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Measurement of Mechanical Properties at the Bottom of a Granular Pack by Wavelength-Scanning Interferometry

By

Yanzhou Zhou

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of

Doctor of Philosophy of Loughborough University

June 2007
Abstract

Granular materials are conglomerations of discrete macroscopic particles. These deceptively simple materials, however, are very different from the ordinary solids, liquids and gases we know from physics textbooks.

One method to understand the static mechanical properties of granular media is to study the contact forces and displacements particle by particle, and generalize the statistical results for the whole pack theoretically and experimentally. Unfortunately, there are a range of interpretations and little experimental data to compare to and subsequently we have made relatively little progress in understanding force transmission in granular media since the 1960s.

Progress requires experimental methods that are accurate down to the single particle level. In this thesis, we introduce a new reflective scanning-wavelength Fizeau interferometer which achieves this and has the advantages of a wide range and is applicable to three dimensional systems. In this system, steel balls of diameter 8 mm and weight 20.01mN are used as our granular material, and are confined within a cylinder with a diameter of 53 beads and height 16 beads. The whole container is positioned over the interferometer allowing the bottom surface to be viewed.

Because our interferometer measures deformation rather than load, in order to determine the load at the boundary of the granular pack, calibration of deformation vs. load is required, which was done by applying known loads to a single 8mm bead on the surface of the optical substrate. The average deformation caused by putting one bead on the top of the optical substrate is 4.7 nm.

Our setup has enabled the mechanical properties at the bottom of the granular pack of 20000 steel balls to be investigated extensively. The probability distributions of normalized contact force were measured and were found to be independent of forces applied to the top surface of the pack (Green's function loads). Additionally, features such as the exponential tails and negative changes to the contact force responses have been noted. Additionally, for the first time distributions of lateral displacements in response to applied loads are reported.
Keyword: Granular material, 3D granular packs, mechanical properties, contact force, response force, response displacement, probability distribution, unequal interval sampling, phase-shifting interferometry, interferometer.
Acknowledgement

Firstly, I would also like to acknowledge financial support from Loughborough University and EPSRC to study “Measurement of Mechanical Properties at the Bottom of Granular Packs by Wavelength-scanning Interferometry.”

I am also very grateful for the support from all the members in our research group, namely Dr. T.W. Martin, Dr. P. Ruiz, D. W. Britton, O. O. Ogundana and H. Viswanathan for their rewarding discussions, friendship and help.

I would like to express my highest appreciation and gratitude to Dr. C. R. Coggrave for his help on phase unwrapping software and methods to control the laser and camera by Matlab and Visual C++ etc..

Lastly, I would like to thank to my supervisors, Prof. J. M. Huntley and Dr. R. D. Wildman for their patience, guidance, modification and comment on my thesis and paper published; and other enthusiastic academic support over the past four years.

Yanzhou Zhou
Loughborough University


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Chapter 1 Force propagation in granular materials

\( P_h, P_v \)  
Horizontal pressure and the vertical pressure respectively.

\( \rho \)  
Density.

\( g \)  
Gravity acceleration.

\( h_j \)  
Height of granular pack.

\( R_j \)  
Radius of the container.

\( \Lambda \)  
Characteristic length.

\( \mu_j \)  
Static coefficient of friction.

\( f_j \)  
Friction.

\( (r,z) \)  
Coordinates.

\( \sigma_r \)  
Shear stress.

\( \sigma_{rr} \)  
Normal stress in the radial direction.

\( \sigma_{zz} \)  
Normal stress in the normal direction.

\( \bar{P}_m, \bar{R}_m \)  
Average stress and the radius of Mohr's circle.

\( \psi \)  
Positive angle between the z-axis and the major principal axis.

\( \phi \)  
Repose angle of the granular pack.

\( S \)  
Scale variable.

\( a, b, c, c_1, c_2 \)  
Constants.

\( \eta, \mu \)  
OSL constitutive coefficients.

\( \bar{u} \)  
Displacement field in vector.

\( \lambda_t, \mu_t \)  
Lamé coefficients.

\( \delta \)  
Polydispersity.

\( F_{ext} \)  
Disturbing force.

\( \bar{m} \)  
Particle mass.

\( I, J, K \)  
Contact forces acting on particle.

\( (i,j,k) \)  
Non-orthogonal coordinates.

\( \theta \)  
Angle between the axes and the horizontal.
Chapter 2 Experimental observations of force propagation in granular materials

\( f \) Normalized contact force.
\( P \) Probability distribution of normalized contact force.
\( \alpha, \beta \) Constants.
\( \rho, g \) Material density, gravity acceleration.
\( R, r, H \) Dimensions of granular pack.
\( P_1, P_2 \) Forces.
\( \phi \) Volume fraction.
\( \sigma_{zz} \) Normal stress in the normal direction.
\( \rho_\theta \) Average angular distribution of contacts.
\( \theta \) Angular distribution.
\( A_d \) Disorder constant.
\( \varepsilon \) Strain.
\( F_t \) Tangential force.
\( F_n \) Normal force.
\( \mu \) Static friction coefficient.
\( S_f \) Variable of active failure.

Chapter 3 Interferometry
\(a\) Amplitude.
\(\phi\) Phase.
\(\Delta \phi\) Phase change.
\((m,n)\) The pixel of CCD array.
\(u_z\) Displacement in the normal direction.
\(\lambda\) Wavelength.
\(E\) Vector of electric field.
\(n\) Refractive index.
\(\theta, \theta, \theta\) Angles of the incident wave, reflected wave the refracted wave.
\(r, t\) Amplitude reflection coefficients and transmission coefficients.
\(R, T\) Intensity reflection coefficients, transmission coefficients.
\(I_R\) Reflected light intensity.

Chapter 4 Automatic fringe pattern analysis

\(t\) Time.
\(\bar{r}\) Space coordinate.
\(I\) Light intensity.
\(\tilde{I}\) Discrete Time Fourier transform of \(I\).
\(I_0\) Background light intensity.
\(V\) Visibility or fringe
\(f\) Unidimensional function of fringe pattern profile.
\(\phi\) Optical phase.
\(\phi_i\) Initial phase of \(l\) order harmonics.
\(\Delta \phi\) Unwrapped phase change.
\(a_k\) Amplitude of the \(k\)th harmonic.
\(M\) Total number of fringe patterns.
\(N\) Number of fringe patterns used per carrier cycle.
\(\nu\) Phase-shifting step.
\(l\) Order of harmonics.
\(w\) window function.
\(\tilde{W}\) Discrete Time Fourier Transform of \(w\).
\( \Delta_{ml} \) Mainlobe width of the window function.
\( \varepsilon \) Tolerance value.
\( u_z \) Displacement in the normal direction.
\( \lambda \) Wavelength.

Chapter 5 Experimental system configuration

\( L \) Laser-cavity length.
\( \lambda_n \) Discrete set of possible wavelengths.
\( N \) Integer.
\( \Lambda \) Groove spacing of the grating.
\( \theta_i, \theta_d \) Incident and diffracted angles of the laser beam.
\( n \) Refractive index.

Chapter 6 Experimental system calibration

\( t \) Time.
\( (m, n) \) The pixel of CCD array.
\( I \) Interference intensity.
\( I_0 \) Average background intensity.
\( V \) Fringe visibility.
\( w \) Window function.
\( k \) Spatial carrier frequency.
\( \theta \) Angle of optical wedge.
\( D_{fov} \) Field of view of the image in \( y \) axis.
\( \lambda \) Wavelength.
\( n \) Refractive index.
\( f \) Frequency.
\( M \) Length of the transform.
\( A \) Complex starting point of CZT.
\( \hat{W}_I \) Complex scalar of CZT.
\( x \) Discrete time sequence.
\( \hat{X} \) Discrete Time Fourier Transform of \( x \).
\( M_{x}, M_{y} \)  
Ranges of the spatial frequencies.

\( M_{x_{0}}, M_{y_{0}} \)  
Complex starting points of CZT.

\( W_{x}, W_{y} \)  
Complex scalars of CZT.

\( \bar{\psi} \)  
Average phase-shifting step.

\( l \)  
Coefficients in the Lorentz Model.

\( \phi \)  
Wrapped phase map.

\( d \)  
Substrate deformation.

\( W \)  
Number of bead loads.

\( a, b, c \)  
Coefficients of calibration of load vs. deformation.

**Chapter 7 Measurement of the mechanical properties of granular packs**

\( \bar{F}_{e} \)  
Sum of the measured forces.

\( F \)  
Contact force.

\( \langle P \rangle \)  
Global mean value of contact force.

\( r \)  
Vector pointing to the \( i^{th} \) particle's location.

\( S \)  
Area of the field of view.

\( N \)  
Number of indentation points in the field of view.

\((x, y)\)  
Spatial coordinates.

\( L \)  
Local disturbing force.

\( d_{b} \)  
Diameter of steel balls.

\( mg \)  
Weight of the steel ball.

\( A, B, C, D, E \)  
Unloaded, loaded states.

\( f \)  
Normalized contact force.

\( \Delta F \)  
Response forces to the local disturbing forces.

\( |\Delta F| \)  
Coarse grained absolute values of the responses.

\( P \)  
Probability distribution.

\( \Phi \)  
Distribution of the fraction of force.

\( D \)  
Displacements of particles for applied loads.

\( a, b, \beta, \alpha \),

\( \beta_1, \beta_2, P_0 \)  
Constants.

\( \zeta = (D/d_{b})i(L/mg) \)  
Non-dimensional displacement.
\[ \xi \quad \text{Non-dimensional response force and displacement.} \]
\[ \nu \quad \text{Maximum normalized displacement.} \]

**Chapter 8 Multiple Surface Phase-shifting Interferometry for measuring force distribution in granular materials**

- \( K \): Number of the surfaces measured.
- \( L \): Transparent surfaces.
- \( \omega \): Angular frequency.
- \( \alpha \): Coefficient of the frequency modulation.
- \( T \): Period of the ramp signal.
- \( C \): Light velocity in vacuo.
- \( \lambda_0 \): Wavelength of the light.
- \( \Delta \lambda, \Delta \omega \): Modulating amplitudes of wavelength and angular frequency.
- \( E \): Complex amplitude of the electric fields.
- \( I \): The interference signal.
- \( (x, y) \): Spatial coordinates.
- \( d \): Distances between surfaces.
- \( \phi \): Phase of the light.
- \( n \): Refractive index.
- \( \tau \): Time delay.
- \( w(t), F(\omega) \): Window function and its Fourier Transform.
- \( T \): Initial relative positions of surfaces.
- \( \Delta T \): Displacement of surfaces.
- \( U_z \): Out of plane displacement of each fringe.
- \( \Omega \): Tilt angle between surfaces.
- \( e \): Discrepancies between the tilt angles measured and calibrated.
Introduction

Granular materials are conglomerations of discrete macroscopic particles. These deceptively simple materials, however, are very different from the ordinary solids, liquids and gases we know from physics textbooks [1-3].

The best way to understand the static mechanical properties of granular media is to study the contact force and displacement particle by particle (at microscopic level), and generalize the statistical results for the whole granular pack (at macroscopic level) theoretically and experimentally.

Most theories of force transmission are classified as building up constitutive relations for closure. The problem is that there are many constitutive relations available due to the difficulty in their derivation from first principles, which lead variously to solutions of hyperbolic, parabolic, and elliptic Partial Differential Equations (PDEs), which are somewhat contradictory to each other. Unfortunately, experiments on granular media have been shown to be highly sensitive to the methodology used, which makes the use of their results as benchmark data against which theoretical problems can be compared uncertain. The result of these challenges is that we have made little progress in understanding force transmission in granular media since 1960s.

This lack of understanding demonstrates the need for novel experimental methods that can be used to probe granular materials and test available models. Ideally, such a method should be accurate down to the order of several beads (8mm steel bead), have a wide load range, be three-dimensional and be robust in the sense that the technique itself should not strongly affect the results.

In this thesis, we introduce a new reflective scanning-wavelength Fizeau interferometer that has all of the advantages described above. This system was used to consider a granular system constructed from steel balls that are confined within a cylinder. A viewing slot, in which an optical wedge resides, is cut into the bottom surface. The whole container is supported over the interferometer, allowing interferometer to view the fringes caused by the interference of the light reflecting
from the upper and the lower surfaces of the wedge and to be scanned underneath the container.

The requirements in order to measure individual contact forces are considerable (for example, the average deformation caused by putting one bead on the top of optical substrate is 4.7 nm, which is smaller than \( \lambda/100 \)) and as such a traditional interferometer could not be used, therefore, we introduce a new temporal phase-shifting method that increases the resolution of our system to about 1 nm, and is able to accommodate problems such as laser mode hopping.

These advances have allowed an important set of experiments to be performed enabling key variables to be measured, such as the contact force distribution, the contact force probability distribution, the contact force correlation in the spatial domain, the response force probability distribution and the displacement probability distributions.

This thesis is classified in the following way. In chapter 1 we will present a review of the literature available for force transmission in granular materials. First of all, we will discuss whether the concept of jamming which provides a unifying picture that could describe a phase transition from the liquid state to the solid state. Furthermore, the models to describe how forces are transmitted in granular media will be discussed in detail. Among them are continuum theories, where the features of static granular materials are modelled by hyperbolic PDE, elliptic PDE, and microscopic theories, which include statistical models that show the same features as models based on parabolic PDEs. All of these approaches are aimed at understanding granular behaviour. However, the precise connection between packing geometry, force propagation, force networks and jamming are still a puzzle.

In order to distinguish between the very different theories of force propagation in the granular pack, a series of experiments was proposed which will be reviewed in chapter 2. Among them, the most important ones are those that consider the microscopic level and include the carbon paper method and the photoelastic method. Due to the disadvantages of the experimental methods above, we see a need for a new method. It is noted that the most promising measurement technique is optical interferometry, for its high accuracy and a large scope of measurement.
Interferometry has been concisely reviewed in chapter 3 and we will focus on its use as an ideal metrology technique to experimental mechanics and in particular, the potential use of a reflective Fizeau interferometer for studying small displacements on the surface of a material. In the end, we present the phenomenon of multi-beam interference in the Fizeau interferometer.

Phase-shifting algorithms are discussed in chapter 4. Two completely independent systematic approaches for designing phase-shifting algorithms are introduced. One is temporal phase-shifting interferometry (TPSI) with equal interval sampling. In order to reduce the errors associated with higher harmonics and phase miscalibration, a data windowing technique is chosen to minimise these errors. The other approach uses temporal phase-shifting interferometry with unequal interval sampling to optimise the performance for a desired set of properties. These properties include insensitivity to the nonlinearity of wavelength scanning of the tunable laser diode and the accommodation of high harmonics from multi-reflection.

In chapter 5, the principle, structure and control method of the measurement system of normal contact force distribution in the bottom of the granular pack are discussed in detail. First, we introduce the external tunable laser diode and its control. Secondly, the cameras and framegrabber cards are presented. Finally, the structure and optical principle for measurement and load calibration are described.

In chapter 6, we compare and contrast the performance of different calibration methods of phase-shifting values and steps. The results show that there is little difference between the two calibration methods chosen, and the system accuracy is ±3.95%.

In chapter 7 the experiments on the granular packs are described. We will see that the probability distributions of normalized contact force and response are independent of the values of the localized disturbing forces applied to the top surface of a granular pack. Features of these distributions will be discussed in relation to jamming in granular media and characteristics of fragile materials will be highlighted. Two key novelties of our experimental approach are the measurement of reduction in the contact forces and small changes in the position during loading, allowing distributions of changes in contact forces and displacement to be presented for the first time. These
will be discussed within the framework of force chains, force propagation and statistical mechanics of the granular packs.

A technique for measuring depth-resolved displacement fields within a three-dimensional (3D) scattering medium based on wavelength scanning interferometry is described in the chapter 8. Sequences of two-dimensional interferograms are recorded whilst the wavelength of the laser is tuned at a constant rate. Fourier transformation of the resulting 3D intensity distribution along the time axis reconstructs the scattering potential within the medium, and changes in the 3D phase distribution measured between two separate scans provide one component of the 3D displacement field. The technique is illustrated with a proof-of-principle experiment involving two independently controlled reflecting surfaces. Advantages over the corresponding method based on low-coherence interferometry include a depth range unlimited by mechanical scanning devices, and immunity from fringe contrast reduction when imaging through dispersive media.
1 Force propagation in granular materials

1.1 Introduction

Granular materials are deceptively simple materials that are very different from the ordinary solids, liquids and gases we know from physics textbooks. Systems that contain granular materials typically exhibit highly nonlinear behaviour, which means the standard thermo-plastic approaches to solving problems of force transmission often break down [1-3]. One of the difficulties in describing stress propagation in granular packs is the indeterminacy of the static forces; there are usually insufficient equations to determine all the forces at work in the pack. This ultimately means that constitutive relations are very difficult to derive from first principles and one must look to experiment to resolve between competing theories that are currently available.

Even though a lot of research has been done on the packing of spheres, force propagation and jamming or unjamming in granular materials, unfortunately, the exact mechanical status of static or quasi-static granular material is still an open and debated issue [1-7]. Up to now, many different approaches have been put forward, including continuum theories [23-31], microscopic models [32-35], statistical approaches [36-42] and others [43-57]. However, there is no consensus on how to express the mechanical properties of a granular medium under various boundary conditions and there is a clear need for the development of precise, accurate experimental techniques to produce benchmark data by which we can judge theoretical predictions.

This project focuses on force propagation in granular materials and so in the following sections we will discuss relevant features of dense granular behaviour. Firstly we will discuss granular packings before looking in detail at models of force propagation in static packs.

1.2 Particle packings and jamming

In granular materials, provided the particles are densely packed, the pack can support a load and resist flow even when tilted at an angle. This stable arrangement, which is
called a jammed state, requires each particle to be in contact with some minimum number of neighbouring particles, but there is a myriad of equally likely particle arrangements that can achieve this situation [7].

If mono-sized spherical particles are put into a container very gently, a relatively small number will fit, with a volume fraction of around $\phi \approx 0.6$ [8-12]. If the particles are then shaken very gently, so that they pack down as much as possible, but still remain completely disordered, their volume fraction increases to about $\phi \approx 0.64$. This is the highest volume fraction of beads packed to retain a random configuration, and is called random close packing $\phi_{RCP}$. There are other important ways that the container of particles can be packed. The first occurs if the container is shaken very hard, allowing the particles to jump up slightly and completely rearrange themselves. Then they begin to order, forming layers of particles packed in a regular lattice of close packing. As shown in Figure 1-1, one of them is called face-centred cubic (FCC), arranged at the spatial structure $FCC = ABCABCA$ or stacking order from plane to plane is such that every third layer lies on top of the first; another is called hexagonal close-packed (HCP), arranged at $HCP = ABABABA$ or stacking order from plane to plane is such that every second layer lies on top of the first. The FCC and HCP arrangements are very close to one another in energy, and it may be difficult to predict which form will be preferred from first principles. This structure is nearly crystalline, and forms the highest volume fraction packing of particles, with $\phi \approx 0.74$.

![Figure 1-1: FCC and HCP arrangements of ordered beads][12].
The second important form of packing occurs if the particles are packed even more gently than for random close packing; in fact, they are first put into fluid that provides nearly neutral buoyancy, so there is almost no gravitational force whatsoever. Then after the particles settle slowly, the packing seems even less dense and the volume fraction is only $\phi \approx 0.56$. This is called random loose packing, and represents the minim packing of particles that can still keep the particles jammed in place.

The volume fraction from ordered close packing, random close packing, to random loose packing reflects a transition from ordered to totally disordered, which can be used to quantify granular packing geometry and construction history indirectly. For example, if a granular pack is constructed by dropping grains from a hopper [13], the stress in granular media is principally shear force due to the friction between particles and the volume fraction varies from 0.58 to 0.62 [13-15].

As the jammed state is approached, usually by increasing the packing intensity, the granular material as a whole starts to stiffen and develops a yield stress, somewhat characteristic of a solid. M. E. Cates and P. Claudin [16, 17] propose that jammed systems are fundamentally different from ordinary solids in that, if the direction of the applied stress changes even by a small amount, then the jam will break up. A canonical example is a pile of sand, which appears solid. Its upper surface slopes and sustains its shape despite the force of gravity. But if one tilts or vibrates the pile, the grains shift and the solid melts, and flows like a liquid. This phase transition material is sometimes referred to as a new class of "fragile matter" [16].

In order to understand jamming, a speculative phase diagram is sketched in Figure 1-2, which depends on temperature, load and density [7, 18-20]. According to this picture, jamming can occur only when the density is high enough. One can then unjam the system either by raising temperature or by applying a stress. The phase diagram raises some interesting questions: for example, a glass may have a lower glass transition temperature under high shear stress. Likewise, a jammed granular material may have a lower yield stress when random motions (that is, thermal fluctuations) are present.
One method to characterize a jammed state is using the statistical description encapsulated in the probability distribution [21-22]. However, is statistical mechanics useful at all in describing these systems? What are the ways that force can propagate within granular media? These and related questions will take years to resolve, and are some of the most interesting conceptual problems that need to be addressed in Physics.

1.3 Force propagation in granular materials

1.3.1 Introduction

Recent laboratory experiments [58-90] have probed the more fundamental response of a system to a point-like source of perturbation. However, neither theory nor experiment has led to a consistent conclusion. Some find wave-like behaviour with hyperbolic PDEs [24]; others predict elastic behaviour with elliptic PDEs [27]; in
some very small systems, even diffusive-like behaviour with parabolic PDEs seems to exist [37], as shown in the Figure 1-3 [5]. Others observe mixed behaviour [27-30]. These discrepancies can be due to the effects of the finite size of the laboratory systems [66], and the idealized particles such as discs or spheres or the two-dimensional nature of some of the experiments. Another possibility is that all types of behaviour of elastic, wavelike and diffusive are present and that geometrical details determine which one is observed [67]. Although most experimental observations can be explained with numerical and theoretical models [7, 28, 55-56], they all depend on certain basic assumptions and again do not provide conclusive answers.

In the following sections, we will discuss the most pertinent theories of force propagation in granular materials.

1.3.2 Continuum approaches

A common approach to predicting granular behaviour is to develop a continuum theory that requires solution of the stress equations through the development of constitutive relations. These constitutive relations, however, are difficult to derive from first principles and usually rely on a set of assumptions that need to be validated experimentally.

1.3.2.1 Janssen effect

Figure 1-4: The structure of granular packing in relation to the Janssen effect [3].
H. A. Janssen discovered that in a vertical cylinder the pressure measured at the bottom does not depend upon the height of the filling [3]. In Figure 1-4, the pressure follows an empirically determined law, given by

\[ P_v(h_j) = \Lambda \cdot \rho \cdot g \cdot (1 - e^{-h_j/\Lambda}) , \]

\[ \Lambda = R_j / 2 \mu_j K_j , \]

\[ K_j = P_h / P_v . \]

where \( P_h \) and \( P_v \) are the horizontal pressure and the vertical pressure respectively, \( \rho \) the density of the granular material, \( g \) the gravity acceleration, \( h_j \) the height of the column of fluid above the level of measurement and \( R_j \) is the radius of the container. \( \Lambda \) is a characteristic length, \( \mu_j \) is the static coefficient of friction.

The interpretation of the law can be given in terms of a simplified model with the following assumptions:

1. The horizontal pressure \( P_h \) is proportional to the vertical pressure \( P_v \).
2. The wall friction \( f_j = \mu_j P_h \) sustains the vertical load at contact with the wall.
3. The density \( \rho \) of the material is constant over all depths.

From Eq. (1-1), we can find that near to the surface \( (h_j < \Lambda) \) the pressure is hydrostatic \( (P_v \propto \rho \cdot g \cdot h_j) \), but at large depth \( (h_j > \Lambda) \), \( P_v \approx \Lambda \cdot \rho \cdot g \) and the weight is mainly carried by the walls.

For granular materials such as sand, soil or snow, material disorder, anisotropy and friction have to be considered. In a fluid, the stress increases linearly with depth. But in a silo, the stress in the granular material saturates at a certain level that is independent of the depth, because static friction causes the weight of the sand to be partially transferred to the silo's walls [3].
1.3.2.2 Hyperbolic model

The stress continuity equation in two dimensions for a static sandpile is as follows [23-26]

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{\theta r}}{\partial \theta} = 0, \\
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{\theta r}}{\partial \theta} = g.
\]  

(1-2)

Here, polar coordinates are used, as shown in Figure 1-5. It is assumed that the granular medium has constant density \( \rho = 1 \), thereby excluding segregation effects.

It will be useful to consider the following two quantities,

\[
P_M = (\sigma_{zz} + \sigma_{rr}) / 2, \\
R_M^2 = (\sigma_{zz} - \sigma_{rr})^2 / 4 + \sigma_{rr}^2.
\]  

(1-3)

where \( P_M, R_M \) are the average stress and the radius of Mohr’s circle, respectively, and \( \psi \) is the positive angle of inclination between the \( z \)-axis and the major principal axis of the stress tensor [23-26], shown in the Figure 1-5.

If the pack is stable, the yield locus must not cut Mohr’s circle, which can be expressed as [24]

\[
\frac{R_M}{P_M \cdot \sin(\phi)} \leq 1,
\]  

(1-4)
where $\phi$ is repose angle of the granular pack.

Eq.(1-2) provides only two relations between the three independent elements of the stress tensor $\sigma_{zz}$, $\sigma_{rr}$ and $\sigma_{rr} = \sigma_{rr}$ and clearly results in an indeterminacy even in the simplest case of a 2D pile: the continuity of the stress does not lead to a closed set of equations. In the hyperbolic model [23-26], however, it is widely assumed that the physics of granular media can be understood purely in terms of rigid particles packed together in frictional contact. The indeterminacy of the stress equations then has a clear origin: for two rigid particles in frictional contact with a specified normal force, the coefficient of static friction defines only the maximum shear force that may be present. Therefore, some definite relation between the average frictional and normal forces emerges. Thus one or more constitutive relations, not between stress and strain, but, among the various components of the stress tensor itself must exist. Ultimately, it is the object of many modellers to derive closure equations for Eq.(1-2) and we present some of their ideas below.

1. Boundary conditions of incipient failure at the free surface.

It is supposed that the surface of a pile, constructed from a point source and at its angle of repose $\phi$, is in a state of incipient slip, shown in the Figure 1-5. Firstly, all the stress components have to vanish on the surface:

$$\sigma_{rr}(S = 1) = \sigma_{rr}(S = 1) = \sigma_{rr}(S = 1) = 0,$$

$$S = r/(cz),$$

$$c = \cot(\phi),$$

where $S$ is a scale variable. The vanishing of the stress at the surface is a direct consequence of the yield criterion, and it also fixes their stress ratios in its immediate neighbourhood.

2. Scaling analysis

The basis of the scaling approach is to assume that the macroscopic material properties of a granular medium under gravity are independent of length scale. The
stress distributions in all piles formed the same way should be similar. Hence, a scaling solution of Eq.(1-2) is of the form:

$$\sigma_{ij} = g \cdot z \cdot s_y(S),$$  \hspace{1cm} (1-6)

where \((i, j)\) are the coordinates \((r, z)\), respectively. In order to close Eq.(1-2), the requirement of locality and the assumption of perfect memory are imposed [23-26]. The most general form is then

$$\frac{\sigma_{rr}}{\sigma_{zz}} = C(U), \quad \frac{\sigma_{rr}}{\sigma_{zz}} = \frac{S_{rr}}{S_{zz}}. \hspace{1cm} (1-7)$$

3. The oriented stress linearity model (OSL)

The oriented stress linearity model (OSL) is defined by assuming a similar linear relationship between normal stresses, not in a \((z, r)\) coordinate system, but in a tilted one \((n, m)\). After the rotation coordinate system, in \((z, r)\) coordinates, The OSL constitutive relation is

$$\frac{\sigma_{nn}}{\sigma_{uu}} = C(U) = \eta + \mu U. \hspace{1cm} (1-8)$$

Eq.(1-8) gives out the relation among stress \(\sigma_{rr}, \sigma_{zz}, \sigma_{nn}\), where \(\eta\) and \(\mu\) are OSL constitutive coefficients. In fact, the coefficients \(\eta\) and \(\mu\) are not independent, for a given repose angle \(\phi\)

$$\eta = \eta_0[1 - \mu \cdot \tan \phi], \hspace{1cm} (1-9)$$

where \(\eta_0\) is the constant. The incipient failure surface (IFS) boundary condition together with Eq.(1-8) and Eq.(1-9) restrict the OSL parameters to the “IFS line” in the \((\mu, \eta)\) plane which are shown in the Figure 1-6 for two values of the friction angle \(\phi = 10^\circ\) (dash line) and \(\phi = 30^\circ\) (full line).
The OSL constitutive relation Eq.(1-8) can be substituted into the stress continuity Eq.(1-2) to give a wave equation,

\[(\partial_z - c_1 \cdot \partial_r) \cdot (\partial_z - c_2 \cdot \partial_r) \cdot \sigma_{ij} = 0,\]  

(1-10)

where \(c_1\) and \(c_2\) are the positive and negative roots respectively of

\[c_{1,2} = \frac{1}{2}(\mu \pm \sqrt{\mu^2 + 4\eta}),\]  

(1-11)

where \(\mu, \eta\) are limited by the "IFS line" shown in the Figure 1-6. The resulting stress propagation equations can be solved without difficulty. There are inner and outer regions, which meet at \(S = S_0 = c_1 / c\).

In the outer region \((S \geq c_1 / c)\),

\[s_\alpha = s_\star \cdot (c - \mu) \cdot (1 - S),\]

\[s_\eta = s_\star \cdot \eta \cdot c \cdot (1 - S),\]

\[s_{\eta} = s_\star \cdot \eta \cdot c \cdot (1 - S),\]  

(1-12)

In the inner region \((0 \leq S \leq c_1 / c)\)
Force propagation in granular materials

\[
\begin{align*}
\sigma_{zz} &= \frac{s_s \cdot (c - c_1)}{c(c_1 - \mu \cdot S)}, \\
\sigma_{rr} &= \frac{s_s \cdot \eta \cdot c_1 \cdot (c - c_1)}{c}, \\
\sigma_{zr} &= \frac{s_s \cdot \eta \cdot (c - c_1)}{cS},
\end{align*}
\]  

(1-13)

where \( s_s \) is given by

\[
\begin{align*}
\sigma_s &= \frac{c}{c^2 - \mu c - \eta} = \frac{cc_1}{(cc_1 + \eta)(c - c_1)}. 
\end{align*}
\]  

(1-14)

According to Eq.(1-12) to Eq.(1-13), Figure 1-6 shows that vertical axis line separates the model showing dual peaks or dip (\( \mu < 0 \)) from those with a hump (\( \mu > 0 \)).

It is important to check that the Coulomb yield criterion Eq.(1-4) isn’t violated anywhere in the pile. However, there is also the possibility of yield at the very centre of the pile. In this neighbourhood, where shear stresses are negligible, Eq.(1-4) is simplified to

\[
\eta_{\min} \leq \eta \leq \eta_{\max},
\]

\[
\eta_{\min} = \frac{1 - \sin \phi}{1 + \sin \phi} = \eta_{\max}^{-1}.
\]  

(1-15)

Thus, the limitation of \( \eta \) lies on the segments of the dash-dotted line in Figure 1-6. Outside this range, there is either too deep a dip or too high a hump, leading respectively to passive (\( \psi(0) = \pm \pi / 2 \)) or active failure (\( \psi(0) = 0 \)) of the material at the centre of the pile, where the horizontal line \( \eta = 1 \) separates active region (\( \eta < 1 \)) from those passive region (\( \eta > 1 \)). In Figure 1-6, the left boundary of dashed – dotted lines \( R_p \) denotes passive failure and the right \( R_a \) active failure there.

If a localized disturbing force \( F \) is applied onto the top of granular slab (see Figure 1-7), then according to the hyperbolic model [42], the normal force in the bottom of the slab is double peaked and the distance between the peaks is \( 2ch_j \), the width of the peak is proportional to \( \sqrt{h_j} \).
1.3.2.3 Elastic model

(a). Stresses in a 2D Granular slab before the application of a localized disturbing force at the upper surface.
(b). Stresses in a 2D Granular slab after the application of a localized disturbing force at the upper surface.

In continuum mechanics, smooth, continuous macroscopic fields are used to describe material behaviour [27]. Isotropic elastostatic problems in the granular medium are solved by combining the equations of static equilibrium with the constitutive relation.
for isotropic elastic materials, to yield the Navier-Cauchy equations for the displacement field $\mathbf{u}$ in vector form,

$$
\mu_i \cdot \nabla^2 \mathbf{u} + (\lambda_i + \mu_i) \cdot \nabla(\nabla \cdot \mathbf{u}) + \rho \cdot \mathbf{b} = 0,
$$

(1-16)

where the body force is given by $\mathbf{b} = -g \cdot \mathbf{z}$ to accommodate gravity, and $\lambda_i$, $\mu_i$ are lamé coefficients [27].

The solution of Navier-Cauchy equations, which are elliptic partial differential equations (PDEs), requires boundary conditions, which pertain to the response to a localized force in a 2D or 3D granular slab, as shown in Figure 1-8. The boundary conditions are a concentrated normal force at $x, y, z = 0$ applied to the top layer and the top of this layer is otherwise force-free.

![Figure 1-9: Green's function response in the granular slab in relation to the elastic model [42].](image)

This kind of model predicts that the normal force, at the base of a pile subjected to a localised disturbing force at the top of granular slab, is single peaked, with the width of the peak proportional to $h_j$, shown in Figure 1-9.
1.3.2.4 Friction enhanced elastic-plastic model

Figure 1-10: Phase diagram for the crossover from a single peaked to a double peaked response [28]. $\delta$ is polydispersity. $F_{\text{ext}}$ is the disturbing force; $m$ is particle mass and $g$ is acceleration due to gravity.

It has been found that friction enhanced elastic-plastic models can provide predictions of the response of a 2D granular slab to an external load that reveals that both approaches (elliptic and hyperbolic models) are valid, albeit on different length scales [27-30] (Figure 1-10). For small systems that can be considered mesoscopic on the scale of the grains, a hyperbolic-like (double-peaked), strongly anisotropic response is expected. However, in large systems, the response is closer to that predicted by traditional isotropic elasticity models (single-peaked response). Static friction, often ignored in simple models, plays a key role. It increases the elastic range and renders the response more isotropic.

1.3.2.5 Double Y model

Figure 1-11: The local splitting and merging of the force chains [25].
Bouchard et al [25] investigated both numerically and analytically the effect of strong disorder on the large-scale properties of the hyperbolic equations in a granular pack subject to a vertical force imposed at the middle of the top surface. The physical mechanism is the local splitting of the force chains ($\Lambda$ processes) or merging ($Y$ processes) at random angles by packing defects (Figure 1-11).

By analogy with the theory of light diffusion in a turbid medium, a Boltzmann-like equation was proposed [31]. For isotropic packings, the resulting large-scale effective equations for the stresses have exactly the same structure as those of an elastic body, despite the fact that no displacement field needs to be introduced at all. Correspondingly, the response function evolves from a two-peak structure at short scales to a broad hump at large scales. Figure 1-12 shows averaged vertical stress response function for different heights. At small scale, the response function has a double-peak shape, which is characteristic of a locally hyperbolic behaviour. For larger heights, these two peaks merge into a single broad peak, comparable to an elastic-like response.

![Figure 1-12: Averaged vertical force response function for different heights [25]](image_url)
1.3.2.6 Single peaked or double peaked response

As we have seen, the prediction of the response of granular slabs to the application of an external vertical force at its top surface is different for different theories.

![Predictions for the response of a granular slab to a localized force](image)

(a). Elliptic model. (b). Hyperbolic model.

The elliptic model is represented here by the results of isotropic elasticity. In this case, a single peaked response on the floor is expected, shown in Figure 1-13 (a). The width of the peak is proportional to the depth of the slab. The shape of the peak is determined by the equations of elasticity, and depends on the boundary conditions at the floor [3, 27].

In the hyperbolic model, the stress propagates along characteristic directions, and two peaks are expected in 2D or a ring in 3D, shown in Figure 1-13 (b). Disorder is expected to give rise to diffusive broadening of the peaks [53], so that their width is proportional to the square root of the depth. In the hyperbolic case, the response is not sensitive to the boundary conditions on the floor.

Both elliptic and hyperbolic models predict that the ways of force transmission in the pack under the local disturbing force don’t depend on the size of granular pack or the strength of the localized disturbing force. The alternative theories (Friction enhanced elastic-plastic model and Double Y model) predict that the Green’s function response can change from double peaked to single peaked due to the different size of the pack or the different strengths of the disturbing forces at the upper surface.
1.3.3 Microscopic approaches

A number of researchers have attempted to describe the behaviour of granular media by probing the behaviour at the level of a single particle. The 3D stress distribution for FCC close packing of frictionless smooth discs has been investigated by a number of authors [32-35]. The main assumptions were that the contacts are frictionless and that there are no horizontal contacts.

As a non-orthogonal coordinate system, \((i,j,k)\), with the axes along the surface diagonal was used, three axes are now required (Figure 1-14). This gives \(I(i,j,k)\), \(J(i,j,k)\), \(K(i,j,k)\) as the contact forces acting on particle \((i,j,k)\). The angle \(\theta\) between the axes and the horizontal is chosen to be less than \(\cos^{-1}(1/\sqrt{3})\), the perfectly close-packed value, and hence no horizontal forces exist between the particles in a given layer. In 3D, each particle experiences six contact forces, three from above and three from below. Resolving the forces vertically and horizontally yields recurrence relations for \(I\), \(J\) and \(K\) as follows:

\[
I(i,j,k) = (i-1) \cdot \frac{W}{3\sin \theta},
\]
\[
J(i,j,k) = (j-1) \cdot \frac{W}{3\sin \theta},
\]
\[
K(i,j,k) = (k-1) \cdot \frac{W}{3\sin \theta},
\]

(1-17)
where $W$ is the weight of the particle. The particles in a given layer obey
$i + j + k = \text{Const}$, and therefore $I + J + K$ is also constant. The vertical force is
proportional to $I + J + K$ and Eq.(1-17) shows that the vertical component of force on
the bottom layer is uniform across the pile.

This model considers a regular packing of spheres with simple transmission laws for
the downward force from one layer to those below, which is not the case for a real
sandpile. It shows a flat stress plateau, a feature that can be viewed as a slightly
generalized continuum limit of hyperbolic model [23, 24].

1.3.4 Statistical microscopic approaches

1.3.4.1 Q model

A scalar Q model was put forward in Refs. [36-42] to explain the inhomogeneities
caused by an unequal distribution of weight in the granular packing. It considers a
regular lattice of sites, each containing a bead of mass unity, shown in the Figure
1-15. Each site $i$ in layer $D$ is connected to exactly $N$ sites $j$ in layer $D+1$. In 2D
systems, $N = 2$, and in a 3D system, $N = 3$. Because it is assumed that the effects of
the horizontal forces can be absorbed in the random variables $q_{ij}$ defined below, only
the vertical components of the forces are considered explicitly. A fraction $q_{ij}$ of the
total weight supported by particle $i$ in layer $D$ is transmitted to particle $j$ in layer

![Figure 1-15: Schematic of force transmission in 2D system [37].](image-url)
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Thus, the weight supported by the particle in layer $D$ at the $i$th site, $w(D,i)$, satisfies the stochastic equation

$$w(D+1,j) = 1 + \sum_i q_{ij}(D) \cdot w(D,i).$$  \hfill (1-18)

The fractions $q_{ij}(D)$ are taken as random variables, independent except for the constraint $\sum_j q_{ij}(D) = 1$. The probability of realizing a given assortment of $q$ at each site $i$ is given by a distribution function

$$\rho_q(q_{i1}, q_{i2}, \ldots, q_{iN}) = \prod_j f(q_{ij}) \cdot \delta(\sum_j q_{ij} - 1).$$  \hfill (1-19)

One of the examples is a uniform distribution, $f(q_{ij}) = \text{const}$. Eq. (1-19), which leads to the induced distribution $\eta_q(q)$,

$$\eta_q(q) = \prod_j \int dq_{ij} \cdot \rho_q(q_{i1}, q_{i2}, \ldots, q_{iN}).$$  \hfill (1-20)

This distribution must satisfy the conditions $\int_0^1 dq \cdot \eta_q(q) = 1$ and $\int_0^1 dq \cdot q \cdot \eta_q(q) = 1/N$.

The main result of the $Q$ model is the calculation of the probability distribution of normal force for different distributions of $q$'s. In the following paragraphs we discuss the details of particular cases.

1. $Q$ model for the critical case

This is the case where each particle transmits its weight to exactly one neighbor in the layer below, so that the variable $q$ is restricted to taking on only the values 0 and 1 ($q_{0,1}$ limit). The weight distribution function $Q_D(w)$ at a depth $D$ has a scaling form for all $N$:

$$Q_D(w) = D^{-a} \cdot g_q(w/D^b),$$  \hfill (1-21)

where $g_q(x) \to x^{-c}$ as $x \to 0$. In a 2D system, the random walk suggests that $a = 2$, $b = 3/2$, $c = 4/3$. In a 3D system, random walks are less likely to coalesce.
According to the mean field theory, one obtains \( a = 3, \ b = 2, \ c = 3/2 \). As \( D \to \infty \), the argument of the scaling function \( g(x) \) in Eq.(1-21) is small for any finite \( w \). Thus, the distribution of weights, \( Q(w) \), is independent of \( D \) as \( D \to \infty \), and is of a power-law form, and hence is infinitely broad. \( Q(w) \) converges to a fixed function.

2. \( Q \) model away from the critical case

When the \( q \) probability distribution is allowed to take values between 1 and 0, it leads to a much faster, typically exponential decay. In addition, the distribution for the normalized weight \( \nu = w/D \), \( P_\nu(\nu) \) converges to a fixed distribution \( P(\nu) \) as \( D \to \infty \). There are three sub-cases that must be considered:

(a). Uniform \( q \) distribution.

A uniform \( q \) distribution is obtained by choosing each of \( q_1, q_2, \ldots, q_{N-1} \) independently from a uniform distribution between 0 and 1, setting \( q_N = 1 - \sum_{j=1}^{N-1} q_j \). Thus, in the limit \( D \to \infty \), the mean field force distribution is:

\[
P(\nu) = \frac{N^N}{(N-1)!} \cdot \nu^{N-1} \cdot e^{-N \cdot \nu}
\]

Eq.(1-22) is used to describe the force distribution probability in 3D granular pack in Ref. [36].

The \( Q \) model’s PDE is a parabolic or diffusion equation [37, 42], and its configuration is shown in the Figure 1-4. Normal force distribution varies with depth,

\[
\sigma_{zz}(x, z) = \frac{F}{2\sqrt{\pi Dz}} \cdot e^{-\frac{x^2}{4Dz}}.
\]

The normal force distribution at the bottom of the slab has a hump, and its width scales approximately as the square root of height.

(b). Asymptotic \( q \) distributions.

If \( \rho_q(q_1, \cdots, q_m) \) is a generic continuous \( q \) probability distribution, which does not have a \( \delta \)-function contribution and probability density satisfied the condition \( q_{ij} \neq 0 \)
for any $q_{ij}$. Within mean field theory, force distribution functions $P(v)$ have the asymptotic form

$$P(v) \propto v^{N-1} \cdot e^{-av} \text{ as } v \to \infty$$

(1-24)

where $a$ is a constant.

(c). Singular $q$ distributions.

It is useful to consider the case where there is a finite probability for some of the $q_{ij}$ functions to be zero, which implies that the induced distribution $\eta_q(q)$ has a $\delta$-function at $q = 0$. Such a choice for $\eta_q(q)$ is also useful in examining the crossover from the critical $q_{0.1}$ limit to the smooth $q$ distributions.

In 3D packing ($N = 3$), it is assumed that $\eta_q(q)$ has the form

$$\eta_q(q) = \sum_{i>0} c_i \cdot \delta(q - \lambda) + (1 - \sum_i c_i) \cdot \delta(q) \cdot \delta(q),$$

(1-25)

with $0 < \lambda < 1$. As $n \to \infty$, $c_0$ remains nonzero and all of the $c_i$'s for $i > 0$ tend to zero, we can approach arbitrary continuous distributions for $\eta(q)$ with $\delta$-function at $q = 0$ and $q = 1$. This results in the asymptotic form

$$P(v) \propto v^{-(1+\frac{1}{m})} \cdot e^{-av}, \text{ with } N \geq m \geq 2.$$  

(1-26)

Here, $P(v)$ decays exponentially, though with a power laws prefactor to the exponential term.

The Q model has some disadvantages, however, which are listed below:

1. The prediction that the force probability distribution goes to zero at small forces is not supported by experiment. The transmission of the force in granular media is highly non-uniform; large forces often propagate along chains of neighbouring particles, leaving whole areas of surrounding particles nearly force free [1], shown in Figure 1-16 (b). Furthermore, there is some evidence [76-77] that the Q model
overestimates the probability distribution at large forces and underestimates it at small forces.

2. The Q model's PDE is parabolic, implying a diffusive-like stress transmission. Such behaviour is not usually considered consistent with the concept of force chains [43] (Figure 1-16).

![Figure 1-16: Observations of forces within a granular pack.](a) by simulation of the Q model and (b) through Photoelastic experiment [43].

3. The prediction that the $q$ distribution in the granular medium is uniform and doesn't depend on construction history and friction contradicts recent experiments [76-77].

The benefit of this model is the introduction of mean field methods into descriptions of static mechanical properties in granular media. It presents the probability distribution function as an important tool for describing granular media at the microscopic level, and though not entirely successful, stimulated considerable research in this area.
1.3.4.2 Tripod model

Based on the scalar Q model, a new tensorial model (known as the Tripod model) [17] was developed. This assumes that each grain has three lower neighbors upon which it rests. One neighbor is on-axis, directly below the grain, and the other two are to either side of that grain at the same angle, shown in the Figure 1-17. This model allows for balance of force vectors and has a tendency to predict that force propagates outward like a wave.

![Figure 1-17: Three-neighbor configuration in 2D tripod model [17].](image)

A key prediction of this model is the presence of negative forces in the Green’s function response, i.e., some particles will experience a reduction in applied force and may induce a mechanical instability. This is a direct consequence of the tensorial nature of the problem and can be interpreted as a signature of fragility of the contact network, which is generically unstable to very small perturbations.

![Figure 1-18: Probability distribution of the normal force in tripod model [17].](image)
As shown in Figure 1-18, the exponential fall-off of the stress distribution at large forces (first found within the scalar model) also holds within a tensorial approach. It does, however, require large disorder, and it is better fitted by a stretched exponential \( \ln P(w) \propto -w^\beta \) with \( \beta \approx 1 \), in the small force region. In Figure 1-18, the three bold (solid, long-dashed, and short-dashed) lines are results from silos where the heights are 1000\( d \), 5000\( d \), 10000\( d \) with large noise, and show exponential tails. The thin line represents a 1000\( d \times 1000d \) silo with small noise, and is nearly Gaussian.

### 1.3.4.3 Force network ensemble model

An ensemble approach for force distributions in static granular packings was developed by J. H. Snoeijer, et al. [46], based on the separation of packing and force scales, together with an \textit{a priori} flat measure in the force phase space under the constraints that the contact forces are repulsive and balance on every particle. It was shown that, for both disordered and regular triangular "snooker ball" configurations \( P(\psi) \) evolves from a jammed distribution with a peak, to an unjammed monotonous distribution as a function of shear stress.

### 1.4 Summary

In this chapter, we have reviewed the research of the force distribution in the granular packs theoretically. First of all, we discussed the concept of jamming which provides a unifying picture that could describe a wide range of different materials that exhibit a phase transition from liquid state to solid state in their response when the temperature drops to a certain level or the density and shear force increases above a certain threshold. Furthermore, the models to describe how forces are transmitted in granular materials have been discussed in detail. Among them are continuum theories, where the features of static granular materials are modelled by hyperbolic PDE, elliptic PDE, and microscopic theories, which include a statistical model that shows the same features as models based on parabolic PDEs. All of these approaches are aimed at understanding granular behaviour inside a pile and to explain why granular material has different behaviour to normal matter. However, the precise connection between packing geometry, force propagation, force networks and jamming is still a puzzle.
It is clear from the above review that most of the works involving granular materials are geared towards theoretical and numerical studies, and some are contradictory to each other. There is however, little experimental work in this field that looks at how macroscopic behaviour emerges from the microscopic (single particle) level. Such experimental work is important, both to validate the models put forward by various researchers, and to give insight into the behaviour of real granular assemblies, which will be introduced in the next chapter.
2 Experimental observations of force propagation in granular materials

2.1 Introduction

The first recorded attempt at measuring the force distribution at the base of a granular pile was during the 1960s [58-59]. In 1981, however, Smid and Novosad [60-61] performed a series of experiments that showed anomalies in the normal stress distributions at the base of a conical sand pile. Intuitively, one would expect the normal stress to be a maximum at the centre, but their result showed that there is a significant depression in the middle of the heap and the neighbouring positions attain pressures that are maximal (Figure 2-1). One explanation for this behaviour is that part of the load is transmitted from the central area to the outer regions of the heap through arching and bridging, but deriving models that are able to predict this behaviour has proved challenging [3].

![Figure 2-1: The pressure at the base of a sandpile, as recorded by Smid and Novosad [60-61].](image)

These challenges, however, have stimulated the development of a number of theories for stress propagation in granular media. Unfortunately, the lack of precise, three-dimensional experimental methods, able to focus on a single particle behaviour, has meant that many of these models have not been validated. Over the past 15 years though a number of experiments have been performed [79-90], initially relying on
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either large averaging areas when measuring the stress, or on unreliable contact
techniques such as carbon paper methods, but more recently, employing more
sophisticated methods to measure the forces due to individual particles. In this chapter
we will discuss the recent developments in experiment techniques and present some
of the more important results.

2.2 Carbon paper method

(a) System configuration. (b) The distribution of forces as a function of normalized weight.

A simple method for determining contact forces at the base of granular pack was
proposed by C.-H. Liu [36] and D. M. Mueth et al [62]. A disordered 3D granular
pack of 55000 soda lime glass spheres with diameter 3 mm was studied. The beads
were confined in an acrylic cylinder of 140 mm inner diameter and the top and bottom
surfaces were provided by close-fitting pistons. Once the cell was filled with beads, a
load, typically 7600 N, was applied to the upper piston while the lower piston was
held fixed. In most experimental runs, the outside cylinder wall was not connected to
either piston so that the cylinder was supported only by friction with the bead pack.
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All constraining surfaces of the system were lined with a layer of carbon paper covering a blank sheet of colour copier paper. Beads pressed the carbon onto the paper in the contact region and left marks whose darkness and area depended on the force on each bead. A region from a typical data set taken from the area over one of the pistons is shown in the Figure 2-2 (a). The resulting normal force distributions $P(f)$ for all system surfaces, averaged over fourteen experimental runs, is shown in Figure 2-2 (b). Interestingly, the form of the distribution appears to be unaffected by changes in the boundary conditions or in the preparation history of the system.

For forces greater than the mean ($f > 1$), the experiments showed that probability decays exponentially,

$$P(f) \propto e^{-\beta f},$$

(2-1)

with $\beta = 1.5 \pm 0.1$. One way of understanding the physical mechanisms behind the exponential tail of the distribution is through the use of the scalar $Q$ model [36-37], where it emerges as a result of a randomization process as the forces are transmitted through the bulk of the bead pack. For forces smaller than the mean, or $0 < f < 1$, it is suggested that $P(f)$ reaches a maximum as $f$ is increased from zero, although the lowest bin contains both measured forces as well as an estimated number of undetectable contacts, giving it a greater uncertainty than other bins [62, 65]. It is noted that the measured distribution there is an absence of either a “dip” or a power law divergence for small forces; instead, the data is most consistently fit by a functional form given by Eq. (2-1) that approaches a finite value as $f \rightarrow 0$. It is difficult, however, to resolve the small force behaviour of the probability distribution by the carbon paper method, as beads that support no weight are indistinguishable from regions in which there are no beads.

Mueggengburg et al. [63] used the carbon paper method to measure local contact forces at the bottom boundaries of a three-dimensional Face Centred Cubic (FCC) and Hexagonal Close Packed (HCP) granular packs, and the response of the pack to an external force of 444.82 N applied to a small area on the top surface.
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Figure 2-3: Intensity patterns at the bottom of FCC, HCP or amorphous crystals [63].

(a). FCC crystal of 9 layers and 20 layers, respectively.
(b). HCP crystal of 9 layers and 20 layers, respectively.
(c). Amorphous crystals 1 inch in depth.

Figure 2-3 (a) shows the patterns of intensities at the bottom of FCC crystals in response to a quick impulse force. Three regions of large force are found, and are arranged in a triangular pattern. In contrast to the triangular patterns found for FCC crystals, HCP crystals show rings of large force at the bottom surface in response to a localized force, known as the Green's function response, shown in the Figure 2-3 (b). The response of an amorphous pack shows a broad central peak centred below the applied force, shown in the Figure 2-3 (c). The low resolution and accuracy of the method makes it difficult to use the technique as a validation tool against which to compare predictive models [66, 67].
M. J. Spannuth et al. [68] explored the effect of stacking fault defects on the transmission of forces in three-dimensional FCC granular crystals. An external force is applied to a small area at the top surface of a crystalline packing of granular beads containing one or two stacking faults at various depths. The response forces at the bottom surface are measured and found to correspond to predictions based on a vector force balance within the geometry of the defects. The elementary stacking fault is a boundary between two pure FCC crystals with different stacking orders, shown in the Figure 2-4. As the number of stacking faults increases, the intensity pattern evolves toward that of an HCP crystal. This leads to the conclusion that the force pattern of that crystal structure can be viewed as the extreme limit of an FCC crystal with a stacking fault at every layer.

The Green’s function response in FCC & HCP pack is consistent with a hyperbolic model [23-26], but the Green’s function response in the disorder granular material is
Experimental observations of force propagation in granular materials consistent with the elliptic model [27]. Currently it is not clear how the response transitions from the hyperbolic model to the elliptic model.

It is important to note that the experimental results of individual runs vary significantly. In order to improve statistics, the locally measured forces were also averaged over symmetries of the crystal.

2.3 **Electronic balance sensor method**

![Diagram of experimental set-up](image1)

![Graph of radial pressure distribution](image2)

![Graph of ln-log plot of calculated probability density function](image3)

Figure 2-5: Experimental set-up of electronic balance methods [64].

(a). Experimental set-up. (b). Radial pressure distribution of the two filling procedures.

(c). Ln-log plot of calculated probability density function.

An alternative method for measuring stress distributions was presented by G. Lovoll *et al.* [64]. Their technique measures the normal force using a high precision
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electronic balance is sketched in Figure 2-5 (a). A cylinder filled with 120000 glass spheres (2 mm) with a width of 40 beads and height of 80 beads was built up. Its base had a 1 mm hole in the centre where a force probe with a resolution of $10^{-3}$ g and a range from 0 to 1200 g was located. This made it possible to measure weights ranging from a few bead weights to nearly the whole system weight. With the probe tip near the table level, the normal forces acting on individual beads in the bottom layer could be measured by sliding the granular system over the table and the probe tip while measuring the pressure and the position.

In the case where the beads were poured in directly (filling 1 in Figure 2-5 (b)), there appears to be a pressure dip in the centre, a more or less constant level for larger distances and finally the pressure drops near the container walls. In the case of filling 2 where the container was filled and then the inner tube was slowly removed, the largest pressure appears in the centre and the pressure falls off monotonously towards the walls.

The measured weights $w$ were scaled with the mean weight $<w>$ in the system and the force distribution function $f = w / <w>$ was calculated in the same way as described in Ref. [36]. The obtained distribution function $P(f)$ was consistent with the form,

$$P(f) \propto \begin{cases} f^\alpha & f < 1 \\ e^{-\beta f} & f > 1 \end{cases}$$

(2-2)

where the indices $\alpha$ and $\beta$ were found to be $0.3 \pm 0.2$ and $1.8 \pm 0.2$, respectively. The value of $\beta$ is in reasonable agreement with the exponent found in the experiments of C.-H. Liu [36] and D. M. Mueth [62]. This lower part of the distribution function, however, was found to have different characteristics to those observed in other experiments [36, 62]. In general these results were found to be in good agreement with the Q model [37, Section 1.3.4].

### 2.4 Capacitive pressure sensor method

L. Vanel et al. [69] developed a method based on capacitance measurements to determine the normal stresses. A single capacitive pressure sensor was placed flush
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with the surface of a base plate, on top of which sits a granular pack, allowing the normal stress at various locations along the radial axis of the conical piles to be measured. The heaps (constructed using sand of diameter 1.2 mm, angle of repose 33° and final height of $H = 8\,\text{cm}$) were constructed by two qualitatively different procedures, shown in Figure 2-6 (a).

![Diagram of experimental setup](image)

Figure 2-6: Dip in the pressure at the base of granular packs [69].

(a). Dimension of sandpile.

(b). System setup and experiment result in measuring the stress distribution of conical sand piles.

The first, a "localized source" procedure used a funnel with the outlet always placed just above the apex of the sandpile. This approach, as opposed to using a fixed funnel height, avoided the deposition of particles with large kinetic energies. The second, a "raining procedure" used a sieve. This method was designed to align the stress
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chains in the vertical direction. The sand was poured from containers with cross-sectional dimensions slightly larger than the platform, and whose bottoms were wire meshes with 0.40 cm diameter holes. The containers were filled while resting on the platform and then raised slowly, allowing a steady rain of sand onto the heap. Excess sand was allowed to avalanche off the platform. For this procedure, the final heap covered the platform.

We can see in Figure 2-6 (b), that there is a clear pressure dip at \( r/R = 0 \), and a maximum in the stress of about \( 0.6 \cdot \rho \cdot g \cdot H \) at position \( r/R = 0.3 \), both of which agree reasonably well with previous conical pile data [70]. The authors suggest that the progressive formation of the pile by successive small avalanches leads to the occurrence of a pressure dip, as a dip does not occur in the profiles of the heaps created by the raining method. Rather, there is a peak pressure of about 0.6 at \( r/R = 0 \), and a steady drop in the pressure moving out towards the edge of the pile.

2.5 Experimental probing of the response to a localized force perturbation

Experiments based on probing the response of static granular assemblies to a localized stress perturbation known as the Green's function were presented by G. Reydellet et al. [67].

Figure 2-7: Experimental system of Green's function method [67].

(a). Experimental system for measuring stress distribution.

(b). Horizontal stress distribution in response to a localized solicitation.
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The spatial distribution of stresses at the bottom of a 3D granular assembly confined within a box is monitored. The actuator applied at the surface of the pile is achieved by a piston $P'$, as shown in the Figure 2-7 (a). More precisely, a local stress modulation was achieved by the modulation of the magnetic field in the coil, which is connected with the piston. The signal of the stress probe $P_l$ at the bottom of the piling is then directed to the lock-in amplifier synchronized by the generator exciting the source. The signal obtained is the convolution of the mechanical response function (the Green’s function) by the width of the source and the width of the probe.

The different pilings tested were very disordered in terms of size, polydispersity, and friction between the grains. The piling procedure was designed to avoid preparation memory effects as much as possible.

The response function $P(x)$ in Figure 2-7 (b) shows one single peak centred at the vertical of the point of application of the force; the two separated bumps as predicted by hyperbolic models were not observed [23-26]. For all granular media studied, the width at half amplitude, $w$, increases linearly with the depth, which is a characteristic of an elliptic model. The slope is independent of the material used, shown in Figure 2-7 (b). This last result rules out parabolic modelling of disordered granular assemblies. Furthermore, in the limit where no “sinking” of the piston inside the pile is observed, the response is linear in the value of the imposed force. The Green’s function response was found to be consistent with predictions of elasticity based models [27].

During the experiments, two effects were observed that should be avoided if possible. The first is that large deformations of the device may drastically change the local force distribution. The second is that the perturbing stress should not create plastic reorganization of the grains, and only weak perturbations should be applied to the upper surface.

2.6 Electronical scale method

A set of precise and reproducible measurements on the static pressure at the bottom of a granular column under different filling procedures is reported by L. Vanel et al. [51]. In order to vary the initial packing fraction, two filling methods are used (see
Experimental observations of force propagation in granular materials

Figure 2-8 (a)). In method 1, the grains fall from a hopper located on the top of the column. Due to the jamming effect of falling grains, a compact packing is obtained. In method 2, the grains first fill an intermediate inner cylinder that initially rests on the piston. Next, the inner cylinder is slowly removed so that the grains gently flow into the outer cylinder and settle rather loosely. Throughout the experiments, the average volume fraction of the grains was estimated by monitoring the total height $H$ of the granular column and the number of grains.

(a) Filling methods. (b) Apparent mass vs. different filling fraction. Sign 1 a tapping experiment; sign 2 a descent experiment with filling method 1; sign 3 a descent experiment with filling method 2.

Figure 2-8 (b) shows that the apparent mass (mass measured) is plotted as a function of the packing fraction for a filling mass of 300g of glass spheres, $\Phi 2\text{mm}$. The arrows indicate the direction of evolution in the course of each experiment. There is clearly a drastic change in the static equilibrium with the experimental procedure followed. Data for the tapping experiments show that a density increase of about 8% induces a pressure decrease of about 20% (curve 1), which shows a compaction effect for initially loose packings. On the other hand, the descent experiments show that density variations of less than 5% induce a pressure increase as large as 50% (see Figure 2-8, curves 1 and 3), which shows a decompaction effect for initially dense packing. In the limit of large deformation a so-called "critical density", independent of the initial stage, is reached. This is probably what could happen around $\phi \approx 0.593$ where both
curves (2, 3) cross. It indicates that an identical density profile might be finally reached from both sides.

But, what happens in curve 1 is less clear. The vibrations produce a compaction effect that "kills" the density gradients developed through the shear bands. After a tap, the displacement of the piston is small enough to avoid the formation of shear bands, and even if a shear band was initiated, the next vibrational shock is likely to destroy it. As a consequence, a more homogeneous granular packing is likely to form. This interpretation is consistent with the fact that the end of curve 1, corresponding to a sequence of tapping and descents, seems to join the beginning of curve 3 which represents an early situation when the shear bands are not yet developed. In the tapping experiment, a decrease of the apparent mass is observed when the density is increased, which suggests a stronger screening effect of the boundaries.

2.7 Elasto-optical method

A simple optical based method was developed by Brockbank et al. [34, 70]. The forces at the base of a granular pile were determined by measuring the elastic deformation of a transparent silicone rubber surface on which the pile was constructed (Figure 2-9). The diameter of the contact region between a given ball and the rubber was measured by using an optical microscope and could be related directly to the normal force acting on the ball.

![Figure 2-9: Experimental device to measure the force profile beneath the piles [70].](image)

The granular materials were chosen for their variation in physical and geometrical properties. These were lead shot, sand, and two sizes of glass beads (mean diameters
Experimental observations of force propagation in granular materials of 180 and 560 mm). The effect of the friction coefficient was studied by etching the larger glass beads with a chemical etchant. The piles were formed by pouring from a funnel at a fixed height.

Figure 2-10: Pressure profiles from a lead-shot pile [70].

Figure 2-11: Pressure profiles averaged over several experiments [70].


The results of pressure measurements comprised of 22 different piles of diameter of ~200 mm are given in Figure 2-10 and Figure 2-11. Figure 2-10 shows a typical lead-shot pile profile as measured from a single experiment. Each point represents a single
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contact-diameter reading. The contact forces show large variations from point to point.

The experiment was repeated several times with each material to reduce the effect of the natural contact force fluctuations. Some of the results are shown in the Figure 2-11, in which the error bars represent the standard deviation in the mean of the pressure values averaged. Significant pressure dips were found to occur with sand (Figure 2-11 (b)), and to a lesser extent with the small glass beads (Figure 2-11 (d)). An increase in glass-bead diameter by a factor of three resulted in almost complete suppression of the dip, regardless of whether the beads were smooth (Figure 1.16(c)) or etched. Likewise, the much larger lead shot (Figure 2-11 (a)) showed no central minimum.

The method is, however, rather time-consuming (about 40s per force measurement), and an automated system would be a useful development.

2.8 Photoelastic method

2.8.1 Introduction

Photoelastic measurements [71-73], involving stress-induced birefringence provides an opportunity to obtain information about the internal structure of 2D granular systems. The experiments use a layer of photoelastic grains consisting of either disks or pentagonal particles. A granular pack is placed between a pair of left- and right-hand circular polarizers as shown in the Figure 2-12. Light passes through this sandwich to produce a polariscope intensity image, which is recorded with a digital camera. When the photoelastic grains are subjected to stresses, they become birefringent; the resulting transmitted intensity is a measure of the applied stress. When viewed through circular polarizers, large forces show up as bright regions. When the force is very large, the polarization is rotated through multiple phases of $\pi$, showing fringes. Figure 2-13 (a) (b) (c) shows a typical intensity and numerical reconstructing picture for disks under diametrical compression observed with polarizers. With the help of this technique, force within granular materials can be characterized at a microscopic level.
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Figure 2-12: Schematic view of the circular polarimeter setup [72].

Figure 2-13: Stress induced birefringence in 2D granular packing [73].


2.8.2 Force distribution in a 2D granular pack

The experiments were carried out with disks made from a photoelastic material [74]. The sample was a mixture of two disk sizes, one with diameter 9 mm (500 disks) and the other with diameter 7 mm (2500 disks). The disks were confined between two Plexiglas sheets. The heap size was about 130 cm at the base, and about 30 cm high. The packs were prepared by pouring particles from (i) a fixed height point-like source (FHPS), (ii) a slowly moving point source (SMPS); and (iii) an extended source (ES).
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Figure 2-14: Comparison of mean-nearest-neighbour-angle distributions [74].

(a). Distributions for three different pouring techniques.
(b). Distributions for the particles on the force chains for three different pouring techniques.

In order to provide a simple measure of the contact orientation, the average angular distribution of contacts $\rho_\theta(\theta)$ was evaluated. The distributions for the fixed height point source (FHPS), slowly moving point source (SMPS), and the extended sources (ES) was contrasted, shown in Figure 2-14 (a). For both the localized-source procedures, there is strong anisotropy, and a clear preferred set of orientations. By contrast, for the extended source procedure, the contacts are much closer to having an isotropic orientation of contacts. The corresponding $\rho(\theta)$ for the stress chain disks is shown in Figure 2-14 (b). In all cases, these large-force-carrying disks have contact angle distributions that break the roughly six-fold symmetry present in the distributions for all disks. This corresponds to stress chains that are inclined at angles $0 < \theta < \pi/2$, i.e., in such a way as to support the heap.
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Figure 2-15: Force distribution for three different pouring techniques [74].

(a). FHPS. (b). SMPS. (c). ES.

The force profiles are affected by preparation history, but not as dramatically as the contact angles. Figure 2-15 shows the force (averaged over 50 samples) as a function of horizontal distance for various heights from the bottom of the heap, measured in small particle diameters. The fixed-height local source method clearly yields a force minimum in the centre of the heap. The slowly lifted point source does not show a convincing dip, at least within the scatter of the data. Similarly, the extended source method shows no dip.

2.8.3 Green's function response in a 2D granular pack

The force response of 2D granular systems to local perturbations was measured under various conditions by the method described in the section 2.8.1 [71-73]. There are
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large variations from realization to realization. Disorder, packing structure, friction were found to affect the average force response in a granular system significantly.

Figure 2-16: Photoelastic response to a point force for bidisperse systems of disks [71-73].

(d). The force vs. the horizontal position at different depths of ordered monodisperse disks.
(e). The force vs. the horizontal position at different depths of bidisperse disks.
(f). The force vs. the horizontal position at different depths of pentagons.
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The different particle types, including disks of 7mm, 9mm and pentagons of a side length of 7mm were used to construct 2D granular pack. For monodisperse disks, a highly ordered packing was obtained. With bidisperse distributions, the disorder in a controlled way [75] was characterized by the parameter \( A_d = \frac{\langle d^2 \rangle}{\langle d^2 \rangle} \). The brackets refer to averages over the sample for powers of the disk diameters \( d \). A sample set, typically about 60 particles wide and 25 particles high, was placed in a nearly vertical plane. The samples were prepared by gently adding particles to the upper surface until the full amount of particles was in place.

Figure 2-16 (a), (d) show the responses with two-peak features that resemble the response structure for ordered monodisperse disks. These data suggest that in a perfectly ordered and frictionless system, the force response would be perfectly sharp force chains, which is consistent with a hyperbolic description of propagating forces [23-26]. However, with increasing the disorder, this feature becomes progressively weaker. In Figure 2-16 (c), (f), it has completely disappeared. The change from a two-peak to a one-peak structure presents clear evidence of the important role of disorder in force responses, a result that is compatible with the predictions of elasticity [27].

2.8.4 Force propagation in 2D sheared and compressed granular packs

Understanding the form of the force distribution when a pack is subject to forces at the system boundaries represents a fundamental goal of granular mechanics. Paper [76] reported measurements of the normal and tangential grain-scale forces inside a 2D system of 2500 bidisperse photoelastic disks that have been subjected to pure shear and isotropic compression. The deformation \( \varepsilon_x = \varepsilon_y = |\Delta L / L| \) for isotropically compressed and sheared systems is 0.016 and 0.042, respectively.
Figure 2-17: Probability distributions of the normal forces, the tangential forces and the mobilized friction for sheared and the isotropically compressed systems [76].

(a). Probability distributions of the normal ($F_n$) and the tangential ($F_t$) forces for the sheared system.

(b). Probability distribution of $S = |F_t| / \mu F_n$ for the sheared system, normalized by its maximum value.

(c). Probability distributions of the normal ($F_n$) and the tangential ($F_t$) forces for the isotropically compressed system.

(d). Probability distribution of $S = |F_t| / \mu F_n$ for the isotropically compressed system.

Figure 2-17 shows the distributions of the normal force, the tangential force and the ratio of tangential to normal force, for a sheared system and an isotropically compressed system. The normal and the tangential forces are normalized by their mean normal forces. The normal force distribution for the sheared system (Figure 2-17(a)) has a peak around the mean, a roughly exponential tail and a dip towards zero for forces lower than the mean. In contrast, for isotropically compressed systems, the normal force distribution (Figure 2-17 (c)) dips towards zero for forces below the mean, is broad around the mean, and decays faster for large forces compared to the sheared system. The tangential force distributions have a nearly exponential tail for forces larger than the mean for both the sheared (Figure 2-17 (a)) and the isotropically compressed system (Figure 2-17 (c)). The mean tangential forces are an order of
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magnitude smaller than the mean normal forces: a feature responsible for a smaller range of maximum tangential forces.

In order to investigate the role of friction in the system, the distribution of the variable $S_f = \frac{|F_t|}{\mu F_n}$ was studied, where $\mu$ is the static friction coefficient, $F_t$ is the tangential force, and $F_n$ is the normal force. The variable $S_f$ gives information about how far away a contact is from the Coulomb failure criterion. If a contact is at the Coulomb failure criterion, $S_f = 1$. Figure 2-17 (b), (d) show the distributions of $S_f$ for the sheared and the isotropically compressed system, respectively. For both types of loading conditions, the distribution of $S_f$ shows that most of the contacts are well below the Coulomb failure condition.

Figure 2-18: Contact orientation in 2D granular pack [76].

(a). Sheared systems. (b). Isotropically compressed system.

The anisotropy induced by an external load has two distinct effects; (1). a purely geometric effect: it is to introduce anisotropy in the contact network. (2). a mechanical effect: it is to develop an anisotropic force chain network and alter the stress distribution in the system. In order to investigate the geometric anisotropy, the distribution of contact angles for contacts carrying forces larger than the mean force was investigated. The sheared system (Figure 2-18 (a)) shows a strongly anisotropic distribution, with a large number of contacts aligned along the direction of the majority of force chains and a small number of contacts aligned in the direction perpendicular to it. In contrast, the isotropically compressed system exhibits a distribution with a six-fold symmetry (Figure 2-18 (b)), indicating that the contacts are distributed evenly along these directions.
2.8.5 Green's function response of a 2D rounded rectangular pack

M. D. Silva et al. studied the response to the local force disturb in a smaller granular system [66]. The packing was composed of rounded rectangular 2D particles. The response appears to exhibit a parabolic envelope shown in the Figure 2-19. This suggests a parabolic or diffusive-like behaviour, as expected from models, which assume an uncorrelated, diffusive propagation of the forces, such as the Q-model [36-37]. However, as mentioned in Ref. [27], this system is quite small in terms of the number of constituents; furthermore, even in this case, it appears that the envelope may be narrowing down near the bottom of the picture in Figure 2-19.

![Figure 2-19: Response of a 2D packing of rounded rectangular particles [66].](image)

2.8.6 Force distribution in a 3D granular pack

In Corwin's experiment, a slowly rotating plunger exerted a constant, shearing force on glass beads filling a cylindrical container [5, 77] (see Figure 2-20 (a)). The motion of the bottom layer of beads revealed that beads near the sides flowed past each other, whereas those near the centre remained jammed and rotated as a solid block. Corwin and colleagues measured the forces on a photoelastic bottom plate, and express the probability of a grain exerting a force of a certain magnitude in a distribution shown in Figure 2-20 (b). Their central finding is that a change in the 'tail' of this distribution (characterizing the particles carrying the largest forces) signals the point
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at which the system jams: jammed grains produce tails with an exponential fall-off, whereas yielded grains produce much steeper tails. Thus flowing grains avoid large forces more effectively than those that are jammed.

Figure 2-20: Force distribution in 3D granular pack by Photoelastic method [77]

(a). Experimental set-up. (b). Change of force probability distributions with local shear strain rate.

The tails of the force distributions established by Majmudar and Behringer hold a surprise [76]: they change from steep for compressed, strongly jammed systems to exponential for sheared, weakly jammed systems. The tails determined by Corwin and colleagues [77], in contrast, become steep only when the grains flow, and are exponential for all jammed cases - irrespective of whether the system has experienced a shear stress. The two experiments probe somewhat different aspects of the force network, but, even so, their results are not easily reconciled [1, 5].

2.8.7 Summary

The photoelastic method is one of the most important experimental works on the force distribution in the granular media. The most distinguished achievements are

1. Disorder, packing structure, friction and texture significantly affect the average force response in granular systems. For the packings with weak disorder, the forces propagate primarily along lattice directions (hyperbolic model). As the disorder
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increases, the two propagation directions of the force merge into a single direction (elastic model).

2. The force networks of shear and normal force systems are very different. The sheared system exhibits strong anisotropies, and force chains are much longer than is the case in a compressed system.

Some problems still remain:

1. Because friction plays an important role in the constitutive relation for theoretical models, the friction in a 2D system should be larger than that in the 3D system, for the contact between the particles in the 2D system forms a line. However, this appears not to be the case.

2. Packing geometry has a significant influence on the force distribution in granular media, which has received considerable attention recently [8-11]. The one parameter that characterises packing geometry in packed granular materials is volume fraction. Unfortunately, volume fraction in 3D system is quite different from that in 2D system, and behaviour in the jamming state is different as well. For example, Ref. [11] reported that a large monodisperse disk random close packings are invariably highly crystalline (φ ≈ 0.88) and only collectively jammed.

3. Relatively small scope of measurement limits its application to measure the force distribution in the whole granular packing, and its resolution does not allow it to measure very small forces.

In conclusion, the photoelastic method helps identify the important factors that affect force propagation in granular media and thus raises the need to incorporate these factors into models. It has been instrumental in examining the predictions of the scalar Q-model [36-41], but unfortunately, cannot resolve sufficiently finely to be able to validate either the hyperbolic [23-26] or elliptic models [27]. Recently, the method has begun to be used to explore the jamming or phase transition in granular materials.
2.9 Scanning wavelength Interferometric method

None of the experimental techniques described above provide an ideal solution to the measurement of stress distribution in static granular materials, due to restrictions on dimensionality, low resolution and low accuracy. An ideal technique would offer:

1. Large dynamic range and high accuracy of force measurement.
2. A matrix of pointwise information.
3. A tool for observing the behaviour of 3D granular material packings.
4. Substrates of high stiffness and with a non-hysteretic response.
5. Easy manipulation and fast processing times.
A method that meets these requirements is optical interferometry [78]. An approach based on phase-shifting interferometry, utilising a wavelength tuneable laser that acts both as the light source and the phase-shifting device, has recently been developed at Loughborough University. The optical arrangement is based on a classic Fizeau arrangement (see Figure 2-21), and was itself a development of the Mach-Zehnder interferometer. The phase-shifting algorithm employed was the fifteen-frame technique, which can reduce the background noise detected during the experiment (see Section 4.2.2).

The apparatus was designed to measure the deformation imposed on a substrate through particle self-weight or applied load. The load or stress on the substrate is then determined through careful calibration. In this case, the substrate (see Figure 2-21 (c)) on which the granular pack was assembled was a double layer elastic substrate with high modulus epoxy constituting the top layer and silicone rubber as the bottom layer, with a gold layer at the interface. Deformations in this film resulted in an optical path difference between the reflected beam and a reference beam that could be analysed to determine the pressure field at the lower surface of the pack.
Figure 2-22: Wrapped phase map for 10 ball bearings above each of 9 points of indentation [78].

Calibration of the substrate was carried out by loading at 9 different locations simultaneously (see Figure 2-22) and determining the relationship between the deformation of the substrate and the size of the load. Spatial and temporal variations of the calibration constants were found to be of order ±20% (see Figure 2-23).
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Figure 2-23: Calibration of displacement vs. number of ball bearing [78].
(a). Repeatability of calibration. (b). Standard deviation error of the calibration.

It was noted that the experimental set-up, as described in [78], is too variable to enable reliable comparison of experimental results with granular theory. The main issues are listed below.

1. This technique involved mounting the interferometer statically on an optical table. This presented difficulties in trying to acquire data from the whole of the sample, as it is restricted to only one field of view.

2. The results from the double layer elastic substrate experiments yield ±20% inconsistencies, because the substrate is made manually and the thickness, elastic modulus, and refractive index are different from place to place.

3. The New Focus Vortex Tunable Laser, was found to be susceptible to mode hops or blurring, wavelength-time driftiness, and light intensity variable when it is modulated, meaning that using commonly accepted ways of analyzing the data were problematic.

4. The method using a Least Squares Fitting technique to provide the values of the PZT voltage (see Section 5.1) needed to give a required phase shift led to large errors on a single measurement.
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In spite of the problems above, these experimental techniques are able to measure small displacements down to the resolution of a few nanometres, and are the first time an interferometry method has been employed to measure the contact force distribution in a granular pack.

### 2.10 Summary

In order to distinguish between the very different theories of force propagation in the granular pack, a series of experiments was proposed. Among them, the most important ones are those that consider the microscopic level and include the carbon paper method and the photoelastic method.

The carbon paper method can be used to measure the force distribution and probability density in the boundary of granular pack and Green’s function response to the local disturbing force [36-41, 56-57]. The experimental results show that the probability distribution in the large force region has an exponential tail. Green function response to the local disturbing force predicts dual peaks in FCC and HCP packs to single peak in random packs.

The advantages of the carbon paper method are that

1. It is able to measure the force distribution beneath 3D granular packs.
2. It can potentially measure up to thousands of indentation points each time.
3. The error caused by statistics is the smaller than that associated with other experimental methods.

Its disadvantages are that

1. Its accuracy and resolution are not as high as optical methods.
2. Hundreds of Newtons disturbing force and pressure force may cause a change of the force distribution inside granular pack.

The photoelastic disk method can be used to measure the force distribution and probability density inside granular packs and Green’s function response to a local disturbing force on the top surface [6, 76-77, 71-72]. Its measurement results show
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that the Green’s function response to the local disturbing force distribution inside packs transitions from the predictions of the hyperbolic model to those of the elliptic model when the disorder in the packs increase. Furthermore, the force chain distribution is still not clear. Recent experiments [6, 76-77] show that the difference between the force probability distributions for shear and compressed systems is significant, which challenges the conclusion of the Q model and calls for more extensive systematic studies especially by experiments.

The advantages of the photoelastic disk method are that

1. It has a high accuracy and resolution.

2. It has an ability to show the force chains and contact orientations inside granular packs.

Its disadvantages are mainly due to the restriction to two dimensions.

Due to the disadvantages of the experimental methods above, we see a need for a method that is accurate down to the order of 1 or 2 beads and able to measure a wide range of loads, is three-dimensional so that we may analyse systems closely related to reality and robust in the sense that the technique itself does not strongly affect the results. It is noted that the most promising measurement technique satisfying the requirements above is optical interferometry, for its high accuracy and large scope of measurement. In the following chapters the development of a new phase-shifting interferometer is updated from the work of Osman et al [78] and will demonstrate the possibility of measuring forces equivalent to less than a particle weight and deformation of the order of Inm.

In the following chapters we will describe advances on the technique of Osman et al [78] that will allow us to measure the deformations due to single particle contacts on a substrate to our desired accuracy. The development of the interferometer will involve the construction of the physical elements of the apparatus (Chapter 3), the capture of phase data and its subsequent analysis to be extract of deformation fields (Chapters 4 and 5). Having established the optical and analysis technique we will go on to describe our methods for calibrating the interferometer to extract contact forces from the deformation fields (Chapter 6). Furthermore, we shall describe techniques for
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automating the determination of the position of a contact and its load using correlation and regression techniques (Chapter 6). Finally, we will apply these methods for firstly looking at the force distribution underneath a granular pile and then secondly, the response characteristics of a flat granular pack (Chapter 7).
3 Interferometry

3.1 Introduction

Optical methods have been used for many years to investigate material properties. It is well known that the interference properties of light waves can be used as a measure of distance, deformation and so on. With the development of the laser, combined with computer technology, whole field interferometry now allows accurate measurement of very small changes at high accuracy. In this chapter, we will firstly introduce the basic properties of light sources, and then we will present various techniques based on interferometry and how one may configure interferometry systems for different applications in the field of solid mechanics.

3.2 Lasers

3.2.1 Introduction

Advances in precision measurement using interferometry have been closely related to the development of the laser: Laser is sources of ultraviolet, visible, or infrared radiation, produced by light amplification of the stimulated emission, with key characteristics [91-92]. Lasers are

1. Coherent. Different parts of the laser beam are related to each other in phase. These phase relationships are maintained over long enough time so that interference effects may be recorded photographically.

2. Monochromatic. The stimulated emission process creates a very narrow band source and allows the beam to be considered as essentially one wavelength.

3. Collimated. Inside the laser, the beam is collimated by the repeated reflection between two mirrors that form a cavity. As a result, laser beam that emerges is very narrow with a very small spreading angle.

The lasers, which are often used by interferometers, are shown in the Figure 3-1 [91].
In recent years, laser diodes have become popular due to their size and the ability to tune their wavelength. The development of this project has focussed on using a diode laser to generate spatially varying phase differences that can be related to the deformation of a body. Therefore in the next section, we will discuss the technical details of the laser diode used.

3.2.2 External cavity tunable diode laser

An external cavity diode laser based on the Littman-Metcalf design is shown in the Figure 3-2, which uses a diffraction grating at grazing incidence to provide wavelength selectivity [96]. Essential to the performance of tunable external-cavity diode lasers (ECDLs) is a high-quality anti-reflection (AR) coating on the front facet.
of the diode. The AR coating turns the diode into a gain element. A collimating lens directs the output of the diode across a diffraction grating at grazing incidence. The end mirror of the laser cavity reflects the first-order diffraction off the grating to provide feedback. Dispersion provided by the grating allows only one longitudinal mode to lase, resulting in a very narrow linewidth. The reflection or zero-order diffraction off the grating serves as the output beam of the laser.

The angle between the grating and the end mirror determines the lasing wavelength. Tuning is achieved by varying the angle using a piezoelectric actuator to rotate the end mirror. Continuous wavelength tuning (mode-hop free) requires selecting an appropriate rotation point.

In this thesis, the light source for the phase-shifting interferometer is the New Focus Vortex 6005 external cavity diode laser and will be discussed in more detail in chapter 5.

3.3 Interferometers and its application in experimental mechanics

Experiments in experimental mechanics frequently involve the need to acquire data simultaneously at many points on the object or material under test. Although traditional point wise transducers can in principle be duplicated many times, the cost typically becomes prohibitive when the required number of sensors reaches the range 10 to 100. By contrast, optical interference techniques based on phenomena such as moiré and speckle encoded in the form of a 2D fringe pattern can provide whole-field information equivalent to upwards of $10^5$ independent sensors.

In this section, several configurations of the interferometer will be introduced [92-95, 99].
3.3.1 Michelson interferometer

A Michelson interferometer is shown in schematic form in Figure 3-3 [99]. Light from a coherent light source is divided into the two beams 1 and 2, which travel along separate paths before being recombined. A photo detector array (CCD) then measures the resultant intensity distribution. It is convenient to represent the complex amplitudes of the interfering beams as a phasor, as shown in Figure 3-4, where $a_1$ and $a_2$ are the amplitudes and $\phi_1$ and $\phi_2$ are the phases of the two beams at a given pixel, whose coordinates will be denoted $(m,n)$. The pixel of CCD array produces a signal, $s$ which is proportional to the intensity of the light falling on to it:
\[ I = |a_1 \exp(i\phi_1) + a_2 \exp(i\phi_2)|^2 = a_1^2 + a_2^2 + 2a_1 \cdot a_2 \cdot \cos \phi, \]  

(3-1)

where \( \phi = \phi_1 - \phi_2 \) is the difference in phase due to the differing optical path lengths encountered by the two beams. For an interferometer involving smooth (i.e. non-speckled) wave fronts, \( a_1 \), \( a_2 \) and \( \phi \) are generally slowly varying functions of position, giving rise to fringes with sinusoidal intensity profiles. The goal of fringe pattern analysis is to measure the distribution \( \phi(m,n) \) independently of \( a_1(m,n) \) and \( a_2(m,n) \), with the smallest possible random and systematic errors. In experimental mechanics, the phase changes that occur over time due to mechanical or thermal deformation are of primary interest and these will be denoted by \( \Delta \phi(m,n) \). The relationship between \( \Delta \phi(m,n) \) and the displacement field at the surface of the specimen are determined by the optical configuration used to acquire the data.

If the specimen has a flat polished surface, then the Michelson interferometer can be used to measure the displacement component \( u_z \) normal to the specimen plane. The beam-splitter acts both to split the beams and then to recombine them. Movement of the specimen or object along the \( z \) direction by the distance \( u_z \) reduces the optical path length for beam 2 by \( u_z \), causing phasor 2 to rotate clockwise through an angle \( 4\pi \cdot u_z / \lambda \). Phasor 1 is unaffected by the object motion (see Figure 3-4). Thus \( u_z \) is related to \( \Delta \phi(m,n) \) by the equation

\[ u_z(x,y) = \frac{\lambda \cdot \Delta \phi(m,n)}{4\pi}. \]  

(3-2)

The mapping from pixel coordinates to position on the specimen \((x,y)\) is governed by the pitch of the pixels and the magnification of the image system.

### 3.3.2 Fizeau interferometer

Common-path interferometers can be used to reduce sensitivity to vibration, because at least part of the vibration is common to both the test and the reference beams [92]. A typical common path interferometer is the Fizeau interferometer, shown in Figure 3-5.
If the reference surface is flat, then the surface flatness of a sample can be measured. In the Figure 3-6 [102], two flat glass surfaces are placed in contact with one another. Interference fringes will appear. If both surfaces are optically flat, the fringes will form straight lines. If the surface being tested is not flat, the fringes will appear to have a local aberration (Figure 3-6). This local disturbance of the fringes can then be related to the topology of the test specimen, usually through a process of unwrapping the fringes in the appropriate manner [109, 140, Section 4.5].

Fizeau Fringes

For a given fringe the separation between the two surfaces is a constant.

Height error = (Δ/2)(Δ/S)

Figure 3-6: A sketch showing typical Fizeau fringes [102].
3.4 Multi-beam interference in Fizeau Interferometer

A complexity of the Fizeau interferometer is that there are multiple reflections at the plate surfaces, with the result that an interfering series of beams of diminishing amplitude emerges on each side of the plate and the intensity distribution in the fringe patterns is modified in a way which depends on light polarization and the properties of reflective medium, particular to metal [94]. In order to be able to use the fringe patterns to extract the topology of a test surface, one then needs to be able to predict the relationship between phase and topology in the presence of multiple reflections and refraction.

3.4.1 Reflection and refraction

When a plane wave falls on to a boundary between two homogeneous media of different optical properties, for example, glass with refractive index $n_g = 1.5$ and air with refractive index $n_i = 1$, it is split into two waves, a transmitted wave proceeding into the second medium, and a reflected wave propagating back into the first medium. The total energy in the reflected and refracted rays is equal to the energy of the incident light, but the proportion of the intensities in these two rays will depend upon the refractive index difference, the angle of incidence and polarization of the light [94].

![Figure 3-7: Refraction and reflection of a plane wave [94].](image)

Polarization is an important property of light, and can either be parallel when the vector of electric field $E_{||}$ lies in the plane of incident ray, or perpendicular when the
vector of electric field $E_\perp$ lies in the plane perpendicular to incident ray, as can be seen from Figure 3-7.

According to Snell's formula [94] the angles $\theta_i$ of the incident wave, $\theta_r$ of the reflected wave and $\theta_t$ of the refracted wave are given by the equations:

$$\theta_i = \theta_r,$$

$$n_i \cdot \sin \theta_i = n_r \cdot \sin \theta_r.$$  

\hspace{1cm} (3-3)

When both of the medium forming the interface are dielectrics that are essentially "nonmagnetic", the amplitude coefficients are described by the Fresnel Equations as follows [94],

$$r_\parallel = \frac{n_i \cos \theta_i - n_r \cos \theta_r}{n_i \cos \theta_i + n_r \cos \theta_r}, \quad t_\parallel = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_r \cos \theta_r},$$

$$r_\perp = \frac{n_i \cos \theta_i - n_r \cos \theta_r}{n_i \cos \theta_i + n_r \cos \theta_r}, \quad t_\perp = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_r \cos \theta_r}. \hspace{1cm} (3-4)$$

Here $r_\parallel$ and $r_\perp$ denote the amplitude reflection coefficients, and $t_\parallel$ and $t_\perp$ are the amplitude transmission coefficients, for parallel and perpendicularly polarized light respectively. The intensity reflection coefficients, $R_\parallel$ and $R_\perp$, and transmission coefficients, $T_\parallel$ and $T_\perp$, are described by the equations [94]

$$R_\parallel = \frac{\tan^2 (\theta_i - \theta_r)}{\tan^2 (\theta_i + \theta_r)}, \quad T_\parallel = \frac{\sin 2\theta_i \cdot \sin 2\theta_r}{\sin^2 (\theta_i + \theta_r) \cdot \cos^2 (\theta_i - \theta_r)},$$

$$R_\perp = \frac{\sin^2 (\theta_i - \theta_r)}{\sin^2 (\theta_i + \theta_r)}, \quad T_\perp = \frac{\sin 2\theta_i \cdot \sin 2\theta_r}{\sin^2 (\theta_i + \theta_r)}. \hspace{1cm} (3-5)$$

If we consider first the propagation of a plane wave from a dielectric into a metal. The complex refraction index of metal $\hat{n}$ can be defined as [94]:

$$\hat{n} = n + k \cdot i,$$  \hspace{1cm} (3-6)
where $k$ is the attenuation index, and $n$ and $k$ are real. The light reflectance is shown in the Figure 3-8.

![Figure 3-8: Metal reflectance [101].](image)

### 3.4.2 Multi-beam interference

![Figure 3-9: Plane waves successively reflected backwards and forwards [98, 100].](image)
Interferometry

When a plane wave or multiple beams interact with the interface one must consider the multiple reflections and their interference. We consider the plane waves travelling from left to right shown successively reflected between the plane parallel surface 1 and 2, in Figure 3-9 [98, 100]. For convenience the electric vector of the incident wave is taken to be linearly polarized in the plane of incident ray $E_\parallel$. The incident ray $I$ of the unit amplitude represents the direction of propagation of the incident plane wave and $\theta$ is its angle of incidence on the surfaces from within the interface. The distance between surface 1 and surface 2 is $d$ and refractive indices of material 1, 2 and 3 are $n_1$, $n_2$, and $n_3$ respectively. Subscripts 1 and 2 represent the reflection or refraction, which happen at the surface 1, or surface 2. For reflected light, a ray that travels from right to left is labelled “+”, and from left to right is labelled “−”. For transmitted light, a ray travels from right to left is labelled “−”, and from left to right is labelled “+”. $r$ is the amplitude reflection coefficient with phase-shifting $\alpha$, $\beta$. $t$ is the transmittance coefficient with phase-shifting $r$. All of the parameters shown in the Figure 3-9 are listed in Table 3-1 [98].

Table 3-1: Parameters appropriate for a multi-beam interferometer [98].

<table>
<thead>
<tr>
<th>Type</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surface 1 External Reflection</td>
<td>$r_1^+$</td>
<td>0.2 (for air to glass interface)</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>0</td>
</tr>
<tr>
<td>Surface 1 Internal Reflection</td>
<td>$r_1^-$, $r_1^+$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\beta_1$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>Surface 1 Transmission +</td>
<td>$t_1^+$</td>
<td>$\sqrt{1 - r_1^+}$</td>
</tr>
<tr>
<td></td>
<td>$r_1^-$</td>
<td>0</td>
</tr>
<tr>
<td>Surface 1 Transmission -</td>
<td>$t_1^-$</td>
<td>$\sqrt{1 - r_1^-}$</td>
</tr>
<tr>
<td></td>
<td>$r_1^+$</td>
<td>0</td>
</tr>
<tr>
<td>Surface 2 Internal Reflection</td>
<td>$r_2^+$</td>
<td>Given in Eq. (3-7)</td>
</tr>
<tr>
<td></td>
<td>$\beta_2$</td>
<td>Given in Eq. (3-7)</td>
</tr>
<tr>
<td>Surface 2 Transmission +</td>
<td>$t_2^+$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$r_2^-$</td>
<td>$-\pi$</td>
</tr>
</tbody>
</table>

In the Table 3-1, only $r_2^+$ and $\beta_2$ are related to the property of interfaces. If medium 1 is air ($n_1 = 1$), medium 2 is glass ($n_2 = 1.5$), and medium 3 is aluminium, whose refractive index $n = 1.35 + 8.6 \cdot i$ [101]. The expression for the amplitude and phase of
reflectance in the interface of medium 2 and medium 3 can then be derived by substituting \( \hat{n} \) into the Eq.(3-3) to Eq.(3-5) [94],

\[
R_\parallel = r_2^* \cdot e^{i\beta_2} = \frac{\tan(\theta - \arcsin \frac{n_2 \cdot \sin \theta}{\hat{n}})}{\tan(\theta + \arcsin \frac{n_2 \cdot \sin \theta}{\hat{n}})},
\]

(3-7)

Therefore, \( r_2^* \), \( \beta_2 \) in the Table 3-1 can be given by Eq.(3-7). Based on the setup of Figure 3-9 and parameters given by the Table 3-1 and Eq.(3-7), the reflective interference is derived as follows.

The optical delay for successive reflections from surface 1 and 2 gives an additional phase lag \( \phi \), which can be shown to be [98]

\[
\phi = \frac{4\pi \cdot n_2 \cdot d \cdot \cos \theta}{\lambda}.
\]

(3-8)

The reflected light intensity is evaluated as following:

\[
I_r = (r_1^*)^2 + \left( \frac{A(\psi)}{(1 - r_1^- r_2^*)^2} \right) \times \left[ (\psi_1^* \psi_2^*)^2 - 2(\psi_1^* \psi_2^*) (\psi_1^- \psi_2^*) \cos \psi_0 + 2\psi_2^* r_1^- r_2^* \cos(\psi + \psi_0) \right],
\]

(3-9)

where

\[
A(\psi) = \left[ 1 + F \cdot \sin^2 \left( \frac{\psi}{2} \right) \right]^{-1},
\]

\[
F = \frac{4r_1^- r_2^*}{(1 - r_1^- r_2^*)^2},
\]

\[
\psi = \phi + \phi_0 = \frac{2\pi \cdot (n_3 \cdot 2d \cos \theta)}{\lambda} + \beta_1 + \beta_2,
\]

\[
\psi_0 = 2\tau_1 - \beta_1 - \alpha.
\]

(3-10)

Based on Eq.(3-9), mathematical simulations of multi-beam reflective interference have been executed in the spatial and temporal domain, which corresponds to our tunable wavelength interferometer where both a tunable wavelength and an optical wedge are used.
3.4.3 Simulation results of multi-beam interference

3.4.3.1 Simulation results in the spatial domain

Using Eq.(3-9), we simulated the interference pattern that would result if a beam of light of wavelength 635.05nm was incident on the surface of an optical wedge at an angle of 0.0033 radians, where thickness of $d$ is linearly modulated from 8.850mm to 8.855mm. The rear side of the optical wedge was coated with Aluminium (complex refractive index $1.35 + 8.6 \cdot i$ [101]), leading to a resultant interference pattern characterised by fringes [172].
Figure 3-10: Interference signal when optical path is linearly modulated about 5μm.

(a). Interference signal in spatial domain.
(b). Interference signal in spatial domain after DC removal.
(c). Amplitude in frequency domain.
(d). Experimental result of amplitude spectrum of the fringe pattern.

Figure 3-10 (a), (b) shows that the reflectance is periodic between 0.86 and 0.93. Figure 3-10 (c) gives out its amplitude spectrum of the non-dimensional frequency. The peak of fundamental frequency and up to 3 order harmonic peaks at integral multiples of the fundamental frequency are clear. The second harmonic peak is about 17% of that at the fundamental frequency and can be compared to that observed in the experimental amplitude spectrum (Figure 3-10 (d)).
3.4.3.2 Simulation results in the temporal domain

One can also simulate time varying signals, e.g., when the wavelength of light source is linearly modulated from 635.00 nm to 635.162 nm at a constant rate. If the light is incident on the surface of an optical plate (again with the rear coated with Aluminium (complex refractive index $1.35 + 8.6 \cdot i$)), thickness 4 mm, at an angle of 0.0033 radians then the resultant fringes are shown in Figure 3-11 [172].
Interferometry

Figure 3-11: Interference signals when wavelength is modulated by 0.162nm.
(a). Interference signal in temporal domain.
(b). Interference signal in temporal domain after DC removal.
(c). Amplitude in frequency domain.

From Figure 3-11 (a), (b) we can see that the interference reflectance has a periodic form and ranges between 0.86 and 0.93. In Figure 3-11 (c) we see that the amplitude spectrum of the non-dimensional frequency shows evidence of up to 2 order harmonic suggesting that in any analysis, contributions from higher orders of harmonics must be considered.

From Figure 3-10 (a) (b) and Figure 3-11 (a) (b), we can see that the interference signals are asymmetrical. This phenomenon is similar to the effect of Fabry-Perot [98] and is a major consideration when considering the results from our Fizeau interferometer. Comparing the spatial and temporal domain fringes we see that introducing a spatially varying phase shift (see Figure 3-10 (c)) normally results in higher harmonic peaks and less spectrum leakages compared to a temporal variation of the phase, (see Figure 3-10 (c)), principally due to the larger number of cycles included in the spectrum calculation. This implies, therefore, the spatial domain is the preferred method for calibrating the phase-shifting steps for our wavelength scanning interferometer [Section 6.2].
3.5 Summary

Interferometry has been introduced in this chapter and we have focussed on its use as an ideal metrology technique because of its excellent sensitivity. Initially, the light source, especially external cavity diode laser based on Littman-Metcalf design was shown and the principle of wavelength tunability was discussed in detail. Subsequently, we demonstrated the applicability of interferometry to experimental mechanics and in particular, the potential use of a reflective Fizeau interferometer for studying small deformations on the surface of materials. Finally, we presented the phenomenon of multi-beam interference in the Fizeau interferometer. The simulation shows that interference profiles are asymmetrical in the spatial and temporal domains, which implies that high harmonics will appear, an effect one must suppress during the measurement of contact displacements.
4 Automatic fringe pattern analysis

The complex requirements of the measurement of deformations due to the weight of granular particles on a boundary (small deformations, large field of view, and repeated experiments) mean that fringe pattern analysis needs to be automated by Phase-Shifting Interferometry (PSI). Although Phase-Shifting Interferometry (PSI) has been used for a long time in electrical engineering, the topic did not become popular in the field of optical interferometry until the 1980s, when good quality CCDs and powerful microcomputers became readily available [103-105]. At the present time the vast majority of interferometry metrology work involves phase-shifting techniques.

Many algorithms for automated fringe analysis are available, but for this project further development was required to develop a new algorithm to improve accuracy by the extension of the number of frames acquired and to cope with the non-linear relationship between the PZT voltages of the laser and the phase-shifting steps.

In this chapter, the general principle and algorithms of Phase-shifting Interferometry and phase unwrapping are discussed in detail. According to the principle of least square method, a new phase-shifting interferometry (TPSI) with unequal interval sampling is put forwards, which is used for evaluating the deformation due to the weight of granular particles on a boundary in the next chapters.

4.1 Introduction

Many whole-field optical measurement techniques collect data in the form of a fringe pattern. These patterns, which usually appear as alternating light and dark bands or fringes, can be mathematically formulated in terms of a time \( t \) and space \( \vec{r} \) dependent intensity \( I(\vec{r},t) \) [106]:

\[
I(\vec{r},t) = I_o(\vec{r},t) \cdot \{ 1 + V(\vec{r},t) \cdot f[\phi(\vec{r},t)] \},
\]  

(4-1)

where \( I_o(\vec{r},t) \) is the background intensity, \( V(\vec{r},t) \) is the visibility or fringe contrast and the periodic nature of the intensity is determined by the unidimensional function \( f \),
that takes values in the interval \([-1, 1]\) and whose argument is the optical phase \(\phi(\vec{r}, t)\). The function \(f\) defines the type of fringe pattern profile, amongst which are sinusoidal (typical of double-beam interferometry) and the Lorentzian (typical of multiple-beam interferometry). The periodic dependence of \(f\) permits it to be expressed as a form of a Fourier series:

\[
I(\vec{r}, t) = \sum_{k=0}^{\infty} a_k \cdot \cos[k \cdot \phi(\vec{r}, t)],
\]

(4-2)

where \(a_k\) is the amplitude of the \(k\)th harmonic.

Generally, the need for greater precision, velocity and measurement-process automation have resulted in the continued development of methods which provide a finer interpolation between the extreme intensity values, perform the cancellation of even gross variations of \(I_0(\vec{r}, t)\) and \(V(\vec{r}, t)\), and eliminate ambiguities in the optical phase \(\phi(\vec{r}, t)\) (because the phase is generally a multiform function of the intensity). The fringe-pattern analysis process generally consists of the following steps:

1. Phase evaluation. This consists of obtaining a spatial distribution of the phase \(\phi(\vec{r}, t)\) or wrapped phase map from many fringe patterns.

2. Phase unwrapping. The previous phase-evaluation stage provides, in most cases, the principal values for the phase \(\phi(\vec{r}, t)\) within \(2\pi\). In such situations, a continuous phase distribution is obtained over its defined domain through the unwrapping of the phase map onto an unbounded interval.

3. Elimination of additional terms. The terms bearing no relationship to the measurement magnitude during the formation of the fringe patterns or during the evaluation stage (the simplest case being that of constant terms or those that depend linearly on spatial coordinates) within the phase \(\phi(\vec{r}, t)\) are removed through an iterative process.

4. Rescaling. In most cases, the correct presentation of the results demands that there be a relationship between the values of the measurable magnitude and the coordinates.
Automatic fringe pattern analysis of the space in which it is defined (a mathematical expression, a table, a graph, relationships with grey-scale levels or color levels, etc.).

Although the process of phase extraction can be explained in a quite general manner above, the combination of different ways of performing phase modulation, sampling of intensity values and demodulation has led to a great variety of Phase Evaluation Methods (PEMs), with their individual algorithms for phase calculation and with their particular needs and performance [107-108].

PEMs can be categorized into temporal and spatial algorithms [109]. In the first case, the data necessary for the evaluation are obtained at different time intervals; whereas in the second case, all of the data are obtained simultaneously. In this project we will use temporal methods primarily and we will discuss in detail Temporal Phase-Shifting Interferometry (TPSI) in the following section.

4.2 Temporal phase-shifting interferometry (TPSI) with equal interval sampling

4.2.1 Introduction

In Temporal Phase-shifting Interferometry (TPSI) the phase is stepped or increased linearly with time [106, 109, 110, 111] to obtain a series of images of fringe patterns $I(\vec{r}, n)$ with a certain phase increment or phase-shifting step $\psi$ between them, where $n$ is the index of fringe patterns $I(n)$ with time, and $\vec{r}$ a position vector. If the rate of stepping requisite for each of the obtained phase-shifting patterns is sufficiently low for a camera to capture, then typically sinusoidal profiles such as,

$$I(\vec{r}, n) = I_o(\vec{r}) \cdot \{1 + V(\vec{r}) \cos[\phi(\vec{r}) + \psi \cdot n]\} \quad n = 0, 1, 2, \cdots, M - 1,$$

is used, where $M$ is the total number of fringe patterns and the phase $\phi(\vec{r})$ at each point is independent of those at other points. This method can provide very high accuracy (about $\lambda/1000$) [141], if the ambient conditions are well controlled.

One of the simplest ways of viewing the TPSI process is that of solving a set of simultaneous equations for the three unknowns $I_o(\vec{r})$, $V(\vec{r})$, and $\phi(\vec{r})$ in Eq.(4-3). In this case it is necessary to have at least three measured intensity values, i.e. $M \geq 3$. One of the simplest cases, which leads to one of the most widely used phase-
Automatic fringe pattern analysis

extraction algorithm (4 Step PSI Technique), is obtained by considering the case

\( M = N = 4 \), where \( N \) is how many fringe patterns used per carrier cycle \([109]\) and a

phase-shifting step of \( \psi = \pi / 2 \). The equations are then

\[
I(\bar{r}, 1) = I(\bar{r}) \cdot \{1 + V(\bar{r}) \cdot \cos[\phi(\bar{r}) + 0]\}, \\
I(\bar{r}, 2) = I(\bar{r}) \cdot \{1 + V(\bar{r}) \cdot \cos[\phi(\bar{r}) + \pi / 2]\}, \\
I(\bar{r}, 3) = I(\bar{r}) \cdot \{1 + V(\bar{r}) \cdot \cos[\phi(\bar{r}) + \pi]\}, \\
I(\bar{r}, 4) = I(\bar{r}) \cdot \{1 + V(\bar{r}) \cdot \cos[\phi(\bar{r}) + 3\pi / 2]\}. \\
\]

(4-4)

The wrapped phase is given by

\[
\phi(\bar{r}) = \tan^{-1} \left[ \frac{I(\bar{r}, 4) - I(\bar{r}, 2)}{I(\bar{r}, 1) - I(\bar{r}, 3)} \right]. \\
\]

(4-5)

4.2.2 General algorithm of TPSI

4.2.2.1 Fourier Transform analysis of TPSI

In phase-shifting interferometry, the most common sources of systematic errors are

nonsinusoidal waveforms of a signal that are due to multiple-beam interference or

nonlinearity and phase-shifting miscalibration due to the expected phase-shifting step

not being made. Therefore, a number of algorithms have been proposed to minimize

the errors, and they are generalized as follows.

The development of simple phase-shifting algorithms that use only a few samples is

based on a method of finding a solution to a set of equations representing sampled

phase-shifting fringe patterns. In many cases \( I_0(r, t) \), \( V(r, t) \) and \( \phi(r, t) \) in Eq.(4-1)

and Eq.(4-3) vary slowly compared to the carrier angular frequency \( 2\pi / N \), where \( N \)

is the number of phase shifting steps per cycle of the carrier and can be considered

constant throughout the capture of the phase maps, and the spatial coordinate \( \bar{r} \) is

neglected for simplicity. The fringe patterns can then be written \([106, 109, 112-120, 130]\)

\[
I(n) = \left\{ \sum_{i=0}^{N} a_i \cdot \cos\left[ \frac{2\pi n}{N} + \phi_i \right] \right\} \quad n = 0, 1, 2, \cdots, M - 1, \\
\]

(4-6)
where $l$ is the order of harmonics; when $l = 1$, it represents the fundamental frequency, where the signal phase value is evaluated. $\phi_i$ is the initial phase of $l$ order harmonics. Experimentally it is not possible to obtain a continuous function for $I$, and one must sample the phase maps accordingly. In this case the Discrete Time Fourier Transform (DTFT) of the set of sampled intensity values $I(n)$ in Eq. (4-6) is given by:

$$\tilde{I}(k) = \sum_{n=0}^{M-1} [I(n) \cdot w(n)] \cdot e^{-i(2\pi n/N)k},$$

(4-7)

where $k$ is a continuous non-dimensional frequency and $w(n)$ is a window function. Hence, we have

$$\tilde{I}(k) = \tilde{W}(k) \star \sum_{j=-\infty}^{\infty} \{a_0 \cdot \delta(k - j \cdot N)
+ \sum_{l=1}^{a} \left[ \frac{a_i}{2} \cdot e^{i\phi} \cdot \delta(k - j \cdot N - l) + \frac{a_i}{2} \cdot e^{-i\phi} \cdot \delta(k - j \cdot N + l) \right]\},$$

(4-8)

where $\delta(k)$ is the Dirac delta function and $\tilde{W}(k)$ is the Discrete Time Fourier transform of $w(n)$. Eq.(4-8) is shown schematically in the Figure 4-1.

It is noted that the spectra of $I(n)$ are the $\delta$ functions in the frequency domain periodically at the interval $N$ irrespective of the frame number $M$ and carrier cycle $M / N$. However, the spectrum of $w(n)$ is highly influenced by $M$ and $M / N$ (its mainlobe width is proportional to $1/M$). In other words, the spectrum of $I(n) \cdot w(n)$ is equal to the spectrum of $I(n)$ smeared by the spectrum of $w(n)$. The more carrier cycles $w(n)$ covers, the clearer the spectrum of $I(n) \cdot w(n)$. If $w(n)$ covers $(-\infty, \infty)$, the spectrum of $I(n) \cdot w(n)$ converges to the spectrum of $I(n)$.

After ignoring the influence of the energy leakages of other harmonics ($a_i = 0$, when $l > 1$), the three terms within the summation of Eq.(4-8) corresponding to $j = 0$ occur at $k = -1, 0, 1$ represent the Fourier transform of a signal consisting of a cosine wave plus DC offset, when the window function is rectangular.
At $k = 1$ in the frequency domain, the Fourier transform of the fundamental signal frequency is

$$\tilde{I}(1) = \tilde{W}(0) \frac{a_1}{2} e^{i\phi}. \quad (4-9)$$

The phase measured is represented by $\phi$ which can be calculated as

$$\phi = \phi_1 + C = \tan^{-1}\left(\frac{\text{Im}[\tilde{I}(1)]}{\text{Re}[\tilde{I}(1)]}\right) + C. \quad (4-10)$$

The constant $C$ is related to the implicit choice of the phase origin, which is determined by a point where the light intensity reaches a maximum. Substituting $\tilde{I}(1)$ from Eq.(4-7) into Eq.(4-10), we obtain the general phase-shifting equation:

$$\phi = -\tan^{-1}\left\{\frac{\sum_{n=0}^{M-1} I(n) \cdot w(n) \cdot \sin\left(\frac{2\pi n}{N}\right)}{\sum_{n=0}^{M-1} I(n) \cdot w(n) \cdot \cos\left(\frac{2\pi n}{N}\right)}\right\} + C \quad (4-11)$$

$$= \tan^{-1}\left\{\frac{\sum_{n=0}^{M-1} I(n) \cdot b(n)}{\sum_{n=0}^{M-1} I(n) \cdot a(n)}\right\} + C$$

Eq.(4-11) was first proposed in Ref. [110], and now forms the basis for the vast majority of phase-shifting interferometers currently in operation around the world. The four-step algorithm given in Eq. (4-4) and Eq.(4-5), for example, is obtained by substituting the values $M = N = 4$ with the rectangular window function.

If $\tilde{W}(k)$ in Eq.(4-8) is constructed so that

$$\tilde{W}(1) = 0, \text{ i.e.,}$$

$$\sum_{n=0}^{M-1} b(n) = 0, \quad \sum_{n=0}^{M-1} a(n) = 0 \quad (4-12)$$

and that

$$\tilde{W}(2) = 0, \text{ i.e.}$$

$$\sum_{n=0}^{M-1} b(n) \cdot \sin\left(\frac{2\pi n}{N}\right) = \sum_{n=0}^{M-1} a(n) \cdot \cos\left(\frac{2\pi n}{N}\right)$$
\[
\sum_{n=0}^{M-1} b(n) \cdot \cos\left(\frac{2\pi n}{N}\right) = \sum_{n=0}^{M-1} a(n) \cdot \sin\left(-\frac{2\pi n}{N}\right). 
\] (4-13)

Then the effect is to place a zero in the magnitude of each of the harmonic spectral functions at the fundamental frequency where the phase is evaluated, as shown in the Figure 4-1.

However, if there is miscalibration of phase-shifting values, the conditions given by Eq.(4-12) and Eq.(4-13) will not be satisfied; therefore, the spectral leakage from other harmonic frequencies will influence the phase evaluation result of Eq.(4-11).

One of the solutions is to introduce a new window function, which has a flatter peak and smaller sidelobes than those of a rectangular one in the frequency domain. Such a window will be presented in the next section.

4.2.2.2 Window function and properties

As far as window functions are concerned [121-123], four examples are given as follows:

1. Rectangular window function.

Rectangular window is expressed by:
Automatic fringe pattern analysis

\[ w(n) = \begin{cases} 
1 & n = 0, \ldots, M - 1 \\
0 & Others 
\end{cases}. \]  \hspace{1cm} (4-14)

The frequency expression for a rectangular window is:

\[ W(k) = M \cdot \text{sinc}(M \cdot k) \cdot e^{-i\pi M k}. \]  \hspace{1cm} (4-15)

The mainlobe width of the window function in frequency domain is:

\[ \Delta_{ML} = \frac{1}{M}. \]  \hspace{1cm} (4-16)

2. Hanning window function.

The time-domain expression is in this case:

\[ w(n) = \frac{1}{2} \left\{ 1 + \cos\left[ \left( \frac{2\pi}{M} \right)(n - \frac{M}{2}) \right] \right\} \quad n = 0, \ldots, M - 1, \]  \hspace{1cm} (4-17)

and its frequency expression is:

\[ \tilde{W}(k) = \frac{M}{2} \cdot \left\{ \text{sinc}(M \cdot k) + \frac{1}{2} \cdot \left[ \text{sinc}M \cdot (k + \frac{1}{M}) \right. \right. \\
\left. \left. + \text{sinc}M \cdot (k - \frac{1}{M}) \right] \right\} \cdot e^{-i\pi M k}. \]  \hspace{1cm} (4-18)

![Graphs showing frequency and magnitude response of different window functions.](image)
3. Dolph-Chebyshev window function.

The Dolph-Chebyshev window function is used to optimize to minimize the width of the mainlobe under the constraints that the window length be fixed and equal levels of the sidelobes should not exceed a given value (see Figure 4-2).

The Chebyshev polynomials are defined by the equation:

$$T_k(x) = \begin{cases} \cos(k \cdot \cos^{-1} x) & |x| \leq 1 \\ \cosh(k \cdot \cosh^{-1} x) & |x| > 1 \end{cases}$$

(4-19)

From the definition, the following recurrence relation follows immediately.

$$T_0(x) = 1, \quad T_1(x) = x,$$

$$T_n(x) = 2x \cdot T_{n-1}(x) - T_{n-2}(x) \quad n \geq 2.$$  (4-20)

The frequency expression of the Dolph-Chebyshev window function is

$$W(k) = \frac{\cos\{(2M + 1)\cos^{-1}[\beta \cos\left(\frac{\pi k}{2M + 1}\right)]\}}{\cosh[(2M + 1)\cosh^{-1} \beta]} \quad -M \leq k \leq M,$$  (4-21)

where:
\[ \beta = \cosh\left(\frac{1}{2M} \cosh^{-1} \frac{1}{\gamma}\right), \]

\[ \gamma = \frac{\text{Amplitude of sidelobe}}{\text{Amplitude of mainlobe}}. \quad (4-22) \]

The Dolph-Chebyshev window may be evaluated from the inverse transform of Eq. (4-21), which has two inputs: the sample length \( M \) and the desired sidelobe level \( \gamma \). It is of the form:

\[ w(n) = \frac{1}{2M + 1} \left[ \frac{1}{\gamma} + 2 \sum_{k=1}^{M} T_k(\beta \cdot \cos \frac{k}{2M+1}) \cdot \cos \frac{2nk\pi}{2M+1} \right] \quad -M \leq n \leq M. \quad (4-23) \]

The resulting window has the minimum main lobe width for a given ripple ratio \( \gamma \) and sample length \( M \). It provides more flexibility than the classical windows because a desired tradeoff between mainlobe width and sidelobe levels can be achieved.


![Kaiser window compared to the rectangular window (M = 40).](image)

The time domain expression of the Kaiser Window function is

\[ w(n) = \frac{I_0(\beta \cdot \sqrt{1 - \left(\frac{n}{M}\right)^2})}{I_0(\beta)} \quad -M \leq n \leq M. \quad (4-24) \]

Its frequency expression is

\[ \tilde{W}(k) = \frac{M \cdot \sinh(\sqrt{\beta^2 -(M\pi k)^2})}{I_0(\beta) \cdot \sqrt{\beta^2 -(M\pi k)^2}}, \quad (4-25) \]
where $I_0(\mu)$ is the modified zeroth-order Bessel function of the first kind:

$$I_0(\mu) = 1 + \sum_{r=1}^{\infty} \left[ \frac{(\mu/2)^r}{r!} \right]^2. \quad (4-26)$$

$\beta$ controls the minimum stop band attenuation of the windowed filter response, and it is estimated using:

$$\beta = \begin{cases} 
0.1102 \cdot (\alpha - 8.7) & \alpha > 50 \\
0.5842 \cdot (\alpha - 21)^4 + 0.07886 \cdot (\alpha - 21) & 21 \leq \alpha \leq 50 \\
0 & \alpha < 21 
\end{cases}, \quad (4-27)$$

where the sidelobe height of the function in frequency domain is $\alpha$ dB.

4.2.2.3 Minimization of the errors in TPSI

Poor performance in a PSI can often be traced to the problems of (a) aliasing of and (b) spectral leakage from higher harmonics in the signal. The former is minimized by increasing the number of samples per cycle of the fundamental (i.e., the value of $N$). The latter is minimized by (a) increasing the number of sampled cycles of the carrier (i.e., the value of $M/N$) and (b) the appropriate choice of a window function, $w(n)$ [109]. For a given number of frames, $M$, there is therefore a trade-off between minimizing the aliasing errors (high $N$) and minimizing the leakage errors (low $N$). The choice of window function has been a subject of major research activity in recent years; several well-known approaches include correlation theory [117], characteristic polynomial theory [113], and the extended averaging technique [124, 142]. In general the aim is to place a zero in the magnitude of each of the harmonic spectral functions of the fundamental frequency where the phase is evaluated. However, phase-shifting miscalibration and other errors mean that it is not always possible to achieve this in practice. A robust way to deal with this problem would be to keep the height of sidelobes of the Fourier transform of the window function smaller than a tolerance value $\varepsilon$, which would in principle allow a maximum phase error to be calculated for any spectral content regardless of the miscalibration error. With this approach, the window function should also have the property that the mainlobe of its transform should be narrower than the value of the fundamental frequency.
4.3 **Temporal phase-shifting interferometry (TPSI) with unequal interval sampling**

Traditional phase-shifting algorithms based on Eq.(4-11) require that the phase-shifting steps should at least be equal, if not, miscalibration errors will result. In this section, a new phase-shifting algorithm is introduced. Its advantage is that the phase-shifting steps could be unequal. It is the mathematical basis for our phase-shifting interferometry experiments used throughout the thesis.

### 4.3.1 Least-squares method in TPSI

The principle of the least-squares method in phase-shifting interferometry (LSPSI) has been described by several authors [125-128, 130]. It allows the reconstruction of wave-front data in phase-shifting interferometry when arbitrary global phase-shifting steps are known. In this section, we develop the method to be a new phase-shifting algorithm, which allows many arbitrary global phase-shifting fringe patterns when up to three orders of harmonics exist.

As before, the \( M \) steps of the phase-shifting are defined as a set of real values \( \{ \psi_n \} \), \( n = 0, \ldots, M - 1 \). When the harmonics up to the third-order are taken into consideration, the expression for the individual interferograms is:

\[
I_n(r) = A(r) + B(r) \cdot \cos(\phi(r) + \psi_n) + C(r) \cdot \cos(2\phi(r) + 2\psi_n) + D(r) \cdot \cos(3\phi(r) + 3\psi_n),
\]

(4-28)

It can be expanded into a new set of coefficients \( \{ a_0, a_1, a_2, a_3, a_4, a_5, a_6 \} \) and \( \tilde{r} \) is omitted for simplicity as follows:

\[
a_0 = A,
\]

\[
a_1 = B \cdot \cos(\phi),
\]

\[
a_2 = -B \cdot \sin(2\phi),
\]

\[
a_3 = C \cdot \cos(2\phi),
\]

\[
a_4 = -C \cdot \sin(2\phi),
\]
\[ a_5 = D \cdot \cos(3\phi), \]

\[ a_6 = -D \cdot \sin(3\phi). \]  

(4-29)

These are the seven unknowns for which we must solve. Because the phase steps are known a priori, the \( \cos\psi, \sin\psi, \ldots, \cos 3\psi, \sin 3\psi \) terms are simply the scalar coefficients of the unknown \( a_1, \ldots, a_6 \) in Eq. (4-29) and \( a_0 \) is the DC term. When we apply the method of least squares separately at each point, the goal is to minimize the error function \( E^2 \), defined as

\[
E^2 = \sum_{n=1}^{M-1} (I_n - a_0 - a_1 \cos \psi_n - a_2 \sin \psi_n - a_3 \cos 2\psi_n - a_4 \sin 2\psi_n - a_5 \cos 3\psi_n - a_6 \sin 3\psi_n). 
\]  

(4-30)

The error function is related to the fit variance, where it is assumed that each measurement point \( I_n \) contains the same uncertainty.

We minimize \( E^2 \) by differentiating Eq. (4-30) with respect to the seven unknowns \( a_0, \ldots, a_6 \). The resultant expression can be written in matrix form:

\[ M_A \cdot X = M_B, \]

\[ X = [a_0, a_1, a_2, a_3, a_4, a_5, a_6]^T, \]
Automatic fringe pattern analysis

\[ M_B = \left[ \sum_{n=0}^{M-1} I_n \cdot \cos \psi_n, \sum_{n=0}^{M-1} I_n \cdot \sin \psi_n, \sum_{n=0}^{M-1} I_n \cdot \cos 2\psi_n, \sum_{n=0}^{M-1} I_n \cdot \sin 3\psi_n, \sum_{n=0}^{M-1} I_n \cdot \sin 3\psi_n \right]^T \]  
(4-31)

The symmetric matrix \( M_A \), called the curvature matrix, depends only on the known phase shifts, whereas the vector \( M_B \) contains the measured interferogram data. \( M_A \) can be calculated just once, yet the calculation of \( M_B \) must be done separately at every point in the measurement domain.

The solution for the coefficient vector \( X \) requires inverting \( M_A \) and premultiplying both sides of Eq.(4-31):

\[ X = M_A^{-1} \cdot M_B. \]  
(4-32)

When there are three or more unique phase steps, the rows will be independent and \( M_A \) will be invertible. Once \( X \) is known, the phase \( \phi(\vec{r}) \) is easily found over the whole domain \( \vec{r} \),

\[ \phi(\vec{r}) = \tan^{-1} \left( \frac{a_2}{a_1} \right). \]  
(4-33)

4.3.2 Irregular FFT theory in TPSI

Fringe patterns in the irregular phase-shifting interferometer can be expressed as [128-129]:

\[ I(n) = \sum_{j=0}^{J} a_j \cdot \cos(\phi_i + l \cdot \psi_n) \quad (n = 0, \ldots, M - 1), \]  
(4-34)

where \( j \) is the order of the harmonics. When \( l=1 \), it represents the fundamental frequency, where the signal phase value is evaluated. \( \phi_i \) is the initial phase of the \( \ell \) th order harmonic. \( n \) is the time index of the fringe pattern recorded by the camera. \( \psi_n \) is the \( n \) th phase-shifting step, which is arbitrary in principle. Eq. (4-34) can be also written as:
\[ I(n) = a_0 + \sum_{i=1}^{j} \left( \frac{a_i}{2} \cdot e^{i (\phi_i + j \psi_n)} + \frac{a_i}{2} \cdot e^{-i (\phi_i + j \psi_n)} \right). \] (4-35)

The Fourier transform of Eq. (4-35) becomes

\[ \tilde{I}(k) = \sum_{n=0}^{N-1} I(n) e^{-i k \psi_n}, \] (4-36)

where \( k \) is a continuous non-dimensional frequency. Substitution of Eq. (4-35) into Eq. (4-36) leads to

\[ \tilde{I}(k) = a_0 \sum_{n=0}^{N-1} e^{-i k \psi_n} + \sum_{l=1}^{j} \left[ \frac{a_l}{2} \cdot e^{i \phi_l} \cdot \sum_{n=0}^{M-1} e^{-i (k-l) \psi_n} + \frac{a_l}{2} \cdot e^{-i \phi_l} \cdot \sum_{n=0}^{M-1} e^{-i (k+l) \psi_n} \right]. \] (4-37)

If we define \( A = a_0 \), \( B_l = \frac{a_l}{2} \cdot e^{i \phi_l} \) and \( C_l = \frac{a_l}{2} \cdot e^{-i \phi_l} \), Eq. (4-37) becomes

\[ \tilde{I}(k) = A \cdot \sum_{n=0}^{N-1} e^{-i k \psi_n} + \sum_{l=1}^{j} \left[ B_l \cdot \sum_{n=0}^{M-1} e^{-i (k-l) \psi_n} + C_l \cdot \sum_{n=0}^{M-1} e^{-i (k+l) \psi_n} \right]. \] (4-38)

Then, we have

\[ A \cdot \sum_{n=0}^{N-1} e^{-i k \psi_n} + \sum_{l=1}^{j} \sum_{n=0}^{M-1} B_l \cdot e^{-i (k-l) \psi_n} + \sum_{l=1}^{j} \sum_{n=0}^{M-1} C_l \cdot e^{-i (k+l) \psi_n} = \sum_{n=0}^{N-1} I(n) e^{-i k \psi_n}. \] (4-39)

If \( k \) is set to be \( 0, 1, \cdots, j, -1, -2, \cdots, -j \), after ignoring the influence of the energy leakages of other harmonics, Eq. (4-39) represents the Fourier transform of fringe pattern up to \( j \) th order harmonics, when the property of window function is rectangular. Eq. (4-39) become

\[
M_A = \begin{bmatrix}
1 & \sum_{n=0}^{M-1} e^{i \psi_n} & \sum_{n=0}^{M-1} e^{i 2 \psi_n} & \cdots & \sum_{n=0}^{M-1} e^{i (j) \psi_n} \\
\sum_{n=0}^{M-1} e^{-i \psi_n} & 1 & \sum_{n=0}^{M-1} e^{-i (1+1) \psi_n} & \cdots & \sum_{n=0}^{M-1} e^{-i (j-1) \psi_n} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\sum_{n=0}^{M-1} e^{i (j-1) \psi_n} & \sum_{n=0}^{M-1} e^{i (j+1) \psi_n} & \cdots & 1 & \sum_{n=0}^{M-1} e^{i (j+1) \psi_n} \\
\sum_{n=0}^{M-1} e^{i (j-1) \psi_n} & \sum_{n=0}^{M-1} e^{i (j+1) \psi_n} & \cdots & \sum_{n=0}^{M-1} e^{i (j+1) \psi_n} & 1 \\
\sum_{n=0}^{M-1} e^{i (j-1) \psi_n} & \sum_{n=0}^{M-1} e^{i (j+1) \psi_n} & \cdots & \sum_{n=0}^{M-1} e^{i (j+1) \psi_n} & \sum_{n=0}^{M-1} e^{i (j+1) \psi_n} & 1 \\
\end{bmatrix}
\]
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\[ M_B = \frac{1}{M} \left[ \sum_{n=0}^{M-1} I(n) \sum_{k=0}^{M-1} e^{-i\psi_k} \cdots \sum_{n=0}^{M-1} I(n) e^{i\psi_0} \cdots \sum_{n=0}^{M-1} I(n) e^{i\psi_{M-1}} \right]^T, \]

\[ X = [A \quad B_1 \quad \cdots \quad B_j \quad C_1 \quad \cdots \quad C_j]^T, \]

\[ X = M_A^{-1} \cdot M_B. \quad (4-40) \]

The wrapped phase map at the \( l \) th order harmonic is obtained by

\[ \phi_l = - \arctan \left( \frac{\text{Im}(B_l)}{\text{Re}(B_l)} \right). \quad (4-41) \]

In fact, Eq.(4-41) is the wrapped phase at harmonic \( l \), which is the more general form of Eq.(4-33).

It should be noted that the number of frames or the number of carrier cycles are important factors to influence the accuracy of TPSI with unequal interval sampling, for the more carrier cycles cover phase-shifting steps, the lower its sensitivity to high-order harmonics.

If there are no mode hops in the whole range of wavelength modulation, the scope of the phase-shifting values should be less than \( 2k\pi \quad (k = 1, 2, 3, \cdots) \), which corresponds to an asymmetric phase-shifting algorithm [113-115]. Because the frame \( n = 0 \) doesn’t practically contribute to TPSI, then the TPSI has \( M - 1 \) frames available. If there are mode hops in the whole range of wavelength modulation, the interference visibility in the mode hop frames are several times lower than its normal value, and the fringes coinciding with mode hops only contribute to increase the noise level. After they are automatically detected, they are excluded from the phase-shifting fringe patterns.

4.4 Minimizing the influence of light intensity modulation

If the phase-shifting in the interferometer is introduced by wavelength scanning of laser diode, it will generally be accompanied by light intensity modulation [131, 137-138], and sometimes by mode hops. The measurement accuracy in phase-shifting interferometry is degraded by intensity modulation among the different fringe
patterns. If the window function is ignored for simplicity, and the DC offset is removed, Eq.(4-3) becomes:

\[ I(\bar{r}, n) = I_0(\bar{r}) \cdot V(\bar{r}) \cdot \cos[\phi(\bar{r}) + \psi \cdot n]. \] (4-42)

The modulation of the fringe pattern is proportional to the light power \( I_0 \) and interference visibility \( V \) independently. When the phase-shifting algorithm is used to generate a wrapped phase map, the influence of the light power and interference visibility should be minimized.

Generally, there are two kinds of solutions to this problem.

(1). Normalizing the intensity profile by the laser power \( I_0 \). The laser power of each fringe is monitored by a photodetector. For example, a photodetector is integrated into the New Focus Vortex 6005 laser source, and can be read directly from the laser controller through the RS232 interface. After the DC offset is removed, each fringe is divided by its normalized laser power during analysis in the phase-shifting algorithm.

(2). Normalizing the intensity profile by the interference visibility \( V \). Interference visibility drifts when wavelength scanning, and especially in the region of mode-hopping. After the DC offset has been removed, each fringe is divided by its normalized interference visibility in the phase-shifting algorithm.

4.5 Phase unwrapping

The previous section describes a range of methods for determining the wrapped phase map lying in the range \(-\pi\) to \(\pi\) evaluated from an arctangent function of a series of fringe patterns. As a result, there are \(2\pi\) phase discontinuities. The process of converting this information into a continuous function is called phase unwrapping [139-140]. The wrapped phase then undergoes an unwrapping process in which an integral multiple of \(2\pi\) is added at each pixel. Figure 4-4 shows the unwrapping process.
Automatic fringe pattern analysis

Figure 4-4: Phase unwrapping along the time axis [139-140].

The process is carried out by unwrapping the phase along a path through the data. This path can either be along a spatial axis or along a time axis. The basic principle is to estimate the phase gradient between two adjacent pixels in an image or between two successive phase values. If the gradient value exceeds the threshold of $\pi$, then a phase discontinuity is assumed to lie between these two points and the phase jump is then corrected by adding or subtracting $2\pi$, according to the sign of the phase gradient. The unwrapped phase at any point is given by $\phi + 2\pi \cdot N_e$, where $N_e$ is the fringe order counter.

A basic problem with these techniques is that they do not address the fundamental cause of the path dependence within a 2D phase map, namely the presence of so-called residues (also known as discontinuity sources, poles and phase singularities in the literature). Figure 4-5 illustrates this problem when unwrapping spatially in 2D. Given the phase at pixel P, the phase at any other pixel (e.g. Q) can be unwrapped by counting the number of $2\pi$ discontinuities along any path (either A or B) linking the two points. In Figure 4-5 (a), all paths chosen give the same answer, which is $6\pi$. However in Figure 4-5 (b), noise in the centre of the field of view has caused a break in one of the $2\pi$ discontinuities, forming two ‘discontinuity sources’ (points 1 and 2). If point Q is to be unwrapped relative to point P, then a spatial unwrapping by path B...
would indicate that $4\pi$ should be added, whereas the correct answer (path A) is the $6\pi$. Thus local noise can propagate to cause global phase errors.

![Figure 4-5: Illustration of spatial phase unwrapping [139-140].](image)

(a) Noise-free case.

(b) Noise at the centre of the field causes unwrapping errors dependent on the path taken.

A family of techniques known as the branch cut method has evolved in which barriers to unwrapping (the branch cut) are placed between residues of opposite sign. The simplest of these is the 'nearest neighbour' algorithm. This method requires that for a given phase at a pixel $(x_0, y_0)$, the phase at any other point $(x, y)$ in the image should be defined uniquely, independent of the unwrapping path. The integral of the phase jumps should be zero. A non-zero value indicates an area in which residues occur. The sign of the integral is defined as charged and it tends to be in pairs of opposite signs called dipoles. Isolated sources may occur near the boundary and cut lines are placed between dipoles in the phase map or from a monopole to the boundary. Each residue is allowed to be at one end of a cut, with the other end attached to a source of the opposite sign, or the boundary of the phase map.

The next step is to minimise the cut-length. Several different cut placement routes will in general give the same minimum cut length. The choice of route will affect only the wrapped phase in the region containing the corrupted phase information. For the
correct cut distribution, this method requires that the separator between dipoles is always larger than the spacing between the sources making up the dipoles.

4.6 Eliminating additional terms and rescaling

When phase-shifting interferometry is used for the measurement of stress in the applications of mechanics, it is necessary that the measurement process be repeated before and after loading. Difference phase maps then eliminate the implicit phase offset $C$ and, for the optical configuration used throughout this thesis, give information on the out-of-plane displacement component, $u_z(x, y)$ where $x$ and $y$ are coordinates representing position on the sample surface. After unwrapping of the phase map, a simple scaling factor can then be used to calculate a displacement map:

$$u_z(x, y) = \frac{\lambda \Delta \phi(x, y)}{4\pi},$$

(4-43)

where $u_z$ is the displacement, $\lambda$ is the wavelength and $\Delta \phi$ is the unwrapped phase change. Eq.(4-43) is strictly applicable only for surfaces separated by a vacuum, and the presence of optical materials will enhance the phase change for a given displacement by a factor of $n$, where $n$ is the refractive index.

4.7 Summary

This chapter has introduced several phase-shifting theories and methods used to obtain the interference phase from intensity distributions.

Two completely independent systematic approaches for designing algorithms are presented. One is temporal phase-shifting interferometry (TPSI) with equal interval sampling. In order to reduce the error of high harmonics and phase miscalibration, a data window technique is chosen to minimise these errors. The other approach uses temporal phase-shifting interferometry with unequal interval sampling to optimise the performance to a desired set of properties. These properties might include insensitivity to the nonlinearity of wavelength scanning of the tunable laser diode and to the higher harmonics from multireflection.
Automatic fringe pattern analysis

The phase unwrapping process is an important step in a whole-field optical technique. The wrapped phase then undergoes an unwrapping process in which an integral multiple of $2\pi$ is added at each pixel. The process converts the wrapped phase map into a continuous function.
5 Experimental system configuration

In this chapter we will describe each of the components of the experimental facility. At the core of our experimental method is the use of a wavelength scanning interferometer based on an external cavity tunable diode laser. We will discuss the control and performance of the laser and techniques. In the next section we will describe the control of the camera and the technical details regarding the efficient manipulation of the data. Finally, we will discuss the experimental rig in which the interferometer resides and our method for reliably obtaining accurate and repeatable measurements of the displacement caused by the load of granular packs.

5.1 Light source: external tunable diode laser

5.1.1 New Focus Vortex 6005

One of the key components of the interferometer is the light source. In these experiments we use a New Focus Vortex 6005, an external-cavity diode laser (ECDL) based on the Littman-Metcalf design as discussed in Section 3.2.2 [96]. This laser was chosen because its output is single longitude mode with a narrow bandwidth and is tunable.

The laser consists of the laser head, controller and cable. The controller contains the electronics for laser-current, laser temperature, and piezo-voltage control, as well as digital-interface electronics. The laser ultra-low noise current controls the optical output power. The piezo voltage controls wavelength tuning by rotating the mirror around a fixed rotation point using a piezoelectric actuator. Increasing the voltage increases the laser frequency or decreases the wavelength. The total piezo voltage (PZT voltage) or the DC level is limited to a range of 0 to 117.5V. Both of these parameters can be adjusted through the front-panel controls, the computer interface of either the parallel IEEE-488 (GPIB) interface or the serial RS-232 interface, or the back-panel BNC connectors.

The system is designed to work in two possible modes:
1. **Constant-current mode:** When operating in constant-current mode, the controller maintains a stable set current with the low-noise current driver. This results in a narrow laser linewidth. Our system works in this mode with a current of 54mA.

2. **Constant-power mode:** When operating in constant-power mode, the controller adjusts the laser current to maintain a stable output power.

### 5.1.2 Laser control and interface

Experimentally we wish to control both the wavelength and the rate at which it changes. To achieve this we make use of the RS232 serial interface between the PC computer and the laser controller. The serial interface RS-232 interfaces the DTE device (Data Terminal Equipment; usually a computer or terminal) and DCE device (Data Communications Equipment, usually a modem). The New Focus Vortex 6005 has a built in RS232 interface as a DCE device and can undertake 30 different operations [96]. For example, the command ":SOUR:VOLT:PIEZ 59" instructs the setting of the PZT voltage to 59V.

Windows API is a set of Application Programming Interfaces available in the Microsoft Windows operating systems, which provides documentation and tools to enable developers to write software associated Windows technologies. The advantage is that the software runs at a higher level rather than that which controls the hardware directly. For example, the developers can program communication between different IO devices and do not need to directly operate the hardware. The disadvantage is that the Window API communication program is not a real time application, for the system itself allocates time to the different applications and processes within the windows [143-144].

We chose to use Windows API serial port communication by Microsoft Visual C++ net to design the control program, as it allows us to open and close the serial port just once during one full-scale wavelength scanning operation (114 PZT voltage steps) and is easily integrated with the software for controlling the CCD camera.
5.1.3 Wavelength tunability and mode hop

5.1.3.1 Wavelength tunability

A key step in the wavelength scanning interferometer is to determine the relationship between the wavelength (and phase) and the PZT voltage input. We use a Fizeau interferometer and a glass optical wedge of average thickness 8.85 mm to measure phase-shifting values, during a laser scan through a linear series of 114 laser PZT voltages from 0V to 113V with a step of 1V. One phase-shifting sequence (from 1 to 114, which corresponds to the PZT voltages) is shown in Figure 5-1. We show the PZT voltage vs. the phase shift in Figure 5-1(a), PZT voltage vs. laser power in Figure 5-1(b) and PZT voltage vs. interference visibility in Figure 5-1(c). From Figure 5-1(a) we can see that the higher the PZT voltage is, the shorter the wavelength of the laser is. The phase value would increase with the PZT voltage.

\[ \text{Phase-shifting Value (rad)} \]
\[ \text{PZT Voltages (V)} \]

(a)

\[ \text{Light Power (mW)} \]
\[ \text{PZT Voltages (V)} \]

(b)
Discontinuous tuning, characterized by periodic "mode-hops" results from two competing wavelength-selection constraints, the mirror grating angle and the laser-cavity length. The laser-cavity length, $L$, defines a discrete set of possible wavelengths or modes, $\lambda_N$, that can lase, given by the equation

$$L = 2N \cdot \lambda_N,$$  \hspace{1cm} (5-1)

where $N$ is an integer. The grating equation insists that

$$\lambda = \Lambda (\sin \theta_i + \sin \theta_d),$$  \hspace{1cm} (5-2)

where $\Lambda$ refers to the groove spacing of the grating while $\theta_i$ and $\theta_d$ refer to the incident and diffracted angles of the laser beam. Rotation of the end mirror causes parameters in both equations to change. An appropriately selected point of rotation synchronizes the two, such that the cavity length remains the same number of half-wavelengths as long as the mirror is being rotated. Thus mode-hop free tuning is achieved. When this condition is not met, the lasing wavelength will periodically hop from one mode to the next (e.g. from $N$ to $N + 1$) as the laser is tuned.
Figure 5-2: Phase shifting steps with mode - hoping.

(a). PZT voltages vs. phase-shifting values. (b). PZT voltages vs. Light power.

(c). PZT voltages vs. interference visibility. Data folder: 19Oct05\D\BL1.
Unfortunately, the system is susceptible to mode-hopping when the environmental temperature is higher than 21.3°. Evidence for the mode-hopping can be seen in Figure 5-2. The discontinuous point or jumps are caused by the changes in modes as the laser is tuned. One finds that it is not only the phase that is affected by the mode-hopping, the laser power also jumps (see Figure 5-2(b)), and its visibility during hopping is significantly lowered from about 0.8 to 0.3, making the interferogram significantly more noisy (see Figure 5-2(c) and Figure 5-3).

![Figure 5-3: Interference fringe patterns.](a). Fringe pattern without mode hop. (b). Fringe pattern during mode hop.

Despite the problems with mode-hopping, we have found that by ensuring that the environmental temperature is below 21.3° and its fluctuations are low, that after allowing the laser to stabilize for about 4 hours, the likelihood of mode-hopping is vastly reduced.

### 5.2 CCD camera and frame grabber

Initially we used a SUN Sparc station based system, connected to a Kodak MegaPlus ES1. However redundancy issues surrounding the SUN Sparc station and potential problems with the SBUS based architecture of the ES 1.0 camera persuaded us to look to a PC based solution.

With a resolution of 1280 x 1024 effective pixels, the VDS Vosskuhler GmbH CCD-1300 QFB is a member of Vosskuhler high resolution CCD family [145]. By means of a progressive interline transfer sensor very short exposure times down to 1/10000
seconds can be achieved at full resolution. The exposure time can be regulated in steps of approximately 76μs. Due to asynchronous operation (image on demand), the exposure starts from 15μs after an external trigger pulse. Moreover, it can work in a continuous operation 12.5 frame/second and 80ms exposure time. The 2/3" sensor offers the possibility of using all C-mount lenses and optics customary in commerce. The performance is shown in the Table 5-1.

Table 5-1: The parameters of VDS Vosskuhler GmbH CCD-1300QFB [145]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCD Resolution</td>
<td>1280*1024</td>
</tr>
<tr>
<td>CCD Active Area</td>
<td>8.58 mm * 6.86 mm</td>
</tr>
<tr>
<td>CCD Pixel Size</td>
<td>6.7 μm</td>
</tr>
<tr>
<td>Saturation Illumination</td>
<td>&gt;25000e</td>
</tr>
<tr>
<td>Dynamic Range</td>
<td>Anti-Blooming</td>
</tr>
<tr>
<td>Continuous Frame Rate</td>
<td>Capture Single Frame</td>
</tr>
<tr>
<td>Image on Demand</td>
<td>12 bit, LVDS</td>
</tr>
<tr>
<td>Output and Interface</td>
<td>C type</td>
</tr>
<tr>
<td>Pixel Clock Rate</td>
<td>0° to 40°C</td>
</tr>
<tr>
<td>Operating Temperature</td>
<td>+12V, max 0.45A</td>
</tr>
</tbody>
</table>

Matrix Meteor-II/Digital/4L is a PCI frame grabber designed to capture from monochrome and component RGB, and frame/line scan sources in LVDS signalling standards, with extensive functionality. Key parts of its performance are listed as follows [146]:

1. Custom-designed Matrox Video Interface ASIC (VIA). It is a sophisticated memory controller for managing real-time acquisition into on-board memory with advanced reformatting capabilities and, in parallel, streaming image data out over the PCI bus without requiring constant host CPU intervention.

Image data can be reformatted by Matrox VIA in real-time prior to transfer to host system or display, and can be reordered into a proper image with little or no host CPU involvement. Features of Matrox VIA include cropping, independent horizontal and vertical sub-sampling from 2 to 16 (by decimation), and independent horizontal and vertical zoom of 2 and 4 (by replication), etc.

2. Real-time capture to system or display. It can transfer acquired images to either system (host CPU) memory for processing or display (VGA) memory for live video-in-a-window at sustained rates up to 130 MB/second.
5.3 Measurement system configuration

5.3.1 Setup of wavelength-scanning interferometer

Figure 5.4: Configuration of the system for the measurement of mechanical properties at the bottom of granular packing.

(a). Schematic diagram. (b). Photographic image of the apparatus.

A 2D reflective scanning-wavelength Fizeau interferometer is employed to measure the normal force distribution in the granular pack, shown in Figure 5-4 (a) and (b). The linearly polarized light beam emitted from the laser is enlarged by the lens OL (30x/0.5, and a negative achromatic lens \( f = -20 \text{mm} \)), and FL (Positive achromatic lens \( f = 400 \text{mm} \), diameter 60 mm), and then redirected onto an optical wedge, by means of the two mirrors M1 and M2. The enlarged parallel beam is reflected by the two surfaces of the wedge, forming interference fringes that are subsequently detected by the CCD-1300QFB camera. The angle between the incident and reflective light is 3.04°.
Experimental system configuration

The light source is an external cavity diode laser (New Focus Vortex 6005) with a wavelength centred at 635.05nm that can be scanned about 0.162 nm by the commands from PC computer. When the wavelength of the laser scans in steps, the correspondent fringe pattern is phase-shifted in steps. Finally, the patterns are transferred to a PC (Intel Pentium IV 3.0 GHz with 1024 Mb memory) for analysis.

When granular beads are put into the container, the normal forces lead to a localized deformation of the upper surface of the optical wedge whilst the lower (glass-air) boundary is unaffected, resulting in a small change in the interference pattern generated by the interfering waves from these two surfaces, when observed from underneath. The interference between top and bottom surfaces in this way results in an essentially common-path interferometer that is largely free from environmental disturbances such as vibration. Traditionally, phase-shifts are introduced by changing the relative path difference between two arms of the interferometer; in this case, however, this is achieved through the tuning of the wavelength of the laser light source.

The mechanical structure is formed from several components that are described in the following paragraphs.

1. The laser and related components are positioned above the camera. This arrangement allows the interferometer to be symmetrical about the movement direction of the translation stage. The camera position is adjustable allowing the angle between the incident and reflective light to be 3.04°.

2. The whole of the interferometer is set on top of a translation stage (ATS 1500 Series), which can be moved using a stepper motor. Each step is equivalent to a movement of 2.54 μm, and the total number of possible steps is 120000 (304.8 mm). In our experiments we scan 15 images covering 59290 steps (150.5966 mm), shown in Figure 5-5, and the field of view of each image is 11.004 mm (10.757 μm/pixel). The field of view of the interferometer is scanned across the length of the substrate allowing us to make measurements as a function distance from the centre of the pack and still maintain the high magnification that allows us to make accurate measurements.
3. The interferometer, the translation stage and the granular pack are supported on a Newport RS-4000 Table and vibration damping I-2000 LabLegs in order to eliminate errors caused external vibrations.

4. One of the most important components of the whole system is the optical wedge, made from optical glass BK7 (refractive index $n = 1.51509$ at 632.8 nm, $n = 1.51472$ at 643.8 nm, elastic modulus 82 GPa) shown in Figure 5-6(a). Its wedge angle made between the upper and lower surfaces is $0.006^\circ$. The wedge is coating by sputtering Ni80/Cr20 onto the upper surface, with thickness about 200nm. It is noted that its reflectance increases with the coating thickness at first and then levels out, shown in the Figure 3-8. Because its reflectivity is about 60%, which is lower than that of Aluminium, the visibility of the twin image in the displacement map is rather low, and will be discussed in Section 6.4.

The optical wedge provides a substrate, which supports the bead pack and provides the means by which we can determine the particle normal forces. This second function is facilitated by the application of the Ni80/Cr20 layer on the top surface of the wedge. Ni80/Cr20 was chosen for its adhesive qualities and its resistance to scratches, as well as its primary purpose to ensure that light is not reflected from the beads themselves, which causes phase distortion in the wrapped phase map.

The dimension of the optical wedge is $200 \times 50 \times 10 \text{ mm}^3$ with available field of view of $160 \times 10 \text{ mm}^2$. From Figure 5-5(a) to (d), we can find that the tilt direction of fringes changes, and it suggests that the surface flatness of the optical wedge is about $7\lambda/2$.

The geometric centre of granular pack is the centre of optical wedge, and the coordinates $(x, y)$ are along the axis of optical wedge. There is a $1.887^\circ$ angle between optical wedge and the movement direction of the translation stage shown in Figure 5-5(e).
5.3.2 Granular pack setup

5.3.2.1 Structure and parameters

The container for the granular materials is supported by four posts, allowing the interferometer to be translated underneath the optical wedge Figure 5-4.

Figure 5-5: Fringe patterns.

Figure 5-6: The container of granular pack and beads acted as granular material.
(a). Optical wedge in the container. (b). Beads in the container.
The container is a cylinder of height 130 mm and diameter, \( \phi \), of 420 mm, which is equivalent to about 53 beads of diameter 8 mm and 18 layers in the height, as shown in Figure 5-6(a). The maximum number that can be contained is about 45,000 ball beads, with a weight of 104 kg. The bottom surface of the container is constructed from steel and has a section cut out for supporting the optical wedge.

Chrome steel balls (AISI 52100 Low Alloy Chrome Steel) of diameter 8 mm and mass 2.001 g served as the granular material and are shown in the Figure 5-6(b). The balls have a maximum spherical error of 2.5 \( \mu m \), a maximum lot diameter variation of 5.0 \( \mu m \) and the mean size between batches is within \( \pm 10 \mu m \).

5.3.2.2 Construction of granular pack

The pack is generated when beads are poured down from a funnel shown in the Figure 5-7, the funnel's geometric centre is located in the centre of optical wedge and its outlet is just above the pack to avoid high velocity impacts and protect the metal coating layer on the top surface of the optical wedge.

![Figure 5-7: Building up the granular pack from a hopper.](image)

5.3.3 Setup of self load experiment

Our initial experiment was to verify whether the weight of the granular pile is equal to the weight of the measurement results, serving as our first test of the experimental method. During construction of the granular pack, the first layer of particles covered the bottom plate without allowing any particles to be supported. Subsequent layers
were then produced by pouring the particles onto the lower layers using the funnel (Figure 5-7). It is observed that the first few layers tended to crystallize and show long-range order, but that this effect is reduced as the number of layers is increased. The pile construction is halted when the pile base width is equal to the longest dimension of the optical wedge, as shown in the Figure 5-8. With this configuration the angle of repose of the pile was found to be 14.3°.

![Fizeau Interferometer](image)

Figure 5-8: Configuration of self-load experiment.

(a). Schematic diagram. (b). Photo.

For these experiments the average number of balls was 1519. The procedures for the experiments were as follows:

1. After the pile has been constructed, the translation stage scans at intervals of 4235 steps, resulting in the collection of 15 sets of images. For each set of images, 114 frame phase-shifting fringe patterns were acquired by the CCD camera and stored in the computer. We refer to this series of captures as BL1.
2. The pile is then removed and process is repeated. We refer to the series without load as BL0.

3. The fringe maps are then processed to extract the deformation field for the whole composite field of view (see chapter 7).

4. The contact regions are then identified manually.

5.3.4 Setup of Green's function response experiment

The granular pack can be observed to respond to a localized force on the upper surface by changes in the contact force distribution, which is called a Green's function response. The rig for generating local disturbing load is shown in the Figure 5-4(b). The loads are shown in the Figure 5-9.

![Figure 5-9: Generators of localized forces on the top surface of granular pack.](image)

(a) Generator 1, weight: 15.2194 N. (b) Generator 2, weight: 3.4385 N. (c) Generator 3, weight: 17.4554 N. (d) Generator 3, weight: 7.8906 N. (e) Generator 3, weight: 6.0214 N.

We follow a similar procedure to that outlined for the granular pack. However, in this case we are interested in the difference between the loaded state of the granular pack and the unloaded. Therefore we collect the 15 series of images when the localized force is applied, and another 15 series of images for when the force has been removed. The data is then processed in exactly the same way as in the last section.
5.4 Rigs for calibration of load vs. substrate deformation

An important component in our system is the optical wedge. One is circular, coated with aluminium on one side; another one is rectangular, coated with Ni80/Cr20 on the side that is used to support the granular pack. The rigs for calibration of load vs. substrate deformation will be discussed in detail in the following sections.

5.4.1 Load calibration system of circular wedge with Aluminium coating

The calibration rig of the circular wedge shown in Figure 5-10, which is used for the calibration of load vs. substrate displacement, is independently supported by four posts on the Newport RS-4000 Table. The chrome steel balls used as granular materials discussed in the last section also acted as the indentator or loads during calibration. These beads are supported in a column on the top of the coating surface of the optical wedge using the rig shown in Figure 5-10(b). The kernel component, the circular optical wedge, has a thickness of 8.85 mm on average, a wedge angle of 0.006°, and available field of view of diameter 40 mm. It is located in the centre of the calibration rig.

![Figure 5-10: Experimental rig for load calibration.](image)

(a). Circular optical wedge. (b) Rig for holding beads. (c) Experimental rig for load calibration.

Because the calibration rig is fixed on the top of the optical table, and the translation stage with the wavelength-scanning interferometer can be moved backwards and forwards underneath with a step of 2.54 μm, the field of view is in the form of the a strip covering part of the optical wedge and can be viewed by the camera with a field of view 11.495mm x 11.495mm for each frame captured (11.23 μm/Pixel), as shown in Figure 5-11.
Figure 5-11: Images corresponding to the sequence of the translation stage movements.

(a). Image 1, the bottom line is AB. (b). Image 2, the bottom line is CD. (c). Image 3, the bottom line is EF. (d). Image 4, the bottom line is GH. (e). Image 5, the bottom line is LI, the upper line is JK.

(f). The whole field of view of images and their locations relative to the optical wedge, with diameter of 3563 pixels (φ40mm).

Figure 5-11 (a), (b), (c), (d), (e) show 5 corresponding images (1024 pixel×1024 pixel), when the translation stage moves 3500 steps 4 times independently. Figure
5-11 (f) and Table 5-2 show that the whole field of view and their locations relative to the optical wedge.

Table 5-2: Corresponding positions and coordinates between the images and optical wedge.

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Position</th>
<th>Coordinates in pixels</th>
<th>Coordinates (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Image 1</td>
<td>A</td>
<td>(0, 0)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>(1027, -25)</td>
<td>(11.53, -0.28)</td>
</tr>
<tr>
<td>Image 2</td>
<td>C</td>
<td>(21, 801)</td>
<td>(0.24, 8.99)</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>(1047, 776)</td>
<td>(11.75, 8.71)</td>
</tr>
<tr>
<td>Image 3</td>
<td>E</td>
<td>(42, 1602)</td>
<td>(0.47, 17.98)</td>
</tr>
<tr>
<td></td>
<td>F</td>
<td>(1086, 1577)</td>
<td>(11.97, 17.70)</td>
</tr>
<tr>
<td>Image 4</td>
<td>G</td>
<td>(63, 2403)</td>
<td>(0.71, 26.98)</td>
</tr>
<tr>
<td></td>
<td>H</td>
<td>(1086, 2377)</td>
<td>(12.19, 26.68)</td>
</tr>
<tr>
<td>Image 5</td>
<td>L</td>
<td>(84, 3204)</td>
<td>(0.94, 35.97)</td>
</tr>
<tr>
<td></td>
<td>I</td>
<td>(1105, 3178)</td>
<td>(12.40, 35.68)</td>
</tr>
<tr>
<td></td>
<td>J</td>
<td>(101, 4227)</td>
<td>(1.13, 47.45)</td>
</tr>
<tr>
<td></td>
<td>K</td>
<td>(1130, 4203)</td>
<td>(12.69, 47.18)</td>
</tr>
<tr>
<td>Centre</td>
<td></td>
<td>(745, 1809)</td>
<td>(8.36, 20.31)</td>
</tr>
</tbody>
</table>

Because the optical wedge is fixed in the centre of the calibration rig, and its position relative to the boundary of granular pack is predetermined by the design of the mechanical structure, therefore, the geometric relations between images due to different positions of translation stage movement and granular material packing are known.

### 5.4.2 Load calibration system of rectangular wedge with Ni80/Cr20 coating

For the load-deformation calibration we use a loading system that allows us to stack a known number of particles on top of each other to create integer number of single particle loads on the substrate surface (Figure 5-12(a))

There are two groups of calibration loads. The first is 16 beads, which are the same as the beads used as granular materials discussed in the previous section. The second is 1 bead indentor and 6 cylinders, as shown in the Figure 5-12(b). The cylinder is made from steel of dimensions, diameter, 8 mm and length, 50 mm and with a weight of 170.961 mN.
5.5 Summary

In this chapter, principle, structure and control method of the measurement system of normal force distribution observed at the bottom of a granular pack were discussed in detail. Firstly, the external tunable laser and its controlling method were introduced; then, the performance of wavelength scanning was assessed with particular regard to mode hopping. Secondly, cameras and frame grabbers were presented, a VDS Vosskuhler GmbH CCD-1300QFB and Matrix Meteor-II/Digital/4L respectively, and their use of a Fizeau interferometer is described. Finally, the structure and optical principle for measurement and load calibration were described. The results of the calibration experiments will be presented in the next chapter.
6 Experimental system calibration

6.1 Introduction

Good measurement relies on the integrity of the measuring equipment. Unfortunately, no matter how sophisticated the equipment is, its accuracy degrades due to thermal, mechanical, electrical, and environmental effects. The effects may be offset by regular calibration.

In this chapter, firstly, PZT voltages vs. interference phase-shifting steps of optical wedge are calibrated. Secondly, the influences of phase-shifting interferometry on accuracy of measuring deformation are discussed in detail. Subsequently, a nonlinear least square fitting method is introduced to increase the accuracy of measurement of contact forces and exact contact positions on the boundary of granular pack. In the end, bead loads vs. substrate deformations are calibrated.

6.2 Calibration of phase-shifting steps in situ using a 2D Fizeau interferometer

6.2.1 Methods for robust extraction of phase from interference patterns

6.2.1.1 Introduction

Since the 1990s, tunable diode lasers have been widely used in metrology, generally due to their convenient size and wavelength tunability [131, 147]. During this period, phase-shifting techniques have also become more and more sophisticated [109]. One of the most popular techniques for introducing a controlled phase difference is by wavelength scanning of a tunable laser. In this thesis, we have employed an external cavity tunable diode laser and consequently we focus our discussions in this section on phase-shifting values or the steps of wavelength scanning interferometry.

In order to minimize systematic error during phase-shifting interferometry, calibrations are necessary. Several interference methods have been developed to calibrate the phase-shifting steps from the tunable diode laser caused by induced-current, temperature and PZT voltage etc. [148-151]. Early on, Chan et al [148] used
Experimental system calibration

an unbalanced Michelson interferometer to obtain the electrical current-wavelength tuning rate by measuring the phase difference and a complicated heterodyne interferometer with high resolution was reported later [149]. Recently, Wu et al [151] presented a spatial synchronous phase measurement technique in order to calibrate the current wavelength-tuning rate of their diode laser. The resolution of the method depends on the straightness of the fringe or the flatness of the two surfaces, where the direction and inclination are also critical in the system.

In 1998, a method of using a 2D Fourier Transform of a sequence of interference images to calibrate a non-linear phase PZT modulator was suggested [152]. The main advantage over previous calibration methods is that in this method both the linear and quadratic phase variation terms are measured, which is equally applicable to smooth wave front and speckled wave front interferometry, allowing convenient in situ calibration for a wide range of interferometer configurations.

![Fourier Transform of the Intensity](image)

**Figure 6-1:** Amplitude in non-dimensional frequency domain [78].

In 2002, Osman et al. [78] proposed a new way to calibrate the phase-shifting values from a tunable diode laser. The system is typical of a reflective Fizeau interferometer (see Figure 2-24). The amplitude in the 1D frequency domain is calculated; two peaks are seen in the positive frequency range, shown in the Figure 6-1. One is due to the carrier or fundamental frequency fringes (where the phase value is acquired) and the other, consisting of higher harmonics, is attributed to multiple reflections.
Experimental system calibration

The drawback of Osman's method is that only 1D information in the image is used. Obviously, accurate and reliable frequency analysis requires efficient use of all the signal energy contained within the fringe pattern. A more natural approach to solve the problem would appear to be 2D Fourier analysis, as described in [152-153]. Although the signal and noise area is dispersed throughout the spatial domain, on calculating the 2D Fourier transform of fringe pattern, however, whilst the noise is spread fairly uniformly over the entire frequency domain, the signal is now peaked and limited to a small region, if the fringes are straight or near straight. Thus, in the region of interest, the signal to noise ratio is considerably enhanced, compared to the 1D analysis.

In order to have high resolution and accuracy in phase-shifting interferometry, new and fully automatic algorithms in Fizeau interferometer for calibrating phase-shifting steps from tunable diode laser in situ are presented in this section, where the phase values of the fringe patterns are easily extracted by 2D FFT and CZT methods. In the following sections we discuss a number of ways in which the system can be calibrated and demonstrate the advantages and disadvantages of each method.

6.2.1.2 Phase acquisition using 2D FFT methods

![Typical fringe pattern](image)

Figure 6-2: Typical fringe pattern caused by the reflection from the two surfaces of a glass wedge (700 pixel x 700 pixel).
Generally, the digitised spatial 2D fringe generated by the interference from an optical wedge (Figure 6-2) has the form \([109, 153]\):

\[
x(m, n) = w(m, n) \cdot \{I_0(m, n) \cdot [1 + V(m, n) \cdot I(m, n)]\},
\]

(6-1)

where \(m, n\) are pixel locations. \(I(m, n), I_0(m, n)\) and \(V(m, n)\) are referred to as interference intensity, average background intensity and visibility respectively and \(w(m, n)\) is the window function.
Figure 6-3: Fourier Transform of the fringe pattern from an optical glass wedge.

(c): Amplitude spectrum and interference visibility without DC. Offset.
(d): Phase map near the carrier frequency (1, 79).

Therefore, the 2D Fourier transform of $x(m,n)$ is given by:

$$\hat{H}(k_x,k_y) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x(m,n) \cdot \exp\left(\frac{-i \cdot 2\pi \cdot (k_x m + k_y n)}{N}\right)$$

(6-2)
\( \hat{H}(k_x, k_y) \) is a complex series, which consists of half the amplitude of the delta function at \( k_x = 0, k_y = 0 \). If the fringes are straight (Figure 6-2), the fringe spatial carrier frequencies are delta functions at \( k_x = f_{x0}, k_y = f_{y0} \) and \( k_x = -f_{x0}, k_y = -f_{y0} \) separately, shown in the Figure 6-3 (b), (c).

The average background light intensity is deducted by implementing distinct block processing (see Figure 6-3 (a)). The image is divided into the blocks of 32-by-32 pixels and the mean light intensity in each block is calculated. The average background light intensity is subtracted and the block normalized, leading to the undesirable effect of the finite width of the zero-order peak in the frequent domain of the fringe pattern being diminished. The amplitude spectrum without the DC offset is shown in the Figure 6-3 (c). It should be noted that the amplitude in the carrier frequency is equal to the interference visibility. In the phase domain (see Figure 6-3 (d)), we find the resolution of 2D FFT method is very limited, but the use of Chirped Z-Transform (CZT, discussed in the next Section) can significantly improve of it [121-123].

The spatial carrier frequency \( k \) at the peak position of the fringes, shown in Figure 6-3 (c), is determined by the tilt angle of optical wedge, \( \theta \), the field of view of the image in y axis, \( D_{\text{FOV}} \) (see Figure 6-2), the wavelength, \( \lambda \), and refractive index of the wedge, \( n \),

\[
k = \frac{2n \cdot D_{\text{FOV}} \cdot \tan \theta}{\lambda}. \quad (6-3)
\]

6.2.1.3 Phase acquisition using 2D CZT methods

In some applications of signal processing, the resolution of the frequency is required to be as high as possible, which is usually achieved by adding zeros to the end of the signal, known as "Zero padding". The more zeros added, the higher the resolution in the frequency domain, at the expense of rather low computing efficiency. Sometimes, we are only interested in the spectrum in a specific band of frequency, instead of the full range. For example, during the calibration of the phase-shifting steps, we only pay attention to a limited band around the fundamental frequency; and
ignore other frequencies. In this case the technique, which is referred to in the literatures [121-123] as a “Chirp-Z Transform”, is a useful alternative to zero-padding FFT in order to perform the necessary interpolation of the data. CZT is a transform method to convert the time domain signal to the specific frequency with a limited bandwidth. Within the frequency bands, interpolation is used to increase the resolution, according to the length of the transform, which is predetermined by the user.

Similar to the DFT (Discrete Fourier Transform), The Chirp Z-Transform, or CZT operates along the unit circle and evaluates the Z-Transform for an input sequence along the contour, shown in Figure 6-4, which is in the Z-plane given by [121-123, 154],

\[ z_k = A \cdot W_i^k, \quad k = 0, \ldots, M - 1, \]

\[ A = e^{i \pi f_1 / f_s}, \]

\[ W_i = e^{i \pi (f_s - f_1) \mu / f_s M}, \quad (6-4) \]

where \( f_1, f_2 \) and \( f_s \) starting frequency, ending frequency and frequency range; \( M \) is the length of the transform; \( A \) is the complex starting point from \( f_1 \) on that contour; \( W_i \) is a complex scalar describing the complex ratio between points on the contour.

Figure 6-4: Contour of Z-plane [121-123, 154].

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Therefore, the Chirp Z-Transform of a time sequence signal $x(n)$ is given by

$$\hat{X}(k) = \sum_{n=0}^{M-1} x(n) \cdot (A \cdot W_i^k)^{-n}.$$  \hspace{1cm} (6-5)

For example, the amplitude spectrum of the signal $x(n)$ is shown in Figure 6-5 (a) [154], and its CZT is used to zoom in on a narrow-band section (100 Hz to 150 Hz) at the resolution of 1024 points, shown in Figure 6-5 (b). The choices of $A$ and $W_i$ correspond to the case where the contour in the $z$ plane, shown in Figure 6-4, consists of equal angular increments around the unit circle.

![Figure 6-5: Amplitude spectrum of time sequence signal $x(n)$ [154].](image)

(a) Amplitude spectrum of FFT. (b) Amplitude spectrum of CZT.

If 2D spatial fringes are given by Eq.(6-1), its CZT is defined by:

$$\hat{H}(k_x,k_y) = \sum_{m=0}^{M_x-1} \sum_{n=0}^{M_y-1} x(m,n) \cdot (A_x W_x^{k_x})^{-m} \cdot (A_y W_y^{k_y})^{-n},$$  \hspace{1cm} (6-6)

$$A_x = e^{i2\pi M_x/M_x}, \quad W_x = e^{i2\pi (M_x-M_0)/(M_x-M)}$$

$$A_y = e^{i2\pi M_y/M_y}, \quad W_y = e^{i2\pi (M_y-M_0)/(M_y-M)},$$  \hspace{1cm} (6-7)

where $M_x0$ and $M_y0$ are the complex starting points; $M_x$ and $M_y$ are the ranges of the spatial frequencies; $W_x$ and $W_y$ are the complex scalars; for $(x,y)$ respectively.
Figure 6-6: CZT of fringe pattern from an optical glass wedge.


We use the 2D CZT to deal with the 2D fringes from the optical glass wedge, shown in Figure 6-2. The spatial carrier frequency is evaluated with 10Hz bandwidth in the spatial frequency domain and has $499 \times 499$ interpolating points. Its amplitude spectrum is shown in the Figure 6-6 (a), and its phase map is given in the Figure 6-6 (b). From them, we can find that the peak is zoomed and details are clearer. Compared with the phase map of FFT method (see Figure 6-3 (d)), the phase map of CZT is smoother and the resolution is improved significantly.

6.2.1.4 Phase acquisition using a Hanning window

The spectrum of a real signal is a convolution of the signal spectrum and the spectrum due to the finite length of the signal, an effect known as leakage. From Figure 6-7 (a), (b), we can find that the spectrum leakage has the influence on the
phase evaluation at the carrier frequency. One way to diminish the influence is to modify the window function. In this section, we show the application of 1D and 2D Hanning window function (Section 4.2.2.2) on limiting the spectrum leakage.
Experimental system calibration

Figure 6-7: Normalized amplitude spectrum at the peak position of the fundamental frequency obtained using a 2D Hanning window function.

(a). Fringe pattern from optical perspex wedge.
(b). Normalized amplitude spectrum of the fringe pattern at the fundamental frequency by CZT.
(c). Fringe pattern from optical perspex wedge using a 1D window function.
(d). Normalized amplitude spectrum of the fringe pattern using a 1D Hanning window function at the fundamental frequency by CZT.
(e). Fringe pattern from optical perspex wedge using a 2D window function.
(f). Normalized amplitude spectrum of the fringe pattern using a 2D Hanning window function at the fundamental frequency by CZT.

Figure 6-7 (c) shows the fringe pattern from optical perspex wedge using a 1D window function, which is vertical to the direction of the fringe. Its amplitude spectrum is given by Figure 6-7 (d). The same fringe pattern under a 2D Hanning window function is given by Figure 6-7 (e), and its amplitude spectrum at the carrier frequency is shown in Figure 6-7 (f). Compared with amplitude spectrum under the rectangular window shown in Figure 6-7 (b), the noise peaks around the carrier frequency are lower, and the influence of spectrum leakage is significantly diminished, at the expense of the main peak being twice as wide as that of a rectangular window.

6.2.2 Acquiring phase-shifting steps from fringe patterns

When using a temporal phase-shifting algorithm with unequal interval sampling (Section 4.3), we request a series of PZT voltages to be sent to the laser controller in
order to scan laser wavelength and induce phase shifts in the fringe patterns, which are grabbed by the camera and saved to hard disk. The phase values are then evaluated by the 2D FFT or CZT method described in the Section 6.2.1.

Subsequently the fringe maps are unwrapped, and the phase-shifting steps are determined (Section 5.1.3). Typical phase-shifts versus PZT voltages are plotted in Figure 5-1, and Figure 5-2 with and without mode-hopping in evidence respectively.

6.3 **Calibration of substrate deformation vs. bead load**

Having determined the relationship between the PZT voltage and the phase shift, we need to calibrate the applied load versus the measured maximum deformation at a contact point. It is noted that changes to the refractive index of the optical materials due to the imposed stress manifest themselves as fictitious deformations. In our case, however, we calibrate this effect out and don’t attempt to correct for the above-mentioned effect.

6.3.1 **Calibration load vs. glass substrate deformation**

6.3.1.1 Least square phase-shifting interferometry (LSPSI)

In our application of phase-shifting interferometry, 114 frame fringe patterns are acquired from Fizeau interferometer, while the laser PZT voltage scans linearly from 0V to 114V in steps of 1V. Because of the strong non-linearity in the relation between PZT voltage and phase-shifting values, a Least Squares Phase-shifting Interferometry (LSPSI) with unequal interval sampling (Eq. (4-33)), was used to determine the wrapped phase map, which corresponds to the substrate deformation before and after loading. There are three factors that should be taken into consideration.

1. Average phase-shifting step. In general, the phase-shifting algorithm is insensitive to the $l$th harmonic in the fringe pattern when the average phase-shifting step $\bar{\psi}$ satisfies the requirement [109, 113]

$$\bar{\psi} \leq \frac{2\pi}{2+l}. \quad (6-8)$$
A feature of our reflective Fizeau interferometer is that harmonics up to the 3rd order cannot be ignored [109, 113]. Therefore, using Eq.(6-8) we deduce that the average phase-shifting step should be equal or smaller than $2\pi/5 \approx 1.257$ rad or 72°.

The phase-shifting steps in a 114-frame LSPSI given by the Figure 5-3 (a) are shown in the Figure 6-8. The average phase-shifting step, $\bar{\varphi}$, is 0.677 rad and the standard deviation is 0.093 rad.

![Graph showing phase-shifting steps](image)

Figure 6-8: Average phase-shifting steps of 114-frame LSPSI.

2. As discussed in Section 4.4, different normalizing methods for the light intensity modulation have been tried. We use the data from the calibration of one bead load vs. glass substrate deformation, while there is no mode-hop in the output of the laser. The analysis region is cropped to [500, 600, 100, 200], at a distance far from the deformation peak. The noise level, maximum peak, minimum peak and average deformation before and after load are listed in the Table 6-1, Table 6-2, Table 6-3, and Table 6-4 respectively.

<table>
<thead>
<tr>
<th>Table 6-1: The effect of normalization of the light intensity with different parameters.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase calibration method: FFT and Rectangular Window.</td>
</tr>
<tr>
<td>Data folder: IrPSIWind_19Oct05D\BL1. Crop region: [500,600,100,200]</td>
</tr>
<tr>
<td>1. Results without normalization of the light intensity.</td>
</tr>
<tr>
<td>Number of frames</td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>112</td>
</tr>
</tbody>
</table>


Experimental system calibration

2. Results after normalization with the measured laser power.

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation</th>
<th>Maximum deformation</th>
<th>Minimum deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
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<tr>
<td>112</td>
<td>0.27</td>
<td>-0.01</td>
<td>2.24</td>
<td>-2.36</td>
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</table>

3. Results after normalization with the interference visibility.

<table>
<thead>
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<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation</th>
<th>Maximum deformation</th>
<th>Minimum deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
</tr>
<tr>
<td>112</td>
<td>0.26</td>
<td>-0.01</td>
<td>2.35</td>
<td>-2.44</td>
</tr>
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</table>

Table 6-2: The effect of normalization of the light intensity with different parameters.

Phase calibration method: CZT and Rectangular window.
Data folder: IrPSIWind_190ct05\DIBL1. Crop region: [500,600,100,200]

1. Results without normalization of the light intensity.

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation</th>
<th>Maximum deformation</th>
<th>Minimum deformation</th>
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<td></td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
</tr>
<tr>
<td>112</td>
<td>0.27</td>
<td>-0.01</td>
<td>2.30</td>
<td>-2.53</td>
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</table>

2. Results after normalization with the measured laser power.

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation</th>
<th>Maximum deformation</th>
<th>Minimum deformation</th>
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<td>nm</td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
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<tr>
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<td>0.27</td>
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<td>-2.42</td>
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3. Results after normalization with the interference visibility.

<table>
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<th>Number of frames</th>
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<th>Average deformation</th>
<th>Maximum deformation</th>
<th>Minimum deformation</th>
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<tbody>
<tr>
<td></td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
</tr>
<tr>
<td>112</td>
<td>0.26</td>
<td>-0.01</td>
<td>2.30</td>
<td>-2.46</td>
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</table>

Table 6-3: The effect of normalization of the light intensity with different parameters.

Phase calibration method: FFT and 2D Hanning Window.
Data folder: IrPSIWind_190ct05\DIBL1. Crop region: [500,600,100,200]

1. Results without normalization of the light intensity.

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation</th>
<th>Maximum deformation</th>
<th>Minimum deformation</th>
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<td>nm</td>
<td>nm</td>
<td>nm</td>
<td>nm</td>
</tr>
<tr>
<td>112</td>
<td>0.27</td>
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<td>2.30</td>
<td>-2.59</td>
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</table>
Experimental system calibration

2. Results after normalization with the measured laser power.

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<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation (nm)</th>
<th>Maximum deformation (nm)</th>
<th>Minimum deformation (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>112</td>
<td>0.27</td>
<td>-0.01</td>
<td>2.26</td>
<td>-2.40</td>
</tr>
</tbody>
</table>

3. Results after normalization with the interference visibility.

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation (nm)</th>
<th>Maximum deformation (nm)</th>
<th>Minimum deformation (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>112</td>
<td>0.26</td>
<td>-0.01</td>
<td>2.33</td>
<td>-2.44</td>
</tr>
</tbody>
</table>

Table 6-4: The effect of normalization of the light intensity with different parameters.

Phase calibration method: CZT and 2D Hanning Window.

Data folder: IrPSIWind_19Oct05\D\BL1. Crop region: [500,600,100,200]

1. Results without normalization of the light intensity.

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation (nm)</th>
<th>Maximum deformation (nm)</th>
<th>Minimum deformation (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>112</td>
<td>0.27</td>
<td>-0.01</td>
<td>2.33</td>
<td>-2.50</td>
</tr>
</tbody>
</table>

2. Results after normalization with the measured laser power.

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation (nm)</th>
<th>Maximum deformation (nm)</th>
<th>Minimum deformation (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>112</td>
<td>0.27</td>
<td>-0.01</td>
<td>2.31</td>
<td>-2.42</td>
</tr>
</tbody>
</table>

3. Results after normalization with the interference visibility.

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation (nm)</th>
<th>Maximum deformation (nm)</th>
<th>Minimum deformation (nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>112</td>
<td>0.26</td>
<td>-0.01</td>
<td>2.33</td>
<td>-2.43</td>
</tr>
</tbody>
</table>

From Table 6-1 to Table 6-4, we find that the CZT phase acquiring method is better than that of the FFT in some circumstances, since the noise levels in the deformation maps are smaller than those associated with the FFT. Furthermore, LPSI normalized by power and interference visibility, generally shows better performance than that without any normalization. Therefore, we use these normalizations for all our analyses below.
3. As discussed in (Section 4.4), the influence of the number of frames is listed in Table 6-5, where we use two total frame numbers, (a). 35 frame phase-shifting sequence with a maximum phase-shifting value of 18.825 rad (about 3 carrier cycles) and (b). 38 frame phase-shifting sequence with a maximum phase-shifting value of 21.013 rad (about 3.35 carrier cycles). The noise level associated with (a) is smaller than that of (b), primarily because the range of the phase-shifting values is nearer to $2 \cdot k \cdot \pi$.

In order to keep the range of the phase-shifting values near to $2k\pi$, an adaptive procedure is used to change the number of frames in the phase-shifting algorithm to ensure that the range of the phase-shifting values approaches the maximum $2\pi k$ carrier cycle.

Table 6-5: LSPSI with different phase-shifting sequence lengths.

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level Standard deviation</th>
<th>Average deformation</th>
<th>Maximum deformation</th>
<th>Minimum deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>0.31</td>
<td>-0.02</td>
<td>3.80</td>
<td>-3.62</td>
</tr>
<tr>
<td>38</td>
<td>0.33</td>
<td>-0.04</td>
<td>3.74</td>
<td>-3.64</td>
</tr>
</tbody>
</table>

The number of frames in Table 6-5 and carrier cycle is much fewer than the 114-frames, which we use during the normal load vs. substrate deformation calibration and measurement of the force distribution in the granular materials as such large numbers of frames are only necessary when significant mode hop is observed.

It is noted that the light power drifts about 17.7% from 7.3mW to 5.1mW (Figure 6-9 (b)), and the interference visibility drifts about 81.8% from 1 to 0.1 (Figure 6-9 (c)), when there are mode-hops in the output of laser. In the region of a mode-hop, the visibility of the fringe patterns is so low that the frames within the region only contribute to an enhancement of the noise level in the LSPSI.
Figure 6-9: Phase-shifting sequences observed with mode hop present.

Data folder: 19Oct05GIBL1. Crop region: [500,600,100,200].

(a). Phase-shifting sequence. (b). Light power. (c). Interference visibility.
Experimental system calibration

Table 6-6: LSPSI with the poor interference visibility frames.

Phase calibration method: 2D Hanning CZT.

The light intensity is normalized by the interference visibility.

Data folder: 19Oct05\GIBL. Crop region: [500,600,100,200].

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level standard deviation</th>
<th>Average deformation</th>
<th>Maximum deformation</th>
<th>Minimum deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>114</td>
<td>0.66 nm</td>
<td>-0.09 nm</td>
<td>10.56 nm</td>
<td>-6.77 nm</td>
</tr>
</tbody>
</table>

Figure 6-10: Noise level of LSPSI with the poor interference visibility frames.

Data folder: 19Oct05\GIBL. Crop region: [500,600,100,200].

We show (Table 6-6 and Figure 6-10) that the standard deviation of the noise level is 0.66 nm and the fluctuation is about ±5 nm in the area far from the loading region, for the case when substrate deformations are measured before and after loading by LSPSI (without removing any poor visibility interference frames). Table 6-7, and Figure 6-11 (d) show that when we remove the poor visibility frames the noise level in LSPSI is significantly lowered, although the number of frames is reduced by 22.

Table 6-7: LSPSI with the poor interference visibility frames removed.

Phase calibration method: 2D Hanning CZT.

The light intensity is normalized by the interference visibility.

Data folder: 19Oct05\GIBL. Crop region: [500,600,100,200].

<table>
<thead>
<tr>
<th>Number of frames</th>
<th>Noise level standard deviation</th>
<th>Average deformation</th>
<th>Maximum deformation</th>
<th>Minimum deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>92</td>
<td>0.30 nm</td>
<td>-0.03 nm</td>
<td>4.58 nm</td>
<td>-3.67 nm</td>
</tr>
</tbody>
</table>
Experimental system calibration

(a) Phase shifting value (rad) vs. PZT Voltages (V)

(b) Light power (mW) vs. PZT Voltages (V)

(c) Interference Visibility vs. PZT Voltages (V)
We conclude that the LSPSI shows excellent performance, primarily because we avoid problems with the light intensity modulation, mode hop, and the range of the phase-shifting value driftness and phase miscalibration errors, which are quite common with tunable laser diodes.

6.3.1.2 2D Nonlinear Lorentz fitting
Figure 6-12: Deformation map of the substrate before and after nonlinear Lorentz fitting.

(a). Deformation map produced by indentating the substrate surface with 2 beads raw map.
(b). The Lorentzian profile produced by fitting to the raw data map in (a).
(c). Error map before and after Lorentz fitting.
Units: (x, y) pixels, 1 pixel = 10.757μm. z: nm.

In order to increase the accuracy of evaluating the peak deformation and find the exact position, first of all, we correlate the deformation map with an adaptable 2D Lorentzian filter to find a rough estimate of the contact point locations. Subsequently, we perform a nonlinear least-square regression [78, 155-156], to find the exact
Experimental system calibration

location and magnitude of the deformation. Our model deformation profile is given by

\[
I(x, y) = l_0 + l_1 \cdot x + l_2 \cdot y - \frac{l_3}{1 + l_4 \cdot (x - l_6)^2 + l_5 \cdot (y - l_7)^2},
\]

(6-9)

where the coefficients \(l_0\), \(l_1\) and \(l_2\) account for the offset and gradient of the background plane, whilst \(l_3\) is the peak deformation, \(l_4\), \(l_5\) are the scaling of deformation and \(l_6\), \(l_7\) are the location of the peak deformation.

In Figure 6-12 (a), (b) we show the raw data for a deformation of the glass substrate due to a load of two 8mm steel beads (40.02mN), and the model profile determined from the data using regression. In Figure 6-12 (c), we show the difference map before and after Lorentz fitting, and its standard deviation error is 0.32 nm. The difference of peak deformation is 1.16 nm and the modification of peak position before and after Lorentz fitting are \(\Delta x = 4.977 \mu m\), \(\Delta y = 6.656 \mu m\).

6.3.1.3 Calibration 1: small load vs. substrate deformation

The load calibration system has been discussed in the Section 5.4 (Figure 5-13 (a)). Here, the small load was a chrome steel bead of diameter 8mm (20.01mN). The maximum applied load is 16 beads (320.12 mN).

We used an LSPSI with up to 114 frames employing compensation for the light intensity modulation to evaluate the wrapped phase map. The noise level and 3D deformation meshes are shown in the Figure 6-13 to Figure 6-20 and Table 6-8 to Table 6-15.

The calibration procedure is as follows:

(a). A column up to 16 beads is put onto the coated surface of the substrate.

(b). 114 frame fringe patterns are captured and stored in the computer. After phase-shifting values are evaluated, the LSPSI is used to determine the phase map (see Section 4.2.1.1). If there are no beads currently on top of the substrate then we go straight to step (f).
Experimental system calibration

(c). 2 beads are removed from the column of the beads.

(d). If there is no bead on the top of the substrate, the translating stage is commanded to move away \(3000\) steps and then \(-3000\) steps to return to the original position again, in order to quantify the influence of movement of the translating stage.

(e). We move to step (b).

(f). The wrapped phase maps \(\phi(z)\) are evaluated before and after loading, where

\[
\phi(z) = \tan^{-1}\left[ \frac{a_2(z) \cdot a_1(0) - a_1(z) \cdot a_2(0)}{a_2(z) \cdot a_2(0) + a_1(z) \cdot a_1(0)} \right],
\]

(6-10)

where \(z = 2, 4, 6, \ldots, 16\). After unwrapping and rescaling, the calibration results are computed. These are shown in Figure 6-13 (b) to Figure 6-20 (b).

In Figure 6-13 to Figure 6-20, the average substrate deformation due to a load of 1 bead is observed to be 4.75 nm, while the noise levels are within the bounds -2 nm and +3 nm and the maximum observed standard deviation of substrate deformation is 0.25 nm. The maximum substrate deformation is 73.88 nm under the load of 16 beads.

(1). 2 bead load (40.02 mN) vs. substrate deformation.

Table 6-8: Experimental results of 2 bead load vs. substrate deformation.

<table>
<thead>
<tr>
<th>Standard Deviation (nm)</th>
<th>Mean Substrate Deformation (nm)</th>
<th>Peak Deformation (nm)</th>
<th>Position (Pixels)</th>
<th>Number of frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.22</td>
<td>0.049</td>
<td>6.91</td>
<td>[696, 320]</td>
<td>112</td>
</tr>
</tbody>
</table>

Data folder: 19OCT05_D.

(a). Noise level in the region [0, 0, 50, 50]. (b). 3D deformation mesh.
(2). 4 bead load (80.03 mN) vs. substrate deformation.

Table 6-9: Experimental results of 4 bead load vs. substrate deformation.

<table>
<thead>
<tr>
<th>Standard Deviation (nm)</th>
<th>Mean Substrate Deformation (nm)</th>
<th>Peak Deformation (nm)</th>
<th>Position (Pixels)</th>
<th>Number of frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.27</td>
<td>-0.12</td>
<td>14.02</td>
<td>[699, 317]</td>
<td>113</td>
</tr>
</tbody>
</table>

Figure 6-14: Substrate deformation under 4 bead load.

Data folder: 19OCT05_D.

(a). Noise level in the region [0, 0, 50, 50]. (b). Deformation mesh.

(3). 6 bead load (120.05 mN) vs. substrate deformation.

Table 6-10: Experimental results of 6 bead load vs. substrate deformation.

<table>
<thead>
<tr>
<th>Standard Deviation (nm)</th>
<th>Mean Substrate Deformation (nm)</th>
<th>Peak Deformation (nm)</th>
<th>Position (Pixels)</th>
<th>Number of frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.24</td>
<td>-0.20</td>
<td>23.31</td>
<td>[698, 319]</td>
<td>112</td>
</tr>
</tbody>
</table>

Figure 6-15: Substrate deformation under 6 bead load.

Data folder: 19OCT05_D.

(a). Noise level in the region [0, 0, 50, 50]. (b). Deformation mesh.

(4). 8 bead load (160.06 mN) vs. substrate deformation.

Table 6-11: Experimental results of 8 bead load vs. substrate deformation.

<table>
<thead>
<tr>
<th>Standard Deviation (nm)</th>
<th>Mean Substrate Deformation (nm)</th>
<th>Peak Deformation (nm)</th>
<th>Position (Pixels)</th>
<th>Number of frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.23</td>
<td>30.99</td>
<td>[698, 319]</td>
<td>113</td>
</tr>
</tbody>
</table>
Experimental system calibration

Figure 6-16: Substrate deformation under 8 bead load.

Data folder: 19OCT05_D.
(a). Noise level in the region [0, 0, 50, 50]. (b). Deformation mesh.

(5). 10 bead load (200.08 mN) vs. substrate deformation.

<table>
<thead>
<tr>
<th>Standard Deviation (nm)</th>
<th>Mean Substrate Deformation (nm)</th>
<th>Peak Deformation (nm)</th>
<th>Position (Pixels)</th>
<th>Number of frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.29</td>
<td>0.31</td>
<td>39.86</td>
<td>[698, 319]</td>
<td>113</td>
</tr>
</tbody>
</table>

Figure 6-17: Substrate deformation under 10 bead load.

Data folder: 19OCT05_D.
(a). Noise level in the region [0, 0, 50, 50]. (b). Deformation mesh.

(6). 12 bead load (240.09 mN) vs. substrate deformation.

<table>
<thead>
<tr>
<th>Standard Deviation (nm)</th>
<th>Mean Substrate Deformation (nm)</th>
<th>Peak Deformation (nm)</th>
<th>Position (Pixels)</th>
<th>Number of frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.27</td>
<td>0.36</td>
<td>49.82</td>
<td>[698, 319]</td>
<td>113</td>
</tr>
</tbody>
</table>
Experimental system calibration

Figure 6-18: Substrate deformation under 12 bead load.

Data folder: 190CT05_D.

(a). Noise level in the region [0, 0, 50, 50]. (b). Deformation mesh.

(7). 14 bead load (280.11 mN) vs. substrate deformation.

Table 6-14: Experimental results of 14 bead load vs. substrate deformation.

<table>
<thead>
<tr>
<th>Standard Deviation (nm)</th>
<th>Mean Substrate Deformation (nm)</th>
<th>Peak Deformation (nm)</th>
<th>Position (Pixels)</th>
<th>Number of frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.29</td>
<td>0.42</td>
<td>63.31</td>
<td>[699, 320]</td>
<td>113</td>
</tr>
</tbody>
</table>

Figure 6-19: Substrate deformation under 14 bead load.

Data folder: 190CT05_D.

(a). Noise level in the region [0, 0, 50, 50]. (b). Deformation map (c). Deformation mesh.

(8). 16 bead load (320.12 mN) vs. substrate deformation.

Table 6-15: Experimental results of 16 bead load vs. substrate deformation.

<table>
<thead>
<tr>
<th>Standard Deviation (nm)</th>
<th>Mean Substrate Deformation (nm)</th>
<th>Peak Deformation (nm)</th>
<th>Position (Pixels)</th>
<th>Number of frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.29</td>
<td>0.50</td>
<td>73.88</td>
<td>[699, 320]</td>
<td>112</td>
</tr>
</tbody>
</table>
Experimental system calibration

Data folder: 19OCT05_D.

(a). Noise level in the region [0, 0, 50, 50]. (b). Deformation mesh.

The experiment is repeated five times at each location. The calibration results of substrate deformation vs. bead loads are shown in the Figure 6-21, the bars at the calibration points represent the standard deviation of the error. From Figure 6-21, we can find that the maximum normalized substrate deformation error in the whole scope of loading (16 beads) is +1.4 to -1.7 beads.

We use a 2nd order least square fit to determine the calibration curve [128], e.g.

\[ W = a \cdot d^2 + b \cdot d + c \]  \hspace{1cm} (6-11)

The coefficients are found to be \( a = 0.000425 \), \( b = 0.215812 \), \( c = 0.009163 \), and \( d \) is the substrate deformation and \( W \) the number of bead loads. The standard deviation error between the calibration curve and the normalized substrate
Experimental system calibration
deformations is ±0.6 beads. Therefore, the accuracy of the measurement system is ±3.42%.

6.3.1.4 Calibration 2: large load vs. substrate deformation

The load calibration system has been discussed in Section 5.4, and shown in Figure 5-13 (a). Initially, an indentor is put on the coated surface of the substrate; then the other cylinder load (170.96mN) in the column is put on the top of another using the rig shown in Figure 5-13 (b). The maximum applied load is 1025.8mN.
Figure 6.22: Bead load vs. substrate deformation.

Data Folder: Calib_Cylind_T6.

(a) BL1 (49.76nm). (b) BL2 (87.30nm). (c) BL3 (110.12nm). (d) BL4 (142.23nm).

(e) BL5 (163.91nm). (f) BL6 (188.01nm).

(g) Bead load vs. substrate deformation. (h) Noise level in the deformation map.
In Figure 6-22 (a) to Figure 6-22 (f), the average substrate deformation due to the cylinder load is 29.67nm. The maximum standard deviation error observed was 0.36nm. The maximum substrate deformation was 188.01nm, obtained under the weight of 6 cylinder loads.

![Graph showing substrate deformation vs. bead load](image)

**Figure 6-23: Calibration of substrate deformation vs. bead load.**

Data folder: Calib_Cylind, T1, T2, T3, T4, T5, T6.

Again we find that the maximum normalized substrate deformation error in the whole scope of loading (6 cylinder loads) is from +3.8 to -4.9 beads.

We use a 2\(^{nd}\) order least square fit to determine the calibration coefficients assuming the form:

\[ W = a \cdot d^2 + b \cdot d + c, \tag{6-12} \]

where \( d \) is the substrate deformation, \( W \) the number of bead loads. The coefficients were found to be \( a = 0.000638 \), \( b = 0.171909 \) and \( c = -0.090856 \). The standard deviation error between the calibration curve and the normalized substrate deformations is ±2.3 beads. Therefore, the accuracy of the measurement system is ±3.95%.
Figure 6-24: Comparison of both calibrations of substrate deformation vs. bead load.

(a). Comparison of both calibrations of substrate deformation vs. bead load.
(b). Comparison of fitted functions determined from (a).

From Figure 6-24, we can find that the difference between Calib-1 and Calib-2 is rather small, and the maximum error between them is \( \sim 3.2 \) beads, which is within the limitation of the system. We use Calib-1 as our standard calibration curve due its greater accuracy at smaller bead loads.
6.4 Twin image [172]

A phenomenon that we have observed is the production of a ‘ghost’ image in the measured deformation fields and is caused by multi-reflection in the Fizeau interferometer [172].

If we consider only the fundamental frequency and the second harmonics in the spatial domain, there are three interfering beams of reflected light, from the surfaces \( L_1, \ L_2 \) coated with aluminium of intensity reflection coefficient \( r^2 \approx 0.95 \) with relative intensities \( I_{R1} \approx 0.04 \cdot I_{IN} \), \( I_{R2} \approx 0.876 \cdot I_{IN} \), and \( I_{R3} \approx 0.033 \cdot I_{IN} \) (Figure 3-10), where \( I_{IN} \) is the incident light intensity. Therefore there are three interference signals, with intensities given by

\[
I_{12} = I_{R1} + I_{R2} + 2\sqrt{I_{R1} \cdot I_{R2}} \cdot \cos[\phi_{R12}(t)]
\]

\[
I_{23} = I_{R2} + I_{R3} + 2\sqrt{I_{R2} \cdot I_{R3}} \cdot \cos[\phi_{R23}(t)]
\]

\[
I_{13} = I_{R1} + I_{R3} + 2\sqrt{I_{R1} \cdot I_{R3}} \cdot \cos[\phi_{R13}(t)]
\]

where \( \phi_{R12}(t), \ \phi_{R23}(t), \ \phi_{R13}(t) \) are the phase differences between \( I_{R1} \) and \( I_{R2} \), \( I_{R2} \) and \( I_{R3} \), \( I_{R1} \) and \( I_{R3} \) respectively.

A properly designed phase-shifting algorithm will remove the higher harmonic interference signal \( I_{13} \). However, in addition to the desired interference signal \( I_{12} \) there is a second signal \( I_{23} \) that is also modulated at the fundamental frequency and appears in the measured deformation fields as a ‘twin image’. It seems likely that this observation is related to the occurrence of ‘hotspots’ previously reported (but not explained) in other studies of Fizeau interferometers [132].

Figure 6-25 shows the calculated deformation field at the maximum load of 66.93 mN in which both the real and the ‘ghost’ indentations are clearly visible (using a perspex optical wedge rather than the glass wedge). The peak deformation of the true indentation is 363 nm whereas that of the fictitious peak deformation is 28 nm, i.e. 7.7% of the peak real deformation.
In our experiments we were able to spatially separate the ghost from the true image of the indentation by arranging for the object beam to illuminate the wedge at a slight angle of approximately 2.15° from the normal. However, for other optical configurations using wavelength tuning this may not be possible and the resultant errors should be borne in mind when using this technique. No phase-shifting algorithm will be able to remove this contribution since it modulates at the same frequency as that of the carrier.

![Deformation field and twin image.](image)

**Figure 6-25: Deformation field and twin image.**

### 6.5 Summary

In this chapter, we introduced and compared the performance of different calibration methods of phase-shifting values vs. PZT voltages and found that phase-shifting evaluated by Chirp-Z Transform was the most effective method for phase acquisition. Subsequent calibrations showed that the accuracy of the measurement system is ±3.95% with a maximum equivalent load of 52 beads.

A phenomenon we call a 'ghost' imaging was observed in the measured deformation fields. Inspection of the data showed that the most likely cause was through the multi-reflection of rays in the Fizeau interferometer. In our experiments we were
able to spatially separate the ghost from the actual image of the indentation and obtain a true measure of the deformation.

The choice of the phase acquisition method and avoiding of any systematic errors caused by the ghost image have enabled us to construct a methodology that is accurate, wholefield and repeatable. As we will discuss in the subsequent chapters, this development is important since it will allow us to measure the deformations due to small changes in particle contact forces and to study the propagation of forces within granular packs.
7 Measurement of the mechanical properties of granular packs

In the following sections, we will present measurements of mechanical properties of 3D granular media acquired using phase-shifting interferometry (Section 4.3). Firstly, we show the force distributions at the bottom of small conical granular packs, and then consider properties of the contact forces in deep granular packs. Subsequently we look at the response of the forces to the local disturbing forces, before finally the particle displacements at the boundaries will be presented in detail.

7.1 Conical granular piles

Our first experiments employed a conical granular pile. This served initially as a test of the experimental technique, since the weight of the pile is known and wall effects could be avoided. Subsequently, we were able to use the method to perform a preliminary investigation on the force distributions at the base of granular piles.

7.1.1 Validation of the methodology

Our first step was to design an experiment to check whether \( F_c \), the sum of the measured forces obtained through experiment, is equivalent to the weight of the particles on top of the optical wedge. To test this we measured the total contact force at the base of a conical pile of particles built up on top of the optical wedge. The contact forces were measured using the methods described in chapter 6. \( F_c \) can be calculated from the individual contact forces using

\[
F_c = \frac{N}{S} \sum_{i=1}^{N} F(r_i) \cdot 2\pi r_i ,
\]

where \( F(r_i) \) is the indentation force at the point \( r_i \), where \( r \) is the vector pointing to the \( i^{th} \) particle's location; \( S \) is the area of the field of view; \( N \) is the number of indentation points in the field of view (see Figure 7-1).
The experiments were repeated 15 times. The total number of beads used in the experiments was 1519, whilst the total number of indentation points was 371. For each experiment, there are 20 to 30 indentation points. The average sum of the measured force is evaluated to be 1405 beads with a standard deviation error of 230 beads, leading to an error of $\pm 15.1\%$.

Figure 7-1: Conical pile and indentation points.

7.1.2 Deformation maps
Figure 7-2: Phase and deformation maps.

(a) Wrapped phase map without loading (BL0). (b) Wrapped phase with the pile (BL1).
(c) Wrapped phase difference map. (d) Deformation map.

Figure 7-2 shows the wrapped phase map and deformation map at the bottom of a granular pile. From Figure 7-2 (c), we can find that the deformation of the substrate before and after the loading is smaller than the laser wavelength (635.12nm). In Figure 7-3 (a), we can see that there are two indentation points.

The 15 deformation maps for one group of experiments (GMSelfM_27Jan06_T17) are shown in the Figure 7-3. In each experiment, the translation stage scans 15 images, which correspond to Figure 7-3(c) to (p). In the Figure 7-3 (a), (b), the two indentation points are pointed out and their background noise is in the range -2nm to 6nm.
Measurement of the mechanical properties of granular pack.
Measurement of the mechanical properties of granular pack

Figure 7-3: The 15 deformation maps for one group of experiments on a conical granular pile.

After all of the 15 Images are combined together, the 3D deformation map and indentation regions are shown in the Figure 7-4. In Figure 7-4 (b), it should be noted that the force distribution is along the axis of the optical substrate, \( y \), and the origin of coordinates \((0,0)\) is the geometric centre of the cylindrical granular pack and under the outlet of the hopper used to construct the granular pack.

### 7.1.3 Force distribution at the bottom of a conical granular pack

Having accumulated the data, we show in Figure 7-5 (a) and (b) the 3D deformation and indentation point maps for all of data. The largest normalized indentation force is less than 25 beads.
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Figure 7-6 (a) shows the indentation force distribution along the axis of the optical substrate, where the distances have been normalized by the bead diameter. Figure 7-6 (b) shows the coarse grained forces and its error bars. As expected the average load increases as we move closer to the axis, but there is some suggestion of a dip in the force distribution where normalized distance is near to -1. Further work was not performed on the pile, and the following sections discuss granular packs rather than piles.

(a) Force distribution. (b) Mean force distribution every 1.5 normalized distance along the axis of the optical substrate.

x: Normalized distance along the axis of the optical substrate in units of particle diameter (8mm).
y: Indentation force, in units of mg (20.01 mN).

7.2 Granular packs

7.2.1 Introduction

As we have discussed, a key test of the available theories of granular stress propagation is contact force distribution and the system response to point loads. When a load is applied to the top surface of the granular pack, the particles transmit their loads to particles below them. The particles resting on the bottom surface transmit this extra load to the support, and it is this change in contact force and position that we propose to measure.

We used three applied forces, of magnitude $L_1 = 4.99 \text{N}$, $L_2 = 7.89 \text{N}$ and $L_3 = 17.46 \text{N}$, (equivalent to 249.4 beads, 394.3 beads and 872.6 beads respectively), placed onto the centre of the top surface of the granular packs. We used steel balls (AISI 52100 Low
Alloy Chrome Steel) of diameter $d_b = 8\text{mm}$ and mass $mg = 20.01\text{mN}$ as our granular medium. In our experiment, the beads were placed within a cylinder of diameter $424\text{mm}$, and height $130\text{mm}$, equivalent to a diameter of $53$ beads and height $16$ beads respectively. A viewing slot, in which the optical wedge resides, was cut into the bottom surface. The whole container is supported over the interferometer, allowing access to the illuminating light. The whole interferometer on the translation stage scans independently underneath the container. The packs were added by pouring beads through a hopper, with its outlet about $2\text{cm}$ above the packs to limit impact damage during deposition. The pack surface was then “flattened” by removing the excess material without disturbing other particles. After deposition was completed, the height of the granular pack was equivalent to $11$ packed layers of beads.

We consider two datasets of Experiment 1 and Experiment 2. Each dataset corresponds to a combination of up to $15$ groups of trials. At each trial, a new granular pack is rebuilt independently. Each trial consists of loaded and the unloaded states when different local disturbing forces are applied to the centre of the top surface of the packs. For each state, the interferometer is scanned across the field of view ($160\times10\text{mm}^2$) of the optical wedge in $15$ steps, and about $300$ contact points are obtained. The loads and the notation for each set of experiments are shown for clarity in Table 7-1.

<table>
<thead>
<tr>
<th>Sequence of Experiment</th>
<th>Experiment 1</th>
<th>Experiment 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Experimental Setup</td>
<td>Pack Only</td>
<td>Pack+ $L_1$</td>
</tr>
<tr>
<td>Symbol of Local Disturbing Force</td>
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<td>$L_1$</td>
</tr>
<tr>
<td>Local Disturbing Force (N)</td>
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<td>$4.99$</td>
</tr>
<tr>
<td>Symbol of Contact Force</td>
<td>$F_a$</td>
<td>$F_b$</td>
</tr>
<tr>
<td>Symbol of Response Force</td>
<td>$N/A$</td>
<td>$\Delta F_{BA}$</td>
</tr>
<tr>
<td>Symbol of Lateral Displacement</td>
<td>$N/A$</td>
<td>$D_{BA}$</td>
</tr>
</tbody>
</table>
7.2.2 Contact force distributions

Figure 7-7: Contact deformation distribution in the whole field of view of the optical wedge.

(a). Contact deformation distribution in the trial T16 of C (see Table 7-1).
(b). Map of contact points. Central points represent contact points, whilst the circles show the circumference of the particles.

Figure 7-7(a) shows the contact deformation distribution and contact point map of one loading state in C \ T16 (see Table 7-1). Figure 7-7(b) shows the location of the contact points in the x-y plane, with the particle cross-sections shown to guide the eye. The facility is able to pick out the majority of the particles within the field of view, but there are some regions where particles are unable to be located. There are two possible reasons for this behaviour. Firstly, some contact loads are less than the resolution of the measurement system (equivalent to the weight of 1.5 beads). Secondly, there are regions of the field of view that are obscured due to aberrations in the optical set up, and we are unable to determine any particle locations in these regions. These regions cover approximately (10%) of the field of view. Despite this, we estimate that we are able to determine the location of 90% of the particles.

7.2.3 Force correlations

In this section, we consider the correlations between the contact forces before the forces on the top surface are applied, $F_{pre}$, and those after, $F_{post}$. Figure 7-8(a) shows $F_{pre}$ in the unloaded state A versus $F_{post}$ in the loaded state B, for the applied load of
$L_1 = 249.4$ beads. We can see from the clustering of points around a gradient close to 1 that the contact forces are not modified significantly. When we increase the load to $L_2 = 872.6$ beads (state E), however, we see some changes (Figure 7-8(b)). The low magnitude contact forces are still, in general, clustered about a gradient $\sim 1$, suggesting that in this range of contact forces, the addition of the localized forces at the surface, results in no major change in behaviour. For larger contact forces, however, the contact forces are now clustered about a gradient noticeably larger than 1. Additionally, we see that there are points that are close to the axes, suggesting a switching from low force to high forces, reminiscent of what might occur if a particle suddenly becomes incorporated into a force chain or removed from one.

![Diagram](a)
7.2.4 Contact force probability distribution

A contact force probability distribution was built up by combining the 15 repetitions for each experiment and binning the measured contact forces. Figure 7-9(a) shows the contact force probability distribution $P(f)$, where the contact force $F$ has been normalized by the global mean value $<F>$ for 5 sets of data (A to E, see Table 7-1), respectively. The distributions are similar and show very little deviation from each other, within the scatter of the data. This similarity suggests that we are justified in combining all the data into one figure to determine an average behaviour (see Figure 7-9(b)).

Previous work in this area has suggested that we should expect an exponentially decaying tail with the decay constant close to 1 [36-37, 62-64, 70, 72]. By examining the region where $f > 1$, we determined that this was broadly true for our system. A fit of the form

$$f(a, b) = \frac{a}{b} e^{-\frac{f}{b}}$$
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\[ P(f) \propto \exp(-\beta \cdot f) \]  

was found to agree closely with the experiment, where the decay constant was found to be \( \beta = 1.16 \pm 0.11 \), and the intercept with the coordinate axis to be \( P(f) = 0.69 \). Mueth [62] proposed a modified form of this distribution

\[ P(f) = a \cdot (1 - b \cdot e^{-f^2}) \cdot e^{-\beta f}, \]  

where \( a = 3 \), \( b = 0.75 \) and \( \beta = 1.5 \). A fit of this form to the experimental data can be seen in Figure 7-9(b), which shows reasonable agreement at low \( f \), and is able to capture the long tail behaviour within the tolerance as well.

Although there is some consensus on what the form of the tail of the contact force probability distribution should look like, there is still considerable debate regarding to its behaviour at small forces. Some theoretical predictions show that a peak is expected to appear in the region around \( f = 1 \) [18, 21, 46-47]. There are other suggestions, however, that the peak disappears even when the pack of hard particles is jammed [15]. In our experiments we were unable to resolve any maximum in the force probability distribution (Figure 7-9(b)), and find that there is a plateau as the forces approach zero, suggesting that the pack is close to the jammed state.
Figure 7.9: Probability distributions of the normalized contact forces.

(a). Normalized contact force probability distributions of sets A to E, respectively.
(b). Combined normalized contact force probability distributions on semi-log axes.

\[ f = \frac{F}{<F>} \]

is the normalized contact force.

### 7.2.5 Fraction of load borne by intervals of contact force

In granular media, the load is supported by the propagation of stresses through the contacts. On a microscopic, single particle scale, this propagation is often thought of as being through chains, where a substantial amount of load is focussed and channelled through the chain. If one considers that the large contact forces observed correspond to the end points of stress chains, and the small contact forces correspond to grains in the matrix, one can gain insight into the fraction of load borne by the chains and that borne by the matrix by considering the distribution of the fraction of load, \( \Phi(f) \), calculated using

\[
\Phi(f) = f \cdot P(f),
\]

where \( f = \frac{F}{<F>} \), is the contact force normalized by the mean contact force across the field of view. Since \( P(f) \) does not appear to be strongly dependent on the
applied load, we combine the data for all the experiments in Table 7-1, and show $\Phi(f)$ in Figure 7-10. The modal value of $\Phi(f)$ is around $f = 1$ and the mean value, as should be expected, is at $f = 1$. This means that most of the load is sustained by particles around the mean contact force, and that the larger contact forces, potentially those in force chains, support a proportionately low amount of the load.

![Figure 7-10: Combined normalized contact force fraction.](image)

$f = F / \langle F \rangle$ is the normalized contact force. The normalized contact force fraction is given by $\Phi = f \cdot P(f)$, where $P(f)$ is the probability density function for $f$.

### 7.2.6 Coarse grained response forces and displacements in the spatial domain

The granular pack can be observed to respond to a localized force on the upper surface by changes in the contact force distribution, and by changes in the location of the contact points. Many researchers report that the response function to an applied load is a double peak at the base, but Goldenberg [27-30] has shown using simulations that there is a crossover from a single-peaked to a two-peaked response forces as the applied load is increased. Geng [72] also reported that the mean forces propagate along double-peaked for packings with weak disorder, as the disorder increases, the two propagation directions merge into a single peak. In our experiments
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we examine the coarse-grained response functions to determine the mode of transmission of the stresses.

Figure 7-11(a) shows the response forces \( \Delta F_{BA} = F_B - F_A \), \( \Delta F_{CA} = F_C - F_A \) and \( \Delta F_{ED} = F_E - F_D \) to the three local disturbing forces, \( L_1 \), \( L_2 \) and \( L_3 \) respectively. Figure 7-11(b) shows the coarse grained absolute values of the responses \( |\Delta F_{BA}| \), \( |\Delta F_{CA}| \) and \( |\Delta F_{ED}| \). For small local disturbing forces we see a small increase in the contact force distribution. As we increase the load to \( L_3 \), we see signs of a new behaviour emerging; there is a suggestion of a dip at the centre in line with the predictions of double peaks [16-17, 23-24, 27-30].

Although granular packs are often modelled as being infinitely stiff, in reality, the application of a load to the upper surface results in small displacements in the particle positions. Our experimental facility is able to detect displacement of the particle contact points.

The coarse-grained responses of the lateral displacements of the beads in the base layer of the granular packs are shown in Figure 7-11(c) for each of the applied loads using,

\[
D_{BA} = \sqrt{(l_6^x - l_6^a)^2 + (l_7^x - l_7^a)^2},
\]

\[
D_{CA} = \sqrt{(l_6^x - l_6^a)^2 + (l_7^x - l_7^a)^2},
\]

\[
D_{ED} = \sqrt{(l_6^x - l_6^b)^2 + (l_7^x - l_7^b)^2},
\]

(7-5)

where \( l_6 \) and \( l_7 \) are given in Eq. (6-9); \( D_{BA}, D_{CA} \) and \( D_{ED} \) are the displacements of particles for applied loads (see Table 7-1).

It is interesting to compare the spatial variation of the displacements with that of the response forces. Figure 7-11(d) to (f) show the coarse grained lateral displacements and response forces as a function of position. For the smallest applied load, the envelope of the distribution of lateral displacements and response forces along the axis of the optical substrate is in the form of a small peak at the centre. As the load is
increased we see evidence of the double peaks in the contact forces, but we observe dips in the lateral displacement distribution. (see Figure 7-11(b) to (c)). This behaviour is repeated when we consider the absolute values of the contact forces and lateral displacements (see Figure 7-11(d) to (f)). Despite this suggestion of an inverse relationship between changes in the contact forces and the lateral displacements, the error bars associated with these measurements were found to be too large to make a clear correlation.

The response force probability distributions can be calculated by binning the changes in contact force following the application of the point load at the upper surface [18, 21, 27, 46-47, 72]. Figure 7-12(a) shows the response distributions for all three applied loads, where the change in contact force has been normalised by the applied load \( f=\Delta F/L \). Using this normalization procedure we see that the distributions lay close to one another, suggesting that the response of the granular medium is scaling with the applied load. From these distributions we can see clearly the positive and negative responses to the applied load; some contact forces have been reduced even though the applied load is compressive. The curves show asymmetries, with the probability of finding a positive response much greater than finding a negative one, with the modal value being close to zero.
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(b)

(c)

(d)
Figure 7-11: Coarse grained mean response forces and lateral displacements.

(a). Coarse grained mean response forces for three applied loads \( L_1, L_2, L_3 \), given in Table 7-1.
(b). Coarse grained mean absolute values of response forces for the three disturbing forces \( L_1, L_2, L_3 \).
(c). Coarse grained mean lateral displacements for the three local disturbing forces \( L_1, L_2, L_3 \).
(d). Coarse grained mean absolute values of response forces and lateral displacements for the local disturbing force \( L_1 \).
(e). Coarse grained mean absolute values of response forces and lateral displacements for the local disturbing force \( L_2 \).
(f). Coarse grained mean absolute values of response forces and lateral displacements for the local disturbing force \( L_3 \).
We define $\Delta F$ as the response force; $D$ is the lateral displacements, $y$ is coordinate along the principal axis of optical wedge, shown in the Figure 7-7(b) and $d_B$ is the diameter of beads (8mm).

7.2.7 Response force probability distribution

The response probability distribution, combined for all three applied loads, is shown in the Figure 7-12(b). The peak of the distribution is located at $f \sim 0$. Its shape is a rather distinctive sharp peak with a slower fall off in the tails. In the region $0 < f < 0.04$,

$$P(f) \propto \exp(-\beta_1 \cdot f),$$  \hspace{1cm} (7-6)

where $\beta_1 = 125 \pm 16$; in the region $-0.02 < f < 0$,

$$P(f) \propto \exp(\beta_2 \cdot f),$$  \hspace{1cm} (7-7)

where $\beta_2 = 273 \pm 7$.
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Figure 7-12: Probability distributions of the response forces.

(a). Probability distributions of the response forces, normalized by the local disturbing forces $L_1$, $L_2$, and $L_3$ (given in Table 7-1).

(b). Combined probability distribution of the response forces for all three sets ($L_1$, $L_2$, and $L_3$).

(c). Combined probability distribution of the absolute values of the response forces all three sets ($L_1$, $L_2$, and $L_3$).
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Figure 7-12(c) shows the distribution of the absolute values of the force where we have plotted \( \ln|\Delta F/L| \) vs. \( \ln(P|\Delta F/L|) \). The linear relationship suggests that it obeys a power law,

\[
P = P_0 \cdot |f|^\alpha.
\]  

A first order regression is also plotted (line without symbols) with the coefficients determined as \( P_0 = 8.34 \times 10^{-4} \) and \( \alpha = -2.22 \).

7.2.8 Displacement probability distributions

The probability distributions of lateral displacements of the beads are shown in Figure 7-13(a) and (b), where we have used the following relation to obtain a non-dimensional displacement, \( \zeta = (D/d_a)/(L/mg) \). In the region \( 0 < \zeta < 2 \times 10^{-5} \), there are few differences between the displacement distributions and it appears that their properties scale with the applied disturbing forces. After recompiling all of the data into one set, the normalized lateral displacement probability distribution is shown in Figure 7-13(a), (b) and appears to be described by the following form,

\[
P(\zeta) = P_0 \cdot (\zeta)^\alpha, \tag{7-9}
\]

where the coefficients were found, by regression, to be \( P_0 \approx 2.59 \times 10^{-9} \) and \( \alpha \approx -2.48 \).

![Displacement probability distributions](image)

(a)
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Figure 7-13: Probability distributions of lateral displacements.

(a) Probability distributions of lateral displacements, for each of the sets for $L_1$, $L_2$ and $L_3$ where $\zeta = (D/d_a)/(L/mg)$.
(b) Probability distributions of lateral displacements, for each of the sets for $L_1$, $L_2$ and $L_3$ where $\zeta = (D/d_a)/(L/mg)$, on semi-log axes.
(b) Probability distributions of lateral displacements, for each of the sets for $L_1$, $L_2$ and $L_3$ where $\zeta = (D/d_a)/(L/mg)$, on log-log axes.

7.2.9 Relations between response forces and displacements

In order to investigate the relation between response forces and lateral displacements, we renormalize the absolute value of the normalized response forces ($|\Delta F/L|$) and lateral displacements ($D/(d_a)/(L/mg)$), by their maximum values respectively.
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Their probability distributions are shown in Figure 7-14. We found that the distributions were similar to each other and obey a power law of the form,

$$P(\xi) = P_0 \cdot (\xi)^\alpha,$$

(7-10)

where the following parameters were found through regression, $P_0 = 0.013$, $\alpha \approx -2.22$ ($\xi = |\Delta F/L|/\max(|\Delta F/L|)$ for the response forces and $\xi = [(D/d_a)/(L/mg)]/\max([(D/d_a)/(L/mg)])$ for the displacements).

Figure 7-14: Probability distributions of renormalized absolute values of normalized response forces and lateral displacements on a log-log plot, where $\xi = |\Delta F/L|/\max(|\Delta F/L|)$ for response forces and $\xi = [(D/d_a)/(L/mg)]/\max([(D/d_a)/(L/mg)])$ for displacements.

Figure 7-15(a) shows that normalized response forces are dependent on the applied loads at the boundary of a granular pack. However, normalized displacements are quite different. Figure 7-15(b) shows that most particle displacements decrease with applied load (which leads to more force chains and jamming of the granular pack) but that the tails at large displacement increase fast with the applied loads (associated with unjamming of the granular pack). From Figure 7-15(c), we can find that the maximum values of normalized displacements, $v = \max([(D/d_a)/(L/mg)])$, increase linearly with the applied loads. This phenomenon reveals that the lateral displacements are the signature of micro-deformations or the volume fraction changes.
of the granular media at the microscopic level and potentially play a more important role than response forces. That the maximum displacement increases faster than the maximum response force when applied load increases suggests that particle displacements are responsible for the unjamming of the granular pack under a relatively large local disturbing force.

(a)

(b)
7.2.10 Discussion

A network of non-cohesive rigid grains with too few contacts is flexible; while if it has the minimum number required to be rigid it is called isostatic or marginally rigid; while if there are more contacts it is stress overconstrained. There have been many theoretical predictions [157-161], but to date, only limited experimental validation is available [86].

Moukarzel [86, 157] reported numerical simulations, with similar results to our own experimental results, which show that if an infinitesimal change in the length of a randomly chosen bond is introduced into 2D granular media, contact forces between particles take exponentially large values. They predict that the probability distribution of their displacements obeys a power law and the probability distribution of the induced vertical displacement and response show very similar characteristics, results
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that we have replicated to some extent. Although our results are close to most of their predictions, there are some differences such as: (1). We observe that the negative response forces decay faster than the positive ones, but theory predicts that they should be symmetrical around zero. (2). Instead of vertical response force and displacement, we measure the vertical force responses and lateral displacements in the bottom of granular packs.

Normalized contact force probability distribution in the bottom of granular pack from our experiments shows no peak around \( f = 1 \), a result similar to the reports of Makse [15] and Mueth [62]. A consequence of this is that, similar to the situation in solid mechanics, the relation between ‘stress’ and ‘strain’ in a granular media can be expressed by a phase transition diagram similar to the presented by Liu (see Figure 1.2) [7], which is used to describe macroscopic behaviour. According to our experimental results and isostatic theory, it is reasonable to assume that there are two concentric spheres in Liu’s diagram; the outer one corresponds to the phase transition states between the fluid and the isostatic state, (which is characterized by no peak in the contact force probability distribution). The inside one represents the boundary between the isostaticity (weak jamming) and hyperstaticity (global jamming) (characterized by a small peak in the contact force probability distribution \( f = 1 \)), although it is still an open question how the probability distribution changes from no peak in weak jamming to a single peak around \( f = 1 \) in global jamming. One of the possible reasons is that whether or not there is a peak around \( f = 1 \) is determined by how large the particle displacements come about before and after the perturbing force is applied to the boundary of granular packs. This is supported in part by Majmudar and Behringer’s [76] experiments; the granular pack is forced to undergo large volume fraction changes, which allows significant particle displacements, and consequently the probability distributions has a peak at \( f = 1 \) and a power law tail.

In many cases, the volume fraction doesn’t change significantly and it is difficult to measure; especially in isostatic granular media since the displacements of hard particles are small. In these situations, it is better to use the differential form of Liu’s diagram with two new microscopic parameters: the responses to the local disturbing force and the particle displacements.

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7.2.11 Conclusion

In order to research the mechanical behaviour of granular packs, a granular pack of about 20,000 beads with an overall height equivalent to an 11 layer FCC pack was built inside a cylindrical container using a hopper feed system, and a range of forces were applied to its top surface. The indentations of beads on the substrate in the bottom of the pack were measured by a phase-shifting Fizeau interferometer along a 20 bead x 2 bead viewing window. After data processing, the force distributions and bead reorganization were determined. We found that,

1. The probability distributions of normalized contact forces and response forces are independent of the values of local disturbing forces.

2. In the region where \( f > 1 \), normalized contact force probability distribution has an exponential tail of the form \( P(f) \propto \exp(-\beta \cdot f) \), where the coefficient was found to be \( \beta = 1.16 \pm 0.11 \). It increases with decreasing applied load with \( \lim_{f \to 0} P(f) \approx 0.69 \). We do not observe a peak in the normalized contact force probability distributions, but there is a peak around \( f = 1 \) in the normalized contact force fraction.

3. There are negative response forces (reductions in the contact force) and lateral displacements when forces are applied to the top surface, commonly recognized as a signature of a 'fragile material'.

4. The probability distribution of the normalized response forces decays from \( |f| = 0 \) with an exponential form. However, probability distribution of renormalized absolute values of response forces, and lateral displacements, are quite similar to each other. They are observed to have power law tails \( P(f) \propto P_0 \cdot (f)^{\alpha} \), where \( P_0 \approx 0.013 \) and \( \alpha \approx -2.22 \).

5. The maximum normalized lateral displacement increases linearly with applied load suggesting that particle displacements are responsible for jamming and unjamming of the granular pack.
7.3 Summary

The building up of a new high-resolution wavelength scanning interferometer has enabled mechanical properties of a granular pack consisting of 20000 steel balls to be investigated. Some key results were that the probability distributions of normalized contact force and response forces are seen to be independent of the values of local disturbing forces. The normalized contact force probability distribution has an exponential tail and instead of a peak in the normalized contact force probability distribution, there is one around $f = 1$ in the distribution of the fraction of load borne which is suggestive of fragility in the stress network. The increasing variation in the change of contact force with increasing applied load points towards reorganization within the granular pack and the inter-particle contacts and the stress chains being redistributed.
Multiple Surface Phase-shifting Interferometry for measuring force distribution in granular materials

In the last chapter, we use phase-shifting interferometry to measure the contact forces on the boundary of granular pack. In this chapter, we present a new depth-resolve scanning-wavelength interferometry, which can potentially be used to detect the position and contact deformation among the particles in the granular pack.

8.1 Introduction

For a long time, phase-shifting interferometry has been mainly confined to the measurement of the two-beam interference signals. Recently, the ability to measure internal displacement fields within a material or structure has become highly desirable in many fields, ranging from alignment of complex optical systems to non-destructive evaluation of composites. More and more attention is paid to the measurement of multiple surface cavities [163-166], with high accuracy of depth resolution.

When considering transparent multiple surface objects (e.g. optical lenses or flats), reflections from surfaces beyond the surface of interest occur but are normally regarded as a nuisance, and wavelength tuning combined with specially designed algorithms, multiple surface phase-shifting interferometry (MSPSI), has been developed to suppress the interference from other surfaces [109-120]. If $K$ surfaces are measured, there will be $K(K-1)/2$ first-order interference peaks in the frequency domain. Demonstrations of multiple surface measurements have been presented in Refs. [163-166]. These experiments were based on wavelength-scanning techniques, which separated interference signals from various surfaces in the frequency domain. Window functions in the frequency analysis were introduced to optimize for maximum tolerance due to material dispersion and scanning nonlinearity, as well as for suppression of noise from other frequency harmonics.

These techniques show that it is possible, in principle, to measure the deformation of multiple surfaces in depths down to the orders of several nanometres. After calibration of deformation vs. applied force, such a technique could in future be used
Multiple Surface Phase-shifting Interferometry for measuring force distribution in granular materials to measure the position and deformation of particles in a granular pack [1-3]. Therefore, the purpose of this chapter is to present the results from a proof-of-principle experiment based on an MSPSI approach to subsurface displacement field measurement. The work was published in Ref. [97].

8.2 System configuration of Multiple Surface Phase-shifting Interferometer

![System configuration](image)

Figure 8-1: System configuration of Multiple Surface Phase-shifting Interferometer.

(a). Schematics of system configuration. (b) Photo of the system.

The system configuration of MSPSI is shown in Figure 8-1, consisting of two independently tiltable transparent surfaces, $L_1$ and $L_2$. A third surface, $L_R$, provides the reference. The optical path difference between surfaces $L_R$ and $L_1$, and between $L_1$ and $L_2$ are approximately 51 and 20 mm, respectively. All three surfaces are glass-air interfaces of glass flats with thickness 5.1 mm and refractive index $n = 1.51$. Antireflection coatings were applied to one side of each plate to suppress
the reflection from the second glass–air interface. The light source used was an external-cavity diode laser (see Table 8-1). The beam is expanded by lens $OL$ and steered by mirrors $M_1$ and $M_2$ towards the collimating lens $FL$. The reflected light from the three glass–air interfaces $L_1$, $L_2$, $L_3$ is imaged by a high-speed camera (see Table 8-2), which records the resulting three-beam interference patterns.

Table 8-1: External cavity diode laser based on Littman-Metcalf design [96]

<table>
<thead>
<tr>
<th>Model</th>
<th>New Focus 6005 Vortex</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wavelength tunable range</td>
<td>0.0598 nm</td>
</tr>
<tr>
<td>Central wavelength</td>
<td>635.05 nm</td>
</tr>
<tr>
<td>PZT voltage range</td>
<td>-2V to 2V</td>
</tr>
<tr>
<td>PZT voltage range for image record</td>
<td>Positive Slope</td>
</tr>
<tr>
<td>Available frame sample time</td>
<td>1.0307 s</td>
</tr>
</tbody>
</table>

Table 8-2: CMOS digital camera details [167]

<table>
<thead>
<tr>
<th>Model</th>
<th>HCC-1000 Vosskuhler GmbH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resolution</td>
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</tr>
<tr>
<td>Crop region</td>
<td>256x256</td>
</tr>
<tr>
<td>Row</td>
<td>1</td>
</tr>
<tr>
<td>Column</td>
<td>596</td>
</tr>
<tr>
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<td>Hardware Trigger</td>
</tr>
<tr>
<td>Exposure time (Shuttle)</td>
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</tr>
<tr>
<td>Record speed</td>
<td>912.50 fps</td>
</tr>
<tr>
<td>Start sumber of image</td>
<td>1</td>
</tr>
<tr>
<td>Number of images</td>
<td>1824</td>
</tr>
<tr>
<td>Available number of images</td>
<td>940</td>
</tr>
<tr>
<td>Field of view</td>
<td>9.981 mm²</td>
</tr>
</tbody>
</table>
8.3 Theory

Suppose that the single longitude-mode laser diode controlled by an external voltage signal emits a beam of light with its wavelength linearly modulated. In this case the time dependent angular frequency will be given by [97, 168, 170]

\[ \omega(t) = \omega_0 + \alpha \cdot t , \]
\[ \alpha = \frac{\Delta \omega}{T} = -\frac{2\pi \cdot C}{\lambda_0^2 \cdot T} \cdot \Delta \lambda . \]  

In Eq.(8-1), \( \omega_0 = 2\pi \cdot C / \lambda_0 \) is the fundamental angular frequency of the light, \( \alpha \) is the coefficient of the frequency modulation. \( T \) is the period of the ramp signal, \( C \) is the light velocity in vacuo; \( \lambda_0 \) is the wavelength of the light. \( \Delta \lambda \) and \( \Delta \omega \) are the modulating amplitudes of wavelength and angular frequency, respectively.

As shown in Figure 8-2, the complex amplitude of the electric fields, \( E \), reflected by \( L_1 \), \( L_2 \), and \( L_R \) separately are:

\[ E_1(x, y) = \sqrt{I_1} \cdot \exp\{i[\omega_0 \cdot (t - \tau_{1R}) + \alpha \cdot (t - \tau_{1R})^2 + \varphi(t - \tau_{1R})]\} , \]
\[ \tau_{1R}(x, y) = \frac{2n \cdot d_{1R}(x, y)}{C} , \]
\[ E_2(x, y) = \sqrt{I_2} \cdot \exp\{i[\omega_0 \cdot (t - \tau_{2R}) + \alpha \cdot (t - \tau_{2R})^2 + \varphi(t - \tau_{2R})]\} , \]
\[ \tau_{2R}(x, y) = \frac{2n \cdot d_{2R}(x, y)}{C} , \]
\[ E_R(x, y) = \sqrt{I_R} \cdot \exp[i(\omega_0 \cdot t + \alpha \cdot t^2 + \varphi(t))] , \]  

(8-2)
where \((x, y)\) are spatial coordinates and are omitted for clarity in subsequent expressions. \(d_{1R}\) and \(d_{2R}\) are the distances between \(L_1\) and \(L_R\), and between \(L_2\) and \(L_R\), respectively. \(\varphi\) is the initial phase of the light; \(n\) is the refractive index in a medium; and \(\tau\) is the time delay. The interference signal can be written as follows,

\[
I = \langle \exp(i \cdot \Delta \varphi) \rangle \cdot \{I_R + I_1 + I_2 \\
+ 2\sqrt{I_1 I_2} \cos(\omega_0 \tau_{1R} + 2\alpha \cdot t \tau_{1R} - \alpha \cdot \tau_{1R}^2) + 2\sqrt{I_1 I_2} \cos(\omega_0 \tau_{2R} + 2\alpha \cdot t \tau_{2R} - \alpha \cdot \tau_{2R}^2) \\
+ 2\sqrt{I_1 I_2} \cos((\omega_0 \tau_{1R} + 2\alpha \cdot t \tau_{1R} - \alpha \cdot \tau_{1R}^2) - (\omega_0 \tau_{2R} + 2\alpha \cdot t \tau_{2R} - \alpha \cdot \tau_{2R}^2)) \}
\]

(8-3)

where \(\langle \rangle\) is a temporal average, \(\tau\) is given by \(\max(\tau_{1R}, \tau_{2R})\) and \(\Delta \varphi\) is the maximum initial phase difference of the light. If \(\tau < \tau_{coh}\), where \(\tau_{coh}\) is the coherence time of the light, \(\langle \exp(i \cdot \Delta \varphi) \rangle \approx 1\) and \(\alpha \cdot \tau^2 \approx 0\).

After removal of the DC part of the signal, the signal received by the camera is as follows,

\[
I(t) = [2\sqrt{I_1 I_2} \cos(\omega_{1R} \cdot t + \phi_{1R}) + 2\sqrt{I_1 I_2} \cos(\omega_{2R} \cdot t + \phi_{2R}) \\
+ 2\sqrt{I_1 I_2} \cos(\omega_{12} \cdot t + \phi_{12})] \cdot w(t)
\]

(8-4)

Suppose \(w(t)\) is a rectangular window function, then the Fourier Transform, \(F(\omega)\), of Eq.(8-4) can be written as

\[
F(\omega) =
4\pi \cdot \{\sqrt{I_1 \cdot I_1} \cdot \sin c\left(\frac{\omega - \omega_{1R}}{2}\right) \cdot e^{i\phi_{1R}} + \sin c\left(\frac{\omega + \omega_{1R}}{2}\right) \cdot e^{-i\phi_{1R}} \} \\
+ \sqrt{I_1 \cdot I_2} \sin c\left(\frac{\omega - \omega_{2R}}{2}\right) \cdot e^{i\phi_{2R}} + \sin c\left(\frac{\omega + \omega_{2R}}{2}\right) \cdot e^{-i\phi_{2R}} \} \\
+ \sqrt{I_1 \cdot I_2} \sin c\left(\frac{\omega - \omega_{12}}{2}\right) \cdot e^{i\phi_{12}} + \sin c\left(\frac{\omega + \omega_{12}}{2}\right) \cdot e^{-i\phi_{12}} \} \}
\]

(8-5)

where,

\[
\omega_{1R} = \alpha \cdot \tau_{1R}, \quad \phi_{1R} = \omega_0 \cdot \tau_{1R}, \\
\omega_{2R} = \alpha \cdot \tau_{2R}, \quad \phi_{2R} = \omega_0 \cdot \tau_{2R}, \\
\omega_{12} = \alpha \cdot \tau_{12}, \quad \phi_{12} = \omega_0 \cdot \tau_{12},
\]

(8-6)
Multiple Surface Phase-shifting Interferometry for measuring force distribution in granular materials

\[ \omega = \alpha \tau = \frac{4\pi \cdot n \cdot \Delta \lambda}{\lambda_0^2 \cdot T} \cdot d. \quad (8-7) \]

Eq.(8-7) reveals three peaks corresponding to each of the cosine terms, centered at frequencies \( \omega_{R1} \), \( \omega_{R2} \) and \( \omega_{12} \). Provided the distance between reference \( L_R \) and sample \( (L_1 \) and \( L_2 \)) is greater than the sample depth, the peaks of interest \( (\omega_{R1} \) and \( \omega_{R2} \)) are separated from the unwanted cross-interference peaks \( \omega_{12} \). The corresponding phases of the peaks of \( \omega_{12} \), \( \omega_{R1} \) and \( \omega_{R2} \) are:

\[ \phi_{12} = \omega_0 \cdot \tau_{12} = \frac{4\pi \cdot n}{\lambda_0} \cdot d_{12}, \]
\[ \phi_{1R} = \omega_0 \cdot \tau_{1R} = \frac{4\pi \cdot n}{\lambda_0} \cdot d_{1R}, \]
\[ \phi_{2R} = \omega_0 \cdot \tau_{2R} = \frac{4\pi \cdot n}{\lambda_0} \cdot d_{2R}. \quad (8-8) \]

Eq.(8-8) describes the wrapped phase map of the relative positions between \( L_1L_2 \), \( L_1L_R \) and \( L_2L_R \), respectively.

8.4 Experimental results

The purpose of the experiment is to prove that MSPSI method can be used to measure depth-resolved displacement fields within a 3D scattering medium. In this experiment, we observe the signal resulting from reflection from surfaces beyond the initial surface (see Figure 8-2), using the operating conditions shown in the Table 8-1, and 8-2, and the optical path differences between surfaces \( L_R \) and \( L_1 \), between \( L_1 \) and \( L_2 \) being approximately 51mm and 20mm, respectively. 1 pixel in the image represents 0.039 mm. The field of view (256 x 256 pixels) is 9.981 x 9.981 mm².

The following procedure was adopted.

1. The initial relative positions \( T_{11}^{L_1L_R} \) between surface \( L_1 \) and surface \( L_R \), \( T_{12}^{L_2L_R} \) between surface \( L_2 \) and surface \( L_R \), and \( T_{11}^{L_1L_2} \) between surface \( L_1 \) and surface \( L_2 \) are measured by MSPSI.
2. Next, after small independent tilts are introduced to surfaces $L_1$ and $L_2$, the relative positions between $L_1$ and $L_R$, between $L_2$ and $L_R$, and between $L_1$ and $L_2$ are denoted as $T_2^{L_1 R}$, $T_2^{L_2 R}$, and $T_2^{L_1 L_2}$, while $L_R$ is kept immobile.

3. Finally, the depth-resolved displacement fields of $L_1$ and $L_2$, or $\Delta T_{L_1}$ and $\Delta T_{L_2}$, are evaluated pixel by pixel using the expression

$$\Delta T_{L_1} = T_2^{L_1 R} - T_1^{L_1 R},$$
$$\Delta T_{L_2} = T_2^{L_2 R} - T_1^{L_2 R}.$$  (8-9)

8.4.1 Experiment T1: Before adjusting the tilt angle of $L_1$ and $L_2$

(a) Image of Frame 1800.
(b) Normalizing light intensity at pixel (256, 203), from Frame 860 to 1800.
Multiple Surface Phase-shifting Interferometry for measuring force distribution in granular materials

Figure 8-3 (a) shows a typical fringe pattern, while Figure 8-3 (b) shows the normalized interference signal measured from one pixel in the field of view in the time domain, before adjusting the tilt angle of $L_1$ and $L_2$.

8.4.1.1 Fourier transform with 4096 points Zero Padding [121-123]

Figure 8-4 and Table 8-3 show the frequency spectrum of $I(t)$ where the peaks of $L_1L_2$, $L_1L_R$ and $L_2L_R$ are present. Prior to the Fourier transform, the mean value of the intensity signal was subtracted and the signal multiplied with a rectangular Window with zero padding of 4096 points to increase the frequency resolution.

![Frequency Domain](image)

Figure 8-4: Amplitude in the frequency domain of Figure 8-3 (b).

Table 8-3: Frequency at the peaks

<table>
<thead>
<tr>
<th>Number</th>
<th>Peak</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$L_1L_2$</td>
<td>5.1211 Hz</td>
</tr>
<tr>
<td>2</td>
<td>$L_1L_R$</td>
<td>15.5859 Hz</td>
</tr>
<tr>
<td>3</td>
<td>$L_2L_R$</td>
<td>20.4844 Hz</td>
</tr>
</tbody>
</table>

Figure 8-5 to Figure 8-7 show the wrapped phase map and 3D map of initial relative positions between $L_1L_2$, $L_1L_R$ and $L_2L_R$, respectively, where the wrapped phases are evaluated pixel by pixel at the frequency of the peaks listed in the Table 8-3, and the 3D maps of initial relative positions are their unwrapping and rescaling results [139-140].
Multiple Surface Phase-shifting Interferometry for measuring force distribution in granular materials

Figure 8-5: Wrapped phase map and 3D map of initial relative position of $L_1$ and $L_2$.
(a). Wrapped phase map. (b). 3D map of initial relative position. (x, y): pixel $z$: μm.

Figure 8-6: Wrapped phase map and initial tilt of $L_1$ and $L_R$.
(a). Wrapped phase map. (b). 3D map of initial relative position. (x, y): pixel $z$: μm.

Figure 8-7: Wrapped phase map and initial tilt of $L_2$ and $L_R$.
(a). Wrapped phase map. (b). 3D map of initial relative position. (x, y): pixel $z$: μm.
8.4.1.2 1024 chirp Z transform with Hanning Window [121-123, 154]

Figure 8-8: Amplitude in frequency domain of Figure 8-3 (b).

Figure 8-8 and Table 8-4 show the frequency spectrum of $I(t)$ where the peaks of $L_1L_2$, $L_1L_R$ and $L_2L_R$ are present. Prior to the CZT transform, the mean value of the intensity signal was subtracted and the signal multiplied with a Hanning Window. Comparing Figure 8-4 with Figure 8-8, we can find that the peaks in the Figure 8-8 are smoother, and the phase values computed using a CZT transform are more accurate.

Table 8-4: Frequency at the peaks

<table>
<thead>
<tr>
<th>Number</th>
<th>Peak</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$L_1L_2$</td>
<td>5.1618 Hz</td>
</tr>
<tr>
<td>2</td>
<td>$L_1L_R$</td>
<td>15.4627 Hz</td>
</tr>
<tr>
<td>3</td>
<td>$L_2L_R$</td>
<td>20.6245 Hz</td>
</tr>
</tbody>
</table>

Figure 8-9 to Figure 8-11 give the wrapped phase maps and 3D maps of initial relative positions between $L_1L_2$, $L_1L_R$ and $L_2L_R$, respectively, where the wrapped phases are evaluated pixel by pixel at the frequency of the peaks listed in the Table 8-4, and the 3D maps of initial relative positions are found from unwrapping their phase maps.
Figure 8-9: Wrapped phase map and 3D map of initial relative position of \( L_f \) and \( L_l \).
(a). Wrapped phase map. (b). 3D map of initial relative position. (x, y): pixel \( z: \mu m \).

Figure 8-10: Wrapped phase map and 3D map of initial relative position of \( L_f \) and \( L_R \).
(a). Wrapped phase map. (b). 3D map of initial relative position. (x, y): pixel \( z: \mu m \).

Figure 8-11: Wrapped phase map and 3D map of initial relative position of \( L_f \) and \( L_R \).
(a). Wrapped phase map. (b). 3D map of initial relative position. (x, y): pixel \( z: \mu m \).
8.4.2 Experiment T2: After adjusting the tilt angle of $L_1$ and $L_2$

Figure 8-12 (a) shows one of the fringe patterns and Figure 8-12 (b) shows the normalized interference signal measured from one pixel in the field of view in the time domain, after adjusting the tilt angle of $L_1$ and $L_2$.

(a)

(b)

Figure 8-12: Interference signal.
(a). Image of Frame 1800. (b). Normalizing light intensity at pixel (256, 203), from Frame 860 to 1800.
Multiple Surface Phase-shifting Interferometry for measuring force distribution in granular materials

Figure 8-13: Amplitude in the frequency domain of Figure 8-12 (b).

<table>
<thead>
<tr>
<th>Number</th>
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<th>Frequency</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>2</td>
<td>L1L_R</td>
<td>15.5992Hz</td>
</tr>
<tr>
<td>3</td>
<td>L2L_R</td>
<td>20.8519Hz</td>
</tr>
</tbody>
</table>

Table 8-5: Frequency at the peaks

Figure 8-14 to Figure 8-16 gives the 3D maps of relative positions between L1L2, L1L_R and L2L_R, respectively, where the wrapped phases are evaluated pixel by pixel at the frequency of the peaks listed in Table 8-5, and the 3D maps are found from unwrapping and rescaling the phase maps.

Figure 8-14: 3D map of relative position between L1 and L2. (x, y): pixel z: μm.
Multiple Surface Phase-shifting Interferometry for measuring force distribution in granular materials

Figure 8-15: 3D map of relative position between $L_1$ and $L_R$ (x, y): pixel $z$: µm.

Figure 8-16: 3D map of relative position between $L_2$ and $L_R$ (x, y): pixel $z$: µm.

8.4.3 Displacement of $L_1L_2$, $L_1L_R$ and $L_2L_R$ between experiment T1 and T2

Figure 8-17: Wrapped phase map by 1024 point CZT.
(a). Wrapped phase difference of $L_1L_R$. (b). Wrapped phase difference of $L_2L_R$.
Black represents $-\pi$ radians and white $+\pi$ radians.
Figure 8-17 show the measured wrapped phase difference maps due to the movement of $L_1$ and $L_2$, respectively.

$$\phi = \text{Wrap} \left[ \text{Unwrap} \left( \phi_2^{L_1L_2} \right) - \text{Unwrap} \left( \phi_1^{L_1L_2} \right) \right], \quad (8-10)$$

where $\phi$ is the wrapped phase difference; $\phi_2^{L_1L_2}$ is the phase of $L_xL_R$ (where $x = 1$ or 2) before adjusting tilt; and $\phi_1^{L_1L_2}$ is the phase of $L_xL_R$ after adjusting tilt, both of which are evaluated from the peaks of the CZT. Each fringe represents an out of plane displacement $u_z = \lambda_0 / 2 = 317\text{nm}$.

After unwrapping and rescaling according to Eq.(8-8), the displacement maps of $L_1$ and $L_2$ are given in Figure 8-18 and Figure 8-19.

**Figure 8-18: Displacement of $L_1$.**

$(x, y):$ pixel $\quad z: \mu m. \ 1024 \text{ Chirp Z Transform with Hanning Window.}$

**Figure 8-19: Displacement of $L_2$.**

$(x, y):$ pixel $\quad z: \mu m. \ 1024 \text{ Chirp Z Transform with Hanning Window.}$
8.4.4 Validation of Tilts of L1, L2 by FT Method [169]

In order to validate the results, the tilts were measured independently using standard two-beam interferometry (FT method) at a fixed wavelength between surfaces $L_R$, $L_I$ and $L_R$, $L_2$ [169]. The two sample surfaces ($L_I$ and $L_2$) were both present (see Figure 8-20). The unusual noise at the edges is an artifact of the Fourier Transform method.

![Figure 8-20: Displacements by FT Method.](image)

(a). Displacement of $L_I$ by FT Method. (b). Displacement of $L_2$ by FT Method.

(x, y): pixel  z: μm.

The tilt angles about the x and y axes were calculated as $\Omega_x = 21 \, \mu$rad and $\Omega_y = 306 \, \mu$rad for surface $L_I$, and $\Omega_x = 112 \, \mu$rad and $\Omega_y = 48 \, \mu$rad for surface $L_2$. The
discrepancies between the tilt angles measured with the two methods were found to be $e_{1x} = 0.9$, $e_{1y} = 5.0 \mu \text{rad}$ for surface $L_1$ and $e_{2x} = 1.9$, $e_{2y} = 1.0 \mu \text{rad}$ for $L_2$ (see Figure 8-21).

![Figure 8-21: Error between FT method and MSPSI method.](image)

(a) Error of displacement of $L_1$ by FT Method and MSPSI method. Average error: 0.2 \mu m.
(b) Error of displacement of $L_2$ by FT Method and MSPSI method. Average error: 0.3 \mu m.
(x, y): pixel z: \mu m.

### 8.4.5 Conclusion

The results suggest that MSPSI is a viable technique for depth-resolved displacement field measurement with at least three significant advantages. Firstly, the depth range of the displacement field is limited only by the coherence length of the laser, rather than by the mechanical scan range of the reference arm of an interferometer. Secondly in systems with broadband light sources, dispersion may be a significant cause of
fringe contrast reduction. In MSPSI, the fringes are produced at high visibility at all times by a single wavelength and therefore the reduction in data quality seen in high bandwidth methods due to dispersion does not arise.

8.5 Summary

We have demonstrated how MSPSI by wavelength scanning can be used to measure depth-resolved displacement fields of different surfaces through transparent media. The approach has a number of potential benefits, in particular the avoidance of mechanical scanning (particularly important for large specimens) and the ability to make measurements even in the presence of significant optical dispersion and image sensors with low dynamic range. The technique could in principle also be used with speckle fields for measuring displacements inside granular packs.


Conclusion and future work

9 Conclusion and future work

9.1 Conclusion

In order to understand the mechanical behaviour of granular packs, we have developed an experimental technique based on the phase-shifting interferometry to observe the contact forces at the base of a system of particles. Our technique has enabled us to measure deformations much smaller than that of the self-weight of a single particle (8mm steel ball, weight of 20.01mN) on a glass surface, and allows us to consider very small changes in granular pack behaviour when it is subjected to perturbation.

The key findings and steps of the project are discussed below.

1. The development of the new reflective scanning-wavelength Fizeau interferometer was a major step forward. We are now able to observe particle deformations on a glass substrate to the accuracy of the order of 1nm. The construction and validation of this technique not only made the experiments described in this thesis possible, but also provides a platform for work in the future.

2. A new variable sampling interval least-square phase shifting algorithm using up to 114 frames was developed and presented. Before the development of this algorithm a considerable amount of time was required to ensure complete stability of the laser was achieved. The new algorithm is far more robust against fluctuations in the visibility and against mode-hopping, allowing more accurate determination of the deformations than previously achievable.

3. With such small displacements and the potential for lateral movement, accurate determinate of the magnitude and position of the contact points was required. For this purpose, a new 2D nonlinear least square fitting method was developed that relied on a two-step process of correlation and regression fitting of a Lorentzian profile. Assessment of the repeat accuracy of the contact force measurement was found to be ±2.3 beads (±3.95%), enabling the probability distributions of contact force and lateral displacement to be built up.
4. For contact forces larger than the mean contact force, the normalized contact force probability distribution has an exponential tail of the general form \( P(f) \propto \exp(-\beta \cdot f) \), where the coefficient was found to be \( \beta = 1.16 \pm 0.11 \). At low forces, the curve intersects at \( \lim_{f \to 0} P(f) \approx 0.69 \). Interestingly, the distribution is not observed to show any clear maximum, a feature, which tends to characterize the granular pack lying in a state between a fluid and an isostatic state.

5. For the first time, we have been able to report both the contact forces and the movements of the particles as a function of perturbations. The small reorganizations of the particles can have a significant effect on the behaviour of the granular pack.

6. A key measurement was the calculation of the probability distribution of the contact forces. Importantly, our technique allowed us to observe reductions in the contact forces in response to a load applied to the upper surface of the granular pack. This is a crucial measurement, since these are characteristic of ‘fragile materials’ and highlight the mechanisms through which the particle force chains are altered during loading.

7. We observed some important trends in the probability distributions. The probability distribution of the normalized response forces is essentially exponential. In contrast, the probability distribution of both the renormalized absolute values of the response forces and the lateral displacements decay with power law tails.

8. The displacement probability distributions appear to be anti-correlated with the contact force response distributions. This highlights the connection between the applications of loads, the dilatancy of the granular pack and the shifts in particle position and stress on a micro-mechanic level, a feature that is often neglected in numerical simulations and theoretical work.

9.2 Future work

In spite of the new experimental results discussed above, some problems are still open. These include the question of whether the bead self-organization has the property of elastic or plastic systems, how response forces propagate in the space under the applied force and what happens to both the force distribution and bead
reorganization when the granular pack “melts” and undergoes a transition from one state to another.

We have identified improvements that could be made to the calibration procedure. The current method was to stack up to weights equivalent to 55 beads on top of each other and measure the deformation of the substrate. Putting or removing a load inevitably disturbs the contact force and contact region [62] so a more reliable method would be to use a continuous loading system, instead of a discrete load, where the full range of the loads could be applied without disturbance.

We believe that automation of the collection of the data will enable both the rate of collection and the number of contact forces obtained to be increased. This latter quantity is important since accumulation of significantly more sampling points will reduce the uncertainties in the data processing and would allow a direct test of the theories that predict stress propagation.

Other ways of depositing the grains in the pack can be considered. For example, one could use a neutral buoyancy fluid to settle the particles in a slow manner and reducing any memory effects to a minimum [72-74, 171]. An alternative method would be to use a net with a grid (∼40×40 mm²) placed at the bottom of the container. The net could be then drawn out from the bottom of the pack through the beads disturbing any force networks that form during deposition.

Our research has focused on the behaviour of the particles in one, static state, but recent work has suggested that strong changes occur as the pack goes from the jammed to the unjammed state. One method for examining this would be to tilt the platform on which the system resides until the pack is in its incipient flow state and examine contact forces, response forces and particle displacements as the system transitions to the unjammed state. This will shed light on an enormously important area: the transition through the jammed state. To date, available evidence has been conflicting on this subject, but our approach has been shown to be accurate and robust and is an ideal method for investigating this problem.
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11 Appendix: papers published


Improved measurement of grain–wall contact forces in granular beds using wavelength scanning interferometry

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Received 2 December 2004, accepted for publication 4 April 2005
Published 13 May 2005
Online at stacks.iop.org/JOptA/7/S453

Abstract

We describe a wholefield optical technique based on a wavelength scanning Fizeau interferometer for measuring the contact forces between a granular bed and a transparent substrate. The substrate material is polymethyl methacrylate (PMMA), aluminium-coated on the internal surface, and changes to the interference pattern formed by reflection from the two surfaces allow the displacement field induced by the contacting grains to be visualized. Quantitative displacement data are obtained by phase shifting the interference pattern using a tunable laser. In order to avoid miscalibration errors associated with the non-linearities of the laser source, a Newton-Raphson iteration scheme is employed to search for the correct PZT voltage required for the linear phase-shifting steps. A new 31-frame phase-shifting algorithm based on the Chebyshev window function was designed to deal with the problems of residual miscalibration and higher harmonics created by multiple reflections within the substrate. The resulting noise in the measurements is below 1 nm, and the repeatability of the load–displacement relationship was found to be approximately 10 nm.

Keywords: granular materials, force measurement, contact mechanics, wavelength scanning interferometry, phase shifting

1. Introduction

Granular materials play an important role in many of our industries [1], for example, in mining, agriculture, civil engineering and pharmaceutical manufacturing, but our fundamental understanding of their behaviour is poor in comparison to say, soil mechanics. Unfortunately, the exact mechanical status of static granular material is still an open and debated issue. For example, one cannot determine from first principles the stress distribution at the base of a sand pile [2], and despite considerable effort in determining the constitutive relations required to solve the problem no conclusive understanding of the problem has been developed. This is compounded by the lack of experimental tests of many of the theoretical ideas proposed in the literature.

This scarcity of experimental evidence to support theoretical predictions has recently stimulated the development of a number of methods to examine the propagation of stresses within granular materials. These have included the use of carbon paper and elastic substrates where the contact diameter between the particle and the bottom surface varies with load to more sophisticated optical techniques using photoelasticity and phase-shifting interferometry [3–6]. In 2000, our group developed a new optical method designed to measure the stress distribution in the boundary of the granular material by phase-shifting interferometry [6], obtaining a resolution in the defonnation of the order of 10 nm. However, its repeatability was lower than that anticipated and it proved unreliable for the measurement of stresses in a granular pile. In this paper, we demonstrate that we can significantly improve the resolution of the phase-shifting interferometry method to a value of around 1 nm, combined with an improvement in the repeatability of the measurements. In the following section we will discuss the configurational improvements that have been made to improve the system. In section 3, the error sources arising in the current setup will be discussed. A new phase-shifting algorithm, designed to deal with problems such as residual miscalibration and harmonics of multi-reflection, is presented...
These stresses lead to a localized deformation of the upper surface whilst the lower surface remains relatively undisturbed. The stability of the deformation is critical in ensuring that the commercially available elastomeric materials, such as polyurethane, show an exponential stress attenuation that is largely free from environmental disturbances such as temperature and vibration, but has the drawback that in order to implement a temporal phase-shifting technique it is necessary to tune the wavelength of the laser light source.

2. System configuration

The configuration of the experimental system is shown in figure 1 and resembles a typical 2D Fizeau heterodyne interferometer. The light source is an external cavity diode laser (Vortex 6005 New Focus Inc, coherence length 6 m, output power > 4 mW). The wavelength can be scanned by about 0.162 nm under the control of the voltage from a 16-bit DA converter. The linearly polarized light beam is enlarged by the lens OL and FL, and then redirected onto a polymethyl methacrylate (PMMA, or ‘Perspex’) plate, by means of the two mirrors M1 and M2. The enlarged parallel beam is reflected by the two surfaces of the Perspex plate, forming interference fringes that are subsequently detected by the CCD camera (Kodak ES 1.0, with a Nikkor lens f/2.8, 135 mm). Finally, the images are transferred to a Sun workstation for analysis.

Granular materials such as steel ball bearings, or sand, can be placed on the upper, aluminized, side of the PMMA plate. The particles at the base of the pile will support the load of those above and thus will transmit the stresses to the plate. These stresses lead to a localized deformation of the upper surface whilst the lower (PMMA-air) boundary is largely unaffected, resulting in a small change in the interference pattern generated by the interfering waves from these two surfaces. The interference between top and bottom surfaces in this way results in an essentially common-path interferometer that is largely free from environmental disturbances such as vibration, but has the drawback that in order to implement a temporal phase-shifting technique it is necessary to tune the wavelength of the laser light source.

Previously, in [6], a double layer substrate was used, enabling the deformation to be controlled through a suitable choice of the elastic modulus of each layer, but difficulties in fabricating layers of uniform thickness meant that the sensitivity was found to drift significantly from place to place, and was a major inconsistency in the deformation-load calibration. It was for this reason that we now use a single layer of PMMA as the substrate material, providing a greater stability both spatially and temporally. The modulus of the material (about 3.5 GPa) ensures deflections of order 100 nm for the bead masses used in these experiments, a value that is large enough to allow accurate displacement measurements whilst minimizing the perturbation of the force chains due to the compliance of the base.

3. Error sources and solutions

3.1. Non-linearity of laser wavelength scanning and linearization

External cavity laser (ECL) technology promises to provide both very narrow line widths and absence of mode hops normally found when tuning semiconductor lasers. Litman-Metcalf (the type used in these experiments: New Focus Vortex 6005) configuration is an example of such ECL sources in which gratings controlled by a PZT are used to provide optical feedback and tune the wavelength [7]. However, the relation between PZT control voltage and the wavelength is non-linear and this can cause a significant error in measuring deformation by phase-shifting interferometry (PSI).

To achieve high resolution and accuracy in PSI, an in situ method of calibrating the phase-voltage relationship using a 2D fast Fourier transform (FFT) method [8] was employed. First the PZT voltage was varied to modify the wavelength of the laser and hence to induce a phase difference between rays reflecting from the bottom and those being transmitted through the bottom surface and reflecting from the top. For each PZT voltage an image of the fringe pattern was recorded.

Figure 2(a) shows a typical set of interference fringes from an undeformed PMMA plate whose faces are relatively flat, but inclined to form a slight wedge. The resultant fringes are therefore generally parallel, but some scratches in the aluminum surface coating are observable. Figure 2(b) is the amplitude spectrum of figure 2(a) after removal of the mean value or DC offset of the image. $k_x$ and $k_y$ refer to spatial frequencies along the horizontal and vertical axes, respectively, in units of fringes across the field of view. At each PZT voltage the phase shift is determined modulo 2π from the real and imaginary parts of the transform, evaluated at one of the peak positions of the fundamental frequency shown in figure 2(b). The phase versus voltage signal is then unwrapped, providing a calibration curve to generate the required phase values for later phase shifting (figure 3). The maximum error between the real and ideal phase-shifting value reaches about 1.5 rad.

After finding all of the phase values for each of the phase-shifting voltages, a Newton–Raphson iteration scheme [9] was used to search automatically for linear phase steps. After many iterations, a new series of PZT voltages was obtained. The improvement compared with the original curves (figure 3) is quantified in figure 4 by plotting the error between the ideal and experimental phase steps, before and after linearization. The maximum non-linear error decreases from about 1.5 to 0.15 rad. Repeatability experiments were also carried out over a 12 h period by using fixed PZT control voltages for that period. Once again the maximum error is approximately 0.15 rad, demonstrating the reasonably repeatable behaviour of the laser over this timescale.

3.2. Multi-reflections in a Fizeau interferometer

Multi-reflections in the Fizeau interferometer arrangement are another potential source of errors. In order to increase sensitivity, and separate optically the test surface from the
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Figure 2. Fringe pattern recorded during calibration of laser. (a) Single fringe pattern from PMMA plate; (b) its Fourier transform after subtraction of the mean intensity.

Figure 3. Measured phase change versus PZT voltage demonstrating non-linear response.

Figure 4. Phase shift error before and after linearization of the phase shifts.

where \( \psi \) is the phase difference, and \( A \) is the Airy function. \( r \) and \( t \) are the amplitude reflection coefficients and amplitude transmittances, respectively. An incident ray travelling upwards is labelled ‘+’, and downwards is labelled ‘−’; \( 1 \) and \( 2 \) refer respectively to the lower and upper surfaces of the PMMA plate. More detailed information is given in [10, 11]. Equation (1) was used to calculate the interference pattern in the time domain as the wavelength was tuned for the case of...
a PMMA plate of thickness 4.529 mm, refractive index 1.48, wavelength 635 nm, incident light angle 0.0033 rad, and metal layer refractive index (A1) 1.35 + 8.6i [12]. The total tuning range was that used for the experiments, giving eight complete cycles of the fundamental. The resulting amplitude spectrum (after subtraction of the DC intensity) is shown in figure 5 after normalization of the frequency axis by the frequency of the fundamental carrier.

From figure 5, we see clearly that there are multiple spectral peaks. The highest peak corresponds to the fundamental frequency, and the others at integral multiples of the fundamental frequency are higher harmonics due to the multiple reflections, if the spectral leakage is ignored. The second harmonic peak is about 17% of that at the fundamental frequency, which is therefore a significant potential error source. Higher harmonic errors are well-known in the field of high accuracy phase-shifting measurement and arise either due to aliasing—if insufficient numbers of frames per cycle of the fundamental are recorded—or due to the leakage from the higher harmonic peaks onto the fundamental peak at which the phase is evaluated. The leakage is exacerbated in the presence of phase shift errors, which is the reason for the care taken to linearize the phase steps as described in section 3.1. Spatial domain fringes (of the type shown in figure 2(a)) normally have more obvious multiple peaks than those from the time domain due to the larger number of cycles included in the spectrum calculation. The second harmonic peak for example is clearly visible in the experimental amplitude spectrum shown in figure 2(b). Generally, the harmonics are again located at integral multiples of the fundamental frequency, and the spectrum leakage is smaller than that in time domain. Therefore, phase information acquired in the spatial domain may be more accurate than that in the time domain, if the PMMA plate is chosen properly.

Another phenomenon that we have observed caused by multi-reflection in a Fizeau interferometer is the production of a 'ghost' image in the measured displacement fields. If we consider only the fundamental frequency and the second harmonics in the spatial domain, there are three interfering beams of reflected light from the surfaces $L_1$, $L_2$ coated with aluminium of intensity reflection coefficients $r_1^2 \approx 0.95$ with relative intensities $I_{R1} \approx 0.04 \cdot I_N$, $I_{R2} \approx 0.876 \cdot I_N$, and $I_{R3} \approx 0.033 \cdot I_N$, as shown in figure 6, where $I_N$ is the incident light intensity. Therefore there are three interferometric signals, with intensities given by

$$
I_{12} = I_{R1} + I_{R2} + 2 \sqrt{I_{R1} \cdot I_{R2} \cdot \cos(\phi_{R12}(t))}
$$

$$
I_{13} = I_{R2} + I_{R3} + 2 \sqrt{I_{R2} \cdot I_{R3} \cdot \cos(\phi_{R23}(t))}
$$

$$
I_{13} = I_{R1} + I_{R3} + 2 \sqrt{I_{R1} \cdot I_{R3} \cdot \cos(\phi_{R13}(t))}
$$

where $\phi_{R12}(t)$, $\phi_{R23}(t)$, $\phi_{R13}(t)$ are the phase difference between $I_{R1}$ and $I_{R2}$, $I_{R2}$ and $I_{R3}$, $I_{R1}$ and $I_{R3}$ respectively.

A properly designed PSI algorithm will remove the higher harmonic interference signal $I_{13}$. However, in addition to the desired interference signal $I_{12}$ the second signal $I_{13}$ that also modulates at the fundamental frequency will therefore appear in the measured displacement fields as a 'ghost image'. It seems likely that this observation is related to the occurrence of 'hotspots' previously reported (but not explained) in other studies of Fizeau interferometers [13].

4. Phase-shifting algorithms

According to phase-shifting interferometry theory, phase maps can be determined from the general equation given by [14]

$$
\phi = \tan^{-1} \left\{ \frac{-\sum_{n=0}^{N-1} I(n) \cdot w(n) \cdot \sin(\frac{2\pi n}{N})}{\sum_{n=0}^{N-1} I(n) \cdot w(n) \cdot \cos(\frac{2\pi n}{N})} \right\} + C \quad (3)
$$

where $n$ is a shift index, $I(n)$ is the intensity distribution of the $n$th frame, $M$ is the total number of phase steps, $N$ is the number of steps per cycle of the temporal carrier, $w(n)$ is a window function for data processing and $C$ is a constant decided by the origin of the phase. This equation can be considered as being equivalent to evaluating the phase from the real and imaginary parts of the windowed Fourier transform of the intensity signal at the temporal carrier frequency introduced by the phase-shifting device [15-17].

Poor performance in PSI can often be traced to the problems of (a) aliasing of and (b) spectral leakage from higher
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harmonics in the signal. The former is minimized by increasing the number of samples per cycle of the fundamental (i.e., the value of $N$). The latter is minimized by (a) increasing the number of sampled cycles of the carrier (i.e., the value of $M/N$) and (b) appropriate choice of a window function, $w(n)$ [14]. For a given number of frames, $M$, there is therefore a trade-off between minimizing the aliasing errors (high $N$) and minimizing the leakage errors (low $N$). In the experimental results section we present results at two $N$ values: $N = 8$ and 4 for which the minimum harmonic frequency that is aliased onto the signal peak occurs at seven and three times the carrier frequency, respectively. The choice of window function has been a subject of major research activity in recent years; several well-known approaches include correlation theory [18], characteristic polynomial theory [19], and the extended averaging technique [16, 20]. In general the main aim is to place a zero in the magnitude of each of the harmonic spectral functions at the fundamental frequency where the phase is evaluated. However, phase-shifting miscalibration and other errors mean that it is not possible to achieve this in practice. A robust way to deal with this problem would be to keep the height of sidelobes of the Fourier transform of the window function smaller than a tolerance value $x$, which would in principle allow a maximum phase error to be calculated for any spectral content regardless of the miscalibration error. With this approach, the window function should also have the property that the main lobe of its transform should be narrower than the value of the fundamental frequency.

The similarity between the requirements above and the property of Chebyshev and Kaiser window functions encouraged us to investigate a new PSI algorithm based on a new window function. Therefore, in the next section we investigate the use of the Dolph–Chebyshev window function which has the property that it is the window function optimized to minimize the width of the main lobe, under the constraints that the window length ($M$) be fixed and the levels of the sidelobes not exceed a given value. It provides more flexibility than the classical windows because a desired tradeoff between mainlobe width and sidelobe levels can be achieved. A 31-point Chebyshev window is shown in figure 7. From figure 7(b), we can see that its sidelobes are 20 dB below the rectangular window of the same length.

By repeating the measurement process before and after loading, difference phase maps can be calculated which give information on the out-of-plane displacement component, $u_z(x, y)$, where $x$ and $y$ are coordinates representing position on the sample surface. The measured phases are wrapped onto the range $-\pi$ to $\pi$. The process of phase unwrapping is therefore required so as to remove the $2\pi$ phase discontinuities. This is done by addition of the correct integral multiple of $2\pi$ to each phase value. From the unwrapped phase map, a simple scaling factor can then be used to calculate a displacement map:

$$u_z(x, y) = \frac{\lambda \Delta \phi(x, y)}{4\pi}$$

where $u_z$ is the displacement, $\lambda$ is the wavelength and $\Delta \phi$ is the unwrapped phase change. Equation (4) is strictly applicable only for surfaces separated by a vacuum, and the presence of the PMMA plate will enhance the phase change for a given displacement by a factor of $n = 1.48$, where $n$ is the refractive index of the PMMA. Furthermore, changes to the refractive index of the PMMA due to the imposed stress will manifest themselves as fictitious displacements. The approach taken here was to calibrate the load versus phase-change response of the plate and in effect to use the resulting calibration constants to convert phase change to force. For simplicity, displacement plots shown in later figures were obtained by the use of equation (4) without attempting to correct for the above-mentioned effects.

5. Experimental results

The first group of load calibration experiments was performed with a PMMA substrate with an average thickness of 4.52 mm. The indenter was a glass bead of diameter 4 mm which was present for both the before-load and after-load measurements. The initial load of 3.619 mN was applied by a stainless steel cylinder (diameter 4 mm, length 5 mm) and subsequent load increments of 2.550 mN with stainless steel beads of diameter 4 mm. This resulted in a maximum additional applied load of 18.917 mN. As discussed in the previous section, a 31-frame PSI algorithm with a Chebyshev window function of $r = 75$ (see figure 7) was used to perform the phase evaluation, where the number of signal cycles was 4, and the phase-shifting step was $\psi = \pi/4$. The displacement fields at minimum and maximum loads are shown in figures 8(a) and (b).
Figure 8. Experimental results from the first group load calibration. (a) Displacement field at minimum load; (b) displacement field at maximum load; (c) force–deflection curve from three independent repeated experiments; (d) deviation between results from runs 1 and 2.

The pixel spacing on the sample in (a) and (b) is 0.018 mm. The corresponding peak deflection is about 34 and 130 nm (run 1), respectively. The pixel spacing on the sample is 0.018 mm along each axis. The load versus peak deflection curve for two separate calibration runs 1 and 2 is shown in figure 8(c). The maximum deviation between the repeat measurement curves was about 10 nm (see figure 8(d)).

The result of the second group of load calibration experiments is shown in figure 8(c) (run 3) and figure 9. The average thickness of the Perspex substrate in this experiment was about 9.95 mm. The load was applied to the indentor by stainless steel cylinders of diameter 4 mm and length 24 mm and providing load increments of 22.310 mN. The phase was evaluated by a PSI algorithm with a Chebyshev window function, where the number of cycles was 8, the phase-shifting step ψ = π/2, and the number of frames was 31. Figure 9 shows the calculated displacement field at the maximum load of 66.93 mN in which both the real and the "ghost" indentations are clearly visible. The peak displacement of the true indentation is 363 nm, whereas that of the fictitious peak deformation is 28 nm, i.e. 7.7% of the peak real displacement. In our experiments we were able to separate spatially the ghost from the true indentation by arranging for the object beam to illuminate the plate at a slight angle of approximately 2.15° from the normal. However, for other optical configurations using wavelength tuning this may not be possible, and the resultant errors should be borne in mind.
when using this technique. No phase-shifting algorithm will be able to remove this contribution since it modulates at the same frequency as that of the carrier. The standard deviation of the phase measurement noise was 0.386 nm over the region away from the indentation and ghost image. The load–deformation curve for run 3 of three load points is shown in figure 8(c). From figure 8(c), we can find that the two independent groups of calibration experiments are in reasonable agreement with one another.

A test of the consistency of the experimental results was performed by loading with the same weight of 6.169 mN. The results are shown in table 1, showing a maximum to minimum range of about 10 nm, and a standard deviation of 3.2 nm.

6. Conclusions

Wavelength scanning interferometry is a powerful technique for measuring individual contact forces at the boundaries of granular packs. Several practical problems were encountered and solutions developed during the development of the system. These included a non-linear response of the commercially available tunable laser which was solved by a Newton–Raphson iteration scheme. Multiple reflections from the load-bearing substrate generated significant harmonic content in the interference signal, the effects of which were reduced by a new 31-frame phase-shifting algorithm based on Chebyshev windows. Finally, the phenomenon of ‘ghost’ images in the measured displacement field was found and explained. The resultant resolution and repeatability in the measurement of the deformation at the contact points was as low as 1 and 10 nm, respectively.

Acknowledgments

We are grateful to Dr C R Coggrave of Phase Vision Ltd for his assistance with several aspects of the project, particularly the 2D phase unwrapping. J M Huntley is also grateful to the Royal Society and Wolfson Foundation for a Royal Society—Wolfson Research Merit Award.

References

Depth-resolved whole-field displacement measurement using wavelength scanning interferometry

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Received 11 February 2004, accepted for publication 5 May 2004
Published 26 May 2004
Online at stacks.iop.org/JOptA/6/679
DOI: 10.1088/1464-4258/6/7/004

Abstract
We describe a technique for measuring depth-resolved displacement fields within a three-dimensional (3D) scattering medium based on wavelength scanning interferometry. Sequences of two-dimensional interferograms are recorded whilst the wavelength of the laser is tuned at a constant rate. Fourier transformation of the resulting 3D intensity distribution along the time axis reconstructs the scattering potential within the medium, and changes in the 3D phase distribution measured between two separate scans provide one component of the 3D displacement field. The technique is illustrated with a proof-of-principle experiment involving two independently controlled reflecting surfaces. Advantages over the corresponding method based on low-coherence interferometry include a depth range unlimited by mechanical scanning devices, and immunity from fringe contrast reduction when imaging through dispersive media.

Keywords: depth-resolved, displacement measurements, wavelength scanning interferometry, phase shifting, wavelength tuning

1. Introduction

The ability to measure internal displacement fields within a material or structure would be highly desirable in many fields, ranging from alignment of complex optical systems to nondestructive evaluation of composites. Standard interferometric techniques (with either speckled or smooth wavefronts) have sufficient sensitivity for such applications but are typically restricted to the measurement of surface deformations. However, only in special cases is it possible to infer the internal deformation state of the structure or material from knowledge of the surface displacements alone.

Depth discrimination with multiple wavelengths has been used in optical profilometry for a number of years. Two basic forms have been developed, depending on whether the multiple wavelengths are present simultaneously [1] or sequentially [2]. In the former case, denoted here as low-coherence interferometry (LCI), one illuminates with a broadband source and scans the reference mirror or sample through the required depth range. In the latter case, which we call wavelength scanning interferometry (WSI), a tunable light source is used thereby avoiding the need for mechanical movement of the sample or the interferometer.

When measuring transparent objects (e.g. optical lenses or flats), reflections from surfaces beyond the surface of interest occur but are normally regarded as a nuisance, and wavelength tuning combined with specially designed phase shifting algorithms have therefore been developed to suppress their effects [3, 4]. Optical coherence tomography (OCT), on the other hand, is a rapidly developing imaging technology, primarily used for medical applications, that exploits the signal from subsurface reflections to provide information on the structure of biological tissues (see, for example, [5] for a recent review of the field). Most OCT systems operate in a
These experiments were based on LCI: the interferometer generator (SG), laser controller (LC), lenses \( L_1, L_2, L_3 \), steering mirror (M), reference surface (R), surfaces under test \( S_1, S_2 \), high-speed camera (C) and personal computer (PC) pointwise manner, with mechanical scanning in one or more lateral directions to build up cross-sectional images, and are based on the LCI technique. The WSI version of OCT was proposed by Fether et al [6] in 1995, and demonstrated by a number of authors (see e.g. [7]).

The first demonstrations of depth-resolved displacement field measurement have been presented very recently [8, 9]. These experiments were based on LCI; the interferometer is only sensitive to the movement of scattering points lying within the slice selected by the reference mirror position, and conventional fringe analysis algorithms can be used to extract the required displacement field.

The purpose of this paper is to present results from a proof-of-principle experiment based on a WSI approach to subsurface displacement field measurement. As far as we are aware, this is the first time that depth-resolved two-dimensional displacement fields have been demonstrated using wavelength scanning interferometry.

2. Wavelength scanning interferometry

For this proof-of-principle experiment we used the simplest possible sample configuration, shown in figure 1, consisting of two independently tiltable reflecting surfaces, \( S_1 \) and \( S_2 \). A third surface, \( R \), provided the reference wave. All three surfaces were glass-air interfaces of glass flats (thickness 5.1 mm), antirefection coated on one side to suppress the reflection from the second glass–air interface. The light source used was a solid-state tunable laser TL (New Focus Vortex 6005), the beam from which was expanded by lens \( L_1 \) and steered by mirror \( M \) towards collimating lens \( L_2 \). The reflected light from the three glass-air interfaces \( R, S_1 \) and \( S_2 \) was imaged by lens \( L_3 \) onto the sensor of a high-speed camera, C (VDS HCC-1000), which recorded the resulting three-beam interference patterns.

The laser wavelength, \( \lambda \), can be tuned in an approximately linear manner around a centre wavelength, \( \lambda_c \), by adjusting a voltage supplied by signal generator SG to the laser controller LC. The time-varying phase difference \( \phi(t) = 4\pi d/\lambda(t) \) between beams reflected back from any pair of surfaces can be expanded around \( \lambda_c \) in a first-order Taylor series approximation as

\[
\phi(t) = \phi_0 + \frac{4\pi d}{\lambda_c} - \frac{4\pi d \Delta \lambda}{(\lambda_c)^2} t
\]

where \( \phi_0 \) is the difference between the phase changes induced on reflection, \( d \) is the optical distance between the surfaces, \( \Delta \lambda \) is the wavelength tuning range, \( T \) is the time it takes to scan through \( \Delta \lambda \) and \( t \) is time \((-T/2 \leq t \leq T/2)\). The second term on the right-hand side of equation (1) is the phase due to the optical path difference between the surfaces. The third term is the phase shift introduced by tuning the wavelength and is equivalent to \( 2\pi f_1 \), where \( f \) is a carrier frequency proportional to \( d \).

The intensity recorded at a particular pixel in the camera sensor, assuming that multiple reflections can be neglected, may be written as

\[
I(t) = I_0 + I_1 + I_2 + 2\sqrt{I_0 I_1} \cos[\phi_{R1}(t)]
+ 2\sqrt{I_1 I_2} \cos[\phi_{R2}(t)] + 2\sqrt{I_0 I_2} \cos[\phi_{R3}(t)]
\]

where \( I_0, I_1 \) and \( I_2 \) are the intensities of the beams coming from surfaces \( R, S_1 \) and \( S_2 \), respectively. \( \phi_{R1}, \phi_{R2}, \phi_{R3} \) are the phase differences between \( R \) and \( S_1 \), \( R \) and \( S_2 \) and \( S_1 \) and \( S_2 \), respectively, given by equation (1) and using the optical path differences \( d_{R1}, d_{R2}, d_{R3} \) between the corresponding surfaces. A Fourier transform of \( W(f)(t) \), where \( W \) is a window function, then reveals (considering only positive frequencies) four amplitude peaks: the dc component and three peaks corresponding to each of the cosine terms, centred at frequencies \( f_{R1}, f_{R2} \) and \( f_{R3} \). Provided that the distance between reference and sample is greater than the sample depth, the peaks of interest \( (f_{R1} \text{ and } f_{R2}) \) are separated from the unwanted peaks \( (f_{R2} \text{ and } \text{dc}) \), resulting in a reconstruction of the true scattering potential of the sample, rather than the autocorrelation of the scattering potential [6].

The Fourier transform of \( W(f)(t) \) in the neighbourhood of the peak from surface \( j \) \((j = 1, 2)\) may be written (neglecting constants and the contributions from other peaks) as

\[
\hat{I}(f) \otimes \hat{W}(f) = \exp(i\phi_j(0)) \hat{W}(f - f_j),
\]

where the tildes denote Fourier transformed variables and \( \otimes \) denotes convolution. The quantity \( \phi_j(0) \), which is the phase at \( t = 0 \), i.e. at the centre wavelength \( \lambda_c \), and which is related to the optical path between the surfaces, can therefore be recovered by evaluating the arctangent of the ratio between the imaginary and real parts of \( \hat{I}(f_j) \otimes \hat{W}(f_j) \). If surface \( j \) is now displaced by an amount \( u_j \), in the out-of-plane direction, equation (1) shows that \( \phi_j(0) \) changes by \( 4\pi u_j/\lambda_c \). The phase difference between \( \phi_j(0) \) evaluated before and after displacement can therefore be used to calculate \( u_j(x, y) \), where \( x \) and \( y \) are the two in-plane coordinates, provided that the phase of \( \hat{W}(f) \) is known and taken into account in the calculation.

The finite window duration has of course the effect of broadening a given spectral line. A rectangular window of duration \( T \), for example, results in a sinc function of width \((\text{measured between the first zeros on either side of the maximum})\) \( \Delta f = 2/T \). It can be shown that this corresponds to an effective depth resolution of \( \delta = \lambda_c^2/\Delta \lambda \) (or a factor of 2 larger if using a Hanning window instead). Given that the coherence length of the laser is sufficient and that a Hanning window is used, it can be shown that the depth range \((\text{maximum distance a surface can be from the reference surface})\) is \( \Delta z = 5N_f/8 \), where \( N_f \) is the number of frames acquired during time \( T \) while tuning the wavelength over the range \( \Delta \lambda \). This is just a consequence of limiting the carrier frequency \( f \) below half the sampling rate of the camera in accordance with the Shannon sampling theorem.
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One minor complication that should be pointed out is that the tilting of surface 1 will induce small phase shifts in the signal from surface 2, even if surface 2 does not itself move. This should be only a small effect in the current setup since the rate of change of phase with angle is zero for a plate initially oriented normal to the optical axis. This prediction is confirmed to be the case by the validation method presented in the next section. For more complex specimen geometries, however, it may be necessary when interpreting the phase change from any given layer to include the measured displacement information for all the preceding layers. Other factors that will also need to be considered when extending the technique to the measurement of true 3D samples are the effect of multiple scattering, phase noise due to speckle decorrelation as the wavelength is tuned and the development of a robust 3D phase unwrapping algorithm [10].

3. Experimental results

The results presented here were obtained by tuning the laser wavelength around $\lambda_c = 635.05$ nm at a rate $\Delta \lambda / T \approx 0.058$ nm s$^{-1}$. A sequence of 940 interferograms was recorded with the camera running at 912 frames s$^{-1}$. The optical path difference between surfaces R and S$_1$, R and S$_2$ and S$_1$ and S$_2$ was approximately 51 and 20 mm, respectively.

Figure 2(a) shows the normalized intensity $I(t)$ measured for one pixel in the field of view, while figure 2(b) shows the portion of the frequency spectrum of $I(t)$ where the peaks of interest are present. Prior to the Fourier transformation, the mean value of the intensity signal was subtracted and the signal multiplied with a Hanning window. The peak frequencies for RS$_1$ and RS$_2$ are within 6% and 3%, respectively, of those expected using the third term of equation (1). The peak widths of approximately 4 Hz correspond to the expected value for a Hanning window of $4/\delta$ where in this case $\delta \approx 1$ s. This corresponds to a depth resolution $\Delta z \approx 14$ mm. The depth range of the system was $\Delta z \approx 1650$ mm. Small independent tilts were then introduced to surfaces S$_1$ and S$_2$ and a second sequence of interferograms was recorded. Finally, the phase difference was obtained by subtracting the phase $\phi_{RS}(0)$ evaluated for all the pixels in sequence 1 from the corresponding values in sequence 2.

Figures 3(a) and (b) show the measured wrapped phase difference maps for the movement of S$_1$ and S$_2$, respectively.
The images show a region of about 10 x 10 mm² at a resolution of 256 x 256 pixels. Each fringe represents an out-of-plane displacement \( u_z = \lambda / 2 \sim 317 \text{ nm} \). Figure 4 shows the displacement field obtained for \( S_1 \) and \( S_2 \) after unwrapping the phase difference maps shown in figure 3. The original spatial resolution of 256 x 256 pixels has been reduced by a factor of ten along each axis for clarity in the mesh plot.

In order to validate the results, the tilts were measured independently using standard two-beam interferometry at a fixed wavelength between surfaces R-S1 and R-S2. Only one of the two sample surfaces (\( S_1 \) and \( S_2 \)) was present at a time. The optical phase difference was evaluated for each case using a spatial phase shifting method [11]. Figure 5 shows four profiles corresponding to the displacements measured for \( S_1 \) and \( S_2 \) at \( x = 5 \) mm with WSI and with two-beam interferometry.

The tilt angles about the \( x \) and \( y \) axes were calculated as \( \Omega_y = 21 \text{ µrad} \) and \( \Omega_z = 306 \text{ µrad} \) for surface \( S_1 \), and \( \Omega_x = 112 \text{ µrad} \) and \( \Omega_y = 48 \text{ µrad} \) for surface \( S_2 \). The discrepancies between the tilt angles measured with the two methods for the horizontal, \( x \), and vertical, \( y \), directions were\( \varepsilon_{yx} = 0.9 \), \( \varepsilon_{xy} = 5.0 \text{ µrad} \) for surface \( S_1 \) and \( \varepsilon_{xz} = 1.9 \), \( \varepsilon_{zx} = 1.0 \text{ µrad} \) for \( S_2 \). Although this can be regarded as good agreement, a small phase offset error was found between the two methods, as can be seen in figure 5, and may be attributed to drifts in \( \lambda \) between successive recording sequences. In some applications a constant phase offset may not be an issue, but in situations where it is important to measure absolute displacements the problem could be overcome by incorporating a reference etalon into the interferometer.

4. Discussion

The results presented in the previous section suggest that WSI is a viable technique for depth-resolved displacement field measurement. In this section we discuss its strengths and weaknesses compared to the LCI version [8, 9].

The main disadvantage is that images need to be acquired for all wavelengths before even a single slice can be selected. The use of a 1 kHz framing camera in this demonstration system nevertheless allowed all data acquisition to be performed in \( \sim 1 \) s, a figure which is compatible with high-volume production testing.

On the other hand, we can foresee at least three significant advantages, which are summarized here. Firstly, the depth range of the displacement field is limited only by the coherence length of the laser, rather than by the mechanical scan range of the reference arm of the interferometer as in low-coherence interferometry. There seems no reason that depth ranges that would be regarded as unfeasible for LCI (of order 1 m or more) cannot be measured using WSI. Secondly, in systems with broadband light sources, dispersion may be a significant cause of fringe contrast reduction. In WSI, the fringes are produced at high visibility at all times by a single wavelength and therefore the reduction in data quality due to dispersion does not arise. Finally, the limited dynamic range of whole-field image sensors based on CCD or CMOS technology (the number of grey levels, \( N_g \), is typically only 256) limits the performance of WSI to a much lesser extent than that of LCI. If \( \delta \) is the slice thickness and \( Z \) the overall thickness of a sample containing uniformly distributed scatterers, then only the fraction \( \delta / Z \) of scattered photons contribute an interferometric signal in the case of LCI, the rest merely producing a dc offset to the intensity image. The modulation depth can therefore never exceed \( N_g \delta / Z \) grey levels and, as a result, attempts to improve the axial resolution (i.e. reduce \( \delta \)) have the unfortunate consequence of reducing the intensity modulation by a corresponding factor. For example, in the case of \( \delta / Z = 1/100 \), with an eight-bit camera, the signal would be no more than 2–3 grey levels deep and the measurements therefore rather susceptible to noise. WSI on the other hand ensures that the full dynamic range of the camera is utilized.

5. Conclusions

We have demonstrated how wavelength scanning interferometry can be used to measure depth-resolved displacement fields of different surfaces through transparent media. The depth resolution is limited by the wavelength range scanned during the whole image acquisition sequence. Like the low-coherence interferometry (LCI) version, the method provides decoupling of the depth resolution and displacement sensitivity. However,
our approach has a number of potential benefits over LCI, in
particular the avoidance of mechanical scanning (particularly
important for large specimens) and the ability to make
measurements even in the presence of significant optical dispersion
and image sensors with low dynamic range.

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