Integrable equations of the dispersionless Hirota type

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Integrable equations of the dispersionless Hirota type

by

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Abstract

Let $u(x, y, t)$ be a function of 3 variables $x, y, t$. Equations of the dispersionless Hirota type have the form

$$F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0.$$ 

Familiar examples include the Boyer-Finley equation $u_{xx} + u_{yy} = e^{u_{tt}}$, the potential form of the dispersionless Kadomtsev-Petviashvili (dKP) equation $u_{xt} - \frac{1}{2}u_{xx}^2 = u_{yy}$, the dispersionless Hirota equation $(\alpha - \beta)e^{u_{xy}} + (\beta - \gamma)e^{u_{xt}} + (\gamma - \alpha)e^{u_{tx}} = 0$, etc.

We study integrability of such systems in the sense of the existence of infinitely many hydrodynamic reductions. The moduli space of integrable equations of the dispersionless Hirota type is proved to be 21-dimensional. In addition, it is shown that the action of the equivalence group $Sp(6)$ on the moduli space has an open orbit.
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Chapter 1

Introduction

It often happens that the understanding of the mathematical nature of an equation is impossible without a detailed understanding of its solutions.

Freeman J. Dyson [28]

The theory of nonlinear differential equations which can be, at least in some sense, solved exactly, is nowadays referred to as the theory of integrable systems. The subject developed rapidly in the past three decades, partly due to its applicability in a wide range of physical situations. It began with the study of shallow water waves in fluid mechanics, and is now popular with both physicists and mathematicians [16]. It is difficult in a short space to recount in any detail the history of integrable systems. However, it would be nice to give a broad idea of how the theory of integrable systems and in particular the theory of solitons has been developed.

The discovery of solitary waves of translation goes back to John Scott Russell in 1834. While conducting experiments to determine the most efficient design for canal boats, Russell observed that when the boat he was using suddenly stopped moving - not so did the mass of water in the channel which was forced to move. Russell conducted some experiments as well as theoretical work [17]. Airy (1845),
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Stokes (1847), Boussinesq (1871) and Lord Rayleigh (1876) made important contributions to the topic. However, it was not until 1895 when Korteweg and de Vries derived the Korteweg-de Vries (KdV) equation, that the one-soliton solution and hence the concept of solitary waves was put on a firm basis [2, 53]. It seems that for some decades, the solitary wave was considered a rather unimportant curiosity in the mathematical structure of nonlinear wave theory [53].

Then in 1965 Zabusky and Kruskal [60] after being motivated by the important work of Fermi, Pasta and Ulam in the 1950's, reported the first numerical computation of solutions of the KdV equation. Their numerical solutions showed remarkable stability of the solitary waves, each of which behaved like a particle. Zabusky and Kruskal named these solitary waves *solitons* [28]. At this point it is worth noting, as Zabusky stressed, the important role that numerical experimentation played in these discoveries [46]. In 1967 Gardner, Greene, Kruskal and Miura showed how the analytical solution of the KdV equation can be obtained. This method was called the *inverse scattering method*. Soon thereafter, in 1968, Peter Lax proposed a more convenient and universal reformulation of the work of Gardner et al. with the so-called Lax formalism. Following Lax’s formulation Zakharov and Shabat (ZS) in 1972 extended the inverse scattering method to solve the *nonlinear Schrödinger equation* (NLS) [27]. Later on in 1972, Ablowitz, Kaup, Newell, and Segur (AKNS) solved the sine-Gordon equation as well as the modified KdV equation. The mathematical technique used by Gardner et al., later by ZS and many others is now referred to as the *Inverse Scattering Transform* (IST). Note here, that some of the above theoretical results were in remarkable agreement with many of the experimental results obtained by Russell more than one hundred and fifty years ago [53]. Furthermore, since the discovery of the IST many important developments have occurred in the theory of integrable equations.
The KdV equation mentioned above is a dispersive (1 + 1)-dimensional equation. It can be cast into the form [45]

\[ u_t = u_{xxx} - 6uu_x, \]

where \( u_{xxx} \) is called the dispersion term. In 1970, Kadomtsev and Petviashvili derived a (2+1)-dimensional generalization of the KdV equation which was called the Kadomtsev-Petviashvili (KP) equation [37].

In addition, taking the KdV equation given above and letting \( x \to \kappa x \) as well as \( t \to \kappa t \), one obtains \( u_t = \kappa^2 u_{xxx} - 6uu_x \). Then taking the limit \( \kappa \to 0 \) one arrives at the (1 + 1)-dispersionless equation

\[ u_t = -uu_x, \]

which contains the simplest type of non-linearity \((uu_x)\). Similarly, if we take the KP equation mentioned above in the form [23],

\[ \left( u_t - uu_x - \frac{\varepsilon^2 u_{xxx}}{12} \right)_x = u_{yy}, \]

and take its dispersionless limit \( \varepsilon \to 0 \), we obtain the dispersionless Kadomtsev-Petviashvili (dKP)

\[ (u_t - uu_x)_x = u_{yy}, \]

also known as the Khokhlov-Zabolotskaya equation [58]. Dispersionless equations made their first appearance in gas dynamics in 1970 and magnetohydrodynamics in 1971 [9, 50]. Since then many researchers have shown a great interest in dispersionless systems (e.g., [29, 30, 55, 59, 6, 24, 21]).

This thesis is devoted to the study of integrable dispersionless systems. In par-
ticular, we will be investigating and classifying integrable second order equations of the form

\[ F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0, \tag{1.1} \]

which constitute a single relation among the second order partial derivatives of a function \( u(x, y, t) \). Equations of this form will be called equations of the dispersionless Hirota type. Equations of this type naturally arise in mechanics, mathematical physics, general relativity and differential geometry. The integrability is understood as the existence of infinitely many hydrodynamic reductions (this approach will be explained in Chapter 3). We prove that the moduli space of integrable equations of the dispersionless Hirota type is 21-dimensional, and the action of the equivalence group \( Sp(6) \) on the moduli space has an open orbit. Some well known examples of integrable equations of the form (1.1) are:

1) dKP equation: \( u_{xt} - \frac{1}{2} u_{xx}^2 = u_{yy}, \)
2) Boyer-Finley equation: \( u_{xx} + u_{yy} = e^{u_{tt}}. \)

In addition we have examples of non-integrable equations of the form (1.1):

1) \( Hess \ u = 1, \)
2) \( Hess \ u = \Delta u, \)

where \( Hess \) is the determinant of the Hessian matrix of \( u \), and \( \Delta \) is the Laplacian.

As mentioned above, our definition of integrability depends upon the existence of infinitely many hydrodynamic reductions. In order to understand this approach we need to have a general idea of the theory of hydrodynamic type systems.
Therefore, we begin Chapter 2 by defining the general form of hydrodynamic type systems:

\[ u_i^j = v_j^j(u)u_x^j, \]

where \( i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad u^j \) is an \( n \)-component column vector and \( v_j^j(u) \) is an \( n \times n \) matrix. We will be considering the cases where the eigenvalues (the characteristic speeds of the system (1.2)) of the matrix \( v_j^j(u) \) are real and distinct. Such systems are called hydrodynamic systems of strictly hyperbolic type.

Furthermore, some systems of type (1.2) are diagonalizable. That means that they can be reduced to Riemann invariants. Any 2x2 system can be transformed to Riemann invariants whereas for 3x3 systems Riemann invariants do not necessarily exist. A standard procedure for the calculation of Riemann invariants exists for 2x2 systems which we address in detail in Sect. 2.2. In addition, the diagonalizability of systems (1.2) is a necessary condition for integrability via the generalized hodograph method [56]. Moreover, we enclose some necessary information about:

1) Commuting flows,
2) Conservation laws,
3) Semi-Hamiltonian property,
4) Generalized hodograph method.

Chapter 3 addresses the problem of integrability of the general class of three-dimensional second order equations of the form (1.1). At this point it is worth noting that equations of type (1.1) have been dealt with by a variety of techniques, however there was no acceptable definition of integrability which would allow the construction of either exact solutions or classification results. Based on the method of hydrodynamic reductions, which was primarily Gibbons and Tsarev's work [29], [30], continued by Ferapontov and Khusnutdinova [21], we
have managed to obtain partial classification results.

In general, the method of hydrodynamic reductions applies to the dispersionless systems

\[ A(u)u_x + B(u)u_y + C(u)u_t = 0, \quad (1.3) \]

where \( u \) is an \( m \)-component column vector of the dependent variables, and \( A, B, C \) are \( l \times m \) matrices. We then seek for multi-phase solutions in the form

\[ u = u(R^1, ..., R^n), \]

where the 'phases' \( R^i(x, y, t) \) are required to solve a pair of commuting diagonal systems

\[ R^i_t = \lambda^i(R) R^i_x, \quad R^i_y = \mu^i(R) R^i_x. \quad (1.4) \]

In addition, the characteristic speeds \( \lambda^i \) and \( \mu^i \) satisfy the commutativity conditions

\[ \frac{\lambda^i_j}{\lambda^i - \lambda^j} = \frac{\mu^i_j}{\mu^i - \mu^j}, \quad i \neq j, \]

see Theorem 1 in Sect. 2.3, as well as [56]. Therefore, the method reduces the three-dimensional PDE (1.1) to a pair of commuting \( n \)-component \((1 + 1)\)-dimensional systems of hydrodynamic type. A three-dimensional equation is thus called integrable if it possesses 'sufficiently many' \( n \)-component reductions of the form (1.4) (for formal definition see the beginning of Chapter 3). We emphasize that equations of the form (1.1) can be put into the form (1.3) by a change of variables, see Chapter 3.

Using the requirement of the existence of \( n \)-component hydrodynamic reductions, we have developed a code which provides integrability conditions for the three-dimensional system (1.1). These conditions constitute a complicated system of PDEs for the function \( F \). Furthermore, we use the same code for sub-cases such
as:

\[ u_{tt} = f(u_{xx}, u_{yy}), \quad u_{tt} = f(u_{xx}, u_{xy}, u_{yy}), \quad u_{xy} = f(u_{zt}, u_{yt}), \]

\[ u_{tt} = f(u_{xx}, u_{zt}, u_{xy}), \]

etc. With these sub-cases the integrability conditions are simple enough to be solved explicitly, providing us with partial classification results.

**Example.** Take the simplest case \( u_{tt} = f(u_{xx}, u_{yy}) \). The integrability conditions are the following:

\[
\begin{align*}
  f_{aaa} &= f_{aa} \left( \frac{f_{ac}}{f_c} + \frac{f_{aa}}{f_a} \right), \\
  f_{aac} &= f_{aa} \left( \frac{f_{ac}}{f_c} + \frac{f_{ac}}{f_a} \right), \\
  f_{acc} &= f_{ac} \left( \frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a} \right), \\
  f_{acc} &= f_{ac} \left( \frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a} \right), \\
  f_{aa} f_{cc} &= (f_{ac})^2.
\end{align*}
\]

Up to the equivalence transformations, the generic solution of this system can be put into either of the following non-equivalent forms,

\[ e^{u_{xx}} + e^{u_{yy}} = e^{u_{tt}} \quad \text{and} \quad u_{xx} + u_{yy} = e^{u_{tt}}. \]

The first equation is apparently new, while the second is the Boyer-Finley equation. Other new examples resulting from the rest of the sub-cases mentioned above are given below.

The sub-case \( u_{tt} = f(u_{xx}, u_{xy}, u_{yy}) \) leads to the following new equations:

1) \( u_{tt} = \alpha u_{xx} + \beta u_{yy} + \frac{2\sqrt{\alpha \beta}}{\gamma} \ln \cosh \gamma u_{xy}, \)

2) \( u_{tt} = \alpha u_{xx} + \gamma \ln u_{xy}, \quad \text{or} \quad u_{tt} = \ln u_{xy}, \)

3) \( u_{tt} = u_{xy} + \beta (u_{xx} - u_{yy}) + \frac{\sqrt{1 + 4\beta^2}}{2\gamma} \ln \cosh \gamma (u_{xx} + u_{yy}), \)

4) \( u_{tt} = u_{xy} + \beta u_{xx} + \phi(u_{yy}). \)
In addition, the sub-case \( u_{xy} = f(u_{xt}, u_{yt}) \) provides us with the following new equations:

1) \( u_{xy} = u_{xt} + e^{u_{yt}} \),

2) \( u_{xy} = u_{xt} \tan(u_{yt}) \).

In the case of the full system (1.1), the integrability conditions allow us to prove one of the main results of this thesis:

- The moduli space of integrable equations of the dispersionless Hirota type is 21-dimensional.

Chapter 4 deals with symmetries of differential equations. Such transformations are groups that depend on continuous parameters and consist of either point transformations (point symmetries), acting on the system's space of independent and dependent variables or, more generally, contact transformations (contact symmetries), acting on the space of independent and dependent variables as well as on all first order derivatives of the dependent variable [7].

We begin by recalling some important aspects of the Lie group theory. We define one-parameter groups which obey the three basic properties of a group, as well as an additional property, namely the smooth dependence of transformations on the group parameter. In particular we discuss concepts such as:

1) The Lie equation,

2) Invariant functions and infinitesimal generators,

3) Invariant equations,

4) Construction of symmetry groups,

5) Prolongation formulae.

Using the theory mentioned above, we calculate infinitesimal symmetries for the
Boyer-Finley equation in the form

\[ u_{xx} + u_{yy} = e^{u_{tt}}, \]

as well as for the new equation

\[ e^{u_{tt}} = e^{u_{xx}} + e^{u_{yy}}. \]

One of our main results comes from the investigation of \( Sp(6) \) as a symmetry group of the integrability conditions. Given a function \( u \), let us consider the 6-dimensional space with coordinates \( x, y, t, u, u_x, u_y \). We are looking at linear transformations which preserve the symplectic form

\[ du_x \wedge dx + du_y \wedge dy + du_t \wedge dt. \]

By definition, these transformations generate the Lie group \( Sp(6) \). If we prolong them to second order derivatives, the structure of equation (1.1) will be preserved. This means that second order derivatives transform via second order derivatives only. Therefore, the class of equation (1.1) is preserved. Moreover, integrability will also be preserved.

Furthermore, the Lie algebra of this group is spanned by 21 vector fields. These 21 vector fields are calculated explicitly and, given a PDE of the form (1.1), we search for its infinitesimal symmetries by solving the determining equation

\[ L_X F|_{F=0} = 0, \]

where \( X \) is a linear combination of the 21 vector fields which generate the Lie algebra of the symplectic Lie group \( Sp(6) \). Our main result here is that:

- The action of the equivalence group \( Sp(6) \) on the moduli space of integrable equations of the dispersionless Hirota type has an open orbit.
Thus, the 'generic' integrable equation has no continuous symmetries, and generates an open orbit of the moduli space.

Finishing off we investigate whether integrability of an equation is related to the number of contact symmetries possessed by the equation. Therefore, we calculate contact symmetries of the equation $e^{uxx} + e^{uyy} = e^{ut}$, the dispersionless KP equation $u_{xt} - \frac{1}{2} f(u_{xx}) = u_{yy}$ and the case where $u_{xt} + f(u_{xx}) = u_{yy}$. Finally, we arrive at the conclusion that:

- Integrability of a differential equation of the dispersionless Hirota type is generically not related to the size of its contact symmetry algebra.
Chapter 2

(1 + 1)-dimensional Hydrodynamic type systems

2.1 General form of Hydrodynamic type systems

In general, systems of the form

\[ u_i = v_j(u) u_x^j \]  \hspace{1cm} (2.1)

or, in matrix notation,

\[
\begin{bmatrix}
  u^1 \\
  u^2 \\
  \vdots \\
  u^n
\end{bmatrix}_t =
\begin{bmatrix}
  v^1_1 & \cdots & v^m_1 \\
  \vdots & \ddots & \vdots \\
  v^1_n & \cdots & v^m_n
\end{bmatrix}
\begin{bmatrix}
  u^1 \\
  u^2 \\
  \vdots \\
  u^n
\end{bmatrix}_x.
\]
are called one-dimensional systems of hydrodynamic type.

Take for example the equations of gas dynamics

\[
\begin{align*}
\rho_t + u_x \rho + u \rho_x &= 0, \\
u_t + uu_x + \gamma \rho^{\gamma-2} \rho_x &= 0.
\end{align*}
\]

Equations (2.2) can be written in the matrix form,

\[
\begin{pmatrix}
\rho \\
u
\end{pmatrix}_t =
\begin{pmatrix}
u & \rho \\
\gamma \rho^{\gamma-2} & u
\end{pmatrix}
\begin{pmatrix}
\rho \\
u
\end{pmatrix}_x.
\]

The systems we will be dealing with are hyperbolic systems of hydrodynamic type which means that the characteristic equation,

\[
det(v(u) - \lambda I) = 0,
\]

has real and distinct roots \(\lambda^1(u), \cdots, \lambda^n(u)\). Furthermore, we will be dealing with systems which possess Riemann invariants. The existence of Riemann invariants for systems of type (2.1) is a requirement for the matrix \(v(u)\) to be diagonalizable - the necessary condition for systems (2.1) to be integrable by the generalized hodograph method [56].

\subsection{2.2 Systems in Riemann Invariants}

We say that the system (2.1) possesses Riemann invariants if we can find suitable variables

\[
R^1(u), \ldots, R^n(u),
\]
such that the system (2.1) becomes diagonalizable,

\[ R^i_t = \lambda^i(R) R^i_x. \] (2.6)

These new variables \( R = (R^1, \cdots, R^n) \) are called Riemann invariants. It is worth noting here that for 2x2 systems Riemann invariants always exist, whereas for 3x3 systems Riemann invariants do not necessarily exist.

In order to transform a 2x2 system to Riemann invariants there exists a standard procedure:

1. Bring the system into the form (2.1) and solve the characteristic equation

\[ \det (v(u) - \lambda I) = 0, \]

which is assumed to have two roots \( \lambda^1(u) \) and \( \lambda^2(u) \).

2. Fix \( \lambda^1(u) \) and \( \lambda^2(u) \) and calculate their corresponding left eigenvectors such that:

\[ (\xi_1, \xi_2)(v(u) - \lambda^1 I) = 0, \]
\[ (\xi_3, \xi_4)(v(u) - \lambda^2 I) = 0. \]

3. Choose \( (\xi_1, \xi_2) \) and \( (\xi_3, \xi_4) \) to be the gradients of \( R^1(u) \) and \( R^2(u) \), respectively, which can be chosen in such a way that the following system is solvable (notice that left eigenvalues are defined up to a scalar multiple):

\[ \left( \frac{\partial R^1}{\partial u^1}, \frac{\partial R^1}{\partial u^2} \right) = (\xi_1, \xi_2), \]
\[ \left( \frac{\partial R^2}{\partial u^1}, \frac{\partial R^2}{\partial u^2} \right) = (\xi_3, \xi_4). \]
Next we show an example of this calculation for a 2x2 system.

**Example.** Calculate Riemann invariants for gas dynamics equations. Consider the system (2.2), bring it into matrix form (2.3) and solve the characteristic equation

\[
\det \begin{pmatrix} u - \lambda & \rho \\ \gamma \rho^{\gamma - 2} & u - \lambda \end{pmatrix} = 0,
\]

(2.7)

to obtain its eigenvalues:

\[
\lambda_{1,2} = u \pm (\gamma \rho^{\gamma - 1})^{1/2}.
\]

(2.8)

Then, we calculate the corresponding left eigenvectors by solving the following systems:

\[
\begin{pmatrix} \xi_1, \xi_2 \end{pmatrix} \begin{pmatrix} (\gamma \rho^{\gamma - 1})^{1/2} & \rho \\ \gamma \rho^{\gamma - 2} & (\gamma \rho^{\gamma - 1})^{1/2} \end{pmatrix} = 0,
\]

\[
\begin{pmatrix} \xi_3, \xi_4 \end{pmatrix} \begin{pmatrix} -(\gamma \rho^{\gamma - 1})^{1/2} & \rho \\ \gamma \rho^{\gamma - 2} & -(\gamma \rho^{\gamma - 1})^{1/2} \end{pmatrix} = 0.
\]

Thus, we obtain the following normalized eigenvectors

\[
(\xi_1, \xi_2) = \left( (\gamma \rho^{\gamma - 3})^{1/2}, 1 \right),
\]

\[
(\xi_3, \xi_4) = \left( -(\gamma \rho^{\gamma - 3})^{1/2}, 1 \right).
\]

Finally, we solve the system of PDE's

\[
\left( \frac{\partial R^1}{\partial \rho}, \frac{\partial R^1}{\partial u} \right) = \left( (\gamma \rho^{\gamma - 3})^{1/2}, 1 \right),
\]

\[
\left( \frac{\partial R^2}{\partial \rho}, \frac{\partial R^2}{\partial u} \right) = \left( -(\gamma \rho^{\gamma - 3})^{1/2}, 1 \right),
\]
to get,

\[ R^1 = u + \frac{2\gamma^{1/2}\rho^{\frac{\gamma-1}{2}}}{\gamma-1}, \quad R^2 = u - \frac{2\gamma^{1/2}\rho^{\frac{\gamma-1}{2}}}{\gamma-1}. \tag{2.9} \]

Note here, that from (2.9) and (2.8) we obtain

\[ \chi^1 = \frac{R^1 + R^2}{2} + \frac{1}{4}(\gamma - 1)(R^1 - R^2), \quad \chi^2 = \frac{R^1 + R^2}{2} + \frac{1}{4}(\gamma - 1)(R^2 - R^1), \tag{2.10} \]

and by direct substitution of (2.10) into (2.6) for \( i = 1, 2 \) we have that

\[ R^1_t + \left( \frac{R^1 + R^2}{2} + \frac{1}{4}(\gamma - 1)(R^1 - R^2) \right) R^1_x = 0, \]
\[ R^2_t + \left( \frac{R^1 + R^2}{2} + \frac{1}{4}(\gamma - 1)(R^2 - R^1) \right) R^2_x = 0. \tag{2.11} \]

We can see that by using the change of variables

\[ u = \frac{R^1 + R^2}{2}, \quad R^1 - R^2 = \frac{4\gamma^{1/2}\rho^{\frac{\gamma-1}{2}}}{\gamma-1}, \]

(2.3) goes to (2.11), and therefore \( R^1 \) and \( R^2 \) are the Riemann invariants of (2.2).

## 2.3 Commuting flows

Consider two PDEs written in the form

\[ u_t = F(u, u_x, u_{xx}, ...), \quad u_\tau = G(u, u_x, u_{xx}, ...), \tag{2.12} \]

where \( u \) is a (vector)-function of \( t, \tau \) and \( x \). In addition, \( t \) and \( \tau \) are the corresponding ‘times’. Then, we say that the PDEs (2.12) commute if they satisfy the consistency condition

\[ u_{t\tau} = u_{\tau t}. \tag{2.13} \]
Although neither the KdV equation, nor the KdVs equation are of hydrodynamic type, we will use them for our next example since the concept of commuting flows also applies to dispersive equations.

**Example.** Show that the following PDEs (KdV and KdVs) commute:

\[
\begin{align*}
    u_t &= u_{xxx} - 6uu_x, \\
    u_r &= -\frac{1}{10}u_{xxxx} + uu_{xx} + 2u_xu_{xx} - 3u^2u_x. 
\end{align*}
\]

(2.14)

Firstly, we have that,

\[
    u_{rr} = u_{xxxx} - 6u_ru_x - 6uu_{xr},
\]

(2.15)

\[
    u_{rt} = -\frac{1}{10}u_{xxxxx} + u_tu_{xx} + uu_{xxx} + 2u_xu_{xx} + 2u_xu_{xxx} - 6uu_xu_x - 3u^2u_{xt}.
\]

(2.16)

Then, using (2.14) we need to calculate higher order derivatives of \(u_t\) and \(u_r\) in terms of \(x\), and substitute into (2.15) and (2.16). For example,

\[
\begin{align*}
    u_{tx} &= u_{xxxx} - 6u_x^2 - 6uu_{xx}, \\
    u_{txx} &= u_{xxxx} - 18u_xu_{xx} - 6uu_{xxx}, \\
    &\vdots \\
    u_{txx} &= -\frac{1}{10}u_{xxxxx} + u_xu_{xxx} + uu_{xxxx} + 2u_x^2 + 2u_xu_{xxx} - 6uu_x^2 - 3u^2u_{xx}, \\
    &\vdots 
\end{align*}
\]

Finally, substituting \(u_t, u_r\) and all partial derivatives calculated above into (2.13) we see that the PDEs (2.14) are consistent with each other.

Next, consider two hydrodynamic type systems of the form

\[
\begin{align*}
    u_t^i &= v_j^i(u)u_x^j, \\
    u_r^i &= w_j^i(u)u_x^j. 
\end{align*}
\]

(2.17)
(1 + 1)-dimensional Hydrodynamic type systems

It is claimed that for systems (2.17) to commute the matrices $V = v^j_i$, $W = w^i_j$ must commute.

**Proof** [56]. Take $u^i_t = v^j_i(u)u^j_x$ and differentiate by $\tau$. We get

$$u^i_{\tau \tau} = v^j_{,k}u^i_s u^j_x + v^j_{,\tau}u^j_x$$

$$= v^j_{,k}w^i_s u^i_x u^j_x + v^j_{,\tau} (w^i_k u^j_x)_x$$

$$= v^j_{,k} w^i_s u^i_x u^j_x + v^j_{,\tau} w^i_k u^j_x u^i_x + v^j_{,\tau} w^i_k u^j_x$$

where $v^j_{,k} = \frac{\partial}{\partial u^k} v^j_i$ and $w^i_{,k} = \frac{\partial}{\partial u^k} w^i_j$. Similarly for $u^i_t = w^i_j(u)u^j_x$ we obtain,

$$u^i_{\tau \tau} = w^j_{,k} v^i_s u^i_x u^j_x + w^j_{,\tau} v^j_x u^i_x u^i_k + w^j_{,\tau} v^j_x u^i_k$$

Then, one needs to equate coefficients of first and second order derivatives of $u$.

We are actually interested only in the coefficients of $u_{xx}$ which gives us:

$$v^j_i w^i_k = w^j_i v^i_k;$$

or in other words

$$[V, W] = VW - WV = 0.$$  

Therefore matrix commutation is a necessary condition for systems of type (2.1) to commute. In addition, consider two diagonal systems of type (2.1). These systems satisfy the necessary commutativity condition mentioned above since diagonal matrices commute.

Furthermore, we claim that the requirement that such diagonal systems commute imposes additional constraints.
Theorem 1  Two systems of the form

\[ R_t^i = \lambda^i(R)R_x^i, \quad R_r^i = \mu^i(R)R_z^i, \quad (2.18) \]

commute, if they satisfy the following condition

\[ \frac{\lambda_j^i}{\lambda^i - \lambda^j} = \frac{\mu^j_i}{\mu^j - \mu^i}, \quad i \neq j, \quad (2.19) \]

where \( \lambda_j^i = \partial_{R^j} \lambda^i \) and \( \mu^j_i = \partial_{R^j} \mu^i \).

Proof [56]. For the systems to commute we need to have,

\[ R_{tr}^i = R_{rt}^i. \quad (2.20) \]

Therefore we need to calculate \( R_{tr}^i, R_{rt}^i \) and equate coefficients of identical derivatives of \( R \). Thus,

\[
R_{tr}^i &= (\lambda^i R_x^i)_r \\
&= \lambda^i R_r^i R_x^i + \lambda^i R_{xr}^i \\
&= \lambda^i \mu^j R_x^i R_x^j + \lambda^i (\mu^i R_x^i)_x \\
&= \lambda^i \mu^j R_x^i R_x^j + \lambda^i \mu^i R_x^i R_x^i + \lambda^i \mu^i R_{xx}^i.
\]

In the same way,

\[
R_{rt}^i = \mu^j \lambda^i R_x^j R_x^i + \mu^i \lambda^j R_x^i R_x^j + \mu^i \lambda^i R_{xx}^i,
\]

Using the fact that diagonal matrices commute, coefficients of \( R_{xx}^i \) cancel and we are left with

\[
\lambda_j^i (\mu^j - \mu^i) = \mu^i_j (\lambda^j - \lambda^i), \quad (2.21)
\]
which is the desired result. Note that (2.21) is called the *commutativity condition* and the diagonal systems (2.18) are said to be *commuting flows*.

### 2.4 Conservation laws

Consider a PDE in the form

\[ u_t = F(u, u_x, u_{xx}, \ldots). \tag{2.22} \]

It is said that a relationship of the form

\[ [f(u)]_t = [g(u)]_x, \tag{2.23} \]

which holds identically by virtue of (2.22), is a conservation law of (2.22). Note that the functions \( f(u) \) and \( g(u) \) are called the conserved density and the flux, respectively, and neither of them involve derivatives with respect to \( t \). In most of the cases, \( f(u) \) and \( g(u) \) are polynomials in \( u \). In this case, assuming that \( u(x) \) tends to zero sufficiently fast when \( x \) tends to infinity, then both \( f(u) \) and \( g(u) \) will also tend to zero, so that integrals are convergent. Therefore, integrating the equation (2.23) over \( x \) we have

\[ \frac{d}{dt} \int_{-\infty}^{+\infty} f \, dx = g(\infty) - g(-\infty) = 0, \]

which shows that the quantity

\[ H = \int_{-\infty}^{\infty} f \, dx \]
is conserved and is usually called an integral of motion.

Next, consider a diagonal system in Riemann invariants (2.6). It is claimed that if

\[ f_{ij} = \frac{\lambda^i_j}{\lambda^i - \lambda^j} f_i + \frac{\lambda^j_i}{\lambda^i - \lambda^j} f_j, \quad i \neq j, \quad (2.24) \]

where \( f_{ij} = \partial_{R^i} \partial_{R^j} f \) etc., then there exist \( g \) such that

\[ [f(R)]_t = [g(R)]_x, \quad (2.25) \]

is a conservation law for (2.6).

**Proof** [56]. Substitute (2.6) into (2.25) to get

\[ f_i \lambda^i R_x^i = g_i R_x^i, \]

where \( R_x^i \)'s cancel out. Therefore we have \( g_i = \lambda^i f_i \), so that

\[ g_{i,j} = \lambda^i_j f_i + \lambda^i f_{i,j}, \quad g_{j,i} = \lambda^j_i f_j + \lambda^j f_{j,i}. \]

Using the fact that partial derivatives commute, \( g_{i,j} = g_{j,i} \), we derive (2.24). Thus, to find conservation laws one has to solve the linear system (2.24).

**Examples.** As mentioned above, the KdV is not of hydrodynamic type. It is however instructive to explain the concept of conservation laws by demonstrating some simple conservation laws possessed by the KdV equation in the form (mentioned above)

\[ u_t + 6uu_x - u_{xxx} = 0. \quad (2.26) \]

We finish by showing that equation (2.26) possesses infinitely many conservation laws.
It is straightforward to see that equation (2.26) can be written in the form

\[ u_t = (3u^2 - u_{xx})_x, \]

so that

\[ \int_{-\infty}^{\infty} u \, dx = \text{constant}. \]  \hspace{1cm} (2.27)

In search of another conservation law of (2.26), expand \((u^2)_t\) and substitute (2.26) back, so that

\[ (u^2)_t = 2uu_t = 2u(6uu_x - u_{xxx}) = 12u^2 u_x - 2uu_{xxx} \]

\[ = (4u^3)_x - 2(uu_{xx})_x + 2u_xu_{xx} \]

\[ = (4u^3 - 2uu_{xx} + u^2)_x, \]

thus,

\[ \int_{-\infty}^{\infty} u^2 \, dx = \text{constant}. \]  \hspace{1cm} (2.28)

A third integral of motion can be constructed by considering the equation

\[ 3u^2(u_t + 6uu_x - u_{xxx}) + u_x(u_t + 6uu_x - u_{xxx})_t = 0, \]  \hspace{1cm} (2.29)

which holds true via (2.26). Equation (2.29) can be written as

\[ (2u^3 + u^2)_t = (-9u^4 + 6u^2u_{xx} - 12u_x^2 + 2u_xu_{xxx} - u_{xxx}^2)_x, \]

therefore,

\[ \int_{-\infty}^{\infty} (2u^3 + u^2) \, dx = \text{constant}. \]  \hspace{1cm} (2.30)
Furthermore, we show that the KdV equation possesses infinitely many conservation laws [45]. Firstly, consider the Gardner equation in the form:

$$w_t = w_{xxx} - 6w(1 + \varepsilon^2 w)w_x. \quad (2.31)$$

Obviously $w$ is a conserved density since (2.31) can be written in the form

$$(w)_t = (w_{xx} - 3w^2 - 2\varepsilon^2 w^3)_x. \quad (2.32)$$

It can be shown that $u$, given by the Gardner transformation

$$u = w + \varepsilon w_x + \varepsilon^2 w^2, \quad (2.33)$$

is a solution of the KdV equation if $w$ is a solution of (2.31). This is done by substituting equation (2.33) into the KdV equation in order to obtain

$$u_t + 6uu_x - u_{xxx} = w_t + \varepsilon w_{xt} + 2\varepsilon^2 w w_t$$

$$+ 6(w + \varepsilon w_x + \varepsilon^2 w^2)(w_x + \varepsilon w_{xx} + 2\varepsilon^2 w w_x)$$

$$- w_{xxx} - \varepsilon w_{xxxx} - 2\varepsilon^2 (w w_x)_{xx}$$

$$= \left(1 + \varepsilon \frac{\partial}{\partial x} + 2\varepsilon^2 w\right) \left(w_t + 6(w + \varepsilon^2 w^2)w_x - w_{xxx}\right).$$

So $u$, given by (2.33), is a solution of the KdV equation if $w$ is a solution of (2.31). Setting $\varepsilon = 0$ equation (2.31) becomes the KdV equation and the transformation (2.33) reduces to $u = w$. Furthermore, using equation (2.33), we observe that $w$ can be written in the form

$$w = \alpha + \varepsilon \beta + \varepsilon^2 \gamma + \varepsilon^3 \delta + \varepsilon^4 \eta + \varepsilon^5 \varphi + \cdots, \quad (2.34)$$
where these $\alpha, \beta, \gamma, \delta, \eta, \varphi$, are functions of $x$ and $t$. Then taking into account equations (2.34) and (2.33) we have that

\[ u = \alpha + \varepsilon \beta + \varepsilon^2 \gamma + \varepsilon^3 \delta + \varepsilon^4 \eta + \varepsilon^5 \varphi + \cdots \]

\[ + \varepsilon \left( \alpha_x + \varepsilon \beta_x + \varepsilon^2 \gamma_x + \varepsilon^3 \delta_x + \varepsilon^4 \eta_x + \varepsilon^5 \varphi_x + \cdots \right) \]

\[ + \varepsilon^2 \left( \alpha^2 + 2\varepsilon \alpha \beta + \varepsilon^2 (\beta^2 + 2\alpha \gamma) + \varepsilon^3 (2\beta \gamma + 2\alpha \delta) + \cdots \right) \]

Equating coefficients of $\varepsilon$ we find that

\[ \alpha = u, \]
\[ \beta = -u_x, \]
\[ \gamma = u_{xx} - u_x^2, \]
\[ \delta = 4u u_x - u_{xxx}, \]
\[ \eta = 2u^3 - 5u_x^2 - 6uu_{xx} + u_{xxxx}, \]
\[ \varphi = -16u^2 u_x + 18u u_{xx} + 8uu_{xxx} - u_{xxxx}, \]

Looking at (2.34) we observe that coefficients of odd powers of $\varepsilon$ such as $\beta$, $\delta$ and $\varphi$, give exact derivatives in $x$ such that,

\[ \beta = (-u)_x, \]
\[ \delta = (2u^2 - uu_{xx})_x, \]
\[ \varphi = \left( -\frac{16}{3} u^3 + 5u_x^2 + 8uu_{xx} - u_{xxx} \right)_x, \]

(2.35)

(2.36)
and therefore do not give us useful information. However, coefficients of even powers of $\varepsilon$ such as $\alpha, \gamma, \eta$ become the three integrals (2.27), (2.28) and (2.30) respectively and therefore produce non-trivial constants of motion. In order to finish off with the proof we must show that every odd power of $\varepsilon$ in (2.34) is an exact derivatives, and that every even power of $\varepsilon$ in (2.34) is never an exact derivatives [17], thus producing every time a non-trivial constant of motion. To do this we need to associate even and odd powers of $\varepsilon$ with the real and imaginary parts of a new $w$. By letting, $\varepsilon \rightarrow i\varepsilon$ and $w = a + ib$ where $a$ and $b$ are real functions, the transformation (2.33) becomes

$$u = a - \varepsilon b_x - \varepsilon^2(a^2 - b^2) + i(b + \varepsilon a_x - 2\varepsilon^2 ab). \quad (2.37)$$

Equating real and imaginary parts of equation (2.37) we obtain

$$b + \varepsilon a_x - 2\varepsilon^2 ab = 0, \quad a = u + \varepsilon b_x + \varepsilon^2(a^2 - b^2). \quad (2.38)$$

Let,

$$a \sim A_e + A_o, \quad b \sim B_e + B_o, \quad as \quad \varepsilon \rightarrow 0, \quad (2.39)$$

where $A_e, B_e$ and $A_o, B_o$ denote asymptotic expansions in even and odd powers of $\varepsilon$ respectively:

$$A_e \sim \sum_{n=0}^{\infty} \varepsilon^{2n} a, \quad A_0 \sim \sum_{n=0}^{\infty} \varepsilon^{2n+1} a,$$

$$B_e \sim \sum_{n=0}^{\infty} \varepsilon^{2n} b, \quad B_0 \sim \sum_{n=0}^{\infty} \varepsilon^{2n+1} b.$$
Collect coefficients of even-power terms in equation (2.38)\textsubscript{1} and odd-power terms in equation (2.38)\textsubscript{2}, to obtain,

\begin{equation}
B_\varepsilon(1 - 2\varepsilon^2 A_\varepsilon) \sim -\varepsilon(A_\varepsilon)_x + 2\varepsilon^2 A_\varepsilon B_\varepsilon, \tag{2.40}
\end{equation}

and

\begin{equation}
A_\varepsilon(1 - 2\varepsilon^2 A_\varepsilon) \sim \varepsilon(B_\varepsilon)_x - 2\varepsilon^2 B_\varepsilon B_\varepsilon, \tag{2.41}
\end{equation}

respectively. Equations (2.40) and (2.41) imply that

\begin{equation}
B_\varepsilon \sim -\varepsilon(A_\varepsilon)_x \sim \varepsilon^2(B_\varepsilon)_{xx}, \quad \text{as} \ \varepsilon \to 0,
\end{equation}

and therefore an exact solution of equations (2.40) and (2.41) is (by iteration) 

\begin{equation}
B_\varepsilon = A_\varepsilon = 0. \quad \text{So,} \quad a \sim A_\varepsilon \quad \text{(even only)} \quad \text{and} \quad b \sim B_\varepsilon \quad \text{(odd only). \ Now, \ by \ writing}
\end{equation}

equation (2.38)\textsubscript{2} in the form

\begin{equation}
\frac{\varepsilon a_x}{1 - 2\varepsilon^2 a} = \frac{1}{2\varepsilon} \frac{\partial}{\partial x} (\log |1 - 2\varepsilon^2 a|), \tag{2.42}
\end{equation}

we observe that \( b \) is exact to all orders in \( \varepsilon \). In addition, the first iterate for \( a \) (using equation (2.38)\textsubscript{1}) is

\begin{equation}
(a)_0 = u,
\end{equation}

which corresponds to the term in \( \varepsilon^0 \). From there and forward the terms will always involve derivatives of \( u \) coming from terms \( \varepsilon b_x \) and \( \varepsilon^2 b^2 \) (see (2.42)). In addition the term \( \varepsilon^2 a^2 \) generates, at each iteration, terms which do not involve any derivatives (together with terms which do). These specific polynomial terms are obtained by the iteration

\begin{equation}
(a)_m = u + \varepsilon^2 \{(a)_{m-1}\}^2, \quad m = 1, 2, ..., \quad (a)_0 = u,
\end{equation}
and so the sequence is given by: $u, u + \varepsilon^2 a^2, u + \varepsilon^2 (\varepsilon^2 a^2)^2, \ldots$ Therefore, we have that for every $\varepsilon^{2n}$ ($n = 0, 1, 2, \ldots$), there will be a term proportional to $u^{n+1}$ and such terms are evidently not exact derivatives. Thus there will be an infinity of conservation laws, each density of which is characterised by the inclusion of a term in $u^{n+1}$ (for $n = 0, 1, 2, \ldots$).

### 2.5 Semi-Hamiltonian property

The diagonal system (2.6) is said to be Semi-Hamiltonian if,

$$
\partial_k \left( \frac{\lambda^i_j}{\lambda^j - \lambda^i} \right) = \partial_j \left( \frac{\lambda^i_k}{\lambda^k - \lambda^i} \right), \quad i \neq j \neq k. \tag{2.43}
$$

Let $a_{ij} = \frac{\lambda^i_j}{\lambda^j - \lambda^i}$, then (2.43) becomes,

$$
\partial_k a_{ij} = \partial_j a_{ik}. \tag{2.44}
$$

Next, we show some results obtained by Tsarev [56]. Tsarev has shown that for Semi-Hamiltonian systems commuting flows and conserved densities depend on $n$ arbitrary functions of one argument. Before continuing with the proofs it is worth noting that calculating $\lambda^i_{jk}$ by expanding equation (2.44) (note here that $\lambda^i_{jk} = \lambda^i_{kj}$) and substituting this into the r.h.s of (2.44) one obtains

$$
\partial_k a_{ij} = a_{ij} a_{jk} - a_{ij} a_{ik} + a_{ik} a_{kj}. \tag{2.45}
$$

**Theorem 2** If (2.43) is satisfied then commuting flows depend on $n$ arbitrary functions of one argument.
**Proof** [56]. A diagonal system (2.6) satisfying (2.43) has commuting flows governed by the linear system,

\[ \mu^i_j = a_{ij}(\mu^i - \mu^j), \quad i \neq j \]  

(2.46)

Changing \( j \) to \( k \) where \( k \neq j \), we have

\[ \mu^i_k = a_{ik}(\mu^k - \mu^i). \]  

(2.47)

We need to take partial derivatives of equations (2.46), (2.47) and show that \( \mu^i_{kj} = \mu^i_{jk} \), where again \( \mu^i_{kj} = \partial_k \mu^i_j \). Using (2.45), (2.46) and (2.47) we obtain,

\[
\begin{align*}
(a_{ij}a_{jk} - a_{ij}a_{ik} + a_{ik}a_{kj})(\mu^i - \mu^j) + a_{ij}(a_{jk}(\mu^k - \mu^j) - a_{ik}(\mu^k - \mu^i)) = \\
(a_{ik}a_{kj} - a_{ik}a_{ij} + a_{ij}a_{jk})(\mu^k - \mu^i) + a_{ik}(a_{kj}(\mu^j - \mu^k) - a_{ij}(\mu^j - \mu^i)).
\end{align*}
\]

(2.48)

Equating coefficients of \( \mu^i, \mu^j, \mu^k \) we observe that everything vanishes. Since we know all mixed partial derivatives from (2.46) as well as that all the partial derivatives are consistent identically in \( \mu^i, \mu^j, \mu^k \), commuting flows depend on \( n \) arbitrary functions of one argument, the functions of integration. Indeed, \( \mu^1 \) can be defined arbitrarily on the \( R^1 \)-axis, since we know \( \mu^1_j \forall j \neq 1 \), \( \mu^2 \) can be defined arbitrarily on the \( R^2 \)-axis, etc.

**Theorem 3** If (2.43) is satisfied then conserved densities depend on \( n \) arbitrary functions of one argument.

**Proof** [56]. Recall the condition (2.24) for diagonal systems (2.6) to possess conservation law (2.25). Now let

\[
\begin{align*}
\frac{df_j}{dt} &= a_{ij}f_i + a_{ji}f_j, \\
\frac{df_k}{dt} &= a_{ik}f_i + a_{ki}f_k.
\end{align*}
\]

(2.49)
and compute $f_{kji} = f_{jki}$. Substituting (2.49) to get rid of second order partial derivatives of $f$, we have

$$a_{ijk}f_i + a_{ij}(a_{ik}f_i + a_{ki}f_k) + a_{jik}f_j + a_{ji}(a_{jk}f_j + a_{kj}f_k) =$$

$$a_{ikj}f_i + a_{ik}(a_{ij}f_i + a_{ji}f_j) + a_{kij}f_k + a_{ki}(a_{kj}f_k + a_{jk}f_j).$$

Equating coefficients of $f_i, f_j, f_k$ we observe that coefficients of $f_i$ immediately cancel each other, while using (2.45) coefficients of $f_j, f_k$ cancel as well. So we know all mixed partial derivatives of all $f$, we also know that all these partial derivatives are consistent. Thus conserved densities depend on $n$ arbitrary functions of one variable, indeed one can define $f$ arbitrarily on any of the coordinate lines.

### 2.6 Generalized hodograph method

Using the generalized hodograph method one can obtain a general solution for semi-Hamiltonian systems of the form (2.6). In this section we state and prove the generalized hodograph formula. Let us first begin by a simple scalar example, the so-called Hopf equation given by,

$$R_t = RR_x. \tag{2.51}$$

The general solutions of (2.51) is a well know result and is given by

$$f(R) = x + Rt, \tag{2.52}$$
where $f$ is an arbitrary function of one variable. Calculating partial derivatives of $f$ with respect to $x$ and $t$ we obtain,

$$f_x = \frac{df}{dR}R_x = 1 + R_xt,$$$$
$$f_t = \frac{df}{dR}R_t = R + R_tt,$$

and solving for $R_x$ and $R_t$ we have,

$$R_x = \frac{1}{\frac{df}{dR} - t},$$
$$R_t = \frac{R}{\frac{df}{dR} - t}.

(2.54)

Then since (2.51) follows from (2.54), we see that indeed (2.52) is the general solution for (2.51). The general hodograph method is given next.

Theorem 4 If $\lambda^i(R)$ satisfies (2.43) then the general solution of the diagonal system

$$R^i_t = \lambda^i(R)R^i_x,$$

(2.55)

is given by

$$\mu^i(R) = \lambda^i(R)t + x,$$

(2.56)

where the characteristic speeds of commuting flows $\mu^i(R)$ are arbitrary functions of $n$ variables:

$$\frac{\mu^i_j}{\mu^i - \mu^i} = \frac{\lambda^i_j}{\lambda^i - \lambda^i}, \quad i \neq j.

(2.57)
Proof [56]. First, use substitution of (2.56) into (2.57) to obtain,

\[ \mu^j_i = \lambda^j_i t. \]  

(2.58)

Then, differentiate (2.56) by \( x \) and \( t \) so that

\[ \mu^j_i R^i_x + \mu^j_i R^i_x = \lambda^j_i R^i_x t + \lambda^i R^i_x t + 1, \]
\[ \mu^j_i R^i_t + \mu^j_i R^i_t = \lambda^j_i R^i_t t + \partial_j \lambda^i t^j t + \lambda^i. \]

(2.59)

Substituting (2.58) into (2.59) we get

\[ R^i_x = \frac{1}{\mu^i - \lambda^i t}, \quad R^i_t = \frac{\lambda^i}{\mu^i - \lambda^i t}. \]

(2.60)

Finally, we prove that (2.56) is a general solution of (2.55) by substituting (2.60) into (2.55). Note here that by general solution we mean that (2.56) has the same amount of freedom as the system (2.55). Taking the data for the initial value problem of (2.55) we have for \( t = 0 \),

\[ R^i(x, 0) = f^i(x), \]

thus we have the freedom of \( n \) arbitrary functions of one variable in the system being solved. In the formula (2.56), \( \mu^i \) also depend on \( n \) arbitrary functions of one variable.
Chapter 3

Multi-dimensional equations of the dispersionless Hirota type

In this chapter we investigate a general class of three-dimensional second order equations of the form

\[ F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0, \]  

(3.1)

where \( u = u(x, y, t) \) is a function of three independent variables. Equations of this type arise in a wide range of applications including non-linear physics, general relativity, differential geometry, integrable systems and complex analysis. For instance, take the well known KP equation [37] in the form

\[ (u_t - 6uu_x + \varepsilon^2 u_{xxx})_x = u_{yy}, \]  

(3.2)

where \( \varepsilon \) is a scalar. Then equation (3.2) can be brought into the form (3.1) as follows. Firstly, take the dispersionless limit \( \varepsilon \to 0 \), so that equation (3.2) becomes

\[ (u_t - 6uu_x)_x = u_{yy}. \]
Then, let \( u = v_x \) in order to obtain

\[ v_{xt} - 6v_x v_{xx} = v_{yy}. \]

In addition, let \( v = w_x \) so that

\[ w_{xt} - 3w_x^2 = w_{yy}, \]

which is in the form (3.1). Up to re-scaling one obtains the dispersionless Kadomtsev-Petviashvili (dispersionless KP) equation

\[ u_{xt} - \frac{1}{2} u_x^2 = u_{yy}, \quad (3.3) \]

also known as the Khokhlov-Zabolotskaya equation, which arises in non-linear acoustics [58]. The Boyer-Finley equation,

\[ u_{xx} + u_{yy} = e^{u_{tt}}, \]

has been extensively discussed in the context of general relativity [6]. The equations

\[ Hess \ u = 1 \quad \text{and} \quad Hess \ u = \Delta u, \]

where \( Hess \) is the determinant of the Hessian matrix of \( u \), and \( \Delta \) is the Laplacian, appear in differential geometry [11, 36]. A subclass of equations of the form (3.1),

\[ u_{tt} = f(u_{xx}, u_{xt}, u_{xy}), \]

was discussed recently in [49] in connection with hydrodynamic chains satisfying the so-called Egorov property.
Equations of the above type have been approached by a whole variety of modern techniques including symmetry analysis, differential-geometric and algebro-geometric methods, dispersionless $\delta$-dressing, factorization techniques, Virasoro constraints, hydrodynamic reductions, etc. However until recently there was no unifying scheme which would explain the integrability of examples taking the form (3.1). Moreover, there was no satisfactory definition of the integrability which would

(a) be algorithmically verifiable,

(b) allow classification results,

(c) provide a scheme for the construction of exact solutions.

We emphasize that equations of the form (3.1) are not amenable to the inverse scattering transform, and require an alternative approach. Such approach, based on the method of hydrodynamic reductions and, primarily, the work [29], see also [12, 22, 24, 42], etc, was proposed in [19]. It was suggested to define the integrability of a multi-dimensional dispersionless system by requiring the existence of 'sufficiently many' hydrodynamic reductions which provide multi-phase solutions known as non-linear interactions of planar simple waves. Technically, one 'decouples' a three-dimensional PDE (3.1) into a pair of commuting $n$-component \((1+1)\)-dimensional systems of hydrodynamic type

\[ R_x^i = \lambda^i(R) R_x^i, \quad R_y^i = \mu^i(R) R_x^i, \quad (3.4) \]

where the characteristic speeds $\lambda^i$ and $\mu^i$ satisfy the commutativity conditions

\[ \frac{\lambda_j^i}{\lambda^j - \lambda^i} = \frac{\mu_j^i}{\mu^j - \mu^i}, \quad i \neq j, \quad (3.5) \]
Multi-dimensional equations of the dispersionless Hirota type

\[ \partial_{R} \lambda^{i} = \lambda^{i}, \] see Chapter 2. The best way to illustrate the method of hydrodynamic reductions is to discuss an example.

**Example.** Let us consider the dispersionless KP equation (3.3),

\[ u_{xt} - \frac{1}{2} u_{xx}^{2} = u_{yy}, \]

and introduce the notation

\[ u_{xx} = a, \ u_{xy} = b, \ u_{xt} = p, \ u_{yy} = p - \frac{1}{2} a^{2} ; \]

then this results in the equivalent quasilinear representation of the dispersionless KP equation,

\[ a_{y} = b_{x}, \ a_{t} = p_{x}, \ b_{t} = p_{y}, \ b_{y} = (p - \frac{1}{2} a^{2})_{x}. \] (3.6)

We seek multi-phase solutions in the form,

\[ a = a(R^{1}, ..., R^{n}), \ b = b(R^{1}, ..., R^{n}), \ p = p(R^{1}, ..., R^{n}), \] (3.7)

where the 'phases' \( R^{i}(x, y, t) \) satisfy the equations (3.4). The substitution of (3.7) into (3.6) implies the relations

\[ b_{i} = \mu^{i} a_{i}, \ p_{i} = \lambda^{i} a_{i}, \]

\[ \lambda^{i} = a + (\mu^{i})^{2}, \] (3.8)

where again \( \partial_{R} b = b_{i}, \ \partial_{R} a = a_{i} \) and \( \partial_{R} p = p_{i} \). Calculating the compatibility conditions \( b_{ij} = b_{ji}, \ p_{ij} = p_{ji} \) (where \( \partial_{R_{i}} b = b_{ij} \) and \( \partial_{R_{i}} p = p_{ij} \)), and substituting (3.8) into the commutativity conditions (3.5), one obtains the Gibbons-Tsarev
system for $a(R)$ and $\mu^i(R)$,

$$\mu^i_j = \frac{a_j}{\mu^j - \mu^i}, \quad a_{ij} = 2\frac{a_i a_j}{(\mu^j - \mu^i)^2}, \quad i \neq j,$$

(3.9)

which was first derived in [29] in the theory of hydrodynamic reductions of Benney’s moment equations. It is remarkable that the Gibbons-Tsarev system is in involution ($\mu^i_{jk} = \mu^k_{ij}$ and $a_{ijk} = a_{ikj}$), and its general solution depends, modulo reparametrizations $R^i \rightarrow \varphi^i(R^i)$, on $n$ arbitrary functions of one variable. Thus, the dispersionless KP equation possesses infinitely many $n$-component reductions parametrized by $n$ arbitrary functions of one variable. We point out that the compatibility conditions involve triples of distinct indices $i, j, k$ only. Thus, for $n = 2$ the Gibbons-Tsarev system is automatically consistent, while its consistency for $n = 3$ implies the consistency for arbitrary $n$. Based on this example, we give the following

**Definition.** An equation of the form (3.1) is said to be integrable if, for any $n$, it possesses infinitely many $n$-component hydrodynamic reductions parametrized by $n$ arbitrary functions of one variable.

We have verified that all examples mentioned above are indeed integrable in this sense, with the exception of the equations $\text{Hess } u = 1$ and $\text{Hess } u = \Delta u$, which do not pass the test (see Sect. 3.4). In addition we found integrability conditions and classification results for particular cases such as:

$$u_{tt} = f(u_{xx}, u_{yy}), \quad u_{tt} = f(u_{xx}, u_{xy}, u_{yy}), \quad u_{xy} = f(u_{xt}, u_{yt}), \quad u_{tt} = f(u_{xx}, u_{xt}, u_{xy}).$$

In Sect. 3.1 we outline the derivation of the integrability conditions (as a system of third order differential relations for the function $F$ in (3.1)), and show the proof of our first main result.
3.1 Derivation of the integrability conditions, and proof of the main theorem

The derivation of the integrability conditions is based on the requirement of the existence of \( n \)-component hydrodynamic reductions. Our main result is

**Theorem 5** The moduli space of integrable equations of the dispersionless Hirota type is 21-dimensional.

Before continuing suppose the derivative of the function \( F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) \) in (3.1) with respect to \( u_{tt} \) is nonzero at some point. Then, in the vicinity of this point, we can (locally) solve for \( u_{tt} \), due to the implicit function theorem [38], and represent the equation in explicit form

\[
u_{tt} = f(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}).
\]

The main steps of the proof can be summarized as follows. Let us first rewrite equation (3.1) in the evolutionary form,

\[
u_{tt} = f(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}).
\]

Our strategy is to derive a set of constraints for the right hand side \( f \) which are necessary and sufficient for the existence of an infinity of hydrodynamic reductions. Let us introduce the notation

\[
u_{xx} = a, \quad u_{xy} = b, \quad u_{yy} = c, \quad u_{xt} = p, \quad u_{yt} = q, \quad u_{tt} = f(a, b, c, p, q).
\]

This provides an equivalent quasilinear representation of our equation:

\[
a_y = b_x, \quad a_t = p_x, \quad b_y = c_x, \quad b_t = p_y = q_x, \quad c_t = q_y,
\]

\[
p_t = \frac{\partial}{\partial x} f(a, b, c, p, q), \quad q_t = \frac{\partial}{\partial y} f(a, b, c, p, q).
\]
We point out that, in the general \((2 + 1)\)-dimensional set-up, the method of hydrodynamic reductions applies to dispersionless systems of the form

\[
A(u)u_x + B(u)u_y + C(u)u_t = 0; \tag{3.11}
\]

here \(u = (u^1, \ldots, u^m)^t\) is an \(m\)-component column vector of the dependent variables, and \(A, B, C\) are \(l \times m\) matrices where \(l\), the number of equations, is allowed to exceed the number of unknowns, \(m\). The method of hydrodynamic reductions consists of seeking multi-phase solutions in the form

\[u = u(R^1, \ldots, R^n)\]

where the 'phases' \(R^i(x, y, t)\) are required to satisfy a pair of consistent \((1 + 1)\)-dimensional systems (3.4). Solutions of this type, known as nonlinear interactions of planar simple waves, can be interpreted as natural dispersionless analogues of finite gap solutions of soliton equations. Notice that the above quasilinear representation (3.10) is of the form (3.11) with \(m = 5, \; l = 8\). Looking for multi-phase solutions in the form

\[a = a(R^1, \ldots, R^n), \; b = b(R^1, \ldots, R^n), \; c = c(R^1, \ldots, R^n), \; p = p(R^1, \ldots, R^n), \; q = q(R^1, \ldots, R^n),\]

where the phases (Riemann invariants) \(R^i\) satisfy equations (3.4), and substituting this ansatz into the quasilinear representation (3.10), we obtain the equations for \(b, p, q, c,\)

\[b_i = \mu^i a_i, \; p_i = \lambda^i a_i, \; q_i = \mu^i \lambda^i a_i, \; c_i = (\mu^i)^2 a_i. \tag{3.12}\]
For example, \[ b_i = \mu^i a_i \] (3.13) is obtained from \[ a_y = b_z, \] (3.14) in (3.10). Looking for multi-phase solutions in the form

\[ a = a(R^1, ..., R^n), \quad b = b(R^1, ..., R^n), \]
equation (3.14) becomes \[ a_i R^i_y = b_t R^t_x, \] (3.15)
where again \( a_i = \frac{\partial}{\partial R^i} a \). Then using equations (3.4) one obtains equation (3.13).

In addition to equations (3.12) one obtains the dispersion relation connecting \( \lambda^i \) and \( \mu^i \),

\[ D(\lambda^i, \mu^i) = f_a + f_b \mu^i + f_c (\mu^i)^2 + f_p \lambda^i + f_q \lambda^i \mu^i - (\lambda^i)^2 = 0. \] (3.16)

In what follows we assume that the dispersion relation (3.16) defines an irreducible conic in the \((\lambda, \mu)\)-plane. This is equivalent to the condition that the expression

\[ \Delta = f_b^2 + f_b f_p f_q - f_a f_q^2 - f_c f_p^2 - 4 f_a f_c \]
does not vanish. Calculating the consistency conditions of (3.12) we obtain the equations for \( a \),

\[ a_{ij} = \frac{\lambda^j}{\lambda - \lambda^i} a_i + \frac{\lambda^j}{\lambda - \lambda^j} a_j, \] (3.17)

\[ \lambda^j a_i + \lambda^i a_j = 0. \]
Rewrite (3.16) in the form

$$\left(\lambda^i\right)^2 = f_a + f_b \mu^i + f_c (\mu^i)^2 + f_p \lambda^i + f_q \lambda^i \mu^i,$$

(3.18)

and apply the operator $\partial_j$ to the dispersion relation (3.18). Therefore we have

$$2\lambda^i \lambda_j^i = f_ab a_j + f_ab b_j + f_ac c_j + f_ap p_j + f_aq q_j$$

$$+ (f_ac a_j + f_bb b_j + f_bc c_j + f_bp p_j + f_bq q_j) \mu^i + f_b \mu_j^i$$

$$+ (f_ac a_j + f_bb b_j + f_bc c_j + f_cp p_j + f_cq q_j) \mu^i + 2 f_c \mu^i \mu_j^i$$

$$+ (f_pa a_j + f_pb b_j + f_pc c_j + f_pp p_j + f_pq q_j) \lambda^i + f_p \lambda_j^i$$

$$+ (f_ac a_j + f_qb b_j + f_qc c_j + f_qp p_j + f_qq q_j) \lambda^i \mu^i + f_q (\lambda_j^i \mu^i + \lambda^i \mu_j^i).$$

(3.19)

Then using (3.5) and (3.12) we obtain $\lambda_j^i$ and $\mu_j^i$ in the form

$$\lambda_j^i = (\lambda^i - \lambda^j) B_{ij} a_j, \quad \mu_j^i = (\mu^i - \mu^j) B_{ij} a_j.$$

(3.20)

In general, $B_{ij}$ are certain rational expressions in $\lambda^i, \lambda^j, \mu^i, \mu^j$ whose coefficients depend on the second order partial derivatives of the function $f(a, b, c, p, q)$. Explicitly, one has

$$B_{ij} = \frac{N_{ij}}{D_{ij}},$$

(3.21)

where,

$$N_{ij} = f_{aa} + f_{ab} (\mu^i + \mu^j) + f_{ac} ((\mu^i)^2 + (\mu^j)^2) + f_{ap} (\lambda^i + \lambda^j) + f_{aq} (\lambda^i \mu^i + \lambda^j \mu^j)$$

$$+ f_{bc} (\mu^i)^2 + f_{bc} (\mu^j)^2 + f_{bp} (\lambda^i \mu^j + \lambda^j \mu^i) + f_{bq} \lambda^i \mu^j (\lambda^i + \lambda^j)$$

$$+ f_{cp} (\mu^i)^2 + f_{cp} (\mu^j)^2 + f_{cq} \lambda^i \mu^j (\lambda^i \mu^j + \lambda^j \mu^i)$$

$$+ f_{pp} \lambda^i \lambda^j + f_{pq} \lambda^i \lambda^j (\mu^i + \mu^j) + f_{qq} \lambda^i \lambda^j \mu^i \mu^j,$$
and

\[ D_{ij} = -2\lambda^i\lambda^j + 2f_a + f_b(\mu^i + \mu^j) + 2f_c\mu^i\mu^j + f_p(\lambda^i + \lambda^j) + f_q(\lambda^i\lambda^j + \lambda^j\lambda^j) \]

\[ = 4D \left( \frac{\lambda^i + \lambda^j}{2}, \frac{\mu^i + \mu^j}{2} \right). \]

Substituting (3.20) into (3.17) we obtain,

\[ a_{ij} = -2B_{ij}a_i a_j; \] (3.22)

notice that the second condition (3.17) will be satisfied identically by virtue of the symmetry \( B_{ij} = B_{ji} \). In addition it is straightforward to see that (3.20) and (3.22) can be considered as a generalization of the Gibbons-Tsarev system (3.9): the case of the Gibbons-Tsarev system corresponds to the choice \( B_{ij} = -1/(\mu^i - \mu^j)^2 \).

Ultimately, we require that the compatibility conditions for the relations (3.20),

\[ \lambda^i_{jk} = \lambda^i_{kj}, \quad \mu^i_{jk} = \mu^i_{kj}, \quad a_{ijk} = a_{ikj}, \] (3.23)

are satisfied identically. One can see that (3.23) are equivalent to the relations

\[ B_{ijk} = (B_{ij}B_{kj} + B_{ij}B_{ik} - B_{kj}B_{ik})a_k, \] (3.24)

which is obtained by differentiating (3.21) with respect to \( R^k \) and which must be satisfied identically by virtue of (3.12), (3.16) and (3.20). As one can obviously see, these derivations are computationally intense and therefore had to be implemented using symbolic calculations. We used Mathematica [44]. However, even with Mathematica, calculations were too intense and we had to overcome memory problems. In order to simplify the derivation of the integrability conditions we
first rewrite (3.24) as

\[ N_{ijk} = N_{ij} \left( \frac{1}{D_{ij}} D_{ijk} + B_{kj} a_k + B_{ik} a_k \right) - D_{ij} B_{kj} B_{ik} a_k. \] (3.25)

Here, the third order derivatives of \( f(a, b, c, p, q) \) are present only in the l.h.s. term \( N_{ijk} \) (see the form of \( N_{ij} \) and \( D_{ij} \) above). Further reduction of the complexity of the expression in the r.h.s. is achieved by representing \( 1/D_{ij} \) in the form

\[
\frac{1}{D_{ij}} = U_{ij} = [2f_a + (\mu_1 + \mu_2)f_b + 2\mu_1 \mu_2 f_c - (\lambda_1 + \lambda_2)f_p - (\lambda_1 \mu_2 + \lambda_2 \mu_1)f_q + f_p^2 + (\mu_1 + \mu_2)f_p f_q + \mu_1 \mu_2 f_p^2 + 2\lambda_1 \lambda_2] / ((\mu_1 - \mu_2)^2 \Delta),
\]

(which holds identically modulo the dispersion relation (3.16)), and the subsequent substitution \( B_{st} = N_{st}/D_{st} = N_{st} U_{st} \). The denominators of the r.h.s. terms in (3.25) cancel out, producing a polynomial in \( \lambda^i, \lambda^j, \lambda^k, \mu^i, \mu^j, \mu^k \) with coefficients depending on the derivatives of the density \( f(a, b, c, p, q) \). This was the crucial simplification of the calculation: the starting expression for the r.h.s. of (3.25) has more than 450,000 terms with different denominators; after properly organized cancellations we get a polynomial expression with less than 6,000 terms.

Using the dispersion relation (3.16), we simplify this polynomial by substituting the powers of \( (\lambda^i)^s, (\lambda^j)^s, (\lambda^k)^s, \), \( s \geq 2 \), arriving at a polynomial of degree one in each of \( \lambda^i, \lambda^j, \lambda^k, \) and degree two in \( \mu^i \)'s. Equating similar coefficients in both sides of (3.25), we arrive at a set of 35 equations for the derivatives of the function \( f(a, b, c, p, q) \), which are linear in the third order derivatives. Solving this linear system we obtain the closed form expressions for all third order derivatives of \( f(a, b, c, p, q) \) in terms of its first and second order derivatives, which we represent symbolically in the form

\[
d^3 f = R(df, d^2 f); \]

(3.26)
Here $R$ depends rationally on the first and second order partial derivatives of $f$. A straightforward calculation shows that the overdetermined system (3.26) is in involution. Thus, the moduli space of integrable systems of the type (3.1) is 21-dimensional: one can arbitrarily specify the values of $f$, $df$ and $d^2f$ at any fixed point $x_0 = (a_0, b_0, c_0, p_0, q_0)$. This amounts to $1 + 5 + 15 = 21$ arbitrary constants. The right hand sides of (3.26) are not presented here because of their complexity: However this general formula can be used in the particular cases which we study in the next sections and for which these formulae become less cumbersome, and are presented (and even solved) in closed form. The conditions (3.26) provide a straightforward computer test of the integrability for any equation from the class under consideration, and allow one to obtain classification results. The *Mathematica* program which calculates the integrability conditions and the program allowing one to check integrability of any given equation from this class is available from

www-staff.lboro.ac.uk/~maevf/fhk-supplementary-materials-2009.tar.gz

and is also attached to this thesis on a CD. At this point it would be instructive for us to see an example of how the integrability conditions are derived for a particular case. Consider the case where

$$u_{tt} = f(u_{xx}, u_{yy}).$$

The dispersion relation simplifies to

$$(\lambda^i)^2 = f_a + f_c(\mu^i)^2.$$
Recall that, according to the notation introduced in Sect. 3.1, \( a = u_{xx}, \ c = u_{yy} \).

Using the rational parametrization of the dispersion relation

\[
\lambda^i = \sqrt{\frac{f_a (s^i)^2 + 1}{(s^i)^2 - 1}}, \quad \mu^i = \sqrt{\frac{f_a}{f_c} \frac{2s^i}{(s^i)^2 - 1}},
\]

and substituting into (3.5) we obtain

\[
s^i_j = \frac{a_{ij}}{2f_c f_a} \left( f_c^2 f_{aa} (1 + (s^j)^4) + f_c (4f_{cc} f_a^2 - 4f_c f_a f_{ac}) + f_c f_{aa} (2s^i(s^j)^3 + 2(s^i s^j)^2) + f_c (2f_{aa} - f_{ac}) (2(s^j)^2 + 2(s^i s^j)^3 + (s^j)^2 + (s^i)^2 (s^j)^4 + s^i s^j + (s^j)^5) \right),
\]

where \( Z = 4((s^j)^2 - 1)^2 \). The substitution of this into (3.17) implies

\[
a_{ij} = \frac{a_{ij}}{B f_c f_a} \left( f_c^2 f_{aa} (1 + (s^j)^4 + 2(s^i)^2 + (s^j)^4 + (s^i s^j)^4) + f_c (2f_{aa} - f_{ac}) (2(s^j)^2 + 2(s^j)^4 (s^j)^2 + 2(s^i)^2 + (s^j)^4) + 8(s^j)^2 f_a (f_{cc} f_a - f_{ac}) \right),
\]

for \( B = 2((s^i)^2 - 1)((s^j)^2 - 1)^2 \). One can see that the consistency conditions for the equations (3.27), that is, \( s^i_{kj} - s^j_{ik} = 0 \), are of the form \( Q a_j a_k = 0 \) where \( Q \) is a complicated rational expression in \( s^i, s^j, s^k \) whose coefficients depend on partial derivatives of \( f(a, c) \) up to third order (to obtain the integrability conditions it suffices to consider 3-component reductions setting \( i = 1, j = 2, k = 3 \)).

Requiring that \( Q \) vanishes identically (where \( a_i \neq a_j \neq 0 \)) we obtain expressions for all third order partial derivatives of \( f(a, c) \) plus an extra second order relation. Similarly, the compatibility conditions for the equations (3.17), that is, \( a_{ijk} - a_{kij} = 0 \), take the form \( K a_i a_j a_k = 0 \) where, again, \( K \) is rational in \( s^i, s^j, s^k \).

Equating \( K \) to zero one obtains exactly the same conditions as at the previous step.
Remark 1. Notice that each compatibility condition (3.23) involves three distinct indices only. This observation immediately implies that

(i) any equation of the form (3.1) possesses infinitely many two-component reductions parametrized by two arbitrary functions of one variable. Indeed, these reductions are governed by equations (3.20) where \(i, j = 1, 2\) and \(\lambda^i, \mu^i\) satisfy the dispersion relation. These equations are automatically consistent, and the general solution depends, modulo reparametrizations \(R^i \to \varphi^i(R^i)\), on two arbitrary functions of one variable. Therefore, the existence of two-component reductions is a common phenomenon which is not related to the integrability.

(ii) On the contrary, the existence of three-component reductions implies the existence of \(n\)-component reductions for arbitrary \(n\). Thus, one can define the integrability as the existence of infinitely many three-component reductions parametrized by three arbitrary functions of one variable.

Remark 2. In two dimensions, any second order PDE of the form

\[
F(u_{xx}, u_{xy}, u_{yy}) = 0
\]

is automatically integrable. Indeed, introducing the parametrization

\[
\begin{align*}
u_{xx} &= a, & u_{xy} &= b, & u_{yy} &= f(a, b),
\end{align*}
\]

so that

\[
F(a, b, f(a, b)) = 0,
\]

one obtains a two-component quasilinear system

\[
\begin{align*}
a_y &= b_x, & b_y &= \frac{\partial}{\partial x} f(a, b);
\end{align*}
\]
any system of this type linearises under a hodograph transformation which inter­
changes dependent and independent variables. This simple trick, however, does
not work in more than two dimensions.

3.2 Examples and classifications results

This section contains partial classification results based on the integrability con­
ditions (3.26). We produce an abundance of non-trivial examples of integrable
equations, both known and new, expressible in elementary functions, theta func-
tions and modular forms.

Integrable equations of the form $u_{tt} = f(u_{xx}, u_{yy})$.

For the case, $u_{tt} = f(u_{xx}, u_{yy})$, the integrability conditions (3.26) simplify to

$$f_{aaa} = f_{aa} \left( \frac{f_{ac}}{f_c} + \frac{f_{ac}}{f_a} \right), \quad f_{aac} = f_{aa} \left( \frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a} \right),$$

$$f_{acc} = f_{cc} \left( \frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a} \right), \quad f_{c ee} = f_{cc} \left( \frac{f_{cc}}{f_c} + \frac{f_{ac}}{f_a} \right),$$

$$f_{aa} f_{cc} = (f_{ac})^2.$$

This system is in involution and its general solution depends on 5 integration
constants. To solve it explicitly one notices that the first two equations imply

$$\frac{f_{aa}}{f_a f_e} = \text{const}.$$  Similarly, the next two equations imply $\frac{f_{ac}}{f_a f_e} = \text{const}$. Further
elementary integration leads to the following general solution,

\[ f(a, c) = s \ln(me^{u_0} + ne^{v_0}) + \text{const.} \]

This corresponds to equations of the form

\[ me^{\mu u_{xx}} + ne^{\nu u_{yy}} + ke^{\kappa u_{tt}} = 0, \]

as well as degenerations thereof,

\[ mu_{xx} + nu_{yy} + ke^{\kappa u_{tt}} = 0, \]

where the coefficients are arbitrary constants. In fact, all these coefficients can be eliminated by appropriate complex rescalings, leading to the two essentially different examples,

\[ e^{u_{xx}} + e^{u_{yy}} = e^{u_{tt}} \quad \text{and} \quad u_{xx} + u_{yy} = e^{u_{tt}}. \]

The first equation is apparently new, while the second is the Boyer-Finley equation.

**Integrable equations of the form** \( u_{tt} = f(u_{xx}, u_{xy}, u_{yy}) \).

This is a generalization of the previous case. Thus, we assume \( f_b \neq 0 \). The
Integrability conditions (3.26) take the form

\[ f_{aa}f_{bb} = f_{ab}^2, \quad f_{aa}f_{cc} = f_{ac}^2, \quad f_{bb}f_{cc} = f_{bc}^2, \]

\[ f_{aa}f_{bc} = f_{ab}f_{ac}, \quad f_{ab}f_{cc} = f_{ac}f_{bc}, \quad f_{ab}f_{bc} = f_{ac}f_{bb}, \]

\[ f_{aaa} = \frac{2f_{aa}(f_{b}f_{ab} - 2f_{c}f_{aa} - 2f_{a}f_{ac})}{f_{b}^2 - 4f_{a}f_{c}}, \quad f_{aab} = \frac{2f_{ab}(f_{b}f_{ab} - 2f_{c}f_{aa} - 2f_{a}f_{ac})}{f_{b}^2 - 4f_{a}f_{c}}, \]

\[ f_{aac} = \frac{2f_{ac}(f_{b}f_{ab} - 2f_{c}f_{aa} - 2f_{a}f_{ac})}{f_{b}^2 - 4f_{a}f_{c}}, \quad f_{abb} = \frac{2f_{ab}(f_{b}f_{ab} - 2f_{c}f_{ab} - 2f_{a}f_{bc})}{f_{b}^2 - 4f_{a}f_{c}}, \]

\[ f_{acc} = \frac{2f_{ac}(f_{b}f_{bc} - 2f_{c}f_{ac} - 2f_{a}f_{cc})}{f_{b}^2 - 4f_{a}f_{c}}, \quad f_{abc} = \frac{2f_{ab}(f_{b}f_{bc} - 2f_{c}f_{ac} - 2f_{a}f_{cc})}{f_{b}^2 - 4f_{a}f_{c}}, \]

\[ f_{bbb} = \frac{2f_{bb}(f_{b}f_{bb} - 2f_{c}f_{ab} - 2f_{a}f_{bc})}{f_{b}^2 - 4f_{a}f_{c}}, \quad f_{bbc} = \frac{2f_{bc}(f_{b}f_{bb} - 2f_{c}f_{ab} - 2f_{a}f_{bc})}{f_{b}^2 - 4f_{a}f_{c}}, \]

\[ f_{bcc} = \frac{2f_{bc}(f_{b}f_{bc} - 2f_{c}f_{ac} - 2f_{a}f_{cc})}{f_{b}^2 - 4f_{a}f_{c}}, \quad f_{ccc} = \frac{2f_{cc}(f_{b}f_{bc} - 2f_{c}f_{ac} - 2f_{a}f_{cc})}{f_{b}^2 - 4f_{a}f_{c}}. \]

Notice that the first six equations imply that the Hessian matrix of the function \( f(a, b, c) \) has rank one. Notice that the rows 1, 2, 3 of the aforementioned Hessian matrix are given by \( \nabla f_a, \nabla f_b \) and \( \nabla f_c \), respectively. Using the fact that the Hessian matrix has rank one, we see that either \( f_b = \text{const} \), or \( f_a = m(f_b) \), \( f_c = n(f_b) \). The substitution into the remaining equations implies \( m'' = n'' = 0 \). Further elementary analysis leads, up to linear transformations of \( x \) and \( y \), to the following canonical forms (here we only list those representatives which contain a nontrivial dependence on \( u_{xy} \)): 
Multi-dimensional equations of the dispersionless Hirota type

(a) Firstly, we have that,

\[ u_{tt} = \alpha u_{xx} + \beta u_{yy} + \varphi(u_{xy}), \]

for which \( \varphi \) satisfies a third order ODE

\[ \varphi''((\varphi')^2 - 4\alpha\beta) = 2\varphi'(\varphi'')^2. \]

The integration leads to the three new canonical forms,

\[ u_{tt} = \alpha u_{xx} + \beta u_{yy} + \frac{2\sqrt{\alpha\beta}}{\gamma} \ln \cosh \gamma u_{xy}, \]
\[ u_{tt} = \alpha u_{xx} + \gamma \ln u_{xy}, \quad \text{or} \quad u_{tt} = \ln u_{xy}, \]

where \( \alpha, \beta, \gamma = \text{const.} \) Note here that the last two cases correspond to \( \beta = 0 \) and \( \alpha = \beta = 0 \), respectively. Furthermore, the first two examples can be viewed as generalizations of the Boyer-Finley equation.

(b) Secondly,

\[ u_{tt} = u_{xy} + \beta(u_{xx} - u_{yy}) + \varphi(u_{xx} + u_{yy}), \]

where \( \beta = \text{const} \) and

\[ \varphi'''(4(\varphi')^2 - 1 - 4\beta^2) = 8\varphi'(\varphi'')^2. \]

This leads to the new equation

\[ u_{tt} = u_{xy} + \beta(u_{xx} - u_{yy}) + \frac{\sqrt{1 + 4\beta^2}}{2\gamma} \ln \cosh \gamma(u_{xx} + u_{yy}). \]

(c) Thirdly,

\[ u_{tt} = u_{xy} + \beta u_{xx} + \varphi(u_{yy}), \]
where \( \varphi \) satisfies a third order ODE

\[
\varphi'''(4\beta \varphi' - 1) = 4\beta (\varphi'')^2.
\]

This results in the new equation

\[
t_{tt} = u_{xy} + \beta u_{xx} + \frac{1}{4\beta} u_{yy} + \alpha e^{u_{yy}},
\]

whose degeneration contains the dKP equation \( u_{tt} = u_{xy} + u_{yy}^2 \).

Integrable equations of the form \( u_{xy} = f(u_{xt}, u_{yt}) \).

Formally, equations from this class are not of the form discussed in this section, however, they can readily be made 'evolutionary' by an appropriate linear change of the independent variables. This can be done by letting \( \xi = x + y \) and \( \eta = x - y \). The resulting set of integrability conditions looks as follows:

\[
\begin{align*}
  f_{pp} &= f_{pp} \left( f_{pq} f_q + f_{pp} \right), & f_{pq} &= f_{pp} \left( f_{qq} f_q + f_{pq} \right), \\
  f_{pq} &= f_{pq} \left( f_{pq} f_q + f_{pp} \right), & f_{qq} &= f_{pq} \left( f_{pq} f_q + f_{pp} \right).
\end{align*}
\]

This system is in involution, and its general solution depends on 6 integration constants. To solve it explicitly one notices that the first two equations imply \( \frac{f_{pp}}{f_{pq} f_q} = \text{const.} \) Similarly, the last two equations imply \( \frac{f_{pq}}{f_{pq} f_q} = \text{const.} \) Further elementary integration gives, under the assumption that both \( f_{pp} \) and \( f_{qq} \) are nonzero, the general solution

\[
f(p, q) = \frac{1}{\kappa} \ln \left( \frac{me^{\mu p} + ne^{\nu q}}{re^{\mu p + \nu q} - k} \right),
\]
which leads to the dispersionless Hirota-type equation for the BKP hierarchy [5],

\[ me^{\mu_{xt}} + n e^{\mu_{xt}} + k e^{\mu_{xy}} = r e^{\mu_{xt} + \mu_{xy}}. \]  

(3.29)

Up to complex rescalings, it can be transformed to

\[ e^{u_{xt}} + e^{u_{yt}} + e^{u_{xy}} = e^{u_{xt} + u_{yt} + u_{xy}}. \]

Degenerations of (3.29), corresponding to \( f_{yy} = 0 \), or \( f_{pp} = f_{qq} = 0 \), result in

\[ u_{xy} = u_{xt} + e^{u_{xt}}, \quad u_{xy} = u_{xt} \tan(u_{yt}), \]

and,

\[ u_{xy} = u_{xt} u_{yt}, \]

respectively.

**Integrable equations of the form** \( u_{tt} = f(u_{xx}, u_{xt}, u_{xy}) \).

Equations of this type arise in the theory of integrable hydrodynamic chains satisfying the additional 'Egorov' property. Here we reproduce the classification result from [49] (see also [8], [18]). The integrability conditions (3.26) take the form

\[ f_{bbb} = \frac{2f_{ab}^2}{f_b}, \quad f_{abb} = \frac{2f_{ab}f_{bb}}{f_b}, \quad f_{ppb} = \frac{2f_{pb}f_{bb}}{f_b}, \]

\[ f_{aab} = \frac{2f_{ab}^2}{f_b}, \quad f_{apb} = \frac{2f_{ab}f_{pb}}{f_b}, \quad f_{ppb} = \frac{2f_{pb}^3}{f_b}, \]
Multi-dimensional equations of the dispersionless Hirota type

\[ f_{ppp} = \frac{2}{f_b^2} (f_p f_{p b}^2 + f_{p b} (f_b f_{pp} + 2 f_{ab}) - f_{bb} (f_p f_{pp} + 2 f_{ap})), \]

\[ f_{app} = \frac{2}{f_b^2} (f_a f_{p b} + f_{ab} (f_b f_{pp} + f_{ab}) - f_{bb} (f_a f_{pp} + f_{aa})), \]  \hspace{1cm} (3.30)

\[ f_{aap} = \frac{2}{f_b^2} (f_{bb} (f_p f_{aa} - 2 f_{a f_{ap}}) - f_{ab} (f_p f_{ab} - 2 f_{b f_{ap}}) - f_{pb} (f_b f_{aa} - 2 f_{a f_{ab}})), \]

\[ f_{aaa} = \frac{2}{f_b^2} ((f_a + f_p^2) f_{a b}^2 + f_{a f_{p b}}^2 + f_{b}^2 (f_{a p}^2 - f_{a a} f_{p p}) - f_{p p} f_{b b} f_{a}^2 \]
\[ + f_{a b} f_b (f_{a a} + 2 (f_a f_{p p} - f_{p f_{ap}})) + 2 f_{p b} (f_p (f_b f_{aa} - f_a f_{ab}) - f_a f_{b f_{ap}}) \]
\[ - f_{b b} ((f_a + f_p ^2) f_{a a} - 2 f_{a f_{p f_{ap}}}); \]

this system is in involution and its general solution depends on 10 arbitrary constants. The integration of these equations leads to the four essentially different canonical forms,

\[ u_{tt} = u_{xy} + \frac{1}{4 A} (A u_{xt} + 2 B u_{xx})^2 + C e^{-A u_{xx}}, \]

\[ u_{tt} = \frac{u_{xy}}{u_{xx}} + \left( \frac{1}{u_{xx}} + \frac{A}{4 u_{xx}^2} \right) u_{xt}^2 + \frac{B}{u_{xx}^2} u_{xt} + \frac{B^2}{4 u_{xx}^2} + C e^{A/u_{xx}}, \]

\[ u_{tt} = \frac{u_{xy}}{u_{xt}} + \frac{1}{6} \eta (u_{xx}) u_{xt}^2, \]

\[ u_{tt} = \ln u_{xy} - \ln \theta_1 (u_{xt}, u_{xx}) - \frac{1}{4} \int \eta (\tau) d\tau, \]

see [49]. Here \( A, B, C \) are arbitrary constants, \( \eta \) is a solution to the Chazy equation [14],

\[ \eta''' + 2 \eta \eta'' = 3 (\eta')^2, \]

and \( \theta_1 \) is the Jacobi theta function.

Further examples in terms of modular forms and theta functions

This section contains further integrable examples which are not expressible in
elementary functions.

Case 1. Let us begin with equations of the form

$$u_{xy} + u_{xt}u_{yt}r(u_{tt}) = 0;$$

in this case the integrability conditions result in a single third order ODE for $r$,

$$r'''(r' - r^2) - r''^2 + 4r^3r'' + 2r'^3 - 6r^2r'^2 = 0,$$

which appeared recently in a different context in the theory of modular forms of level two: compare with the equation (4.7) from [1]. This equation possesses a remarkable $SL(2, \mathbb{R})$-invariance,

$$
\tilde{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \tilde{r} = (\gamma z + \delta)^2 r + \gamma(\gamma z + \delta), \quad \alpha \delta - \gamma \beta = 1; \\
\text{here } z = u_{tt}. \text{ Modulo this } SL(2, \mathbb{R})\text{-action, the generic solution is given by the series}
$$

$$r(u_t) = 1 + 8e^{4u_{tt}} - 8e^{8u_{tt}} + 32e^{12u_{tt}} - 40e^{16u_{tt}} + 48e^{20u_{tt}} - 32e^{24u_{tt}} + ...,$$

which, upon setting $e^{4u_{tt}} = q$, coincides with the Eisenstein series

$$E(q) = 1 - 8 \sum_{n=1}^{\infty} \frac{(-1)^n n q^n}{1 - q^n},$$

associated with the congruence subgroup $\Gamma_0(2)$ of the modular group.

Case 2. As a generalization of Case 1, let us consider equations of the form

$$u_{xy} + u_{xt}r(u_{yt}, u_{tt}) = 0.$$
In a somewhat different representation, equations of this type were discussed in [10]. It was demonstrated that the integrability conditions imply \( r = 2v'/v \) where prime denotes differentiation by \( u_{yt} \), and \( v(u_{yt}, u_{tt}) = \theta \left( \frac{u_{yt}}{2\pi}, -\frac{u_{tt}}{\pi} \right) \) is the Jacobi theta function:

\[
\theta(z, \tau) = 1 + 2 \sum_{n=1}^{\infty} e^{\pi in^2 \tau} \cos(2\pi nz).
\]

Case 3. Further generalization,

\[
u_{xy} + f(u_{zt}, u_{yt}, u_{tt}) = 0,
\]

was discussed in [18], where it was shown that the generic solution is given by the ratio of two Jacobi theta functions:

\[
f(u_{zt}, u_{yt}, u_{tt}) = -\frac{1}{4} \ln \frac{\theta_1(u_{tt}, u_{yt} - u_{zt})}{\theta_1(u_{tt}, u_{yt} + u_{zt})}.
\]

Symplectic Monge-Ampère equations

Let us consider a function \( u(x^1, ..., x^k) \) of \( k \) independent variables and introduce the \( k \times k \) Hessian matrix \( U = [u_{ij}] \) of its second order partial derivatives. The symplectic Monge-Ampère equation is a PDE of the form

\[
M_k + M_{k-1} + ... + M_1 + M_0 = 0,
\]

where \( M_l \) is a constant-coefficient linear combination of all \( l \times l \) minors of the matrix \( U \), \( 0 \leq l \leq k \). Thus, \( M_k = \det U = \text{Hess } u \), \( M_0 \) is a constant, etc. Equivalently, these PDEs can be obtained by equating to zero a constant-coefficient \( k \)-form in the \( 2k \) variables \( x^i, u_i \). This class of equations is invariant under the
natural action of the symplectic group $Sp(2k)$. In the case $k = 2$ one obtains a standard Monge-Ampère equation,

$$u_{11}u_{22} - u_{12}^2 + \alpha u_{11} + \beta u_{12} + \gamma u_{22} + \delta = 0,$$

which can be interpreted as the equation of a 'sphere' corresponding to the pseudo-Euclidean metric $du_{11}du_{22} - du_{12}^2$. Monge-Ampère equations (3.31) can be characterized as the only equations of the form $F(u_{11}, u_{12}, u_{22}) = 0$ which are linearizable by a transformation from the equivalence group $Sp(4)$.

The case $k = 3$ is also understood completely: one can show that, for $k = 3$, any symplectic Monge-Ampère equation is either linearizable (in this case it is automatically integrable), or $Sp(6)$-equivalent to either of the two essentially different canonical forms [41, 4],

$$\text{Hess } u = 1, \quad \text{Hess } u = u_{11} + u_{22} + u_{33}.$$

Based on the integrability conditions (3.26), we have verified directly that both PDEs (3.32) are not integrable by the method of hydrodynamic reductions. Thus, a 3-dimensional symplectic Monge-Ampère equation is integrable if and only if it is linearizable (this is no longer true in more than three dimensions). The linearizability condition constitutes a single relation among the coefficients of the equation: for a Monge-Ampère equation of the form

$$\epsilon \det \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{bmatrix}$$
Multi-dimensional equations of the dispersionless Hirota type

\[ h_1(u_{22}u_{33} - u_{23}^2) + h_2(u_{11}u_{33} - u_{13}^2) + h_3(u_{11}u_{22} - u_{12}^2) \]
\[ + g_1(u_{11}u_{23} - u_{12}u_{13}) + g_2(u_{22}u_{13} - u_{12}u_{23}) + g_3(u_{33}u_{12} - u_{13}u_{23}) \] (3.33)
\[ + s_1u_{11} + s_2u_{22} + s_3u_{33} + \tau_1u_{23} + \tau_2u_{13} + \tau_3u_{12} + \nu = 0, \]

the linearizability condition specifies a quartic hypersurface in the space of coefficients,

\[ h_1^2s_1^2 + h_2^2s_2^2 + h_3^2s_3^2 + g_1^2s_2s_3 + g_2^2s_1s_3 + g_3^2s_1s_2 \]
\[ -2(h_1h_2s_1s_2 + h_1h_3s_1s_3 + h_2h_3s_2s_3) + 4\epsilon s_1s_2s_3 + 4\nu h_1h_2h_3 \]
\[ + \epsilon\tau_1\tau_2\tau_3 - \nu g_1g_2g_3 - \epsilon^2\nu^2 - \nu(g_1^2h_1 + g_2^2h_2 + g_3^2h_3) \] (3.34)
\[ -(g_1\tau_1 + g_2\tau_2 + g_3\tau_3 + 2\epsilon\nu)(h_1s_1 + h_2s_2 + h_3s_3 - \epsilon\nu) \]
\[ + 2(g_1h_1s_1\tau_1 + g_2h_2s_2\tau_2 + g_3h_3s_3\tau_3) + \tau_1^2h_2h_3 + \tau_2^2h_1h_3 + \tau_3^2h_1h_2 \]
\[ - \epsilon(\tau_1^2s_1 + \tau_2^2s_2 + \tau_3^2s_3) + s_1\tau_1g_2g_3 + s_2\tau_2g_1g_3 + s_3\tau_3g_1g_2 \]
\[ -(g_1h_1\tau_2\tau_3 + g_2h_2\tau_1\tau_3 + g_3h_3\tau_1\tau_2) = 0. \]

Notice that, if \( \epsilon \neq 0 \), one can always eliminate second order minors in (3.33) by adding to \( u \) an appropriate quadratic form. In this case the equation takes the
form

\[
\begin{vmatrix}
  u_{11} & u_{12} & u_{13} \\
  u_{21} & u_{22} & u_{23} \\
  u_{31} & u_{32} & u_{33}
\end{vmatrix} + s_1 u_{11} + s_2 u_{22} + s_3 u_{33} + \tau_1 u_{23} + \tau_2 u_{13} + \tau_3 u_{12} + \nu = 0,
\]

while the linearizability condition simplifies to

\[4s_1 s_2 s_3 + \nu^2 + \tau_1 \tau_2 \tau_3 - s_1 \tau_1^2 - s_2 \tau_2^2 - s_3 \tau_3^2 = 0.\]

### 3.3 Integrability in more than three dimensions

The integrability of three-dimensional equations of the form (3.1) was defined as the existence of \(n\)-component hydrodynamic reductions (3.4) parametrized by \(n\) arbitrary functions of a single variable. This approach readily generalizes to any dimension: a \(d\)-dimensional PDE

\[
F(u_{x^ix^j}) = 0 \quad (3.35)
\]

for a function \(u\) of \(d\) independent variables \(x^1, ..., x^d\) is said to be integrable if it possesses infinitely many \(n\)-component hydrodynamic reductions parametrized by \((d-2)n\) arbitrary functions of a single variable. In this case equations (3.4) are replaced by \(d-1\) commuting \((1+1)\)-dimensional systems of hydrodynamic type [24, 21]. Among the known four-dimensional integrable examples one should primarily mention the 'first heavenly' equation,

\[u_{xy}u_{zt} - u_{xt}u_{zy} = 1,\]
Multi-dimensional equations of the dispersionless Hirota type

as well as its equivalent forms,

\[ u_{tx} + u_{zy} + u_{xz} u_{yy} - u_{xy}^2 = 0 \]

and

\[ u_{tt} = u_{xy} u_{xt} - u_{xt} u_{xy}, \]

known as the 'second heavenly' and the 'Grant' equations, respectively [51, 31]. It was demonstrated in [24, 21] that \( n \)-component reductions of these equations are parametrized by \( 2n \) arbitrary functions of a single variable.

An interesting six-dimensional integrable generalization of the heavenly equation,

\[ u_{tt} + u_{z\bar{z}} + u_{tx} u_{zy} - u_{ty} u_{zx} = 0, \]

arises in the context of \( sdiff(\Sigma^2) \) self-dual Yang-Mills equations [52]. Its \( n \)-component reductions are parametrized by \( 4n \) arbitrary functions of a single variable [21]. Notice that all these examples belong to the class of symplectic Monge-Ampère equations as introduced at the end of Sect. 3.2.

Although the general problem of classification of multi-dimensional integrable equations can be approached in a similar way, the method of Sect. 3.1 leads to quite complicated analysis. One way to bypass lengthy calculations is based on the following simple idea: suppose we want to classify four-dimensional integrable equations of the form (3.35) for a function \( u(x, y, z, t) \). Let us look for travelling wave solutions in the form

\[ u(X, Y, Z) = u(x + \alpha t, y + \beta t, z + \gamma t). \]
The substitution of this ansatz into (3.35) leads to a three-dimensional equation which must be integrable for \textit{any} values of constants \(\alpha, \beta, \gamma\). Since, in three dimensions, the integrability conditions are explicitly known, this provides strong restrictions on the original function \(F\), which are \textit{necessary} for the integrability. The philosophy of this approach is well familiar from soliton theory: symmetry reductions of integrable systems must be themselves integrable.

Thus, for the first heavenly equation, traveling wave solutions are governed by

\[
\alpha(u_{XY}u_{XZ} - u_{XX}u_{YZ}) + \gamma(u_{XY}u_{ZZ} - u_{XZ}u_{YZ}) = 1,
\]

which is a three-dimensional symplectic Monge-Ampere equation.
Chapter 4

Symmetry properties

4.1 Lie groups and Lie algebras

The point symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables of the system with the property that it transforms solutions of the system to other solutions.

Sophus Lie has shown that by using the general theory of Lie groups of transformations when finding solutions to partial differential equations (PDEs), it turns out that a much wider class of transformations, other than scalings, translations, or rotations, can leave PDEs invariant. Lie groups of transformations are characterized by infinitesimal generators. Using Lie's algorithm one can find all infinitesimal generators of point transformations and contact transformations admitted by a given differential equation. One only needs to calculate the admitted infinitesimal generators of a given differential equation [47].
One-parameter groups of transformations

Every one-parameter group of transformations depending on a real parameter, is completely determined by the first term of its Taylor expansion in the group parameter \( a \), in other words by the *infinitesimal transformation* or the corresponding *tangent vector field*. The later is also referred to as the *infinitesimal operator* or *generator* of the group.

Definition and examples

Consider the transformation \( T \):

\[
z' = f(z),
\]

where the new point \((z' = z'^1, \ldots, z'^N)\) in the Euclidean space \( \mathbb{R}^N \) is the displacement of the point \((z = z^1, \ldots, z^N)\) in the same space \( \mathbb{R}^N \). Assuming that the transformation \( T \) is invertible and denoting its inverse by \( T^{-1} \), \( z' \) can be carried to its original position \( z \). Also, the successive application of \( T \) and \( T^{-1} \) in any order yields the identical transformation \( I \) that leaves every point \( z \) unaltered.

Now, a one-parameter family \([T_a]\) of transformations is given by

\[
z' = f(z, a),
\]

where \( a \) is a real parameter continuously varying in a given interval \( S \subset \mathbb{R} \). Every particular value of the parameter \( a \) determines a definite transformation \( T_a \) of the family. We presuppose that the value \( a = 0 \) corresponds to the identity mapping, i.e. \( T_0 = I \) for any \( a \in S \) other than zero. If the identity transformation occurs
for some other value $a_0 \neq 0$ where $a_0 \in S$, then the previous condition $T_0 = I$ is achieved by a simple shift of the parameter, $a = \bar{a} + a_0$. We also assume that there exists an inverse, in order for the family under consideration to contain the inverse transformation $T_a^{-1}$ of every member $T_a$ of the family.

For example, the dilation $z' = az$ becomes the identity transformation when $a = 1$. Having made the above mentioned shift of the parameter, one has the transformation

$$z' = z + az,$$  \hspace{1cm} (4.2)

that meets the condition $T_0 = I$. Furthermore, we see that the inverse transformation $T_{f(a)}$ is defined for any value of $a$ from the interval $-1 < a < \infty$, here $f(a) = -a/(1 + a)$. Let us take $a$ and $b$ from the above interval and consider the successive transformations (4.2) with these values of the parameter. The first transformation carries the point $z$ to $z' = z + az$, the latter being mapped by the second transformation to

$$z'' = z' + bz' = z + az + b(z + az) = z + (a + b + ab)z.$$

It follows that the result of application of two successive transformation of the family (4.2) is identical with the third transformation of the same family with the parameter $c = a + b + ab$. Symbolically, $T_b T_a = T_{a+b+ab}$. Accordingly, the transformations (4.2) are said to form a one-parameter group.

**Definition 1** [33] Transformations (4.1) are said to form a one-parameter group if they satisfy the condition

$$T_b T_a = T_{\phi(a,b)},$$  \hspace{1cm} (4.3)

where $\phi(a, b)$ is a sufficiently differentiable function.
Let the group property (4.3) and the 'initial' condition $T_0 = I$ be satisfied. Then $T_0 T_a = T_a$ and $T_b T_0 = T_b$, whence

$$\phi(a, 0) = a, \quad \phi(0, b) = b. \quad (4.4)$$

For instance, the conditions (4.4) are manifestly satisfied for the transformations (4.2) with $\phi(a, b) = a + b + ab$.

Summarizing the basic properties of one-parameter groups we have:
1) $T_0 = I$ (or $T_{a_0} = I$) (existence of the identity);
2) $T_a^{-1} = T_{f(a)}$ (existence of the inverse element);
3) $T_c(T_b T_a) = (T_c T_b) T_a$ (associativity of multiplication).

In abstract group theory these three properties are taken as the definition of a group. Apart from these properties there is an additional property, namely, the smooth dependence of transformations on the group parameter.

The Lie equation

Let the transformation (4.1) define a group $G$ and let the condition (4.3) expressing the group property have the simple form $T_b T_a = T_{a+b}$, so that $\phi(a, b) = a + b$.

In other words, let

$$f(f(z, a), b) = f(z, a + b). \quad (4.5)$$

Here it is obvious that $f(a) = -a$. Let us expand the function $f(z, a)$ into the Taylor series with respect to the parameter $a$ in the neighbourhood of $a = 0$. Since $T_0 = I$, one has $f(z, 0) = z$. Thus, denoting

$$\xi(z) = \frac{\partial f(z, a)}{\partial a} \bigg|_{a=0} \quad (4.6)$$
one obtains from (4.1) the \textit{infinitesimal transformation}

\[ z' = f(z, a) = z + \xi(z)a + o(a) \approx z + \xi(z)a, \]  

(4.7)

where \( o(a) \) stands for the higher-order terms in \( a \).

The following theorem due to Lie, states that the function \( f(z, a) \) obeying the condition (4.5) is uniquely defined by the infinitesimal transformation (4.7). Note that the formula (4.6) defines the tangent vector to the curve described by the points \( z' = f(z, a) \). Therefore, the group \( G \) is also said to be defined by its tangent vector \( \xi(z) \).

\textbf{Theorem 6} [33] \textit{Let the function} \( f(z, a) \) \textit{satisfy the group property} (4.5) \textit{and have the expansion} (4.7). \textit{Then it solves the first-order ordinary differential equation} (referred to as the Lie equation) \textit{with the initial condition}:

\[ \frac{\partial f}{\partial a} = \xi(f), \quad f \mid_{a=0} = z. \]  

(4.8)

\textit{Conversely, given a vectors field} \( \xi(z) \), \textit{the solution of} (4.8) \textit{(the solution exists and it is unique) satisfies the group property} (4.5).

At this point it is important to note that the Lie equation defines, strictly speaking, not a group but rather a \textit{local group} meaning that the multiplication of the elements \( T_a \) and \( T_b \) of the family \( [T_a] \) of transformations (4.1) is possible only for \( a \) and \( b \) from some subinterval \( S' \subset S \) containing \( a = 0 \). One can chose this subinterval so that every transformation \( T_a \) with \( a \in S' \) possesses the inverse \( T_a^{-1} = T_{a^{-1}} \) with \( a^{-1} \in S \).
Invariant functions and infinitesimal generators

**Definition 2** A function $F(z)$ is called an invariant function of the transformation group (4.7),

$$z' = f(z, a) = z + \xi(z)a + o(a),$$

if the following equation holds for every (admissible) values of $z$ and $a$:

$$F(f(z, a)) = F(z). \quad (4.9)$$

**Theorem 7** [33] A function $F(z)$ is an invariant if and only if it solves the partial differential equation

$$\xi^i(z) \frac{\partial F(z)}{\partial z^i} = 0. \quad (4.10)$$

The invariance condition (4.10) is a homogeneous linear partial equation of the first order. Therefore any one-parameter transformation group in $\mathbb{R}^N$ has $N - 1$ functionally independent invariants and any other invariant is a function of these $N - 1$ basic invariants.

Let us now introduce the linear differential operator

$$X = \xi^i(z) \frac{\partial}{\partial z^i}. \quad (4.11)$$

Then the invariance condition (4.10) is written as follows:

$$XF = 0.$$ 

The operator $X$ is referred to as the *infinitesimal generator* (or *operator*) of the group $G$ of transformations (4.7). The operator (4.11) is more convenient in practical applications than the tangent vector $\xi = (\xi^1, ..., \xi^N)$. Given an operator
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$X$, the transformations of the corresponding group are determined by solving the Lie equation.

**Example.** Let us find the transformation corresponding to the infinitesimal generator

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$ 

Comparing it with the formula (4.11) one can see that $\xi = (y, -x)$, and hence the Lie equation (4.8) has the form

$$\frac{dx'}{da} = y', \quad \frac{dy'}{da} = -x'.$$

This system and the initial conditions $x'|_{a=0} = x, y'|_{a=0} = y$ yield

$$x' = x \cos a + y \sin a, \quad y' = y \cos a - x \sin a.$$

**Example.** Lorentz transformation: $x' = x \cosh a + y \sinh a, \quad y' = y \cosh a + x \sinh a,$

$$X = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$ 

**Invariant equations**

Consider a surface $M$ in $\mathbb{R}^N$ defined by simultaneous equations

$$F_1(z) = 0, \ldots, F_s(z) = 0, \quad s \leq N. \quad \text{(4.12)}$$

**Definition 3** The surface is said to be invariant with respect to the group $G$ of
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transformations (4.7),

\[ z' = f(z, a) \equiv z + a\xi(z) + o(a), \]  

(4.13)

if the transformations (4.7) carry every point \( z \) of the surface along this surface. In other words, if \( z \) is a solution of the system (4.12), then \( z' \) is also its solution, i.e.

\[ F_1(z') = 0, \ldots, F_s(z') = 0. \]

Accordingly, the equations (4.12) are said to be invariant under the group \( G \), or to admit \( G \).

**Theorem 8** [33] The system of equations (4.12) is invariant with respect to the group \( G \) of transformations (4.7) with the infinitesimal generator

\[ X = \xi^i \frac{\partial}{\partial z^i}, \]

if and only if,

\[ XF_k|_M = 0, \ k = 1, \ldots, s. \]  

(4.14)

Construction of symmetry groups

The section below shows the necessary analytical methods for the calculation of infinitesimal symmetries of differential equations. We will see how we can find admissible groups which characterize symmetry properties of differential equations and which are used for complete integration (or construction of certain classes of exact solutions) as well as qualitative investigation of the differential equation. Any transformation mapping a differential equation into an equivalent
equation of the same form is said to be admitted by the differential equation under consideration.

Before continuing it is important to note that in what follows the variables $t$, $u$, $u_t$, $u_x$, $u_{xx}$ are considered to be independent.

For example consider the general evolutionary partial differential equation of the second order

$$u_t - F(t, x, u, u_1, u_2) = 0,$$

where $u$ is a function of the independent variables $t$ and $x = (x^1, ..., x^n)$, and $u_1, u_2$ are the sets of its first and second order partial derivatives: $u_1 = (u_{x^1}, ..., u_{x^n})$, $u_2 = (u_{x^1 x^1}, u_{x^1 x^2}, ..., u_{x^n x^n})$. Recall that invertible transformations of the variables $t, x, u,$

$$\tilde{t} = f(t, x, u, a), \quad \tilde{x}^i = g^i(t, x, u, a), \quad \tilde{u} = h(t, x, u, a), \quad i = 1, ..., n,$$

depending on a continuous parameter $a$ are said to be symmetry transformations if the equation (4.15) has the same form in the new variables $\tilde{t}, \tilde{x}, \tilde{u}$:

$$\tilde{u}_t - F(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_1, \tilde{u}_2) = 0.$$

The set $G$ of all symmetry transformations of a given equation forms a continuous group, i.e. $G$ contains the identity transformation,

$$\tilde{t} = t, \quad \tilde{x}^i = x^i, \quad \tilde{u} = u,$$

as well as the inverse to any transformation from $G$ and the composition of any two transformations from $G$. The symmetry group $G$ is also known as the group admitted by the equation (4.15).
According to the Lie theory, the construction of the symmetry group $G$ is equivalent to determination of its infinitesimal transformations (4.7):

$$\vec{t} \approx t + a_0(t, x, u), \quad \vec{x}^i \approx x^i + a_0^i(t, x, u), \quad \vec{u} \approx u + a_\eta(t, x, u),$$

(4.18)

or, the infinitesimal generator

$$X = \xi^0(t, x, u) \frac{\partial}{\partial t} + \xi^i(t, x, u) \frac{\partial}{\partial x^i} + \eta(t, x, u) \frac{\partial}{\partial u},$$

(4.19)

which is also referred to as an operator admitted by the equation (4.15).

The group transformations (4.16) corresponding to the infinitesimal transformations with the generator (4.19) are found by solving the Lie equations

$$\frac{d\vec{t}}{da} = \xi^0(\vec{t}, \vec{x}, \vec{u}), \quad \frac{d\vec{x}^i}{da} = \xi^i(\vec{t}, \vec{x}, \vec{u}), \quad \frac{d\vec{u}}{da} = \eta(\vec{t}, \vec{x}, \vec{u}),$$

(4.20)

with the initial conditions $\vec{t}|_{a=0} = t, \; \vec{x}^i|_{a=0} = x^i, \; \vec{u}|_{a=0} = u$.

By definition, the transformation (4.16) form a symmetry group $G$ of the equation (4.15) if the function $\vec{u} = \vec{u}(\vec{t}, \vec{x})$ satisfies the equation (4.17) whenever the function $u = u(t, x)$ satisfies the equation (4.15). Recall that the quantities $\vec{u}_t$, $\vec{u}_{(1)}$, $\vec{u}_{(2)}$ involved in the equation (4.17) are obtained from the equation (4.16) by the chain rule. The infinitesimal form of the latter formulae is

$$\vec{u}_t \approx u_t + a_\zeta_0(t, x, u, u_t, u_{(1)}), \quad \vec{u}_x \approx u_{x^i} + a_\zeta^i(t, x, u, u_t, u_{(1)}),$$

$$\vec{u}_{xx} \approx u_{xx^i} + a_\zeta_{ij}(t, x, u, u_t, u_{(1)}, u_{(2)}),$$

(4.21)
where the functions $\zeta_0, \zeta_i, \zeta_{ij}$ involve $\xi^0, \xi^i, \eta$ together with their derivatives, and are given by the prolongation formulae:

$$\zeta_0 = D_t(\eta) - u_tD_t(\xi^0) - u_{x^j}D_t(\xi^j), \quad \zeta_i = D_t(\eta) - u_tD_t(\xi^0) - u_{x^j}D_t(\xi^j), \quad \zeta_{ij} = D_j(\zeta_i) - u_{x^j}x^kD_j(\xi^k) - u_{x^j}D_j(\xi^0).$$

Here $D_t$ and $D_i$ denote the total differentiations with respect to $t$ and $x^i$, respectively, viz.

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx^j} \frac{\partial}{\partial u_{x^j}} + \cdots,$$

$$D_i = \frac{\partial}{\partial x^i} + u_{x^i} \frac{\partial}{\partial u} + u_{tx^j} \frac{\partial}{\partial u_t} + u_{x^j}x^k \frac{\partial}{\partial u_{x^k}} + \cdots.$$

Substitutions of (4.18) and (4.21) into the left-hand side of the equation (4.17) yields:

$$\bar{u}_t - F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_1, \bar{u}_2) \approx u_t - F(t, x, u, u_1, u_2) + a(\zeta_0 - \frac{\partial F}{\partial u_{x^j}} \zeta_{ij} - \frac{\partial F}{\partial u_{x^i}} \zeta_i - \frac{\partial F}{\partial \eta} \zeta_i - \frac{\partial F}{\partial \xi^i} \eta - \frac{\partial F}{\partial \xi^0} \xi^0).$$

Therefore, by virtue of the equation (4.15), the equation (4.17) yields

$$\zeta_0 - \frac{\partial F}{\partial u_{x^j}} \zeta_{ij} - \frac{\partial F}{\partial u_{x^i}} \zeta_i - \frac{\partial F}{\partial \eta} \zeta_i - \frac{\partial F}{\partial \xi^i} \eta - \frac{\partial F}{\partial \xi^0} = 0,$$

where $u_t$ is replaced by $F(t, x, u, u_1, u_2)$ in $\zeta_0, \zeta_i, \zeta_{ij}$.

The equation (4.23) defines all infinitesimal symmetries of the equation (4.15) and therefore it is called the determining equation. Conventionally, it is written in the compact form

$$X_2(u_t - F(t, x, u, u_1, u_2)|_{u_t=F} = 0,$$
where \( X_{(2)} \) is the *prolongation* of the operator (4.19) up to the second order derivatives:

\[
X = \xi^0(t,x,u) \frac{\partial}{\partial t} + \xi^i(t,x,u) \frac{\partial}{\partial x^i} + \eta(t,x,u) \frac{\partial}{\partial u} + \zeta_0 \frac{\partial}{\partial u_t} + \zeta_i \frac{\partial}{\partial u_x^i} + \zeta_j \frac{\partial}{\partial u_x^i u_x^j}.
\]

The determining equation (4.23) (or its equivalent (4.24)) is a linear homogeneous partial differential equation of the second order for unknown functions \( \xi^0(t,x,u), \xi^i(t,x,u) \) and \( \eta(t,x,u) \) of the ‘independent variables’ \( t, x, u \). It may seem that this equation is even more complicated than the original differential equation (4.15). However, in practice the determining equation is linear, and unlike the original differential equation (4.15) is solvable analytically. This is due to the fact that the left-hand side of the determining equation involves the derivatives \( u_{x^i}, u_{x^ix^j} \), along with the variables \( t, x, u \) and functions \( \xi^0, \xi^i, \eta \) of these variables. Since the equation (4.23) is valid identically with respect to all variables involved, the variables \( t, x, u, u_{x^i}, u_{x^ix^j} \) are treated as ‘independent’ ones. It follows that the determining equation decomposes into a system of several equations. As a rule, this is an overdetermined system (it contains more equations than the number \( n + 2 \) of the unknown functions \( \xi^0, \xi^i, \eta \)).

**Derivation of first, second and third Prolongation formulae**

Note here, that compared to the general evolutionary PDE \( t \) is now included in \( x = [x^i] \). The components of the vector \( z \) are chosen from the following different sets of variables:

\[
x = [x^i], \ u = [u^a], \ u_{(1)} = [u_t^a], \ u_{(2)} = [u_{t1}^a], \ldots,
\]
where the index $i = 1, \ldots, n$ and $\alpha = 1, \ldots, m$. These variables are considered to be algebraically independent but connected by the differential relations

$$u^\alpha_i = D_i(u^\alpha), \quad u^\alpha_{ij} = D_j(D_i(u^\alpha)) = D_iD_j(u^\alpha), \cdots,$$

where,

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots. \quad (4.26)$$

The variables $u^\alpha_{ij}$, etc. are manifestly symmetric in subscripts, i.e. $u^\alpha_{ij} = u^\alpha_{ji}$. The operator $D_i$ in known as the total differentiation with respect to $x^i$, and it should not be mistaken for the operator of the partial differentiation $\frac{\partial}{\partial x^i}$. The operator $D_i$ is a formal sum of an infinite number of terms. However, it truncates when acting on any function of a finite number of variables $x, u, u_1, \cdots$. In consequence, the total differentiations $D_i$ are well defined on the set of all functions depending on a finite number of $x, u, u_1, \cdots$. In what follows these functions are supposed to be analytic.

The quantities $x^i$ are called independent variables, whereas $u^\alpha$ are termed 'differential variables' with successive derivatives $u^\alpha_i, u^\alpha_{ij}, \cdots$ of the first, second, etc. orders. Any analytic function of a finite set of variables $x, u, u_1, \cdots$ is referred to as a differential function. The maximal order $p$ of a derivative involved in the differential function $f = f(x, u, u_1, \cdots, u_p)$ is termed the order of this function and is denoted by $\text{ord}f$. The set of all differential functions of finite order is denoted by $A$.

Let $[F] \in A$ be a differential function of the order $p$. The equation

$$F(x, u, u_{(1)}, \cdots, u_{(p)}) = 0,$$  

(4.27)
defines a surface in the space of variables \(x, u, \ldots, u_p\). Consider the equation (4.27) together with its differential consequences,

\[ D_i F = 0, \quad D_i D_j F = 0, \ldots, \]

and say that the equation (4.27) defines a differential surface \([F]\). Likewise we shall consider the surface defined by systems of equations

\[ F_1(x, u, \cdots, u_p) = 0, \cdots, F_s(x, u, \cdots, u_p) = 0. \quad (4.28) \]

Let us assume that \(z = (x, u)\) and write the transformations (4.1) in the form

\[ x'^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i, \quad (4.29) \]

\[ u'^\alpha = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \quad (4.30) \]

These transformations are referred to as point transformations. Assuming that the transformations (4.29) form a one-parameter group with the group property (4.5), let us write the infinitesimal operator in the form

\[ X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad (4.31) \]

where \(\xi^i\) and \(\eta^\alpha\) are given by the formula (4.6):

\[ \xi^i = \frac{\partial f^i}{\partial a}|_{a=0}, \quad \eta^\alpha = \frac{\partial \phi^\alpha}{\partial a}|_{a=0}. \quad (4.32) \]

The usual formulae for the change of derivatives are distinguished by the fact that they preserve the differential relations (4.25). Accordingly these formulae can be easily obtained as follows. First of all we note that the change of independent
variables (4.29) implies the following relation between total differentiations with respect to non-primed variables:

\[ D_i = D_i(f^j)D'_j. \]  

(4.33)

Furthermore we write the relations (4.25) in the primed variables:

\[ u'^\alpha = D'_i(u'^\alpha), \ldots. \]

Now we differentiate both sides of the equation (4.30) by using the above equations to obtain:

\[ D_i(\phi^\alpha) = D_i(f^j)D'_j(u'^\alpha) = u'^\alpha(D_i(f^j)). \]  

(4.34)

Thus, the change of the derivative of the first order under the point transformations (4.29)-(4.30) is determined by

\[ u'^\alpha D_i(f^j) = D_i(\phi^\alpha), \]  

(4.35)

or,

\[ \left( \frac{\partial f^j}{{\partial x}^i} + u'_{\beta} \frac{\partial f^j}{{\partial u}^\beta} \right) u'^\alpha = \frac{\partial \phi^\alpha}{{\partial x}^i} + u'_{\beta} \frac{\partial \phi^\alpha}{{\partial u}^\beta}. \]

The above equation defines the quantities \( u'^\alpha \) as functions of \( x, u, u(1) \) and \( a \) for sufficiently small \( a \). Indeed, when \( a = 0 \) the initial condition \( f^j|_{a=0} = x^j \) yields that \( D_i(f^j) = \delta^j_i \). Consequently the matrix \( ||D_i(f^j)|| \) is invertible when \( a \) is small.

The second differentiation of the equation (4.35) yields transformations of derivatives of the second order, etc.

In what follows we shall need the prolongation (extension) of the infinitesimal operator (4.31) rather than of the transformations (4.29)-(4.30) themselves. Let
us write the prolongation of the infinitesimal operator to the first derivatives in the form

\[ X_{(1)} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \quad (4.36) \]

where \( \zeta_i^\alpha = (\partial u_i^\alpha / \partial a)|_{a=0} \) are the additional coordinates to be determined. To this end, let us differentiate both sides of the equation (4.35) with respect to the parameter \( a \) at \( a = 0 \). Invoking the formulae (4.32) and the initial conditions in (4.29)-(4.30), and taking into account that the differentiations \( D_i \) and \( \partial / \partial a \) are permutable one obtains

\[ D_i(\eta^\alpha) = \zeta_i^\alpha D_i(x^j) + u_{ij}^\alpha D_i(\xi^j) = \zeta_i^\alpha \delta_i^j + u_{ij}^\alpha D_i(\xi^j) = \zeta_i^\alpha + u_{ij}^\alpha D_i(\xi^j). \]

This yield the desired formula for the first prolongation:

\[ \zeta_i^\alpha = D_i(\eta^\alpha) - u_{ij}^\alpha D_i(\xi^j). \quad (4.37) \]

The above formula manifests that the extended operator (4.36) is uniquely determined via the coordinates \( \xi^i, \eta^\alpha \) of the initial operator \( X \).

The formula for the second prolongation is derived in the following way. Firstly, differentiate (4.35) applying (4.33) and repeat the procedure of the derivation of the first prolongation formula (4.37):

\[ D_m \left( u_j^\alpha D_i(f^j) \right) = D_m D_i(\phi^\alpha), \quad \text{or} \quad D_m(f^k)D_k^j \left( u_j^\alpha D_i(f^j) \right) = D_m D_i(\phi^\alpha). \]

Then differentiate both sides with respect to \( a \) modulo \( a = 0 \),

\[ \left. \frac{\partial}{\partial a} \left( u_{jk}^\alpha D_m(f^k)D_i(f^j) + u_{ij}^\alpha D_m D_i(f^j) \right) \right|_{a=0} = \left. \frac{\partial}{\partial a} \left( D_m D_i(\phi^\alpha) \right) \right|_{a=0}, \]
giving,

$$\zeta_{ij} = D_i(\zeta_j) - u_{ij}^\alpha D_i(\xi^\alpha),$$

(4.38)

which finally gives,

$$\zeta_{ij} = D_i(\zeta_j) - u_{ij}^\alpha D_i(\xi^\alpha),$$

where \( m \) is replaced with \( j \) for convenience. Note that

$$D_m(\zeta_i) = D_m D_i(\eta^\alpha) - u_{jm}^\alpha D_i(\xi^j) - u_{ij}^\alpha D_m D_i(\xi^j).$$

Using similar calculations, it can be shown that the third prolongation is given by,

$$\zeta_{ijk} = D_k(\zeta_{ij}) - u_{ijm}^\alpha D_i(\xi^m).$$

Let us turn to the problem of calculation of point transformation groups admitted by the systems of differential equations (4.28). For the sake of brevity, we shall write sometimes this system in the form (4.27) assuming that \( F \) is a vector, \( F = (F_1, \cdots, F_s) \). Furthermore, the maximal order of the derivatives involved in \( F \) is denoted by \( p \). Recall that \([F]\) denotes a differential surface determined by the differential equations under consideration together with their differential consequences, \( F = 0, D_i F = 0, D_i D_j F = 0, \cdots \).

Since the equations (4.28) involve derivatives up to the order \( p \) inclusively (although some of \( F_k \), where \( k = 1, \cdots, s \), may depend on derivatives of lower order) it suffices to extend the transformations (4.29)-(4.30) to the derivatives up to the order \( p \). It is important that the prolongation of a one-parameter group \( G \) of transformation (4.29)-(4.30) forms again a one-parameter group. It is denoted by \( G(p) \) and acts on all the variables \( x, u, u(1), \cdots, u(p) \). The infinitesimal operator
of the group \(G(p)\) is equal to

\[
X_{(p)} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \cdots + \zeta_{i_1\cdots i_p}^\alpha \frac{\partial}{\partial u_{i_1\cdots i_p}^\alpha},
\]  
(4.39)

where the additional coordinates are given by the following general prolongation formula similar to the second prolongation formula (4.38)

\[
\zeta_{i_1\cdots i_h}^\alpha = D_{ik} (\zeta_{i_1\cdots i_{k-1}}^\alpha) - u_{j_{i_1\cdots i_{k-1}}^\alpha} D_{ik} (\xi^j).
\]  
(4.40)

**Definition 4** A system of differential equations (4.28) is said to be invariant under a group \(G\) of point transformation (4.29)-(4.30), or to admit a group \(G\) if the differential surface \([F]\) defined by (4.28) is invariant with respect to the extended group \(G(p)\). The group \(G\) is also termed a symmetry group and its generator is called an infinitesimal symmetry.

Theorem 8 furnishes the following invariance test convenient for calculating infinitesimal operators of groups admitted by differential equations.

**Theorem 9** [33] The system of differential equations (4.28) admits the group \(G\) with the generator \(X\) if and only if the following equation hold:

\[
X_{(p)} F_k_{[F]} = 0, \quad k = 1, \ldots, s.
\]  
(4.41)

This theorem together with the Lie theorem 6 reduces the problems of searching for all one-parameter groups admitted by a given system of differential equations to solution of the equations (4.41). This fact justifies the term *determining equation* commonly used in group analysis when referring to the equations (4.41).

The prolongation formulae (4.37), (4.38) and (4.40) manifest that the determining equations (4.41) furnish a system of linear homogeneous differential equations with respect to the coordinates \(\xi^i\) and \(\eta^\alpha\) of the operator (4.31). Notice that
these coordinates depend only on $x$ and $u$, while the determining equation involves also the derivatives $u_{(1)}, \ldots, u_{(p)}$. Therefore, the obtained system of differential equations for $\xi^i(x,u), \eta^a$ will be overdetermined. This property of determining equations simplifies their solution and many consider it as the core of Lie group analysis.

4.2 Point Symmetries: Boyer-Finley and $e^{utt} = e^{uxx} + e^{uyy}$

Example. Consider the Boyer-Finley equation in the form

$$u_{yy} = e^{utt} - u_{xx}, \quad (4.42)$$

and search for the admissible operator in the form

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u},$$

where $\xi^1, \xi^2, \xi^3$ and $\eta$ are unknown functions of $x, y, t$ and $u$. The second prolongation of the operator $X$ is given by:

$$X_{(2)} = X + \zeta_1 \frac{\partial}{\partial u_x} + \zeta_2 \frac{\partial}{\partial u_y} + \zeta_3 \frac{\partial}{\partial u_t} + \zeta_{11} \frac{\partial}{\partial u_{xx}} + \zeta_{12} \frac{\partial}{\partial u_{xy}} + \zeta_{13} \frac{\partial}{\partial u_{xt}} + \zeta_{22} \frac{\partial}{\partial u_{yy}} + \zeta_{23} \frac{\partial}{\partial u_{yt}} + \zeta_{33} \frac{\partial}{\partial u_{tt}}.$$

Hence $X_{(2)}(u_{xx} + u_{yy} + e^{utt}) = \zeta_{11} + \zeta_{22} + e^{utt} \zeta_{33}$ and the determining equation (4.41) is written:

$$(\zeta_{11} + \zeta_{22} + e^{utt} \zeta_{33}) \bigg|_{u_{yy} = e^{utt} - u_{xx}} = 0. \quad (4.43)$$
It can be easily seen that the left hand side will give a polynomial with respect to the variables \( u_x, u_y, u_t, u_{xx}, u_{xy}, u_{xt}, u_{yt} \) and \( u_{tt} \). Since the function \( \xi^1, \xi^2, \xi^3 \) and \( \eta \) depend only \( x, y, t, u \) the equations (4.43) are satisfied identically only if all coefficients of the polynomial vanish. Accordingly, we isolate the terms contained in the polynomial mentioned above and those free of these variables and set each term equal to zero. What we have obtained is an overdetermined system which can be easily solved. The general solution of the determining equation (4.43) is given by,

\[
\xi^3 = C_1 t + C_2, \quad \eta = (C_1 + \xi^1 + \xi^2) t^2 + C_3(x, y) t + C_4(x, y) + 2C_1 u,
\]

where \( \xi^1, \xi^2 \) are solutions of the Laplace equations related to each other via the Cauchy-Riemann equations:

\[
\xi^1_x = \xi^2_y, \quad \xi^1_y = -\xi^2_x.
\]

Therefore we get the following obvious point-symmetries:

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = t \frac{\partial}{\partial t} + (t^2 + 2u) \frac{\partial}{\partial u},
\]

plus symmetries resulting from \( C_3, C_4 \) as well as \( \xi^1, \) and \( \xi^2 \).

**Example.** Similarly, the equation

\[
e^{utt} = e^{uxx} + e^{uyy},
\]

admitting the second prolongation of the operator

\[
X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u},
\]
Symmetry properties

gives the determining equation

\[ (e^{u_{xx}} \xi_{11} + e^{u_{yy}} \xi_{22} + e^{u_{tt}} \xi_{33}) \bigg|_{u_{yy} = \ln(e^{u_{tt}} - e^{u_{xx}})} = 0. \] (4.44)

Then, its general solution is given by:

\[ \xi^1 = C_1 x + C_2, \xi^2 = C_1 y + C_3, \xi^3 = C_1 t + C_4, \]

\[ \eta = C_1 u + A(y - t - x) + B(y - t + x) + D(y + t - x) + E(y + t + x), \]

where \( C_i \) are constants, and \( A, B, C, D \) are arbitrary functions of the indicated arguments. Hence, we obtain the following obvious point-symmetries:

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial t}, \]

\[ X_4 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + t \frac{\partial}{\partial t}, \]

plus point-symmetries resulting from the functions \( A, B, C \) and \( D \).

4.3 Lie algebras of vector fields

Consider two arbitrary operators of the form (4.11)

\[ X_1 = \xi_1^1 \frac{\partial}{\partial z^1}, \quad X_2 = \xi_2^i \frac{\partial}{\partial z^i}, \]

and define their commutator \([X_1, X_2]\) by the formula

\[ [X_1, X_2] = X_1 X_2 - X_2 X_1. \] (4.45)
An alternative definition of the commutator is

\[ [X_1, X_2] = (X_1(\xi_2^i) - X_2(\xi_1^i)) \frac{\partial}{\partial x^i}. \]

Now, using this definition one can prove that the commutator obeys three properties:

1) bilinearity: \([cX_1, X_2] = [X_1, cX_2] = c[X_1, X_2], \ c=\text{const}, \]
\[ [X, X_1 + X_2] = [X, X_1] + [X, X_2], \]
\[ [X_1 + X_2, X] = [X_1, X] + [X_2, X]; \]

2) skew-symmetry: \([X_1, X_2] = -[X_2, X_1]; \]

3) Jacobi's identity: \([X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0. \]

A vector space endowed with a bilinear multiplication law is called an algebra. Thus, property 1) shows that any vector space of operators (4.11) with the commutator (4.45) is an algebra. In addition, the algebra satisfying properties 2) and 3) is called a Lie algebra.

**Definition 5 [33]** A Lie algebra is a vector space \( L \) of operators (4.11) such that \( L \) contains the commutator \([X_1, X_2]\) of any operator \( X_1, X_2 \in L \), and it is denoted by the same \( L \) which is used to denote the vector space. The dimension of the Lie algebra is identified with that of the vector space, and if finite (say equal to \( r \)) it is denoted by \( L_r \).

Given an \( r \)-dimensional Lie algebra \( L_r \), let us consider a basis \( X_1, ..., X_r \) of the vector space \( L_r \) and take the commutators \( X_\mu, X_\nu \) of the basic operators. Since \( L_r \) is the linear span of the basic operators, i.e. every \( X \) from \( L_r \) has the representation

\[ X = \sum_{\mu=1}^{r} c^\mu X_\mu, \ c^\mu = \text{const}. \]
Symmetry properties

The knowledge of all \([X_\mu, X_\nu]\) allows one to find the commutator of arbitrary operators of the algebra \(L_r\). It follows that an \(r\)-dimensional vector space \(L_r\) with the basis \(X_1, \ldots, X_r\) is a Lie algebra if and only if the commutators of all basic operators belong to \(L_r\), i.e. if

\[
[X_\mu, X_\nu] = \sum_{\lambda=1}^{r} c_{\mu\nu}^{\lambda} X_\lambda, \quad \mu, \nu = 1, \ldots, r. \tag{4.46}
\]

The coefficients \(c_{\mu\nu}^{\lambda}\) of these equations are real numbers and are referred to as the structure constants of the Lie algebra.

The structure of Lie algebras also helps to find when a group is commutative, in particular to clarify the question why all one-parameter groups are commutative. Namely, a group \(G_r\) is commutative if and only if its Lie algebra \(L_r\) with the basis \(X_\nu = \xi_\nu^i(z) \frac{\partial}{\partial z^i}, \quad \nu = 1, \ldots, r,\)

has only vanishing structure constants, i.e \([X_\mu, X_\nu] = 0, \quad \mu, \nu = 1, \ldots, r.\) Then, \([X, Y] = 0\) is evident for all \(X, Y \in L_r\), and the algebra is referred to as commutative or Abelian. In particular, any one-parameter group is commutative since its Lie algebra has a single basic operator \(X_1\) and hence \([X_1, X_1] = 0.\)

4.4 Differential forms

In this section the aim is to give a brief review of the theory of differential forms. We first deal with the concept of differential 1-forms. We then move to introduce the so-called wedge product (or exterior product). We discuss the main properties of the wedge product and show how these properties are used in order to construct differential \(p\)-forms, where \(p > 2\). In addition, we discuss the exterior derivative which when acted on a differential \(p\)-form, produces a differential \((p+1)\)-form [54].
Differential 1-forms

A differential 1-form on an open set in $\mathbb{R}^2$ is an expression $F(x, y)dx + G(x, y)dy$, where $F$ and $G$ are real valued functions on the open set. In general, one can say that $\omega$ is a differential 1-form on an open set in $\mathbb{R}^n$ if,

$$\omega = \sum_{i=1}^{n} a_i dx_i,$$

where $a_i$ are real differentiable functions in $\mathbb{R}^n$.

Wedge product

For us to proceed to higher degree differential forms we need to introduce the so-called wedge product, which is denoted by $\wedge$. The wedge product is used to produce higher degree forms. For example, consider two differential 1-forms $\omega_1$ and $\omega_2$. Then,

$$\Omega = \omega_1 \wedge \omega_2,$$

is called a differential 2-form.

The wedge product possesses some important properties. Let $f, g, \omega$ be any differential forms, then we have:

1) associativity: $\omega \wedge (f \wedge g) = (\omega \wedge f) \wedge g$,

2) bilinearity: $\omega \wedge (c_1 f + c_2 g) = c_1 (\omega \wedge f) + c_2 (\omega \wedge g)$,

$$ (c_1 \omega + c_2 f) \wedge g = c_1 (\omega \wedge g) + c_2 (f \wedge g).$$

Furthermore, let $f$ be a differential $k$-form and $g$ be a differential $l$-form, then

$$f \wedge g = (-1)^{kl} g \wedge f.$$
Example. Let $f$ and $g$ be differential 1-forms, then

$$ f \land g = -g \land f, \tag{4.47} $$

and therefore,

$$ f \land f = 0. \tag{4.48} $$

Differential $p$-forms

Given two differential 1-forms $\alpha dx + \beta dy$ and $\gamma dx + \delta dy$, we show how one obtains a differential 2-form $\Omega$ using the above properties. Let

$$ \Omega = (\alpha dx + \beta dy) \land (\gamma dx + \delta dy). $$

By bilinearity we obtain,

$$ \Omega = \alpha \gamma dx \land dx + \alpha \delta dx \land dy + \beta \gamma dy \land dx + \beta \delta dy \land dy, $$

and using (4.47) and (4.48),

$$ \Omega = (\alpha \delta - \beta \gamma) dx \land dy, $$

which is a differential 2-form.

In general, a differential $p$-form $\Omega$ has the representation

$$ \Omega = \sum_{f_1, f_2, \ldots, f_p=1}^{p} F_{f_1 f_2 \ldots f_p} dx^{f_1} \land dx^{f_2} \land \cdots \land dx^{f_p}, $$

where the coefficients $F_r(x) = F_{f_1 f_2 \ldots f_p}(x)$ are smooth functions [26]. Let us now
Symmetry properties

consider \( \omega \) and \( \xi \), a differential \( p \)-form and \( q \)-form respectively, so that

\[
\omega = \sum_{h_1,h_2,\ldots,h_p=1}^{p} a_{h_1 h_2 \ldots h_p} dx^H, \quad \xi = \sum_{k_1,k_2,\ldots,k_q=1}^{q} b_{k_1 k_2 \ldots k_q} dx^K,
\]

where \( dx^H = dx^{h_1} \wedge \cdots \wedge dx^{h_p} \) and \( dx^K = dx^{k_1} \wedge \cdots \wedge dx^{k_q} \). Taking the wedge product of \( \omega \) and \( \xi \) we obtain,

\[
\omega \wedge \xi = \sum a_H b_K dx^H \wedge dx^K,
\]

which is a differential \((p + q)\)-form.

Interior product

Let \( v = \frac{\partial}{\partial x^i} \) be a vector field and \( \omega = dx^{h_1} \wedge \cdots \wedge dx^{h_k} \) a \( k \)-form. Then their interior product which we denote as \( \circ \) is given by

\[
v \circ \omega = \frac{\partial}{\partial x^i} \circ (dx^{h_1} \wedge \cdots \wedge dx^{h_k})
\]

\[
= (-1)^{n-1} dx^{h_1} \wedge \cdots \wedge dx^{h_{n-1}} \wedge dx^{h_{n+1}} \wedge \cdots \wedge dx^{h_k},
\]

for \( i = h_n \), and equal to zero for \( i \neq h_n \).

Example. For brevity let \( \frac{\partial}{\partial x^i} = \partial_x^i \). Then, \( \partial_x \circ dx \wedge dy = dy \), \( \partial_x \circ dz \wedge dx = -dz \) and \( \partial_x \circ dy \wedge dz = 0 \).

The differential

Let \( \omega = \sum a_H(x) dx^H \) (differential \( k \)-form). Its differential \( d \) (or exterior deriva-
Symmetry properties

tive) is given by the \((k + 1)\)-form,

\[ d\omega = da_H \wedge dx^H = \sum \frac{\partial a_H}{\partial x^j} dx^j \wedge dx^H, \]

where we have summation over repeated indices. In addition, the differential \(d\) obeys the following rules:

1) Linearity: \(d(c\omega + c'\omega') = cd\omega + c'd\omega'\) for \(c, c'\) constant.

2) Anti-derivation: \(d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^k \omega \wedge (d\theta)\), for \(\omega\) a \(k\)-form, \(\theta\) an \(l\)-form.

3) Closure: \(d(d\omega) = 0\).

Example. Let \(f(x, y)\) be a \(C^1\) real valued function on an open set \(U\). Its differential is given by,

\[ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy, \]

which is a differential 1-form.

Lie derivative of a function

Let \(v\) be a vector field and \(f\) a real valued function, so that \(v = \xi^i \frac{\partial}{\partial x^i}\) and \(f = f(x^i)\) for \(i = 1, \ldots, n\). The Lie derivative of a function \(f\) is given by

\[ \mathbf{v}(f) = \mathbf{L}_v(f) = \xi^i \frac{\partial f}{\partial x^i}, \]

where we have summation over repeated indices.

Lie derivative of a vector field

Let \(v = \xi^i \frac{\partial}{\partial x^i}\) and \(\omega = \theta^i \frac{\partial}{\partial x^i}\) be vector fields. The Lie derivative of \(\omega\) w.r.t
Symmetry properties

\( v \) is given by

\[
 v(\omega) = L_v(\omega) = [v, \omega],
\]

where the so-called Lie bracket is defined as

\[
 [v, \omega] = \left( \xi^j \frac{\partial \theta^i}{\partial x^j} - \theta^i \frac{\partial \xi^j}{\partial x^j} \right) \frac{\partial}{\partial x^i},
\]

where again we have summation over repeated indices.

Lie derivative of a differential form

Let \( \omega = a_H(x) dx^H \) be a differential form and \( v = \xi^i \frac{\partial}{\partial x^i} \) be a vector field. Then the Lie derivative of the differential form \( \omega \) is given by:

\[
 v(\omega) = L_v(\omega) = d(v \circ \omega) + v \circ (d\omega).
\]

4.5 \( Sp(6) \) as a symmetry group of the integrability conditions

In this section we investigate the action of the equivalence group \( Sp(6) \) on the moduli space of integrable equations of the dispersionless Hirota type. Note that \( Sp(6) \) and the moduli space have coinciding dimensions equal to 21. Our main result states that this action has an open orbit.

Let us consider a 6-dimensional symplectic space with canonical coordinates \( x = (x^1, x^2, x^3)^t \) and \( p = (p_1, p_2, p_3)^t \) (viewed as 3-component column vectors) which preserves the symplectic form \( dp_1 \wedge dx^1 + dp_2 \wedge dx^2 + dp_3 \wedge dx^3 \). By definition, on Lagrangian submanifolds the symplectic form restricts to zero. Therefore,
Lagrangian submanifolds can be parametrized in terms of a generating function $u(x, y, t): x = (x, y, t)^t$ and $p = (u_x, u_y, u_t)^t$.

Lagrangian planes are defined by the equation $dp = Udx$ where $U$ is a $3 \times 3$ symmetric matrix (the Hessian matrix of $u$). Thus, the Lagrangian Grassmannian $\Lambda$ is 6-dimensional, and can (locally) be identified with the space of $3 \times 3$ symmetric matrices. The equation (3.1) defines a 5-dimensional hypersurface $M^5 \subset \Lambda$; the corresponding solutions $u(x, y, t)$ can be interpreted as Lagrangian submanifolds whose Gaussian images belong to the hypersurface $M^5$.

The action of the linear symplectic group $Sp(6)$,

$$
\begin{pmatrix}
\begin{array}{c}
dp \\
dx
\end{array}
\end{pmatrix} =
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
dp \\
dx
\end{array}
\end{pmatrix},
$$

naturally extends to $\Lambda$:

$$
\tilde{U} = (AU + B)(CU + D)^{-1}.
$$

(4.49)

Here $A, B, C, D$ are $3 \times 3$ matrices such that $A^t C = C^t A$, $B^t D = D^t B$, $A^t D - C^t B = id$; notice that the extended action is no longer linear. The transformation law (4.49) suggests that the action of $Sp(6)$ preserves the class of equations (3.1), indeed, second order derivatives transform through second order derivatives only. Moreover, since any changes of variables obviously preserve the integrability, the group $Sp(6)$ can be viewed as a natural ‘equivalence group’ of the problem: it maps integrable equations to integrable. Thus, $Sp(6)$ is a point symmetry group of the integrability conditions (3.26) derived in Chapter 3. This has also been verified directly using the invariance condition (4.41). The classification of integrable equations of the form (3.1) has to be performed modulo this equivalence: two $Sp(6)$-related equations should be regarded as ‘the same’.
The identity
\[ \det d\tilde{U} = \frac{\det(A - (AU + B)(CU + D)^{-1}C)}{\det(CU + D)} \det dU, \]
which readily follows from (4.49), implies that the conformal class of the order 3 tensor \( \det dU \) is invariant under the action of \( Sp(6) \). The converse is also true:

**Theorem 10** [13] The group of conformal automorphisms of the symmetric tensor of third order \( \det dU \) is isomorphic to \( Sp(6) \).

The proof consists of a direct calculation of conformal automorphisms of the the order 3 tensor \( \det dU \). As pointed before, \( U \) is a \( 3 \times 3 \) symmetric matrix \( U = [u_{ij}] \).

Letting \( \Omega = \det dU, \) \( a = u_{xx}, b = u_{xy}, c = u_{xt}, p = u_{yy}, q = u_{yt} \) and \( f = u_{tt} \) we have that
\[ \Omega = \det dU = \det dU = (1 + \varepsilon g(a, b, c, p, q, f)) \det dU, \]
where \( g \) is an unknown function. Equation (4.51) gives a system of first order differential equations for \( A, B, C, P, Q \) and \( F \). Integrating this systems gives us
21 vector fields. These conditions are equivalent to

\[ L_x \Omega = \theta \Omega, \]

where,

\[ X = A \frac{\partial}{\partial a} + B \frac{\partial}{\partial b} + \cdots + F \frac{\partial}{\partial f}. \]

Thus, for a 3 x 3 symmetric matrix \( U = [u_{ij}] \), the Lie algebra of the group of conformal automorphisms of the cubic form \( \det dU \) is spanned by 21 vector fields which generate the Lie algebra of the symplectic Lie group \( Sp(6) \):

\[
\begin{align*}
X_{11} &= \frac{\partial}{\partial u_{11}}, \quad X_{12} = \frac{\partial}{\partial u_{12}}, \quad X_{13} = \frac{\partial}{\partial u_{13}}, \\
X_{22} &= \frac{\partial}{\partial u_{22}}, \quad X_{23} = \frac{\partial}{\partial u_{23}}, \quad X_{33} = \frac{\partial}{\partial u_{33}}, \\
J_1 &= 2u_{11} \frac{\partial}{\partial u_{11}} + u_{12} \frac{\partial}{\partial u_{12}} + u_{13} \frac{\partial}{\partial u_{13}}, \\
J_2 &= 2u_{22} \frac{\partial}{\partial u_{22}} + u_{21} \frac{\partial}{\partial u_{21}} + u_{23} \frac{\partial}{\partial u_{23}}, \\
J_3 &= 2u_{33} \frac{\partial}{\partial u_{33}} + u_{31} \frac{\partial}{\partial u_{31}} + u_{32} \frac{\partial}{\partial u_{32}}, \\
L_{12} &= 2u_{12} \frac{\partial}{\partial u_{11}} + u_{22} \frac{\partial}{\partial u_{12}} + u_{23} \frac{\partial}{\partial u_{13}}, \\
L_{13} &= 2u_{13} \frac{\partial}{\partial u_{11}} + u_{33} \frac{\partial}{\partial u_{13}} + u_{32} \frac{\partial}{\partial u_{12}}, \\
L_{21} &= 2u_{21} \frac{\partial}{\partial u_{22}} + u_{11} \frac{\partial}{\partial u_{21}} + u_{13} \frac{\partial}{\partial u_{23}},
\end{align*}
\]
Given a PDE of the form $F(u_{11}, u_{12}, u_{13}, u_{22}, u_{23}, u_{33}) = 0$, we will look for its infinitesimal symmetries by solving the determining equation $L_X F|_{F=0} = 0$, where $X$ is a linear combination of the 21 vector fields presented above. Notice that the answer may not coincide with the full algebra of Lie-point symmetries: we
consider only those symmetries which belong to the equivalence group $Sp(6)$. Below we list particular examples of integrable equations which possess symmetry algebras of different dimensions (it is worth noting that any two equations with different symmetry algebras are automatically non-equivalent).

**Example 1.** The 2-dimensional linear wave equation, $u_{11} + u_{22} - u_{33} = 0$, possesses nine infinitesimal symmetries:

$$X_{12}, \ X_{13}, \ X_{23}, \ X_{11} + X_{33}, \ X_{22} + X_{33}, \ J_1 + J_2 + J_3, \ L_{12} - L_{21}, \ L_{13} + L_{31}, \ L_{23} + L_{32}.$$ 

**Example 2.** The dKP equation, $u_{22} - u_{13} + \frac{1}{2} u_{11}^2 = 0$, possesses seven infinitesimal symmetries:

$$X_{12}, \ X_{23}, \ X_{33}, \ X_{13} + X_{22}, \ X_{11} + L_{31}, \ J_1 + 2J_2 + 3J_3, \ 2L_{32} + L_{21}.$$ 

Note here that our efforts to find an integrable example with exactly eight symmetries were not successful.

**Example 3.** The Boyer-Finley equation, $u_{11} + u_{22} - e^{u_{33}} = 0$, possesses six infinitesimal symmetries:

$$X_{12}, \ X_{13}, \ X_{23}, \ X_{11} - X_{22}, \ J_1 + J_2 + 2X_{33}, \ L_{12} - L_{21}.$$ 

**Example 4.** The degeneration of the dispersionless Hirota equation, $u_{12} - u_{13} - e^{u_{23}} = 0$, possesses five infinitesimal symmetries:

$$X_{11}, \ X_{22}, \ X_{33}, \ J_1 + X_{23}, \ X_{12} + X_{13}.$$
Example 5. The equation $e^{u_{11}} + e^{u_{22}} - e^{u_{33}} = 0$ possesses four infinitesimal symmetries:

$$X_{12}, \ X_{13}, \ X_{23}, \ X_{11} + X_{22} + X_{33}.$$ 

Example 6. The dispersionless Hirota-type equation for the BKP hierarchy, $e^{u_{13}} + e^{u_{13}} + e^{u_{23}} = e^{u_{13}+u_{13}+u_{23}}$, possesses three infinitesimal symmetries:

$$X_{11}, \ X_{22}, \ X_{33}.$$ 

Remark. We point out that the existence of ‘many’ symmetries is not related to the integrability: for instance, both equations (3.32), which are not integrable, possess 8-dimensional symmetry algebras isomorphic to $SL(3, R)$ and $SU(3, R)$, respectively [4]. Thus, the equation $Hess \ u = 1$ possesses eight infinitesimal symmetries:

$$L_{12}, \ L_{13}, \ L_{21}, \ L_{23}, \ L_{31}, \ L_{32}, \ J_{1} - J_{2}, \ J_{1} - J_{3}.$$ 

The equation $Hess \ u = u_{11} + u_{22} + u_{33}$ also possesses eight infinitesimal symmetries:

$$X_{22} - X_{11}, \ X_{33} - X_{11}, \ P_{1} + X_{23} \ P_{2} + X_{13},$$

$$P_{3} + X_{12}, \ L_{12} + L_{21}, \ L_{13} + L_{31}, \ L_{23} + L_{32}.$$ 

The main result of this section is the following:

**Theorem 11** The action of the equivalence group $Sp(6)$ on the moduli space of integrable equations of the dispersionless Hirota type has an open orbit. 

This fact is, in a sense, surprising: it establishes the existence of a ‘universal’ equation with no symmetries, which generates an open part of the moduli space.
under the action of $Sp(6)$. In particular, one should be able to obtain all equations with non-trivial symmetries by taking appropriate degenerations of this universal equation.

**Proof of Theorem 11**

The main idea of the proof is to prolong the 21 infinitesimal generators $X_{11} - P_3$ to the moduli space of solutions of the involutive system (3.26). We point out that, since third order derivatives of $f$ are explicitly known, this moduli space can be identified with the values of $f$ and its partial derivatives $f_i$, $f_{ij}$ up to second order (21 parameters altogether). The prolongation can be calculated as follows:

(1) Following the standard notation adopted in the symmetry analysis of differential equations [35, 47], we introduce the variables

$$x^1 = u_{11}, \quad x^2 = u_{12}, \quad x^3 = u_{13}, \quad x^4 = u_{22}, \quad x^5 = u_{23}, \quad u = u_{33},$$

and rewrite the above 21 generators in the form

$$\xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u},$$

here $\xi^i$ and $\eta$ are certain functions of $x$ and $u$. In this notation, a dispersionless Hirota-type equation is represented in the form $u = u(x^1, ..., x^5)$ (the function $u$ is denoted by $f$ in Chapter 3).

(2) Prolong infinitesimal generators to the second order jet space [48] with coordinates $u, u_i, u_{ij}$,

$$\xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_i} + \zeta_{ij} \frac{\partial}{\partial u_{ij}},$$
where $\zeta_i$ and $\zeta_{ij}$ are calculated according to the standard prolongation formulae:

$$\zeta_i = D_i(\eta) - u_k D_i(\xi^k), \quad \zeta_{ij} = D_j\zeta_i - u_{ik} D_j(\xi^k);$$

here $D_i$ are the operators of total differentiation.

(3) To eliminate the $\frac{\partial}{\partial x^i}$-terms, subtract the linear combination of total derivatives $\xi^i D_i$ from the prolonged operators. It is sufficient to keep only the following terms in $D_i$:

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + u_{ijk} \frac{\partial}{\partial u_{jk}};$$

notice that, since $u_{ijk}$ are explicit functions of lower order derivatives, the resulting operators will be well-defined on the 21-dimensional space with coordinates $u, u_i, u_{ij}$. Although these operators will depend on the variables $x^i$ as parameters (indeed, the isomorphism of the moduli space with the space $u, u_i, u_{ij}$ depends on the choice of a point in the $x$-space), all algebraic properties of these operators will be $x$-independent.

(4) Finally, the dimension of the maximal $Sp(6)$-orbit equals the rank of the $21 \times 21$ matrix of coefficients of these operators. It remains to point out that this rank equals 21 for any ‘random’ choice of numerical values for $x^i, u, u_i, u_{ij}$ (however, it equals $21 - r$ for any example with $r$ symmetries).

### 4.6 Lie algebra of $Sp(6, \mathbb{R})$

Given our 21 operators (see Sect. 4.5) we can see that condition (4.46) holds true, by calculating all commutators. The Lie algebra of generators is given in Appendix 1. in the form of a table. Note that the notation used in the table is
the following:

\[ X_1 = X_{11}, X_2 = X_{12}, X_3 = X_{13}, X_4 = X_{22}, X_5 = X_{23}, X_6 = X_{33}, \]

\[ X_7 = J_1, X_8 = J_2, X_9 = J_3, \]

\[ X_{10} = L_{12}, X_{11} = L_{13}, X_{12} = L_{21}, X_{13} = L_{23}, X_{14} = L_{31}, X_{15} = L_{32}, \]

\[ X_{16} = H_1, X_{17} = H_2, X_{18} = H_3, \]

\[ X_{19} = P_1, X_{20} = P_2, X_{21} = P_3. \]

### 4.7 Contact symmetries

Using the knowledge from the previous section as well as some theory given next, we shall find and compare contact symmetries of the Hirota equation, the dKP equation as well as the equation \( u_{xt} + f(u_{xx}) = u_{yy} \). Furthermore we investigate integrability vs contact symmetries. Will integrable equations possess more symmetries that non-integrable one?

#### Prolongation formulae

In this section we begin with some theory on groups of contact (or tangent) transformations which will be required. We only discuss the multidimensional case with arbitrary number \( n \) of independent variables \( x = (x^1, \ldots, x^n) \) and one dependent variable \( u \), since it is the only case when one can have true contact symmetries.

Let \( p \) denote the set of first derivatives \( p_i = \frac{\partial u}{\partial x^i} \), and consider a one-parameter
group of transformations

\[ \tilde{x}^i = \phi^i(x, u, p, \alpha), \quad \tilde{u} = \psi(x, u, p, \alpha), \quad \tilde{\bar{p}}_i = \omega_i(x, u, p, \alpha) \quad (4.52) \]

in the \((2n + 1)\)-dimensional space of variables \((x, u, p)\). Transformations (4.52) are called contact transformation when \(\bar{p}_i = \frac{\partial u}{\partial x_i}\). In terms of infinitesimal transformations we have that,

\[ \tilde{x}^i = x^i + \xi^i(x, u, p) \alpha, \quad \tilde{u} = u + \eta(x, u, p) \alpha, \quad \tilde{\bar{p}}_i = u_i + \zeta_i(x, u, p) \alpha. \]

Next, we prove the prolongation formulae for finding contact symmetries. Consider the contact form \(\omega = du - p_i dx^i = 0\), where \(p_i = \frac{\partial u}{\partial x_i}\). Then using the operator

\[ X = \xi^i(x^i, u, p_i) \frac{\partial}{\partial x^i} + \eta(x^i, u, p_i) \frac{\partial}{\partial u} + \zeta^i(x^i, u, p_i) \frac{\partial}{\partial p_i}, \quad (4.53) \]

we calculate the Lie derivative of \(\omega\) modulo \(\omega\). Therefore we have the determining equation

\[ L_x(\omega) \bigg|_{\omega=0} = 0, \quad (4.54) \]

which gives,

\[ X \circ \left( d\omega \right) + d \left( X \circ \omega \right) \bigg|_{\omega=0} = 0, \]

or,

\[ X \circ \left( -dp_i \wedge dx^i \right) + d \left( \eta - p_i \xi^i \right) \bigg|_{\omega=0} = 0. \]

Then, by expanding the interior and exterior products we get,

\[-\zeta_i dx^i + \xi^i dp_i + \eta_i dx^i + \eta u du + \eta_p dp_i - \xi^i dp_i - p_i (\xi^i dx^i + \xi^i du + \xi^i dp_i) \bigg|_{\omega=0} = 0. \]
Substituting \( du = \omega + p_i dx^i = 0 \) we have that,

\[-\zeta_i dx^i + \eta_i dx^i + \eta_u p_i dx^i + \eta_i dp_i - p_i \xi^i_j dx^i - p_i \xi^j_i p_i dx^i - p_i \xi^i_p p_i dp_i = 0.\]

Since \( dx^i \) and \( dp_i \) are independent we can collect their coefficients separately and equate them to zero. Thus,

\[
\begin{align*}
\eta_i - p_i \xi^i_{p_i} &= 0, \\
-\zeta_i + \eta_i + p_i \eta_u - p_i \xi^j_i - (p_i)^2 \xi^i_u &= 0. \\
\end{align*}
\] (4.55)

Introducing a function \( W(x, u, p) \) and setting \( \xi^i = -W_{p_i} \), by substitution in (4.55) we get \( \eta = W - p_i W_{p_i} \) and \( \zeta = W_{x_i} + p_i W_u \). Therefore we have that,

\[
X = -W_{p_i} \frac{\partial}{\partial x^i} + (W - p_i W_{p_i}) \frac{\partial}{\partial u} + (W_{x_i} + p_i W_u) \frac{\partial}{\partial p_i},
\] (4.56)

which represents a generic contact vector field. The functions \( W \) occurring here was called by Lie the characteristic function of the contact transformation group [34].

The process of finding contact symmetries of differential equations is quite similar with the one studied in the previous section where we calculated infinitesimal symmetries of differential equations. This is done by acting with the operator (4.56) on the differential equation \( F = 0 \) modulo the differential equation itself, i.e.,

\[
X(F) \bigg|_{F=0} = 0.
\] (4.57)

In the cases where we are able to solve the determining equation we find \( W \). Once we find \( W \) we can find the contact symmetries of the differential equation by calculating equation (4.56).
Equation $e^{u_{xx}} + e^{u_{yy}} = e^{u_{tt}}$

Following the technique described at the end of the last section we have the determining equation:

$$X(e^{u_{xx}} + e^{u_{yy}} - e^{u_{tt}})|_{e^{u_{xx}} + e^{u_{yy}} - e^{u_{tt}} = 0} = 0.$$ (4.58)

Since $W$ is a function of $t, x, y, u, u_t, u_x, u_y$ only we can take coefficients of different terms in $u_{ij}$ where $i, j = t, x, y$ and equate them to zero. Therefore we have:

- $e^{u_{yy}}u_{yy}^2 : W_{u_{yy}} = 0, e^{u_{yy}}u_{xy}u_{yy} : W_{u_{uxy}} = 0, e^{u_{xx}}u_{xx}^2 : W_{u_{xxu}} = 0,$
- $e^{u_{yy}}\ln(e^{u_{xx}} + e^{u_{yy}})u_{ty} : W_{u_{tyu}} = 0, e^{u_{yy}}\ln(e^{u_{xx}} + e^{u_{yy}})u_{tx} : W_{u_{txu}} = 0,$
- $e^{u_{xx}}u_{xy} : u_{t}W_{u_{uyy}} + 2W_{tu_{y}} = 0, e^{u_{yy}}u_{tx} : u_{t}W_{u_{uxx}} + W_{tu_{x}} = 0,$
- $e^{u_{yy}}u_{ty} : -u_{t}W_{u_{uyy}} + u_{y}W_{u_{uyt}} + W_{uyt} + W_{tu_{y}} = 0,$
- $e^{u_{xx}}u_{tx} : -u_{t}W_{u_{uxx}} + u_{x}W_{u_{uxt}} + W_{uxt} + W_{tu_{x}} = 0,$
- $e^{u_{yy}}\ln(e^{u_{xx}} + e^{u_{yy}}) : W_{u} + 2u_{t}W_{u_{u}} + 2W_{tu_{t}} = 0,$
- $e^{u_{xx}}u_{xx} : W_{u} + 2u_{x}W_{u_{uxx}} + 2W_{u_{xx}} = 0, e^{u_{yy}}u_{yy} : W_{u} + 2u_{y}W_{u_{uyy}} + 2W_{uyy} = 0,$
- $e^{u_{yy}} = u_{y}^2W_{uu} - u_{t}^2W_{uu} + 2u_{y}W_{wy} + W_{yy} - 2u_{t}W_{tu} - W_{tt} = 0,$
- $e^{u_{xx}} = u_{x}^2W_{uu} - u_{t}^2W_{uu} + 2u_{x}W_{ux} + W_{xx} - 2u_{t}W_{tu} - W_{tt} = 0.$

Obviously the first six relations tell us that $W$ is linear in $u_t, u_x, u_y$. Then further work shows that $W$ is also linear in $u$ and the final form of $W$ is given by:

$$W = (C_1t + C_2)u_t + (C_1x + C_3)u_x + (C_1y + C_4)u_y - 2C_1u + f(t, x, y),$$ (4.59)
where \( C_1, ..., C_4 \) are arbitrary constants. Furthermore, \( f(t, x, y) \) can be calculated from the system of wave equations:

\[
f_{tt} = f_{xx}, \quad f_{tt} = f_{yy},
\]

resulting in \( f = A(y - t - x) + B(y - t + x) + D(y + t - x) + E(y + t + x) \), where \( A, B, D, E \) are arbitrary functions of their argument. In addition, substituting the characteristic function (4.59) into equation (4.56) gives the following independent generators:

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y},
\]

\[
X_4 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - 2u \frac{\partial}{\partial u} - u_t \frac{\partial}{\partial u_t} - u_x \frac{\partial}{\partial u_x} - u_y \frac{\partial}{\partial u_y},
\]

\[
X_5 = \frac{\partial}{\partial u}, \quad X_6 = \frac{\partial}{\partial u_t}, \quad X_7 = \frac{\partial}{\partial u_x}, \quad X_8 = \frac{\partial}{\partial u_y},
\]

plus the infinite part resulting from the function \( f(t, x, y) \).

**dKP equation** \( u_{xt} - \frac{1}{2}u_{xx}^2 - u_{yy} = 0 \)

Here, the determining equation is given by:

\[
X \left( u_{xt} - \frac{1}{2}u_{xx}^2 - u_{yy} \right) \bigg|_{u_{xt} = u_{xy} + \frac{1}{2}u_{xx}^2} = 0.
\]

Since \( W \) is a function of \( t, x, y, u, u_t, u_x, u_y \) we can take coefficients of different polynomials in \( u_{ij} \) where \( i, j = t, x, y \) and equal them to zero. Therefore we have:

\[
u_{xx}^2 u_{xy} : W_{u_{xy}} = 0, \quad u_{xx}^3 u_{yy} : W_{u_{xy}u_t} = 0,
\]

\[
u_{xy}^2 u_{xx} : W_{u_{xy}u_y} = 0, \quad u_{xx}^2 u_{xy} : W_{u_{xy}u_y} = 0,
\]
\[
\begin{align*}
\frac{u_{xx}^3}{W_{uxu_x}} &= 0, \quad \frac{u_{xx}^2}{u_{yy}} : W_{uyu_y} = 0, \\
\frac{u_{xx}u_{xy}}{2u_xW_{uu_y} + 2W_{xu_y}} &= 0, \quad \frac{u_{xx}u_{yy}}{u_xW_{uu_t} + W_{xu_t}} = 0, \\
\frac{u_{xx}^2}{5u_xW_{uu_x} - u_tW_{uux} + 3W_{xu_x} - W_{tux} + W_u} &= 0, \\
\frac{u_{yy}}{2u_yW_{uuy} - u_xW_{uux} - u_tW_{uu_t} + 2W_{yu_y} - W_{xu_x} - W_{tuy}} &= 0, \\
\frac{u_{xy}}{-u_tW_{uu_y} + 2u_yW_{uux} + 2W_{yu_x} - W_{tuy}} &= 0, \\
\frac{u_{ty}}{-u_xW_{uuy} + 2u_yW_{uux} + 2W_{yu_t} - W_{wuv}} &= 0,
\end{align*}
\]

where it is obvious that \( W \) will be linear in \( u_t, u_x, u_y \). Furthermore, we are left with terms which do not contain second order derivatives of \( u \) and which are all equated to zero together giving:

\[
u_x^2W_{uux} - u_tu_xW_{uu} + 2u_yW_{uy} + W_{yy} - u_xW_{ux} - u_xW_{ut} - W_{tx} = 0.
\]

Further manipulation of the above results gives:

\[
W = \left( \frac{1}{2}C_1t^3 + \frac{3}{4}C_2t^2 + \frac{3}{8}C_3t + C_7 \right)u_t + \left( \frac{1}{2}(C_1t^2 + C_2t + C_3)x \\
+ \frac{1}{4}(C_1t^2 + C_2t + C_3)y^2 + \frac{1}{2}(C_1t^2 + C_3t + C_6)y + \mu(t) \right)u_x \\
+ \left( (C_1t^2 + C_2t + C_3)y + \frac{1}{2}C_4t^2 + C_5t + C_6 \right)u_y \\
+ \frac{1}{6}(C_1t + \frac{1}{2}C_2)x^3 + \frac{1}{3}(C_1y^2 + C_4y + 2\mu_t)x^2 \\
+ \left( \frac{1}{2}\mu(t)y^2 + n(t)y + m(t) \right)x \\
+ \frac{1}{24}\mu_t y^4 + \frac{1}{6}n_t y^3 + \frac{1}{2}m_t y^2 + \nu(t)y + \kappa(t),
\]

(4.63)

where \( \mu, n, m, \nu, \kappa \) are arbitrary functions of \( t \) and \( C_i \) for \( i = 1, \ldots, 7 \) are arbitrary constants. Therefore, the substitution of the function (4.63) into equations (4.56) yields the following linearly independent generators:

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = -\frac{1}{2}y \frac{\partial}{\partial z} - \frac{\partial}{\partial y} + \frac{1}{2}u_x \frac{\partial}{\partial u_y},
\]
\[\begin{align*}
X_3 &= -\frac{1}{2}ty\frac{\partial}{\partial t} - t\frac{\partial}{\partial y} + \left(\frac{1}{2}yu_x + uy\right)\frac{\partial}{\partial u} + \frac{1}{2}tu_x\frac{\partial}{\partial u}, \\
X_4 &= -\frac{1}{4}t^2y - \frac{1}{2}t\frac{\partial}{\partial t} + \frac{1}{4}y^2\frac{\partial}{\partial y} + \left(\frac{1}{2}yu_x + tu_y\right)\frac{\partial}{\partial u} + \frac{1}{2}y\frac{\partial}{\partial u} + \frac{1}{4}(t^2u_x + x^2)\frac{\partial}{\partial u}, \\
X_5 &= -\frac{3}{2}t\frac{\partial}{\partial t} - \left(\frac{1}{2}x + \frac{1}{2}y^2\right)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{3}{2}u_t\frac{\partial}{\partial u} + \frac{1}{2}u_x\frac{\partial}{\partial u} + \left(\frac{1}{2}yu_x + uy\right)\frac{\partial}{\partial u}, \\
X_6 &= -\frac{3}{4}t^2\frac{\partial}{\partial t} - \left(\frac{1}{2}tx + \frac{1}{2}ty^2\right)\frac{\partial}{\partial x} - ty\frac{\partial}{\partial y} + \frac{1}{12}x^3\frac{\partial}{\partial u} \\
&+ \left(\frac{3}{2}tu_t + \left(\frac{1}{2}x + \frac{1}{2}y^2\right)u_x + yu_y\right)\frac{\partial}{\partial u} + \left(\frac{1}{2}tu_x + \frac{1}{3}x^2\right)\frac{\partial}{\partial u}, \\
X_7 &= -\frac{1}{2}t^3\frac{\partial}{\partial t} - \left(\frac{1}{2}t^2x + \frac{1}{4}t^2y^2\right)\frac{\partial}{\partial x} - t^2y\frac{\partial}{\partial y} + \left(\frac{1}{6}tx^3 + \frac{1}{3}y^2x^2\right)\frac{\partial}{\partial u} \\
&+ \left(\frac{3}{2}t^2u_t + \left(tx + \frac{1}{2}ty^2\right)u_x + 2tyu_y + \frac{1}{6}x^3\right)\frac{\partial}{\partial u} \\
&+ \left(\frac{3}{2}(tu_x + tx^2 + y^2x)\frac{\partial}{\partial u} + \left(t^2yu_x + t^2u_y + \frac{1}{2}yx\right)\frac{\partial}{\partial u}, \\
X_8 &= -\frac{\partial}{\partial x} + \frac{1}{2}xy\frac{\partial}{\partial u} + \frac{1}{2}x^2\frac{\partial}{\partial u} + xy\frac{\partial}{\partial u}, \\
X_9 &= xy\frac{\partial}{\partial u} + y\frac{\partial}{\partial u} + x\frac{\partial}{\partial u}, \quad X_{10} = x\frac{\partial}{\partial u} + \frac{\partial}{\partial u}, \quad X_{11} = \frac{\partial}{\partial u} + \frac{\partial}{\partial u}, \quad X_{12} = \frac{\partial}{\partial u},
\end{align*}\]

plus infinite part coming from the terms: \(\mu(t), n(t), m(t), \nu(t), \kappa(t)\).

**Equation:** \(u_{xt} + f(u_{xx}) = u_{yy}\)

Firstly, let \(u_{xx} = a\) so that \(f(a) = f\). In order to find the contact symmetries of \(u_{xt} + f = u_{yy}\) case we need to calculate first the determining equation. In the same way as the previous cases we obtain:

\[\begin{align*}
&u_{tx}u_{tt} : W_{u_{tt}} = 0, \quad u_{ty}u_{tx} : W_{u_{ty}} = 0, \\
&u_{xy}^2 : W_{u_{xu}} = 0, \quad u_{xy}u_{tx} : W_{u_{xy}} = 0, \\
&u_{ty}u_{xy} : -W_{u_{uy}} + 2W_{u_{tx}} = 0, \quad u_{tx}^2 : -W_{u_{ux}} - W_{u_{ut}} = 0,
\end{align*}\]

therefore, \(W\) is linear in \(W_{u_{ij}}\) where \(i, j = t, x, y\). However, with the rest of the terms we need to be careful because there exist terms such as \(f, f', a\) such that combinations of these terms may equal zero. For example we might have a case where \(f = f'\) or \(f = f'a\). In each case, coefficients of similar terms need to
be added and then equated to zero. Let us first note the coefficients and then proceed with the analysis. The terms and theirs coefficients are listed below:

\[ f : -W_u - 2uyW_{uy} - 2W_{yy} = 0, \quad f' : W_u + 2u_x W_{ux} + 2W_{xu} = 0, \]
\[ u_{xy} : -u_t W_{uy} + 2u_y W_{ux} + 2W_{y} - W_{xy} = 0, \quad f'_{xy} : 2u_x W_{uy} + 2W_{xy} = 0, \]
\[ u_{ty} : -u_x W_{uy} + 2u_y W_{ux} + 2W_{yt} - W_{xy} = 0, \quad a : -u_t W_{ux} - W_{tu} = 0, \]
\[ u_{tx} : -u_x W_{ux} - u_t W_{ux} + 2u_y W_{yy} + 2W_{y} - W_{xy} - W_{xt} = 0, \]
\[ f'_{tx} : 2u_x W_{xt} + 2W_{xt}, \quad u_{tt} : -u_x W_{ut} - W_{xt}, \quad f' : u_x^2 W_{ux} + 2u_x W_{ux} + W_{xx}, \]
\[ u_y^2 W_{uy} - u_t u_x W_{uy} + 2u_y W_{uy} + W_{yy} - u_t W_{ux} - u_x W_{tu} - W_{tx} = 0. \]

From the above results we need to find all possible relationships between the terms \( f, f', f', a \). All possible cases can be captured in the equation

\[ f'(Aa + B) + Cf + Da + E = 0, \quad (4.64) \]

where \( A, B, C, D, E \) are arbitrary constants. Equation (4.64) is then split in case 1 where \( A = 0 \) and case 2 where \( A \neq 0 \).

Case 1 gives \( f' = Cf + Da + E \) which can then be split to the cases where \( C = 0 \) and \( C \neq 0 \). Then when \( C = 0 \) we have the case 1(a) where we recover the dKP case with \( f = a^2 \). Furthermore, when \( C \neq 0 \), we have the case 1(b) where we find \( f = e^a \).

With case 2 we have \( f' = a \), where an appropriate transformation has been made without loss of generality. Next, from the case where \( C = 1 \) we have that \( f' - f = Da \) which gives \( f'' = D \) and therefore we get case 2(a) where \( f = a \ln a - a \). We are then left with the case where \( C \neq 1 \) which gives \( f' = Cf + E \) and which can then be split in two cases where \( C = 0 \) and \( C \neq 0 \).

\( C = 0 \) gives the case 2(b) where \( f = \ln a \) and \( C \neq 0 \) gives the case 2(c) with \( f = a^c \). Note here that we are interested only in the cases where \( f \) being non-
linear.

Putting all the cases together we have the $dKP$, for which we already know its contact symmetries, $f = e^a$, $f = a \ln a - a$, $f = \ln a$ and $f = a^x$. Let us now take each of the above cases separately.

With case 1(b) where $f = e^a$, it is obvious that $f = f'$. Therefore we need to take coefficients of $f, f'$ together, while taking coefficients of $f'a, a$ separately and equate them to zero. Manipulation of the resulting equations gives:

$$W = \left( \frac{5}{2} C_1 t^2 + C_6 t + C_7 \right) u_x + \left( -\frac{1}{2} C_1 t + 2 C_2 - C_6 \right) x + \frac{1}{4} C_1 y^2 + \frac{1}{2} C_3 y + C_6 \right) u_x$$

$$+ \left( (C_1 t + C_2) y + C_3 t + C_4 \right) u_y + \left( C_1 t + 2 C_6 - 4 C_2 \right) u + \left( \frac{3}{2} C_1 t + C_6 - C_2 \right) x^2$$

$$+ \left( \frac{3}{2} C_1 y^2 + \mu(t) y + \nu(t) \right) x + \frac{1}{6} \mu u y^3 + \frac{1}{2} \nu t y^2 + m(t) y + n(t),$$

where $\mu, \nu, m, n$ are arbitrary functions of $t$ and $C_i$ for $i = 1, ..., 7$ are arbitrary constants. As a result we have the following generators:

$$X_1 = -\frac{\partial}{\partial t}, \ X_2 = -\frac{\partial}{\partial x}, \ X_3 = -\frac{\partial}{\partial y},$$

$$X_4 = -\frac{1}{2} y \frac{\partial}{\partial x} - t \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u_x} + \frac{1}{2} u_x \frac{\partial}{\partial u_y},$$

$$X_5 = -t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + (2u + x^2) \frac{\partial}{\partial u} + 3u_t \frac{\partial}{\partial u_t} + (2x + u_x) \frac{\partial}{\partial u_x} + 2u_y \frac{\partial}{\partial u_y},$$

$$X_6 = -2x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - (4u + x^2) \frac{\partial}{\partial u} - 4u_t \frac{\partial}{\partial u_t} - 2(x + u_x) \frac{\partial}{\partial u_x} - 3u_y \frac{\partial}{\partial u_y},$$

$$X_7 = (\frac{1}{2} t x - \frac{1}{2} y^2) \frac{\partial}{\partial x} - \frac{1}{2} t^2 \frac{\partial}{\partial t} - ty \frac{\partial}{\partial y} + (tu + \frac{3}{2} t x^2 + \frac{3}{2} y^2 x) \frac{\partial}{\partial u}$$

$$+ \frac{1}{2} (tu - x u_x + 2 y u_y + 3 x^2 + 2 u) \frac{\partial}{\partial u_x},$$

$$+ (\frac{1}{2} t u_x + 3 t x + \frac{3}{2} y^2) \frac{\partial}{\partial u_x} + (\frac{1}{2} y u_x + 2 t u_y + 3 y x) \frac{\partial}{\partial u_y},$$

$$X_8 = x y \frac{\partial}{\partial u} + y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \ X_9 = x \frac{\partial}{\partial u} + \frac{\partial}{\partial u_x}, \ X_{10} = y \frac{\partial}{\partial u} + \frac{\partial}{\partial u_y}, \ X_{11} = \frac{\partial}{\partial u},$$

plus the infinite part.

For the case 2(a) ($f = a \ln f - a$) we need to add coefficients of $f$ and $f'a$ and equate them to zero as well as subtract the coefficient of $f$ from the coefficient of $a$ and again equate to zero. The rest of the coefficients are equated to zero.
separately. Working on this system of equations results in:

\[ W = (C_1 t + C_2) u_t + (C_1 x + \frac{1}{2} C_3 y + C_5 t + C_6) u_x 
+ (C_1 y + C_3 t + C_4) u_y - 2(C_1 + C_5) u + (\mu(t) y + \nu(t)) x \]
\[ + \frac{1}{6} \mu t y^3 + \frac{1}{2} \nu t y^2 + m(t) y + n(t), \]

where \( \mu, \nu, m, n \) are arbitrary functions of \( t \) and \( C_i \) for \( i = 1, \ldots, 6 \) are arbitrary constants. These yield the generators presented below:

\[
X_1 = -\frac{\partial}{\partial t}, \quad X_2 = -\frac{\partial}{\partial x}, \quad X_3 = -\frac{\partial}{\partial y},
X_4 = t \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} + (u - u_t) \frac{\partial}{\partial u_t} - u_x \frac{\partial}{\partial u_x} - u_y \frac{\partial}{\partial u_y},
X_5 = -\frac{1}{2} y \frac{\partial}{\partial x} - t \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u_y} + \frac{1}{2} u_x \frac{\partial}{\partial u_x},
X_6 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2 u \frac{\partial}{\partial u} + u_t \frac{\partial}{\partial u_t} + u_x \frac{\partial}{\partial u_x} + u_y \frac{\partial}{\partial u_y},
X_7 = x y \frac{\partial}{\partial u} + y \frac{\partial}{\partial u_x} + x \frac{\partial}{\partial u_y},
X_8 = x \frac{\partial}{\partial u} + y \frac{\partial}{\partial u_x},
X_9 = y \frac{\partial}{\partial u} + x \frac{\partial}{\partial u_y},
X_{10} = \frac{\partial}{\partial u},
\]

plus the infinite part.

Next we consider the case 2(b). In this case, \( f = \ln a \), where obviously \( f \neq f' \neq a \neq f'a \), and therefore we need to equate separately each of their coefficients to zero. Again manipulating the resulting equations we come up with:

\[ W = (-\frac{1}{4} C_1 t^2 + C_6 t + C_7) u_t + ((\frac{5}{2} C_1 t + 2 C_2 - C_6) x + \frac{1}{4} C_3 y^2 + \frac{1}{2} C_5 t + C_6) u_x 
+ ((C_1 t + C_5) y + C_3 t + C_4) u_y - 2(C_1 t + C_2) u + (\mu(t) y + \nu(t)) x 
+ \frac{1}{6} \mu t y^3 - (\frac{1}{2} \nu t + \frac{3}{2} C_1 t + C_2 - C_6) y^2 + m(t) y + n(t), \]

where \( \mu, \nu, m, n \) are arbitrary functions of \( t \) and \( C_i \) for \( i = 1, \ldots, 7 \) are arbitrary constants. Therefore we get the following generators:

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y},
X_4 = t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial u} + u_t \frac{\partial}{\partial u_t} - u_x \frac{\partial}{\partial u_x} + 2 y \frac{\partial}{\partial u_y},
\]
\[ X_5 = -\frac{1}{2} y \frac{\partial}{\partial x} - t \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u_t} + \frac{1}{2} u_x \frac{\partial}{\partial u_y}, \]
\[ X_6 = -2x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - (2u + y^2) \frac{\partial}{\partial u} - 2u_t \frac{\partial}{\partial u_x} - (2y + u_y) \frac{\partial}{\partial u_y}, \]
\[ X_7 = \frac{1}{4} t^2 \frac{\partial}{\partial t} - \left( \frac{1}{2} t x + \frac{1}{4} y^2 \right) \frac{\partial}{\partial x} - ty \frac{\partial}{\partial y} - (2t + \frac{1}{2} t y^2) \frac{\partial}{\partial u} \]
\[ - \frac{1}{2} (5u_t - 5x u_x - 2y u_y + 4u + 3y^2) \frac{\partial}{\partial u_t} + \frac{1}{2} t u_x \frac{\partial}{\partial u_x} + (\frac{1}{2} y u_x - 3y - t u_y) \frac{\partial}{\partial u_y}, \]
\[ X_8 = x y \frac{\partial}{\partial u} + y \frac{\partial}{\partial u_x} + x \frac{\partial}{\partial u_y}, \]
\[ X_9 = x \frac{\partial}{\partial u} + \frac{\partial}{\partial u_x}, \]
\[ X_{10} = y \frac{\partial}{\partial u} + \frac{\partial}{\partial u_y}, \]
\[ X_{11} = \frac{\partial}{\partial a}, \]

plus the infinite part.

Lastly, we have the case 2(c) with \( f = \alpha^\kappa \) for \( \kappa \neq 0, 1, 2 \) where obviously \( f \kappa = f' \alpha \) which means that we need to add the coefficients of \( f \) to the coefficients of \( f' \alpha \) times \( \kappa \) and then equate it to zero. The rest of the coefficients can be equated to zero separately. All these will give:

\[ W = \left( \frac{5\kappa - 1}{4(\kappa + 1)} C_1 t^2 + C_6 t + C_7 \right) u_t \]
\[ + \left( \frac{5-\kappa}{2(\kappa + 1)} C_1 t + 2C_2 - C_6 \right) x + \frac{1}{4} C_1 y^2 + \frac{1}{2} C_3 y + C_5 \right) u_x \]
\[ + ((C_1 t + C_2) y + C_3 t + C_4) u_y + \frac{2}{\kappa - 1} \left( \frac{C_1 (\kappa^2 - 3\kappa + 2)}{2(\kappa + 1)} t + C_2 (1 - 2 \kappa) + \kappa C_5 \right) u \]
\[ + \left( (\mu(t) y + \nu(t)) x + \frac{1}{2} \mu \gamma y^2 + \frac{1}{2} \nu \gamma y^2 + m(t) y + n(t), \right) \]

where \( \mu, \nu, m, n \) are arbitrary functions of \( t \) and \( C_i \) for \( i = 1, \ldots, 7 \) are arbitrary constants. This gives:

\[ X_1 = \frac{\partial}{\partial t}, \]
\[ X_2 = \frac{\partial}{\partial x}, \]
\[ X_3 = \frac{\partial}{\partial y}, \]
\[ X_4 = -\frac{1}{2} y \frac{\partial}{\partial x} - t \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u_t} + \frac{1}{2} u_x \frac{\partial}{\partial u_y}, \]
\[ X_5 = -A(\kappa) t^2 \frac{\partial}{\partial t} - (B(\kappa) t x + \frac{1}{4} y^2) \frac{\partial}{\partial x} - ty \frac{\partial}{\partial y} + (\frac{\kappa - 2}{\kappa + 1}) t u \frac{\partial}{\partial u} \]
\[ + ((2A(\kappa) + D(\kappa)) t u + B(\kappa) t u_x + y u_y) \frac{\partial}{\partial u_t} \]
\[ + B(\kappa) t u_x \frac{\partial}{\partial u_x} + (\frac{1}{2} y u_x + D(\kappa) t u_y) \frac{\partial}{\partial u_y}, \]
\[ X_6 = -2x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + (1 - 2 \kappa) u \frac{\partial}{\partial x} + \frac{2 \kappa - 4 \kappa}{\kappa - 1} u_t \frac{\partial}{\partial u} + \frac{2 \kappa - 1}{\kappa - 1} u_x \frac{\partial}{\partial u_x} + \frac{1 - 3 \kappa}{\kappa - 1} u_y \frac{\partial}{\partial u_y}, \]
\[ X_7 = -t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \left( \frac{2 \kappa}{\kappa - 1} \right) u \frac{\partial}{\partial x} + \left( \frac{3 \kappa - 1}{\kappa - 1} \right) u_t \frac{\partial}{\partial u} + \left( \frac{\kappa + 1}{\kappa - 1} \right) u_x \frac{\partial}{\partial u_x} + \left( \frac{2 \kappa}{\kappa - 1} \right) u_y \frac{\partial}{\partial u_y}, \]
\[ X_8 = x y \frac{\partial}{\partial u} + y \frac{\partial}{\partial u_x} + x \frac{\partial}{\partial u_y}, \]
\[ X_9 = x \frac{\partial}{\partial u} + \frac{\partial}{\partial u_x}, \]
\[ X_{10} = y \frac{\partial}{\partial u} + \frac{\partial}{\partial u_y}, \]
\[ X_{11} = \frac{\partial}{\partial a}, \]
plus the infinite part. Note here that $A, B, D$ are given by:

$$
A = \frac{5\kappa - 1}{4(\kappa + 1)}, \quad B = \frac{5 - \kappa}{2(\kappa + 1)}, \quad D = \frac{\kappa^2 - 3\kappa + 2}{2(\kappa + 1)}.
$$

The above calculations then imply the following conclusion:

- Integrability of a differential equation of the dispersionless Hirota type is generically not related to the size of its contact symmetry algebra.

Notice here that by size we mean the number of arbitrary functions and the number of arbitrary constants appearing in the generic symmetry generator.
Chapter 5

Conclusion and Further work

In this chapter we begin by summarizing the ideas of this thesis. We then move on to outline possibilities to extend the work presented above.

In this thesis we have investigated second order equations of the dispersionless Hirota type,

\[ F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0. \]  \hspace{1cm} (5.1)

Equations of such type have been approached by a variety of techniques. However, classification results have not been obtained until recently, since there was no acceptable definition of integrability of such systems.

Extending the work of Gibbons and Tsarev [29], [30], as well as Ferapontov and Khusnutdinova [21], we used the existence of infinitely many hydrodynamic reductions as a definition of integrability. Based on this method we were able to classify integrable second order equations of the dispersionless Hirota type. Technically, this method allows one to reduce the three-dimensional PDE (5.1) to a pair of commuting \( n \)-component \((1+1)\)-dimensional systems of hydrodynamic type. A three-dimensional equation is thus called \textit{integrable} if it possesses ‘sufficiently many’ \( n \)-component reductions to a pair of consistent hydrodynamic type
Conclusion and Further work

systems

\[ R_t^i = \lambda^i(R) \, R_x^i, \quad R_y^i = \mu^i(R) \, R_z^i. \]

Such reductions are governed by a complicated overdetermined system. When calculating its integrability conditions, the Mathematica package played an important role. We managed to prove that

- The moduli space of integrable equations of the dispersionless Hirota type is 21-dimensional.

Moving on, we investigated \( Sp(6) \) as a symmetry group of the integrability conditions. The class of equations under study is form-invariant under the action of the contact group \( Sp(6) \) generated by the linear symplectic transformations \( x, y, t, u_x, u_y, u_t \). As we have seen these transformations map integrable equations to integrable. Our second main result then states:

- The action of the equivalence group \( Sp(6) \) on the moduli space of integrable equations of the dispersionless Hirota type has an open orbit.

Many questions have arisen during our study of equations of the dispersionless Hirota type. Obviously, these questions form possible areas of further investigation.

Firstly, it would be interesting to understand the geometry behind our equations which are related to the Lagrangian Grassmannian. This amounts to the classification of geometric invariants of hypersurfaces which do not change under the action of \( Sp(6) \).

Secondly, one could try and obtain explicitly the form of the generic solution of the dispersionless Hirota type. However, since particular solutions have been expressed in terms of modular forms, one should not expect a simple answer.
Finally, it would be nice to construct exact solutions of our equations and find their physical interpretation.
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Figure 5.1: Lie algebra of \( Sp(6, \mathbb{R}) \)
References

Bibliography


[44] Mathematica and Wolfram Mathematica are trademarks of Wolfram Research Inc. www.wolfram.com


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