The production of droplets from liquid jets by capillary and electrohydrodynamic instabilities

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THE PRODUCTION OF DROPLETS FROM LIQUID JETS

BY CAPILLARY AND ELECTROHYDRODYNAMIC INSTABILITIES

by

CHARLES STEPHEN PARKIN  B.Sc., D.I.S., M.Sc.

A Doctoral Thesis submitted in partial fulfilment of the requirements for the award of

Doctor of Philosophy of Loughborough University of Technology.

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Director of Research:– Supervisor:–
Prof. D.C. Freshwater  B. Scarlett, M.Sc.

Department of Chemical Engineering

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TO MAGGIE
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1. Introduction

Spraying, and in particular the process known as atomisation, is an important particle production method being extensively used in the combustion, agricultural and process industries. It involves the reaction of a liquid to a variety and combination of forces whose effect in a commercial spraying device may not be clear.

In his famous treatise "The Theory of Sound (95) Lord Rayleigh stated "A large class of phenomena, interesting not only in themselves but also throwing light on others jet more obscure, depend for their explanation upon the transformations undergone by a cylindrical body of liquid....". The break-up of a cylinder of liquid, or jet, may be regarded as the simplest case of spraying and can, as Lord Rayleigh suggested, throw light on the more obscure phenomena that occur in the more complex methods of spraying.

In a typical commercial atomiser an unstable sheet of liquid is produced. This breaks-up into cylindrical ligaments which in turn break-up into droplets (15). In the particular example of a spinning disc atomiser, however, the formation of an unstable sheet may not be necessary as at low liquid flowrates the flow across the disc can directly form ligaments at the disc rim (39). Hence, it can be seen that the break-up of a liquid jet can be considered as an important step in the atomisation process.

The production of droplets from jets formed directly at nozzles was used in the lead shot production process where a nozzle was placed in the base of a tank of molten lead. A similar process is now used to obtain fertilizer prills where solid particles are produced from fertilizer melts by liquid jet break-up. This method could be extended to other
materials such as metals and polymers where it is advantageous to obtain the material as solid spheres. The extension of this method to such materials requires a knowledge of the break-up behaviour of a wide range of liquids exhibiting both Newtonian and non-Newtonian behaviour. Since the more viscous liquids are difficult to break-up (127), for the method to be successful in handling viscous liquids stronger forces than the naturally occurring capillary forces may be required. It is for this reason that the application of electrohydrodynamic forces to jets is considered in this thesis.

It is of prime importance in most applications of jet break-up, after determining the conditions for instability, to determine the size of droplets formed. It is for this reason that this thesis is primarily concerned with the prediction of the size of droplets formed and also concerned with possible methods of monosize droplet production by liquid jet break-up.

Chapter Two of this thesis is devoted to a literature review of the previous work on the subject of jet break-up. It covers a wide area reviewing the current state of knowledge on the behaviour of jets under the influence of most common forces. For those readers with interests in the wider aspects of spraying the bibliographies of de Juhasz (13) and Lapple et al (49) and the monograph of Marshall (57) are recommended.

The production of droplets from capillary instability is discussed in Chapter Three. A unique application of the current knowledge of capillary jet behaviour is given in Chapter Four, where the production mechanism of the glass spheres found in the lunar fines recovered by the Appollo space program is discussed.

Chapter Five reports an experimental investigation into the electrohydrodynamic instability of water jets. A comparison between the results and existing theory is given. Chapter Six presents a
Theoretical analysis of the stability of liquid jets to alternating electric fields.

The conclusions derived from the work presented in this thesis, and the author's suggestions for the direction of further work in the field of jet break-up, are given in Chapter Seven.
CHAPTER TWO
CHAPTER TWO

2. Literature Survey

When a liquid jet is influenced by some force it is important to establish under what conditions this force will cause the jet to break-up and what degree of instability the jet will have. It is also equally important having established that the jet is unstable and will break-up, what size of droplets will result.

This survey reviews the previous work concerning both the stability of jets and the prediction and measurement of the sizes of the resultant droplets. The effect of all common forces is reviewed, but special attention is paid to capillary and electrostatic forces. The former is a large naturally induced force and the latter one that can be easily brought to bear on the jet if it is thought to be advantageous to do so.

2.1. Early Investigations

Savart was one of the first to investigate the problem of the break-up of liquid jets, but the first important theoretical analysis was that of Plateau. Plateau's problem was that of the break-up of a laminar liquid jet influenced by capillary forces. The jet was considered as being unaffected by its surrounding media as well as by gravity.

Plateau's analysis regarded the jet as a cylinder of liquid moving at the velocity of the jet. His analysis was concerned with the static stability of a short cylinder of liquid. Plateau's result was that the maximum stable length of such a cylinder of liquid under capillary forces is equal to the circumference of the cylinder and that such a cylinder is stable to all but axisymmetric disturbances. Plateau then assumed that a jet would behave in a similar manner to a
short cylinder and divide into lengths equivalent to its circumference. However, as Bouasse (5) points out, this form of analysis is inadequate and what is required is a dynamic analysis which takes into account the properties of the jet fluid.

Several interesting phenomena whose significance was not fully realised until later were observed in the early investigations. Firstly Savart (62) noted that the striking of drops into his receiving vessel could cause disturbances to travel back to the nozzle and cause a regular series of drops to be formed. This is reported in Rayleigh's classic work (95). Secondly Plateau (56) reported that a jet did not break-up into a series of monosized droplets but that between the larger drops were smaller drops which have been called "Plateau's spherules". These secondary drops are now more usually called satellite drops.

2.2. Dynamic Theories of Jet Stability

The dynamic stability of liquid jets has been an area of considerable interest since the latter half of the nineteenth century. It is intended to present the various special cases of jet stability in a unified manner, showing how various forces affect the stability and hence droplet production.

Where it is not explicitly stated, the jets are projected into a non-reacting medium and they are Newtonian in behaviour.

2.2.1. Basic Stability Theory and the General Equations of Motion

Dynamic theories of jet stability are concerned with fluid mechanic, or as it is more usually called hydrodynamic, stability theory. Stability theory is concerned with the growth of system parameters when subject to a perturbation away from their steady state values. If the system parameters revert to their original steady state values then the system is considered as being stable. If the perturbations grow in magnitude with time then the system is unstable.
For a system to be stable all possible modes of disturbance must be stable. There exist also states of neutral stability on the boundary between stable and unstable regions.

Considering the steady state condition all the system parameters \(X_1, \ldots, X_j\) remain unaltered with time. If it is now assumed that one parameter is altered and subject to a perturbation several important features of stability theory will become evident. The first feature is the question of whether the products of the perturbation amplitude \(c\) can be ignored in the stability analysis. If they are ignored then only an infinitesimal disturbance is being considered and the analysis is linearised. Only non-linear analyses can take into account the finite amplitude of the initial disturbance level.

Linearised analyses have achieved a large amount of success explaining fluid mechanic stability phenomena but recently several non-linear theories have been successful in explaining other phenomena\(^{125,131}\).

If the form of a linearised solution is examined further features will become evident. The disturbance is assumed to be of an arbitrary, but periodic, nature and can be represented as a superimposition of modes. The case which is relevant to jet stability is that which occurs in cylindrical geometry where

\[
\phi = \phi(r, z, \theta, t)
\]

Where \(r\) is the radial axis, \(z\) the axial distance and \(\theta\) the azimuthal angle, \(t\) is time. All possible modes of an arbitrary disturbance are represented by

\[
c = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} c_m(k) e^{ikz + im\theta} dk
\]
As can be seen from equation (2.2) the jet surface can be represented by

$$r = a_o + c_{mk} e^{i(kz + m\theta)}$$  \hspace{1cm} (2.6)

The parameter $a_o$ is defined by considering the volume of fluid under one wavelength of disturbance. This is invariable and hence

$$\text{Volume} = V/2 \int \int r^2 d\theta dz$$  \hspace{1cm} (2.7)

Hence if $a_o$ is the radius of the undisturbed jet or steady state value

$$a^2 = a_o^2 + 1/4c_{mk}^2$$  \hspace{1cm} (2.8)

which gives approximately for $m = 1,2,\ldots$

$$a_o = a \left(1 - \left(1/8\right)c_{mk}^2/a^2\right)$$  \hspace{1cm} (2.9)

and when $m = 0$

$$a_o = a \left(1 - \left(1/4\right)c_{mk}^2/a^2\right)$$  \hspace{1cm} (2.10)

Hence equations (2.9), (2.10) and (2.6) define the jet surface and are a boundary condition.

Another feature of the stability theory which is particularly important is the determination of the mode of maximum instability.

The importance of this mode can be explained by an argument which is after Chandrasekhar (8). If a system is characterised by a series of unstable modes $k_1, k_2, \ldots, k_n$ the amplitude of which will increase as

$$R_t^p e^{p t}$$

(p being positive real) then the mode of maximum instability will be that mode which has the largest real part. It is by this mode $k^*$ that the system will tend to break-up when subject to an arbitrary disturbance since its amplitude will be

$$e^{(R_o^p - p^*)t}$$

relative to any disturbance and this tends to infinity with time. Therefore unless there is a superimposed disturbance of a large initial amplitude which at the break-up time $t_b$ has an amplitude
To investigate the system stability all possible modes represented by \( m, k \) (the azimuthal and axial wave numbers) must be investigated. As can be seen from Equation (2.2) \( m \) is a discrete variable and has integer values. This must be so for a continuous surface. \( k \) is a continuous variable. \( c \) is subscripted to show its dependance on \( m \) and \( k \).

Since only linearised analyses are being considered, the time dependance of \( c \) can be eliminated by seeking solutions of the form

\[
c_{m,k}(r,t) = c_{m,k}(r)e^{P_{m,k}t}
\]

(2.3)

The exponent \( p \), which is the linearised growth rate, is subscripted to show that it is a function of \( m \) and \( k \). The problem is now reduced to solving the general equations of motion of fluids for \( P_{m,k} \) will be generally complex. For stability \( \text{Real} (P_{m,k}) \) will be negative and for instability \( \text{Real} (P_{m,k}) \) will be positive. The neutral stability case occurs when \( \text{Real} (P_{m,k}) = 0 \). If there exists neutral stability there will be a stationary pattern if \( \text{Real} (P_{m,k}) = 0 \) means that \( \text{Imag} (P_{m,k}) = 0 \) and oscillatory motion if \( \text{Imag} (P_{m,k}) \) need not be zero.

In this thesis we are concerned with the break-up of a liquid jet within another medium which may, or may not, influence its stability. Hence we are concerned with solutions of the general equations of motion for liquids. These equations are firstly the continuity equation for an incompressible fluid.

\[
\nabla \cdot \vec{v} = 0
\]

(2.4)

and secondly the momentum equation

\[
\rho \frac{D\vec{v}}{Dt} = \nabla \cdot \vec{T} - \nabla \cdot \vec{\bar{P}} + \vec{F}
\]

(2.5)

Where \( \vec{T} \) is the shear stress tensor and \( \rho \) the liquid density, \( \vec{\bar{P}} \) the and \( \vec{F} \) the force density or a combination of force densities.
greater than that of \( k^* \) then the maximum growth rate disturbance will tend to be the controlling mode of instability.

The forces which may be brought to bear on a jet to influence its stability will now be examined and the special case solutions will be given and compared with the experimental evidence that is available.

### 2.2.2. Capillary Instability

(a) Not affected by the Surrounding Medium

The capillary instability of a liquid jet is the most fully researched area of jet stability theory. The capillary force is introduced into the equations of motion by the inclusion of a boundary condition that the excess pressure at the fluid interface is

\[
\Delta \bar{P} = T \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]  

Where \( R_1 \) and \( R_2 \) are the principle radii of curvative. This can be expressed for small perturbations as

\[
\Delta \bar{P} = \left( \frac{1}{a} - \frac{1}{a^2} \right) \left( c + \frac{\partial^2 c}{\partial z^2} \right) - \frac{\partial^2 c}{\partial z^2}
\]

(Lord Rayleigh in 1879 produced an inviscid linearised analysis by a Lagrangian method. This considered the jet as a conservative system and equated potential and kinetic energies. His analysis proved to be the basis of all subsequent work on jet stability. Rayleigh's analysis agreed with Plateau's that a jet is stable to all but axisymmetric disturbances of wavelength greater that the jet's circumference; that is, disturbances whose dimensionless wave number is less than unity (\( K < 1.0 \)). However, Rayleigh examined the modes of instability and found that the characteristic equation of growth rates for axisymmetric disturbances was given by

\[
P_0^2 = \frac{T}{\rho a^2} K \frac{I_1(K)}{I_0(K)} (1 - K^2)
\]

Where \( I_1 \) and \( I_0 \) are first and zero order modified Bessel functions.
of the first kind. The above function has a maximum at \( K = 0.697 \) which is the mode of maximum instability.

Rayleigh (9) investigated his theory experimentally using stroboscopic illumination and spark photography. A periodic vibration was applied to the nozzle producing a regular mode of break-up. The applied disturbance was of large enough initial amplitude to swamp all naturally occurring vibrations. The idea of using controlled vibrations had been tried earlier by Savart (102) and Magnus (56).

Recently Pierce (87) has shown that by using controlled vibrations with a liquid jet the jet is acting as a preferential wave amplifier. C.V. Boys (6) did in fact construct a device in which the jet acted as an amplifier and demonstrated its use in his renowned series of lectures to children.

A feature to note is that the growing waves are stationary with respect to the jet and as such the velocity of propagation for the wave is that of the jet. This point was emphasised by Vivdenko and Shabalin (122).

The analysis into the stability of Newtonian viscous liquids was begun by Rayleigh (94) who considered the limiting case when viscous forces predominate over the fluid's inertia. His result was that the jet had a maximum growth rate disturbance at \( K = 0.0 \). The result suggests that infinitely long wavelength disturbances will tend to break up highly viscous jets. This is consistent with the observation of such liquids as treacle and molten glass which are not easily resolved into droplets and break-up into long ligaments.

The first qualitative analysis based on an arbitrary value of viscosity was carried out by Weber (127). This was based on some experimental work by Haenlein (32). A characteristic equation for the growth rate was obtained. This was
Where \( n^2 - k^2 = \frac{p_0}{\mu a^2} \) and \( \mu \) is the liquid viscosity.

Equation (2.14) is not explicit in terms of the growth rate \( p_0 \) because \( n \) is contained in the arguments of the Bessel functions. Weber overcame this problem by a series of approximations to the values of the Bessel functions and obtained a quadratic equation.

\[
\frac{p_0^2}{2} + \frac{K L_0(K)}{I_1(K)} \mu a K^2 \left[ 2 \frac{L_0(K)}{n!^2} \frac{2K^2 (L_0(K)K - L_0(K)K)}{I_1(K)K} \right]
\]

(2.14)

The right hand side of equation (2.15) became zero at \( K = 0 \) and \( K = 1 \) when the values of \( p_0 \) are zero and a negative value. It can thus be seen that viscous jets are stable at \( K = 0 \) and \( K = 1 \) and that between \( K = 0 \) and \( K = 1 \), since \( p_0 \) has one positive real value and one negative value, the jet is unstable. Viscosity can be seen not to affect the stability criteria for break-up. An increase in viscosity was, however, shown to decrease the wavenumber of the maximum growth rate disturbance.

A similar analysis was carried out by Chandrasekhar (8) who obtained a characteristic equation similar to that of Weber. Chandrasekhar did, however, directly use this equation and by interpolation obtained a series of growth rate curves for varying values of the parameter \( J \). This dimensionless parameter is defined by

\[
J = \frac{T a \rho}{\mu^2} = \frac{Re^2}{We}
\]

(2.16)

Where \( We \) is the jet Weber number and \( Re \) its Reynolds number. \( J \) defines the jet system. When \( J = \infty \) the system can be regarded as inviscid and the solution reverts to Rayleigh's solution (9a).

Rutland (9b) indicated that for \( J \geq 100 \) jets can be regarded as inviscid.
When $J \approx 0$ viscous forces predominate. The growth rate curves obtained by Chandrasekhar are given in Fig. (2.1.) It can be seen that there is a reduction in growth rates with increasing viscosity and, as stated earlier, the maximum growth rate disturbance wavenumber is reduced. A similar analysis to Weber's is given by Levich (50).

The stability of non-Newtonian liquid jets, particularly visco-elastic jets, has been studied by several workers. Middleman (73) was the first to produce a theoretical prediction of visco-elastic jet stability using an analysis similar to that of Weber (27). Middleman used a linearised analysis with a linearised visco-elastic model for the fluid. A three constant Olroyd fluid (70) was assumed, having a defining equation of

$$T_{i,j} + \lambda \frac{\partial T_{i,j}}{\partial t} = -\mu_0 \left( E_{i,j} + \sigma \frac{\partial E_{i,j}}{\partial t} \right)$$

(2.17)

Where $\lambda_i$ and $\sigma_i$ are visco-elastic constants and $\mu_0$ represents the long term fluid viscosity or zero shear rate viscosity. $\lambda_i$ the relaxation time for the removal of shear stress and $\sigma_i$ is a retardation time, and represents the decay of strain if the stress are removed. Equation (2.17) has been applied to describe suspensions of elastic solids and dilute polymer solutions (28).

The theoretical result of Middleman was that, contrary to what is found experimentally, the addition of elasticity increased instability. Middleman did note, however, that the important effect of normal stress generation in the nozzle and the relaxation of this stress afterwards could not be included with his linear model which strictly requires the fluid to be relaxed as it leaves the nozzle.

Goldin et al (27) produced a generalised linear visco-elastic model using the concept of complex viscosity (4) and used this in Weber's theory (127). The complex viscosity for a linear visco-elastic fluid is defined as
Capillary Instability Growth Rates for Viscous Jets.
\[ \mu_c(a) = \frac{\mu \prod_{j=1}^{N-1} (a + \lambda_j)}{a \prod_{j=1}^{N-1} (a + \sigma_j)} \] (2.18)

\( \lambda_i \) and \( \sigma_j \) are relaxation and retardation times and \( a \) is a complex variable which upon substitution into stability theory become the linearised growth rate exponent \( \rho \).

Discussing the general case of a linearised visco-elastic model, Goldin et al concluded that its degree of instability would be higher than that of a Newtonian fluid with an equivalent zero shear rate complex viscosity \( \mu_c(0) \), but not as high as an inviscid liquid. Goldin et al (27) experimented with several dilute polymer solutions but did not model their liquids. Their results with weakly elastic fluids confirmed their predictions but more strongly elastic fluids proved to be more stable than the corresponding Newtonian fluid. They considered this lack of agreement to be a result of normal stress generation. This, being a non-linear phenomena, is not able to be considered by linearised theories. Photographs of their jets breaking up showed highly non-linear waveforms and highly stable inter-crest filaments.

Kroesser and Middleman (47) modelled their fluids (dilute polymer solutions) on the Maxwell model which is represented by

\[ T_{ij} = \lambda_i T_{ij} - \frac{\mu_c(0)}{\rho a^4} (E_{ij}) \] (2.19)

This is equivalent to the Olroyd model as examined by Middleman (73) with \( \sigma_i = 0 \). Using this model they introduced a parameter to describe elastic effects.

\[ E_l = \frac{\lambda_i \mu_c(0)}{\rho a^4} \] (2.20)

The difficulty of measuring the relaxation time of weakly elastic
fluids was by-passed by using information from a molecular theory of visco-elasticity. However, doubt must be expressed over data obtained in such a manner where experimental evidence is poor. Kroesser and Middleman (47) used the elasticity parameter $E_1$ for the correlation of break-up data over three orders of magnitude of $\lambda$, and found by varying the length of the nozzle used, that normal stress generation played a significant part in the increased stability of elastic jets.

Lenczyk and Kiser (50) also found that normal stress generation played an important part in the increased stability of elastic jets. They examined the break-up lengths of non-Newtonian but inelastic fluids and compared these lengths with visco-elastic fluids. They were unable to measure the elasticity of their fluids, attempting to use a Wiessenberg Rheogoniometer.

Goldin et al (28) in a later paper examined the break-up of very viscous non-Newtonian liquids which were shear thinning and had a finite yield stress. In their earlier paper (27) one such set of solutions (Carbapol 9434 in water) had given a break-up length less than that of water. The break-up length of their shear thinning non-Newtonian fluids was found to be similar to Newtonian fluids of similar average viscosity based on flow through the nozzle. The fluids used were thixotropic and had a time of reformation to their original structure, but this reformation was carried out without the normal stress generation associated with elastic fluids. Goldin et al (28) postulated that this recovery time for shear thinning liquids is important in determining the stability of such liquids, and such liquids that are difficult to break-up have a short recovery time compared with the jet break-up time, and those which are easily broken-up have a slower recovery time.

In the early 1960's two independent experimental investigations were carried out which renewed interest in Newtonian jet break-up and
in particular interest in the role on non-linear effects. Crane, Birch and McCormack (11) revived the method used by Rayleigh (95) and provided controlled perturbations to the nozzle to investigate the stability of water jets. Crane et al (11) used a frequency generator and an electrical vibrator to provide regular perturbations. They noticed that the jet appeared to exhibit non-sinusoidal waveforms near to break-up. Linearised theories are strictly concerned with small initial disturbances and cannot allow for other than sinusoidal waveforms.

Crane et al (11) plotted break-up times, and lengths for varying frequency and wavelength and found only reasonable agreement with Rayleigh's theory (93). Water can be regarded as an inviscid fluid in this application. By plotting the natural logarithmic disturbance amplitude against distance along the jet a non-linear relationship was obtained, although the mean, exponential growth rate did approximate to Rayleigh's theory. Their photographs showed another non-sinusoidal characteristic in the form of "bunching" at high disturbance accelerations. McCormack et al (60) investigated both these effects and produced analyses of them.

The non-sinusoidal nature of the surface wave, when small disturbances were used, was thought by McCormack et al to be the result of velocity modulation. A second order approximation was proposed which accounted for this modulation. A theoretical analysis for the "bunching" at high accelerations was also proposed, this too was accounted for by velocity modulation. In a later paper McCormack et al (61) showed the similarity between water jets under high accelerations and the effect produced by Hartmann (36) who produced discs or plates on mercury jets by electromagnetic fields. Hartmann's analysis was used to assess the percentage velocity modulation.
Donnelly and Glaborson (16) investigated the growth rate of disturbance at varying wavenumbers to obtain information about the theories of Rayleigh (93) and Chandrasekhar (8). Their photographs showed a marked non-linear effect although close agreement with these theories was claimed. This was shown later by Yuen (131) to be a result of their measurement method. Donnelly and Glaborson (16) perturbed their jets by a loudspeaker attached to the frame of their apparatus but with more viscous jets they adopted the more direct approach of perturbation by a pin attached to the loudspeaker cone which penetrated the jet surface.

Wang (125) was the first to produce a non-linear capillary instability theory. This was an inviscid theory extending Rayleigh's linear analysis (93) to higher orders. Rayleigh's first order analysis was included as a boundary condition. The solution was a power series in the initial disturbance amplitude up to the third order. Wang's result showed that the maximum growth rate disturbance had a finite amplitude effect and could be given by

\[ K = 0.697 - \eta_0^2 3.255/4 \]  

(2.21)

Where \( \eta_0 \) is the initial disturbance amplitude for non-linear theories which is dimensionless with respect to the jet radius. Wang's analysis was not valid over all wavenumbers and fails at \( K = 1.0 \) and 0.5. Wang also considered the effect of finite amplitude on the stationary states. That is waves which are stationary with respect to an observer and as such have a velocity of propagation equal and opposite to that of the jet.

Yuen (131) developed a similar third order non-linear theory using the method of strained coordinates (121). The analysis complied with Rayleigh's to the first order but contained the effects of higher harmonics of the disturbance wave. The second order term contained
both the first harmonic and a purely time dependent term to affect the addition of volume from the fundamental. This conserved the volume contained by the wave to the second order. The third order term contained the second harmonic. Each harmonic was assumed to be sustained by the transference of energy from lower harmonics and, up to some cut off point for that harmonic, by its inherent instability. The equation of the jet surface is

\[
\eta = \eta_o \cos K \zeta \cosh \omega \tau + \eta_o^2 \left[ B_{2a}(\bar{\tau}) \cos 2K \zeta \right. \\
\left. - \frac{1}{8} \left( \cosh \omega \tau + 1 \right) \right] \eta_o^3 \left[ B_{3a} \tau \cos K \zeta \right. \\
\left. + B_{3a}(\bar{\tau}) \cos 3K \zeta \right] \tag{2.22}
\]

\( \eta \) is the disturbance wave, dimensionless with respect to the undisturbed jet radius, \( K, \zeta, \tau \) are strained coordinates of wavenumber, axial distance, and time, all of which are dimensionless. Time is dimensionless with respect to \( (\tau^2 a^{-2} \rho^{-\frac{1}{2}}) \) and \( \eta, \omega, K, \zeta, \tau, B \) are all coefficients whose values are given by Yuen (131).

From Equation (2.22) it can be seen that the third harmonic is sustained up to \( K = 1/3 \) by both inherent instability and transference of energy from lower harmonics. Above this point only transference of energy occurs. \( K = 1 \) defines the cut off point for the fundamental. This is defined as

\[
K = \frac{K}{K_c} \tag{2.23}
\]

Where \( K_c \) is defined as

\[
K_c = 1 + \frac{9}{16} \eta_o^2 \tag{2.24}
\]

and so defines the cut off wave number in normal dimensionless form.
The wave profile plotted by Yuen showed non-sinusoidal characteristics.

The non-linear theory as given in Reference (131) contained several typographical errors in the coefficients of his solution. Two of these errors were corrected and published by Rutland and Jameson (98). A third error has been noted by the author (132) and is given here

\[ \frac{d}{\delta t} = \left[ 2 \kappa I_b - 3 - 3\kappa / I_a \right] \beta / (32) - \left[ 24 \left( \begin{array}{c} b_2 - 1 \\ \kappa^2 + 3\kappa I_a (8b + 5) \\ + \left( 8b_2 - 11 \right) \kappa / I_a \end{array} \right) \right] \beta / (256) + \left( 10 + \right. \\
\left. 3\kappa^4 - \kappa^2 - 16b_2 \left( 1 - \kappa^2 \right) \right) \kappa / (256) \]

(2.25)

The correction is printed in red.

Yuen (131) explained Donnelly and Glaberson's (16) agreement with the linearised Rayleigh theory (93) as being due to their method of measurement. Donnelly and Glaberson measured growth rates by the difference between the diameter of neighbouring wave necks and crests. (see Fig 2.2.) Yuen (131) showed that this method would cancel any second order effects and only third order effects would be present. Yuen points out that these would be of minor importance even up to jet lengths of one metre with water jets. The Yuen theory does in fact show a difference in growth rate of the necks and crests and this
Method of Growth Rate Measurement used by Donnelly and Glaberson (18).
was verified experimentally by Goedde and Yuen (26).

Goedde and Yuen (26), using an apparatus similar to that
of Donnelly and Glaberson (16), showed that initially the neck grows
faster than the crest. This is the result of the conservation of
volume. Later on, non-linear effects became important and the crest
grows faster. Goedde and Yuen (26) were unable to verify Yuen's
prediction as to the variation of cut off wave number with initial
amplitude. This was because the magnitude of the initial
disturbance was small and near to \( K = 1 \) \( K_c \) could not be measured
accurately. They did show that the mean of the neck and crest
growth rates agreed with the linearised theories of Rayleigh (93)
and Chandresekhar (8).

Yuen's theory (131) showed an intermediate swelling on the wave
at \( T = 22 \) and \( K = 0.3 \) which was considered by Yuen to be the result
of a breakdown in his theory. Rutland and Jameson (98) observed
such intermediate swellings and considered them to be the cause of
the production of satellite drops (89). They showed that the
swellings were not the result of higher harmonics in their perturbation
system. Rutland and Jameson (99) directly compared a wave profile
they obtained with the Yuen theory at \( T = 46.7 \). Agreement was not
good as the Yuen theory predicted four crests per wavelength and the
experimental evidence showed only three.

Nayfeh (76) critically reviewed Yuen's theory and suggested
that Yuen's theory is not strictly valid near to the wave cut off point
as at \( K = 1 + (9/16) \eta_o^2 \) the wave was shown to exhibit growth.

Nayfeh suggested that the cut off point should be represented by
a series which to the second order is

\[
K_c = 1 + (3/4) \eta_o^2
\]  

(2.26)

Nayfeh (76) proposed the use of multiple time scales as a method
of solution. If the time scales are \( T_o , T_1 , T_2 \), then we have

\[
T_o = T \quad T_1 = \eta_o T \quad T_2 = \eta_o^2 T
\]  

(2.27)
The time scales $\tau_0$ and $\tau_1$ are valid near to $K = 1.0$ and $\tau_0$ and $\tau_2$ valid elsewhere. Nayfeh (76) showed how the analysis proceeds to the second order. A third order theory would require the use of a fourth time scale $\tau_3 = \eta_0^3 \tau_0$.

Dumbleton (19) calculated the cut off wavenumber by integrating the surface area over one wavelength. This gave a value up to the order of $\eta_0^4$ for the surface area $S$. The criteria of instability being $\frac{dS}{d\eta} \leq 0$ and from this the value of $K_0$ was found to be in agreement with Yuen's (131).

2.2.2. Capillary Instability.

(b) Affected by the surrounding media

The analyses of Weber (127) and Tomotika are important in this field. Weber examined the stability of jets projected into gases whilst Tomotika considered liquid-liquid systems.

Weber (127) examined the influence of air on the capillary instability of a jet by axisymmetric waves. The air was considered to be an inviscid but compressible fluid. Weber's result was the addition of a term to his linearised viscous theory

$$- \frac{\rho}{\rho'} \frac{V_j^2}{2a^2} K \frac{H_1(iy)}{H_0(iy)}$$

Where $\gamma = \frac{V_j}{V_s}$, where $V_s$ is the velocity of sound, $V_j$ the jet velocity, $\rho'$ the density of air and $H_0$, $H_1$ are Hankel functions.

Grant and Middleman (31) incorrectly quoted this result and subsequently used this result to predict the maximum in the break-up length curve. They found that their application of the Weber theory did not adequately predict this point but Weber (127) himself was also unsuccessful in predicting this point when he used Haenlein's (32) data.

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Haenlein (32) noted that transverse waves were formed on high velocity jets projected into air. This led Weber to produce a further analysis for transverse $m = 1$ mode waves using similar assumptions to those for the axisymmetric case (127). Weber's result was that the growth rate could be described by

$$P_i^2 + \frac{3\mu K^2}{\rho a^2} + P_i = \frac{10\mu}{3\rho a^2} + \frac{3\mu K^2}{\rho a^2}$$

$$= -\frac{\rho}{\rho'} v_j^2 \frac{K}{a^2} H_i(\gamma) - \frac{2T K^2}{a^2 \rho}$$

(2.29)

Where $y$ is as defined previously and $H_i'(Z)$ is the differential of the Hankel function with respect to its argument $Z$. Weber used his theory to investigate the growth rate dispersion curves and found that at the point where air begins to effect the jet the maximum growth rate shifts towards higher wave numbers at first rapidly, and then more slowly.

Levich (51) has presented an approximate analysis for jets projected into air. Chow and Hermans (9) have produced an analysis based on the assumption of a flowing gas surrounding the jet which can be represented by a logarithmic velocity profile.

Tomotika (118) considered a viscous jet projected at a low velocity into another viscous liquid. He derived the generalised equations relating jet/surrounding media interaction and discussed two important limiting cases. The first case corresponded to Rayleigh's result for a highly viscous jet (94) and assumed

$$\frac{\mu}{\mu'} \rightarrow \infty$$

and the second\n
$$\frac{\mu}{\mu'} = 0.$$ 

In the first case Rayleigh's result was recovered, and in the second case it was found that the wavelength of maximum instability was at $K = 0$. This indicated that very long wavelengths would also control
break-up in the second case. A case when the ratio $\frac{\mu}{\mu'} = 0.91$
was also examined but with both fluids assumed to be highly viscous.
A dispersion curve was obtained for the effect of $\frac{\mu}{\mu'}$ on the maximum
growth disturbance but since the fluids inertia was neglected this
too is only relevant to highly viscous fluids.

Christiansen and Hixson (10) investigated the projection of a
low viscosity liquid and a linearised analysis based on inviscid flows
was produced. The growth rate dispersion curve for axisymmetric
disturbances was given as

$$ p = \frac{T K(1 - \omega^2) \rho}{\alpha^2 \left[ \rho' \mu'(K) - \frac{K_0'(K)}{K_0(K)} \rho \right]} $$  \hspace{1cm} (2.30)

Meister and Scheele (64) produced a generalised computer solution of
Tomotika's general equations and showed how the many special cases of
liquid-liquid jet instability can be regarded as special cases of
Tomotika's equations. In a later paper they investigated break-up
lengths of liquid-liquid systems both theoretically and experimentally
(65).

When in liquid-liquid systems the relative jet velocity is
increased the relative motions of the fluid phases become important.
Ranz and Drier (91) showed how the relative motion of the fluids can
cause instabilities of a wavelength much shorter than the circumference
of the jet and modified an analysis by Taylor (115) to correlate
this effect.

2.2.3 Electrical Instabilities

If equation (2.5) is examined then the term can be used to
express the force densities of electrical origin and the equation of
momentum is modified by the replacement of
The above force density $\vec{F}_e$ can be substituted directly for in Equation (2.5) only if incompressible fluids are considered. This is because the use of Equation (2.31) requires a redefinition of the pressure term (2.32).

$$\vec{P} = \vec{Ph} - \frac{1}{2}(\vec{E} \cdot \vec{E}) \xi - \frac{1}{2}(\mu_m \vec{H} \cdot \vec{H}) \chi$$

Where $\xi = \frac{\partial}{\partial \rho} \frac{d \mu_m}{d \rho}$ and $\chi = \frac{\partial}{\partial \rho} \frac{d \xi}{d \rho}$

Where $\xi$ and $\chi$ are the parameters of magneto- and electro- striction which is incompressible dynamics are zero. $\vec{Ph}$ is pressure as defined normally.

The forces which are referred to in Equation (2.31) act only in the region of the jet surface and, using the terminology of Melcher
(68), are surface coupled. As such they do not operate in the liquid bulk and may be alternatively represented as a stress tensor (67) (103).

\begin{equation}
\mathbf{T}_{e,j} = \varepsilon \mathbf{E} \cdot \mathbf{E}_j - \frac{1}{2} \delta_{ij} \varepsilon \mathbf{E} \cdot \mathbf{E} + \mu \mathbf{H} \cdot \mathbf{H}_j - \frac{1}{2} \delta_{ij} \mathbf{H} \cdot \mathbf{H}
\end{equation}

(2.33)

Where \( \mathbf{T}_e \) is the stress tensor of electrical origin, \( \delta_{ij} \) is the Kronecker delta. This then must satisfy the boundary condition at the surface that

\begin{equation}
\mathbf{n} \Delta \mathbf{P} = \mathbf{n} \cdot [\mathbf{T}_e + \mathbf{T}_m]
\end{equation}

(2.34)

\( [ \cdot ] \) represents the jump across the interface \( \mathbf{n} \) is the unit vector normal to the surface \( \mathbf{T}_m \) is the mechanical stress tensor at the surface which normally represents the capillary force.

The other equations which are important in problems of electrical stability are the electro-magnetic field equations of Maxwell (113) which are required for the description of the field. They are

\begin{equation}
\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \nabla \times [\varepsilon (\mathbf{E} - \varepsilon \mathbf{E} \times \mathbf{V})]
\end{equation}

(2.34)

\begin{equation}
\nabla \times \mathbf{E} = \mathbf{J} + \mu \frac{\partial \mathbf{H}}{\partial t}
\end{equation}

(2.35)

\begin{equation}
\nabla \cdot \mu \mathbf{H} = 0
\end{equation}

(2.36)

\begin{equation}
\nabla \cdot \varepsilon \mathbf{E} = \mathbf{q}
\end{equation}

(2.37)

The last term in Equation (2.34) represents the mechanical coupling because of polarization currents. These have been shown to be unimportant (68). The field of magnetohydrodynamics jet stability concerns the above Equations (2.34) to (2.37) in the form

\begin{equation}
\nabla \times \mathbf{H} = \mathbf{J}
\end{equation}

(2.38)

\begin{equation}
\nabla \cdot \mu \mathbf{H} = 0
\end{equation}

(2.39)

\begin{equation}
\nabla \times \mathbf{E} = \mathbf{J} + \mu \frac{\partial \mathbf{H}}{\partial t}
\end{equation}

(2.40)

Electrohydrodynamics concerns

\begin{equation}
\nabla \times \mathbf{E} = 0
\end{equation}

(2.41)
\[ \nabla \cdot \mathbf{E} = \rho \]  
\[ \nabla \cdot \mathbf{J} + \frac{\partial \mathbf{B}}{\partial t} = 0 \]

The last Equation (2.43) guarantees conservation of charge.

### 2.2.4. Magnetohydrodynamic Stability

In magnetohydrodynamics it is convenient to have Maxwell's field equations expressed solely in terms of the magnetic field intensity \( \mathbf{H} \) and the free current density \( \mathbf{J} \). Chandrasekhar (8) has shown how this may be achieved by using the electrical conductivity \( \sigma \) and by substitution Equation (2.40) becomes

\[ \frac{\partial \mathbf{H}}{\partial t} = \nabla \times (\mathbf{\nabla} \times \mathbf{H}) - \nabla \times (\mathbf{R} \mathbf{\nabla} \mathbf{H}) \]  
\[ (2.44) \]

Where \( \mathbf{R} \) is the resistivity = \( \frac{1}{\sigma} \mathbf{H} \).

Chandrasekhar (8) has produced an analysis of the stability of a jet in a uniform axial magnetic field undergoing either gravitational or capillary instability. Since the former is only of astronomical interest only the effect on capillary instability will be presented.

For the case of an inviscid fluid Chandrasekhar's result was

\[ p = \frac{T}{a^2 \rho} \left( K \frac{I_k(K)}{L_k(K)} (1 - \frac{2}{K}) - \frac{H_z}{a \mu_m} K \left( \frac{T}{a \mu_m} \right)^2 I_k(K) \right) \]  
\[ (2.45) \]

From the above growth rate dispersion relationship it can be seen that the magnetic field increases the wavelength at which instability occurs and will stabilize the jet for all \( K \) when

\[ H_z > \left( \frac{T}{2 a \mu_m} \right) \]  
\[ (2.46) \]

Chandrasekhar (8) calculated that a field as small as 100 Weber meter\(^{-2}\) would stabilize a mercury jet.

However, with the inclusion of a finite resistivity the field required was found to be of the order of 100 Weber meter\(^{-2}\) or greater.

Alterman (1) considered a similar problem to Chandrasekhar, that is the magnetohydrodynamic stability of a hollow jet. Alterman...
also included the effects of rotating the jet in the magnetic field.

2.2.5. Electrohydrodynamic Stability

Solutions of the equations concerning the electrohydrodynamic stability of jets have been chiefly centred on the limiting cases of perfectly conducting and perfectly insulating jets. The simplified field equations which govern the stability of a perfect conductor are

\[ \nabla \cdot \mathbf{E} = 0 \]  
\[ \nabla \times \mathbf{E} = 0 \]

and the electric force density is defined as \( \mathbf{F}_e = \mathbf{q} \cdot \mathbf{E} \)

If Equation (2.47) is examined it can be seen that it can be redefined as

\[ \nabla^2 \Phi = -\mathbf{q} \]

Where \( \Phi \) is electric potential. The above equation is Poisson's equation. For regions where no charge can exist the equation degenerates to

\[ \nabla^2 \Phi = 0 \]

For perfect insulators (or dielectrics) the relevant field equations become

\[ \nabla \cdot \mathbf{E} = 0 \]  
\[ \nabla \times \mathbf{E} = 0 \]

and the electric force density

\[ \mathbf{F}_e = \frac{1}{2} (\mathbf{E} \cdot \mathbf{E}) \nabla \mathbf{c} \]

Whether a particular fluid can be regarded as a perfect conductor, or insulator, for the purposes of a particular experiment, centres around the ratio of the charge relaxation time to the characteristic time for fluid movements. The charge relaxation time is defined as

\[ t_r = \frac{\mathbf{c}}{\mathbf{y}} \]

This was derived by Melcher and Taylor (67) from considering a fluid as obeying Ohm's law. If the charge relaxation time is much shorter than the characteristic time for the fluids motion, then the fluid may
be regarded as a perfect conductor.

Bassett (3) was the first to produce a stability analysis of an electrified jet and by assuming an inviscid liquid he obtained a linearised analysis similar to that of Rayleigh (93) with the addition of a term

$$- \frac{q_j^2}{\eta} \left[ 1 - \frac{K_{1}(K)}{K_{0}(K)} \right] \frac{I_{1}(K)}{I_{0}(K)}$$

(2.55)

It should be pointed out that in Bassett's original text the modified Bessel functions of the second kind had a different definition from that now generally accepted (126). This results in a change of sign for Equation (2.55) from the original.

The term presents an increase in stability for $K < 0.6$ and increased instability for $K > 0.6$. Since under normal conditions we assume the maximum growth rate disturbance controls break-up, the reported enhanced stability of weakly electrified jets (95) was not explained by Bassett's theory. The theory did, however, qualitatively agree with the evidence from more powerful electrifications which show an increase in instability.

Taylor (116) noted that Bassett's analysis contained a numerical error and gave the electrification term as

$$- \frac{q_j^2}{4\pi \varepsilon_0 a \rho} \left[ 1 - \frac{K_{1}(K)}{K_{0}(K)} \right] \frac{I_{1}(K)}{I_{0}(K)}$$

(2.56)

The error did not effect Bassett's qualitative conclusions (3) but would alter any evaluation of growth rates.

Magarvey and Outhouse (64) investigated photographically the break-up of a charged water jet within a closed cylindrical electrode which was kept at ground potential. Since water can be regarded as a perfect conductor, and an inviscid fluid, the experimental technique could be said to approximate to Bassett's analysis. However, since the cylinder was closed at one end, quantitative information could not be
obtained due to the complex nature of the field surrounding the jet. Their experimental evidence did show that at high electrifications break-up was marked by a violent "whipping" action and a rapid decrease in the jet diameter. The latter is often referred to as length extension.

A point to be emphasised is that the author considers electrohydrodynamic models of the instability of a cylinder of liquid inadequate to describe the phenomena when a jet is "electrically driven" (using Taylor's terminology (116)). That is, when the jet would not be formed but for the presence of an electric field which causes the liquid to be removed as either a very fine stream, or directly into droplets in the manner described by Drozin (17), Zeleny (133) and Vonnegut and Neubauer (123) (24). This phenomenon will not be discussed as the the electrohydrodynamic stability of jets, which, by its very nature, requires a continuous jet to be formed. The phenomenon described previously is probably better described in terms of electrostatic atomisation from a surface (84).

Taylor (116) extended Bassett's analysis to investigate the stability of the mode \( m = 1 \) which moves the jet laterally. He obtained the following equation for the growth rates

\[
p_i^2 = -\frac{K^2}{a^3 \rho} \frac{I_i'(K)}{I_i(K)} \left( 1 - \frac{q_j^2}{4\pi \epsilon_o a^2 \rho} \right) + \frac{K}{K_i(K)} \frac{I_i'(K)}{I_i(K)}
\]

(2.57)

This result reverts to Rayleigh's (93) analysis when the surface charge is zero and gives stability of the transverse mode. Instability can occur in this mode when

\[
q_j^2 > \frac{4\pi \epsilon_o a^2 \rho \left( K \frac{K_i(K)}{K_i'(K)} \right)}{2
\]

(2.58)

But whether this mode or the \( m = 0 \) mode grows at any \( K \) depends on whether \( p_i > p_o \). Taylor (116) investigated the stability of a cylindrical soap film co-axial with an electric field in order to
evaluate his theory. He was unable to explain the stability of his soap film by the addition of the \( m = 1 \) mode. Michael and O'Neill (72) showed that the half wavelength of the \( m = 1 \) mode could explain the stability of the film. However, the investigation of the static stability of a soap film cannot yield any useful information on the stability of jets where the half wavelength mode has no physical meaning.

Melcher (68) investigated theoretically the mode \( m = 0 \) to 2 for the case of an earthed jet co-axial with a charged cylindrical electrode. The results obtained for the first two modes were similar to that of Taylor (116) but with the substitution of

\[
q_j = \frac{\Phi_0 \varepsilon_0 2\pi}{\ln(b/a)}
\]

(2.59)

Where \( b \) is the radius of the cylindrical electrode. The mode \( m = 2 \) was found not to produce any wave growth until \( \Gamma \), the electrification constant, was greater than 2.9. The electrification constant for a charged cylinder co-axial with an earthed jet is

\[
\Gamma = \frac{\Phi_0 \varepsilon_0}{a \left( \ln \left( \frac{b}{a} \right) \right)^2}
\]

(2.60)

The first three modes of instability (\( m = 0 \) to 2) and their growth rate curves are shown in Fig. (2.3).

As can be seen in Fig. (2.3) the effect of electrification is to increase the wave cut off points of the modes \( m = 0 \) and \( m = 1 \), thus causing shorter wavelength disturbances of wavenumber \( K > 1 \) to grow.

The complimentary case of an earth cylinder co-axial with a charged jet was given by Schneider et al (108) for the \( m = 0 \) mode using a Lagrangian method. The mode \( m = 1 \) was given by Huebner and Chu (43) using a similar method. Both these analyses are incorrect, by a numerical factor, in their electrification constants. This should be identical to Equation (2.60) and not differing by a factor
The First Three Modes of Instability and their Electrohydrodynamic Growth Rates. (68)
This is because in equating energies they ignored the fact that energy is required to keep the potential difference between the jet and cylinder constant for changes in charge. A more correct analysis would be obtained using a method similar to Michael (70) who also equated energies.

Taylor (116) investigated the stability of jets of water and dilute sodium chloride, using longitudinal electric fields. The fields were produced by either placing a charged plate below the nozzle or by the improved method of a plate above and below the nozzle. The latter producing a more uniform field. Length extension and a transverse mode were exhibited by his jets which showed a marked increase in stability with increasing potential gradient. This was contrary to Taylor's theory of the $m = 1$ mode.

Subsequently Saville (109) investigated the stability of a finite conductivity, and finite dielectric constant, liquid in a longitudinal field. He found that under certain conditions the effect of surface charge was to stabilise the jet, but agreement with the conditions used by Taylor was not obtained.

The stability of a viscous perfectly conducting jet within a cylindrical electric field has been investigated by Michael and O'Neill (71). Their analysis was an extension of that of Chandrasekhar (8). They showed that the fluids viscosity would have no effect on the stability criteria of a jet which is subject to electrohydrodynamic forces. This does not imply that the growth rate is unaffected.

An analysis of a perfect dielectric jet was given by Glonti (25) who considered the effects of viscosity. Glonti considered axisymmetric disturbances, and showed that a longitudinal electric field would have a stabilising effect on a jet. He also stated that a transverse electric field would have no effect as a dielectric jets' stability. Glonti's inclusion of viscous effects gave similar results to that of Michael and
O'Neill (71), namely no change in stability criteria but a decrease in wave growth rates. Nayyar and Murty (77) repeated Glonti's analysis assuming an inviscid fluid and using a Lagrangian method of solution. Glonti's result, in the limit of no viscosity, was recovered.

2.2.6. Gravitational Instability

A linearised solution of the gravitational stability of an infinite cylinder has been obtained by Chandrasekhar (8) for both inviscid and viscous fluids. It is not, however, applicable to terrestrial liquid jets and is only of astronomical interest. However, the solution is important as it has several features in common with other cases. Firstly it was shown by Chandrasekhar that an inviscid cylinder is stable to all but axisymmetric disturbances. The solution for these disturbances being

$$ F_0^2 = 4\pi G \rho K \frac{I(K)}{I_0(K)} \left[ K_0(K) I_1(K) - \frac{1}{2} \right] $$

Where G is the gravitational constant, K is wave number dimensionless with respect to the jet circumference. This has a mode of maximum instability at $K = 0.58$ and a cut off wave number at $K = 1.0668$.

When the effect of viscosity is included, Chandrasekhar (8) found that the stability criteria was not altered by viscosity but the growth rate dispersion curve was affected in a similar manner to that of capillary instability.

2.2.7. Rotational Instability

By rotating a jet a radial pressure gradient is set up within the jet, thus applying another force influencing jet break-up. Pedley (81) reviewed such rotating flows but an analysis for a jet undergoing solid body rotation was produced by Gillis and Kaufman (24). They showed that the criteria, established by Hocking (41), that a jet undergoing solid body rotation is unstable is

$$ 0 < K < \left( \frac{1 + L}{L} \right)^{1/2} $$

(2.62)
Where $L = \frac{T}{\rho \Omega^2}$ where $\Omega$ is the angular velocity. This criteria also applies for viscous jets.

Rutland and Jameson (97) solved the general case of an arbitrary viscosity liquid using an iterative procedure. They plotted growth rate curves for functions of $L$ and $Re_Q$. Where $Re_Q$ is the angular Reynolds number

$$Re_Q = \frac{\rho a^2 \Omega}{\mu} \quad (2.63)$$

The general solution was compared with the special cases of inviscid and high viscosity jets, and also for zero rotation. Experiments were carried out by Rutland and Jameson (97) to verify the calculated growth rates, using an apparatus similar to that of Donnelly and Glaberson (16). The effect of rotation was to increase the growth rates. The agreement with theory was good except at higher wave numbers.

2.3. Other Factors affecting Jet Stability

So far this thesis has been concerned only with the stability of liquid jets which have already been resolved into a cylinder of liquid. This section concerns the resolution of liquid issuing from a nozzle into a cylindrical configuration.

A criteria for deciding whether a continuous jet is formed when liquid issues from a nozzle was proposed by Linblad and Schneider (52). They equated the kinetic energy of the liquid leaving the nozzle with the potential energy of the newly formed surface. This gives the minimum jet velocity to be

$$v_j = \left( \frac{4T}{\rho a} \right)^{1/2} \quad (2.64)$$

This of course ignores the effect of dissipation of heat by viscosity.

Once the liquid issues from the nozzle two effects can be present which affect the size of the jet, namely the effects of profile relaxation and gravitational acceleration. On exit from a nozzle the removal of
viscous shear stress causes the velocity profile to rearrange into a flat velocity profile at some distance away from the nozzle. This causes a change in diameter of the jet. Harmon (35) used a mass and momentum balance to show that an initial jet radius of \( a_n \) at the nozzle will decay to a final value of \( \sqrt{3/2} a_n \), if an initially fully developed parabolic velocity profile is assumed. The uniform velocity achieved at some distance from the nozzle was found to be 4/3 of the mean nozzle velocity. A parabolic laminar velocity profile will occur in a capillary of its length to diameter ratio obeys (29).

\[
\frac{L_n}{d_n} = 0.115 \, Re
\]  

(2.65)

Middleman and Gavis (74) and Gavis (23) showed how viscous dissipation and surface tension can alter the profile rearrangement for Newtonian liquids. Middleman and Gavis (75) also investigated the same effect for non-Newtonian liquids. For Newtonian liquids Middleman and Gavis (74) correlated the ratio \( d_j / d_n \) as a function of Reynolds number \( Re_n \) based on the average nozzle velocity. The value of \( d_j / d_n \) at large \( Re_n \) approached the \( 3/2 \) predicted by Harmon (35), but they noted that at \( Re_n < 16 \) the jets expanded on issue from the nozzle. This was explained in terms of the viscous dissipation of heat.

Goren and Wronski (30) used a boundary layer theory near to the nozzle, and perturbation analysis away from the nozzle, to investigate the shape of jets at low velocities ( \( Re_n < 50 \) ). Brun and Leinhard (7) used a simplified linearised model, neglecting both surface tension and gravity, and produced a numerical analysis that gave reasonable agreement with experiment. From the theory of Brun and Leinhard (7) the distance from the nozzle at which 95% profile relaxation is achieved is

\[
\frac{L}{d_n} = 0.025 \, Re_n
\]  

(2.66)
For the limits $50 \leq Re_n \leq 1050 \quad We_n \leq 100$

Duda and Vrentas (18) produced a similar but more complex analysis introducing the effects of both surface tension and gravity, but their result varies little from that of Brun and Leinhard (7).

Gravity will have a significant effect on the jet if its Froude number is sizable:

$$ Fr_n = \frac{\tau_n^2}{L_b g} \quad (2.67) $$

Equation (2.67) is for a vertically projected jet. For a horizontally projected jet $L_b$ is replaced by $a_n$. Brun and Leinhard (7) discussed the effects of gravity on a horizontally projected jet. For a vertically projected jet the degree of contraction due to gravity may be approximately found by a simple momentum balance.

2.4. Droplet Production

2.4.1. Capillary Instability

It was thought by Plateau (89) that a jet would divide into droplets of size equivalent to a cylinder whose length was the jet's circumference. This was based on his work on the maximum stable length of a cylinder of liquid. Rayleigh (93) subsequently showed that the maximum growth rate disturbance for an inviscid jet was $K = 0.697$. This disturbance is assumed to break-up the jet under all but controlled conditions. (16). This assumed disturbance gives a droplet of diameter $1.89d_j$. Harmon (35) related this to the nozzle diameter by assuming complete decay from a fully parabolic velocity profile at the nozzle, to a flat profile near break-up. This gave the relationship

$$ d_j = 1.63d_n \quad (2.68) $$

for the inviscid case, and

$$ d_j = 1.63d_n \left( 1 + 3Z \right)^{\gamma_c} \quad (2.69) $$
The Method of Rutland and Jameson (98) used to calculate Main and Satellite Drops.
for the viscous Newtonian case using the theory of Weber (127). \( Z \) is the Ohnesorge number.

The above analyses ignore the fact that more than one droplet per disturbance wavelength is generally formed. Plateau (89) and Savart (102) both noted the production of smaller drops between the main drops and these were named by Plateau "spherules". Rayleigh (95) also reported the production of satellite drops when using tuning forks to provide controlled perturbations but, since his linearised analyses failed to predict any wave deformation, Rayleigh considered that the satellite drops were produced by higher harmonics present in the forks. Rayleigh (95) observed satellite drops stroboscopically when using tuning forks of 370 Hz downwards. When this is related to \( K \) it appears he noted the production of satellite from \( K = 0.48 \) downwards.

The first theoretical prediction of more than one drop per wavelength was made by Rutland and Jameson (98) who used the Yuen non-linear theory (131) as a model for break-up. The Yuen theory gave a secondary wave between the wave crests at longer times. Rutland and Jameson (98) assumed that when the wave profile as predicted by Yuen reached the axis of the jet break-up would occur, and they predicted the relative sizes of the main and satellite drops by the method shown in Fig. (2.4). The volume beneath the secondary wave was assumed to provide one satellite drop and the volume beneath the main wave to provide the main drop. No volume rearrangement during break-up was assumed. The predicted sizes of the main and satellite droplets are shown in Fig. (2.5.).

These represent smoothed data from two initial amplitudes of \( \eta_0 = 0.1 \) and 0.01. The initial amplitude was found to have little effect on dropsize. The theoretical curves were smoothed by the addition of a correction for lack of agreement with volume conservation, since with the Yuen theory volume is not conserved above the second order.
As can be seen from Fig. (2.5.) Rutland and Jameson's theory predicts that the size of the satellite drop would exceed that of the main drop at lower wave numbers. Using an apparatus similar to that of Donnelly and Glaberson (16), Rutland and Jameson investigated these curves using water jets of 4 m.m. nozzle diameter. It was found that at lower wave numbers the size of the satellite drops did exceed that of the main. Where several satellite drops occurred the total volume of the drops was included as one for verification of the model. Rutland and Jameson's predicted curves, shown in Fig. (2.5.), show no satellite drops above K = 0.7. This was not found experimentally, as under all conditions used satellite drops were produced.

This production of satellite drops under all conditions conflicts with the claimed evidence of several workers ((100), (52), (80), for example) whose aim was to produce a stream of monosized droplets from a vibrating capillary. These workers claim that by the alteration of flowrate or disturbance frequency the production of satellite drops could be eliminated. Rutland and Jameson (88) explained this lack of agreement as being due to either the operation of these devices at the satellite/main droplet cross over point, or to some scaling effect present with the small jets used by these workers.

Several of the devices which have been constructed to produce monosize droplets have however utilised a pendent drop mechanism, such as characterised by Harkins and Brown (33), rather than break-up from a continuous jet. Under this mechanism the drops will be removed from the nozzle when the force applied to the drop, by gravity or the acceleration due to the vibration of the device, becomes equivalent to the capillary forces. In Table (2.1.) those references which the author considers to be utilizing this mechanism, together with those who appear to use the break-up of a continuous jet, are given.
Predicted Drop Sizes of Rutland and Jameson (98).
Table (2.1.)

Devices which claim to produce Monosized droplets

<table>
<thead>
<tr>
<th>Mechanisms</th>
<th>Continuous Jet</th>
<th>Pendant Drop</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dabora</td>
<td>(12)</td>
<td>Magarvey and Taylor (55)</td>
</tr>
<tr>
<td>Atkinson and Miller</td>
<td>(2)</td>
<td>Mason, Jarayatne and Woods (58)</td>
</tr>
<tr>
<td>Linblad and Schneider</td>
<td>(52)</td>
<td>Rasbash (92)</td>
</tr>
<tr>
<td>Ryley and Wood</td>
<td>(100)</td>
<td>Samuels and Sparks (101)</td>
</tr>
<tr>
<td>Schotland</td>
<td>(109)</td>
<td></td>
</tr>
<tr>
<td>Schneider and Hendricks</td>
<td>(107)</td>
<td></td>
</tr>
<tr>
<td>Park and Crosby</td>
<td>(80)</td>
<td></td>
</tr>
<tr>
<td>Ström*</td>
<td>(114)</td>
<td></td>
</tr>
<tr>
<td>Wisseima and Davies*</td>
<td>(129)</td>
<td></td>
</tr>
</tbody>
</table>

* all but these used hypodermic needles.
2.4.2. Capillary Instability with effect of Surrounding Media

The size of droplets formed from the capillary instability of jets projected into gases when the surrounding media affects break-up does not appear to have been the subject of much research. Merrington and Richardson (69) carried out experiments of jet break-up from nozzles on aircraft and on high towers where air resistance effects break-up. They found that the dropsizes produced were independent of surface tension and depended only upon liquid viscosity and relative fluid motions. These results correlated to

\[
\frac{v_n d_d}{\frac{1}{2} \rho R_s} = 500
\]  

(2.70)

Hence it is likely that break-up was by purely air friction and turbulence and not by capillary instability.

Dropsizes from jets in liquid-liquid systems have been subject to more research since the surface area of the droplets is an important parameter in liquid extraction systems. Keith and Hixson (46) found that nozzle velocity increased the total surface area up to some maximum which was slightly lower than the maximum break-up length. They produced a correlation for this maximum surface area condition based on dimensional analysis. Christiansen and Hixson (10) using their inviscid liquid theory and assuming one drop per disturbance wavelength obtained an expression for the dropsizes in low viscosity liquid-liquid systems. This was that

\[
d_d = 1.92 d_j
\]  

(2.71)

They found this to be so for a range of liquids ± 7%.

Meister and Scheele (66) studies more viscous systems and included the effect of interfacial velocity. They also pointed out that when a jet is short the formation of a drop is more a matter of a force balance than instability theory, and produced a dropsize equation based
on this mechanism using the theory of Scheele and Meister (105).

2.4.3. Electrohydrodynamic Stability

Huebner (42) investigated the production of droplets from a charged water jet surrounded by an earthed cylinder with one closed end. Later (43) the work was repeated with a collector which more closely followed the infinite cylindrical geometry assumed for the theory. The results of both experiments were correlated with the electrification constant and compared with the theoretical values assuming one drop per disturbance wavelength at the maximum growth rate disturbance. It appears, that the satellite drops which appeared were ignored in the correlation, as in the earlier paper (42) a bi-modal drop size distribution was reported. Break-up was by the \( m = 0 \) mode at lower electrifications and the \( m = 1 \) mode at higher electrifications. The size of droplets was lower than predicted for both sets of experiments (43).

A similar series of experiments was carried out by Peskin and Raco (85) for dielectric jets in longitudinal electric fields. Using the same assumptions concerning droplet formation as Huebner (43) a prediction of droplet size was made. Fair accuracy was achieved but presumably satellite drops were ignored.

Schneider et al (108) investigated the break-up of a charged water jet in an earthed cylinder in a similar manner to Huebner but provided controlled perturbations to the jet by means of a piezo-electric transducer. No attempt was made to measure growth rates or variation of cutoff frequency, with electrification but, the charge per drop was measured and compared with the theoretical value. Wave numbers were restricted to a region where satellite drops were claimed to be absent.

2.4.4. Rotating Jets

The production of droplets from rotating jets was investigated experimentally by Rutland and Jameson (97) who found that in the ranges

\[
3 \leq \text{Re} \leq 30
\]

\[
0.5 \leq L \leq 6.3
\]
The main dropsizes correlated to
\[ d_d = 1.6 K^{-0.313} d_j \]  
(2.72)
and the satellite drops to
\[ d_s = 2.21 e^{-1.65 K} \]  
(2.73)
Rutland and Jameson (97) noted that at \( L < \) the main crests grew very rapidly due to the instability being dominated by rotational effects. This produced a wave form of almost spherical bulbs of liquid joined by long ligaments. The effect was found to increase the size of the satellite drops and at \( L = 0.43 \) they found that the satellite drops were larger than the main. When compared directly with non-rotating jets the dropsizes were affected by an increase in the size of the satellite drops with rotation.

2.5. Break-up Length

A great deal of experimentation has been devoted to investigating the prediction of the stable length, or break-up length, of a liquid jet using the linearised theories discussed in Section (2.2).

The argument which is used to show how the break-up length may be calculated proceeds as follows. Firstly it is assumed that break-up occurs when the amplitude of the growing wave is comparable to the jet radius. Secondly the disturbance is considered to be the maximum growth rate disturbance \( \rho^* \). Since this is an exponential growth rate than the following relationship holds.

\[ t_b = \frac{1}{\rho^*} \left( \ln \left( \frac{a}{c^*} \right) \right) \]  
(2.74)
Where \( t_b \) is the jet break-up time and \( c^* \) is the initial disturbance amplitude. The value of \( \rho^* \) can be obtained from the relevant linearised theories as discussed in Section 2.2. The break-up length for constant jet velocity is
\[ L_b = v_j t_b \]  
(2.75)
Where \( v_j \) is the jet velocity and \( L_b \) the break-up length. What remains unknown is the value of \( c^* \), the initial disturbances level amplitude.
This must be found by experimentation. Haenlein (32) found the value of $\ln \left( \frac{a}{c} \right)$ to be 12 for jets projected into air, and Meister and Scheele (65) found it to be 6 for liquid-liquid systems. However the value of $c^t$ must be regarded as a constant of the particular apparatus concerned, and since the disturbance level can also be affected by mechanical vibrations from nearby machinery, the environment near the nozzle is also important in determining the factor experimentally.

The above procedure was first used by Smith and Moss (112) in 1917. They used mercury jets projected in air and showed that there was an initially linear portion in the $L_b - v_j$ curves as is predicted using Rayleigh's theory (93). Tyler (119) investigated the maximum growth rate disturbance using water jets. He found that the wavelength which controlled break-up in the capillary instability region of the $L_b - v_j$ curves was slightly longer than that predicted by Rayleigh's theory. This has been thought by Rutland (98) to be because of the small diameter nozzles used causing viscous effects, which lengthen the wavelength of maximum instability.

So far only jets which are not affected by aerodynamic effects have been discussed. As can be seen in Section (2.2.2.) at higher velocities these effects become important and control break-up. A typical $L_b - v_j$ curve is shown in Fig. (2.6.). The region near to the origin is where there is no continuous jet. Then from A to B is a linear portion obeying either Rayleigh's theory (93) or for viscous jets Weber's (127). At some point C is a maximum caused by aerodynamic effects and from D to E these aerodynamic effects show an increasing affect, shortening the jet's break-up length.

Ohnesorge (78) classified the different forms of droplet formation from nozzle into four regimes. These are now presented in order of increasing jet velocity.
A Typical Jet Break-up Curve.
1. Pendant Drop regime
2. Capillary Instability
3. Aerodynamic Effects with Capillary Instability
4. Disintegration or Atomisation

The first three regimes were specified by Ohnesorge using Haenlein's data (32), by plotting the Ohnesorge number $Z$ against Reynolds number $Re_N$. The advantage of this plot is that for a given liquid and nozzle the Ohnesorge number is constant and the plot should predict in what characteristic regime the jet should break-up for a given velocity. Haenlein (32) used only one form of nozzle and as such the initial disturbance level did not vary significantly. Because of this, Ohnesorge's plot is found to be of little use.

Weber's analysis of the effect of aerodynamic forces on capillary instability (127) postulated that the maximum in the growth rate curve occurs because aerodynamic forces begin to have an effect on the $m = 0$ mode of break-up near this point. Weber (127) also showed that at increased velocities when aerodynamic forces control break-up then the $m = 1$ mode will cause break-up.

Grant and Middleman (31) used the theory Weber developed for the influence of aerodynamic forces on the $m = 0$ mode to predict the maximum in the break-up length and found it to be inaccurate. They proposed two empirical amendments to Weber's theory. They also noted that after the maximum in the \[ L_b - V_j \] curve, break-up still proceeded by the mode $m = 0$. This has been noted by Haenlein (127) and was in fact used by him in his analysis. They did, however, wrongly attribute to Haenlein the idea that the maximum was the point where break-up changed from $m = 0$ to the $m = 1$ mode because, it was in fact, assumed by Haenlein to be the point when aerodynamic forces significantly affect the $m = 0$ mode.
Fenn and Middleman (21) examined the effect of ambient pressure on the maximum break-up length and found by experimentation that the ambient air pressure had no effect with $We' < 5.3$. Above this value the maximum point was associated with a change in mode from $m = 0$ to $m = 1$. Phinney (86) attempted to explain the maximum point by a change in the initial disturbance level. He plotted the parameter $\ln\left(\frac{a_n}{c}\right)$ against $Re_n$. The plot showed a break-away point at a critical $Re_n$. No physical explanation was given for this point although he thought that, since the critical $Re_n$ was of the same order as the transition from laminar to turbulent flow in a pipe, there might be some connection.

At some point after the onset of the transverse mode ($m = 1$) the violent disintegration or atomisation reported by many workers (78) (32) (110) occurs and the concept of break-up length has no meaning.

At any point in the "$L_b - \nu_j$" curve transition to turbulence may occur. The prediction of the point at which this occurs could prove difficult since it is dependent on nozzle design. Schweitzer (110) points out that there is also the possibility of a semi-turbulent jet occurring when short tube nozzles are used (e.g. orifice plates). Semi-turbulent jets appear turbulent near the nozzle but contain a turbulent core within a laminar envelope. The value of the critical Re for pipes could be used to predict turbulence if the nozzle length/diameter ratio was greater than the value given in Equation (2.63).

Grant and Middleman (31) correlated break-up lengths for turbulent jets in the region $200 < (We') > 10$

$$\frac{L_b}{d_n} = 8.51 (\frac{\nu_j}{We})^{0.64}$$

(2.76)

At higher We this may give a break-up length which has little physical significance.
The prediction of break-up lengths in liquid-liquid jet systems has been discussed by Meister and Scheele (65) and experimented evidence has been obtained by Keith and Hixson (46) and Tyler and Watkin (120). Attempts at using a similar approach to Equation (2.74), with the substitution of the relevant $p^*$, to give a prediction of break-up length have not met with a great deal of success. Meister and Scheele (65) consider the jet break-up length to be very dependant on interfacial velocity and to be described by an integral approach modifying Equation (2.74) to

$$\int_0^{L_b} \frac{dz}{v_i} = \frac{1}{p^*} \ln \left( \frac{a_i}{c_i} \right)$$

(2.77)

Where $v_i$ is the interfacial velocity. Meister and Scheele (65) modified the analysis of Duda and Vrentas (18) for the jet velocity profile to produce an analysis for liquid-liquid systems. This predicted the break-up length by

$$L_b = \left[ \frac{a_i^2 v_i}{a_n^2} \right]_{z=5a_n}^{z=L_b} + \left( \frac{a_i^2 v_i}{a_n^2} \right)_{z=L_b} \frac{1}{2p^*} \ln \left( \frac{a_i}{c_i} \right)$$

(2.78)

The interfacial velocity $v_i$ was calculated as the average from 5 nozzle diameters and at break-up, and the above equation was solved by iteration. $v_i$ was obtained from graphs given in their paper. Reasonable agreement with experiment was obtained and the value of $\ln \left( \frac{a_i}{c_i} \right)$ was found to be 6. Droplet merging after "break-up", giving longer jet lengths, was noted.
CHAPTER THREE

3. Droplet Formation from Capillary Instability

Although the stability of jets to capillary forces has been well investigated, as was seen in Chapter 2, the size of the droplets formed was little investigated until recently. This chapter describes the author's work in the field of capillary instability, the aim of which was to investigate more closely the mechanism of break-up and its effect on droplet size, using the technique of high speed ciné photography and also to examine the predictions of Rutland and Jameson (98) concerning the sizes of droplets formed.

The work of Rutland and Jameson (98), as reported in detail in Section (2.4.1.), indicated that the sizes of the main and satellite droplets could be predicted using the Yuen theory (131) as a model for break-up. It was also indicated by Rutland and Jameson that it should be possible to operate a jet, provided with controlled disturbance, at the main-satellite drop cross-over point where a monosized stream of droplets would result. Hence, it was decided to repeat the theoretical calculations of Rutland and Jameson and investigate experimentally the production of droplets from water jets to test the theory.

It was also decided to examine the reason why several workers have claimed to produce monosized drops by the use of small nozzles. An attempt would be made to discover if this results from a scaling effect, or by operation at the main-satellite drop cross-over point, both of which were reasons proposed by Rutland and Jameson (98), or whether in fact it was the result of some other effect.

3.1. Theory

The Yuen theory (131) together with corrections noted by Rutland and Jameson (98) and the author in Section (2.4.1.) is used in this section.
To obtain a dropsize prediction the wave profile is required at the earliest point where some part of the jet surface approaches the jet axis. It is at this point that break-up is assumed to occur. The values of the dimensionless break-up time $T_b$ for initial perturbations $\eta_0 = 0.01$ and $0.1$ were computed and are given in Table (3.1) and plotted on Fig. (3.1). The values were obtained by the computer program Master JET2 which was written in FORTRAN IV code and run on an I.C.L. 1904A. A copy of this program is given in Appendix A for reference. The values of break-up time and the wave profiles at break-up agreed closely with those obtained by Rutland and Jameson (98).

By assuming that all the volume enclosed by the main wave becomes the main drop and all the volume enclosed by the secondary becomes a satellite as in Fig. (2.4), a prediction of dropsizes can be made. To obtain the droplet values, and hence diameters, the volume under the jet surface needs to be obtained by integration. Since the wave is both axisymmetric and (in the theory) symmetric about the half wavelength, the following integrations were carried out

\[ V_m = 2\pi \int_0^{z_c} (1 + \eta) \, dz \]  
\[ V_s = 2\pi \int_{z_c}^{V_m} (1 - \eta) \, dz \]  

Where $V_m$ and $V_s$ are the respective main and satellite drop volumes. $z_c$ is the point on the jet axis, $z$, where the surface approaches the jet axis. These integrations were carried out using a FORTRAN IV computer program Master DROP which was run on an I.C.L. 1904A computer. Use was made in the program of the I.C.L. subroutine F4INTSMP which gives numerical integration by Simpson's rule with difference correction and self adjusting step length. This computer program is also given in Appendix A.
Table 3.1.

Dimensionless Break-up Times

<table>
<thead>
<tr>
<th>K</th>
<th>$\eta_0 = 0.01$</th>
<th>$\eta_0 = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>129.14</td>
<td>68.40</td>
</tr>
<tr>
<td>0.10</td>
<td>65.47</td>
<td>39.01</td>
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<tr>
<td>0.15</td>
<td>44.57</td>
<td>27.11</td>
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<tr>
<td>0.20</td>
<td>34.42</td>
<td>21.02</td>
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<tr>
<td>0.25</td>
<td>28.59</td>
<td>17.45</td>
</tr>
<tr>
<td>0.30</td>
<td>24.92</td>
<td>15.18</td>
</tr>
<tr>
<td>0.35</td>
<td>22.50</td>
<td>13.68</td>
</tr>
<tr>
<td>0.40</td>
<td>20.81</td>
<td>12.57</td>
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<td>0.45</td>
<td>19.37</td>
<td>11.54</td>
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<td>0.50</td>
<td>31.26</td>
<td>16.55</td>
</tr>
<tr>
<td>0.55</td>
<td>17.02</td>
<td>9.50</td>
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<td>0.60</td>
<td>16.25</td>
<td>9.11</td>
</tr>
<tr>
<td>0.65</td>
<td>15.64</td>
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<td>0.95</td>
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<td>11.58</td>
</tr>
<tr>
<td>1.00</td>
<td>374.74</td>
<td>26.85</td>
</tr>
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</table>
Fig. (3.1)

Predicted Dimensionless Break-up Times.
As with Rayleigh's linearised theory the volume of liquid contained under one wavelength is not strictly conserved, the theory conserves mass, only to the second order. The volume error is shown in Table (3.2) along with the error defined by Rutland and Jameson (98). That is

\[ \% \text{ Error} = 100 \left( \frac{d_i - d_j}{d_j} \right) \] (3.3)

The smoothed dropsizes were obtained by allocating the volume error in proportion to the predicted dropsize. These dropsizes were also calculated by Master DROP. The theoretical curve given in Fig. (3.2) shows the mean value of dropsizes as predicted for the two perturbation values. This was done because in practice it is not possible to specify the initial perturbation magnitude. It can be seen, however, from Table (3.2) that an order of magnitude change in initial perturbation amplitude has little effect on the dropsizes produced.

At the point \( K = \left( 1 + \frac{9}{16} \eta_0^2 \right) \) 0.5 the coefficient in the Yuen theory tends to infinity. This has the effect of causing a discontinuity in the break-up time as shown in Fig. (3.1). At this point stability is predicted by the theory. This disagrees with Rayleigh's first order solution (93) and is contrary to observation and therefore must be regarded as a failure of the theory at this point. The theory fails at \( K = 0.5 \) which corresponds to the failure of the Wang (125) third order theory at \( K = 0.5 \), bearing in mind that the Yuen theory has strained coordinates. This singularity has an effect on neighbouring wave numbers and gives anomalous dropsizes at \( K = 0.5 \) where it is predicted that no satellite drop is formed. The point \( K = 0.5 \), and the narrow band of values effected by the singularity, were omitted when Fig. (3.2) was plotted. The inclusion of the value of dropsizes at \( K = 0.5 \) can be seen to be the cause of the local minimum in Rutland
Experimental and Predicted Dropsizes.
### Table 3.2 (a)

**Predicted Drop sizes**

\[ \eta_0 = 0.01 \]

<table>
<thead>
<tr>
<th>K</th>
<th>Unsmoothed Main</th>
<th>Smoothed Main</th>
<th>Unsmoothed Satellite</th>
<th>Smoothed Satellite</th>
<th>% Error*</th>
<th>% Volume Error</th>
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As defined by Rutland and Jameson (98).
Table (3.2) (b)

Predicted Dropsize

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* As defined by Rutland and Jameson (98).
Predicted Drop Sizes of Rutland and Jameson (98).
and Jameson's predicted dropsize curves. Their theoretical results are shown in Fig. (3.3) and Table (3.3).

Rutland and Jameson's predicted dropsizes can be seen to differ markedly with those obtained by the author. In particular Rutland and Jameson predicted a cross-over point in the sizes of the main and satellite drops where the size of the satellite drop begins to exceed that of the main, whereas the author's predicted curve (Fig. (3.2) ) show no such a point. The trends of the data from Master DROP were checked by hand calculation and a similar check was carried out on the wave profile given by Rutland and Jameson (98) in their paper and these trends confirmed the author's prediction as shown in Fig. (3.2).

From a copy of Rutland and Jameson's computer program (45) it appears that the difference in the predicted dropsizes has occurred because in their program Rutland and Jameson (98) allowed two different values of $Z_0$ to be used. This has the effect of allowing the limits of integration to overlap causing an increase in the size of the satellite drops and a change in the volume error experienced.

The volume error with the author's use of the Yuen theory can be seen to be of the order of 50%. This shows the reverse trend to that shown by Rutland and Jameson (98), that is, that the volume error was found to be the highest at the wavenumbers with the shortest break-up time. This large error is undoubtedly the result of the theory not strictly conserving volume. This becomes very important when the Yuen theory is extended to break-up where it is not strictly valid because of the large values of time.
Table (3.3)

Dropsizes Predicted by Rutland and Jameson (98)

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3.2. **Experiments with Water Jets.**

A series of experiments with jets of water was carried out. Water can be considered as behaving inviscidly since its Chandrasekhar parameter $J$ is of the order of $10^4$ even with small jets. Slightly smaller bore nozzles than those used by Rutland and Jameson (98) were used. Since the largest size of hypodermic needle commonly available is around 3 m.m. diameter this was chosen as being the basis of one set of experiments.

To investigate if there is any scaling effect present in break-up an apparatus using an order of magnitude smaller nozzle was constructed. The two sets of apparatus are referred to in this thesis as the large and small scale apparatus.

3.2.1. **Large Scale Apparatus.**

The apparatus is shown in schematic form in Fig. (3.4) and a general view in Fig. (3.5). The apparatus consisted of an air pressure pumping system which feeds a flow quietening chamber and hence the nozzle. An audio perturbation system was placed below the nozzle. The water was collected in a receiving vessel and could be returned to the pressure vessel if required.

The system was designed so as to produce break-up by the controlled perturbations and hence, it was a design requirement that all uncontrolled disturbances, that is naturally occurring, should be minimised.

The parts of the apparatus which were in contact with the water were constructed from either brass, copper, polyethylene or duralumin so as to reduce the possibility of contamination of the water by corrosion products. The water used was previously distilled and had a surface tension of $71.5 \text{ dyn cm}^{-1}$ when measured by de Nouy tensiometer.

To provide both a steady flow rate and a smooth flow pattern, the air pressure pumping system was employed. The water was contained in a 100 litre copper pressure vessel which was maintained at 23 p.s.i.g.
Schematic Diagram of Large Scale Apparatus.
Fig. (3.5)

A General View of the Large Scale Apparatus.
The effective head of water decreased as the level of the water dropped during a run, but because of the large cross-sectional area and the pressure used, this decrease was less than 1% over a run of thirty minutes. The flowrate for each run was set up by a rotameter.

After being pumped from the pressure vessel by air pressure the water passed through a filter which removed any coarse suspended material that could affect the flow. A length of polyethylene flexible tubing led to the remainder of the apparatus which was fixed to a "vibration free pad". This pad was isolated from the floor of the laboratory and had separate foundations. The flexible tubing and the quietening chamber were suspended by soft springs which had a period of natural vibration of \( \frac{1}{2} \) second. This period was such that it did not allow vibrations in the frequency band for growth to be transmitted to the nozzle. The perturbation equipment was mounted via the laboratory floor and so was isolated from the nozzle mountings.

A detailed drawing of the quietening chamber is shown in Fig. (3.6). The function of the chamber is to remove any flow disturbances by, firstly, acting as a reservoir and secondly by smoothing out any disturbances by means of a packed bed of spheres contained in its upper half. This packed bed also served to remove any entrained air. This was collected in the domed portion of the chamber and periodically removed by means of a blank nozzle and the bleed valve on top of the chamber.

The water then passed into the more tapered section which ensured laminar flow on entry to the nozzle. The nozzles were 10 gauge Hamilton hypodermic needles with specially machined ends. Attachment to the quietening chamber was by means of a Luer lock and an adaptor which was bored out to the nozzle diameter.

Although several nozzles were tried, all measured runs were carried out using the longest needle (2 inches) which gave the longest
Fig. (3.6)

Detail of Flow Quieting Chamber.

Air Bleed Valve

Packed Bed

of Spheres

Inlet

Luer Lock

Nozzle
stable length. The Ln/dn ratio of this capillary was 20. This was not high enough to give fully parabolic flow if considered alone, but since the flow into the nozzle was essentially laminar it is very likely that there was a fully parabolic profile in the nozzle. The length of the nozzle probably assisted in damping out flow disturbances. Unperturbed jets of at least 40 c.m. length were observed when no disturbance was imposed on the jets.

The perturbation apparatus was a Taylor 192 A audio frequency generator which drove a 12 inch 8 Ω loudspeaker through a Sinclair Z50 amplifier. The amplifier in this application was rated at 10 watts R.M.S. The loudspeaker was mounted in close proximity to the jet, but down stream from the nozzle so its sound field could not directly affect the nozzle. The waveform produced by the perturbation apparatus was checked at amplifier output by oscilloscope to ascertain if any harmonics were present. After break-up the droplets were collected in a polyethylene vessel and the water returned to the pressure vessel on completion of a run.

3.2.2. Small Scale Apparatus

The flow rate required for an order of magnitude change in nozzle diameter fell outside the range of the larger apparatus, therefore, a small scale apparatus was constructed to operate with these smaller nozzles. This is shown in schematic form in Fig. (3.7) and a general view in Fig. (3.8).

The apparatus consisted of the same basic units as the larger scale apparatus with the exception of any facility for returning the used water.

An air pressure pumping system similar to the larger apparatus was used. The water in this case was held above a Millipore prefilter through which it passed when the air pressure was applied. Typical air pressures were about 15 p.s.i.g. The pressure cylinder was
Fig. (3.7)

Schematic Diagram of Small Scale Apparatus.
Fig. (3.8)

A General View of Small Scale Apparatus.

Fig. (3.9)

Detail of Small Scale Apparatus showing arrangement of Earphone & Nozzle.
constructed from "Perspex".

The water flowed through the filter into the bore of a syringe which had had its piston removed. The nozzles used were again hypodermic needles. This time 24 and 27 guage and attached to the syringe by a Luer lock fitment. These sizes of needles correspond to inside diameters of 0.3 and 0.2 m.m. Both nozzles were two inches in length and were examined by microscope to ensure that they had 90° ends. The ends were made burr free by electro-chemical etching in a chromic acid solution.

The perturbation apparatus was also similar to that of large scale, but the loudspeaker was replaced by an 8 Ω earphone. This was because the 12 inch loudspeaker had too large a sound field to be useful. A small loudspeaker was tried, but, at the high frequencies required by the small jets, attenuation by the air was too great. To overcome this problem an earphone was used with a glass rod attached by epoxy resin to its diaphragm. The rod was then placed in contact with the Luer lock fitment of the nozzle. The arrangement of the earphone is shown in Fig. (3.9).

3.2.3. Measurement and Photographic Observation.

The jets from the large scale apparatus were observed by high speed cine photography and jets from both sets of apparatus were measured using high speed still photography.

The photographic system for measurement on the large apparatus consisted of a Kodak Specialist 3 4 x 5 inch camera using a 150 m.m. f5.6 Schneider - Krenzauch lens with Ilford commercial ortho film. A transparent rule was placed in the plane of the jet for reference. Illumination was provided by an E.M.I. Type 6 stroboscope placed behind a diffusing screen. The stroboscope was linked to the shutter release of the camera which was set a 1/30th second. This allowed the film to be exposed by a single flash from the stroboscope triggered by the
shutter release. The jet was observed stroboscopically during the setting up of the camera shot.

A signal from the frequency generator was fed to the stroboscope, thus linking the frequency of the imposed wave with that of the stroboscopic illumination. The regular motion of the jet was thus "frozen" by this process and this allowed accurate camera positioning to be obtained, as well as giving information of break-up by visual observation.

Measurements of both jet wavelength and its resultant dropsize were made from these photographs. The negatives were enlarged to 20 times the original onto Kodak Projection Print Film 4588 which has very high dimensional stability. Measurements were made by a P.C.D. type ZA£ digital data reader, referring on each photograph to the rule which was included as part of the enlargement. Wavelengths were measured from crest to crest measurements of several waves. The diameter of the jet was calculated from a knowledge of the wave frequency and flowrate. This method was found to be in agreement with the direct measurement of an undisturbed jet.

Because of their size (48), the drops formed from the break-up of these larger jets did not immediately form spheres but oscillated about a spherical form exhibiting ellipsoidal shapes. The size of the drops was obtained by measuring their major and minor axes and obtaining their equivalent diameter by assuming the shapes to be oblate or prolate spheroids. Since the break-up occurs in an axisymmetric mode (3) the equivalent diameter is then expressed as

\[ d_d = \left( d_s \ d_r \right)^{\frac{2}{3}} \]  

(3.4)

Care was taken to measure only those drops whose form approximated to these assumed shapes and "dumb-bell" shapes, in particular, were avoided.

For the observation of the jets by high speed cine film a 16 m.m.
Fastax camera was used. This was operated at around 5,000 pictures per second. Continuous illumination was used, provided by a 1.K.W. Quartz Iodine lamp placed behind the diffusing screen. Setting up of the camera shot was by stroboscope. Kodak Plus – X reversal film was used and the film viewed by a Vanguard M – 16C projection head fitted to the digital data reader. The projection head was capable of advancing the film at single or multiple frame intervals.

The measurement system for the smaller jets was similar to that of the large scale apparatus. An Exacta 35 m.m. camera, fitted with a Tessar f2.8 50 m.m. lens was used. It was found necessary to use a bellows extension. Illumination was again provided, for both photography and observation, by the stroboscope and this was operated in a similar manner. Kodak Pan-F film was used and the negatives were enlarged onto Kodak Projection Print 4588 or onto Kodak Bromide paper WSG.3S which is a waterproof paper and also has high dimensional stability. A glass rod of 1 m.m. diameter was used to provide a scale for enlargements and enlargements were typically 70 times overall.

Measurement from the positives and processing of the results were as with the larger apparatus.

3.2.4. Method

Both sets of apparatus were operated in a similar manner. The pressure vessels were pressurised and the flow set at the start of each run. The flow was set, in the case of the larger apparatus by rotometer and in the case of the smaller apparatus by fine adjustment of the pressure. Jet velocities for both sets of experiments were 200-300 c.m. sec^-1. This was considered as being high enough to reduce the effects of acceleration due to gravity.

The frequency generator was set to the required value and the amplitude of the disturbances altered until the imposed wave broke-up the jet in a significantly shorter distance than the undisturbed jet.
This ensured break-up was controlled by the imposed wave.

The waveform of the disturbance was checked by oscilloscope to ensure it was sinusoidal and the flow rates were measured by direct collection of the jet before and after each photograph or cine film.

The setting up of the photograph was carried out with the stroboscope driven by the frequency generator, but the stroboscope was switched to being triggered by the shutter release when the still photographs were being taken.

At the end of each run the air pressure was removed and in the case of the larger scale apparatus the water was pumped back into the pressure vessel for re-use.

3.2.5. Results

3.2.5.4. A Description of Jet Break-up.

In this section it is intended to present the qualitative results obtained from the high speed ciné films of the jets produced by the larger nozzles and to describe the phenomena that occur during jet break-up by capillary instability.

A sequence of enlargements from a typical ciné film are shown in Fig. (3.10) to illustrate some of the complex phenomena that takes place unseen by the naked eye and which even stroboscopic illumination does not fully reveal.

The typical result, shown in Fig. (3.10) will now be described. Although the number of satellite drops, and relative length of the inter-crest ligament, is particular to the wave number used, it is intended that this example should indicate to the reader what phenomena occur at break-up.

The jet conditions for the film were:-

\[
\begin{align*}
\text{Flowrate} & \quad \text{c.m.}^3 \text{Sec.}^{-1} \\
\text{Frequency} & \quad \text{Hz} \\
\text{Jet Diameter} & \quad \text{dj} \quad \text{m.m.}
\end{align*}
\]

\[
\begin{align*}
= 3.249 \\
= 130 \\
= 1.45
\end{align*}
\]
A Typical Sequence of Jet Break-up.

Fig. (3.10)
Jet Velocity \( V_j \) c.m. Sec.\(^{-1} \) = 197

Wavenumber \( K \) = 0.302

The film speed was 4800 (p.p.s.), and, hence, the time interval between each picture shown (12 picture intervals) was \( 2.5 \times 10^{-3} \) sec.

Progression of time is from left to right on the diagram. It is possible to scan diagonally and follow the progress of one wavelength of the jet as it breaks up. The particular wavelength under consideration is marked on the first picture, \( P_1 \). It should be noted that a sequence of events can also be seen on each individual photograph, as each wavelength can be considered to show the same phenomena as the wavelength above at a time interval equal to the period of the applied disturbance. Hence it can be seen that the marked ligament in \( P_1 \) will eventually form two satellite drops. These are to be seen in the lower portion of \( P_1 \).

How this occurs will now be described.

As can be seen from the first picture the shape of the wave prior to break-up is non-sinusoidal and at the wavenumber used in the illustration there is a long inter-crest ligament on which there is a secondary wave as predicted by Yuen (131). The wave is barely perceptible on this enlargement but was seen more clearly using the P.C.D. reader and Vanguard projection head. At the point in time represented by the first picture the ligament is about to become detached from the down stream crest.

In the second picture \( (P_2) \) the ligament, because of capillary forces, has begun to roll back towards the upstream crest. The sharp end of the ligament has now been replaced by an almost spherical end and a capillary wave can be detected on the ligament surface. The ligament then detaches itself from the upstream crest, as can be seen in \( P_3 \), due to the main disturbance wave continuing to grow. Several capillary waves have now grown on the ligament.

In \( P_4 \) the ligament is now fully detached at either end. Capillary
waves have now appeared at the upstream end of the ligament, and at the lower end the waves are beginning to show a marked growth. P5 shows that the two sets of waves grow and, in this case, a total of six capillary waves are present. The waves in this picture have begun to interfere at some point nearer the upstream end of the ligament. The main drop in this picture is now beginning to alter towards a more prolate form.

Eventually, as shown in P7, the ligament can become split up because of the capillary waves and in this case it has formed two satellite drops. The main drop has now become prolate and because of its size will continue to oscillate for a long time. Shorter ligaments have been seen to roll up into only one satellite drop. Presumably, the longer ligaments allow the capillary waves more time to develop as they roll up and, hence, break-up the ligament before it has time to revolve itself into a sphere and so produce more than one satellite drop.

3.2.5(b) Quantitative Results

The dropsize results obtained from high speed still photographs of both large and small diameter jets are given in Table (3.4) and plotted on the predicted curve on Fig. (3.2) along with Rutland and Jameson's (98) results. If more than one satellite drop was formed from the secondary wave they were taken as being one drop for the purpose of comparison with theory, and the equivalent diameter was plotted.

Between $K = 0.25$ and 0.5 the agreement with the theoretical curves is good. However, about $K = 0.2$ the agreement is poor and as can be seen from Fig. (3.2) near this point the size of the satellite drops exceeds that of the main. An explanation for this lack of agreement can be found if the predicted wave profile at break-up $K = 0.2$ (Fig. 3.11) is compared with the actual profile. It must be noted that the predicted profile is exaggerated in scale since the abscissa is
Table (3.4)

Dropsizes from Experiments with Water Jets

<table>
<thead>
<tr>
<th>K</th>
<th>Frequency</th>
<th>Flowrate</th>
<th>$d_j$</th>
<th>$v_j$</th>
<th>Main</th>
<th>Satellite</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.184</td>
<td>100</td>
<td>3.470</td>
<td>1.37</td>
<td>235</td>
<td>2.10</td>
<td>2.39</td>
</tr>
<tr>
<td>0.188</td>
<td>115</td>
<td>3.356</td>
<td>1.30</td>
<td>251</td>
<td>2.26</td>
<td>2.17</td>
</tr>
<tr>
<td>0.209</td>
<td>125</td>
<td>3.789</td>
<td>1.37</td>
<td>257</td>
<td>2.18</td>
<td>2.09</td>
</tr>
<tr>
<td>0.212</td>
<td>155</td>
<td>3.958</td>
<td>1.35</td>
<td>285</td>
<td>2.61</td>
<td>1.86</td>
</tr>
<tr>
<td>0.315</td>
<td>210</td>
<td>4.236</td>
<td>1.37</td>
<td>283</td>
<td>2.39</td>
<td>1.29</td>
</tr>
<tr>
<td>0.342</td>
<td>200</td>
<td>4.540</td>
<td>1.47</td>
<td>270</td>
<td>2.13</td>
<td>1.24</td>
</tr>
<tr>
<td>0.345</td>
<td>210</td>
<td>4.286</td>
<td>1.42</td>
<td>270</td>
<td>2.21</td>
<td>1.43</td>
</tr>
<tr>
<td>0.345</td>
<td>210</td>
<td>4.286</td>
<td>1.42</td>
<td>270</td>
<td>2.19</td>
<td>1.21</td>
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<tr>
<td>0.354</td>
<td>170</td>
<td>3.351</td>
<td>1.41</td>
<td>204</td>
<td>2.11</td>
<td>1.37</td>
</tr>
<tr>
<td>0.379</td>
<td>155</td>
<td>3.958</td>
<td>1.58</td>
<td>202</td>
<td>1.89</td>
<td>1.19</td>
</tr>
<tr>
<td>0.395</td>
<td>200</td>
<td>3.951</td>
<td>1.47</td>
<td>233</td>
<td>1.99</td>
<td>1.24</td>
</tr>
<tr>
<td>0.443</td>
<td>230</td>
<td>4.195</td>
<td>1.49</td>
<td>242</td>
<td>1.99</td>
<td>1.07</td>
</tr>
<tr>
<td>0.443</td>
<td>230</td>
<td>4.195</td>
<td>1.49</td>
<td>242</td>
<td>1.87</td>
<td>1.07</td>
</tr>
<tr>
<td>0.452</td>
<td>230</td>
<td>4.524</td>
<td>1.53</td>
<td>245</td>
<td>1.82</td>
<td>1.04</td>
</tr>
<tr>
<td>0.470</td>
<td>1950</td>
<td>0.187</td>
<td>0.26</td>
<td>342</td>
<td>2.30</td>
<td>1.00</td>
</tr>
<tr>
<td>0.470</td>
<td>240</td>
<td>4.441</td>
<td>1.52</td>
<td>244</td>
<td>1.96</td>
<td>1.07</td>
</tr>
<tr>
<td>0.480</td>
<td>2000</td>
<td>0.187</td>
<td>0.26</td>
<td>344</td>
<td>2.16</td>
<td>0.97</td>
</tr>
<tr>
<td>0.490</td>
<td>1500</td>
<td>0.238</td>
<td>0.24</td>
<td>227</td>
<td>1.92</td>
<td>0.92</td>
</tr>
<tr>
<td>0.490</td>
<td>2100</td>
<td>0.187</td>
<td>0.26</td>
<td>350</td>
<td>2.27</td>
<td>0.96</td>
</tr>
<tr>
<td>0.529</td>
<td>235</td>
<td>4.382</td>
<td>1.59</td>
<td>222</td>
<td>1.80</td>
<td>0.88</td>
</tr>
<tr>
<td>0.529</td>
<td>250</td>
<td>5.453</td>
<td>1.67</td>
<td>248</td>
<td>1.78</td>
<td>0.87</td>
</tr>
<tr>
<td>0.540</td>
<td>1500</td>
<td>0.246</td>
<td>0.25</td>
<td>213</td>
<td>2.03</td>
<td>0.89</td>
</tr>
<tr>
<td>0.543</td>
<td>250</td>
<td>3.434</td>
<td>1.45</td>
<td>209</td>
<td>1.91</td>
<td>0.86</td>
</tr>
<tr>
<td>0.610</td>
<td>1900</td>
<td>0.110</td>
<td>0.24</td>
<td>238</td>
<td>2.11</td>
<td>0.58</td>
</tr>
<tr>
<td>0.614</td>
<td>300</td>
<td>4.069</td>
<td>1.50</td>
<td>230</td>
<td>1.91</td>
<td>0.87</td>
</tr>
<tr>
<td>0.614</td>
<td>300</td>
<td>4.069</td>
<td>1.50</td>
<td>230</td>
<td>1.73</td>
<td>0.94</td>
</tr>
<tr>
<td>0.618</td>
<td>275</td>
<td>3.414</td>
<td>1.46</td>
<td>204</td>
<td>1.82</td>
<td>0.61</td>
</tr>
<tr>
<td>0.640</td>
<td>322</td>
<td>3.470</td>
<td>1.41</td>
<td>222</td>
<td>1.82</td>
<td>0.93</td>
</tr>
<tr>
<td>0.700</td>
<td>1900</td>
<td>0.101</td>
<td>0.25</td>
<td>210</td>
<td>2.03</td>
<td>0.28</td>
</tr>
</tbody>
</table>
$K = 0.2$
$	au = 21.02$
$\eta_o = 0.1$

Fig. (3.11)

Predicted Wave Profile for $K = 0.2$. 
dimensionless. The theory predicts two secondary waves at this point whilst the experimental evidence in Fig. (3.12) shows only one. The theory at this point does therefore not accurately predict the profile and below $K = 0.2$ it continues to predict two secondary waves whilst one is found experimentally.

Throughout the work described here, and that of Rutland and Jameson (98) both main and satellite drops were produced. This result is in disagreement with several workers, who are listed in Table 2.1, who claimed that a monosized distribution could be produced from fine bore nozzles.

Some of these experiments claimed that the production of satellite drops could be stopped by an alteration of flowrate, or frequency. Rutland and Jameson (98) suggested that this could be due to either a scaling effect causing the ligament to roll back into the main drop, or that the main and satellite drops were formed but were of equal size. The later explanation does not seem likely, as near to $K = 0.2$ the total volume of the satellites was close to that of the mains. The ligament was long and broke up into three satellite drops. Also, the region in which Park and Crosby (80) observed monosized drops was $K = 0.35 - 0.70$. No viscous effect seems likely as Park and Crosby's jets give a $J$ of the order of $10^4$, well outside the limit of $J = 100$ set by Rutland (96) as being the lower limit for inviscid behaviour. Park and Crosby (80) used flow modulation to provide perturbations and it is likely that there was significant velocity modulation in the axial direction.

Park and Crosby did not provide any photographic evidence from their results but Schneider and Hendricks (107) gave photographs of monosized droplet production when using longitudinal perturbations. From Schneider and Hendricks' evidence a possible explanation of monosized drops being formed is that velocity modulation in the axis of the
Break-up of Water Jet at $K = 0.209$. 
jet can cause the ligament to be rolled back. Another explanation is that the satellite drop is formed but coalesces with the main drop within one wavelength's distance or coalesces with the jet itself due to the relative velocity of the jet and drop because of velocity modulation. Some of the small jet experiments carried out by the author produced satellite droplets that coalesced with the main drop soon after break-up. Hence, this seems the more likely explanation. Examples of this phenomena are shown in Fig. (3.14c) and Fig. (3.14d). The effect is more pronounced with higher wavenumbers where the relative difference in size of the main and satellite drop is greater. Typical photographs from the large jets are shown in Fig. (3.13) and from small jets in Fig. (3.14).

An interesting phenomena that can account for many of the claims of monosized droplet production by vibrating capillaries was noticed when small nozzles were used. It was found that it was possible to operate the apparatus at a resonance frequency of the hypodermic needles. At such frequencies monosized drops could be produced from a mode of break-up that resembled a linear combination of the $m = 1$ transverse mode and the $m = 0$ mode. This mode was due to high amplitude vibrations causing the jet to be diverted in the plane of the perturbations. By an alteration of the flowrate either one, two or four streams of droplets could be produced. Examples of this are shown in Figs. (3.15 - 3.17). With one or two streams the divergence of the jet reduced over the jet length and monosized drops were produced. With four streams of droplets the jet continued to diverge over its length and a tri-modal drop size distribution resulted. Table (3.5) gives the experimental details of the runs illustrated in Figs. (3.15 - 3.17).

To check that the phenomena observed was associated with the resonance frequencies a small weight was added to the needle tips. This
Fig. (3.13)
Break-up of Small Water Jets

Fig. (3.14)
### Table (3.5)

**Details of Runs Exhibiting a Resonance Effect**

<table>
<thead>
<tr>
<th>Fig.</th>
<th>Fig.</th>
<th>Fig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.15</td>
<td>3.16</td>
<td>3.17</td>
</tr>
<tr>
<td>Frequency (Hz)</td>
<td>2040</td>
<td>2040</td>
</tr>
<tr>
<td>dj (mm)</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>dd (mm)</td>
<td>0.38</td>
<td>0.34</td>
</tr>
<tr>
<td>No. of Streams</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Flowrate (m³ sec⁻¹)</td>
<td>0.0616</td>
<td>0.0735</td>
</tr>
</tbody>
</table>
Fig. (3.15)

Resonance controlled Break-up.
Resonance controlled Break-up.
Resonance controlled Break-up.
caused the nozzle to exhibit the more normal form of axisymmetric instability at the same frequency independent of flowrate.

The phenomena at its most simple is the effect of high acceleration vibrations on a jet, and can be seen to be the reason why the many workers such as Ryley and Wood (100), Schotland, and Atkinson and Miller, who are listed in Table (2.1), have claimed that monosized drops could be formed by an alteration of frequency or flow rate when using vibrating needles. Wissema and Davies (129) used large nozzles with large amplitude vibrations and they claimed monosized droplet production. They called the point at which satellites no longer occur the "minimum resonance frequency". Their jets also diverged in a similar manner to the author's. Ström (114) also claimed monosized droplet production using an extremely fine nozzle. This too could be explained in terms of high acceleration disturbances.
CHAPTER FOUR

4. The Formation of the Glass Spherules Found in Lunar Fines

The existence of spherical glass particles in the lunar fines recovered by the Appollo space program has been widely reported. For example in the papers by Tolansky (117), Fulchignoni et al (22) and McKay et al (63). It has been thought that these spheres originated from splashes caused by local impact melting when meteorites collide with the lunar surface. However, Tolansky (117) suggested that volcanic blow out could also be a cause.

At the time the research reported in this thesis was being carried out, the Particle Technology research group in the Department of Chemical Engineering was in possession of two Appollo 12 samples, namely samples 12070/45 and 12057/72. Extensive work on the particle size distribution and shape factors of sample 12057/72 was carried out by the late Professor Heywood and is reported in reference (38). It was decided that the knowledge gained from the author's research into the field of droplet formation from capillary jets should be used to investigate what possible processes are involved in the formation of lunar glass spheres.

The presence of dumb-bell and cylindrical shaped particles in the lunar samples provides almost conclusive proof that the spheres were formed from the liquid phase rather than by vapour condensation, this shape of particle being formed by a droplet freezing in a partially resolved state. An example of a dumb-bell shaped glass particle is shown in Fig. (4.1), the length of the particle being 350\(\mu\)m.

The spheres in the sample varied widely in appearance, effective density, and composition (which was determined by electron probe measurement). This inhomogeneity supports the theory of meteorite impact melting rather than volcanic blow out and rules out the possibility of vapour condensation. Volcanic blow out would produce a more uniform
A Dumb-Bell Shaped Lunar Glass Particle.

A Selection of Five Lunar Glass Spheres.
composition since the spheres would be the result of a well inviscid process. A selection of five of the spheres from sample 12070/45 is shown in Fig. (4.2), the colour of which varies from opaque grey, red, yellow to silver, indicating their inhomogenity.

4.1. Particle Size Distribution.

Sample 12070/45 was examined to obtain the size distribution and incidence of spheres. The sample was sieved and each size fraction of the coarser size fractions (>70 µm) was counted by microscope, every particle being examined. The smaller size fractions were sampled and the results normalised. The smaller particles (<20 µm) were counted by scanning electron microscope and the larger (20-70 µm), counted by optical microscope. This laborious work was carried out by Mr. R. Buxton and is reported in Scarlett and Buxton (104).

This incidence and size distribution of spheres is shown in logarithmic form on Fig. (4.3). The crosses represent the incidence of spheres and the solid line the frequency size distribution. The spheres can be seen to be about one in every two hundred particles and are distributed throughout the size ranges. The largest spherical particle was 650 µm and the smallest 0.1 µm, although some of the larger particles were hollow (104). The number of spheres in each size range varies, varying from a few in the range >100 µm to several hundred around 20 µm. The slope of the frequency curve is -3 indicating that a frequency distribution by weight would be almost flat. This can have occurred in a well mixed regolith the spheres in the sample coming from many separate events.

4.2 Glass Sphere Production by Impact Melting.

The formation of the glass spheres by an impact melting process will now be discussed considering not only the conditions required for the splash but also what other conditions would limit the size of the spheres found.
Fig. (4.3)

**INCIDENCE OF SPHERES**

in 0.6760 gms sample no. 12070/45

Frequency Distribution of Spheres

Relative Incidence of Spheres

Size (Micrometres)
4.2.1. Jet Break-up and Sphere Formation.

It is postulated that upon impact a meteorite would liquify the local area around it and cause a splash. There have been no investigations of splashing caused by impact melting although many investigations of the splashing of droplets on a liquid surface have been carried out (130) (34). Engel (20) did in fact use droplet-liquid surface splashes to investigate the depths of craters likely to be formed by meteorite-space vehicle impacts. She assumed that the meteorite and the impact surface would liquify on contact. A photographic sequence of a water drop splash is shown in Fig. (4.4). This was obtained by Hobbs and Kezweeny (40).

As can be seen from Fig. (4.4) a wide range of ligaments and jets can occur from a single impact as not only the "coronet" splash forms jets but there is also a large rebound jet which can be seen to break-up with capillary waves. Hence, one would expect that from a wide range of impacts that there is on the lunar surface, each giving a wide range of drops, that a broad size distribution would result. It is therefore the aim of this investigation to obtain the physical limitations of sphere formation by impact melting.

The theory of Weber (127) can be expressed in terms of the break-up time $t_b$ in the form.

$$ t_b = d_j \ln \left( \frac{d_j}{2C_r} \right) \left[ \frac{12}{3 \text{Re}} \right] $$

This can be seen to be independent of jet velocity. To obtain values of the break-up time a value of $\ln \left( \frac{d_j}{2C_r} \right)$ needs to be assumed, since it is difficult to obtain any experimental data. The value of 12 was assumed as was used by Haenlein (32) for nozzles.

Isard (44) plotted an estimate of the viscosity of lunar glass against temperature. By knowing this variation with temperature the break-up time can be expressed as a function of jet diameter for various temperatures. These are shown on Fig. (4.5) plotted as continuous lines.
A Water Drop Splash

(obtained by Hobbs and Kezweeny (40)).
Jet Break-up and Cooling Times.

--- Time to jet break-up

--- Time to jet cool

Jet diameter in mm

1000K
1200K
1300K
1400K

10
1
0.1
0.01
0.001
0.0001

100
10
1
0.1
0.01
0.001
0.0001

--- Time to jet break-up

--- Time to jet cool

T_K to T_K time to cool

T_1 = 1
T_2 = 10
T_3 = 100
T_4 = 1000

--- Time to jet break-up

--- Time to jet cool

time seconds

10
(1400°K) can be taken as an estimate of the maximum jet diameter. For example, if the initial temperature was 1540°K the maximum jet diameter is 200\( \mu \)m, whilst a jet diameter of \( 9 \times 10^3 \)\( \mu \)m would require an initial temperature of 2800°K. Temperatures over 3000°K are not thought to be probable, as at this temperature de Maria et al. (14) have shown that a large portion of the glass would be vapourized. An initial temperature of 2000°K was taken as it is at this temperature the \( \text{SiO}_2 \) component in the glass begins to vapourize (14).

This gives a limiting diameter of \( 2 \times 10^3 \)\( \mu \)m and this is independent of jet velocity. Hence an upper physical limit to the size spheres has been established, namely that spheres from a jet in excess of \( 2 \times 10^3 \)\( \mu \)m cannot occur since the jet would freeze in an unbroken form.

Another physical limitation is that below a certain velocity a jet cannot be formed. This may be calculated by assuming that the kinetic energy of the jet must be greater than the surface energy created. The equation which gives this was given by Linblad and Schneider (52) and is

\[
\nu_{m,j} = \left( \frac{8T}{d_j \rho} \right)^{1/2}
\]

This gives, for the properties of a typical lunar glass

\[
\nu_{m,j} = 31 \frac{d_j}{\rho} \nu_2
\]

The analysis also ignores any viscous effects which would increase the velocity. Equation (4.5) and the maximum jet diameter due to cooling are plotted on Fig. (4.6).

4.2.2. Other Physical Limitations

One obvious restriction that can be placed on a jet is that its velocity is less than the lunar escape velocity. If this were not so the particles would fail to return to the lunar surface. The escape velocity can be taken as \( 2.24 \times 10^5 \) cms. sec\(^{-1}\) and this is plotted on Fig. (4.6).
Physical Limitations on Sphere Production.
As can be seen, the behaviour of the glass is essentially inviscid for temperatures greater than $1400\,^\circ$K, since the second term in Equation (4.1) is small. It is probable that the spheres are formed above $1200\,^\circ$K since at this temperature the break-up times are long. If it is assumed that the behaviour of the jet is inviscid, and that the maximum growth rate disturbance controls break-up, the size of the primary, secondary and tertiary drops can be estimated using the data obtained in Chapter 3. This data is given in Table (4.1) along with some data of Rutland (96) as an example of viscous break-up.

As can be seen from Table (4.1) the largest particle is 1.8 times the diameter of the jet from which it was formed and the ratio of the largest to smallest particle is 10:1. In the viscous region a more complete study, with varying viscosities and surface tensions is needed to fully investigate the drop size range. However, from the work of Rutland (96) it appears that the likely range is about 3:1.

For solid spherical particles to form the break-up time must be greater than the cooling time of the jet. It will be assumed that the heat is conducted through the cylindrical jet and lost at the surface by radiation. The heat loss by radiation is therefore

$$h = \sigma_s \varepsilon_r (\Theta_i^4 - \Theta_f^4)$$  \hspace{1cm} (4.2)

Where

- $\sigma_s$ is Stefan's constant
- $\varepsilon_r$ the emissivity
- $\Theta_i$ the initial temperature
- $\Theta_f$ the final temperature

Equation (4.2) may be linearised to

$$h = \sigma_s \varepsilon_r (\Theta_i - \Theta_f) \Theta_i^3$$  \hspace{1cm} (4.3)

An estimate of the change in surface temperature was obtained using the charts due to Schneider (106). The broken lines in Fig. (4.5) show the cooling times as a function of the ratio of initial to final temperature. The points where the cooling curves cross the inviscid break-up line..."
Table (4.1)

**Dropsizes from Liquid Jets**

<table>
<thead>
<tr>
<th></th>
<th>INVISCID</th>
<th>VISCOUS*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \frac{d_d}{d_j} )</td>
<td>( \frac{d_d}{d_j} )</td>
</tr>
<tr>
<td>MAIN</td>
<td>1.8</td>
<td>3.1</td>
</tr>
<tr>
<td>SECONDARY</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>TERTIARY</td>
<td>0.2</td>
<td>-</td>
</tr>
<tr>
<td>REFERENCE</td>
<td>Parkin (Chapter 3)</td>
<td>Rutland (96)</td>
</tr>
</tbody>
</table>

* We = 237 and Re = 6.3
If the particles are formed by jet break-up then on return to the lunar surface they must return with a velocity that is equal and opposite to the jet velocity. An experimental investigation by Priemer (90) into the impact fracture of glass spheres was used to plot the relevant line on Fig. (4.6). This line is a function of both particle size and velocity.

The final consideration is that the liquid droplet, once formed, must remain in flight long enough to solidify. Hence, there is a minimum cooling time. The cooling time for the largest droplet formed by jet break-up will be of interest. Hence the cooling time for a sphere of diameter twice that of the jet will be considered. It is also assumed that the height of travel of the particle is small in comparison with the radius of the moon. Hence, by assuming the lunar gravitational acceleration to be $1.6 \times 10^2 \text{ c.m. sec}^{-1}$ the time of flight may be found by

$$t_f = \frac{2v_j}{1.6 \times 10^2} \quad (4.6)$$

The sphere was considered as being solid when at 0.8 times its radius the temperature had fallen from 2000-1000$^0\text{K}$ and the charts of Schneider were used to obtain the cooling time. This cooling time was plotted on Fig. (4.6).

4.2.3. Hollow Spheres

The size distribution of the lunar glass has been affected by the fact that some are hollow. An example of such a sphere whose shell has been broken is shown in Fig. (4.7). The particle is 260 $\mu\text{m}$ diameter with a wall thickness of 5 $\mu\text{m}$ which gives an equivalent solid particle of 125 $\mu\text{m}$ diameter. The spheres could have been made hollow by the volatilisation of some of its lighter components, sodium and magnesium for example. Only 2 c.m. of $\frac{1}{2}$ excess pressure would have been necessary to form the particle in Fig. (4.7) if magnesium oxide had volatised. This would have occurred at 2000$^0\text{K}$, a temperature which can easily be achieved by meteoric impact. This possibility that this sphere
Scanning Electron Micrographs of Hollow Glass Spheres.
was formed by occluded gas is remote as Scarlett and Buxton (104) pointed out that this would require a temperature of 50,000°K.

Another sphere is shown in Fig. (4.8). This is also hollow but is punctured by what appears to be a blow out hole caused by a volatile component. Basing this on magnesium oxide a temperature of 3000°K and resulting excess pressure of 50 c.m. of H₂ would have been necessary. It could, however, have been punctured by a projectile causing impact melting and hence a smooth edged puncture would also result.

4.3. Comparison of the Physical Limitations of Impact Melting and the Size Distribution of Spheres.

From Fig. (4.5) the maximum size of jet likely is around 1 m.m. diameter. If it is assumed that the size of the largest drop resulting from this jet is twice its diameter then this gives a maximum sphere size of around 2 m.m. The largest sample observed in sample 12070/45 was 650 m.m. However, the sample was a "less than 1 m.m." pre-sieved sample from N.A.S.A. There have, however, been no reports of spheres greater than 1 m.m. diameter. This validates the upper physical limitations. Further support is added to this limit by the fact that dumb-bell shaped particles have been observed, suggesting that freezing, in some cases, has occurred very near to the break-up point.

Hence the facts from analysis of lunar spheres supports the physical limitations proposed, and the theory that the spheres were caused meteoric impact.
CHAPTER FIVE
CHAPTER FIVE

5. Stability and Droplet Formation from the Electrohydrodynamic Instability of Water Jets.

The work of Melcher (68) and others (116) (43) (108); described in Section (2.2.5) has indicated that a jet of a good conducting liquid may be increased in its instability by the addition of a radial electric field. The effect of such fields on the disturbance growth rates, and whether this effect is consistent with existing theories, has not been previously investigated.

An important feature of the predicted effects of electrification, is that electrification can extend the range of unstable wavenumbers beyond $K = 1$ for the axisymmetric mode and also increase the growth rates of wavenumbers $K > 0.6$. This feature is shown in Fig. (2.3). Thus, shorter wavelengths and smaller droplets could be produced by controlling the break-up of a good conducting jet in a radial electric field. The effect of allowing shorter wavelengths to grow is also important in another respect. The work of the author as described in Chapter 3, and the work of Rutland and Jameson (98), has shown that the size of satellite drops produced from capillary instability is reduced with increasing wavenumber. A similar effect is expected here, leading, perhaps, to monosize droplet formation.

It was also decided to measure the dropsizes resulting from controlled electrohydrodynamic break-up, since the effects of electric field on controlled break-up have not been previously investigated. It was thought that data on the size of drops formed by disturbances which cannot be used with capillary instability would be forthcoming. Such disturbances are those with small growth rates, such that they cannot be superimposed above the level of naturally occurring disturbances, and also those disturbances where $K > 1.0$ which are stable to capillary forces.
5.1. Apparatus

The large apparatus described in Section (3.2.1.) was modified to enable a set of experiments to be carried out on the effect of a radial electric field on the instability of water jets. A schematic diagram of the apparatus is shown in Fig. (5.1.) and a general view of the alterations is shown in Fig. (5.2.)

The apparatus consisted, as before, of an air pressure pumping system feeding a flow quietening chamber and a nozzle, but a new perturbation system and an electrode were added.

It was decided that the experiments should be carried out with the jet earthed and a charged cylinder co-axial with the jet. This would ensure a safer mode of operation than the reverse case of a charged jet and an earthed cylinder which would necessitate a large volume of water, and a large part of the apparatus, being charged to lethal voltages.

Both the \( m = 0 \) and \( m = 1 \) mode can be obtained with jets in electric fields, hence it is important only to superimpose the required mode. Since the axisymmetric mode was required the superimposed disturbances had to be axisymmetric. This was achieved by perturbing the jet electrically, as an acoustic perturbation would provide disturbance from which either mode could grow depending on conditions. An alternating current, with a direct current bias, was placed on a thin brass plate through which the jet was projected. Since the force on the jet is proportional to the square of the potential difference, a d-c bias was provided to ensure that there was a force applied to the jet throughout the perturbation oscillation so giving a more positively controlled disturbance than with pure a-c. The perturbation plate was attached to the quietening chamber by four P.T.F.E. pillars to insulate it from the earthed quietening chamber. The field on the plate was provided by a 1.K.W. Derritron Electronics amplifier whose output circuit was modified so as to supply 2,000 volts a.c. with a current in the micro-amp range. The low current was for safety reasons.
A Schematic Diagram of Apparatus used in Electrohydrodynamic Stability Experiments.
Fig. (5.2)

General View of Alterations to Large Scale Apparatus for Electrohydrodynamic Experiments.
The amplifier was driven by a Taylor 192A frequency generator which was linked as in Section (3.2.1.), to an E.M.I. type 6 stroboscope. The stroboscope was used for setting up and general observation.

It was thought necessary to provide a radial electric field by a method that could allow for visual observation without the use of slits which would give a non-uniform field inside the cylinder. For this reason a method similar to that of Melcher (68) was used. A glass cylinder of 7 cm. inside diameter and 30 cm. long was coated on the inside by a thin tin oxide layer which was optically transparent. The length of the cylinder was considered to be sufficient to give no significant end effects over the length of the jet. The method of coating the cylinder is given in detail in Appendix B. The uniformity of the coating was checked by a resistance meter. The cylinder was mounted in an insulated sheath and placed co-axially with the jet.

The potential of the cylinder was provided by a Brandenberg 707R high voltage supply which was capable of giving up to 2 milli-amps current at 15 K.V. A moderately large current was required as preliminary runs, with a supply which had a smaller available current, had indicated that the striking of drops onto the cylinder at higher voltages could draw large amounts of current. With a small available current this could cause large voltage fluctuations. The Brandenberg supply was fitted with an ammeter which was used to check the current drawn. A cut-out was fitted to switch off the supply if the 2 milli-amps was exceeded. The supply proved stable under all the conditions used. Electrical contact between the high tension supply cable and the surface of the glass cylinder was by a ring of fine wire and alcohol based silver paint. A guard was placed around the brass perturbation plate and large warning lamps, to indicate whether any of the high voltage supplies were live, were placed in a prominent position on the apparatus.
5.2. Measurement

Measurements of wave growth rates and dropsizes were made by high speed cine photography. A Fastax or a Hycam high speed cine camera was used with Kodak Pan-X reversal film run at around 5,000 pictures per second (p.p.s.). Illumination for cine photography was by a 1 K.W. Quartz Iodine continuous source. Analysis of the film was made by the P.C.D. digital data reader fitted with a Vanguard M16-C projection head.

The wave growth rates were measured using a method similar to that of Donnelly and Glaberson (16), that is measuring the difference between adjacent neck and crests as shown in Fig. (2.2.). This method was shown by Yuen (131) to remove higher order effects and approximate to the linearised growth rate. Hence, this criteria was used to obtain a linearised growth rate for comparison with theory. The difference between necks and crests was measured for one wavelength at several frame intervals. This gave a growth rate when the film speed was known. Film speeds were measured by timing marks and from a knowledge of the jet velocity.

Exponential growth rates were obtained from these results by a linear least squares correlation of the natural logarithm of the difference against time. This was carried out using a computer program written in BASIC code on a P.D.P. - 11 Computer. The number of difference measurements per growth rate determination varied slightly as a fixed interval of five pictures was used. However, the number was typically seven. The mean of at least three of the growth rate determinations was calculated for comparison with theory.

The dropsizes resulting from the break-up were obtained by a similar procedure to that described in Section (3.2.3.), but measurement was from the cine film. Since the overall enlargement of the cine file was less than that of the still photographs, and because more droplets were available on the cine film, the accuracy of the measurements were increased by taking the mean value of a least ten droplets. All the droplets
measured were within the area of the film used for obtaining the growth rates.

The potential applied to the electrode was measured by the built-in meter on the Brandenberg supply and was accurate to ± 1%. The surface tension of the water was found by du Noly tensiometer and found not to vary reasonably during the period of the experiments.

5.3. Method

The method of operating the apparatus was similar to that described in Section (3.2.4.) for the large scale apparatus. Similar jet velocities were used.

The cine film was taken only when the voltage on the cylinder was steady and care was taken to remove any droplets adhering to the charged cylinder that could interfere with the photography. Several films were taken consecutively, in order that with a fixed frequency the voltage could be varied, and the effect of electrification assessed. It was not possible to operate at a completely fixed wavenumber since during the taking of a series of films; which could take over one hour, the flowrate would vary slightly and cause a small change in jet diameter and hence wavenumber.

5.4. Results

It was the first requirement of the experiments with water jets to determine the charge relaxation time $t_r$ of the water used. It could then be determined whether the assumption of perfect conduction was valid. The conductivity of the water used was determined by a Portland conductivity meter and was found to be $162 \mu \text{hos cm}^{-1}$. This gave a charge relaxation time of $4.4 \times 10^{-4}$ secs.

If the characteristic time for the fluids motion is considered to be the period of the imposed disturbance, as was suggested by Michael and O'Neill (71), then the charge relaxation is over an order of magnitude smaller than the period of the highest frequency used. If, however, the
criteria of Schneider et al (108) is used, which is that the characteristic time is the time taken for the jet to pass through the electrode, then it is two orders of magnitude smaller.

The characteristic time used by Saville (103) was that time for the disturbance to grow by the ratio $e : 1$. This gives a characteristic time of 7 milli seconds which is still greater than the charge relaxation time. Hence, it can be seen that the assumption of perfect conduction is valid. A series of runs were carried out with the potential difference between the electrode and the jet at values of 0, 3, 6, 7, and 9 K.V. and are run at 10 K.V. The relationships between the electrification constants $\Gamma$ and the potential difference are given in Table (5.1). The electrification constant is given at a mean jet diameter of 1.45 mm and a surface tension of 71.5 dynes cm$^{-1}$.

Table (5.1)

<table>
<thead>
<tr>
<th>$\Phi$ (K.V.)</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>0.1006</td>
</tr>
<tr>
<td>6</td>
<td>0.4025</td>
</tr>
<tr>
<td>7</td>
<td>0.5478</td>
</tr>
<tr>
<td>9</td>
<td>0.9056</td>
</tr>
<tr>
<td>10</td>
<td>1.1181</td>
</tr>
</tbody>
</table>

The theoretical growth rate curves of the $m = 1$ and $m = 0$ mode, for the electrification constants of Table (5.1) are shown on Fig. (5.3). The theory of Melcher (68) was used to evaluate these curves. The growth rates are dimensional with respect to $\left(\frac{\tau}{\rho a^3}\right)^{\frac{1}{2}}$.

The predicted $m = 0$ and $m = 1$ growth rates of this
Table (5.2)

Growth Rates from the Electrohydrodynamic Instability of Water Jets

<table>
<thead>
<tr>
<th>Run No.</th>
<th>( \Gamma )</th>
<th>K</th>
<th>( P_0 ) CAL</th>
<th>( P_1 ) CAL</th>
<th>( P ) EXP</th>
<th>ERROR*</th>
</tr>
</thead>
<tbody>
<tr>
<td>FE7</td>
<td>0.0</td>
<td>0.354</td>
<td>0.232</td>
<td>0.0</td>
<td>0.274</td>
<td>-15.3</td>
</tr>
<tr>
<td>FE6</td>
<td>0.0</td>
<td>0.543</td>
<td>0.317</td>
<td>0.0</td>
<td>0.348</td>
<td>-9.0</td>
</tr>
<tr>
<td>FE3</td>
<td>0.0</td>
<td>0.618</td>
<td>0.336</td>
<td>0.0</td>
<td>0.329</td>
<td>+2.0</td>
</tr>
<tr>
<td>HE14</td>
<td>0.0984</td>
<td>0.585</td>
<td>0.328</td>
<td>0.0</td>
<td>0.322</td>
<td>+2.0</td>
</tr>
<tr>
<td>HE22</td>
<td>0.1011</td>
<td>0.293</td>
<td>0.193</td>
<td>0.0</td>
<td>0.257</td>
<td>-24.7</td>
</tr>
<tr>
<td>HE17</td>
<td>0.1018</td>
<td>0.378</td>
<td>0.242</td>
<td>0.0</td>
<td>0.286</td>
<td>-15.5</td>
</tr>
<tr>
<td>HE10</td>
<td>0.1036</td>
<td>0.585</td>
<td>0.329</td>
<td>0.0</td>
<td>0.287</td>
<td>+14.5</td>
</tr>
<tr>
<td>HE13</td>
<td>0.3988</td>
<td>0.548</td>
<td>0.314</td>
<td>0.0</td>
<td>0.314</td>
<td>-0.1</td>
</tr>
<tr>
<td>HE18</td>
<td>0.4059</td>
<td>0.389</td>
<td>0.237</td>
<td>0.0</td>
<td>0.261</td>
<td>-9.2</td>
</tr>
<tr>
<td>HE9</td>
<td>0.4069</td>
<td>0.663</td>
<td>0.351</td>
<td>0.0</td>
<td>0.310</td>
<td>+13.2</td>
</tr>
<tr>
<td>HE5</td>
<td>0.4109</td>
<td>0.760</td>
<td>0.366</td>
<td>0.0</td>
<td>0.365</td>
<td>+0.1</td>
</tr>
<tr>
<td>HE23</td>
<td>0.4111</td>
<td>0.270</td>
<td>0.168</td>
<td>0.0</td>
<td>0.242</td>
<td>-30.7</td>
</tr>
<tr>
<td>HE12</td>
<td>0.5220</td>
<td>0.648</td>
<td>0.349</td>
<td>0.0</td>
<td>0.314</td>
<td>+11.2</td>
</tr>
<tr>
<td>HE4</td>
<td>0.5442</td>
<td>0.839</td>
<td>0.377</td>
<td>0.0</td>
<td>0.339</td>
<td>+11.1</td>
</tr>
<tr>
<td>HE8</td>
<td>0.5467</td>
<td>0.704</td>
<td>0.364</td>
<td>0.0</td>
<td>0.326</td>
<td>+11.8</td>
</tr>
<tr>
<td>HE19</td>
<td>0.5549</td>
<td>0.383</td>
<td>0.228</td>
<td>0.0</td>
<td>0.289</td>
<td>-21.0</td>
</tr>
<tr>
<td>HE11</td>
<td>0.8587</td>
<td>0.662</td>
<td>0.360</td>
<td>0.0</td>
<td>0.317</td>
<td>+13.6</td>
</tr>
<tr>
<td>FE10</td>
<td>0.8871</td>
<td>0.284</td>
<td>0.156</td>
<td>0.174</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>HE7</td>
<td>0.8966</td>
<td>0.729</td>
<td>0.386</td>
<td>0.0</td>
<td>0.341</td>
<td>+13.3</td>
</tr>
<tr>
<td>HE3</td>
<td>0.9102</td>
<td>0.764</td>
<td>0.398</td>
<td>0.0</td>
<td>0.407</td>
<td>-2.2</td>
</tr>
<tr>
<td>HE25</td>
<td>0.9251</td>
<td>0.270</td>
<td>0.145</td>
<td>0.1855</td>
<td>0.203</td>
<td>-28.3</td>
</tr>
<tr>
<td>HE16</td>
<td>1.2665</td>
<td>0.453</td>
<td>0.245</td>
<td>0.329</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

* Error = \left(\frac{P_{\text{calc}} - P_{\text{exp}}}{P_{\text{exp}}}\right) \times 100
run are close and presumably the imposition of the \( m = 0 \) mode was enough to ensure its sole control of the break-up at this wavenumber.

As can be seen from Fig. (5.3) there is no predicted \( m = 1 \) growth for electrifications up to 3 K.V. and the \( m = 0 \) mode dominates the growth rate curve giving the maximum growth rate wavenumber at all the electrifications used.

The was born out experimentally as, except for runs FE10 and HE16, the axisymmetric mode was dominant at break-up. Details of these two runs are given in Table (5.2). The first of these runs (FE10) was carried out at \( K = 0.284 \) and 9 K.V. and exhibited both the imposed \( m = 0 \) mode and the naturally induced \( m = 1 \) mode. When the theoretical growth rates for these conditions are compared, the point being shown on Fig. (5.3) by a star, it can be seen that the \( m = 1 \) mode has the greater growth rate. Hence, although the \( m = 0 \) mode has been imposed on the jet the \( m = 1 \) mode with its greater growth rate has contributed significantly to the break-up. No direct comparison with the predicted growth rates was possible because of the combination of modes as can be seen in Fig. (5.4). The second run (FE16) was carried out at 10 K.V. and \( K = 0.451 \) and this also exhibited a combination of modes as could be predicted from its respective \( m = 1 \) and \( m = 0 \) growth rates.

The experimental and theoretical growth rates of the \( m = 0 \) mode as shown in Table (5.2) varied by from -30 to +15%. This represents only fair agreement with theory. However, the trend of increasing instability for increasing electrification above \( K = 0.6 \) and decreasing instability with increasing electrification below \( K = 0.6 \) can be seen on Fig. (5.3).

It was not found possible to control break-up above \( K = 1.0 \) as is predicted. Attempts to do so at 9 K.V. where the mode of maximum instability is around \( K = 1.0 \), met with failure due to there being insufficient growth of the imposed disturbance. This would indicate that the maximum growth rate point is less than \( K = 1.0 \) and the curves exhibit an
Figures are potentials in Kilovolts

The Mean Predicted Electrohydrodynamic Growth Rate Curves and Experimental Results.
Electrohydrodynamic Instability Exhibiting a Combination of
the \( m = 0 \) and \( m = 1 \) modes
even more sharply decreasing slope above their maximum than is predicted. It was possible to operate at $K = 0.84$ and 7 K.V. with a comparatively small amplitude disturbance. The author found it impossible to operate at this wavenumber with capillary instability due to inadequate growth, although Rutland and Jameson (98) were able to do so possibly by using a higher amplitude disturbance than the author.

The size of the droplets formed by electrohydrodynamic instability is given in Table (5.3) and plotted on Fig. (5.5). As can be seen, particularly from comparing Fig. (5.5) with Fig. (3.2.) there is no significant change in the size of droplets formed at the same wavenumber for either electrohydrodynamic or capillary instability.

The remarks in Section (3.2.5.2) concerning the suitability of the Yuen theory as a model for jet break-up seem applicable here, as between $K = 0.25$ and 0.5 the predicted dropsizes from Chapter 3 correlate well with the experimental results.

It was found that once detached from the continuous jet, the droplets became charged and due to mutual repulsion formed diverging streams. The satellite drops because of their small mass diverges more rapidly. They presented a limit to electrification, as when a large number of drops because of their divergence impinged on the electrode, large current was drawn. This caused fluctuations in the voltage supplied. If the current drawn exceeded the maximum of 2 milli amps then the supply to the electrode was ceased. In practice it proved difficult to operate above 10 K.V. and hence, the maximum voltage of most runs was limited to 9 K.V. The divergence of the streams could of course, be useful in supplying a stream of monosized drops, since any of the diverging streams could be collected or electrostatically diverted.
Table (5.3)

Dropsizes from Electrohydrodynamic Instability of Water Jets

<table>
<thead>
<tr>
<th>Run No.</th>
<th>Frequency Hz</th>
<th>K</th>
<th>( d_j ) m.m.</th>
<th>( \phi ) K.V.</th>
<th>Main</th>
<th>Satellite</th>
</tr>
</thead>
<tbody>
<tr>
<td>HE25</td>
<td>130</td>
<td>0.270</td>
<td>1.389</td>
<td>9.0</td>
<td>2.06</td>
<td>1.57</td>
</tr>
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Experimental Dropsizes from Electrohydrodynamic Instability.
CHAPTER SIX
6. A Theoretical Analysis of the Stability of a Liquid Jet in an Alternating Electric Field

There exist several theoretical analyses of the stability of jets subject to fixed polarity electric fields. These have been reviewed in Section (2.2.5). There have, however, been no attempts at producing a theoretical analysis predicting the effects of alternating electric fields.

Peskin and Raco (83) produced a theoretical analysis of ultrasonic atomisation in which they postulated a plane liquid surface coupled with a sinusoidally varying perpendicular force field. Their linearised stability analysis led to Mathieu's differential equation (59). A simplified solution of the equation was obtained considering only the first unstable region. It was considered by the author that their problem was similar to that of a liquid jet in an alternating radial electric field and that a similar analysis requiring the solution of a form of Mathieu's differential equation should result. A subsequent literature search on the subject of ultrasonic atomisation revealed that Peskin and Raco (84) in another paper had considered the problem of a-c atomisation where a plane liquid surface was subjected to an alternating electric field. This also had led to a solution of Mathieu's equation being required and a simplified solution was adopted. This gave the result that a low frequency alternating field required twice the electric field that was required for d-c atomisation. This result appears to have been found experimentally (53).

It was thus decided that a theoretical analysis of jet stability in an alternating radial electric field should be carried out, to obtain, if possible both stability criteria and disturbance growth rates.

6.1 Problem Formulation

A linearised stability analysis will now be formulated. The jet
liquid is considered as being a perfect conductor. This can be assumed if the charge relaxation time $t_r$ is small compared with the period of oscillation of the alternating field. If the period of the alternating field is smaller than the charge relaxation time then the jet can be considered as behaving dielectrically and the analyses of Glonti (25) and Nayyar and Murty (77) for a dielectric in a non alternating field are appropriate.

Consider a jet of radius $a$ projected along the axis of a cylindrical electrode of radius $b$, where $b > a$. Considering the case of charged electrode and earthed jet, then the cylinder is assumed to be maintained at a potential $\Phi_o$ above the jet where

$$\Phi_o = \Phi_0 \cos(\omega t)$$

(6.1)

Where $f$ is the frequency and $\Phi_0$ the peak to peak potential.

The coordinate system and geometric arrangement for the problem are shown in Fig. (6.1). $Z$ is the distance along the jet axis, the azimuthal angle and $r$ the radial distance, $q$ is the charge per unit length existing on the cylindrical electrode. If it is assumed that free space exists between $r = a$ and $r = b$ then for the undisturbed jet

$$\Phi = q \frac{\ln(r/a)}{2\pi \varepsilon_o}$$

(6.2)

and

$$q = \frac{2\pi \varepsilon_o \Phi_o}{\ln(b/a)}$$

(6.3)

Let the jet be subject to a small disturbance and the jet surface thus represented as

$$r = a_o + c_m \cos k z \cos m \theta$$

(6.4)

Where $c_m \ll a$ and $k$ is the axial wavenumber and $m$ the azimuthal wave number. $a_o$ is related to the undisturbed jet radius $a$ by conservation of volume. The volume enclosed by one wavelength is conserved and hence,
Geometry and Arrangement of Electrode for Alternating Field Analysis.
\[ \pi \int_0^\lambda \int_0^{2\pi} r \, d\theta \, dz = \pi a^2 \lambda \]  

(6.5)

Where \( \lambda = \frac{2\pi}{K} \), and \( K = ka \) and is the dimensionless wavenumber.

Hence,

\[ a_o^2 = a - \frac{1}{4} c_m \quad \text{for} \quad m > 0 \]

\[ a_o^2 = a - \frac{1}{2} c_m \quad \text{for} \quad m = 0 \]  

(6.6)

This gives

\[ a_o = a \left( 1 - \frac{1}{8} c_m \right) \quad \text{for} \quad m > 0 \]

\[ a_o = a \left( 1 - \frac{1}{4} c_m \right) \quad \text{for} \quad m = 0 \]  

(6.7)

The problem will be formulated using a method similar to that used by Rayleigh (93). The viscosity of the fluid will be neglected and the jet will be regarded as inviscid. The system will thus be conservative and the problem can be formulated by equating the changes in energy due to the disturbance.

The surface of the jet is given, to the first order (93) as

\[ S = \int \int \left[ 1 + \left( \frac{dr}{dz} \right)^2 \left( \frac{dr}{rd\theta} \right)^2 \right] r \, d\theta \, dz \]  

(6.8)

Hence, the change in potential energy due to surface tension, which is proportional to the change in surface area is

\[ E_s = -\frac{1}{4} T \pi \left( 1 - K - m^2 \right) \frac{c_o^2}{a} \left( 1 - \delta_m \right) \]  

(6.9)

Where \( \delta_m = 1 \) for \( m = 0 \) and zero for \( m > 0 \).
Equation (6.9) is similar to that obtained by Rayleigh (93).

It is necessary to obtain an expression for the change in electrical energy due to the disturbance.

Let \( \phi(r) \) be the change in potential due to the disturbance. Then,

\[
\phi = \phi_1(a) + \phi_2(a)^2
\]

(6.10)

Where \( \phi_1 \) and \( \phi_2 \) are the first and second order changes in potential. The first order change in potential \( \phi_1 \) must obey Laplace's equation, namely

\[
\nabla^2 \phi_1 = 0
\]

(6.11)

Hence, a solution is

\[
\phi_1 = \left[ A I_m(kr) + B K_m(kb) \right] e^{i(kz+m\theta)}
\]

(6.12)

Where \( A \) and \( B \) are arbitrary constants to be determined from the boundary conditions. The boundary conditions are, that the potential remains unchanged at \( r = b \) and that the jet remains earthed.

The first boundary condition gives

\[
A I_m(ka) + B K_m(kb) = 0
\]

(6.13)

and the second

\[
c_m \cos kz \cos m \theta \frac{\partial \phi_1}{\partial r} \bigg|_{r=a} + \phi_1(a) = 0
\]

(6.14)

Hence since

\[
\left. \frac{\partial \phi}{\partial r} \right|_{r=a} = -\frac{q}{2\pi \varepsilon_o a}
\]

(6.15)

Thus \( A \) and \( B \) are given by

\[
A = -\frac{q}{2\pi \varepsilon_o} K_m(kb)
\]

(6.17)
\[ B^* = \frac{q}{2\pi \varepsilon_o} I_m(kb) I \]  \hspace{1cm} (6.18)

Where

\[ I = \left[ I_m(ka) K_m(kb) - I_m(kb) K_m(ka) \right]^{-1} \]  \hspace{1cm} (6.19)

The charge on the cylinder is thus unchanged, as to the first order in \((\varphi_a)\) it is periodic in \(z\) and \(\theta\). Hence, the electrostatic energy of the system, which is measured by \(\frac{1}{2} q \Phi_o\), is also unchanged. There can however, exist second order changes and these must now be examined.

The boundary condition that the potential on the jet is zero gives

\[ \Phi + \left( \frac{c}{a} \right) \phi_1 + \left( \frac{c}{a} \right) \phi_2 = 0 \]  \hspace{1cm} (6.20)

at \(r = a_o + c \cos k z \cos m \theta\)

If the non-periodic terms in \(z\) and \(\theta\) are examined, then up to the second order from equations (6.20), (6.7) and (6.2)

\[ \phi_2 = \frac{q}{8\pi \varepsilon_o} \left[ \frac{k a}{4} \left( A I_m'(ka) - B K_m'(ka) \right) \right] (1 + \delta_{\phi}) \]  \hspace{1cm} (6.21)

But the first boundary condition that the potential on the cylinder remains unchanged requires that

\[ \left( \phi_2 \right)_{r=b} = 0 \]  \hspace{1cm} (6.22)

Thus the second order mean potential change is

\[ \phi_2 = q \frac{\ln(r/b)}{8\pi \varepsilon_o \ln(a/b)} \left[ 1 + \frac{ka K_m(kb) I_m'(ka) - I_m(kb) K_m'(ka)}{I_m(ka) K_m(kb) - I_m(kb) K_m(ka)} \right] (1 + \delta_{\phi}) \]  \hspace{1cm} (6.23)
Hence the change in charge on the cylinder $\Delta q$ is from Equation (6.3)

\[
\Delta q = \frac{c^2}{a^4} q \pi \varepsilon \ln \left( \frac{b}{a} \right) \left[ 1 + K \frac{K_m'(K)}{K_m(K)} \right] (1 + \delta_{m,n})
\]  

\hspace{1cm} (6.24)

Where the approximation for large ratios $b/a$ that

\[
\frac{K_m'(K)}{K_m(K)} = \frac{K_m(kb)\text{Im}(ka) - K_m'(kb)\text{Im}'(ka)}{\text{Im}'(ka)K_m(kb) - \text{Im}(kb)K_m'(ka)}
\]

\hspace{1cm} (6.25)

was used. Thus the change in electrostatic energy which is $\frac{1}{2} \Delta q \Phi_e$ becomes

\[
\frac{c^2}{a^4} \frac{q^2}{\pi + 6 \varepsilon} \left[ 1 + K \frac{K_m'(K)}{K_m(K)} \right]
\]

\hspace{1cm} (6.26)

Since, however, as Michael (70) points out, energy is supplied at the rate $\Delta q \Phi_e$ to the cylinder the energy available to build up the disturbance $E_e$ is

\[
\frac{c^2}{a^4} \frac{\Phi_e \pi \varepsilon}{4 \text{Im}(b/a)} \left[ 1 + K \frac{K_m'(K)}{K_m(K)} \right]
\]

\hspace{1cm} (6.27)

The change in kinetic energy $E_k$ due to the disturbance was given by Rayleigh (93). This is

\[
\left( \frac{\partial c}{\partial t} \right)^2 \pi \frac{\rho a^2}{4} \frac{\text{Im}(ka)(1 + \delta_{m,n})}{\text{Im}'(ka)}
\]

\hspace{1cm} (6.28)

Hence, using Lagrange's equation of motion (88)

\[
\frac{d^2 c}{dt^2} + \left[ \frac{E_s + E_e}{E_k} \right] c = 0
\]

\hspace{1cm} (6.29)

or

\[
\frac{d^2 c}{dt^2} + \left[ \frac{T}{\rho a^2 (1 - m^2 - K^2)} \frac{\text{Im}'(K)}{\text{Im}(K)} + \frac{2 \Phi_e \cos (ft)}{\rho a^2 \text{ln}(b/a) \text{Im}(K)} \left( 1 + K \frac{K_m'(K)}{K_m(K)} \right) \right] c = 0
\]

\hspace{1cm} (6.30)
It is apparent, when the substitution
\[ \cos^2(\psi t) = \frac{\cos(2\psi t)}{2} + \frac{1}{2} \]  
(6.31)
is made that Equation (6.30) is a form of Mathieu's equation
\[ \frac{d^2c}{d\gamma^2} + \left[ A - 2Q \cos 2\gamma \right] c = 0 \]  
(6.32)
Where
\[ A = \frac{2\pi}{\mu_p a^3} \left( 1 - m^2 - k^2 \right) k \frac{\text{Im}'(k)}{\text{Im}(k)} \]  
(6.33)
and
\[ Q = -\frac{\mu_p^2 k}{f^2_p a^4 4 \ln(b/a) \text{Im}(k)} \left( 1 + k \frac{K_m(k)}{K_m(k)} \right) \]  
(6.34)
and
\[ \gamma = ft \]  
(6.35)
It can be seen that the coefficients of A and Q are dimensionless and that the \( \gamma \) is dimensionless with respect to the field frequency.

6.2. The Solution of Mathieu's Equation

Mathieu's equation has stable, unstable and periodic solutions. The latter may be regarded as solutions exhibiting neutral stability. A general solution of Mathieu's equation may be given by
\[ P(\gamma) \]  
(6.36)
Where \( P(\gamma) \) is a periodic solution having periods of \( \pi \) or \( 2\pi \). \( p' \) is the exponential growth rate and in common with most stability theory can have either positive real or negative complex values giving either instability of stability.

The conditions of stability and instability for Mathieu's equation exist in discrete areas bounded by periodic solutions (62). The periodic solutions are given by Mathieu functions \( \text{ce}_n(\gamma, Q) \) where \( n = 0, 1, 2 \ldots \) and \( \text{se}_n(\gamma, Q) \). \( \text{ce}_n \) and \( \text{se}_n \) are called cosine elliptical and sine elliptical. The stability areas can be plotted on A, Q axis by plotting the functions \( a_n, b_n \) which represent the
Stability Diagram for Mathieu's Equation.
characteristic numbers of the Mathieu functions $ce^n$ and $se^{n+1}$. The areas between $a_{n+1}$ and $b_{n+1}$ and the area beneath $a_e$ are areas of instability, and those areas between $a_2$ and $b_2$ are areas of stability. The functions $a_{2n}$ and $b_{2n+2}$ are symmetric about the $A$ axis, but $a_{2n+1}$ and $b_{2n+1}$ are asymmetric. The whole diagram is, however, symmetric about the $A$ axis since $a_{2n+1}$ and $b_{2n+1}$ are mutually symmetric. This symmetry allows for solutions to be valid for both $-Q$ and $+Q$. A diagram showing the stable and unstable areas is shown on Fig. (6.2).

For the solution of the particular problem considered here it was found necessary to establish the likely values of $A$ and $Q$ so as to decide upon the particular method of solution to be adopted. It was for this reason that $A$ and $Q$ were plotted using the form

$$A = -C_f (1 - \kappa^{-m^2}) \frac{I_m'(\kappa)}{I_m(\kappa)} - 2Q,$$

where

$$Q = -\Gamma_f \frac{I_m'(\kappa)}{I_m(\kappa)} \left[ 1 + \kappa \frac{K_m'(\kappa)}{K_m(\kappa)} \right],$$

(6.37)

(6.38)

Where $C_f$ and $\Gamma_f$ are the capillary and electrification constants and are dimensionless. They are defined as

$$C_f = \frac{T}{a^2 \rho}$$

(6.39)

$$\Gamma_f = \frac{\Phi_0^2 \varepsilon_s}{f^2 \rho \ln(y_a) a^4}$$

(6.40)

The values of $C_f$ and $\Gamma_f$ were appropriate to the conditions met during the experimental work reported in Chapter 5 except that the diameter of the charging electrode was taken as one inch. The conditions were those for water of 71.5 dynes cm.\(^{-1}\) surface tension and density of 1 g/m cm\(^{-3}\) and jets of 1.45 m.m. diameter. The plots of $A$ against $Q$ for the modes $m = 0$ and $m = 1$ and peak voltages of 3, 5, 7, 10 and 15 K.V. are shown on
Figs. (6.3) - (6.6) for the frequencies 50 and 100 Hz.

From the graphs it is clear that the instability of jets in a-c fields will exist in discrete areas, not in a single area as found for jets in d-c fields.

It is necessary to solve the problem to establish both stability criteria and growth rates as functions of the wavenumber K.

Considering, firstly the question of stability criteria, there are no simple expressions for the values of $a_n$ and $b_{n+1}$, although at low values of $Q$ a power series is available (37). For higher values accurate determination of $a_n$ and $b_{n+1}$ must be by recursion formulae. To avoid the use of recursion formulae a simplified approach was adopted making use of the tables given by McLachlan (62). The tables give values of $a_n$ and $b_{n+1}$ for $n = 0$ to 5 and $Q = 0$ to 40. A standard computer program* was used to fit an eighth order polynomial to the results in McLachlan tables using the values of $Q$ from 0 to 25. This enabled a simple expression to be used to evaluate stability criteria by computer.

The values of the exponential growth rate $p'$ occur on the $A, Q$ stability diagram as lines of constant $p'$. These are referred to as iso-$p'$ contours and are asymptotic to the functions $a_n, b_{n+1}$. The functions $a_n, b_{n+1}$ may be considered as special cases of the iso-$p'$ contours, namely the iso-$p'$ contour where $p' = 0$. Iso-$p'$ contours do not cross and are thus single valued. Considering a segment across an unstable region, with $Q$ constant, the value of $p'$ increases from zero until some maximum and then returns to zero. Hence the traverse of an unstable region by a function such as one of those shown in Figs. (6.3) - (6.6) will give rise to an increase in $p'$ from zero to some maximum and a decrease until $p' = 0$ at the intersection of the next $a$, or b.

* Program Z201 of the Loughborough University Computer Library.
The computer program written to establish instability conditions was adapted to evaluate the exponential growth rates. It is necessary to evaluate the growth rates over a wide range of unstable areas since a major interest in jet instability is the wavenumber of maximum growth rate $K^*$ and this may not occur in the first unstable region met by the function given by Equations (6.37) and (6.38).

Because of the wide range of values of $A$ and $Q$, as is shown in Figs. (6.3) - (6.6), it was not found possible to use one single method to obtain growth rates. Two separate methods were adopted. The first method applied in those areas where the values of $A$, $Q$ lay between $a_{n+1}$ and $b_{n+1}$. In those areas, which for this problem were the narrow portions of the unstable regions, the following approximate formulae (due to McLachlan (62)) was used

$$ p = \left[ \frac{(a_n(Q) - A)(A - b_n(Q))}{2n+1} \right]^{\frac{1}{2}} $$

(6.41)

Where the point $A$, $Q$ lies between $a_{n+1}$ and $b_{n+1}$.

In the other unstable area beneath $a_n$ the formulae given above is inapplicable and the method given in Section (4.74) of McLachlan (62) was used. The accuracy of this method is low when $Q \approx A$ and so in this region interpolation was used as no other method appears to be available.

The computer program referred to in this section is Master A.C. and this is given in Appendix C together with the values of the polynomial coefficients used to define $a_n$ and $b_{n+1}$.

6.3 Results

The program Master A.C. was run on an ICL 1904A computer using equations (6.37) and (6.38) defined for both the $m = 0$ and the $m = 1$ modes. The values of $C_f$ and $\Gamma_f$ corresponded to the field frequencies of 50, 100, 500 and 10000 Hz and voltages of 3, 5, 7, 10 and 15 K.V. A graph of the growth rate results for each value was plotted.

The results for the $m = 0$ mode at 50 Hz are given in Figs. (6.7) -
Figs. (6.3) - (6.6) A vs. Q Plots for Alternating Field Theory
Fig. (6.3)

\[ m = 0 \]
\[ f = 50 \text{ Hz} \]
Fig. (6.4)

$m = 0$

$f' = 100\, \text{Hz}$
Fig. (6.5)

m = 1
f = 50 Hz
Fig. (6.6)

\[ m = 1 \]
\[ f = 100 \text{ Hz} \]
It can be seen from these graphs that the main growth rate curves are of substantially the same form as those for d-c jet instability except there are now a number of subsidiary growth rate curves at higher wavenumbers when high electrifications are used. A reference to Fig. (6.3) will show that the main curve is the result of the $A, \Omega$ plot crossing the area beneath $a_0$. The subsidiary areas are the result of the $A, \Omega$ plot crossing the narrow unstable areas corresponding to $a_1$, $a_2$, and $a_3$ etc. From Fig. (6.3) it can be seen that the increasing number of subsidiary areas is the effect of the values of $\Omega$ increasing for larger electrifications.

In F.M. radio transmission the existence of sidebands (111) can be explained by a similar analysis and solution of Mathieu's equation to that carried out here (62) with the unstable areas corresponding to $a_1$, $a_2$, etc. being responsible for those bands. It is for this reason that the nomenclature of radio transmission will be adopted here, and these growth curves will be referred to as sidebands.

The main growth rate curves can be seen to dominate the break-up as the wavenumber of maximum instability $K^*$ occurs within the main curve for $m = 0$ and 50 Hz.

The results for the $m = 1$ mode at 50 Hz are given in Figs. (6.12) - (6.15). No growth rates were observed for this mode until 5 K.V. The $m = 1$ growth rates determined can be seen to be less than those for the $m = 0$ mode, except at lower wave numbers and with high electrification. This agrees with the d-c result (68) where the $m = 0$ dominates break-up except at high electrifications and low wavenumbers.

The results for 100 Hz are plotted on Figs. (6.16) - (6.26). Considering the $m = 0$ mode, it can be seen that the main curves are substantially the same shape as those for 50 Hz. The height of the main curves is, however, reduced. It was found that $K^*$ for the 100 Hz results occurred at substantially the same wavenumber as for the 50 Hz results and that the $p^*$ growth rates were half those of the 50 Hz results. Considering
again Equations (6.35) and (6.36) it is clear that since the growth rate $p'$ is dimensionless with respect to the field frequency, the dimensional maximum growth rate has remained unchanged with an increase from 50 to 100 Hz. The width of the main curve has, however, been increased, its cut off wavenumber increasing.

Considering the sidebands, they too have been reduced in height and increased in width by the increase in frequency. Other trends can also be identified. The sidebands become situated further away from the main curve with an increase in electrification at fixed frequency and they increase in dimensional growth rate.

A comparison of the $m = 1$ mode results for 50 and 100 Hz reveals that an increase in frequency also reduces their height and increases their width. The wavelength of maximum instability is not, however, constant although there is a recurrence of the trend for $K^*$ to increase for increasing electrification. Although the height of the $m = 1$ mode curves was reduced by an increase in frequency there does not appear to be a simple relationship between the $p'^*$ for 50 and 100 Hz.

Master A.C. was used to evaluate the growth rates of two further frequencies, namely 500 and 1000 Hz. These results are shown on Figs. (6.25) - (6.30). Considering the $m = 0$ mode the trends of widening main curves was repeated but no sidebands were found in the wavenumbers considered. It appears that these sidebands, in line with the trend shown by the results from 50 and 100 Hz, have moved to higher wavenumbers, this time outside the range considered. The values of $K^*$ and the dimensional maximum growth rate continued to remain substantially constant for a given electrification although a slight increase in $K^*$ and the growth rates were observed at higher electrifications.

The trends shown by the $m = 1$ modes were repeated at 500 Hz with reduced $p'$ with frequency and the mode $m = 1$ having larger growth rates than the $m = 0$ mode at higher electrifications and low wavenumbers.
Figs. (6.7) - (6.30) Dimensionless Growth Rates vs. Wavenumber
for Alternating Field Theory
Fig. (6.7)

\[ m = 0 \]
\[ f = 50 \text{ Hz} \]
\[ 3 \text{ KV} \]
$m = 0$

$f = 50 \text{Hz}$

$5 \text{ KV}$
Fig. (6.9)

\[ R = \frac{3.6}{3.2} \]

\[ P = \frac{2.4}{2.0} \]

\[ 0.8 \]

\[ 0.4 \]

\[ K = 0 \]

\[ f = 50 \text{ Hz} \]

\[ V = 7 \text{ KV} \]
Fig. (6.10)

\[ m = 0 \]
\[ f = 50\,\text{Hz} \]
\[ 10\,\text{KV} \]
Fig. (6.11)

\[ m = 0 \]
\[ f = 50 \text{Hz} \]
\[ 15 \text{KV} \]
\( f = 50 \text{ Hz} \)

\( 5 \text{ KV} \)

\( p' \)

\( m = 1 \)
Fig. (6.13)

$m = 1$

$f = 50 \text{ Hz}$

$7 \text{ KV}$
$m = 1$

$f = 50\text{ Hz}$

$10\text{ KV}$
m = 1
f = 50 Hz
15 KV
Fig. (6.16)

\[ m = 0 \]
\[ f = 100 \]
\[ 3 \text{KV} \]
Fig. (6.17)

\[ m = 0 \]
\[ f = 100 \text{ Hz} \]

\[ 0.5 \text{ KV} \]
Fig. (6.18)

\[ m = 0 \]
\[ f = 100 \text{ Hz} \]
\[ 7 \text{KV} \]
Fig. (6.19)

\[ m = 0 \]
\[ f = 100 \text{ Hz} \]
\[ 10 \text{ KV} \]
Fig. (6,20)

\[
m = 0
f = 100 \text{ Hz}
15 \text{ KV}
\]
\( m = 1 \)
\( f = 100 \text{ Hz} \)

5 kV
Fig. (6.22)

\[ m = 1 \]
\[ f = 100 \text{ Hz} \]

\[ 7 \text{ KV} \]
Fig. (6.23)

\[ f = 100 \text{Hz} \]

\[ m = 1 \]

\[ 10 \text{ KV} \]
Fig. (6.24)

\[ m = 1 \]
\[ f = 100\, \text{Hz} \]
\[ 15\, \text{KV} \]
$$f = 500 \text{ Hz}$$

$$m = 0$$
Fig. (6.26)

\[ K \]

\[ p_i' \]

- \( m = 1 \)
- \( f = 500\text{Hz} \)
- \( 5\text{KV} \)
Fig. (6.27)

\[ m = 1 \]
\[ f = 500 \text{ Hz} \]

7 KV
Fig. (6.28)

\[ p' \]

\[ m = 1 \]
\[ f = 500 \text{ Hz} \]
\[ 10 \text{ KV} \]
Fig. (6.29)

\[ m = 1 \]

\[ f = 500 \text{ Hz} \]

15KV
Fig. (6.30)

\[ m = 0 \]

\[ f = 1000 \text{ Hz} \]
There did, however, appear to be growth at \( K \cong 1.2 \) where the \( m = 0 \) mode exhibits no growth. The results for the \( m = 1 \) mode at 1000 Hz were not considered at valid. This was because it was found that at higher electrifications, where growth might be expected, all the unstable values of \( A \) were negative and such that the method used was not accurate.

The maximum growth rates \( p' \) * and their wavenumbers \( K* \) for all the computer runs reported above are given in Table (6.1).

Since \( K* \) and the dimensional maximum growth rate appear to be constant for the \( m = 0 \) mode, it was decided to examine what values of d-c electrification would correspond to this effect. The electrification constant needed to evaluate such an equivalent d-c case can be found by examining Equations (6.39) and (6.40) and Equation (2.60). From these equations

\[
\Gamma = \frac{\Gamma_f^4}{C_f} \quad (6.42)
\]

From examining the growth rates obtained from Equations (2.13) and (2.56), using the electrification constant \( \Gamma \) defined as above, it appears that the dimensional growth rate and wavenumber of maximum instability appear to correspond to an electrification of \( \frac{\Gamma}{2} \). This corresponds to an electrification of the root mean square value of the potential. Hence, it appears that the substitution of the R.M.S. value of \( \Phi_o \) into the d-c theory will predict \( K* \) and \( p* \) quite accurately. Table (6.2) gives the dimensionless and dimensional values of \( p* \) and the values of \( K* \) as calculated by using the d-c theory. It should be noted that the d-c growth rate is dimensionless with respect to \( (T \rho^{-1} a^{-3})^\frac{1}{2} \). The values of this parameter are those used to calculate \( C_f \).

At the highest electrification and at lower frequencies there appears to be some deviation of both \( K* \) and the dimensional maximum growth rate from the d-c R.M.S. value. This effect may be increased at still lower frequencies.
### Table (6.1)

**Maximum Growth Rate Instabilities from a-c Theory**

#### m = 0

<table>
<thead>
<tr>
<th>K.V.</th>
<th>50 Hz</th>
<th>100 Hz</th>
<th>500 Hz</th>
<th>1000 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( p^* )</td>
<td>( K^* )</td>
<td>( p^* )</td>
<td>( K^* )</td>
</tr>
<tr>
<td>3</td>
<td>2.99</td>
<td>0.70</td>
<td>1.49</td>
<td>0.71</td>
</tr>
<tr>
<td>5</td>
<td>3.01</td>
<td>0.72</td>
<td>1.52</td>
<td>0.72</td>
</tr>
<tr>
<td>7</td>
<td>3.06</td>
<td>0.74</td>
<td>1.53</td>
<td>0.74</td>
</tr>
<tr>
<td>10</td>
<td>3.19</td>
<td>0.79</td>
<td>1.60</td>
<td>0.79</td>
</tr>
<tr>
<td>15</td>
<td>3.68</td>
<td>0.90</td>
<td>1.85</td>
<td>0.91</td>
</tr>
</tbody>
</table>

#### m = 1

<table>
<thead>
<tr>
<th>K.V.</th>
<th>50 Hz</th>
<th>100 Hz</th>
<th>500 Hz</th>
<th>1000 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( p^* )</td>
<td>( K^* )</td>
<td>( p^* )</td>
<td>( K^* )</td>
</tr>
<tr>
<td>3</td>
<td>0.024</td>
<td>0.11</td>
<td>0.015</td>
<td>0.24</td>
</tr>
<tr>
<td>5</td>
<td>0.076</td>
<td>0.13</td>
<td>0.053</td>
<td>0.25</td>
</tr>
<tr>
<td>7</td>
<td>0.21</td>
<td>0.16</td>
<td>0.13</td>
<td>0.29</td>
</tr>
<tr>
<td>10</td>
<td>0.77</td>
<td>0.26</td>
<td>0.39</td>
<td>0.38</td>
</tr>
<tr>
<td>15</td>
<td>2.36</td>
<td>0.38</td>
<td>1.52</td>
<td>0.62</td>
</tr>
</tbody>
</table>

*a* - not accurate, see text

Dashes represent no growth
### Table (6.2)

**Maximum Growth Rate Instabilities**

for d-c (R.M.S.) Theory

| K.V. | K*  | Dimensionless p* | Dimensional p* |
|------|-----|------------------|                |
| 3    | 0.70| 0.345            | 149            |
| 5    | 0.72| 0.348            | 151            |
| 7    | 0.74| 0.353            | 153            |
| 10   | 0.79| 0.369            | 160            |
| 15   | 0.96| 0.440            | 191            |

b - The dimensions of p* are secs.
6.4 Summary of Results

Within the ranges

\[ 0 \approx K \leq 1.5 \]
\[ 50 \approx f \leq 1000 \text{ Hz} \]
\[ 0 \approx \Gamma_f \leq 37 \]
\[ 18 \approx C_f \leq 75 \]

The dimensional maximum growth rate of the \( m = 0 \) mode is substantially independent of frequency and may be calculated with little loss of accuracy from the d-c theory (68 ) by the substitution of the R.M.S. potential difference. Within the above range the \( m = 0 \) mode dominates break-up containing the mode of maximum instability.

The following trends were observed

(1) Increasing frequency increases the width and spacing of the side bands.

(2) Increasing electrification increases the width and height of the sidebands.

The \( m = 1 \) mode growth rates do not exceed those of the \( m = 0 \) mode except at high electrifications and low wavenumbers (there was one exception at high electrification and high wavenumber).

The following trends were observed

(1) Increasing frequency increases the dimensional maximum growth rate of the \( m = 1 \) mode.

(2) Increasing frequency increases the width and spacing of the growth rate curves.

(3) Increasing electrification increases the width and height of the growth rate curves.
CHAPTER SEVEN
CHAPTER SEVEN

7. Conclusions and Suggestions for Further Work

7.1 Conclusions

The work presented in this thesis concerns four areas of jet instability phenomena. Conclusions from the work carried out in each of these areas will now be drawn.

7.1.1 Droplet Formation from Capillary Instability

(a) The Yuen non-linear theory can be used to predict the sizes of main and satellite drops in the range of wavenumbers \(0.25 \geq K \leq 0.5\).

(b) Satellite drops are always produced from capillary instability although coalescence between main and satellite drops, close to the break-up point, can occur with small diameter jets.

(c) Contrary to the author's use of the Yuen theory, but in agreement with Rutland and Jameson's experiments, the total volume of satellite drops can exceed that of the main.

(d) A stream of monosize droplets can be formed from a liquid jet by the use of a hypodermic needle vibrated at a resonance frequency. This effect of large amplitude vibrations on a liquid jet can account for many of the claims of monosized droplet production from liquid jets.

7.1.2 Formation of Glass spheres found in Lunar Fines

The application of jet break-up theory to the problem of the production mechanism of the lunar glass spheres can give the physical limitations of an impact melting mechanism. The limitations are consistent with existing experimental evidence.

7.1.3 Stability and Droplet formation from the Electrohydrodynamic Instability of Water Jets

For electrifications up to \(\Gamma = 0.925\) it may be concluded that:

(a) The growth rates, in a radial electric field, of an axisymmetric disturbance are in fair agreement with the theories of Melcher and Taylor.

(b) The dropsizes produced from the electrohydrodynamic instability.
of a water jet, when subject to an axisymmetric disturbance, do not differ significantly from those obtained by capillary instability.

7.1.4 The Stability of a Liquid Jet in an Alternating Electric Field

For the range of values given in Section (6.4), a theoretical analysis considering an inviscid good conducting fluid predicts that:

(a) The maximum growth rate disturbance is independent of frequency and occurs in the axisymmetric mode.

(b) The maximum growth rate of the axisymmetric mode as determined by a complete a-c theory is equivalent to that predicted by a d-c theory using the root mean square potential difference.

(c) The existence of 'sideband' growth rate curves for the axisymmetric mode.

7.2 Suggestions for Further Work

The author considers that the following areas should be considered as worthy of further study.

7.2.1 Capillary Instability

The phenomena observed by the author when small nozzles resonate is important as it produces a monosized stream of droplets. It appears to be more complex than the instability of a cylinder of liquid and to be concerned with both the forces which move the jet laterally and the instability of the resulting liquid mass. An adequate explanation of why this phenomena produces monosized drops and a prediction of the size of the drops is required.

As indicated in Chapter One of this thesis the behaviour of jets of both Newtonian and non-Newtonian fluids are important. Chapter Three of this thesis was essentially concerned with low viscosity Newtonian liquids which are, in terms of the liquids handled by spraying devices, somewhat special cases. The question of what size drops result from a given viscous liquid jet cannot be answered for the full range of viscosities except by the assumption that only one drop per wavelength
is produced. A full experimental investigation of the effect of viscosity on the size of satellite drops formed appears to be required. Controlled perturbations would be needed but operation near the mode of maximum instability would be of prime importance.

The current knowledge of the behaviour of non-Newtonian, and in-particular visco-elastic fluids, is limited. The linearised visco-elastic models used to date do not appear to describe with any great success the experimental evidence. Previous experimental investigations have used the concept of break-up length to determine the degree of instability. The author considers that future investigations should use the more direct method of using controlled perturbations and measurement of growth rates. A recent paper by Gordon et al (134) which was published after Chapter Two had been written, does in fact use controlled perturbations to study visco-elastic break-up for the first time. The investigations to date concerning visco-elastic fluids have been chiefly concerned with weakly elastic fluids. This appears to be because they obey linearised fluid models and do not exhibit significant normal stress generation. These weakly elastic fluids have however presented problems of elasticity measurement. The effect of normal stress generation is undoubtedly important in determining visco-elastic jet stability and since the characteristics of the fluid depend upon its 'history' the flow characteristics through the nozzle are important. A complete analysis of visco-elastic jet stability is required which must include consideration of nozzle flow characteristics and subsequent normal stress generation and decay.

7.2.2 The Production Mechanism of Lunar Glass Spheres

As was remarked by the author in Section (4.2.1) there exist no experimental investigation of impact melting. It is therefore suggested that simulations of meteoric impact melting should be carried out, projecting a high velocity particle onto a powder surface.
7.2.3 Electrohydrodynamic Instability

Although the author has investigated experimentally the growth rates due to electrohydrodynamic instability and found agreement with theory, this was at only one ratio of jet diameter to electrode diameter ratio. It is therefore suggested that an investigation of other jet-electrodes ratios should be made.

The author's theoretical analysis of a-c stability requires verification by an experimental investigation, especially with regard to the prediction that the maximum growth rate is independent of frequency and that there exist "sideband" growth rate curves for the axisymmetric mode. This could easily be achieved by using an apparatus similar to that described in Section (5.1)
### BIBLIOGRAPHY

<table>
<thead>
<tr>
<th>No.</th>
<th>Author(s)</th>
<th>Title</th>
<th>Source</th>
</tr>
</thead>
</table>


41. HOCKING, M. The stability of a rigidly rotating column of liquid. Mathematika, 7, 1960, 1-.


45. JAMESON, G.J. Private communication, 1971.


86. PHINNEY, R.E. Stability of a laminar viscous jet - the influence of the initial disturbance level. A.I.Ch.E. J., 18 (2), 1972, 432-434.


90. PREIMMER, J. Forts.-Berichte, 3 (8), 1965.


104. SCARLETT, B., BUXTON, R.E. Particle size distribution of spherical particles in Apollo 12 samples. Submitted for publishing in Earth Planet Sci. Let.

105. SCHEELE, G.F., MEISTER, B.J. Drop formation at low velocities in liquid-liquid systems. A.I.Ch.E. J., 14, 1968, 9-.


119. TYLER, E. The instability of liquid jets. Phil. Mag., 16, 1933, 504-.

120. TYLER, E., WATKIN, F. Experiments with capillary jets. Phil. Mag., 14, 1932, 849-.


133. ZELENY, J., The role of surface instability in electrical discharges from drops of alcohol and water in air at atmospheric pressure. J. Franklin Inst., 219 (6), 1935, 659-675.

NOTATION

A Mathieu equation constant.
A* Arbitrary constant.
B* Arbitrary constant.
\( B_{22}, B_{23}, B_{33} \) Coefficients defined by Yuen (131).
Cf Dimensionless capillary constant.
\( \tilde{E} \) Electric field intensity.
El Elasticity parameter.
\( \bar{F} \) Force vector.
\( \bar{F}_E \) Electrical force vector.
Fr Froude number.
G Gravitational constant.
\( H_m \) Hankel function of order \( m \).
\( I_m \) Modified first kind Bessel function of order \( m \).
\( I_{1/m}, I_{2/m} \) Modified Bessel function ratios defined by Yuen (131).
J Chandrasekhar parameter. \( (T \rho a / \mu^2) \)
J* Free current density.
Km Modified second kind Bessel function of order \( m \).
L An inverse Weber number. \( (T/\rho a^2 \omega^2) \)
Lb Jet break-up length.
Ln Nozzle length.
\( \bar{P} \) Pressure.
\( \bar{P}_h \) "Hydrostatic" pressure.
Q Mathieu equation constant.
R Resistivity.
Re Reynolds number.
R,R* Principal radii of curvature.
S Surface area.
T Surface tension.
\( V_m, V_S \) Main and satellite droplet volume.
\( W_e \) Weber number. \( (v^2 a \rho / T) \)
X Arbitrary system parameter.
Z Ohnesorge number. \( (\mu / T^4 \rho a^2) \)

a Undisturbed jet radius.
\( a_0 \) Wave equation parameter.
\( a_d \) Droplet radius.
sn Nozzle radius.
b Charging electrode radius.
b_{2a} Coefficient defined by Yuen (131).
c Wave perturbation amplitude.
c_i Initial perturbation amplitude.
d_{2a} Coefficient in Yuen theory.
dn Nozzle diameter.
dj Jet diameter.
dd Droplet diameter.
ds Satellite droplet diameter.
f Frequency.
g Acceleration due to gravity.
h Heat loss by radiation.
k Axial wavenumber.
m Azimuthal wavenumber.
n Modified wavenumber (viscous theory).
p Exponential wave growth rate.
q Charge density.
q_j Jet surface charge density.
r Radius.
t Time.
t_r Charge relaxation time.
t_b Break-up time.
t_f Time of flight.
v Velocity.
v_s Velocity of sound.
v_j Jet velocity.
v_n Nozzle velocity.
v_m Minimum jet velocity.
y Modified wavenumber.
z Jet axis.

Greek Symbols

Γ Electrification constant.
E Electric field intensity.
ε_E Electrical energy.
\( \varepsilon_K \)  Kinetic energy.
\( \varepsilon_S \)  Surface energy.
\( H \)  Magnetic field intensity.
\( \Theta \)  Temperature (degrees Kelvin).
\( T \)  Shear stress tensor.
\( \gamma \)  Dimensionless time (with respect to field frequency).
\( \Phi \)  Electric potential.
\( \Phi_c \)  Alternating electric field potential.
\( \Phi_o \)  Charging electrode potential.
\( \Omega \)  Angular velocity.

\( \alpha \)  Complex variable.
\( \gamma \)  Electrical conductivity.
\( \delta \)  Kroneker delta, or unit tensor.
\( \varepsilon \)  Permittivity.
\( \varepsilon_o \)  Permittivity of free space.
\( \varepsilon_r \)  Emissivity.
\( \zeta \)  Strained jet axis coordinate.
\( \eta \)  Non-linear disturbance amplitude.
\( \eta_0 \)  Initial non linear disturbance amplitude.
\( \theta \)  Azimuthal angle.
\( \kappa \)  Strained dimensionless wavenumber.
\( \lambda \)  Wavelength.
\( \lambda_1 \)  Visco-elastic constant.
\( \mu \)  Viscosity.
\( \mu_c \)  Complex viscosity.
\( \xi \)  Magneto striction parameter.
\( \rho \)  Density.
\( \sigma \)  Visco elastic constant.
\( \sigma_\beta \)  Stefan's constant.
\( \tau \)  Dimensionless time (with respect to \( (T \rho^1 a^{-3})^{1/2} \))
\( \tau_0, \tau_1, \tau_3 \)  Multiple time scales.
\( \chi \)  Electrostriction parameter.
Appendix A

MASTER JET2
REAL NNU,KK,N,Z,SMVN,SMYN,NMYN
INTEGER JJ1, JJ3, J1, J3, J4, J5
DIMENSION T(1000), Z(1000), N(1000), NNO(50), KK(50), S(5)

C***********************************************************************
C THIS PROGRAM CALCULATES THE DIMENSIONLESS BREAK-UP TIME
C USING NON-LINEAR ANALYSIS
C***********************************************************************

COMMON NNO,KK,J1,J3
READ (1,101) JJ1,JJ3
100 FORMAT(2(J4))
READ(1,101) (NNO(J1), J1=1,JJ1)

C NNU IS THE INITIAL DISTURBANCE AMPLITUDE
C READ(1,101) (KK(J3), J3=1,JJ3)

C KK IS THE WAVE NUMBER
C
101 FORMAT(10(F8.5))
DO 200 J1=1,JJ1
   WRITE(2,501) NNO(J1)
200    FORMAT(Z,501) NNO(J1)
301    FORMAT(10H1,2X,2HINITIAL DISTURBANCE = ,F8.6)
DO 200 J5=1,JJ5
   T(1)=1.0
   S(1)=5.0
   S(2)=1.0
   S(3)=0.1
   S(4)=0.0
   CALL ZMIN(T(1),N(1),7(1))
   DO 4 J4=1,1000
       T(J4+1)=T(J4)+S(1)
       CALL ZMIN(T(J4+1),N(J4+1),Z(J4+1))
       WRITE(2,600) T(J4+1), N(J4+1)
   4 CONTINUE
   WRITE(6,600) T(J4+1), N(J4+1)
   IF(N(J4+1)+1.03,3,15)
5 CONTINUE
   N(1)=N(J4)
   Z(1)=Z(J4)
   T(1)=T(J4)
6 CONTINUE
4 CONTINUE
   TMVN=T(1000)
C TMYN IS THE BREAK-UP TIME
C ZMHN IS THE POSITION OF ZC
C NMHN IS THE DISTANCE FROM THE JET AXIS OF THE MINIMUM
C
C FORMAT(5X,1HK,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7)
WRITE(2,501)
501 FORMAT(9X,THK,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7,1H7)
WRITE(2,500) KK(J3), TMHN, MHN, MHN
C
C THIS SUBROUTINE CALCULATES THE POSITION OF THE LOCAL MINIMUM
C
C*****************************************************************************
C
DIMENSION Z(100), K(100), N(100), SS(5), NU(50), KS(50)
COMMON NU, KS, J, J3
PIE=3.1415926535897932
Z(1)=0.1*PIE/KS(J3)
SS(1)=0.1*PIE/KS(J3)
SS(2)=0.01*PIE/KS(J3)
SS(3)=0.01*PIE/KS(J3)
CALL YUEN(NU(J1), K, KS(J3), Z(J1), NN(1))
N(1)=REAL(NA(1))
DO 30 1=1, J3
DO 10 J=1, 1000
Z(J+1)=Z(J)+SS(1)
CALL YUEN(NU(J1), K, KS(J3), Z(J+1), NN(J+1))
N(J+1)=REAL(NN(J+1))
WRITE(2,400) N(J+1), N(JK), Z(JK+1)
10 CONTINUE
30 CONTINUE
IF(Z(JK+1)-PIE/KS(J3))50, 50, 40
50 CONTINUE
IF(K(1)-100)70,40,20
20 CONTINUE
21 K(1)=K(JA-1)
NN(1)=NN(JA-1)
Z(1)=Z(JA-1)
GO TO 29
40 K(1)=N(JA)
NN(1)=NN(JN)
Z(1)=Z(J6)
2V CONTINUE
Z(1000)=Z(J8+1)
N(1000)=N(J8+1)
3V CONTINUE
ZMYN=Z(1000)
NMIN=N(1000)
RETURN
END
SUBROUTINE YUEN(N0,TS,XS,Z,N)
COMPLEX W1,W2,WS,
       C22,P22,P2,ALPHA,BETA,GAMMA,V3,V,T,A22,A822,
       YB31,B31,C31,A31,B431,B331,DS31,FS3,A53,N,C1,C2,C3,BR53
REAL NO,
       7,TS,KS,K,K*,KS,K5,IA,IR,IC,113,103,112,102,11,10,E10,
       1E11,F102,E112,F113,E108,L,3,NS,N4
C**********************************************************************
C THIS SUBROUTINE EVALUATES THE WAVE PROFILE FROM YUEN'S THEORY
C**********************************************************************

KC=1.0+NO+NO+Y./16.
K=KS/KC
E=KC*?

CALL F4IUACC(K,E10)
I0=EXP(K)*E10
CALL F4IUACC(K,E11)
I1=EXP(K)*E11
IA=10!/11

K2=2.*K
CALL F4IUACC(K2,E102)
I02=EXP(K2)*E102
CALL F4IUACC(K2,E112)
I12=EXP(K2)*E112
IR=10!/112

K3=3.*K
CALL F4IUACC(K3,E103)
I03=EXP(K3)*E103
CALL F4IUACC(K3,E113)
I13=EXP(K3)*E113
IC=113/103

L=1.*IA+1R
J=1.-3.*K+1A

C1=Y*((1.0,0,0)-K*K)/1A
C2=Z.*K*((1.0,0,0)-4.*K*K)/10
C

\[
\begin{align*}
C_{ss} & = (3 + k \times 1 + (N^4 - W^9 + w^7) - 5 \times k \times N^4 + (m^4 + W^3 + w^3) + 2 \times W^3 + 3 \times k \times 1 \times (2 \times W^3 + W^5 + 4 \times (1 + k \times x) + 1 \times (1.5 - W^3 + W^2 + W^1)) \times (1.2 + W^2 + W^1) / (2 + (W^3 + W^5 + W^3) + (W^3 + W^3 = 4 + 3 - 4 \times W^1 + W^2 + W^3))
\end{align*}
\]

C

\[
\begin{align*}
E_{ss} & = (1.1 + 2 + (W^3 + W^5 + w^7 + w^1)) \times (m + W^1 + W^3 / \eta \times (1 + 3 \times (W^L + T^d))) + W^1 \times W^2 \times (W^1 + W^3 - \eta \times (1 + 3 \times (W^3 + W^5 + W^3) + (W^3 + W^3 = 4 + 3 - 4 \times W^1 + W^2 + W^3))
\end{align*}
\]

C

\[
\begin{align*}
A_{ss} & = (E_{ss} + 0.3 + F_{ss})
\end{align*}
\]

C

\[
\begin{align*}
A_{ss} & = A_{ss} + C \times \sinh(W^1 + T^d) + A_{ss} \times \cosh(W^1 + T^d) + C \times \sinh(W^2 + T^d) + A_{ss} \times \cosh(W^2 + T^d) + C \times \sinh(W^3 + T^d) + A_{ss} \times \cosh(W^3 + T^d)
\end{align*}
\]

C

\[
\begin{align*}
N & = N \times \cos(k + T) + C \times \sinh(W^1 + T) \times N \times N \times (W^2 + 2-\cos(2 \times k + T) - 1) / \eta \times (\cosh(W^1 + T)
\end{align*}
\]

\begin{align*}
&+ N \times N \times N \times (W^2 + 2-\cos(2 \times k + T) - 1)
\end{align*}

RETURN

END

COMPLEX FUNCTION \( \sinh(x) \)

COMPLEX \( X \)

\( \sinh = (\exp(x) - \exp(-x)) / 2 \),

RETURN

END

COMPLEX FUNCTION \( \cosh(x) \)

COMPLEX \( X \)

\( \cosh = (\exp(x) + \exp(-x)) / 2 \),

RETURN

END

FINISH

184
MASTER DROP
REAL NQ,(K,1,7)
DOUBLE PRECISION MNVOL,STVOL,ERVOL,ERAD,PRMAIN,PSAT,RTOI,ERSAT,ERMAT
IN,PIE,ACC,FUNK,AB
INTEGER K,J,J1,J2,J3,J4,J5,J6
DIMENSION T(20),NNO(5),KK(20),MNVOL(20),STVOL(20),ERVOL(20),ERAD(20),PRMAIN(20),PSAT(20),RTOI(20),ERSAT(20),ERMAT(20)
C*********************************************************************/
C THIS PROGRAM FINDS THE DROP SIZE OF THE MAIN AND SECONDARY DROPS
C USING YUENS NON-LINEAR ANALYSIS TO PREDICT THE WAVE PROFILE AT
C BREAK-UP
C*********************************************************************/
COMMON MNO,T,KK,J1,J2
READ(1,101) J1,J2
101 FORMAT(2(I4))
READ(1,101) (MNO(J1),J1=1,J1)
C NNO IS THE INITIAL DISTURBANCE AMPLITUDE
C
10 FORMAT(3(F4.5))
DO 700 J1=1,J1
WRITE(2,501) NNO(J1)
701 FORMAT(1H1,2X,22HINITIAL DISTURBANCE = ,F8.6)
WRITE(2,501)
501 FORMAT(8X,14K,7X,SHRMAIN,6X,HERMAIN,7X,4RSAT,6X,5ERSAT,7X,7ERA
40,6K,6HERVOL)
READ(1,191) (KK(J2),T(J2),Z(J2),J2=1,J2)
C Z IS THE POSITION ON THE JFT AXIS OF Z
C KK IS THE WAVE NUMBER
C T IS THE BREAK-UP TIME
C
DO 600 J2=1,J2
PIE=3.1415926535897932
J=20
ACC=0.00001
EXTERNAL FUNK
A=-Z(J2)
B=Z(J2)
C=Z(J2)
D=Z^2.*PIE/KK(J2)-Z(J2)
CALL FINTSYM(A,B,FUNK,ACC,J,MNVOL(J2))
MNVOL(J2)=MNVOL(J2)+PIE
IF(J2)<A,S=5
WRITE(2,600) J
600 FORMAT(11HRETURN J = ,I4)
5 CONTINUE
C THIS SUBROUTINE EVALUATES THE WAVE PROFILE FROM YUKES THEORY

C*****************************************************************************

C

KC=1.+(K-0.5)*Y/10.
K=K5/6C
E=KC*7
C
CALL F410ACC(Y,10)
10=EXP(K)*F10
CALL F411ACC(K,F11)
11=EXP(K)*F11
IA=10/11
C
K2=2.*K
CALL F410ACC(K2,E10)
102=EXP(K2)*F102
CALL F411ACC(K2,E11)
112=EXP(K2)*F112
IA=102/112
C
K3=3.*K
CALL F410ACC(K3,E10)
103=EXP(K3)*E103
CALL F411ACC(K3,E11)
113=EXP(K3)*E113
IC=113/103
C
L=1.+IA*10
J=1.-3.*X*1A
C
C1=K*((1.,U,0.)-K*K)/1A
C2=K*((1.,U,0.,0.4.*K*K))/1n
C3=K*((1.,U,0.,0.9.*K*K))/1f
W1=CSQRT(C1)
W2=CSQRT(C2)
W3=CSQRT(C3)
N3=REAL(W2+W2+W1+W1)
N4=REAL(W2+W2-W1+W1)
C
C22=(2.*K*K+W1*W1+(1.+IA*1A))/(K.*(1.-4.*K*K))
C
R22=W1*(1.-2.*K*1A)/4.
C
R22=W1*(1.+(1.-2.*K*1A))+(2.*K*K+W1*W1*(3.-IA*1A))*K)/(4.*1B
1(W2+W2-W1+W1))
N = N0*Cos(*F) + Cosh(W*1) + N0*N0 + (B*B2+COS(7.*X*F)-1./H.*(COSH(W*1) + 1.)) + N0*N0*N0*(B3+B3*COS(Y*F)+B3s2*COS(5.*X*F))

N IS THE JET WAVE PROFILE

RETURN
END
COMPLEX FUNCTION S1WH(X)
COMPLEX X
S1NH= (CExp(X)-CExp(-X))/2.
RETURN
END
COMPLEX FUNCTION COSH(Y)
COMPLEX X
COSH = (CExp(X)+CExp(-X))/2.
RETURN
END
FINISH
IF(2(JJ)=PIF/KK(JJ))1,3,3
1 CONTINUE
CALL FLINTSMC(C,D,FUNY,AC,J,STVOL(JJ))
STVOL(JJ)=STVOL(JJ)*PIF
IF(JJ)6,6,9
6 WRITE(2,500) J
9 CONTINUE
3 REMAIN(JJ)=(MNVAL(JJ)+S/PIF/4.)**(1./3.)
RSAT(JJ)=(STVOL(JJ)+S/PIF/4.)**(1./3.)
RTOT(JJ)=((MNVAL(JJ)+STVOL(JJ))*(KK(JJ)/2./PIF/PIF))**(1./2.)
ERFAD(JJ)=100.*((STOT(JJ)-1.)
ERVOL(JJ)=(STVOL(JJ)+MNVAL(JJ))/2.*PIF/PIF/KK(JJ)
ERSAT(JJ)=((STVOL(JJ)-(ERVOL(JJ))*(STVOL(JJ))/(MNVAL(JJ)+STVOL(JJ))
6)))*3./PIF/4.)**(1./3.)
ERMAIN(JJ)=(MNVAL(JJ)-(ERVOL(JJ)+MNVAL(JJ))/(MNVAL(JJ)+STVOL(JJ))
4)))*3./PIF/4.)**(1./3.)
*ERVOL(JJ)=ERVOL(JJ)+(MV)/(2.+PIF/PIF/KK(JJ))
WRITE(2,500) JV(JJ),REMAIN(JJ),ERMAIN(JJ),RSAT(JJ),ERSAT(JJ),ERFAD(JJ),ERVOL(JJ)

WX IS THE WAVELENGTH
REMAIN IS THE RADIUS OF THE MAIN DROP
ERMAIN IS THE VOLUME OF THE MAIN DROP
RSAT IS THE RADIUS OF THE SATURATED DROP
ERSAT IS THE VOLUME OF THE SATURATED DROP
ERFAD IS THE ERROR AS DEFINED BY WITLAND AND JAMESON
ERVOL IS THE VOLUME ERROR

500 FORMAT(7(2X,F9.5))
200 CONTINUE
700 CONTINUE
STOP
END

DOUBLE PRECISION FUNCTION FUNK(Z)
REAL Z,AC,T,KS
DOUBLE PRECISION N,PIF,FUNY
COMPLEX NN
INTEGER JJ,JJ
DIMENSION NV(S),T(JJ),KS(?)
COMMON NV(1),KS,J,J
CALL YUEN(NV(JJ),T(JJ),KS(JJ),Z,NN)
N=NFA(NN)
FUNK=1.+Z*NN
RETURN
END

SUBROUTINE YUEN(NV,TS,KS,ZN)
COMPLEX W1,W2,WN, C22,P22,PH,ALPHA,BETA,GAMMA,V5,V,T,A22, kB22,
YB31,031,C31,A31,BS31,B33,C53,E53,A33,N1,N2,C3,B33
Appendix B

The Application of a Transparent Conducting Surface to Glass

This appendix describes how a transparent, but electrically conducting, surface was applied to the glass cylinder used in the work described in Chapter 5. The coating material was tin oxide, which, when applied in a sufficiently thin coating, is transparent. The coating was applied to the surface as dimethyl tin dichloride\(^*\), \((\text{CH}_3\text{)}_2\text{SnCl}_2\), which when heated on a glass surface provides a tin oxide layer.

Using the apparatus shown in Fig. (B.1), the D.T.D. was sublimed onto the surface of the glass cylinder. The cylinder was held at a temperature of 600°C in a muffle furnace. Since only the inner surface of the cylinder required coating, the inlet into the furnace was led into the centre of the cylinder and vented at the other end.

The first trial samples using this method proved to be opaque. This was because an excess of D.T.D. was used. On the final coating much less than 1 gm was used to coat more than 600 cm. It was found to be important with this method to cool the glass slowly so that it could become annealed and hence mechanically tough. The coating was fully transparent as the photographs reproduced in Chapter 5 testify, although some interference patterns were observed. When checked with a resistance meter the coating was very uniform, giving a resistance of \(3 \, \Omega \text{ cm}^2\). The coating showed no signs of deterioration with use and should be mechanically strong as it is used in the glass bottle industry to provide low friction coatings.

There are no toxicity problems involved in the handling of D.T.D., but the furnace should be adequately vented to atmosphere as fumes of D.T.D. are choking. D.T.D. is normally supplied as needle like crystals with a sublimation temperature of 106°C.

\(^*\) The D.T.D. was kindly supplied by the Redfearn National Glass Co., Barnsley, Yorkshire.
Schematic Diagram of Apparatus for Coating Glass Electrode.
Appendix C

MASTER AC
DOUBLE PRECISION V1
REAL K, I0, I1, K0, K1
COMPLEX R, CB
REAL K1D, I1D
REAL PP
REAL P
DIMENSION PP(200)
DIMENSION AA(10, 200), BB(10, 200), A(200), A1(200), Q(200), K(200)
DIMENSION P(9), Z(9)
DIMENSION X(6, 10), Y(6, 10)
DIMENSION W(108)
DIMENSION C(20), E(20)
COMMON W, Q, A, I, AA

C**************************************************************************************
C THIS PROGRAM INVESTIGATES THE STABILITY OF AN INVISID JET
C IN A CYLINDRICAL A.C. FIELD (ELECTROSTATIC)
C BY EXAMINING THE STABILITY OF MATHIEUS EQUATION
C WARNING: THIS PROGRAM REQUIRES A FINITE ELECTRIFICATION CONSTANT E
C**************************************************************************************

READ (1, 102) N
READ (1, 102) N2
READ (1, 103) B
READ (1, 101) (C(J), E(J), J = 1, N2)
READ (1, 104) ((X(J, I), I = 1, 9), J = 1, 6)
READ (1, 104) ((Y(J, I), I = 1, 9), J = 1, 5)
CALL UT POP
AMAX = 35.0
AMIN = -15.0
QMIN = 0.0
QMAX = 25.0
DO 170 J2 = 1, N2
WRITE (2, 202)
WRITE (2, 203) C(J2), E(J2)
WRITE (2, 201)
DO 100 I = 2, N + 1
A(I) = 0.0
Q(I) = 0.0
K(I) = 0.0
K(I) = K(I - 1) + B/N
CALL F410ACC(K(I), I0)
CALL F411ACC(K(I), I1)
CALL F4K0(K(I), K0)
CALL F4K1(K(I), K1)
CALL F4K0(K(I), K0)
CALL F4K1(K(I), K1)
I1D = 10 - I1/K(I)
K1D = K0 - K1/K(I)
Q(I) = -E(J2)*K(I)*I1D/I1*(1 + K(I)*K1D/K1)
A(I) = +C(J2)*K(I)*K(I)*I1D/I1 - 2.0*Q(I)
IF(Q(I), LE, 0.0) Q(I) = -Q(I)
DO 600 J = 1, 6

192
AA(J,I)=X(J,1)+X(J,2)*Q(I)+X(J,3)*Q(I)*Q(I)+X(J,4)*Q(I)*Q(I)*Q(I)
1+X(J,5)*Q(I)*Q(I)+X(J,6)*Q(I)*Q(I)*Q(I)+X(J,7)*Q(I)*Q(I)*Q(I)*Q(I)
1+X(J,8)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)+X(J,9)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)
600 CONTINUE
DO 700 J=1,5
BB(J,I)=Y(J,1)+Y(J,2)*Q(I)+Y(J,3)*Q(I)*Q(I)+Y(J,4)*Q(I)*Q(I)*Q(I)
1+Y(J,5)*Q(I)*Q(I)*Q(I)*Q(I)+Y(J,6)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)
1+Y(J,7)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)+Y(J,8)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)
3*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)+Y(J,9)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)
700 CONTINUE
DO 60 J=1,5
IF(A(I).GE.36.0) GO TO 900
IF (A(I).LE.AA(J,I).AND.A(I).LE.BB(J,I)) GO TO 900
IF (A(I).LE.AA(J+1,I)) GO TO 67
P(I)=((AA(J+1,I)-A(I))/(A(I)-BB(J,I)))
JJ=J
IF(A(I).GE.BB(J,I).AND.A(I).LE.AA(J+1,I)) GO TO 66
60 CONTINUE
GO TO 900
66 CONTINUE
P(I)=SQRT(P(I))/2./JJ
WRITE(2,400) K(I),A(I),Q(I),P(I)
GO TO 70
67 CONTINUE
CC=A(I)-A(I)-1.)/(2.-(A(I)-1.)*(A(I)-1.)*Q(I)*Q(I))Q(I)*Q(I)-Q(I)
1+Q(I)*Q(I)*Q(I)*(Q(I)-5.)
1.A(I)-7.)/(32.-(A(I)-1.)*(A(I)-1.)*Q(I)*Q(I))Q(I)*Q(I)*Q(I)*Q(I)
1+Q(I)*Q(I)-Q(I)*Q(I)-Q(I)*Q(I)-Q(I)*Q(I)*Q(I)*Q(I)-Q(I)*Q(I)
1+Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)*Q(I)
1.3=CMPLX(CC,0.0)
R=CSQRT(R)
P(I)=AIMAG(R)
WRITE(2,400) K(I),A(I),Q(I),P(I)
400 FORMAT(3(2X,E14.6),8HUNSTABLE,2X,E14.6)
GO TO 70
900 WRITE(2,500) K(I),A(I),Q(I)
L=N+1
70 CONTINUE
PP(I)=P(I)
100 CONTINUE
CALL UTP4A(QMIN,QMAX,AMIN,AMAX,5.0,10.0,1,1,1)
CALL UTP4B(Q,A,L,2)
CALL UTP4A(Q,A,L,2)
CALL UTP4B(K,PP,L,2)
170 CONTINUE
CALL UTPCL
101 FORMAT(2(F10.4))
103 FORMAT(F10.4)
102 FORMAT(14)
500 FORMAT(2X,E14.6,2X,E14.6,2X,E14.6,2X,E14.6,2X,HUNSTABLE)
104 FORMAT(2X,E18.12)
801 FORMAT(2X,E14.6,2X,E14.6,2X,E14.6)
201 FORMAT(7X,H1,H2,H3,H4,H5,H6)
202 FORMAT(1H1,4SHMATHIEUS EQUATION IS C"+(A-2*Q*COS(2*T))*C=0)
203 FORMAT(//7X,1H1,4SHCAPILLARY CONSTANT =,F10.4,2X,THELECTRIFICATION CO
STOP
END
### Mathieu Functions

The first number gives the polynomial order, the last the coefficient.

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