1}^{1} T_{2n-1}(x) dx$$

This proves Lemma 2.14. Now combining Lemma 2.14 with (2.27) and (2.28) we have Theorem 2.12.

**Remark.** As a corollary we have the following representation of the Bernoulli number $B_{2n}$ as a sum of the fractions of the type

$$B_{2n} = \frac{1}{2^{2n}} \left( \frac{k_0}{1} + \frac{k_1}{3} + \frac{k_2}{5} + \cdots + \frac{k_n}{2n+1} \right),$$

where the denominators are consecutive odd numbers and the numerators are the coefficients of the polynomials $T_{2n-1}(x)$. Comparison with the formula (6.93) from the book [26] leads to the following relation with the tangent numbers:

$$\int_{-1}^{1} T_n(x) dx = \frac{2n+2}{2^{n+1} - 1} T_n(0)$$

(2.37)

Let us look now at the properties of the elliptic Bernoulli numbers in the general case.

### 2.4.2 Properties of the elliptic Bernoulli numbers

**Proposition 2.15.** The elliptic Bernoulli numbers satisfy the following recurrence relation

$$B_{2n} = (-1)^{n-1}(2n-1)! [B_{n+1}^{(n+1)} \omega - (B_{n+1}^{(n+1)} - B_n^{(n)}) \eta] - (2n-1)! \sum_{r=2}^{n-1} \frac{(-1)^r B_r^{(n)}}{(2n-2r-1)!} B_{2n-2r},$$

where $B_r^{(n)}$ are the Halphen coefficients.

Indeed, integrating by parts, we have

$$\int (\varphi(z)^{(n-1)})^2 dx = (-1)^{n-1} \int \varphi(z)^{(2n-2)} \varphi(z) dz,$$

so

$$B_{2n} = -\frac{(-1)^{n-1}}{2} \int \varphi(z)^{(2n-2)} \varphi(z) dz$$

(2.38)
Re-arranging the formula (2.15) gives
\[
\varphi^{(2n-2)} = (2n - 1)! \left[ \varphi^n - \frac{B_2^{(n)}}{(2n - 5)!} \varphi^{(2n-6)} - \ldots - \frac{B_{n-2}^{(n)}}{3!} \varphi^{(2)} - B_{n-1}^{(n)} \varphi - B_n^{(n)} \right]
\]

Multiplying the above formula by \( \varphi(z) \) and taking the contour integral over a cycle gives
\[
B_{2n} = (-1)^{n-1} \frac{2n-1}{2} \left( \int \varphi^{n+1}(z) dz - \int B_n^{(n)} \varphi(z) dz \right) - (2n-1)! \sum_{r=2}^{n-1} \frac{(-1)^r B_r^{(n)}}{(2n - 2r - 1)!} B_{2n-2r}
\]

Using the relation \( K_{n+1} = \int \varphi^{n+1}(z) dz = 2B_{n+1}^{(n+1)} \omega - 2B_n^{(n+1)} \eta \) and \( \int \varphi(z) dz = -2\eta \), we obtain the proposition.

The recurrence relation in this proposition can be considered as an elliptic analogue of the recurrence relation for the usual Bernoulli numbers:
\[
\sum_{j=0}^{m} \binom{m+1}{j} B_j = 0,
\]
where \( \binom{m+1}{j} \) are the binomial coefficients.

There is another way to compute the elliptic Bernoulli numbers if we assume that we know already the integrals \( K_n \). Let us use the following very convenient notation for a polynomial \( D(x) = \sum_{j=0}^{N} d_j x^j \) and a sequence \( K_0, K_1, \ldots \), we define \( D[K] \) as follows:
\[
D[K] := \sum_{j=0}^{N} d_j K_j.
\]

Let \( D_n(x) = x A_{n-1}(x, g_2, g_3) = x A_{n-1}^*(x, 0, g_2, g_3) \), where \( A_n^* \) are the polynomials defined in Lemma 2.5 by the recurrence relation (2.11). Then from (2.38) we immediately have the following

**Proposition 2.16.** The elliptic Bernoulli numbers can be expressed through \( K_n \) as follows:
\[
B_{2n} = \frac{(-1)^{n-1}}{2} D_n[K]
\]
Here are the first few elliptic Bernoulli numbers

\[ E_2 = \frac{1}{12} g_2 \omega, \quad E_4 = -\frac{3}{5} g_3 \omega + \frac{2}{5} g_2 \eta, \quad E_6 = \frac{2}{7} g_2^2 \omega - \frac{36}{7} g_3 \eta, \]

\[ E_8 = -\frac{36}{5} g_2 g_3 \omega + \frac{24}{5} g_2^2 \eta, \quad E_{10} = \frac{72}{11} (g_2^3 + 18 g_3^2) \omega - \frac{2160}{11} g_2 g_3 \eta, \]

\[ B_{12} = -\frac{298512}{455} g_2^2 g_3 \omega + \frac{2592}{455} (49 g_2^3 + 750 g_3^2) \eta, \quad B_{14} = 864 g_2 (g_2^3 + 36 g_3^2) \omega - 36288 g_2 g_3 \eta, \]

\[ B_{16} = -\frac{15552}{85} g_3 (1039 g_2^3 + 4500 g_3^2) \omega + \frac{10368}{85} g_2 (539 g_2^3 + 18000 g_3^2) \eta. \]

The special lemniscatic \((g_3 = 0)\) and equianharmonic \((g_2 = 0)\) cases can be seen straight from here.

**Remark.** In the classical case the coefficients of the Faulhaber polynomials \(F_m(\lambda)\) can be computed using the property that in the corresponding polynomial \(F_m(\frac{x^2 + x}{2}) = \frac{1}{2m} (B_{2m}(x + 1) - B_{2m}) = x^{2m-1} + \frac{1}{2m} (B_{2m}(x) - B_{2m})\), all the coefficients of odd powers of \(x\) except \(x^{2m-1}\) are zero (see [40]). This is related to the fact that the coefficients of the Bernoulli polynomials are proportional to the Bernoulli numbers and all the odd Bernoulli numbers except \(B_1 = -1/2\) are zero. A simple check shows that this is not true in the elliptic case: already for \(F_3^W\) the coefficient at \(x^3\) in \(F_3^W(\frac{x^2 + x}{2})\) is \(3 g_2 \eta - 2 g_3 \omega\) which is of course zero if \(g_2 = g_3 = 0\) but not in general. This means that the elliptic Bernoulli numbers do not play the same role in the elliptic case as they do in the usual one.

It is worthy to mention that there is another elliptic generalisation of Bernoulli numbers: the so-called *Bernoulli-Hurwitz numbers* \(BH_{2k}\) [33, 38, 55]. They are defined as

\[ BH_{2k} = 2k(2k - 2)! c_k, \]

where \(c_k\) are the coefficients of the Laurent series of the Weierstrass \(\wp\)-function at zero.

\[ \wp(z) = \frac{1}{z^2} + c_2 z^2 + c_3 z^4 + ... \]

These coefficients \(c_k\) satisfy the recurrence relation

\[ c_k = \frac{3}{(2k + 1)(k - 3)} \sum_{m=2}^{k-2} c_m c_{k-m} \]
for $k \geq 4$ with $c_2 = g_2/20, c_3 = g_3/28$ (see [2]), which can be easily established using the relation $\varphi'' = 6\varphi^2 - \frac{1}{2}g_2$.

The Bernoulli-Hurwitz numbers are related to the Halphen coefficients in the following way. Recall that the principal part of a Laurent series is the sum of the terms with negative powers. Since the principle part of $\varphi^{(2n)}$ is $\frac{(2m+1)!}{2^{2m+1}}$, we have from the relation (2.15) the following

**Proposition 2.17.** The principal part of the Laurent series

$$\varphi^n = \left(\frac{1}{z^2} + c_2 z^2 + c_3 z^4 + \ldots \right)^n$$

can be written in terms of Halphen coefficients as

$$\sum_{r=0}^{n-1} \frac{B_r^{(n)}}{z^{2n-2r}}$$

Since $\left(\frac{1}{z^2} + c_2 z^2 + c_3 z^4 + \ldots \right)^n = \frac{1}{z^{2n}} + \frac{c_2}{z^{2n-2}} + \frac{nc_2}{z^{2n-4}} + \frac{n(n-1)c_2^2}{z^{2n-6}} + \ldots$, this allows us to express the Bernoulli-Hurwitz numbers through the Halphen coefficients and vice versa. $B_0^{(n)} = 1, B_1^{(n)} = 0$, $B_2^{(n)} = nc_2 = ng_2/20 = \frac{n}{8}BH_4$,

$B_3^{(n)} = nc_3 = ng_3/28 = \frac{n}{144}BH_6$,

$B_4^{(n)} = nc_4 + \frac{n(n-1)c_2^2}{2} = \frac{n(3n-1)c_4}{11520}BH_8$,

$B_5^{(n)} = nc_5 + n(n-1)c_2c_3 = \frac{n(11n-8)c_5}{1209600}BH_{10}$

Note that these expressions for $B_r^{(n)}$ are only valid when $r < n$. 
Chapter 3

Elliptic Bernoulli polynomials

We introduce now another class of polynomials, which can be considered as an elliptic generalisation of the odd Bernoulli polynomials $B_{2k+1}(x)$. They are related to the quantum Euler top and to the classical Lamé operator.

Recall that for integer $s$ the Lamé operator $L_s = -\frac{d^2}{dz^2} + s(s+1)\varphi(z)$, considered on the real line shifted by the imaginary half-period, has exactly $s$ gaps in its spectrum. The ends of the gaps are given by the zeroes of the Lamé spectral polynomials $R_{2s+1}(E)$ (see section 1.2).

Here we consider a related but different problem: we want to find explicit expressions for the coefficients $b_k$ of the Lamé spectral polynomials $R_{2s+1}(E) = E^{2s+1} + b_1 E^{2s} + b_2 E^{2s-1} + \cdots + b_{2s}$ as functions of $s$ (and thus for all values of parameter $s$). We show that in this relation naturally appear some new polynomials, generalising the odd Bernoulli polynomials.

Note that once $b_k(s)$ are known for $k = 0, 1, \ldots, 2s$ one can find the eigenvalues of the quantum Euler top in the representation with spin $s$ (integer or half-integer) by solving the corresponding algebraic equation $R_{2s+1}(E) = 0$.

3.1 Lamé equation and quantum Euler top

The observation that the Lamé equation is closely related to the quantum top was first made by Kramers and Ittmann [41] (see also [66]) who showed that the corresponding Schroedinger equation is separable in the elliptic coordinate system and
that the resulting differential equations are of Lamé form.

More precisely, let us consider the quantum Hamiltonian of the Euler top (see [43])

\[ \hat{H} = a_1 \hat{M}_1^2 + a_2 \hat{M}_2^2 + a_3 \hat{M}_3^2, \]

acting in the space of functions on the unit sphere

\[ q_1^2 + q_2^2 + q_3^2 = 1, \]

using the standard representation of the components of the angular momentum as first order differential operators

\[ \hat{M}_1 = -i(q_2 \partial_3 - q_3 \partial_2) \]
\[ \hat{M}_2 = -i(q_3 \partial_1 - q_1 \partial_3) \]
\[ \hat{M}_3 = -i(q_1 \partial_2 - q_2 \partial_1). \]

Let us introduce the elliptic (or spherico-conical) coordinates \( u_1, u_2 \) on this sphere as the roots of the quadratic equation

\[ \frac{q_1^2}{a_1 - u} + \frac{q_2^2}{a_2 - u} + \frac{q_3^2}{a_3 - u} = 0, \]

where the parameters \( a_1, a_2, a_3 \) are the same as in the top’s Hamiltonian. One has then the following expressions for the cartesian coordinates in terms of \( u_1, u_2 \):

\[
q_1^2 = \frac{(a_1 - u_1)(a_1 - u_2)}{(a_1 - a_2)(a_1 - a_3)} \\
q_2^2 = \frac{(a_2 - u_1)(a_2 - u_2)}{(a_2 - a_1)(a_2 - a_3)} \\
q_3^2 = \frac{(a_3 - u_1)(a_3 - u_2)}{(a_3 - a_1)(a_3 - a_2)} \tag{3.1}
\]

The system has an obvious quantum integral (Casimir) \( \hat{M}^2 = \sum \hat{M}_i^2 \), which is the square of the angular momentum operator.

\[ [\hat{M}^2, \hat{M}_i] = 0 \]
One can check (see Appendix D) that in the elliptic coordinate system the operators $\hat{H}$ and $\hat{M}^2$ have the form

$$\hat{M}^2 = -\frac{4}{u_1 - u_2} \left[ \sqrt{-P(u_1)} \frac{\partial}{\partial u_1} (\sqrt{-P(u_1)} \frac{\partial}{\partial u_1}) + \sqrt{P(u_2)} \frac{\partial}{\partial u_2} (\sqrt{P(u_2)} \frac{\partial}{\partial u_2}) \right]$$

$$\hat{H} = -\frac{4}{u_1 - u_2} \left[ u_2 \sqrt{-P(u_1)} \frac{\partial}{\partial u_1} (\sqrt{-P(u_1)} \frac{\partial}{\partial u_1}) + u_1 \sqrt{P(u_2)} \frac{\partial}{\partial u_2} (\sqrt{P(u_2)} \frac{\partial}{\partial u_2}) \right]$$

where $P(u) = (u - a_1)(u - a_2)(u - a_3)$ Note that the operator $\hat{M}^2$ corresponds to the standard Laplacian $-\Delta$ on the unit sphere

Since $\hat{M}^2$ and $\hat{H}$ commute, one can look for joint eigenfunctions. The spectral problem $\hat{M}^2 \psi = \mu \psi$ is well-known in the theory of spherical harmonics (see e.g. [51]). It is known that the spectrum has the form $\mu = s(s + 1)$ for non-negative integer values of $s$. The dimension of the corresponding eigenspace $V_s$ is $2s + 1$ and $V_s$ is an irreducible representation of dimension $2s + 1$ of the rotation group $SO_3$ called representation with spin $s$.

It turns out that the joint eigenvalue problem

$$\hat{M}^2 \phi = s(s + 1)\phi,$$

$$\hat{H} \phi = E \phi$$

is separable in the elliptic coordinates $u_1, u_2$ (see [41, 66]) Namely, if we substitute $\phi(u_1, u_2) = \phi_1(u_1) \phi_2(u_2)$ into this system we find that each of the functions $\phi_1(u_1), \phi_2(u_2)$ satisfies the same differential equation

$$(4[P(u)]^{\frac{1}{2}} \frac{d}{du} ([P(u)]^{\frac{1}{2}} \frac{d}{du}) - s(s + 1)u + E)\psi = 0,$$

which can be rewritten as

$$\frac{d^2}{du^2} \psi + \frac{1}{2} \left[ \frac{1}{u - a_1} + \frac{1}{u - a_2} + \frac{1}{u - a_3} \right] \frac{d}{du} \psi = \frac{1}{4} \frac{s(s + 1)u - E}{(u - a_1)(u - a_2)(u - a_3)} \psi \quad (3.4)$$

A remarkable fact is that this is the algebraic form of the following slightly generalised version of the Lamé differential equation

$$-\frac{d^2}{dz^2} \psi + s(s + 1)\varphi_*(z)\psi = E \psi \quad (3.5)$$
where $\varphi_*(z)$ is a solution of the differential equation

$$
(\varphi_*')^2 = 4P(\varphi_*) = 4(\varphi_* - a_1)(\varphi_* - a_2)(\varphi_* - a_3)
$$

(3.6)

Indeed, after the change of variables $u = \varphi_*(z)$ the equation (3.4) coincides with (3.5) (see [68]).

When the sum $a_1 + a_2 + a_3 = 0$ the equation (3.6) determines the Weierstrass elliptic function $\wp(z)$. Otherwise it differs from it by adding a constant. It is well-known (see e.g. [16]) that for $\phi$ to be a regular solution on the sphere the corresponding $\phi_*$ must be doubly-periodic, which implies that $E$ must have one of the $2s + 1$ characteristic values $E_m(s) = E^m_2$. For each $E_m(s)$ there exists exactly one doubly-periodic solution to the Lamé equation, which is (up to a factor) the Lamé function $E^m_s(u)$. Therefore the basis of the eigenfunctions of the operator $\hat{H}$ in the invariant subspace $V_s$ consists of $2s + 1$ solutions $\phi(u_1, u_2)$ of the form $E^m_s(u_1)E^m_s(u_2)$. They are sometimes called elliptoidal harmonics (see [68]).

Thus, we come to the following result [41, 66]

**Theorem 3.1.** The characteristic polynomial of the quantum top Hamiltonian $\hat{H}_s$ in the representation space with integer spin $s$ coincides with the Lamé spectral polynomial $R_{2s+1}(\lambda) = \prod_{m=0}^{2s}(\lambda - E_m(s))$ of the corresponding generalized Lamé equation (3.5).

**Remark.** A simple relation between the quantum Euler top and the Lamé equation mentioned above is a bit misleading. Indeed there are several spectral problems related to the Lamé equation. For example if we would consider $x$ just real without the imaginary half-period shift, we would have a singular version of the Lamé operator (since $\varphi_*$ has poles on the real line) whose spectrum has nothing to do with the quantum top. In its turn, the quantum Euler top in the representation with half-integer spin $s$ has eigenvalues which are just some special double eigenvalues of the periodic Lamé operator which in this case has infinitely many gaps.
3.2 Elliptic Bernoulli polynomials

We define now the elliptic Bernoulli polynomials $B_{2k+1}(s) = B_{2k+1}(s; g_1, g_2, g_3)$ as the traces of the powers of $\hat{H}_s$, where $\hat{H}_s$ is as before the quantum top operator $\hat{H}$ in the representation with spin $s$.

$$B_{2k+1}(s; g_1, g_2, g_3) = tr \hat{H}_s^k, \quad k = 0, 1, 2, \ldots \quad (3.7)$$

Here the parameters $g_1 = 4(a_1 + a_2 + a_3), g_2 = -4(a_1a_2 + a_2a_3 + a_1a_3), g_3 = 4a_1a_2a_3$ are defined by the relation

$$4(z - a_1)(z - a_2)(z - a_3) = 4z^3 - g_1z^2 - g_2z - g_3$$

**Theorem 3.2.** The trace of the $k$-th power of the quantum Euler top Hamiltonian in the representation with spin $s$ is a polynomial in $s$ of degree $2k + 1$, anti-symmetric with respect to $s = -\frac{1}{2}$, and whose coefficients are polynomials in $g_1, g_2, g_3$ with rational coefficients. When $g_2 = g_3 = 0$, these polynomials reduce to the classical odd Bernoulli polynomials up to a constant factor:

$$B_{2k+1}(s; g_1, 0, 0) = \frac{g_1^k}{(2k + 1)2^{2k-1}}B_{2k+1}(s + 1). \quad (3.8)$$

Here are the first elliptic Bernoulli polynomials, for $k = 0$ to $k = 4$.

$B_1(s) = 2s + 1,$

$B_3(s) = \frac{1}{12} g_1 \quad s(s + 1)(2s + 1),$

$B_5(s) = \frac{1}{240} g_1^2 \quad s(s + 1)(2s + 1)(3s^2 + 3s - 1)$

$+ \frac{1}{60} g_2 \quad s(s + 1)(2s - 1)(2s + 1)(2s + 3),$

$B_7(s) = + \frac{1}{1344} g_1^3 \quad s(s + 1)(2s + 1)(3s^4 + 6s^3 - 3s + 1)$

$+ \frac{1}{120} g_1 g_2 \quad s(s + 1)(2s - 1)(2s + 1)(2s + 3)(6s^2 + 6s - 5)$

$+ \frac{1}{280} g_3 \quad s(s + 1)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5),$

$B_9(s) = + \frac{1}{11520} g_1^4 \quad s(s + 1)(1 + 2s)(5s^6 + 15s^5 + 5s^4 - 15s^3 - s^2 + 9s - 3)$

$+ \frac{1}{3360} g_1 g_2 \quad s(s + 1)(2s - 1)(2s + 1)(2s + 3)(5s^4 + 10s^3 - 5s^2 - 10s + 7)$

$+ \frac{1}{840} g_1 g_3 \quad s(s + 1)^2(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5)$

$+ \frac{1}{1680} g_2^2 \quad s(s + 1)(2s - 1)(2s + 1)(2s + 3)(4s^4 + 8s^3 - 11s^2 - 15s + 21)$

The explicit form of the next 3 elliptic Bernoulli polynomials can be found in appendix C.
ELLIPTIC BERNOULLI POLYNOMIALS

To prove Theorem 3.2, consider the standard basis in $V_s$ consisting of the eigen-vectors $|j>$ of $\hat{M}_3$: $\hat{M}_3|j> = j|j>$, $j = -s, -s + 1, \ldots s - 1, s$. In this basis, the Hamiltonian $\hat{H}$ is a tri-diagonal symmetrical matrix $H = H_s$ with the following elements (see e.g. Landau-Lifshitz [43], page 417)

\[
< j|H|j> = \frac{1}{2}(a_1 + a_2)[s(s + 1) - j^2] + a_3 j^2, \tag{3.9}
\]

\[
< j|H|j + 2 >=< j + 2|H|j > = \frac{1}{4}(a_1 - a_2)\sqrt{(s - j)(s - j - 1)(s + j + 1)(s + j + 2)}.
\]

Note that both expressions are symmetrical with respect to $s = -\frac{1}{2}$; they are also homogeneous polynomials of degree 1 in $a_1, a_2, a_3$. Now, consider any diagonal element of $H^k$; it has the form

\[
< j|H^k|j> = \sum_{i_1, i_2, \ldots, i_{k-1}} < j|H|i_1>< i_2|H|i_2 > \ldots < i_{k-1}|H|j>,
\]

where the distance between 2 consecutive indices $i_t, i_{t+1}$ is either 0 or $\pm 2$. Since the starting point and the ending point coincide, if the matrix element $< i_1|H|i_1 + 2 >$ appears along the path so does the element $< i_1 + 2|H|i_1 >$. This proves that the diagonal matrix elements of $H^k$ are polynomials of degree $2k$ in both $s$ and $j$. Moreover, from (3.9) they are symmetric with respect to $s = -\frac{1}{2}$ and homogeneous polynomials of degree $k$ in $a_1, a_2, a_3$. Now summing over $j = -s, -s + 1, \ldots s - 1, s$ and taking into account that the sums of the odd powers of $j$ are zero while the sums of even powers $2l$ are the odd Bernoulli polynomials $B_{2l+1}(s + 1)$ (multiplied by $\frac{2^l}{2l+1}$) we have the first statement of Theorem 3.2. The anti-symmetry of $B_{2k+1}(s)$ with respect to $s = -\frac{1}{2}$ follows from the well-known property of the Bernoulli polynomials $B_m(1 - s) = (-1)^mB_m(s)$. The symmetry with respect to $a_1, a_2, a_3$ is clear from the definition of the elliptic Bernoulli polynomials (3.7).

In the case when $a_1 = a_2 = 0$, we have $g_2 = g_3 = 0$, $g_1 = 4a_3$ and $\hat{H} = a_3 \hat{M}_3^2$. The spectrum of $H_s$ is then very simple: $\lambda_j = a_3 j^2$ for $j = -s, -s + 1, \ldots s - 1, s$. Since the sum $\sum_{j=1}^{s} j^{2k} = \frac{1}{2k+1}B_{2k+1}(s + 1)$, we thus obtain:

\[
B_{2k+1}(s, g_1, 0, 0) = \frac{g^k}{(2k + 1)2^{2k-1}}B_{2k+1}(s + 1)
\]
This completes the proof.

Note that from the point of view of the elliptic curve $\Gamma$ given by the equation

$$y^2 = 4x^3 - g_1x^2 - g_3x - g_3,$$

the last case $g_2 = g_3 = 0$ corresponds to the limit when one of the periods of $\varphi_4(z)$ goes to infinity. There are two more interesting special cases: lemniscatic, when $g_1 = g_3 = 0$, and equianharmonic, when $g_1 = g_2 = 0$, corresponding to the elliptic curves with additional symmetries.

It is natural also to consider the Weierstrass reduction $g_1 = 0$, we will call the corresponding polynomials $B_{2k+1}^W(s, g_2, g_3) = B_{2k+1}(s; 0, g_2, g_3)$ the reduced elliptic Bernoulli polynomials

**Theorem 3.3.** The elliptic Bernoulli polynomial $B_{2k+1}(s)$ has the following properties

1. as a polynomial in $g_1, g_2, g_3$, $B_{2k+1}(s)$ is homogeneous of weight $2k$, where the weights of $g_1, g_2$ and $g_3$ are assumed to be 2, 4 and 6 respectively

2. $B_{2k+1}(s)$ for $k \geq 1$ is divisible by $s(s+1)(2s+1)$

3. in the reduced case $B_{2k+1}^W(s)$ is divisible by $s(s+1)(2s-1)(2s+1)(2s+3)$ for all $k$ and by $s(s+1)(2s-1)(2s+1)(2s+3)(2s-3)(2s+5)$ for all odd $k$

4. in the lemniscatic case $B_{2k+1}(s; 0, g_2, 0) = 0$ for all odd $k$

5. in the equianharmonic case $B_{2k+1}(s; 0, 0, g_3) = 0$ if $k$ is not divisible by 3

6. in the isotropic case where $a_1 = a_2 = a_3 = a \ i.e. g_1 = 12a, g_2 = 12a^2$ and $g_3 = 4a^3$, $B_{2k+1}(s) = a^k(2s+1)s^k(s+1)^k$.

The proof of the first two claims follows from the definition and the anti-symmetry property of $B_{2k+1}(s)$. To prove the third one, consider the representation with spin $s = \frac{1}{2}$. It is easy to check that $\hat{H}$ acts as the 2 by 2 scalar matrix $\frac{1}{4}(a_1 + a_2 + a_3)\text{Id}$, which is zero in the reduced case. Therefore $B_{2k+1}^W(\frac{1}{2}) = 0$ for all $k$. By anti-symmetry with respect to $s = -\frac{1}{2}$, we also have $B_{2k+1}^W(-\frac{3}{2}) = 0$.


For a half integer \( s \), we know from Kramers' theorem (see [43]) that the eigenvalues are no longer distinct but are double roots. For the particular case \( s = 3/2 \), these eigenvalues take the values \( \pm \sqrt{3(a_1^2 + a_2^2 + a_3^2)}/2 \) (see [43], page 419) therefore for all odd \( k \), \( B_{2k+1}^W(3/2) = 0 \) and again by anti-symmetry \( B_{2k+1}^W(-5/2) = 0 \). The lemniscatic and equianharmonic cases follow from the first claim. In the isotropic case \( \tilde{H}_s = a s(s + 1)I d \), which implies the last statement.

In the general case the elliptic Bernoulli polynomials are not zero and their highest coefficients are described by the following

**Theorem 3.4.** The highest term \( A_0 s^{2k+1} \) of the elliptic Bernoulli polynomial \( B_{2k+1}(s) = A_0 s^{2k+1} + A_1 s^{2k} + \ldots + A_{2s} \) can be written

\[
A_0 s^{2k+1} = 2 \int_0^s \text{Res} \, \xi^{-1} [\gamma(s^2 - j^2)\xi + (\alpha s^2 + \beta j^2) + \gamma(s^2 - j^2)\xi^{-1}]^k \, dj, \quad (3.10)
\]

where \( \alpha = \frac{1}{2}(a_1 + a_2) \), \( \beta = \frac{2a_3 - a_1 - a_2}{2} \), \( \gamma = \frac{1}{4}(a_1 - a_2) \).

Indeed, for a large \( s \) and \( j \) the leading behaviour of the matrix elements of \( \tilde{H} \) is

\[
< j|\tilde{H}|j > = \frac{1}{2} (a_1 + a_2) [s^2 - j^2] + a_3 j^2 = \alpha s^2 + \beta j^2,
\]

\[
< j|\tilde{H}|j + 2 > = < j + 2|\tilde{H}|j > = \frac{1}{4} (a_1 - a_2) (s^2 - j^2) = \gamma(s^2 - j^2).
\]

Therefore the leading term of the diagonal element \( < j|\hat{H}|j > \) coincides with the constant term of the Laurent polynomial \( [\gamma(s^2 - j^2)\xi + (\alpha s^2 + \beta j^2) + \gamma(s^2 - j^2)\xi^{-1}]^k \) in the auxiliary variable \( \xi \). Replacing the summation over \( j \) by the integration, which is correct in the leading order, we come to the formula (3.10)

Note that from this formula the fact that the final result is a symmetric function of \( a_1, a_2, a_3 \) (and thus is a polynomial in \( g_1, g_2, g_3 \)) is not obvious at all.

**Remark.** From the quasi-classical arguments we can write the highest coefficient \( A_0 \) as the following integral over the unit sphere

\[
A_0 = \frac{1}{2\pi} \int_{|M|^2=1} H^k d\Omega = \frac{1}{2\pi} \int_{|M|^2=1} (a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2)^k d\Omega, \quad (3.11)
\]

where \( d\Omega \) is the area element on the unit sphere. Thus the formula (3.10) gives an expression for this integral. It would be interesting to compare it with the calculation of this integral using elliptic coordinates.
3.3 Effective way to compute the elliptic Bernoulli polynomials

Although the definition of the elliptic Bernoulli polynomials themselves gives a way to compute them as traces of powers of the given matrices $H_s$, it does not seem to be as effective as the following procedure based on the fact that the matrix $H_s$ is tri-diagonal.

Indeed, in the standard basis of $V_s$ consisting of the eigenvectors $|j>$ of $\tilde{M}_s$, $\tilde{M}_s|j> = j|j>$, $j = -s, -s+1, \ldots, s-1, s$ (see section 3.2), the eigenvalue problem $\tilde{H}\psi = \lambda\psi$ leads to the following difference equation:

$$c_{n-2}\psi_{n-2} + v_n\psi_n + c_n\psi_{n+2} = \lambda\psi_n,$$  \hspace{1cm} (3.12)

where

$$c_n(s) = \frac{a_1 - a_2}{4}\sqrt{(s-n)(s-n-1)(s+n+1)(s+n+2)},$$

$$v_n(s) = \frac{1}{2}(a_1 + a_2)[s(s+1) - n^2] + a_3n^2.$$

For such an equation one can use the standard procedure (see e.g. [18]) from the theory of solitons to find the local spectral densities, which are difference analogues of the KdV densities discussed in chapter 2. In our case it works as follows:

Let $\chi_n = \frac{c_n\psi_{n+2}}{\psi_n}$, then the equation (3.12) becomes

$$c_{n-2}^2 + (v_n - \lambda)\chi_{n-2} + \chi_n\chi_{n-2} = 0$$ \hspace{1cm} (313)

We look for a solution in the form $\chi_n = \lambda - \sum_{k=0}^{\infty} \chi_{n,k}\lambda^{-k}$ Substitution of this expression into the equation (313) gives $\chi_{n,0} = v_n, \chi_{n,1} = c_{n-2}^2, \chi_{n,2} = c_{n-2}^2 v_{n-2}$, and for general $k \geq 1$ the recurrence relation:

$$\chi_{n,k+1} = \sum_{i=1}^{k} \chi_{n,i}\chi_{n-2,k-1}.$$ \hspace{1cm} (314)

Let $X = \sum_{k=0}^{\infty} \chi_{n,k}\lambda^{-(k+1)}$ so that $\chi_n = \lambda(1 - X)$ and $\log \chi_n = \log \lambda - \sum_{k=1}^{\infty} \frac{X_k}{k}$. Thus we have

$$\log \chi_n - \log \lambda = -\sum_{k=1}^{\infty} \frac{I_{n,k}}{\lambda^k},$$ \hspace{1cm} (315)
where \( I_{n,1} = v_n, \ I_{n,2} = c_n^2 + \frac{v_n^2}{2}, \ I_{n,3} = c_n^2 v_{n-1} + v_n c_{n-2} + \frac{v_n^3}{3}, \ldots \)

On the other hand one can check that \( \prod_n x_n = \prod_m (1 - E_m^2) \) where \( E_m = E_m(s) \) are the eigenvalues of \( \hat{H}_s \). Thus

\[
\sum_n (\log x_n - \log \lambda) = -\sum_m \sum_{k=1}^{\infty} \frac{(E_m)^k}{k\lambda^k} = -\sum_{k=1}^{\infty} \frac{Tr \hat{H}_s^k}{k\lambda^k}
\]

Comparing this with the equations (3.15), we obtain

\[
Tr \hat{H}_s^k = k \sum_n I_{n,k} = k \sum_{n=-s}^{s} I_{n,k}
\]

**Theorem 3.5.** The elliptic Bernoulli polynomials \( B_{2k+1} = B_{2k+1}(s) \) can be computed as

\[
B_{2k+1} = k \sum_{n=-s}^{s} I_{n,k}, \tag{3.16}
\]

where \( I_{n,k} \) are the local densities determined by the relations (3.14, 3.15)

This gives a very effective way to compute the elliptic Bernoulli polynomials since the local densities are polynomials in \( c_n^2 \) and \( v_n \) (and hence in \( n \)) and thus the summation over \( n \) can be done with the use of the standard Bernoulli polynomials. We have applied this procedure to find the first 9 elliptic Bernoulli polynomials using Mathematica.

### 3.4 Application: coefficients of the Lamé spectral polynomials

We will consider again the generalised version of the Lamé equation (3.5). The coefficients \( b_k = b_k(s) \) of the corresponding Lamé spectral polynomial

\[
R_{2s+1}(E) = \prod_{i=0}^{2s} (E - E_i(s)) = E^{2s+1} + b_1 E^{2s} + b_2 E^{2s-1} + \ldots + b_{2s} E^{2s-k+1} + \ldots + b_{2s+1}
\]
up to a sign are the elementary symmetric functions of the eigenvalues

\[ b_k(s) = (-1)^k \tilde{\varepsilon}_k(s), \]

where \( \tilde{\varepsilon}_1(s) = \sum_i E_i(s), \) \( \tilde{\varepsilon}_2(s) = \sum_{i<j} E_i(s)E_j(s), \) \( \tilde{\varepsilon}_3(s) = \sum_{i<j<k} E_i(s)E_j(s)E_k(s) \)

The elementary symmetric functions are related to power sums \( B_{2k+1}(s) = \sum_i [E_i(s)]^k \)

by the following well-known relations

\[ k\tilde{\varepsilon}_k = \sum_{j=1}^{k} (-1)^{j-1} B_{2j+1} \tilde{\varepsilon}_{k-j} \]

with \( \tilde{\varepsilon}_0 = b_0 = 1 \) (see e.g. [45]) This implies the following.

**Theorem 3.6.** The coefficients \( b_k \) of the Lamé spectral polynomial \( R_{2s+1}(E) \) are related to the elliptic Bernoulli polynomials \( B_{2j+1} \) by the recurrence relation

\[ b_k = -\frac{1}{k} \sum_{j=1}^{k} B_{2j+1}(s)b_{k-j}. \]

The coefficient \( b_k \) is a polynomial in \( s, g_1, g_2, g_3 \) with rational coefficients. As a polynomial in \( s \) it has degree \( 3k \) and is divisible by \((s+1) s (s-1) \cdot(s-\lfloor \frac{k-3}{2} \rfloor)\)

The last part of Theorem 3.6 results from the properties of \( B_{2j+1} \) and the fact that \( b_k = 0 \) for \( k > 2s+1 \).

One can apply this result also to the case of a half-integer spin \( s \): all the roots of the polynomial \( R_{2s+1}(E) \) are then double and correspond to the doubly-periodic solutions of the Lamé equation.

In the reduced case \((g_1 = 0)\) the degree of \( b_k \) drops to \( \lfloor \frac{5k}{2} \rfloor \) (for \( k > 1 \)). Using the explicit form of the elliptic Bernoulli polynomials given in Appendix C one can find the first seven coefficients \( b_k \), which in reduced case are

\[ b_1 = 0, \]

\[ b_2 = -\frac{g_2}{120} s(s+1)(2s-1)(2s+1)(2s+3), \]
\[ b_3 = -\frac{g_3}{840} s(s + 1)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5), \]

\[ b_4 = \frac{g_5^2}{201600} s(s - 1)(s + 1)(2s - 1)(2s + 1)(2s + 3)(56s^4 + 76s^3 - 94s^2 + 201s + 630), \]

\[ b_5 = \frac{g_5 g_3}{1108800} s(s - 1)(s + 1)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5)(88s^4 + 68s^3 - 302s^2 + 663s + 1890), \]

\[ b_6 = \frac{g_5^2}{201801600} (s - 2)(s - 1)s(s + 1)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5) \times \]
\[ (4576s^5 + 12944s^4 - 20720s^3 + 48312s^2 + 597150s + 779625) - \]
\[ \frac{g_5^2}{10378368000} (s - 2)(s - 1)s(s + 1)(2s - 5)(2s - 1)(2s - 3)(2s + 1)(2s + 3) \times \]
\[ (16016s^6 + 89232s^5 + 197160s^4 + 544280s^3 + 2033829s^2 + 3858813s + 2619540), \]

\[ b_7 = \frac{g_5^2 g_3}{242161920000} (s - 3)(s - 2)(s - 1)s(s + 1)(2s - 5)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5) \times \]
\[ (32032s^6 + 189072s^5 + 463440s^4 + 1682920s^3 + 7301418s^2 + 15249213s + 11351340) \]
Chapter 4

Density of states of Lamé operators

We apply now the theory of elliptic Faulhaber and Bernoulh polynomials to compute the density of states for the Lamé operators. The density of states is one of the most important notions in the spectral theory of the Schrodinger operators with periodic potential (see e.g. [56]). In the finite-gap case, in particular for the Lamé operator

\[ L_n = -\frac{d^2}{dx^2} + n(n+1)\varphi(x + \omega'), \]

the density of states has the form

\[ \rho_n(E) = \frac{1}{2\pi} \frac{P_n(E)}{\sqrt{R_{2n+1}(E)}}, \quad (4.1) \]

where \( P_n(E) = E^n + a_1 E^{n-1} + a_2 E^{n-2} + \cdots + a_n \) for some coefficients \( a_1, a_2, \ldots, a_n \) and \( R_{2n+1}(E) = \prod_{m=0}^{2n}(E - E_m) = \sum_{k=0}^{2n+1} b_k E^{2n+1-k} \) is the Lamé spectral polynomial (see [14, 54] and the remark at the end of this chapter). We are going to explain how to find the coefficients \( a_k \) as a function of \( n \) in terms of the elliptic Faulhaber polynomials and the coefficients of the Lamé spectral polynomials found in section 3.4.

**Theorem 4.1.** The coefficient \( a_k = a_k(n) \) in the numerator \( P_n(E) = E^n + a_1 E^{n-1} + a_2 E^{n-2} + \cdots + a_n \) of the Lamé density of states (4) is a polynomial in \( n \) of degree \( \left[ \frac{5k}{2} \right] \). Its coefficients are polynomials with rational coefficients of \( \varphi, g_2, g_3 \) and \( \bar{\varphi} = -\frac{n}{\omega} \), homogeneous of weight \( 2k \), where the weights of \( \varphi, g_2, g_3 \) are 2, 4, 6 respectively.

Let us start with the known results about the density of states for the Schrodinger operator \( L = -\frac{d^2}{dx^2} + u(x) \) on \( \mathbb{R} \) with real periodic potential \( u(x) \) of period \( T \) (see
The density of states of the Lamé operators \( \text{e.g. [54, 56]} \). The spectrum of the operator \( L \) is known to be continuous and has a band structure with in general an infinite number of gaps. Consider the restriction \( L^{(R)} \) of \( L \) on \( (-R, R) \), for \( R > 0 \), with Dirichlet boundary conditions at \( \pm R \). The spectrum of \( L^{(R)} \) is then discrete. Denote the eigenvalues as \( \mathcal{E}_n(L^{(R)}) \) and define the integrated density of states \( N(E) \) as the limit

\[
N(E) = \lim_{R \to \infty} \frac{\# \{ \mathcal{E}_n(L^{(R)}) < E \}}{2R},
\]

where \( \# \{ \mathcal{E}_n(L^{(R)}) \} \) is the counting function of the discrete spectrum of \( L^{(R)} \). It is known that such limit exists (see e.g. [56]) and has the following asymptotic expansion as \( E \to \infty \)

\[
N(E) = \frac{\sqrt{E}}{\pi} - \frac{1}{2\pi T \sqrt{E}} \sum_{k=0}^{\infty} \frac{I_{k+1}[u]}{(4E)^k},
\]

(4.2)

where \( I_k[u] = \int_0^T T_k(u,u,u_2, \ldots, u_{k-2})dx \) are the KdV integrals (2.2) evaluated over period \( T \).

The density of states is defined as the derivative of the integrated density of states

\[
\rho(E) = \frac{dN(E)}{dE}.
\]

It is known (see [56]) that for a periodic potential \( u(x) \) the integrated density of states is related in a very simple way to the so-called quasi-momentum \( p(E) \)

\[
N(E) = \frac{p(E)}{\pi}
\]

(4.3)

Recall that the quasi-momentum \( p(E) \) appears naturally in relation to the Bloch (Floquet) solutions of the Schrödinger equation \( L\psi = E\psi \). These solutions have the form \( \psi_E(x) = \exp^{ip(E)x} \phi(x) \), where \( \phi(x) \) is a periodic function of period \( T \).

If \( u(x) \) is a finite gap potential with \( n \) gaps then the derivative of the quasi-momentum \( p(E) \) has the form [54]

\[
\frac{dp(E)}{dE} = \frac{1}{2} \frac{P_n(E)}{\sqrt{R_{2n+1}(E)}},
\]
where \( P_n(E) = E^n + a_1 E^{n-1} + a_2 E^{n-2} + \ldots + a_n \) for some coefficients \( a_1, a_2, \ldots, a_n \) and \( R_{2n+1}(E) = \prod_{m=0}^{2n}(E - E_m) \), where \( E_0, E_1, \ldots, E_{2n} \) are the end points of the spectrum intervals of \( L \).

Thus in that case, the density of states has the following form

\[
\rho(E) = \frac{1}{2\pi} \frac{P_n(E)}{\sqrt{R_{2n+1}(E)}}. \tag{4.4}
\]

Note that from (4.2) it has the following expansion at infinity

\[
\rho(E) = \frac{1}{2\pi \sqrt{E}} + \frac{1}{\pi \lambda \sqrt{E}} \sum_{k=1}^{\infty} \frac{(2k-1)}{E^k}. \tag{4.5}
\]

Consider the case of the Lamé operator \( L_n = -\frac{d^2}{dx^2} + n(n+1)\varphi(x + \omega') \) The integrals \( I_k[u] \) are then the elliptic Faulhaber polynomials \( F_k^W(\lambda) \), where \( \lambda = \frac{n(n+1)}{2} \) and \( T = 2\omega \). Thus, the high-energy asymptotics for the densities of states of the Lamé operator can be given in terms of the elliptic Faulhaber polynomials as follows

\[
N_n(E) = \frac{\sqrt{E}}{\pi} - \frac{1}{4\pi \omega \sqrt{E}} \sum_{k=0}^{\infty} \frac{F_{k+1}^W(\lambda)}{(4E)^k}, \quad \lambda = \frac{n(n+1)}{2}, \tag{4.6}
\]

\[
\rho_n(E) = \frac{1}{2\pi \sqrt{E}} \left[ 1 + \frac{1}{2\omega} \sum_{k=1}^{\infty} \frac{2k-1}{E^k} \right], \quad \lambda = \frac{n(n+1)}{2} \tag{4.7}
\]

Thus in principle, if we know all the Faulhaber polynomials then we know the density of states of the Lamé operators \( L_n \) for all \( n \).

The most effective way to compute \( \rho_n(E) \) is to express first the coefficients \( b_k \) of the Lamé spectral polynomial using the elliptic Bernoulli polynomials in the reduced case \( g_1 = 0 \) (see section 3.4)

\[ b_1 = 0, \]

\[ b_2 = -\frac{g_2}{120} n(n+1)(2n-1)(2n+1)(2n+3), \]

\[ b_3 = -\frac{g_3}{840} n(n+1)(2n-3)(2n-1)(2n+1)(2n+3)(2n+5), \]
\[ b_4 = \frac{g_2^2}{201600} n(n-1)(n+1)(2n-1)(2n+1)(2n+3)(56n^4+76n^3-94n^2+201n+630). \]

Re-writing the high energy asymptotics of the density of states of the Lamé operator \( L_n \) in the form

\[ \frac{1}{2\pi} \frac{P_n(E)}{\sqrt{R_{2n+1}(E)}} = \frac{1}{2\pi\sqrt{E}} \left[ 1 + \frac{a_1}{E} + \frac{a_2}{2E} + \frac{a_3}{2^2E^2} + \frac{a_4}{2^3E^3} + \cdots \right] \]

and equating the coefficients term by term with the expansion (4.7) gives

\[ a_1 = \frac{\mathcal{F}_1^W(\lambda)}{4\omega} = -\frac{n(n+1)}{\omega} \]

\[ a_2 = \frac{b_2}{2} + \frac{3\mathcal{F}_2^W(\lambda)}{16\omega} \]

\[ a_3 = \frac{a_1b_2}{2} + \frac{b_3}{2} + \frac{5\mathcal{F}_3^W(\lambda)}{2^6\omega} \]

\[ a_4 = \frac{a_2b_2}{2} + \frac{3b_2^2}{8} + \frac{a_1b_3}{2} + \frac{b_4}{2} + \frac{7\mathcal{F}_4^W(\lambda)}{2^8\omega} \]

From this we find recursively

\[ a_1 = \frac{n(n+1)}{2} \tilde{\varphi}, \]

\[ a_2 = -\frac{g_2}{480} (n-1)n(n+1)(6+25n+16n^2), \]

\[ a_3 = -\frac{g_3}{3360} (n-2)(n-1)n(n+1)(45+243n+247n^2+64n^3) \]

\[ -\frac{g_4}{960} (n-2)(n-1)n(n+1)(27+16n), \]

\[ a_4 = \frac{g_2^2}{3225600} (n-3)(n-2)(n-1)n(n+1)(-2520-12942n-10315n^2+4565n^3+6880n^4+1792n^5) \]

\[ -\frac{g_3\tilde{\varphi}}{13440} (n-3)(n-2)(n-1)n(n+1)^2(600+563n+128n^2), \]

where \( \tilde{\varphi} = -\frac{g_1}{\omega} \) is the average of \( \varphi(x) \) over a period. Note that the coefficient \( a_k \) is divisible by \((n+1)n(n-1)(n-2)\cdot(n-k+1)\), which is related to the fact that \( L_n \) has \( n \) gaps for integer \( n \). All this implies Theorem 4.1.
Thus the density of states of the Lamé operator
\[ \rho_n(E) = \frac{1}{2\pi} \frac{P_n(E)\,dE}{\sqrt{R_{2n+1}(E)}} = \frac{1}{2\pi} \frac{E^n + a_1E^{n-1} + a_2E^{n-2} + \ldots + a_n}{\sqrt{E^{2n+1} + b_1E^{2n} + \ldots + b_{2n+1}}} \]
can be written, in principle, explicitly in terms of \(E, n, g_2, g_3, \phi\).

For \(n = 1\) the potential \(u(x) = 2\varphi(x)\) and
\[ \rho_1(E) = \frac{1}{2\pi} \frac{E + \phi}{\sqrt{E^3 - \frac{g_2}{4} E + \frac{g_3}{4}}} = \frac{1}{2\pi} \frac{E + \phi}{(E + e_1)(E + e_2)(E + e_3)}. \]

For \(n = 2\), \(u(x) = 6\varphi(x)\) and
\[ \rho_2(E) = \frac{E^2 + 3\phi E - \frac{3}{2} g_2}{\pi \sqrt{4E^3 - 21g_2E^3 - 27g_3E^2 + 27g_4^2 E + 81g_2g_3}}. \]

For \(n = 3\), \(u(x) = 12\varphi(x)\) and
\[ \rho_3(E) = \frac{E^3 + 6\phi E^2 - \frac{45}{4} g_2 E - \frac{135}{4} g_3 - \frac{45}{2} g_2 \phi}{2\pi \sqrt{E^7 - \frac{63}{2} g_2 E^5 - \frac{207}{2} g_3 E^4 + \frac{4185}{16} g_2^2 E^3 + \frac{18225}{8} g_2 g_3 E^2 - \frac{3375}{16} (g_2^3 - 27g_3^2) E}}. \]

One can see the explicit form of the coefficients \(a_5, a_6, a_7\) as well as \(\rho_4(E)\) and \(\rho_5(E)\) in Appendix B. Our results agree with Belokolos-Enolski and Takemura calculations [7, 57], who used a modified version of the classical approach to this problem originated by Krichever in his paper [42].

Remark. The differential \(\rho(E)dE = \frac{1}{2\pi} \frac{P_n(E)\,dE}{\sqrt{R_{2n+1}(E)}}\) has a simple algebro-geometric meaning: it is an Abelian differential with a second order pole at infinity normalised by the condition that all the periods over gaps (b-periods) are zero (see [14, 54]). This means that once the equation of the spectral curve is known one can find the coefficients of the numerator by solving a linear system of equations with the coefficients being some standard hyperelliptic integrals on this curve. From our results it follows that in the Lamé case these coefficients can actually be expressed polynomially in terms of the standard elliptic integrals and the parameters \(g_2, g_3\) of the elliptic curve defining the potential. This is of course related to the reduction problem for the Lamé spectral curves (see [7, 17, 57]).
Part II

Periodic continued fractions and hyperelliptic curves
The usual continued fraction expansions of real numbers

\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} \]

have been known essentially since Euclid but the terminology "continued fractions" was introduced by John Wallis in the 17th century only.

Throughout the 19th century continued fractions have been used to represent analytic functions. In his book [67] H S. Wall lists continued fraction expansions for many special functions, in particular as \( J \)-fractions

\[ \frac{a_0}{b_1 + \lambda + \frac{a_1}{b_2 + \lambda + \frac{a_2}{\ddots}}} \]

A very interesting application of continued fractions in the theory of elliptic integrals was found by Abel and Chebyshev [11], [12]. For a more recent result in this direction we refer to a recent paper [62]. We would like to mention also a note by Jacobi [37] who used the elliptic functions to find explicitly the continued fraction expansion of a function of the form \( \frac{J + \sqrt{R}}{N} \), where \( R \) is a polynomial of degree at most four.

The main object of our investigation is the following continued fraction, which we call an \( \alpha \)-fraction:

\[ \phi = b_0 + \frac{\lambda - \alpha_1}{b_1 + \frac{\lambda - \alpha_2}{b_2 + \ddots}} = [b_0, b_1, \ldots, \alpha]_\alpha, \quad (4.8) \]

where \( \alpha = (\alpha_i), \alpha_i \in \mathbb{C} \) is a given sequence, \( b_i \) are arbitrary complex numbers, and \( \lambda \) is a formal parameter. This kind of fraction appears in the theory of integrable systems, in particular in the theory of the periodic dressing chain [65].

We examine a special case of \( N \)-periodic \( \alpha \)-fractions

\[ \phi = [b_0, b_1, b_2, \ldots, b_N]_\alpha, \quad (4.9) \]

when the sequences \( \alpha_i \) and \( b_i \) are periodic with period \( N \)

\[ \alpha_{i+N} = \alpha_i, \ b_{i+N} = b_i \ for \ all \ i \geq 1 \]
In the particular case when $b_N = b_0$ we have $\phi = [b_0, b_1, \ldots, b_{N-1}]_\alpha$, which we call a purely $N$-periodic $\alpha$-fraction.

We restrict our study to the case of odd period $N = 2g + 1$. We also assume that all the parameters $\alpha_i$ are distinct.

We will discuss the following questions:

**Question 1.** Which algebraic functions admit $N$-periodic $\alpha$-fraction expansions?

**Question 2.** How many such expansions may exist for a given algebraic function and how do we find them?

**Question 3.** What is the geometry of the set of functions which admit periodic $\alpha$-fraction expansions?

We will show that the answers are closely related to the classical theory of hyper-elliptic curves.
Chapter 5

Periodic $\alpha$-fractions

Consider a $N$-periodic $\alpha$-fraction of period $N = 2g + 1$

$$\phi = b_0 + \frac{\lambda - \alpha_1}{b_1 + \frac{\lambda - \alpha_2}{b_2 + \cdots + \frac{\lambda - \alpha_{N-1}}{b_{N-1} + \frac{\lambda - \alpha_N}{b_N + \frac{\lambda - \alpha_1}{b_1 + \cdots}}}}} = [b_0, \overline{b_1, \ldots, b_{N-1}, b_N}]_\alpha \quad (5.1)$$

Because of periodicity we can write formally (5.1) as

$$\phi = b_0 + \frac{\lambda - \alpha_1}{b_1 + \frac{\lambda - \alpha_2}{b_2 + \cdots + \frac{\lambda - \alpha_{N-1}}{b_{N-1} + \frac{\lambda - \alpha_N}{b_N + \frac{\lambda - \alpha_1}{b_1 + \cdots}}}}}$$

which implies a quadratic relation

$$A(\lambda)\phi^2 + 2B(\lambda)\phi + C(\lambda) = 0, \quad (5.2)$$

where $A, B, C$ are certain polynomials in $\lambda$ with coefficients polynomially depending on $b_i$. Thus to any periodic $\alpha$-fraction (5.1), there corresponds an algebraic function

$$\phi(\lambda) = \frac{-B(\lambda) + \sqrt{R(\lambda)}}{A(\lambda)}, \quad (5.3)$$

where

$$R(\lambda) = B(\lambda)^2 - A(\lambda)C(\lambda) \quad (5.4)$$
PART II: Periodic $\alpha$-fractions

is the discriminant of (5.2). In that case we will say that (5.1) is a periodic $\alpha$-fraction expansion of the algebraic function (5.3) from the hyperelliptic extension $C(\lambda, \sqrt{R(\lambda)})$ of the field of rational functions $C(\lambda)$.

Recall that a polynomial $R$ of degree $N$ is $\alpha$-admissible if

$$R(\lambda) = S^2(\lambda) + \mathcal{A}(\lambda)$$

(5.5)

where $\mathcal{A}(\lambda) = \prod_{i=1}^{N}(\lambda - \alpha_i)$ and $S(\lambda)$ is some polynomial of degree $g$ or less, where as before $N = 2g + 1$

A polynomial is called monic if its highest coefficient is 1 and anti-monic if it is equal to $-1$. Note that $\alpha$-admissible polynomials $R$ are automatically monic.

**Theorem 5.1.** The algebraic functions $\phi(\lambda)$ admitting an $N$-periodic $\alpha$-fraction expansion have the form (5.2, 5.3) with the polynomials $A, B, C$ satisfying the following conditions

1. $\deg B \leq g$, $A(\lambda)$ and $C(\lambda)$ are monic and anti-monic polynomials of degree $g$ and $g+1$ respectively

2. the discriminant $R(\lambda) = B^2 - AC$ is $\alpha$-admissible.

Conversely, for an open dense subset of such triples $(A, B, C)$ the corresponding function (5.3) has exactly two $N$-periodic $\alpha$-fraction expansions. The corresponding $b_\alpha$ are rational functions of both the coefficients of $A, B, C$ and the parameters $\alpha_i$, and can be found by an effective matrix factorisation procedure

We see that (at least formally) $\phi$ is a fixed point of the fractional linear transformation

$$s(\phi) = b_0 + \frac{a_1}{b_1 + b_2 + \cdots + b_{N-1} + \frac{a_N}{P_{N+k}}}$$

(5.6)

with $b_N = b_N - b_0$ and $a_k = \lambda - \alpha_k$, $k = 1, \ldots, N$ The function $s(\phi)$ can be written as $s(\phi) = \frac{P_{N-1}\phi + P_N}{Q_{N-1}\phi + Q_N}$, where the quantities $P_k, Q_k$ are determined by the standard recurrence relations (see e.g. [67], page 14):

$$P_{-1} = 1, \quad Q_{-1} = 0,$$
\[ P_0 = b_0, \quad Q_0 = 1, \]
\[ P_{k+1} = b_{k+1}P_k + a_{k+1}P_{k-1}, \quad Q_{k+1} = b_{k+1}Q_k + a_{k+1}Q_{k-1} \] (5.7)
for \( k < N - 1 \) and
\[ P_N = b_N^*P_{N-1} + a_NP_{N-2}, \quad Q_N = b_N^*Q_{N-1} + a_NQ_{N-2} \]
Thus we have
\[ \phi = \frac{P_{N-1}\phi + P_N}{Q_{N-1}\phi + Q_N}, \] (5.8)
which can be written as a quadratic equation
\[ Q_{N-1}\phi^2 + (Q_N - P_{N-1})\phi - P_N = 0. \] (5.9)
It is easy to see from the recurrence relations that \( P_k \) and \( Q_k \) are polynomials in \( \lambda \) of the form
\[ P_{2k} = (b_0 + b_2 + \ldots + b_{2k})\lambda^k + \ldots, \quad P_{2k+1} = \lambda^{k+1} + \ldots, \]
\[ Q_{2k} = \lambda^k + \ldots, \quad Q_{2k-1} = (b_1 + b_3 + \ldots + b_{2k-1})\lambda^{k-1} + \ldots, \]
for \( k \leq g \) and \( Q_{2g+1} = (b_1 + b_3 + \ldots + b_{2g-1} + b_{2g+1} - b_0)\lambda^g + \ldots \), where the dots denote the lower degree terms. From (5.9) we have
\[ A(\lambda) = Q_{N-1}(\lambda), \quad B(\lambda) = \frac{1}{2}(Q_N(\lambda) - P_{N-1}(\lambda)), \quad C(\lambda) = -P_N(\lambda). \] (5.10)
Thus the polynomial \( A \) is monic of degree \( g \) and \( C \) is anti-monic of degree \( g + 1 \). The polynomial \( B \) has degree \( g \) or less and the coefficient \( \beta \) of its highest term \( \beta\lambda^g \) is
\[ \beta = -b_0 + \frac{1}{2} \sum_{k=1}^{N} (-1)^{k+1}b_k. \]
Let us now show that the discriminant \( R = B^2 - AC \) is \( \alpha \)-admissible. We have
\[ R = \frac{1}{4}(Q_N - P_{N-1})^2 + P_NQ_{N-1} = \frac{1}{4}(P_{N-1} + Q_N)^2 + P_NQ_{N-1} - P_{N-1}Q_N. \]
We claim that
\[ P_NQ_{N-1} - P_{N-1}Q_N = \prod_{i=1}^{N}(\lambda - \alpha_i) \]
Indeed, the determinant

$$
\begin{vmatrix}
P_N & P_{N-1} \\
Q_N & Q_{N-1}
\end{vmatrix}
= \begin{vmatrix}
b_N P_{N-1} + a_N P_{N-2} & P_{N-1} \\
b_N Q_{N-1} + a_N Q_{N-2} & Q_{N-1}
\end{vmatrix} = -a_N \begin{vmatrix}
P_{N-1} & P_{N-2} \\
Q_{N-1} & Q_{N-2}
\end{vmatrix} = \cdots = (-1)^N a_N a_{N-1} \cdots a_1 \begin{vmatrix}
b_0 & 1 \\
1 & 0
\end{vmatrix}
$$

Since $N$ is odd,

$$
\begin{vmatrix}
P_N & P_{N-1} \\
Q_N & Q_{N-1}
\end{vmatrix} = a_N a_{N-1} \cdot a_1 = \prod_{i=1}^N (\lambda - \alpha_i) = \mathfrak{A}
$$

Now by taking $S(\lambda) = \frac{P_{N-1} + Q_N}{2}$, which is a polynomial of degree $g$ or less, we see that $R(\lambda) = S^2 + \mathfrak{A}$, so $R$ is $\alpha$-admissible. This proves the first part of Theorem 5.1 in the periodic case.

To prove the second part let us introduce the following matrix

$$
M(\lambda) = \begin{bmatrix}
1 & b_0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
0 & \lambda - \alpha_1 \\
1 & b_1
\end{bmatrix} \cdots \begin{bmatrix}
0 & \lambda - \alpha_N \\
1 & b_N
\end{bmatrix},
(5.11)
$$

with $b_N^* = b_N - b_0$. One can check that it can be rewritten also as

$$
M = \begin{bmatrix}
b_0 & \lambda - \alpha_1 \\
1 & 0
\end{bmatrix} \cdots \begin{bmatrix}
b_{N-1} & \lambda - \alpha_N \\
1 & 0
\end{bmatrix} \begin{bmatrix}
1 & b_N^* \\
0 & 1
\end{bmatrix}
(5.12)
$$

The following Lemma explains its importance for our problem

**Lemma 5.2.** Vector $\left( \begin{array}{c} \phi \\ 1 \end{array} \right)$ with $\phi = [b_0, b_1, \ldots, b_{N-1}, b_N]_\alpha$ is an eigenvector of the matrix $M(\lambda)$.

The proof follows from the fact that $\phi$ is a fixed point of the fractional linear transformation (5.6). The product of matrices (5.11) corresponds to the representation of $s(\phi)$ as a superposition $s_0 \circ s_1 \circ \cdots \circ s_N(\phi)$, where $s_0(\phi) = b_0 + \phi$, $s_k(\phi) = \frac{\lambda - \alpha_k}{b_k + \phi}$ for $k = 1, 2, \ldots, N - 1$ and $s_N(\phi) = \frac{\lambda - \alpha_N}{b_N + \phi}$.

Let $T(\lambda) = \frac{1}{2} tr M$ be half of the trace of the matrix $M(\lambda)$, which is a polynomial of degree $g$ or less. Note that the determinant of $M$ is equal to $-\mathfrak{A} = -\prod_{i=1}^N (\lambda - \alpha_i)$ as it follows immediately from (5.11).
Lemma 5.3. The matrix (5.11) has the form

\[ M(\lambda) = \begin{bmatrix} T(\lambda) - B(\lambda) & -C(\lambda) \\ A(\lambda) & T(\lambda) + B(\lambda) \end{bmatrix}, \quad (5.13) \]

where \((A, B, C)\) is the \(\alpha\)-tuple of polynomials corresponding to \(\phi\). The discriminant \(R = B^2 - AC\) equals to \(T^2 + \alpha\).

Indeed

\[ M(\lambda) = \begin{bmatrix} P_{N-1}(\lambda) & P_N(\lambda) \\ Q_{N-1}(\lambda) & Q_N(\lambda) \end{bmatrix}, \]

where \(P_k, Q_k\) are defined above by (5.7). Now the first claim follows from the relations (5.10). Taking the determinant of both sides of (5.13) we have \(-\alpha = T^2 - B^2 + AC\), which implies \(B^2 - AC = T^2 + \alpha\).

Now we need the following result on the factorisation of such matrices. This kind of problem often appears in the theory of discrete integrable systems (e.g. [4] and [50]).

**Proposition 5.4.** Let \(M(\lambda)\) be a polynomial matrix of the form (5.13), where \(A\) is a monic polynomial of degree \(g\), \(C\) is an anti-monic polynomial of degree \(g + 1\), \(T\) and \(B\) are polynomials of degree \(g\) or less. Assume also that \(\det M(\lambda) = -\prod_{i=1}^{N} (\lambda - \alpha_i)\). Then for an open dense set of such \(M\) there exists a unique factorisation of the form

\[ M(\lambda) = \begin{bmatrix} b_0 & \lambda - \alpha_1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_{N-1} & \lambda - \alpha_N \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & b_N - b_0 \\ 0 & 1 \end{bmatrix}. \]

We describe the procedure which allows to find \(b_i\) uniquely assuming at the beginning that the factorisation exists.

Consider the transpose \(M^T\) of the matrix \(M\). For \(\lambda = \alpha_1\) the matrix \(M^T(\lambda)\) is degenerate (since \(\det M^T(\lambda) = \det M(\lambda) = -\prod_{i=1}^{N} (\lambda - \alpha_i)\)). Find the null-vector \(e_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\) of \(M^T(\alpha_1)\), which is by definition any non-zero vector such that

\[ M^T(\alpha_1)e_1 = 0, \quad (5.14) \]
or explicitly
\[
\begin{bmatrix}
T(\alpha_1) - B(\alpha_1) & A(\alpha_1) \\
-C(\alpha_1) & T(\alpha_1) + B(\alpha_1)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

It must satisfy the relation
\[
\begin{bmatrix}
b_0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
since all other factors are non-degenerate when \( \lambda = \alpha_1 \) This determines \( b_0 \) uniquely as
\[
b_0 = \frac{T(\alpha_1) - B(\alpha_1)}{A(\alpha_1)} = -\frac{C(\alpha_1)}{T(\alpha_1) + B(\alpha_1)}. \tag{5.15}
\]

Consider now the matrix \( M_1 = \begin{bmatrix} b_0 & \lambda - \alpha_1 \\ 1 & 0 \end{bmatrix}^{-1} M(\lambda) \). It is polynomial in \( \lambda \) because of the following elementary Lemma

**Lemma 5.5.** Let \( M \) be the polynomial matrix, \( \lambda = \alpha \) be a simple root of its determinant and \( e = \begin{bmatrix} 1 \\ -b \end{bmatrix} \) be a null vector of \( M^T(\alpha) \) Then the matrix \( \begin{bmatrix} b & \lambda - \alpha \\ 1 & 0 \end{bmatrix}^{-1} M(\lambda) \) is polynomial.

Indeed, let \( M = \begin{bmatrix} X(\lambda) & Y(\lambda) \\ Z(\lambda) & W(\lambda) \end{bmatrix} \) then

\[
\begin{bmatrix} b & \lambda - \alpha \\ 1 & 0 \end{bmatrix}^{-1} M(\lambda) = \begin{bmatrix} Z(\lambda) & W(\lambda) \\ X(\lambda) - bZ(\lambda) & Y(\lambda) - bW(\lambda) \end{bmatrix}
\]

From \( M^T(\alpha) \begin{bmatrix} 1 \\ -b \end{bmatrix} = 0 \) it follows that \( \lambda = \alpha \) is a root of the polynomials \( X(\lambda) - bZ(\lambda) \) and \( Y(\lambda) - bW(\lambda) \). Therefore these polynomials are divisible by \( \lambda - \alpha \), which proves the claim.

Repeat now the procedure by taking \( \lambda = \alpha_2 \) and so on. After \( N \) steps we will come to the following polynomial matrix with determinant 1

\[
M_N(\lambda) = \left[ \begin{bmatrix} b_{N-1} & \lambda - \alpha_N \\ 1 & 0 \end{bmatrix}^{-1} \times \cdots \times \begin{bmatrix} b_0 & \lambda - \alpha_1 \\ 1 & 0 \end{bmatrix}^{-1} \right] M(\lambda).
\]
To complete the proof of Proposition 5.4 we need to show that $M_N$ is of the form
\[
\begin{bmatrix}
1 & b_N \\
0 & 1
\end{bmatrix}
\]

Recall that the matrix $M(\lambda)$ is of the form
\[
\begin{bmatrix}
a_0 \lambda^g + \ldots + \lambda^{g+1} + \ldots & d_0 \lambda^g + \ldots \\
\lambda^g + \ldots & d_0 \lambda^g + \ldots
\end{bmatrix},
\]
where the dots mean terms of lower degree, and the coefficients $a_0$ and $d_0$ may be zero.

It is easy to show that the matrix $M_2(\lambda) = \begin{bmatrix} b_1 & \lambda - \alpha_2 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} b_0 & \lambda - \alpha_1 \\ 1 & 0 \end{bmatrix}^{-1} M(\lambda)$
is of the form
\[
\begin{bmatrix}
a_0 \lambda^g + \ldots & \lambda^g + \ldots \\
\lambda^g + \ldots & d_0 \lambda^g + \ldots
\end{bmatrix}
\]
and by induction $M_{2k}(\lambda)$ is of the form
\[
\begin{bmatrix}
a_k \lambda^{g-k} + \ldots & \lambda^{g-k+1} + \ldots \\
\lambda^{g-k} + \ldots & d_k \lambda^{g-k} + \ldots
\end{bmatrix}.
\]
Therefore the matrix $M_{N-1}$ is of the form
\[
\begin{bmatrix}
a_g & \lambda + c \\
1 & d_g
\end{bmatrix}
\]
where $a_g, c, d_g$ are constant. The matrix $M_N = \begin{bmatrix} b_{N-1} & \lambda - \alpha_N \\ 1 & 0 \end{bmatrix}^{-1} M_{N-1}$
equals to
\[
\begin{bmatrix}
\frac{1}{\lambda - \alpha_N} & \frac{d_g}{\lambda - \alpha_N} \\
\frac{a_g - b_{N-1}}{\lambda - \alpha_N} & \frac{d_g}{\lambda - \alpha_N}
\end{bmatrix}.
\]
Since $M_N$ is a polynomial matrix, we have $a_g = b_{N-1}$ and $b_{N-1}d_g - c = \alpha_N$. Thus $M_N$ has the required form.

We see that the procedure will not work only if at some stage the first component of the null vector of $M_k^{T}(\alpha_{k+1})$ vanishes. Clearly this happens only for a closed algebraic subset of codimension 1, so for generic triples $(A, B, C)$ the matrix decomposition exists and is unique. This completes the proof of Proposition 5.4.

Now we are ready to finish the proof of Theorem 5.1 in the periodic case. Let $(A, B, C)$ be an $\alpha$-triple, then by definition there exists a polynomial $S$ of degree $g$ or less such that the discriminant $R = B^2 - AC$ is equal to $S^2 + 2$. Clearly the polynomial $S$ is defined up to a sign. Consider two corresponding matrices $M$ given by (5.13) with $T(\lambda) = \pm S(\lambda)$. Each of them generically has a unique factorisation given by Proposition 5.4. One can easily check that this gives two $N$-periodic $\alpha$-fractiOn representations of the corresponding function $\phi(\lambda) = \frac{-B(\lambda) + \sqrt{R(\lambda)}}{A(\lambda)}$ and thus completes the proof in this case.
Chapter 6

Purely periodic $\alpha$-fractions

Let now $\phi = [b_0, b_1, \ldots, b_{N-1}]_\alpha$ be a purely periodic $\alpha$-fraction. This is a particular case of the previous situation with $b_0 = b_N$. But since the corresponding $b_N^* = b_0 - b_N = 0$, this case is actually degenerate and needs special consideration.

**Theorem 6.1.** The algebraic functions $\phi(\lambda)$ admitting a purely $N$-periodic $\alpha$-fraction expansion have the form $\phi(\lambda) = \frac{-B(\lambda) + \sqrt{R(\lambda)}}{A(\lambda)}$ with the polynomials $A, B, C$ satisfying the following conditions:

1. $\deg B \leq g$, \quad $A(\lambda)$ and $C(\lambda)$ are monic and anti-monic polynomials of degree $g$ and $g + 1$ respectively

2. the discriminant $R(\lambda) = B^2 - AC$ is $\alpha$-admissible

3. $C(\alpha_N) = 0$

Conversely, for an open dense subset of such triples $(A, B, C)$ the corresponding function $\phi(\lambda) = \frac{-B(\lambda) + \sqrt{R(\lambda)}}{A(\lambda)}$ has an unique $N$-periodic $\alpha$-fraction expansion.

The proof is similar to the proof of Theorem 5.1. First of all as before, $\phi$ satisfies the relation

$$\phi = \frac{P_{N-1}\phi + P_N}{Q_{N-1}\phi + Q_N},$$

where $P_k, Q_k$ satisfy the relations (5.7), but now because $b_N^* = 0$ we have

$$P_N = (\lambda - \alpha_N)P_{N-2}, \quad Q_N = (\lambda - \alpha_N)Q_{N-2}$$

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Now from (5.10) we have
\[ A(\lambda) = Q_{N-1}(\lambda), \quad B(\lambda) = \frac{1}{2}((\lambda - \alpha_N)Q_{N-2}(\lambda) - P_{N-1}(\lambda)), \quad C(\lambda) = - (\lambda - \alpha_N)P_{N-2} \]

Since \( Q_{N-1} \) and \( P_{N-2} \) are monic this shows that \( A, B, C \) satisfy the property (1) of Theorem 6.1 with the additional condition (3) \( C(\alpha_N) = 0 \). The proof of the second property (\( \alpha \)-admissibility of \( R \)) goes unchanged.

Now as in the previous case in order to find the purely periodic \( \alpha \)-fraction one should factorise the matrix

\[ M(\lambda) = \begin{bmatrix} T(\lambda) - B(\lambda) & -C(\lambda) \\ A(\lambda) & T(\lambda) + B(\lambda) \end{bmatrix} \]

as the product

\[ \begin{bmatrix} b_0 & \lambda - \alpha_1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_{N-1} & \lambda - \alpha_N \\ 1 & 0 \end{bmatrix} \]

The main difference is that in the purely periodic case, the trace \( T(\lambda) \) of the matrix \( M(\lambda) \), which was known before only up to a sign, now is determined uniquely by the condition

\[ T(\alpha_N) = -B(\alpha_N). \quad (6.1) \]

Indeed \( T(\alpha_N) = \frac{1}{2}(P_{N-1}(\alpha_N) + Q_N(\alpha_N)) = \frac{1}{2}(P_{N-1}(\alpha_N)) = -B(\alpha_N) \) since \( Q_N(\alpha_N) = 0 \). If \( B(\alpha_N) \neq 0 \), which is a generic case, this determines \( T(\lambda) \) uniquely (and thus the matrix \( M \)) by a triple \((A, B, C)\). This completes the proof of Theorem 6.1.

**Example.** Consider the simplest case \( N = 1, g = 0 \). Then the \( \alpha \)-triples have the form

\[ A = 1, \quad B = \beta, \quad C = - (\lambda + \gamma) \]

with arbitrary \( \beta, \gamma \in \mathbb{C} \).

In the periodic case the corresponding \( \phi \) has the general form

\[ \phi = -\beta + \sqrt{\lambda + \gamma + \beta^2}. \quad (6.2) \]

In this particular case, \( \phi = [b_0, b_1, b_2]_\alpha \) satisfies the quadratic equation

\[ \phi^2 + (b_1^* - b_0)\phi - (\lambda - \alpha_1 + b_0b_1^*) = 0, \]
where $b_1^* = b_1 - b_0$. Thus to find a periodic continued $\alpha$-fraction expansion of (6.2) one should solve the system of equations

$$b_1^* - b_0 = 2\beta, \quad b_0 b_1^* = \alpha_1 + \gamma,$$

(6.3)

which has two solutions

$$b_0 = -\beta \pm \sqrt{\beta^2 + \alpha_1 + \gamma}, \quad b_1^* = \beta \pm \sqrt{\beta^2 + \alpha_1 + \gamma}$$

One can easily check that these two solutions correspond to the two solutions of the factorisation problem from the previous section.

In the purely periodic case the additional condition $C(\alpha_1) = 0$ implies $\gamma = -\alpha_1$ and the general form of $\phi$ is

$$\phi = -\beta + \sqrt{\lambda - \alpha_1 + \beta^2}$$

(6.4)

In that case we have also that $b_1^* = b_1 - b_0 = 0$, so the system (6.3) reduces to just one equation $b_0 = -2\beta$, which determines the purely periodic $\alpha$-fraction expansion of (6.4) uniquely in agreement with our previous consideration.
Chapter 7

Birational action of $\mathbb{Z}_2 \times S_N$

A surprising corollary of Theorem 5.1 is the invariance of the set of $N$-periodic $\alpha$-fractions under the permutations $\sigma \in S_N$ of the set $\alpha$:

$$\sigma(\alpha)_k = \alpha_{\sigma(k)}.$$

This is not obvious from the very beginning and in fact is not true in the purely periodic case.

In this chapter we explain how to use this symmetry to describe all $2N!$ periodic $\alpha$-fractions for a given algebraic function $\phi$. We are going to show that the corresponding symmetric group is the product $G = \mathbb{Z}_2 \times S_N$.

To show this, consider the following permutation $\pi \in S_N$, which reverses the order $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N$ to $\alpha_N, \alpha_{N-1}, \ldots, \alpha_2, \alpha_1$ and the involutions $\sigma_k$ swapping $\alpha_k$ and $\alpha_{k+1}$, where $k = 1, \ldots, N - 1$.

Define the action of $\sigma_k$ on $b = (b_i), i = 0, \ldots, N$ with $b_k \neq 0$ as follows

$$\bar{b}_{k-1} = b_{k-1} + \frac{\alpha_{k+1} - \alpha_k}{b_k}, \quad \bar{b}_{k+1} = b_{k+1} - \frac{\alpha_{k+1} - \alpha_k}{b_k},$$

(7.1)

the rest of $b_i$ remain the same. This determines the action of the symmetric group $S_N$ since the $\sigma_k$ generate it. The generator $\iota$ of $\mathbb{Z}_2$ simply interchanges the two different $\alpha$-fraction expansions of Theorem 5.1. To describe the action of $\mathbb{Z}_2$ it is enough to describe the action of the involution $\iota \pi \in G$, which turns out to be quite simple

$$\bar{b}_j = -b_{N-j}, j = 1, \ldots, N - 1, \quad \bar{b}_0 = b_0 - b_N, \quad \bar{b}_N = -b_N$$

(7.2)
PART II: Birational action of $\mathbb{Z}_2 \times S_N$

Now the action of the group $G$ is described by the following

**Theorem 7.1.** The formulae (7.1) and (7.2) define a birational action of the group $G = \mathbb{Z}_2 \times S_N$ on the set of $N$-periodic $\alpha$-fractions. Its orbits consist of all $2N!$ possible periodic $\alpha$-fraction expansions for a given $\alpha$-triple $(A, B, C)$ and any permutation of the parameters $\alpha_i$

In the purely periodic case the symmetric group is broken down to $S_{N-1}$ generated by $\sigma_k$ with $k = 1, \ldots, N-2$ given by (7.1).

Let us start with the action of $\sigma_k$ first. One can check directly the following matrix identity

$$\begin{bmatrix} b_{k-1} & \lambda - \alpha_k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_k & \lambda - \alpha_{k+1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{k+1} & \lambda - \alpha_{k+2} \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} b_{k-1} & \lambda - \alpha_{k+1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_k & \lambda - \alpha_k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{k+1} & \lambda - \alpha_{k+2} \\ 1 & 0 \end{bmatrix}$$

Similarly for $k = N-1$ we have

$$\begin{bmatrix} b_{N-2} & \lambda - \alpha_{N-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{N-1} & \lambda - \alpha_{N} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & b_N - b_0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} b_{N-2} & \lambda - \alpha_{N} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{N-1} & \lambda - \alpha_{N-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & b_N - b_0 \\ 0 & 1 \end{bmatrix}$$

Taking into account the results of the previous section we see that the action of $\sigma_k$ is indeed given by the formula (7.1). Note that this action preserves the trace of the monodromy matrix $M(\lambda)$ which fixes one of the two possible choices for $\sigma_k$.

To prove the remaining part of Theorem 7.1 recall that $\phi = [b_0; b_1, \ldots, b_{N-1}, b_N]_\alpha$ is a fixed point of the fractional linear transformation (5.6), and therefore it is a fixed point of its inverse, which as one can easily check is given by

$$s^{-1}(\phi) = -b_N + b_0 + \frac{a_N}{-b_{N-1} + \frac{a_{N-1}}{-b_{N-2} + \frac{a_{N-2}}{-b_1 + \frac{a_1}{-b_0 + \phi}}}}$$

Thus

$$\phi = [b_0 - b_N; -b_{N-1}, \ldots, -b_1, -b_N]_{\alpha_N}, \alpha_1, \alpha_0 \quad \text{(7.3)}$$

is a periodic $\alpha$-fraction corresponding to the sequence $\pi(\alpha) = \alpha_N, \alpha_1, \alpha_0$. 
Now note that the trace of the corresponding matrix \((5\,12)\) is equal to \(P_{N-1} + Q_N = (b_1 + b_2 + \cdots + b_N) \lambda^9 + \cdots\). (in the notations of the previous section) If we replace \(b_1, \ldots, b_N\) by \(-b_{N-1}, \ldots, -b_1, -b_N\) its highest coefficient clearly changes sign. This means that the new periodic \(\alpha\)-fraction \((7.3)\) corresponds to the action of the element \(\tau\pi \in G\). Since \(\tau\pi\) and \(\sigma_k\) generate the group \(G\) we have described the full action.

In the purely periodic case because of the additional condition \(C(\alpha_N) = 0\) the symmetric group is reduced to \(S_{N-1}\), permuting \(\alpha_i\) with \(i = 1, \ldots, N-1\) This group is generated by \(\sigma_k\) with \(k = 1, \ldots, N-2\) with the action given by the same formula \((7.1)\) Theorem 7.1 is proved.

**Example.** Let \(N = 3\), \(g = 1\), \(\alpha = (1, 3, 4)\) and

\[
\phi = \frac{3x - 7 + \sqrt{4x^3 - 31x^2 + 62x + 1}}{2(x - 6)}.
\]

We have \(A(x) = x - 6\), \(B(x) = -\frac{1}{4}(3x - 7)\), \(C(x) = -x^2 + 4x - 2\) and \(R(x) = \frac{1}{4}(4x^3 - 31x^2 + 62x + 1) = \left(\frac{x - 7}{2}\right)^2 + (x - 1)(x - 3)(x - 4)\) is \(\alpha\)-admissible.

To each permutation of the sequence \((1, 3, 4)\) we have the following two periodic continued \(\alpha\)-fraction representations of \(\phi\):

\[
\phi = [1; -3, 1, 3]_{1,3,4} = [-\frac{1}{5}; -\frac{15}{6}, \frac{6}{5}, \frac{3}{10}]_{1,3,4}
\]
\[
\phi = [1, -2, 1, 2]_{1,4,3} = [-\frac{1}{5}; -\frac{5}{3}, \frac{6}{5}, -\frac{8}{15}]_{1,4,3}
\]
\[
\phi = [\frac{1}{3}, -3, \frac{5}{3}, \frac{7}{3}]_{1,3,4} = [-1, -\frac{15}{6}, \frac{2}{3}, -\frac{1}{2}]_{3,1,4}
\]
\[
\phi = [\frac{1}{5}, -6, \frac{5}{3}, \frac{8}{15}]_{3,4,1} = [-1, -\frac{15}{6}, 2, -\frac{1}{2}]_{3,4,1}
\]
\[
\phi = [\frac{1}{2}, -\frac{6}{5}, \frac{15}{6}, -\frac{3}{10}]_{4,3,1} = [-2, -\frac{1}{3}, 3, -\frac{3}{3}]_{4,3,1}
\]
\[
\phi = [\frac{1}{2}, -2, \frac{15}{6}, \frac{1}{2}]_{4,1,3} = [-2, -\frac{5}{3}, 3, -\frac{7}{3}]_{4,1,3}
\]
Chapter 8

Relation with the affine hyperelliptic Jacobi varieties

In this section we discuss the geometric aspects of the periodic α-fractions. Let us first note that strictly speaking a function

$$\phi(\lambda) = \frac{-B(\lambda) + \sqrt{R(\lambda)}}{A(\lambda)}$$

is not a function of $\lambda \in \mathbb{C}$, but a function on the corresponding hyperelliptic curve $\Gamma(R)$ given by the equation

$$\mu^2 = R(\lambda)$$  \hspace{1cm} (8.1)

This curve has a natural involution $\tau \quad (\lambda, \mu) \rightarrow (\lambda, -\mu)$, interchanging the branches of the square root.

Let us fix now an α-admissible polynomial $R(\lambda)$ and assume that all its roots are distinct, so that the corresponding hyperelliptic curve $\Gamma_R$ is non-singular and has genus $g$. Let us ask the following natural question. which functions on $\Gamma_R$ admit periodic α-fractions? According to Theorem 5.1 the variety of such functions is birationally equivalent to the variety $M_R^g$ of the triples of polynomials $(A, B, C)$ such that

$$A(\lambda) = \lambda^g + \ldots, \quad C(\lambda) = -\lambda^{g+1} + \ldots, \quad \deg B \leq g,$$  \hspace{1cm} (8.2)

satisfying the relation

$$B^2(\lambda) - A(\lambda)C(\lambda) = R(\lambda)$$  \hspace{1cm} (8.3)
The triples satisfying conditions (8.2) form an affine space $\mathbb{C}^{3g+2}$, while the relation (8.3) is equivalent to $2g+1$ algebraic equations on the coefficients of $A$, $B$, $C$. Thus $M^R_R$ is an affine algebraic variety.

We claim that $M^R_R$ is nothing else but the product $J(\Gamma_R)^{aff} \times \mathbb{C}$, where $J(\Gamma_R)^{aff}$ is the affine part of the Jacobi variety of the corresponding hyperelliptic curve (8.1). Let us denote $\tilde{M}^R_R$ the subvariety of $M^R_R$ given by the additional condition $C(\alpha_N) = 0$.

We have the following:

**Theorem 8.1.** There exists a bijection between the set $M^R_R$ and the extended affine Jacobi variety $J(\Gamma_R)^{aff} \times \mathbb{C}$. The corresponding algebraic functions (5.3) can be characterised as meromorphic functions $\phi \in L(D + P_\infty)$ on $\Gamma_R$ with the non-special pole divisors $D + P_\infty$ and with $\phi \sim \sqrt{\lambda}$ at infinity.

In the purely periodic case under the assumption that $R(\alpha_N) \neq 0$ there exists a natural $2:1$ map from the set $\tilde{M}^R_R$ to $J(\Gamma_R)^{aff}$. The corresponding $\phi$ from $L(D + P_\infty)$ are fixed by the condition that one of the two values of $\phi(\alpha_N)$ is zero.

The proof is essentially due to Jacobi [36], who found the following elementary description of the hyperelliptic Jacobi variety. We follow here Mumford's lectures [52].

The affine Jacobi variety $J(\Gamma_R)^{aff}$ is defined as the set of the positive divisors $D = P_1 + \cdots + P_g$, where $P_1, \ldots, P_g$ are points on the affine curve (8.1) (not necessarily distinct) such that $P_i \neq r(P_j)$ for any $i \neq j$.

Consider the following Jacobi triples $(U, V, W)$ of polynomials in $\lambda$, where $U$ and $W$ are monic polynomials of degree $g$ and $g + 1$ respectively, and $V$ is a polynomial of degree less than or equal to $g - 1$, such that the following relation is satisfied

$$V^2(\lambda) + U(\lambda)W(\lambda) = R(\lambda).$$

(8.4)

We denote the corresponding variety $N_R$.

**Theorem (Jacobi).** There is a natural bijection between the set $N_R$ of the Jacobi triples and the affine Jacobi variety $J(\Gamma_R)^{aff}$.

We sketch here the proof, which is actually not difficult. Let $D = P_1 + \cdots + P_g$ be a divisor from $J(\Gamma_R)^{aff}$. Let us assume at the beginning that all points $P_i \in \Gamma_R$...
are distinct. We would like to associate with $D$ a Jacobi triple $(U, V, W)$. This can be done as follows.

Let $(\mu_i, \lambda_i)$ be the coordinates of the points $P_i$. Then the polynomial $U$ is defined as

$$U(\lambda) = \prod_{i=1}^{g}(\lambda - \lambda_i).$$

The polynomial $V$ is defined by the Lagrange interpolation formula from the conditions $V(\lambda_i) = \mu_i$, $i = 1, \ldots, g$ as

$$V(\lambda) = \sum_{i=1}^{g} \frac{\mu_i \prod_{j \neq i}(\lambda - \lambda_j)}{\prod_{j \neq i}(\lambda_i - \lambda_j)}$$

(note that by assumptions all $\lambda_i$ are distinct). Now $W$ is defined uniquely by the relation (8.4). It is a polynomial because, by construction, $R - V^2$ vanishes at all the zeroes of $U(\lambda)$. If some of the points $P_i$ collide one should use a natural generalisation involving also the derivatives of the polynomial $V(\lambda)$ (see [52] for details). This defines a natural map from $J(\Gamma_R)^{aff}$ to $N_R$.

The inverse map is also quite natural: the $\lambda$-coordinates of $P_i$ are given by the zeroes of $U(\lambda)$ while the $\mu$-coordinates are the values of the polynomial $V$ at these zeroes. The fact that the corresponding points $P_i$ belong to $\Gamma_R$ follows from the relation (8.4).

Now let us return to our set $M_R^g$. To every Jacobi triple $(U, V, W)$ and complex number $\beta \in \mathbb{C}$ we can relate a unique $\alpha$-triple $(A, B, C)$ defined by

$$A = U, \quad B = V + \beta U, \quad C = -W + 2\beta V + \beta^2 U \quad (8.5)$$

Indeed

$$B^2 - AC = (V + \beta U)^2 - U(-W + 2\beta V + \beta^2 U) = V^2 + UW = R.$$  

Conversely, for a given $\alpha$-triple $(A, B, C)$ there exists a unique pair $((U, V, W), \beta)$ where $\beta$ is the coefficient of $\lambda^\beta$ in $B$ and
PART II: Relation with the affine hyperelliptic Jacobi varieties

\[ U = A, \ V = B - \beta A, \ W = -C + 2\beta B - \beta^2 A \]  \hspace{1cm} (8.6)

is a Jacobi triple.

In the purely periodic case we have a well-defined 2 \cdot 1 map \( \tilde{M}_R^g \to J(\Gamma_R)^{aff} \) defined by the same formulae (8.5). Indeed for a given Jacobi triple \((U, V, W)\) there are generically 2 \( \alpha \)-triples \((A, B, C)\) given by (8.6), where \( \beta \) must satisfy the quadratic equation

\[ U(\alpha_N)\beta^2 + 2V(\alpha_N)\beta - W(\alpha_N) = C(\alpha_N) = 0, \]

so that \( \beta = -\frac{V(\alpha_N)\pm\sqrt{R(\alpha_N)}}{U(\alpha_N)} \) if \( U(\alpha_N) \neq 0 \) or \( \beta = \frac{W(\alpha_N)}{2V(\alpha_N)} \) if \( U(\alpha_N) = 0 \) and \( V(\alpha_N) \neq 0 \). Note that if both \( U(\alpha_N) = 0 \) and \( V(\alpha_N) = 0 \), then \( R(\alpha_N) = 0 \) which we have excluded.

In other words, to any divisor of degree \( g \) of the form \( D = \sum_{i=1}^g P_i \), we construct the polynomial \( A(\lambda) \) in the same way as \( U(\lambda) \). The condition \( C(\alpha_N) = 0 \) implies that \( B(\alpha_N) = \sqrt{R(\alpha_N)} \), which means that the polynomial \( B(\lambda) \) is now required to pass through the \( g \) points \( P_i \) plus the additional point \( P(\alpha_N, \sqrt{R(\alpha_N)}) \in \Gamma_R \). Generically there are two such points depending on the choice of square root, which determines the corresponding \( B(\lambda) \) and thus the \( \alpha \)-triple uniquely.

Conversely, to any \( \alpha \)-triple \((A, B, C)\) with \( C(\alpha_N) = 0 \) we can associate a divisor \( D \) and a point \( P \in \Gamma_R \) such that \( \Lambda(P) = \alpha_N \), where \( \Lambda : \Gamma_R \to \mathbb{C} \) is the projection defined by \( \Lambda(\lambda, \mu) = \lambda \).

Let us discuss now the corresponding meromorphic functions \( \phi \) on \( \Gamma_R \).

Let \( D = P_1 + \ldots + P_g, \ P_i \in \Gamma_R^{aff} \) be a positive non-special divisor, and \( L(D + P_\infty) \) be the corresponding linear space of meromorphic functions with poles at most at \( P_1, \ldots, P_g \) and \( P_\infty \). It has dimension 2. Indeed by the classical Riemann-Roch theorem [27]

\[ \dim L(D + P_\infty) = 2 + \dim \Omega(D + P_\infty), \]

where \( \Omega(D + P_\infty) \) is the linear space of holomorphic 1-forms \( \omega \) on \( \Gamma_R \) with zeroes at \( P_1, \ldots, P_g \) and \( P_\infty \). Any holomorphic 1-form on the hyperelliptic curve \( \Gamma_R \) has the
form
\[ \omega = \frac{P(\lambda)d\lambda}{\sqrt{R(\lambda)}} \]

where \( P(\lambda) \) is a polynomial of degree less than or equal to \( g - 1 \). Since \( D \) is non-special, such \( \omega \) must be identically zero so that \( \dim \Omega(D + P_{\infty}) = 0 \) and \( \dim L(D + P_{\infty}) = 2 \).

Consider the functions from the linear space \( L(D + P_{\infty}) \) which are equivalent to \( \sqrt{\lambda} \) at infinity; these functions differ from each other by the addition of a constant. One can easily see that, up to this freedom, they have the form (5.3) given by the Jacobi construction above. Note that the formulae (8.5) correspond to the shift \( \phi \rightarrow \phi + \beta \) in the purely periodic case, \( \phi \) has a zero at one of the two points of \( \Gamma_R \) where \( \lambda = \alpha N \), which reduces the shift to two values. This completes the proof of Theorem 8.1.
Summary and perspectives

Let us summarize first the main results of the thesis.

- Two elliptic versions of the Bernoulli polynomials $B_n(x)$ depending on the parity of $n$ are introduced and investigated. The results are applied to the computation of the densities of states of the classical Lamé operators.

- A theory of a new class of periodic continued fractions ($\alpha$-fractions) is developed in relation with the classical theory of hyperelliptic curves and their Jacobi varieties.

We find remarkable that the even and odd Bernoulli polynomials have elliptic generalizations which are quite different in form but at the same time have a very similar origin. Indeed both of them come from the trace of the resolvent

- $(\mu I - L_s)^{-1}$ in the elliptic Faulhaber case,

- $(\mu I - \hat{H}_s)^{-1}$ in the elliptic Bernoulli case,

and $L_s$ and $\hat{H}_s$ are closely related (see section 3.1)

We believe that these polynomials are very important and deserve further investigation. We mention only some of the natural questions still to be answered.

The coefficients of the classical Faulhaber polynomials are known to have alternating signs (see [40]). We conjecture that the same is true for the reduced elliptic Faulhaber polynomials. More precisely, recall that in the Weierstrass form these polynomials have the form $F^W_m = A_m(\lambda, g_2, g_3)\omega + B_m(\lambda, g_2, g_3)\eta$, where

$$A_m = \sum A_{k,l}^{(m)}(\lambda) g_2^k g_3^l, \quad B_m = \sum B_{k,l}^{(m)}(\lambda) g_2^k g_3^l.$$ 

Conjecture 1 ([29]). The coefficients of the polynomials $A_{k,l}^{(m)}(\lambda)$ and $B_{k,l}^{(m)}(\lambda)$ have alternating signs.
SUMMARY AND PERSPECTIVES

For the first 8 polynomials one can check this from their explicit form given in the appendix A. But this property does not hold in the general case, see e.g. $F_4$ in Chapter 2. An ideal proof would be to find a combinatorial interpretation for these coefficients (cf. Gessel andViennot [22] and Knuth [40] in the classical case). It might be easier to prove it first for the elliptic Bernoulli numbers.

**Conjecture 2 ([29]).** The elliptic Bernoulli numbers $B_{2m}$ have the form

$$B_{2m} = (-1)^{m-1}(\hat{A}^{(m)}(g_2, g_3), \hat{B}^{(m)}(g_2, g_3)),$$

where the polynomials $\hat{A}^{(m)}(g_2, g_3), \hat{B}^{(m)}(g_2, g_3)$ have positive rational coefficients.

Another interesting question is the behaviour of the elliptic Faulhaber polynomials on a real line. Actually it is more instructive to look at the corresponding polynomials $F_\eta(x^{2+\eta})$ as functions of $x$ rather than $\lambda = x^{2+\eta}$. Indeed, we know that the integer values $x = n$ should play a special role here, being related to the "finite-gap" values of the parameter in the Lamé equation. Recall also that in the usual case these polynomials coincide with $\frac{1}{2m}(B_{2m}(x + 1) - B_{2m})$, so their graphs are simply shifted graphs of the corresponding Bernoulli polynomials.

**Conjecture 3 ([29]).** As $m$ tends to $\infty$ for $x$ in a finite interval on the real line

$$\frac{F_\eta(x^{2+\eta})}{2B_{2m-2}} \to \frac{1 - \cos 2\pi x}{2\pi^2}$$

In the hyperbolic limit $g_2 = g_3 = 0$ (i.e. for the usual Bernoulli polynomials) it is known to be true (see e.g. [64]). The normalisation constant $2B_{2m-2}$ is chosen to guarantee the correct second derivative at zero. Another justification for this conjecture is that the asymptotic expansion (4.7) should be convergent for integer $n$ (due to the finite-gapness property), which means that the numbers $F_\eta(x^{2+n})$ should be relatively very small.

In the figure 8.1 we show the graphs of the normalised polynomials

$$\Phi_\eta(x) = \frac{F_\eta(x^{2+\eta})}{2B_{2m-2}}$$
SUMMARY AND PERSPECTIVES

Figure 8.1: Normalised elliptic Faulhaber polynomial in the usual case $\Phi_8(x)$ (left) and normalised elliptic Faulhaber polynomial in the elliptic lemniscatic case $\Phi_8^W(x)$ (right)

for $m = 8$ in the usual and lemniscatic elliptic case $g_1 = g_3 = 0$

The elliptic Faulhaber polynomials have further generalisation related to the so-called Treibich-Verdier potentials [61]

$$u = m_0(m_0 + 1)\varphi(x) + \sum_{i=1}^{3} m_i(m_i + 1)\varphi(x - \omega_i),$$

where $\omega_i$ are the half-periods of the Weierstrass elliptic $\wp$-function. The corresponding KdV integrals $I_k[u]$ are polynomials in four variables $\lambda_i = m_i(m_i+1)$, $i = 0, 1, 2, 3$ with the coefficients depending on $e_1, e_2, e_3, \omega, \eta$. It would be interesting to investigate them.

We also believe that the elliptic Faulhaber polynomials and elliptic Bernoulli numbers are interesting from a number-theoretic point of view and would like to mention the papers [38, 39, 55] in this relation.

We have shown that for any given $k$ the coefficient $b_k(s)$ of the Lamé spectral polynomial $R_{2s+1}(E)$ can be computed effectively for all values of parameter $s$ using the elliptic Bernoulli polynomials. However, we believe that these polynomials are of interest by themselves. In particular, one can expect interesting links with the arithmetic of the corresponding elliptic curves and the representation theory. In this relation, we would like to mention the elliptic generalisation of the Bernoulli numbers - the so-called Bernoulli-Hurwitz numbers $BH_{2k}$, whose arithmetic was investigated in [38, 55].

Another interesting possible relation is with the zeta-function $\zeta_H(z) = tr\hat{H}^{-z}$ of
the quantum top and its special values. The lemniscatic case when \( a_3 = \frac{a_1+a_2}{2} \) could be particularly interesting from the arithmetic point of view.

Recall that the parameter \( s \) was originally an integer or a half-integer (spin). A natural question to ask is the role of these values in the theory of the elliptic Bernoulli polynomials. We conjecture that like in the case of the usual Bernoulli polynomials (see e.g. [64]) these values are the asymptotic positions of the real roots of the polynomials \( B_{2k+1}(s) \) for large \( k \). More precisely, we conjecture that for a real \( s \) in the bounded interval the ratio

\[
\frac{B_{2k+1}(s)}{B_{2k+1}(0)} \to \frac{\sin 2\pi s}{2\pi}
\]

as \( k \) tends to infinity. Actually, we believe that this is true for each component of \( B_{2k+1}(s) \), which is a coefficient at monomial \( g_1^p g_2^q g_3^r \).

It is interesting to look at the graphs. In Figure 8.2 we show the graphs \( y(s) \) of the coefficients of the polynomial \( B_{15}(s) \) at: (a) \( g_1^7 \), (b) \( g_1^3 g_2^2 \), (c) \( g_1^2 g_2 g_3 \), and (d) \( g_2^2 g_3 \). Each polynomial has been normalised i.e. it has been divided by its first derivative at zero and then multiplied by \( 2\pi \). The sinusoidal behavior for small \( s \) looks quite plausible.

![Figure 8.2. Components of \( B_{15}(s) \) ](image-url)
In relation to the second part of the thesis we would like to mention that the theory of periodic continued $\alpha$-fractions can be applied to the theory of the discrete KdV equation (see e.g. [4]) and the Yang-Baxter maps [63].

Another intriguing direction would be to consider the non-periodic case. As it follows from the results of Shabat and Veselov [65] this could be useful for the theory of the Painlevé equations.
Appendix A

Reduced elliptic Faulhaber polynomials

The lemniscatic and equianharmonic cases can be read off by putting $g_3 = 0$ and $g_2 = 0$ respectively.

\[ \mathcal{F}_1^w = -\eta [4\lambda] \]

\[ \mathcal{F}_2^w = + g_2 \omega \left[ \frac{2}{3}\lambda^2 \right] \]

\[ \mathcal{F}_3^w = - g_2 \eta \left[ \frac{5}{6}\lambda^2(3\lambda - 2) \right] + g_3 \omega \left[ \frac{5}{6}\lambda^2(2\lambda - 3) \right] \]

\[ \mathcal{F}_4^w = + g_2^2 \omega \left[ \frac{7}{21}\lambda^2(25\lambda^2 - 40\lambda + 24) \right] - g_3 \eta \left[ \frac{32}{7}\lambda^2(5\lambda^2 - 15\lambda + 9) \right] \]

\[ \mathcal{F}_5^w = - g_2^2 \eta \left[ \frac{8}{15}\lambda^2(49\lambda^3 - 140\lambda^2 + 168\lambda - 72) \right] + g_2 g_3 \omega \left[ \frac{16}{15}\lambda^2(28\lambda^3 - 105\lambda^2 + 126\lambda - 54) \right] \]
\[ \mathcal{F}_6^\omega = \frac{\lambda^2}{11}(45\lambda^4 - 200\lambda^3 + 416\lambda^2 - 400\lambda + 144)]

\[ - \frac{\lambda^2}{55}(87\lambda^4 - 515\lambda^3 + 1179\lambda^2 - 1206\lambda + 450)] + \frac{\lambda^4}{55}(56\lambda^4 - 420\lambda^3 + 1197\lambda^2 - 1368\lambda + 540)]

\[ - \frac{\lambda^2}{11}(847\lambda^5 - 5390\lambda^4 + 17248\lambda^3 - 30536\lambda^2 + 26928\lambda - 9072)]

\[ + \frac{\lambda^4}{455}(9526\lambda^5 - 71995\lambda^4 + 250404\lambda^3 - 472428\lambda^2 + 433224\lambda - 149256)] - \frac{\lambda^4}{51}(220\lambda^5 - 2310\lambda^4 + 10395\lambda^3 - 22770\lambda^2 + 22572\lambda - 8100)]

\[ + \frac{\lambda^5}{21}(1521\lambda^6 - 13104\lambda^5 + 59696\lambda^4 - 165568\lambda^3 + 269568\lambda^2 - 224640\lambda + 72576)]

\[ \mathcal{F}_7^\omega = \frac{\lambda^2}{55}(2171\lambda^6 - 22477\lambda^5 + 113295\lambda^4 - 336492\lambda^3 + 570492\lambda^2 - 485784\lambda + 158760)] + \frac{\lambda^6}{5}(1821\lambda^6 - 2184\lambda^5 + 12285\lambda^4 - 38844\lambda^3 + 68094\lambda^2 - 58968\lambda + 19440)]
Appendix B

Lamé densities of states

B.1 The first 7 coefficients \( a_k(n) \) of the numerator \( P_n(E) \) of the Lamé density of states \( \rho_n(E) \)

Here we use the notations \( \bar{\rho} = \frac{n}{\omega} \), \( \hat{a}_k(n) = a_k(n)/u_k(n) \), where \( u_k(n) = (n+1)n(n-1)\ldots(n-k+1) \).

\( \hat{a}_1 = \frac{1}{2} \bar{\rho} \),

\( \hat{a}_2 = -\frac{\bar{\rho}}{480}(6 + 25n + 16n^2) \),

\( \hat{a}_3 = -\frac{\bar{\rho}}{3060}(45 + 243n + 247n^2 + 64n^3) - \frac{2\bar{\rho}}{960}n(n+1)(27 + 16n) \),

\( \hat{a}_4 = \frac{\bar{\rho}^2}{3225600}(-2520 - 12942n - 10315n^2 + 4565n^3 + 6880n^4 + 1792n^5) - \frac{2\bar{\rho}}{13440}n(n+1)(600 + 563n + 128n^2) \),

\( \hat{a}_5 = \frac{2a_3}{17740800}(-28350 - 145305n - 98919n^2 + 130400n^3 + 185250n^4 + 78480n^5 + 1126n^6) + \frac{\bar{\rho}^2}{6451200}n(n+1)(-22050 - 19707n + 3217n^2 + 7328n^3 + 1792n^4) \),

\( \hat{a}_6 = \frac{\bar{\rho}^3}{3228825600}(585728n^7 + 6077568n^6 + 24055710n^5 + 42381080n^4 + 22989372n^3 - 21506058n^2 - 26135595n - 4677750) - \frac{\bar{\rho}^2}{664215552000}(4100096n^8 + 23905024n^7 + 14017296n^6 - 19221924n^5 - 520562096n^4 - 391295859n^3 + 180468864n^2 + 310981356n + 62868960) + \frac{\bar{\rho}^3}{79963200}n(n+1)(22528n^5 + 171920n^4 + 427045n^3 + 215715n^2 - 568458n - 612360) \),

\( \hat{a}_7 = -\frac{\bar{\rho}^3}{1549836288000}(164000384n^9 + 160552704n^8 + 520180864n^7 + 153752039n^6 - 2909673459n^5 - 6672918014n^4 - 4259587899n^3 + 2522284551n^2 + 3574729990n + 681080400) + \frac{\bar{\rho}^3}{6457051200}n(n+1)(585728n^6 + 6709056n^5 + 29129901n^4 + 54097586n^3 + 19647288n^2 - 62676225n - 92)$

\[ \rho_n(E) = \frac{\hat{a}_k(n)}{u_k(n)} P_n(E) \]
APPENDIX B. LAMÉ DENSITIES OF STATES

\[ \rho_n(E) = \frac{1}{2\pi} \frac{E + \varphi}{\sqrt{E^3 - \frac{g_2}{4} E + \frac{g_3}{4}}} = \frac{1}{2\pi} \frac{E + \varphi}{(E + e_1)(E + e_2)(E + e_3)}. \]

**B.2 The explicit form of the first 5 Lamé densities of states**

Here as before $\varphi = -\frac{\pi}{\omega}$ is the average of $\varphi(x)$ over a period. The explicit form of the Lamé spectral polynomials for $n = 4$ and $n = 5$ are borrowed from Halphen [32] and Belokolos-Enolski [7] (a small discrepancy with [7] is due to a minor misprint in that paper).

For $n = 1$ the potential is $u(x) = 2\varphi(x)$. The density of states is well-known in this case

\[ \rho_1(E) = \frac{E + \varphi}{\sqrt{E^3 - \frac{g_2}{4} E + \frac{g_3}{4}}}. \]

For $n = 2$ we have $u(x) = 6\varphi(x)$ and

\[ \rho_2(E) = \frac{E^2 + 3\varphi E - \frac{3}{2}g_2}{\pi \sqrt{E^5 - 21g_2E^3 - 27g_3E^2 + 27g_2^2E + 81g_2g_3}}. \]

For $n = 3$ the potential is $u(x) = 12\varphi(x)$ and the density of states $\rho_3(E)$ is

\[ \frac{E^3 + 6\varphi E^2 - \frac{45}{4}g_2E - \frac{135}{4}g_3 - \frac{45}{2}g_2\varphi}{2\pi \sqrt{E^7 - \frac{45}{2}g_2E^5 - \frac{297}{2}g_3E^4 + \frac{4185}{16}g_2^2E^3 + \frac{18225}{8}g_2g_3E^2 - \frac{3375}{16}(g_2^3 - 27g_3^2)E}}. \]

For $n = 4$ the potential is $u(x) = 20\varphi(x)$ and

\[ \rho_4(E) = \frac{E^4 + 10\varphi E^3 - \frac{181}{4}g_2E^2 - \left(\frac{1295}{4}g_3 + \frac{455}{2}g_2\varphi\right)E + \frac{273}{2}g_2^2 - 875g_3\varphi}{2\pi \sqrt{(E^3 - 52g_2E - 560g_3) \prod_{k=1}^{4}(E^2 - 10e_kE - 35e_k^2 - 7g_2)}}. \]

For $n = 5$ we have $u(x) = 30\varphi(x)$ and the density of states $\rho_5(E)$ is

\[ \frac{E^5 + 15\varphi E^4 - \frac{531}{4}g_2E^3 - \left(\frac{6615}{4}g_3 + \frac{481}{2}g_2\varphi\right)E^2 + \left(\frac{18117}{8}g_3^2 - \frac{43525}{4}g_3\varphi\right)E + \frac{178605}{8}g_2g_3 + \frac{133855}{2}g_2^2\varphi}{2\pi \sqrt{(E^2 - 27g_2) \prod_{k=1}^{5}(E^3 + 15e_kE^2 + (315e_k^2 - 132g_2)E - 675e_k^3 - 540g_3)}}. \]
Appendix C

The first 8 elliptic Bernoulli polynomials

\[ B_1 = 2s + 1 \]
\[ B_3 = \frac{1}{12} s^2 + 2 \]
\[ B_5 = \frac{1}{360} g_1^2 s(s + 1)(2s + 1)(3s^2 + 3s - 1) + \frac{1}{60} g_2 s(s + 1)(2s - 1)(2s + 1)(2s + 3) \]
\[ B_7 = \frac{1}{1344} g_1^3 s(s + 1)(2s + 1)(3s^4 + 6s^3 - 3s + 1) + \frac{1}{1280} g_1 g_2 s(s + 1)(2s - 1)(2s + 1)(2s + 3)(6s^2 + 6s - 5) + \frac{1}{280} g_3 s(s + 1)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5) \]
\[ B_9 = \frac{1}{11520} g_1^4 s(s + 1)(1 + 2s)(5s^5 + 15s^5 + 5s^4 - 15s^3 - s^2 + 9s - 3) + \frac{1}{3360} g_1^2 g_2 s(s + 1)(2s - 1)(2s + 1)(2s + 3)(5s^4 + 10s^3 - 5s^2 - 10s + 7) + \frac{1}{840} g_1 g_3 s^2(s + 1)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5) + \frac{1}{1080} g_2^2 s(s + 1)(2s - 1)(2s + 1)(2s + 3)(4s^4 + 8s^3 - 11s^2 - 15s + 21) \]
APPENDIX C. THE FIRST 8 ELLIPTIC BERNOUlli POLYNOMIALS

\[ B_{11} = \begin{align*}
\frac{1}{33792} & \ g_1^5 \ s(s+1)(2s+1)(s^2+s-1)(3s^6+9s^5+2s^4-11s^3+3s^2+10s-5) \\
+ \frac{1}{50688} & \ g_1^3g_2 \ s(s+1)(2s-1)(2s+1)(2s+3)(20s^6+60s^5-10s^4-120s^3+44s^2+114s-75) \\
+ \frac{1}{29568} & \ g_1^2g_3 \ s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)(10s^4+20s^3-4s^2-14s+21) \\
+ \frac{1}{20736} & \ g_1g_2^2 \ s(s+1)(2s-1)(2s+1)(2s+3)(40s^6+120s^5-86s^4-372s^3+242s^2+448s-315) \\
+ \frac{1}{7392} & \ g_2g_3 \ s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5)(8s^4+16s^3-34s^2-42s+63) \\
\end{align*} \]

\[ \begin{align*}
\frac{5509040}{g_1^6} & \ s(s+1)(2s+1)(105s^{10}+525s^9+525s^8-1050s^7-1190s^6+2310s^5+1420s^4-3285s^3-287s^2+2073s-691) \\
+ \frac{1}{521280} & \ g_1^4g_2 \ s(s+1)(2s-1)(2s+1)(2s+3) \\
(525s^8+2100s^7+350s^6-6300s^5-70s^4+12810s^3-4105s^2-11910s+7601) \\
+ \frac{1}{2842840} & \ g_1^2g_3 \ s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
(350s^6+1050s^5-100s^4-1950s^3+1433s^2+2583s-1650) \\
\end{align*} \]

\[ \begin{align*}
B_{13} = + \frac{1}{2562560} & \ g_1^2g_2^2 \ s(s+1)(2s-1)(2s+1)(2s+3)(1400s^8+5600s^7-1450s^6-23950s^5+5438s^4+57326s^3-24627s^2-58215s+41481) \\
+ \frac{1}{320320} & \ g_1g_2g_3 \ s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
(200s^6+600s^5-670s^4-2340s^3+1922s^2+3192s-2475) \\
+ \frac{1}{960960} & \ g_2^3 \ s(s+1)(2s-1)(2s+1)(2s+3)(400s^8+1600s^7-1640s^6-10520s^5+8193s^4+35786s^3-28282s^2-48195s+43659) \\
+ \frac{1}{160160} & \ g_3^2 \ s(s+1)(2s-3)(2s-1)(2s+1)(2s+3)(2s+5) \\
(80s^6+240s^5-840s^4-2080s^3+4401s^2+5481s-7425) \\
\end{align*} \]
APPENDIX C. THE FIRST 8 ELLIPTIC BERNOULLI POLYNOMIALS

\[ \frac{1}{737280} \ g_1^7 \ s(s + 1)(2s + 1)(3s^2 + 2 + 18s + 2 + 24s^{10} + 45s^9 - 81s^8 + 144s^7 + 182s^6 \]
\[ \quad - 345s^5 - 217s^4 + 498s^3 + 44s^2 - 315s + 105) \]
\[ + \frac{1}{1597440} \ g_1^5 g_2 \ s(s + 1)(2s - 1)(2s + 1)(2s + 3)(42s^{10} + 210s^9 + 105s^8 - 840s^7 \]
\[ \quad - 364s^6 + 2730s^5 + 205s^4 - 5540s^3 + 1650s^2 + 5078s - 3185) \]
\[ + \frac{1}{15178880} \ g_1^4 g_3 \ s(s + 1)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5)(315s^8 + 1260s^7 \]
\[ \quad + 140s^6 - 3990s^5 + 1265s^4 + 10650s^3 - 5152s^2 - 11352s + 9009) \]
\[ + \frac{1}{15178880} \ g_1^3 g_2^2 \ s(s + 1)(2s - 1)(2s + 1)(2s + 3)(2520s^{10} + 12600s^9 + 1750s^8 - 68600s^7 \]
\[ \quad - 13130s^6 + 253630s^5 - 14558s^4 - 557066s^3 + 206601s^2 + 542619s \]
\[ \quad - 360360) \]
\[ B_{15} = \]
\[ + \frac{1}{1088640} \ g_1^2 g_2 g_3 \ s(s + 1)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5)(280s^8 + 1120s^7 - 670s^6 \]
\[ \quad - 5930s^5 + 3047s^4 + 17284s^3 - 11237s^2 - 21054s + 18018) \]
\[ + \frac{1}{3294720} \ g_1 g_2^3 \ s(s + 1)(2s - 1)(2s + 1)(2s + 3)(1120s^{10} + 5600s^9 - 2400s^8 - 43200s^7 \]
\[ \quad - 8814s^6 + 201162s^5 - 60127s^4 - 517124s^3 + 256797s^2 + 557766s \]
\[ \quad - 405405) \]
\[ + \frac{1}{274560} \ g_1^3 g_3^2 \ s^2(s + 1)^2(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5) \]
\[ \quad (80s^5 + 240s^4 - 840s^3 - 2080s^2 + 4401s^2 + 5481 - 7425) \]
\[ + \frac{1}{274560} \ g_2^2 g_3 \ s(s + 1)(2s - 3)(2s - 1)(2s + 1)(2s + 3)(2s + 5)(80s^8 + 320s^7 \]
\[ \quad - 600s^6 - 2920s^5 + 4037s^4 + 13314s^3 - 16959s^2 - 24156s + 27027) \]
\[ g_1^8 = s(1 + s)(1 + 2s)(-3617 + 10851s - 1519s^2 - 17145s^3 + 7485s^4 + 11835s^5 - 6275s^6 - 4845s^7 + 2775s^8 + 1365s^9 - 805s^{10} - 315s^{11} + 175s^{12} + 105s^{13} + 15s^{14}) \]

\[ + g_{92}^9 = s(1 + s)(-1 + 2s)(1 + 2s)(3 + 2s)(10851 - 17490s - 5485s^2 + 19266s^3 - 1009s^4 - 9570s^5 + 1535s^6 + 2898s^7 - 567s^8 - 630s^9 + 105s^{10} + 126s^{11} + 21s^{12}) \]

\[ + g_{93}^9 = s(1 + s)(-3 + 2s)(-1 + 2s)(1 + 2s)(3 + 2s)(5 + 2s)(-900900 + 1210209s + 550405s^2 - 1114200s^3 - 8270s^4 + 462315s^5 - 8477s^6 - 110250s^7 + 8820s^8 + 24255s^9 + 4851s^{10}) \]

\[ + g_{929}^9 = s(1 + s)(-3 + 2s)(-1 + 2s)(1 + 2s)(3 + 2s)(5 + 2s)(900999909 - 140435295s - 49621851s^2 + 1484979238s^3 - 949091s^4 - 69878555s^5 + 7244275s^6 + 19727890s^7 - 2440445s^8 - 3847725s^9 + 297675s^{10} + 582120s^{11} + 97020s^{12}) \]

\[ B_{17} = \]

\[ + g_{9293}^9 = s(1 + s)(-3 + 2s)(-1 + 2s)(1 + 2s)(3 + 2s)(5 + 2s)(-3378375 + 4270266s + 213723s^2 - 3687256s^3 - 475828s^4 + 1379210s^5 + 49280s^6 - 298900s^7 - 8575s^{10} + 9800s^9 + 173372s^8 + 144300s^9 + 8820s^{10}) \]

\[ + g_{9193}^9 = s(1 + s)(-3 + 2s)(-1 + 2s)(1 + 2s)(3 + 2s)(5 + 2s)(21173724 - 30694950s - 12502065s^2 + 3046963s^3 + 1266893s^4 - 13169862s^5 + 441306s^6 + 3302520s^7 - 151290s^8 - 254000s^9 + 2800s^{10} + 4095s^{11} + 10080s^{12}) \]

\[ + g_{9293}^9 = s(1 + s)(-3 + 2s)(-1 + 2s)(1 + 2s)(3 + 2s)(5 + 2s)(131107977 - 163378215s - 84398157s^2 + 137791221s^3 + 2158050s^4 - 4837775s^5 - 3008423s^6 + 952609s^7 + 346332s^8 - 1184000s^9 - 644000s^{10} + 94080s^{11} + 15680s^{12}) \]

\[ + g_{9293}^9 = s(1 + s)(-3 + 2s)(-1 + 2s)(1 + 2s)(3 + 2s)(5 + 2 s)(131107977 - 163378215s - 84398157s^2 + 137791221s^3 + 2158050s^4 - 4837775s^5 - 3008423s^6 + 952609s^7 + 346332s^8 - 1184000s^9 - 644000s^{10} + 94080s^{11} + 15680s^{12}) \]

\[ + g_{9293}^9 = s(1 + s)(-3 + 2s)(-1 + 2s)(1 + 2s)(3 + 2s)(5 + 2 s)(131107977 - 163378215s - 84398157s^2 + 137791221s^3 + 2158050s^4 - 4837775s^5 - 3008423s^6 + 952609s^7 + 346332s^8 - 1184000s^9 - 644000s^{10} + 94080s^{11} + 15680s^{12}) \]
Appendix D

Quantum top in sphero-conical coordinates

Let us introduce the elliptic (or sphero-conical) coordinates $u_1, u_2$ on the sphere

$$q_1^2 + q_2^2 + q_3^2 = 1$$

as the roots of the quadratic equation

$$\frac{q_1^2}{a_1 - u} + \frac{q_2^2}{a_2 - u} + \frac{q_3^2}{a_3 - u} = 0,$$

for some positive parameters $a_1, a_2, a_3$. These coordinates were first used in 1859 by C. Neumann in the problem of motion on a sphere in the (harmonic) field with potential $U(x) = \frac{1}{2}(a_1q_1^2 + a_2q_2^2 + a_3q_3^2)$.

Consider the first component of the angular momentum operator

$$\hat{M}_1 = -i(q_2\partial_{q_3} - q_3\partial_{q_2}).$$

We are going to show that $\hat{M}_1$ can be written in the sphero-conical coordinate system as

$$\hat{M}_1 = -\frac{2i}{u_1 - u_2} \frac{(u_1 - a_2)(u_1 - a_3)(u_2 - a_2)(u_2 - a_3)}{(a_1 - a_2)(a_3 - a_1)} \left[\frac{1}{2}(u_1 - a_1)\partial_{u_1} - (u_2 - a_1)\partial_{u_2}\right].$$

Consider the function

$$\phi_u(q_1, q_2, q_3) = \frac{q_1^2}{a_1 - u} + \frac{q_2^2}{a_2 - u} + \frac{q_3^2}{a_3 - u}$$

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The function $\phi_u = \phi_u(q_1, q_2, q_3)$ can also be written as

$$
\phi_u = \frac{(u - u_1)(u - u_2)}{(a_1 - u)(a_2 - u)(a_3 - u)} \tag{D.6}
$$

Indeed as a function of $u$ it admits two zeroes $u = u_1$ and $u = u_2$ and the leading coefficient in the numerator must be one due to (D.1).

Note that $q_1^2$, $q_2^2$ and $q_3^2$ are the residues of the function $\phi_u$, considered as a function of $u$, at the points $u = a_1$, $u = a_2$ and $u = a_3$ respectively. This leads to the following expressions for the cartesian coordinates in terms of the elliptic coordinates

$$
q_1^2 = \frac{(a_1 - u_1)(a_1 - u_2)}{(a_1 - a_2)(a_1 - a_3)},
q_2^2 = \frac{(a_2 - u_1)(a_2 - u_2)}{(a_2 - a_1)(a_2 - a_3)},
q_3^2 = \frac{(a_3 - u_1)(a_3 - u_2)}{(a_3 - a_1)(a_3 - a_2)} \tag{D.7}
$$

Since $\partial_u$, $\partial_{u_2}$ is a basis of the tangent plane to the sphere, we can write

$$
\hat{M}_1 = -\iota(X_1 \partial_{u_1} + Y_1 \partial_{u_2}), \tag{D.8}
$$

where $X_1 = X_1(u_1, u_2)$ and $Y_1 = Y_1(u_1, u_2)$.

Applying this operator to $(\log \phi_u)$, we have

$$
\hat{M}_1(\log \phi_u) = \iota\left(\frac{X_1}{u - u_1} + \frac{Y_1}{u - u_2}\right) \tag{D.9}
$$

On the other hand, using (D.3), (D.5) and (D.6) we have

$$
\hat{M}_1(\log \phi_u) = -2uq_2q_3(a_2 - a_3)\frac{a_1 - u}{(u - u_1)(u - u_2)}. 
$$

Replacing $q_2, q_3$ by their expression (D.7), we obtain

$$
\hat{M}_1(\log \phi_u) = -2\sqrt{\frac{(a_2 - u_1)(a_2 - u_2)(a_3 - u_1)(a_3 - u_2)}{(a_1 - a_2)(a_3 - a_1)}}\frac{a_1 - u}{(u - u_1)(u - u_2)}. \tag{D.10}
$$
Taking the residues of (D.10) at \( u = u_1 \) and \( u = u_2 \) we get
\[
X_1 = \frac{2}{u_1 - u_2} \sqrt{\frac{(a_2 - u_1)(a_2 - u_2)(a_3 - u_1)(a_3 - u_2)}{(a_1 - a_2)(a_3 - a_1)}} (u_1 - a_1),
\]
and
\[
Y_1 = \frac{2}{u_1 - u_2} \sqrt{\frac{(a_2 - u_1)(a_2 - u_2)(a_3 - u_1)(a_3 - u_2)}{(a_1 - a_2)(a_3 - a_1)}} (a_1 - u_2),
\]
which leads to the expression (D.4). Similar expressions for the other components \( \hat{M}_2 \) and \( \hat{M}_3 \) of the angular momentum can be obtained by changing the indices of the \( a_i \) cyclicly.

The square of the angular momentum operator
\[
\hat{M}^2 = \sum_i \hat{M}_i^2
\]
has the following expression
\[
\hat{M}^2 = -\sum_i [(X_i \frac{\partial X_i}{\partial u_1} + Y_i \frac{\partial X_i}{\partial u_2}) \partial u_1 + (X_i \frac{\partial Y_i}{\partial u_1} + Y_i \frac{\partial Y_i}{\partial u_2}) \partial u_2 + X_i^2 \partial^2_{u_1} + Y_i^2 \partial^2_{u_2} + 2X_i Y_i \partial^2_{u_1 u_2}] \tag{D.11}
\]

The quantum Hamiltonian operator of the Euler top
\[
\hat{H} = \sum a_i \hat{M}_i^2
\]
is
\[
\hat{H} = -\sum_i [(a_i X_i \frac{\partial X_i}{\partial u_1} + a_i Y_i \frac{\partial X_i}{\partial u_2}) \partial u_1 + (a_i X_i \frac{\partial Y_i}{\partial u_1} + a_i Y_i \frac{\partial Y_i}{\partial u_2}) \partial u_2 + a_i X_i^2 \partial^2_{u_1} + a_i Y_i^2 \partial^2_{u_2} + 2a_i X_i Y_i \partial^2_{u_1 u_2}] \tag{D.12}
\]

Let \( P(u) = (a_1 - u)(a_2 - u)(a_3 - u) \) It can easily be shown from the explicit expression of \( X_i, Y_i \) that
\[
\sum_i X_i Y_i = 0,
\]
\[
\sum_i a_i X_i Y_i = 0,
\]
\[
\sum_i X_i^2 = -\frac{4P(u_1)}{u_1 - u_2},
\]
\[
\sum_i a_i X_i^2 = -\frac{4P(u_1)}{u_1 - u_2} u_2,
\]
\[
\sum_i X_i \frac{X_i}{u_1 - a_i} = -\frac{4P(u_1)}{(u_1 - u_2)^2},
\]
\[
\sum_i a_i X_i \frac{X_i}{u_1 - a_i} = -\frac{4P(u_1) u_2}{(u_1 - u_2)^2}.
\]

and similar relations for \( Y_i(u_1, u_2) = X_i(u_2, u_1) = -\frac{u_2 - u_1}{u_1 - a_i} X_i(u_1, u_2) \).

Hence we deduce the following relations
\[
\sum_i X_i \frac{\partial X_i}{\partial u_1} = \frac{1}{2} \sum_i \frac{\partial X_i^2}{\partial u_1} = -2 \frac{P'(u_1)}{u_1 - u_2} + 2 \frac{P(u_1)}{(u_1 - u_2)^2},
\]
\[
\sum_i Y_i \frac{\partial Y_i}{\partial u_2} = 2 \frac{P'(u_2)}{u_1 - u_2} + 2 \frac{P(u_2)}{(u_1 - u_2)^2},
\]
\[
\sum_i Y_i \frac{\partial X_i}{\partial u_2} = -\sum_i \frac{u_2 - a_i}{u_1 - a_i} X_i \frac{\partial X_i}{\partial u_2} = \sum_i \left( \frac{u_1 - u_2}{u_1 - a_i} - 1 \right) X_i \frac{\partial X_i}{\partial u_2} = \frac{u_1 - u_2}{2} \sum_i \frac{\partial}{\partial u_2} \left( \frac{X_i^2}{u_1 - a_i} \right) - \frac{1}{2} \sum_i \frac{\partial X_i^2}{\partial u_2} = -2 \frac{P(u_1)}{(u_1 - u_2)^2},
\]
\[
\sum_i X_i \frac{\partial Y_i}{\partial u_1} = -2 \frac{P(u_2)}{(u_1 - u_2)^2},
\]
\[
\sum_i a_i X_i \frac{\partial X_i}{\partial u_1} = \frac{1}{2} \sum_i \frac{\partial}{\partial u_1} \left( a_i X_i^2 \right) = -2 \frac{P'(u_1)}{u_1 - u_2} u_2 + 2 \frac{P(u_1)}{(u_1 - u_2)^2} u_2,
\]
\[
\sum_i a_i X_i \frac{\partial Y_i}{\partial u_2} = 2 \frac{P'(u_2)}{u_1 - u_2} u_1 + 2 \frac{P(u_2)}{(u_1 - u_2)^2} u_1,
\]
\[
\sum_i a_i Y_i \frac{\partial X_i}{\partial u_2} = -\sum_i \frac{u_2 - a_i}{u_1 - a_i} a_i X_i \frac{\partial X_i}{\partial u_2} = \sum_i \left( \frac{u_1 - u_2}{u_1 - a_i} - 1 \right) a_i X_i \frac{\partial X_i}{\partial u_2} = \sum_i \left( \frac{u_1 - u_2}{u_1 - a_i} - 1 \right) a_i X_i \frac{\partial X_i}{\partial u_2}.
\]
Substituting all this in (D 11) and (D.12) we obtain the following expressions for the square of the angular momentum operator

\[ \hat{M}^2 = 2 \frac{P'(u_1)}{u_1 - u_2} \frac{P'(u_2)}{u_1 - u_2} \frac{\partial Y_i}{\partial u_1} - 2 \frac{P(u_1)}{u_1 - u_2} \frac{\partial^2}{\partial u_2} - 4 \frac{P(u_2)}{u_1 - u_2} \frac{\partial^2}{\partial u_2}, \]

or equivalently

\[ \hat{M}^2 = -\frac{4}{u_1 - u_2} \left[ \sqrt{-P(u_1)} \frac{\partial}{\partial u_1} \left( \sqrt{-P(u_1)} \frac{\partial}{\partial u_1} \right) + \sqrt{P(u_2)} \frac{\partial}{\partial u_2} \left( \sqrt{P(u_2)} \frac{\partial}{\partial u_2} \right) \right], \]

and for the Hamiltonian operator

\[ \hat{H} = 2 \frac{P'(u_1)}{u_1 - u_2} u_1 \frac{\partial Y_i}{\partial u_1} - 2 \frac{P'(u_2)}{u_1 - u_2} u_2 \frac{\partial Y_i}{\partial u_2} + 4 \frac{P(u_1)}{u_1 - u_2} u_1 \frac{\partial^2}{\partial u_2} - 4 \frac{P(u_2)}{u_1 - u_2} u_2 \frac{\partial^2}{\partial u_2}, \]

or equivalently

\[ \hat{H} = -\frac{4}{u_1 - u_2} \left[ u_2 \sqrt{-P(u_1)} \frac{\partial}{\partial u_1} \left( \sqrt{-P(u_1)} \frac{\partial}{\partial u_1} \right) + u_1 \sqrt{P(u_2)} \frac{\partial}{\partial u_2} \left( \sqrt{P(u_2)} \frac{\partial}{\partial u_2} \right) \right]. \]

This proves the formulae (3.2) and (3.3) from Chapter 3.
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