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THE EFFECT OF ELECTRICAL NOISE ON THE
SPECTRAL PURITY OF OSCILLATORS

by

MICHAEL JOHN MEAD B. Tech.

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of
Doctor of Philosophy of the Loughborough University of Technology
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Engineering.

<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CHAPTER 1. Introduction</strong></td>
<td></td>
</tr>
<tr>
<td><strong>CHAPTER 2. The Theory of Noise in Oscillators</strong></td>
<td></td>
</tr>
<tr>
<td>2.1 Different Types of Oscillator Noise</td>
<td>11</td>
</tr>
<tr>
<td>2.2 Comparison Between Different Oscillator Configurations</td>
<td>13</td>
</tr>
<tr>
<td>2.2.1 RC and LC Oscillators</td>
<td>14</td>
</tr>
<tr>
<td>2.2.2 Minimising the Effect of Shot Noise</td>
<td>19</td>
</tr>
<tr>
<td>2.2.3 Series and Parallel Loss Resistance</td>
<td>21</td>
</tr>
<tr>
<td>2.3 The Non-linear Aspect of an Oscillator</td>
<td>22</td>
</tr>
<tr>
<td>2.3.1 Essential and Undesirable Types of Non-linearities</td>
<td>22</td>
</tr>
<tr>
<td>2.3.2 Instantaneous and Delayed Amplitude Limiting</td>
<td>26</td>
</tr>
<tr>
<td><strong>CHAPTER 3. The Effect of White Noise Inside an Oscillator</strong></td>
<td></td>
</tr>
<tr>
<td>3.1 Instantaneous Limiting Oscillator using Van der Pol's Model</td>
<td>31</td>
</tr>
<tr>
<td>3.1.1 The Van der Pol Oscillator</td>
<td>31</td>
</tr>
<tr>
<td>3.1.2 Noise in the Van der Pol Oscillator</td>
<td>34</td>
</tr>
<tr>
<td>3.1.3 Limitations to the Van der Pol Model</td>
<td>42</td>
</tr>
<tr>
<td>3.2 Instantaneous Limiting Oscillator using Hard Limiting Model</td>
<td>47</td>
</tr>
<tr>
<td>3.2.1 The Hard Limiting Oscillator Model</td>
<td>47</td>
</tr>
<tr>
<td>3.2.2 Noise in the Hard Limiting Oscillator</td>
<td>55</td>
</tr>
<tr>
<td>3.3 Phase, Amplitude and RF Spectra of an Oscillator</td>
<td>60</td>
</tr>
<tr>
<td>3.3.1 Oscillator Phase</td>
<td>60</td>
</tr>
<tr>
<td>3.3.2 Oscillator Amplitude</td>
<td>64</td>
</tr>
<tr>
<td>3.3.3 Complete Oscillator Signal</td>
<td>68</td>
</tr>
<tr>
<td>3.4 Delayed Amplitude Limiting Oscillator</td>
<td>74</td>
</tr>
<tr>
<td><strong>CHAPTER 4. The Effect of Other Types of Oscillator Noise</strong></td>
<td></td>
</tr>
<tr>
<td>4.1 Noise Added to the Output of an Oscillator</td>
<td>88</td>
</tr>
<tr>
<td>4.2 The Effect of Flicker Noise</td>
<td>93</td>
</tr>
<tr>
<td>4.2.1 Amplitude Modulation</td>
<td>93</td>
</tr>
<tr>
<td>4.2.2 Frequency Modulation</td>
<td>96</td>
</tr>
<tr>
<td>4.3 The Combined Effect of Different Types of Noise</td>
<td>100</td>
</tr>
</tbody>
</table>
### CHAPTER 5. Characterisation and Measurement of Oscillator Noise

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1 Characterisation of Oscillator Noise</td>
<td>103</td>
</tr>
<tr>
<td>5.2 Methods of Measuring Oscillator Noise</td>
<td>108</td>
</tr>
<tr>
<td>5.2.1 The RF Spectrum</td>
<td>103</td>
</tr>
<tr>
<td>5.2.2 Amplitude and Frequency Noise Modulation Measurement</td>
<td>110</td>
</tr>
<tr>
<td>5.2.3 Measurement of Phase Noise Modulation</td>
<td>112</td>
</tr>
<tr>
<td>5.3 A Practical Test Set for Measuring Oscillator Noise</td>
<td>114</td>
</tr>
</tbody>
</table>

### CHAPTER 6. Comparison Between Theoretical and Experimental Results

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1 Philosophy of Experimental Work</td>
<td>128</td>
</tr>
<tr>
<td>6.2 Experimental Instantaneous Limiting Oscillator</td>
<td>130</td>
</tr>
<tr>
<td>6.2.1 Design of Oscillator</td>
<td>130</td>
</tr>
<tr>
<td>6.2.2 Internal Noise in Experimental Oscillator</td>
<td>134</td>
</tr>
<tr>
<td>6.3 White Noise in the Instantaneous Limiting Oscillator</td>
<td>135</td>
</tr>
<tr>
<td>6.3.1 Oscillator Noise Due to Noise Generator</td>
<td>135</td>
</tr>
<tr>
<td>6.3.2 Effect of Thermal Noise in Tank Circuit</td>
<td>143</td>
</tr>
<tr>
<td>6.3.3 Relaxation Time of Oscillator</td>
<td>145</td>
</tr>
<tr>
<td>6.4 The Delayed Amplitude Limiting Oscillator</td>
<td>151</td>
</tr>
<tr>
<td>6.5 Effect of Flicker Noise</td>
<td>162</td>
</tr>
</tbody>
</table>

### CHAPTER 7. Conclusions

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1 How to Make a Low Noise Oscillator</td>
<td>169</td>
</tr>
<tr>
<td>7.2 White Noise Inside an Oscillator</td>
<td>172</td>
</tr>
<tr>
<td>7.3 Noise Added to Oscillator Output</td>
<td>175</td>
</tr>
<tr>
<td>7.4 Flicker Noise</td>
<td>176</td>
</tr>
</tbody>
</table>

### APPENDIX

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1 Spectra of Sinewaves with Noise</td>
<td>178</td>
</tr>
<tr>
<td>8.2 The Hard Limiting Model of a Hartley Oscillator</td>
<td>182</td>
</tr>
<tr>
<td>8.3 List of Principal Symbols</td>
<td>187</td>
</tr>
</tbody>
</table>
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CHAPTER 1

INTRODUCTION

This thesis is a study of the effect of electrical noise in oscillators. An ideal noiseless oscillator would produce a line spectrum with all the oscillator power being generated at discrete frequencies. A real oscillator generates power over a wide band of frequencies, although most of it is close to the fundamental and harmonic frequencies. The power density spectrum of a noisy oscillator is shown in Figure 1.1, with the effects of the noise exaggerated so that it can easily be seen. The power generated at the fundamental frequency of oscillation lies in a very narrow band, with a 3 dB bandwidth usually much less than 1 Hz for a 100 MHz oscillator. The power density spectrum well away from the fundamental and harmonic frequencies is usually only a little above that due to thermal noise in the output impedance of the oscillator. Thus the effect of noise on the spectrum of most oscillators is very slight.

For many purposes it is satisfactory to consider that an oscillator produces power only at discrete frequencies. There are however, some situations where the spectral purity of an oscillator is important. One of these situations occurs with the testing of VHF mobile radio receivers where a signal generator of high spectral purity is required. It was for this reason that Marconi Instruments suggested it would be worthwhile studying the effect of noise in oscillators. In this thesis variable frequency VHF oscillators of high spectral purity will mainly be considered. Much of the theoretical work however, is applicable to the effect of noise in other types of oscillators.
Figure 1.1: Power density spectrum of a noisy oscillator

Figure 1.2: The essential elements of a feedback oscillator
It is only in recent years that the effects of noise in oscillators has become important in the testing of mobile radio receivers. This is because the demand for more channels in mobile radio communications has been met by progressively reducing the channel spacing. Consequently the selectivity of the receivers has had to be increased. The spectral purity of most signal generators has been found inadequate for measuring the adjacent channel rejection of the receivers to the British Post Office specification (1.1). When a signal generator is tuned to a channel frequency adjacent to the receiver channel frequency, then some of the unwanted power, which noise in the signal generator has produced, falls into the pass band of the receiver. If this is too great then the response of the receiver is a measure of the oscillator noise of the signal generator rather than the adjacent channel rejection of the receiver.

Oscillator noise has also become important in other situations where it is necessary to achieve the ultimate in performance. It affects the accuracy of atomic clocks (1.2) and the resolution of radar systems (1.3). It is important in radio communications and many kinds of electronic measuring instruments. Thus the work in this thesis may be of fairly general use.

It is possible to visualise the effect of noise in an oscillator in several different ways. We will first consider how oscillations build up when an oscillator is switched on. Figure 1.2 shows any sinusoidal feedback oscillator which consists of three essential elements; an amplifier, a limiter and a filter. A single valve or transistor often acts as both amplifier and limiter and its output is fed back via the filter to its input. Suppose the
oscillator is switched on by making some connection which does not
generate any transients, then noise in the oscillator is amplified and
filtered many times by the regenerative action of the oscillator and starts
to build up into a sinusoidal signal. The oscillator signal can thus be
considered to be generated by noise. As the oscillations grow the active
device is driven into non-linear operation and less energy is fed back into
the tank circuit. The amplitude of oscillation eventually reaches a steady
value when the energy being dissipated by the tank circuit equals the energy
being fed in by the active device. This model of an oscillator emphasises
that the signal and noise are inseparable. It is also essential to include
the limiting action of the active device in any model of an oscillator for a
complete understanding of its operation.

Now consider an oscillator from a different viewpoint. An ideal
noiseless oscillator would produce a perfectly periodic signal with a con­
stant amplitude and frequency and also a perfect line spectrum. If we
let the oscillator become a real one then noise will disturb the signal and
it will no longer be perfectly periodic. The line spectrum will be broadened
by the noise and the amplitude and frequency will fluctuate randomly about
their mean values. The oscillator signal can be considered to be frequency
and amplitude modulated by noise. This leads one to think of an oscillator
signal as a carrier with noise modulation sidebands. This model of an
oscillator generating a signal which is disturbed by noise lends itself more
easily to analytical treatment than the model in which the signal is generated
by noise.
We have seen how the effect of noise can be observed in several different ways. The power spectrum of the oscillator signal can be measured with a spectrum analyser. Alternatively the oscillator signal can be passed into an amplitude detector, phase detector or frequency discriminator and the power spectrum of the noise modulation measured. The effect of noise can also be observed in the time domain by using a frequency counter. A different frequency will be measured by the counter each time it is used, because the oscillator signal is not perfectly periodic. Any of the above mentioned effects can be used to characterise the noise performance of an oscillator. Usually a method of characterisation is chosen which can be directly related to the performance of the system in which the oscillator is used. There is no universally accepted way of characterising oscillator noise and the different methods which are used cannot always be related to one another.

When measuring oscillator noise, care must be taken to ensure that noise generated by the measuring equipment does not affect the results. If a spectrum analyser is used, oscillator noise produced by its local oscillator is added to the oscillator noise of the signal being measured. Commercial spectrum analysers are usually too noisy in this respect. The noise sidebands of variable frequency oscillators can be measured with a specially built spectrum analyser employing a crystal local oscillator. Oscillator noise is usually an order of magnitude less in a crystal oscillator than the best variable frequency oscillator. Other methods of measuring oscillator noise usually make use of a second oscillator and thus similar difficulties arise.
The effect of noise on the performance of an amplifier is well understood and widely applied. The same cannot be said for an oscillator which is a more complicated type of mechanism. An amplifier can be considered as a linear system and the signal and noise dealt with separately by linear analysis. The noise performance of the amplifier can be simply characterised by the signal to noise ratio. In an oscillator, as we have seen, the signal and noise are inseparable and the effects of the noise can be characterised in many different ways. The essential non-linear properties of an oscillator make it necessary to use non-linear differential equations when analysing the performance of an oscillator.

There are several types of oscillator noise which are generated in different ways by different types of electrical noise. This makes it difficult to draw conclusions from experimental results unless the type of oscillator noise involved is known. While redesigning old valve signal generators Marconi Instruments found that bipolar transistor oscillators were much noisier than valve oscillators, even though both devices had the same noise figure when used as an RF amplifier. The noise figure of a valve or transistor at RF frequencies is mainly determined by the amount of shot noise. Usually flicker noise rather than shot noise is responsible for the noise performance of an oscillator. The effect of flicker noise is usually greater in bipolar transistor oscillators than valve oscillators. The flicker noise varies the input capacitance of the valve or transistor which, because it is part of the capacitance of the frequency determining network of the oscillator, causes noise frequency modulation.
The importance of flicker noise was not appreciated at the start of this research project. It was thought that the difference between valve and transistor oscillators was due to differences in the amplitude limiting. The limiting action in a transistor oscillator is much harder than for a valve oscillator. The very abrupt change between the active region and saturation of a transistor makes it a hard limiter. Non-linearities in an oscillator were thought to be undesirable because they caused the noise to be mixed in with the oscillator signal. It was thus thought that a perfectly linear oscillator with a feedback level control system to maintain a constant amplitude, should be less noisy because the noise is not mixed in with the signal. This has been found to be incorrect because the amplitude tends to be unstable in a perfectly linear oscillator. Some non-linearities in an oscillator are undesirable because they do mix noise in with the signal but others are essential for stabilising the amplitude and suppressing amplitude noise.

To understand an oscillator it must be considered as a non-linear system. Unfortunately, this makes the analysis of an oscillator difficult. Even the simple oscillator model used by Van der Pol, (1.5) in which the non-linearity is approximated by a cubic law, leads to a non-linear differential equation. When the effect of noise is also included then the non-linear differential equation has a stochastic forcing function. Such equations can be considered to represent a Markov process, provided the correlation time of the noise is small compared to the correlation time of the oscillator signal. Then it is possible to solve the equations by use of the Fokker–Planck equation. This method of solution although fairly
simple in concept and mathematically very sound involves the solution of some difficult partial differential equations. It is easier to use the linearisation method (1.4) for finding the effect of noise on an oscillator. This method does not place any restriction on the correlation time of the noise but the noise must be relatively small. The departure of the amplitude of oscillation from its steady state value is then very small and can be represented by a linear stochastic differential equation which is easier to deal with than a non-linear one. The linearisation method is lengthy and difficult but the mathematical techniques used are probably easier and more familiar to the Electrical Engineer than those needed for use of the Fokker-Planck equation.

Many difficult equations can easily be solved numerically by using a digital computer for simulation. Equations with non-linearity can be dealt with and the effects of noise added by using fast random number generators. Unfortunately this attractive method is not suitable for studying the behaviour of oscillators. Both very short and very long time constants are involved in the operation of a sinusoidal oscillator. The short time constant is the period of oscillation and the long one the relaxation time of the oscillator amplitude. Long execution times are required for simulation programs involving both short and long time constants. For a typical sinusoidal oscillator, several months of computer time would be needed for one simulation. This could be reduced if it was possible to simulate only the amplitude and phase rather than the complete oscillator signal. This cannot be done because the complete signal is needed for the
effects of the non-linearities to be properly included.

In the next three chapters the effect of noise in an oscillator is analysed. There are many different types of oscillators and several types of oscillator noise which require consideration. An attempt has been made to deal with all of these possibilities without presenting a confusing collection of many different pieces of analysis. Chapter 2 shows that all VHF sinusoidal oscillators can be considered to be of two basic types, which exhibit only three types of oscillator noise. Thus the detailed analyses which follows in Chapters 3 and 4 can thus be applied to many different types of oscillators.

Chapter 5 shows how oscillator noise can be measured and characterised. It describes a test set for measuring oscillator noise which has been built for this research. In Chapter 6 the theoretical results are compared with the performance of some experimental oscillators. A reasonable agreement is obtained between the theoretical and experimental results. The theoretical noise performance cannot be found precisely because it is difficult to measure accurately the necessary parameters of a real oscillator. However, the theoretical and experimental results contained in this thesis should enable low noise oscillators to be designed in a much more systematic way. At present a lengthy process of trial and error is used to make a low noise oscillator.


CHAPTER 2

THE THEORY OF NOISE IN OSCILLATORS

The effect of noise on an oscillator depends on the type of noise, its location in the oscillator circuit and the type of oscillator. There are a great many types of oscillators and several different types of noise. Hence the problem being studied is complicated because of the large number of possibilities which require consideration. A detailed study of every possibility would take too long and is fortunately unnecessary. Many apparently dissimilar oscillators are affected by noise in a similar way and can be dealt with by the same analysis. This chapter attempts to clarify this situation by dealing with oscillator noise in a general way.

The different types of oscillator noise are described in Section 2.1 so that in later chapters each type of oscillator noise can be analysed separately. In Section 2.2 different oscillator configurations are compared while in Section 2.3 the non-linear aspect of an oscillator is considered.

2.1 Different Types of Oscillator Noise

Three basic types of oscillator noise can be identified by considering the way noise can act in an oscillator. This is a better method of classification than one based on the type of noise source. For example, shot noise and thermal noise can have the same effect if they occur in the same part of the oscillator circuit. Two types of oscillator noise are shown in Figure 2.1 which represents any type of sinusoidal feedback oscillator.
Noise which is added to the oscillator output signal is the simplest type of oscillator noise. The effect of this is relatively easy to establish because the oscillator can be considered as a linear system. This type of noise is responsible for the wide band low level part of the power density spectrum of an oscillator.

Noise which is added to the oscillator inside the feedback loop is of a second and more complicated type. It can only be understood when the oscillator is considered as a non-linear feedback system. The effect of the noise can be visualised by considering the noise as a series of random impulses which disturb the oscillator signal so that it is not perfectly periodic. Noise pulses occurring near to a maximum or minimum of the oscillator signal mainly affect the amplitude of the oscillations. They cause the amplitude to deviate from its steady state value in a random manner. The amplitude limiting mechanism, which every oscillator has, tends to correct these random deviations and reduce the effect of the noise on the amplitude. Noise pulses occurring near to the zero crossings of the oscillator
signal affect the phase of the oscillator. There is no mechanism in an oscillator to correct this, so that the phase performs a random walk due to the effect of the noise.

The third type of noise also acts inside the feedback loop but it modulates the signal by varying the parameters of the oscillator rather than just adding to the signal. Frequency modulation is caused by noise which varies the capacitance or inductance of the oscillator tank circuit. Amplitude modulation is caused by noise which varies the loop gain or limiting mechanism of the oscillator. Fortunately most of this noise modulation appears to be due to low frequency noise which is easier to deal with than noise at frequencies comparable with the oscillator frequency. The noise modulation is responsible for the part of the oscillator power spectrum closest to the carrier frequency.

We will look at all three types of oscillator noise in later chapters, but we will devote most of our attention to the second type of noise. Noise which is added to the oscillator signal inside the feedback loop is mathematically the most difficult to analyse and thus requires more attention. Much of the published theoretical work on oscillator noise deals exclusively with this second type of noise. One is thus inclined to think that it is the most important type of oscillator noise in practical oscillators. However, its effect is often dominated by the other types of oscillator noise.

2.2 Comparison Between Different Oscillator Configurations

In this section we will compare different types of oscillator configurations assuming that only noise added inside the feedback loop is important. So that this can be done without using difficult mathematics, the oscillators are assumed
to be linear with unity loop gain. This is acceptable if one wants to compare the magnitude of oscillator noise for different types of oscillators. It will be shown that the analysis of many apparently dissimilar oscillators leads to the following second order equation.

\[ V \left( p^2 + \omega_0^2 \right) = E_{n1} + pE_{n2} \quad \ldots \quad (2.1) \]

The voltage in the tank circuit is \( V \) and the angular frequency of oscillation is \( \omega_0 \). The terms \( E_{n1} \) and \( E_{n2} \) are due to noise and for an 'ideal' noiseless oscillator they are zero. Noisy oscillators can be identified because their values for \( E_{n1} \) and \( E_{n2} \) are large compared with the values for good low noise oscillators.

Equation 2.1 can be derived for many types of oscillators, so that the effect of noise need only be fully analysed once to establish the noise properties of many oscillators. With some of the more complicated types such as Colpitts and Hartley oscillators, third order equations are obtained which have a \( p^3 \) term. The effect of noise in these types of oscillators cannot be found by the methods which are generally used for the simpler oscillators with second order equations. We shall concentrate most of our attention on oscillators with second order equations but later in Chapter 3 one special method of analysis will be used to extend the analysis to third order equations.

2.2.1 RC and LC Oscillators

Sinusoidal oscillators can be divided roughly into two types. Those which have inductors and capacitors to determine their frequency and those which have resistors and capacitors. One of each type will be compared.
Figure 2.2: A simple type of LC oscillator

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ \bar{e}_n^2 = 4kT\tau \Delta f \]

Figure 2.3: Wien - bridge oscillator

\[ \bar{e}_{ni}^2 = \bar{e}_{n2}^2 = 4kT\tau \Delta f \]

\[ \omega_0 = \frac{1}{RC} \]
Each will be treated as a linear system since this is sufficient for a comparison of the relative magnitudes of the effects of noise. A simple type of LC oscillator is shown in Figure 2.2. It is assumed that thermal noise in the tank circuit is the only significant noise source.

The power supply decoupling and biasing components have been left out to simplify the circuit. A field effect transistor is chosen for this oscillator but a valve or bipolar transistor could have been used. Analysis of this circuit leads to the equation shown below.

\[ \sqrt{\left[p^2 + \rho \left(rC - Mq_m\right)\omega_0^2 + \omega_o^2 \right]} = \omega_o^2 E_n \]

The oscillator is considered to be a linear system so that for oscillations of constant amplitude we have :-

\[ rC = Mq_m \]

\[ \therefore \sqrt{\left(p^2 + \omega_o^2 \right)} = \omega_o^2 E_n \]  \( \ldots \ldots \) (2.2)

This is an over-simplification but will suffice for the moment. The following numerical values are typical of a 10 MHz oscillator and will be used for estimating the magnitude of \( e_n \).

\[ C = 100 \mu F \quad Q = \frac{1}{(\omega_0 rC)} = 100 \quad \tau = 1.6 \mu s \]

Now let us consider the Wien bridge oscillator shown in Figure 2.3 as a typical example of the many kinds of RC oscillators. It is assumed that the amplifier used is noiseless with zero input impedance and a transfer impedance \( A_z \).
Analysis of this circuit gives the following equation.

\[ V\left[ p^2 + p\left(3RC - A_2C\right)\omega_o^2 + \omega_o^2 \right] = \omega_o^2 E_{n1} + \omega_o^2 E_{n2} + p\omega_o E_{n2} \]

When the gain is adjusted for oscillations of constant amplitude we have:

\[ V\left[ p^2 + \omega_o^2 \right] = \omega_o^2 E_{n1} + \omega_o^2 E_{n2} + p\omega_o E_{n2} \quad \ldots (2.3) \]

The following numerical values are typical for a 10 MHz oscillator.

\[ C = 100 \text{ pF} \quad R \equiv 160 \Omega \]

The same frequency and capacitance is chosen for the Wien bridge and LC oscillator in an attempt to make a fair comparison between two different types of oscillators. If we compare equations 2.2 and 2.3 we see that they have the same form but the Wien bridge oscillator equation 2.3 has two extra noise terms. Let us compare the \( \omega_o^2 E_n \) term in 2.1 and the \( \omega_o^2 E_{n1} \) term in 2.3.

\[ \overline{e_n^2} = 4kTR\Delta f \quad \overline{e_{n1}^2} = 4kTR\Delta f \]

For typical 10 MHz oscillators \( r \) is 1.6 \( \Omega \) and \( R \) is 160 \( \Omega \). Thus the RMS noise is some 10 times bigger in the Wien bridge oscillator than the LC oscillator. When the other noise terms are also included then the Wien bridge oscillator is much worse than the LC oscillator.

It is not surprising to find that an oscillator using resistors instead of inductors is noisier simply because the thermal noise voltage from the resistor is greater than the inductor. Practical experience has shown that there is a definite order of oscillator types, according to the magnitude of noise effects in them (2.1).
Figure 2.4: Valve L C oscillator with tank circuit at grid

\[ i_q = v g_m + i_s(t) \]
\[ \omega_0 = \frac{1}{\sqrt{LC}} \]
\[ \overline{i_n^2} = \frac{4kT\Delta f}{R} \]

Figure 2.5: Valve L C oscillator with tank circuit at anode

\[ i_q = v g_m + i_s(t) \]
\[ \omega_0 = \frac{1}{\sqrt{LC}} \]
\[ \overline{i_n^2} = \frac{4kT\Delta f}{R} \]
Crystal oscillators are less noisy than LC oscillators and RC oscillators are the noisiest. Since we are interested in how low noise variable frequency oscillators can be made, then the LC oscillator is the type which will be considered in detail.

2.2.2 Minimising the Effect of Shot Noise

So far we have considered that only thermal noise is significant. Let us now include noise from the active device which for this example is a valve with shot noise. It will also be instructive to choose this time a parallel loss resistance representation. The circuit of the simple LC oscillator is shown in Figure 2.4. This circuit gives:

\[ V \left[ \frac{g_m}{2} + p \left( \frac{1}{R} - M \omega_c \right) \omega_c^2 + \omega_0^2 \right] = \frac{P}{C} \left( I_n + \frac{M}{L} I_s \right) \quad \cdots (2.4) \]

Consider now the relative magnitudes of \( i_n(t) \) and \( i_s(t) \). Van der Ziel (2.3) gives the following formulae for shot noise.

\[ \overline{i_s^2} = 4kT \lambda g_m \Delta f \]

where \( \lambda \approx 2.5 \) for an average triode.

If the oscillator used a field effect transistor instead of a valve, then the following similar expression can be used.

\[ \overline{i_s^2} = 4kT 0.7g_m \Delta f \]
For a constant amplitude of oscillations we have:

\[ \frac{L}{R} = M g_m \]

\[ \therefore \quad i_s^2 \sim \frac{4kT}{R} \frac{L}{M} \Delta f \]

The shot noise in equation 2.4 is reduced by a factor of \( M/L \) compared with the thermal noise. In order to compare the magnitudes of their effects we write:

\[
\left( \frac{M}{L} i_s(t) \right)^2 = \frac{4kT}{R} \frac{\lambda M}{L} \Delta f
\]

\[ = \frac{\lambda M}{L} i_n^2 \]

The following numerical values are typical of a 60 MHz oscillator and will be used to estimate the magnitude of \( \lambda M/L \).

\[ C = 10\,\mu F \quad \quad Q = \omega_0 RC = 100 \]

\[ R = 26.5\,k\Omega \quad g_m = 5\,mA/\nu \]

\[ \therefore \quad \frac{\lambda M}{L} = \frac{\lambda}{R g_m} = \frac{1}{50} \]

For this circuit the effect of shot noise is much less than thermal noise but this is not the case if the tank circuit is placed at the anode as shown in Figure 2.5.
This circuit gives:

\[ \sqrt{P^2 + P\left(\frac{L}{R} - Mq_m\right)\omega_0^2 + \omega_0^2} = \frac{P}{C} \left(I_n + I_s\right) \ldots (2.5) \]

Now the full shot noise occurs on the right hand side of the equation and its effect will be greater than that of thermal noise. For other types of oscillators such as Colpitts or Clapp it can also be shown that the effect of shot noise can be made negligible by placing the tank circuit at the grid. More complicated oscillators may require some analysis in order to see if the effect of shot (or other types of device noise) can be neglected. If they cannot, then an additional noise generator must be included in the analysis which is presented later on. Provided the additional noise can be considered to be stationary then this does not involve any extra difficulty. If the oscillatory current through the device is large, compared with the standing current, then it might not be satisfactory to consider the device noise as stationary. In a transistor for example the mean squared value of the shot noise is proportional to the emitter current, so that in a transistor oscillator, the noise varies with the oscillator signal. This effect modifies the oscillator noise but it is relatively small and will not be considered here. Generally thermal noise can be made predominant by placing the tank circuit at the input of the active device. Thus the effect of non-stationary noise is not of interest unless it is required to make very noisy oscillators.

2.2.3 Series and Parallel Loss Resistance Representations

Now compare equation 2.2 which was obtained from an oscillator with a series loss resistance representation of the tank circuit, with equation 2.4 which was obtained from a parallel loss resistance representation. The terms
due to thermal noise on the right hand side of these equations are not the same. This would at first appear to be a problem since we might choose either representation and expect to get the same results. It can be shown that the difference is superficial as long as the tank circuit has a high Q.

The RHS of equation 2.2 is $\omega_o^2 E_n$, which has a spectral density of $\omega_o^4 4kT$. The thermal noise term on the RHS of equation 2.4 is $\frac{P}{C} I_n$ and this has a spectral density of $\omega^2 4kT/(RC^2)$. In order to have the same Q for the series and parallel loss resistance representations, we must make $R = \omega_o^2 L^2/r$.

Using this the spectrum of $\frac{P}{C} I_n$ becomes $\omega^2 \omega_o^2 4kT \rho$ which is the same as the spectrum of $\omega_o^2 E_n$ for frequencies near to the oscillator frequency where $\omega = \omega_o$. It is only the noise near to the oscillator frequency which is important because noise at other frequencies is filtered out by the tank circuit.

Thus the use of either the series or parallel loss resistance representation will lead to the same end result although different equations are obtained at the start of the analysis.

2.3 The Non-linear Aspect of an Oscillator

2.3.1 Essential and Undesirable Types of Non-linearities

Linear circuit theory is frequently used for the analysis and design of oscillators (2.3). With this approach one can obtain the frequency of oscillation and the conditions for unity loop gain. The oscillator can then be built with its loop gain substantially greater than unity, so that it will always oscillate, even if its parameters change slightly with temperature, supply voltage or ageing.

The amplitude of the oscillations will always be limited by some non-linearity usually in the active device, even if no attention is paid to how this happens. The limiting action deserves more attention than it is usually given because it
can have a profound effect on the harmonic content, long term frequency stability, amplitude stability and noise of the oscillator. One phenomenon which can result from a neglect of the limiting mechanism is a periodic interruption of the oscillations called 'squegging'. This occurs in a valve oscillator when the grid goes into forward conduction on the peaks of oscillation and charges up the grid capacitance so that it biases the valve off. The oscillations then die down until the charge has leaked away and then the oscillations start building up again.

Consider the operation of the amplitude limiting mechanism when the amplitude of oscillation is altered by some transient in its power supply. If the oscillator is properly designed then the amplitude should be quickly brought back to its steady state value by the limiting mechanism. Noise can be considered to be a series of random transients which produce random disturbances of the oscillator amplitude. This is counteracted by the limiting action in the oscillator. Thus the non-linearities in the oscillator which are responsible for limiting the amplitude of oscillations are beneficial. Other non-linearities which have a mixing action are undesirable. Noise at frequencies far away from the frequency of oscillation is mixed in with the signal by certain types of non-linearities.

In order to demonstrate which types of non-linearities are undesirable and which are beneficial, we shall use a power series representation for the non-linear active device in the oscillator. If $v_1$ is the input and $i_o$ the output of the active device then $i_o = \alpha v_1 + \beta v_1^2 - \gamma v_1^3 + \delta v_1^4 - \epsilon v_1^5 \ldots \ldots$ (2.6)
The output of the active device is fed back to its input via the tank circuit, so that the arrangement of the oscillator is as shown in Figure 2.6.

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

**Figure 2.6 : Non-linearities in a feedback oscillator**

Due to the filtering action of the tank circuit the input to the active device is approximately a sine wave of angular frequency \( \omega_0 \). Therefore we have:

\[ v_i = A \sin \omega_0 t \]

In order to see how flicker noise is mixed in with this signal, we represent part of the flicker noise spectrum occurring at the input of the active device by a sine wave of angular frequency \( \omega_1 \). Now we have:

\[ v_i = A (\sin \omega_0 t + B \cos \omega_1 t) \quad \cdots 2.7 \]

where \( B \ll 1 \quad \& \quad \omega_1 \ll \omega_0 \)

The output of the active device can be found by using equations 2.6 and 2.7.
Terms involving $B^2$ and higher powers of $B$ can be neglected so that:

$$i_o = \alpha A (\sin \omega_0 t + B \sin \omega_1 t) + \beta A^2 \left[ \frac{1 - \cos 2\omega_0 t + B \cos (\omega_1 - \omega_0) t}{2} \right] - B \cos (\omega_1 + \omega_0) t$$

$$- \gamma A^3 \left[ \frac{3}{4} \sin \omega_0 t - \frac{1}{4} \sin 3\omega_0 t + \frac{3}{4} B \sin (2\omega_0 - \omega_1) t \right] - \frac{3}{4} B \sin (2\omega_0 + \omega_1) t + \frac{3}{2} B \sin \omega_1 t$$

$$+ \delta A^4 \left[ \frac{3}{8} - \frac{1}{2} \cos 2\omega_0 t + \frac{1}{8} \cos 4\omega_0 t + 3B \cos (\omega_1 - \omega_0) t \right] - 3B \cos (\omega_1 + \omega_0) t - B \cos (3\omega_0 - \omega_1) t + B \cos (3\omega_0 + \omega_1) t$$

$$- \varepsilon A^5 \left[ \cdots \right]$$

The tank circuit will remove all frequencies except those around $\omega_o / 2\pi$ so that $v_o$ can be written as:

$$v_o = Z A \left( \alpha - \gamma A^2 \frac{3}{4} - \varepsilon A^4 \cdots \right) \sin \omega_0 t$$

$$+ Z A^2 B \left( \beta + 3\delta A^2 + \cdots \right) \left[ \cos (\omega_0 - \omega_1) t - \cos (\omega_0 + \omega_1) t \right] \cdots \tag{2.8}$$

This expression contains amplitude modulation sidebands in addition to the wanted signal frequency of $\omega_o / 2\pi$, both above and below the wanted signal at difference frequencies of $\omega_1 / 2\pi$. If the full flicker noise spectrum was added to the input signal then at the output the signal would have continuous noise modulation sideband. The shape of the noise sideband spectrum would be the same as the flicker noise spectrum. It can be seen by examining
equation 2.8 that the existence of the noise modulation sidebands depends
on $\beta$, $\gamma$, ... the coefficients of the even powers in the power law represen-
tation of the non-linear device. If all the coefficients of even powers are zero then there is no modulation due to low frequency noise or inter-
ference and we get:

\[ v_o = Z A \left( \alpha - \gamma A^2/4 - \varepsilon A^4 \ldots \right) \sin \omega t \quad \ldots \tag{2.9} \]

Consider now how the non-linearity limits the amplitude of oscillations.

When $A$ is small the gain of the active device is $\alpha$. As the signal amplitude $A$ grows the gain falls due to the negative coefficients $\gamma$, $\varepsilon$ of the odd powers of $V_i$. The steady state amplitude is reached when the gain of the active device has fallen to equal the loss in the tank circuit. Clearly the existence of the coefficients of the odd powers is essential for correct limiting in the oscillator.

It has been shown that the ideal non-linear active device has a power series representation with only the coefficients of odd powers non-zero. Such a device has perfectly symmetrical characteristics. The relationship between the input and output is a perfect odd-function when all the coefficients of even powers are zero.

2.3.2 Instantaneous Limiting and Delayed Amplitude Limiting

The amplitude of oscillations in an oscillator can be controlled by instantaneous non-linearities as we have already seen. It can also be con-
trolled by delayed amplitude limiting. In a low frequency oscillator this can be done by placing a lamp or thermistor in the feedback loop. The heating
effect of the oscillator signal changes the resistance of the lamp or thermistor. This can be arranged to control the level of oscillation (2.4). This form of amplitude limiting does not act instantaneously because of the thermal capacity of the lamp or thermistor. Lamps and thermistors are not used for radio frequency oscillators so we will not wish to consider them further. They are unsuitable because of their high stray reactance. They tend to be noisy devices which should be avoided in a low noise oscillator.

Delayed amplitude limiting is frequently used in LC oscillators. It can be included very easily in a valve oscillator by using grid-leak biasing. This form of delayed amplitude limiting is not always fool proof and can cause 'squegging' as mentioned earlier. A more complicated delayed amplitude limiting system can be obtained by detecting the amplitude of oscillations in the tank circuit, low-pass filtering this voltage and using it with a servo amplifier to control the gain of the active device. This kind of system has particular advantages with crystal oscillators where it is important to keep the oscillations to a very low level to avoid temperature rises in the crystal. Instantaneous non-linearities cannot always be used to limit the amplitude at a low level because the characteristics of most devices are linear for signals up to about half a volt.

Delayed amplitude limiting does not generate harmonics like instantaneous limiting and for this reason it is always used in low distortion oscillators. We are primarily concerned with low noise radio frequency oscillators. It is not clear whether instantaneous or delayed amplitude limiting produces the best low noise oscillator and so both will need to be analysed in detail later on.
It may be argued that with delayed amplitude limiting the oscillator is perfectly linear so that noise at frequencies far away from the frequency of oscillation cannot be mixed in with the signal. However, this advantage might be offset by the possibility of noise in the servo amplifier producing noise in the oscillator signal. Also, because disturbances to the amplitude are corrected more slowly in a delayed amplitude limiting oscillator, then the instantaneous limiting oscillator may have less amplitude noise.

When instantaneous limiting is used the loop gain can be made just large enough for oscillations to start and then some slight non-linearities in the active device will limit the amplitude. This is a 'soft' limiting oscillator and it will have a low harmonic content. Alternatively the loop gain can be made very much larger than is necessary for oscillations to start, and the active device biased so that its non-linearities are very abrupt. This can be done by letting the gate of an FET (or grid of a valve) go into forward conduction on the peaks of oscillations. This kind of oscillator has 'hard' limiting and a high harmonic content. A hard and soft limiting oscillator can have the same amplitude of oscillations in the tank circuit but exhibit very different properties. The soft limiting oscillator is in a marginally stable state so that any change in the parameters of the active device due to a temperature change could stop it oscillating or radically change its amplitude. The hard limiting oscillator is less likely to be affected by temperature changes and we will see later that it is less affected by noise. The hard limiting oscillator produces larger harmonics which is sometimes a disadvantage.
When an oscillator is required with good amplitude stability and low harmonics then a combination of soft limiting and slow delayed amplitude limiting can be advantageous (2.5). This type of oscillator has all the advantages of a soft limiting oscillator but without its disadvantages. With soft instantaneous limiting the harmonic content is small. The slow delayed amplitude limiting keeps the loop gain of the oscillator slightly greater than unity. Changes in the parameters of the oscillator, due to temperature variations, do not affect the operation of the oscillator as in a simple soft limiting oscillator. The effect of noise, transients and power supply harmonics on the amplitude is controlled by the soft instantaneous limiting. These effects can usually be made smaller with soft instantaneous limiting than with delayed amplitude limiting.

In this thesis only the simple instantaneous limiting and delayed amplitude limiting oscillators are analysed in detail. The author has not found sufficient time to consider oscillators which employ a combination of the two types of limiting.
References for Chapter 2


2.3 FRASER, W., 'Telecommunications', MacDonald, 1967, Chapter 4.


CHAPTER 3

THE EFFECT OF WHITE NOISE INSIDE AN OSCILLATOR

The previous chapter has prepared the ground for the following detailed analysis. This chapter will deal with white noise which is added inside the feedback loop of the oscillator. Most of the analysis concentrates on one simple type of oscillator configuration but as shown in the last chapter, the results obtained for this one oscillator are similar to those which would be obtained for many other oscillators. Both instantaneous and delayed amplitude limiting will be considered and several methods of analysis will be used. Some of these methods are not new but the significance of their results can be seen more clearly here when the different methods of analysis are compared.

Normally it is difficult to compare the results of different methods of analysis because the results are presented in different ways and apply to different types of oscillators. Here the results will be presented in a form which allows a direct comparison to be made between the results for different oscillator models.

3.1 Instantaneous Limiting Oscillator using Van der Pol's Model

3.1.1 The Van der Pol Oscillator

In the Van der Pol model of an oscillator (3.1) the non-linear characteristics of a valve are represented by a power series. If the grid voltage is \( V \) then the anode current \( i_a \) is given by the following equation.
Figure 3.1  The Van der Pol oscillator
The oscillator circuit we will study is shown in Figure 3.1 and analysis of this gives the following equation.

\[ \frac{d^2v}{dt^2} + \frac{dv}{dt} \left( rC - M\alpha' + 2M\beta'v + 3M\gamma'v^2 \right) \omega_0^2 + \omega_0^2 v = \omega_0^2 e_n(t) \]

Before looking at the effect of the noise it will be necessary to review some of the results obtained by Van del' Pol. The differential equation can be written in general terms without the noise as shown below.

\[ \ddot{v} - \dot{v} (\alpha - \beta v - \gamma v^2) \omega_0^2 + \omega_0^2 v = 0 \quad \ldots (3.2) \]

\[ \alpha = M\alpha' - rC \quad \beta = 2M\beta' \quad \gamma = 3M\gamma' \]

From a physical knowledge of an oscillator we know that \( v \) is approximately sinusoidal of frequency \( \frac{\omega_0}{2\pi} \) and its amplitude changes slowly. Thus Van der Pol solves equation 3.2 by trying the following solution.

\[ v(t) = A(t) \cos \omega_0 t \quad \ldots (3.3) \]

When the oscillator is started at time \( t=-\infty \) with zero amplitude then Van der Pol found that the solution was approximately :—
\[ v(t) = \frac{2\sqrt{\frac{\alpha}{\alpha}}}{\sqrt{1 + e^{-\alpha t}}} \cos \omega_0 t \quad \ldots (3.4) \]

When \( \alpha t \ll -1 \) we get the exponentially rising cosine wave which linear analysis gives

\[ v(t) = 2\sqrt{\frac{\alpha}{\alpha}} e^{\frac{\alpha t}{2}} \cos \omega_0 t \quad \ldots (3.5) \]

When \( \alpha t \gg +1 \) we get:

\[ v(t) = 2\sqrt{\frac{\alpha}{\alpha}} \left[ 1 - \frac{1}{2} e^{-\alpha t} \right] \cos \omega_0 t \quad \ldots (3.6) \]

Thus if the oscillator is disturbed by a small amount after its amplitude has reached its steady state value, then the amplitude returns to the steady state value with a time constant of \( \frac{1}{\alpha} \). Noise can be considered to be a series of random impulses which will disturb the amplitude of the oscillator. Thus we would expect the equation for the response of the amplitude of the oscillator to noise to contain the time constant \( \frac{1}{\alpha} \). Later on we will see that this is so.

3.1.2 Noise in the Van der Pol Oscillator

Now we are ready to consider the effect of noise so that equation 3.2 is re-written with the noise \( \mathcal{E}_n \) added.

\[ \ddot{v} - (\alpha - \beta v - \gamma v^2)\omega_0^2 v + \omega_0^2 v = \omega_0^2 \mathcal{E}_n(t) \quad \ldots (3.7) \]
This is a second order non-linear stochastic differential equation which has been solved by many people who have studied the effect of noise in oscillators. (3.2) (3.3) (3.4) (3.5). Most of the methods used are essentially the same and start by transforming the second order equations into two first order equations. This is done by making the following substitutions.

\[ x = \nu \quad \& \quad y = \frac{1}{\omega_0} \frac{d\nu}{dt} \]

Therefore

\[ \dot{x} = \omega_0 y \quad \ldots \ldots \quad (3.8) \]

\[ \dot{y} = -\omega_0 x + \left(\alpha - \beta x - \gamma x^2\right)\omega_0^2 y + \omega_0 e_n(t) \quad (3.9) \]

Two new independent variables \( A(t) \) and \( \Theta(t) \) are now defined by the following equations.

\[ x = A \cos\Theta \quad \& \quad y = A \sin\Theta \quad \ldots \ldots \quad (3.10) \]

It can be shown that these functions are completely independent by finding their Jacobian which is not zero. (3.6). Looking back at equation 3.3 it can be seen that for a noiseless oscillator \( A(t) \) is the amplitude and \[ \frac{1}{2\pi} \frac{d\Theta(t)}{dt} \] the frequency of oscillation. For a noisy oscillator which is not perfectly periodic it is not clear how one can precisely define the amplitude and frequency. We will define the amplitude and frequency by equation 3.10 although this will lead to a difficulty later on. Substituting \( x \) and \( y \) into 3.8 and 3.9 gives :-
\[
\frac{dA}{dt} = (\alpha - \beta A \cos \Theta - \gamma A^2 \cos^2 \Theta) A \omega^2 \sin^2 \Theta + \omega_0 e_n(t) \sin \Theta
\]

\[
\frac{d\Theta}{dt} = -\omega_0 + (\alpha - \beta A \cos \Theta - \gamma A^2 \cos^2 \Theta) \omega_0 \sin \Theta \cos \Theta + \frac{\omega_0}{A} e_n(t) \cos \Theta
\]

We expect the noise to cause only small variations of \(A\) and \(\frac{d\Theta}{dt}\) from their steady state values \(A_0\) and \(\omega_0\) so we can write:

\[
A = A_0 + \alpha(t) \quad \Theta = -\omega_0 t + \phi(t)
\]

Therefore

\[
\dot{\alpha} = (\alpha - \beta A \cos \Theta - \gamma A^2 \cos^2 \Theta) A \omega^2 \sin^2 \Theta + \omega_0 e_n \sin \Theta
\]

\[
\dot{\phi} = (\alpha - \beta A \cos \Theta - \gamma A^2 \cos^2 \Theta) \omega_0^2 \sin \Theta \cos \Theta + \frac{\omega_0}{A} e_n \cos \Theta
\]

This is rewritten with the powers of trigonometrical functions replaced by multiple angles.

\[
\frac{\dot{\alpha}}{\omega_0^2} = A \left( \frac{\alpha}{2} - \frac{A^2 \gamma}{8} \right) - \frac{A^2 \beta}{4} \cos \Theta - \frac{A \alpha}{2} \cos 2\Theta
\]

\[
+ \frac{A^2 \beta}{4} \cos 3\Theta + \frac{A^3 \gamma}{8} \cos 4\Theta + \frac{\omega_0}{\omega_0} e_n \sin \Theta \quad \ldots 3.114
\]

\[
\frac{\dot{\phi}}{\omega_0^2} = -\frac{\beta A}{4} \sin \Theta + \left( \frac{\alpha}{2} - \frac{\gamma A^2}{4} \right) \sin 2\Theta - \frac{\beta A}{4} \sin 3\Theta
\]

\[
- \frac{\gamma A^2}{8} \sin 4\Theta + \frac{e_n \cos \Theta}{A \omega_0} \quad \ldots 3.11b
\]
We must try to interpret equations 3.11a and 3.11b and see how they can be used. One gives an expression for the rate of change of amplitude while the other an expression for the rate of change of phase. There are many complicated terms on the right hand sides of these equations but they are only of three types. Each equation has a 'stochastic' term which consists of the noise $e_n$ and a trigonometrical function of $\Theta$. In these stochastic terms the trigonometrical function acts only as a weighting function which varies the magnitude of the noise throughout the period of oscillation. In equation 3.11a for example the noise has no effect on $\psi(t)$ when the oscillator signal $\nu(t)$ is passing through its maximum because $\sin \Theta$ is zero but when the oscillator signal is near to zero and $\sin \Theta$ is one, then the noise has its greatest effect. The second type of term contains a trigonometrical function of $\Theta$ together with $A(t)$. These we will call 'oscillatory' terms because they cause rapid oscillations in $\psi(t)$ and $\phi(t)$. The third type of term contains only $A(t)$ and occurs only in equation 3.11a.

Let us consider a perfect noiseless oscillator which has been oscillating for some time and has reached a steady state. Then we have $e_n(t)$ zero and also expect $\psi(t)$ and $\phi(t)$ to be zero because there is no noise to disturb the amplitude and phase from their steady state values. An examination of 3.11 shows that $\phi(t)$ and $\psi(t)$ are not zero even if $e_n$ is zero but are oscillatory due to the presence of the 'oscillatory' terms. If we let the 'oscillatory' terms be zero then equation 3.11 becomes:
\[ \dot{a} = A \left( \frac{\alpha}{2} - \frac{A^2 \delta}{8} \right) \omega_0^2 \]

\[ \dot{\phi} = 0 \]

Now we can have \( a(t) \) and \( \dot{\phi}(t) \) zero provided that:

\[ A = \sqrt{\frac{4\alpha}{\delta}} \]

This result for the steady state amplitude agrees with the result of Van der Pol's analysis given by equation 3.4. It is only the 'oscillatory' terms in equation 3.11 which present any difficulty in the interpretation of the equation.

The 'oscillatory' terms have arisen because of the way \( A \) and \( \Theta \) were defined by equation 3.10. It was tacitly assumed that the oscillator signal did not contain any harmonic frequencies. Looking at equation 3.3 and 3.10 it is clear that \( A(t) \) and \( \frac{d\theta(t)}{dt} \) can only be constant if the oscillator signal \( \nu(t) \) which is equal to \( \chi \) is sinusoidal. If the oscillator signal contains harmonics as it must with non-linearities in the oscillator, then \( A \) and \( \Theta \) must be 'oscillatory'. Van der Pol's analysis neglects the harmonics of the oscillator signal because they are small and do not affect the utility of equation 3.4 which is concerned with the build-up of the oscillations. Since we are concerned with very small changes in \( A \) and \( \Theta \) then the harmonics become important even though they are small.
The 'oscillatory' terms in equation 3.11 must somehow be removed before the equation can be solved. Blaquiere (3.2) does this by 'smoothing' the equation because it is the slow random changes in A(t) and Θ(t) which are required and not their rapid oscillations. The 'smoothing' is accomplished by averaging the 'oscillatory' terms over one cycle which make them zero, provided we assume that A and Θ only change by a very small amount during one cycle. Stratonovich (3.3) uses a more rigorous mathematical technique for removing the 'oscillatory' terms. He uses an asymptotic method developed by Bogoliubov (3.7) which transforms equation 3.11 into a new amplitude A*(t) and phase Θ*(t) which are not oscillatory. These are not defined explicitly but are found by successive approximations and correspond to the amplitude and phase of the fundamental of the oscillator signal. It is only the fundamental which we are concerned with because this corresponds to the narrow band model which will be used in Chapter 5 to characterise the oscillator noise. Stratonovich's first approximation for A*(t) and Θ*(t) gives the same results as Blaquiere's method. For almost all practical purposes this result is sufficiently accurate so that we will not use Stratonovich's method.

Taking equation 3.11 and time averaging, we get :-
\[
\dot{a} = \omega^2 A \left( \frac{\alpha}{2} - \frac{A^2 \gamma}{8} \right) + \omega_0 e_n \sin \Theta \quad \ldots \ (3.12a)
\]

\[
\dot{\phi} = \frac{\omega_0 e_n}{A} \cos \Theta \quad \ldots \ (3.12b)
\]

Since we are concerned with oscillators which have only weak internal noise then the departure from the steady state due to noise is very small and it is possible to linearise equation 3.12a by using Taylor's series.

\[
F(A) = F(A_0) + \frac{\partial F(A)}{\partial A} \bigg|_{A=A_0} a(t)
\]

\[
F(A) = \omega^2 A \left( \frac{\alpha}{2} - \frac{A^2 \gamma}{8} \right) \quad \& \quad A_0 = \frac{4\alpha}{\gamma}
\]

Therefore
\[
F(A) = \omega^2 \sqrt{\frac{4\alpha}{\gamma}} \left[ \frac{\alpha}{2} - \frac{\gamma}{8} \left( \sqrt{\frac{4\alpha}{\gamma}} \right)^2 \right] + \omega_0^2 a(t) \left[ \frac{\alpha}{2} - \frac{3\gamma}{8} \left( \sqrt{\frac{4\alpha}{\gamma}} \right)^2 \right]
\]

\[
F(A) = -\alpha \omega_0^2 a(t)
\]

Substituting this result into equation 3.12 gives

\[
\dot{a} = -\alpha \omega_0^2 a + \omega_0 e_n \sin \Theta \quad \ldots \ (3.13a)
\]

\[
\dot{\phi} = \frac{\omega_0}{A} e_n \cos \Theta \quad \ldots \ (3.13b)
\]
Figure 3.2: Anode current plotted against grid voltage for an ECC88 valve, with the cubic equation used to represent it.
Stratonovich deals with the situation where strong external noise is added to the oscillator and under these conditions the linearization method breaks down. It is then necessary to make use of the Fokker–Planck equation and restrict the correlation time of the noise. This must be small compared to the relaxation time of the oscillator. We will not consider this situation since we are concerned with low noise oscillators.

By solving equation 3.13 we can write expressions for the amplitude and phase.

\[
\phi = \int_0^t \frac{\omega_0}{A_o} \cos \Theta e_n(t) \, dt \quad \ldots \quad (3.14a)
\]

\[
a = e^{-\alpha \omega_0^2 t} \int_0^t e^{\alpha \omega_0^2 t} \omega_0 \sin \Theta e_n(t) \, dt \quad \ldots \quad (3.14b)
\]

These results can be used to find the spectrum of \(a\) and \(\phi\) and the RF spectrum of the oscillator. Before this is done we will consider a different model of an instantaneous limiting oscillator which will give a similar equation to 3.14.

3.1.3 Limitations of the Van der Pol Model

We will now look at how the Van der Pol model of an oscillator can be applied to a real oscillator. Let the oscillator shown in Figure 3.1 be made with half of an ECC88 doubletriode valve with an anode voltage of 150 volts. Over a limited range of grid voltage the following cubic equation agrees closely with the characteristics of the valve as shown in Figure 3.2.
\[ i_a = 0.062 + 0.00975v_g - 0.004v_g^2 - 0.00075v_g^3 \] ... (3.15)

Where \( i_a \) is the anode current and \( v_g \) the grid voltage. It is assumed that the oscillatory voltage at the anode is sufficiently small to be neglected.

The cubic equation was chosen so as to agree exactly with the real characteristics of the valve at grid voltages of \(-1\), \(-2\), \(-3\), and \(-4\) volts. Within the range \(-0.5\) volts to \(-4\) volts the cubic equation follows closely to the real characteristics but outside this range the two curves diverge.

It is the relation between the signal voltage at the grid \( v \) and the signal current out of the anode \( i \) which we require. Let the valve be biased with \(-2\) volts on the grid so that the standing current through the valve is 32.5 mA. Then we have:

\[ v_g = v - 2 \quad \& \quad i_a = i + 0.0325 \]

Equation 3.15 can now be written in terms of the signal so that we obtain:

\[ i = 0.01175v - 0.001v^2 - 0.00075v^3 \]

Looking back to equation 3.1 we see that we have the following numerical values:

\[ \alpha' = 0.01175 \quad \beta' = 0.001 \quad \gamma' = 0.00075 \]
From equation 3.4 the steady state amplitude can be found.

\[ A_\infty = 2 \sqrt{\frac{\alpha'}{\delta}} = 2 \sqrt{\frac{M\alpha' - rC}{3M\delta'}} \]

If we assume that \( M\alpha' \gg rC \) then the maximum possible amplitude is obtained, which is given by:

\[ A_\infty = 2 \sqrt{\frac{\alpha'}{3\delta'}} \approx 4.56 \text{ volts} \]

Looking at Figure 3.2 we can see that the cubic equation follows fairly closely to the characteristics of the valve when the grid voltage varies between 0 and -4 volts. If the Van der Pol oscillator is used to represent a real oscillator and the grid is biased with -2 volts, then the amplitude of oscillation should be less than 2 volts. This is to ensure that the behaviour of the Van der Pol model is the same as that of the real oscillator. If the amplitude is to be less than 2 volts, then the difference between \( M\alpha' \) and \( rC \) must be relatively small. Unfortunately this makes the amplitude very sensitive to changes in \( \alpha' \) and \( r \). Changes in thermionic emission will affect \( \alpha' \) and changes in the Q of the tank circuit affect \( r \). Thus a real oscillator only corresponds to the Van der Pol oscillator when its limiting is soft. Such an oscillator is normally undesirable because its amplitude
is sensitive to changes in its parameters. Possibly the only use which has been made of such an oscillator is for nuclear resonance absorption measurements (3.8).

If one attempts to fit a cubic equation to the characteristics of the ECC88 by using the points at grid voltages of -2, -3, -4 and -5 then an entirely different equation is obtained. For these points the coefficient of the cubic term $\gamma'$ is found to be positive rather than negative. According to the Van der Pol model this result allows the amplitude of oscillation to grow exponentially to infinity. This clearly shows that the Van der Pol model does not give reliable quantitative result. It is however very useful in giving a quantitative explanation of many of the effects occurring in valve oscillators. The phenomenon of locking, hysteresis and oscillation and two frequencies can be explained by using the Van der Pol model (3.1). The noise behaviour of an oscillator is also well explained in a quantitative manner by the Van der Pol model as we shall see. The limiting action in most instantaneous limiting valve oscillators is due to forward conduction of the grid rather than the gentle curvature of its characteristics. This is a 'hard' type of limiting compared to the soft limiting of the Van der Pol Oscillator and thus results in a more stable oscillator. The Van der Pol model can be used for oscillators which use semiconductor devices but as with valves it is usually some abrupt change rather than a change of curvature which is responsible for limiting the amplitude.
\[ |i(t)| = I_o \]

Figure 3.3: The hard limiting oscillator
We have used the Van der Pol model for a simple type of oscillator and obtained a second order non-linear differential equation. This we could solve by splitting it into two first order non-linear differential equations. If the Van der Pol model is used with the more complicated types of oscillators then third or higher order non-linear differential equations are obtained. These cannot be split into two first order equations so that the Van der Pol model is only useful for oscillators with second order equations. Many important types of oscillators such as Colpitts Clap and Hartley cannot be analysed by using the Van der Pol model. It is possible that the effect of noise in these more complicated types of oscillators is fundamentally different to the effect of noise in the simple oscillator. Thus a method of analysis is required which can be used for the more complicated types of oscillators.

3.2 Instantaneous Limiting Oscillator using Hard Limiting Model

3.2.1 The Hard Limiting Oscillator Model

In this section a hard limiting oscillator model is used to represent an oscillator in which the limiting action is due to abrupt changes in the characteristic of the active device. This type of model has as far as the author knows, not been used before for analysing the effect of noise in oscillators. If the active device in an oscillator is driven fairly hard then it will feed pulses back into the oscillator rather than a sine-wave. The active device is then not unlike a high gain limiter which changes state at every zero crossing of its input signal. Figure 3.3
Figure 3.4: Instantaneous limiting in a real oscillator
shows the hard limiting oscillator model we shall use.

Robinson (3.8) has built oscillators which deliberately incorporate a high gain limiter and correspond almost exactly to the hard limiting model in Figure 3.3. These oscillators use three or four valves in cascade to give a high gain limiter which gives an almost perfect square wave output. Most high frequency oscillators only use one valve or transistor and thus do not correspond exactly to the hard limiting model in Figure 3.3. The current fed into the tank circuit is usually a distorted series of pulses. Consider the oscillator shown in Figure 3.4. The voltage waveform measured at the drain of the field effect transistor shows the waveform of the current through the device. The behaviour of this oscillator lies between that of the Van der Pol and hard limiting oscillator models and both models are probably equally appropriate for analysis of the real oscillator.

From the circuit in Figure 3.3 the following differential equation can be obtained.

\[ i_n(t) - i(t) \frac{M}{L} + C \frac{dV}{dt} + \frac{V}{R} + \frac{1}{L} \int V \, dt = 0 \]  (3.20)

If the oscillator is started at time \( t = t_0 \) by the limiter changing from \( -I_0 \) to \( +I_0 \) and for the moment the limiter is kept at \( +I_0 \), then a simple linear analysis can be used to solve equation 3.20. In order to find the effect of the noise \( i_n(t) \) we let one impulse of noise occur at time \( t = t_K \). This enables the noise to be dealt with very simply for the moment.
Later on we will find the effect of \( i_n(t) \) as a white noise current generator by considering \( i_n(t) \) to be a series of many random pulses.

The following expressions for \( i(t) \) and \( i_n(t) \) can be written:

\[
i_n(t) = a_k \delta(t - t_k) \quad \ldots \quad (3.21)
\]

\[
i(t) = 2I_o H(t - t_o) - I_o \quad \ldots \quad (3.22)
\]

For convenience we can drop \( I_o \) in 3.22 because \( i(t) \) is coupled into the tank circuit by mutual inductance and thus a constant current of \( I_o \) has no effect. Since the initial conditions at time \( t = 0 \) are zero the Laplace transform of equation 3.20 is easily found to be:

\[
I_n(p) - I(p) \frac{M}{L} + V(p) \left[ \frac{1}{pC} + \frac{1}{pL} \right] = 0 \quad \ldots \quad (3.23)
\]

Taking the Laplace transforms of equations 3.21 and 3.22 gives:

\[
I_n(p) = a_k e^{-p t_k} \quad \ldots \quad (3.24)
\]

\[
I(p) = \frac{2I_o}{p} e^{-p t_o} \quad \ldots \quad (3.25)
\]

From 3.23, 3.24 and 3.25 \( V(p) \) can be found.

\[
V(p) = \frac{2MI_o}{LC} e^{-p t_o} - \frac{a_k}{C} pe^{-p t_k} \frac{1}{p^2 + \frac{1}{pRC} + \frac{1}{LC}} \quad \ldots \quad (3.26)
\]
The inverse Laplace transform is then:

\[ v(t) = \left( \frac{2MI_o}{LC\omega_1} \right) e^{-\frac{1}{2RC}(t-t_o)} H(t-t_o) \sin \omega_1(t-t_o) \]

\[ - \left( \frac{a_k}{c} \right) e^{-\frac{1}{2RC}(t-t_k)} H(t-t_k) \left[ \frac{\cos \omega_1(t-t_k)}{2RC\omega_1} \sin \omega_1(t-t_k) \right] \]

... \(3.27\)

where \( \omega_1 = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}} \Rightarrow \omega_o = \sqrt{\frac{1}{LC}} \)

For a high Q tank circuit we can, to a very good approximation, let \( \omega_1 \) be \( \omega_o \). Without any loss of generality we can let \( t_o = 0 \). Thus the disturbance in the tank circuit of the oscillator caused by the limiter changing from \(-I_o\) to \(+I_o\) at time \( t = 0 \) is as written below.

\[ v(t) = 2\omega_o MI_o e^{-\frac{\omega_o t}{2Q}} \sin \omega_o t \]

\[ + H(t-t_k)e^{-\frac{\omega_o (t-t_k)}{2Q}} \left( \frac{-a_k}{C} \right) \left[ \frac{\cos \omega_o (t-t_k)}{2Q} \sin \omega_o (t-t_k) \right] \]

... \(3.28\)
To find the noise free oscillator signal we let the noise pulse be zero and allow the limiting amplifier to change over every time the signal in the tank circuit passes through zero. Each time the limiter changes state a new exponentially decaying sine wave is started. Thus the voltage in the tank circuit at a time $T$ is given by:

$$\nu(t) = Ke^{\frac{-\omega_0 t}{2Q}} \sin \omega_0 t - KH(t-\frac{\pi}{\omega_0})e^{\frac{-\omega_0}{2Q}(t-\frac{\pi}{\omega_0})} \sin \omega_0 (t-\frac{\pi}{\omega_0})$$

$$+ KH(t-\frac{2\pi}{\omega_0})e^{\frac{-\omega_0}{2Q}(t-\frac{2\pi}{\omega_0})} \sin \omega_0 (t-\frac{2\pi}{\omega_0})$$

$$- \ldots$$

where

$$K = 2\omega_0 MI_0$$

Now if the oscillator was started at $t = -\infty$ we would have:

$$\nu(t) = Ke^{\frac{-\omega_0 t}{2Q}} \sin \omega_0 t \left[ H(t+\frac{n\pi}{\omega_0})e^{\frac{-n\pi}{2Q}} + \ldots + H(t+\frac{\pi}{\omega_0})e^{\frac{-\pi}{2Q}} \right]$$

$$+ H(t) + \ldots$$

$$+ H(t-\frac{n\pi}{\omega_0})e^{\frac{n\pi}{2Q}}$$

$$n = 1, 2, 3, \ldots$$
If $0 < t < \frac{\pi}{\omega_o}$ we can drop the H functions and only retain terms up to $H(t)$.

\[ v(t) = K e^{-\frac{\omega t}{2Q}} \sin \omega_o t \left( 1 + e^{-\frac{\pi}{2Q}} + e^{-\frac{2\pi}{2Q}} + e^{-\frac{3\pi}{2Q}} + \ldots \right) \]

\[ \sum_{n=0}^{\infty} e^{-n\frac{\omega t}{2Q}} = \frac{1}{1 + e^{-\frac{\omega t}{2Q}}} \]

Hence:

\[ v(t) = \frac{2KQ}{\Pi} e^{-\frac{\omega t}{2Q}} \sin \omega_o t, \quad 0 < t < \frac{\pi}{\omega_o} \]

\[ v(t) = \frac{2KQ}{\Pi} e^{-\frac{\omega_o}{2Q}(t-\frac{\pi}{\omega_o})} \sin \omega_o t, \quad \frac{\pi}{\omega_o} < t < \frac{2\pi}{\omega_o} \]

\[ v(t) = v(t + \frac{2\pi}{\omega_o}) \quad \ldots \ldots (329) \]
This expression can be used to find the Fourier series representation of the oscillator signal by the usual method. For large values of $Q$ the oscillator signal is:

$$v(t) = A_0 \sin \omega_0 t + \frac{A_0}{4Q} \cos \omega_0 t - \frac{A_0}{24Q} \cos 3\omega_0 t - \frac{A_0}{60Q} \cos 5\omega_0 t - \ldots$$

As one would expect from the rotational symmetry of the signal only odd harmonics are present.

It has already been pointed out that one serious objection to the Van der Pol model is that it can only be used for simple types of oscillators. It cannot be used for most of the important and frequently used types of oscillators. The hard limiting model we are using here is far more flexible and can be used for Hartley, Clap and Colpitts oscillators. Appendix 8.1 shows how the Hartley oscillator can be analysed. The results are very similar to those obtained here for a simple tuned grid oscillator. Comparing equation 3.28 with equation 8.14 it can be seen that the main difference is in the magnitude of the coefficients. Thus all the
following work could equally well be applied to Hartley, Clap or Colpitts oscillators and the results would only differ in the magnitude of various coefficients. Note that the coefficient \((-a_k/c)\) for the simple oscillator becomes \((-a_k/c)(L_2 + M)/L\) for the Hartley oscillator. It is important to remember this change later on, although often \(L_2\) is almost equal to \(L\) so the error in using \((-a_k/c)\) for a Hartley might not be significant.

3.2.2 Noiise in the Hard Limiting Oscillator

In order to investigate the effect of noise, consider the oscillator to have been started at time \(t = 0\) and allow one noise impulse to cause a disturbance. Then we have equation 3.28 for the voltage between time \(t = 0\) and \(t = \frac{\pi}{\omega_0}\) which can be written as:

\[
\nu(t) = e^{\frac{-\omega_0 t}{2Q}} \left[ e^{\frac{\omega_0 t}{2Q}} H(t-t_k)(-\frac{a_k}{c})(\sin\omega_0 t_k - \frac{\cos\omega_0 t_k}{2Q}) + K \right] \sin\omega_0 t \\
+ e^{\frac{-\omega_0 t}{2Q}} \left[ e^{\frac{\omega_0 t}{2Q}} H(t-t_k)(-\frac{a_k}{c})(\cos\omega_0 t_k - \frac{\sin\omega_0 t_k}{2Q}) \right] \cos\omega_0 t
\]

Now if we consider the effect of many noise impulses we get:

\[
\nu(t) = \left[ a(t) + Ke^{\frac{-\omega_0 t}{2Q}} \right] \sin\omega_0 t + b(t) \cos\omega_0 t
\]
Where

\[ a(t) = \sum_{t_k=0}^{t} e^{-\frac{\omega_0}{2Q}(t-t_k)} \left( -\frac{q_k}{c} \right) \left[ \sin \omega_0 t_k - \frac{1}{2Q} \cos \omega_0 t_k \right] \]

\[ \ldots (3.31) \]

and

\[ b(t) = \sum_{t_k=0}^{t} e^{-\frac{\omega_0}{2Q}(t-t_k)} \left( -\frac{q_k}{c} \right) \left[ -\frac{1}{2Q} \sin \omega_0 t_k + \cos \omega_0 t_k \right] \]

\[ \ldots (3.32) \]

Since \( a(t) \) and \( b(t) \) are due to noise and thus very small we can write the voltage in the tank circuit as:

\[ v(t) = \left[ Ke^{-\frac{\omega_0 t}{2Q}} + a(t) \right] \sin \left( \omega_0 t + \frac{b(t)e^{\frac{\omega_0 t}{2Q}}}{K} \right) \]

\[ \ldots (3.33) \]

We see that the phase error \( \phi(t) \) accumulated after time \( t \) is given by:

\[ \phi(t) = \frac{b(t)e^{\frac{\omega_0 t}{2Q}}}{K} \]

\[ \ldots (3.34) \]
This phase error alters the zero crossing time and thus controls the timing of the next step function from the limiter. Any phase error accumulated during the first half cycle does not decay but is retained and must be added to the phase errors accumulated in subsequent half cycles. Now let the oscillator be started at time \( t = -\infty \). While we consider the effect of noise on the phase of the oscillator signal we can neglect \( a(t) \) which is always small even when \( t \) becomes infinite and does not affect the zero crossings. We can write the oscillator signal for \( 0 < t < \frac{\pi}{\omega_0} \) as:

\[
\mathcal{V}(t) = A_o e^{-\frac{\omega_0 t}{2Q}} \sin(\omega_0 t + \varphi + \phi(t))
\]

Here \( \varphi \) is the phase error accumulated since the oscillator was started up to the time \( t = 0 \). Thus it is a stochastic variable lying with equal probability between 0 and \( 2\pi \). The phase error accumulated since time \( t = 0 \) is \( \phi(t) \) and this is given by:

\[
\phi(t) = \frac{b(t) e^{\frac{\omega_0 t}{2Q}}}{A_o}
\]

Equation 3.34 is similar to 3.35 except that \( K \) is replaced by the steady state amplitude \( A_o \) and \( b(t) \) must now be modified so that all the trigonometrical functions in it contain the accumulated phase error \( \varphi \). The series of random noise impulses can also be replaced by a continuous
stochastic noise function $i_n(t)$ so that the summation for $b(t)$ goes to an integral. Taking 3.32 and making these changes gives us:

$$b(t) = \int_0^t e^{-\frac{\omega_0 (t-t_1)}{2Q}} \left( \frac{i_n(t_1)}{C} \right) \left[ \cos(\omega_0 t_1 + \varphi) - \frac{S \sin(\omega_0 t_1 + \varphi)}{2Q} \right] dt_1 \quad \cdots (3.36)$$

Since $Q$ is large and we are limited to small values of $t$ and $t_1$ (i.e. $0 < t < \frac{\pi}{\omega_0}$ and $0 < t_1 < \frac{\pi}{\omega_0}$) we get:

$$\phi(t) = \int_0^t \left[ \frac{i_n(t_1)}{A_n C} \right] \cos(\omega_0 t_1 + \varphi) dt_1 \quad \cdots (3.37)$$

If we compare equation 3.37 with the equivalent expression for the Van der Pol oscillator given by equation 3.14a we see that they are similar. For the Van der Pol oscillator a series loss resistance was used while for the hard limiting oscillator a parallel loss resistance was used. Provided both tank circuits have the same high-Q then the $i_n(t)/C$ in 3.37 is equivalent to the $e_n(t) \omega_0$ in 3.14a. For example if only thermal noise is significant then $(i_n(t)/C)^2 = 4kT\Delta f/(RC^2)$ and $(e_n(t) \omega_0)^2 = 4kT \omega_0^2 \Delta f$.

These are identical when the $Q$ of both tank circuits are the same so that $R = \omega_0^2 L^2 / r$.

We now consider the effect of noise on the amplitude of the oscillator which is essentially very different to its action on the phase. The phase error accumulated after one half cycle is permanently retained because it controls the timing of the output of the limiter. The amplitude error accumulated after one half cycle is passed on to the next and subsequent cycles, but it is continuously decaying so that a long time after the first
half cycle the amplitude error due to the first half cycle is zero.

The mean squared phase error goes on increasing with time while the mean squared amplitude error reaches a limiting value when the accumulation of new errors balances the decay of old ones.

If we include now the random amplitude terms in the oscillator signal we have:

\[ v(t) = \left[ A_0 + a(t) \right] \sin \left[ \omega_0 t + \phi(t) + \varphi \right] \quad \ldots \ldots (338) \]

Following a similar argument to that preceding equation 3.36 we find that:

\[ a(t) = \int_0^t e^{-\omega_0^2 (t-t_1)} \left[ \frac{i_n(t_1)}{C} \right] \left[ \sin (\omega_0 t_1 + \varphi) - \frac{\cos (\omega_0 t_1 + \varphi)}{2Q} \right] dt_1 \]

The exponential in this expression cannot be removed as was done for \( b(t) \), because here \( t \) and \( t_1 \) are not restricted to being very small. For a high Q tank circuit the cosine term can be neglected to give:

\[ a(t) = \int_0^t e^{-\omega_0^2 (t-t_1)} \left[ \frac{i_n(t_1)}{C} \right] \sin (\omega_0 t_1 + \varphi) dt_1 \quad \ldots \ldots (339) \]

Comparing this with 3.14b we see that it is similar to the result obtained for the Van der Pol oscillator. The term \( i_n(t)/C \) in 3.39 is equivalent to the term \( e_n(t) \omega_0 \) in 3.14b. We have already seen that \( e_n(t) \omega_0 \) corresponds to \( i_n(t)/C \) in the phase equations 3.37 and 3.14a. The time constant \( 2Q/\omega_0 \)
in 3.39 is also equivalent to the time constant $\frac{1}{\alpha'}$ in 3.14b because both are the time constants of their respective oscillator models.

Equations 3.37 and 3.39 are the principle results of this section. They are equivalent to equations 3.14a and 3.14b which were obtained for the Van der Pol oscillator. Thus essentially the same results have been obtained by using two completely different models of an oscillator and different mathematical procedures. These results are applicable to any high Q feedback oscillator with instantaneous limiting. In the next section these results will be converted into more useful expressions. The effect of noise on an oscillator signal can be most conveniently observed by measuring the power spectral densities of its phase, amplitude and the complete oscillator signal. These spectra are derived in the following pages.

3.3 Phase, Amplitude and RF Spectra of an Oscillator

3.3.1 Oscillator Phase

We will start by considering the phase of the oscillator. The equation 3.37 obtained for the hard limiting oscillator will be used, although we could equally well choose equation 3.14a which is essentially the same.

$$\phi(t) = \int_0^t \frac{i_o(t)}{A_0 C} \cos (\omega_o t_1 + \varphi) \, dt_1 \quad \ldots (3.40)$$
Since \( i_n(t) \) is a stochastic variable we can only evaluate this integral in a mean and mean squared sense. (3.9)

\[
\varphi(t) = \int_0^t f(t_i) i_n(t_i) \, dt_i
\]

i.e. if

\[
\langle \varphi(t) \rangle = \int_0^t f(t_i) \langle i_n(t_i) \rangle \, dt_i
\]

then

\[
\langle \varphi^2(t) \rangle = \int_0^t \int_0^t f(t_i) f(t_j) \langle i_n(t_i)i_n(t_j) \rangle \, dt_i \, dt_j
\]

and

\[
\langle \varphi^2(t) \rangle = \int_0^t \int_0^t f(t_i) f(t_j) \langle i_n(t_i)i_n(t_j) \rangle \, dt_i \, dt_j
\]

Here the brackets refer to ensemble averages

We have \( \langle i_n(t_1) \rangle = 0 \) and \( \langle i_n(t_1) i_n(t_2) \rangle = \frac{N_0}{2} \delta(t_1-t_2) \)

It is assumed that \( i_n(t) \) is white noise of one sided spectral density \( N_0 \).

Therefore

\[
\langle \varphi(t) \rangle = 0
\]

and

\[
\langle \varphi^2(t) \rangle = \int_0^t \int_0^t \frac{N_0}{2A_s^2C^2} \cos(\omega t_1 + \varphi) \cos(\omega t_2 + \varphi) \delta(t_1-t_2) \, dt_1 \, dt_2
\]

\[
\langle \varphi^2(t) \rangle = \frac{N_0}{4A_s^2C^2} \left[ t + \frac{\sin(2\omega t + 2\varphi)}{2\omega} \right] \quad \ldots (3.41)
\]

This expression for the mean squared phase error is plotted in Figure 3.5.

The variance accumulated during one half cycle of time \( \pi/\omega_0 \) is

\[
\pi N_0/(\omega_0 A_s^2C^2). \quad \text{This is very small for a good oscillator but because it accumulates over many cycles and does not decay then a large error is eventually obtained.}
\]
Figure 3.5: Mean squared phase error $\langle \phi^2(t) \rangle$

The periodic part of $\langle \phi^2(t) \rangle$ can be neglected because it becomes very small compared with the variance of the phase error accumulated after many cycles.

Therefore

$$\langle \phi^2(t) \rangle = D_0 t \quad \ldots \quad (342)$$

where

$$D_0 = \frac{N_0}{4A^2C^2}$$

The process approximates to a pure diffusion process and this result will be used later on to find the RF spectrum. Throughout this thesis one-sided power spectral densities will be used unless otherwise stated. That is the mean square value of a signal $g(t)$ is given by

$$\int_{-\infty}^{\infty} S_j(f) df$$
Power Spectrum of $i_n(t) \cos \Theta$

If the noise current $i_n(t)$ has a single sided spectral density $N_0$ then we can represent it by a large number of discrete sine waves each of amplitude $\sqrt{2N_0 \Delta f}$ spaced $\Delta f$ hertz apart and with randomly distributed phases. Then we can write:

$$i_n(t) \cos \Theta = \sqrt{2N_0 \Delta f} \sum_{l=0}^{l=\infty} \sin(2\pi l \Delta f t + \Theta_l) \cos[\omega_0 t + \varphi + \phi(t)]$$

where $l = 0, 1, 2, \ldots \infty$ $\Delta f \to 0$

This can be rewritten as:

$$i_n(t) \cos \Theta = \sqrt{2N_0 \Delta f} \sum_{l=0}^{l=\infty} \frac{1}{2} \sin[(\omega_0 + 2\pi l \Delta f) t + \varphi + \phi(t) + \Theta_l] + \frac{1}{2} \sin[(\omega_0 - 2\pi l \Delta f) t + \varphi + \phi(t) - \Theta_l]$$

For a given value of $l$ we have a sine wave of frequency $\omega_0 + 2\pi l \Delta f$ and another of frequency $\omega_0 - 2\pi l \Delta f$ which both contribute to the power spectrum of $i_n(t) \cos \Theta$. The power spectrum of the summation of sine waves of frequency $\omega_0 + 2\pi l \Delta f$ is zero below $\omega_0$ and $\frac{N_0}{4}$ above $\omega_0$. If we could use negative frequencies then the power spectrum of the summation of sine waves of frequency $\omega_0 - 2\pi l \Delta f$ would be zero above $\omega_0$ and $\frac{N_0}{4}$ below $\omega_0$. However it is the single sided power spectrum which we want and this is $\frac{N_0}{2}$ between $0$ and $\omega_0$ and $\frac{N_0}{4}$ above $\omega_0$. The power spectrum then of the complete signal $i_n(t) \cos \Theta$ must be $\frac{N_0}{2}$ for all frequencies.
It is customary to call $S_s(f)$ a power spectral density even though its units are not watts/hertz and we will thus follow this practice.

In order to find the phase and frequency noise modulation spectra we start with equation 3.40 which upon differentiation gives:

$$\frac{d\phi(t)}{dt} = \frac{i_n(t)}{A_o C} \cos(\omega_0 t + \psi)$$

The instantaneous frequency $\nu(t)$ of an oscillator signal $\nu(t)$ is defined by:

$$\nu(t) = \frac{1}{2\pi} \frac{d\Theta}{dt} \quad \text{where} \quad \nu(t) = A \sin \Theta$$

Equation 3.38 gives:

$$\nu(t) = [A_0 + a(t)] \sin[\omega_0 t + \psi + \phi(t)]$$

Therefore

$$\nu(t) = \frac{\omega_0}{2\pi} + \frac{i_n(t)}{2\pi A_o C} \cos \Theta$$

Let the instantaneous frequency deviation be $\Delta \nu(t)$

$$\Delta \nu(t) = \nu(t) - \frac{\omega_0}{2\pi} = \frac{i_n(t)}{2\pi A_o C} \cos \Theta$$

We have $S_{\nu}(f) = N_o$ and opposite we see that the power spectrum of $i_n(t) \cos \Theta$ is $\frac{N_o}{2}$.

Therefore

$$S_{\Delta \nu}(f) = N_o \left(\frac{1}{2\pi A_o C}\right)^2 \frac{1}{2}$$

$$S_{\Delta \nu}(f) = \frac{D_o}{2\pi^2} \quad \cdots 343$$

Having found the frequency deviation spectrum, the phase spectrum can be found by noting that differentiation in the time domain is equivalent
to multiplication by \( j\omega \) in the frequency domain.

\[
S_{\varphi}(f) = \omega^2 S_{\theta}(f) = 4\pi^2 S_{\Delta\varphi}(f)
\]

Therefore

\[
S_{\theta}(f) = \frac{2D_o}{\omega^2} \quad \ldots (3.44)
\]

Equations 3.43 and 3.44 are important results. These spectra are frequently used measures of the frequency stability of an oscillator and can easily be measured for a real oscillator.

3.3.2 Oscillator Amplitude

The following equation determines the behaviour of the amplitude of the oscillator.

\[
a(t) = e^{-\mu t} \int_0^t e^{\mu t} f_n(t_1) S_{\sin \Theta} dt_1 \quad \ldots (3.45)
\]

If the oscillator is of the Van der Pol type then this equation corresponds to equation 3.14b and the values of \( \mu \) and \( f_n(t) \) are:

\[
\mu = (M\alpha' - rC)\omega_o^2 \quad \& \quad f_n(t) = e_n(t)\omega_o
\]

For a hard limiting type of oscillator then equation 3.45 corresponds to equation 3.39 and we have:

\[
\mu = \frac{\omega_o}{2Q} \quad f_n(t) = \frac{i_n(t)}{C} \quad \& \quad \Theta = \omega_o t + \phi
\]
The quantity \( \frac{1}{\mu} \) is the relaxation time of the oscillator. This is the time constant of return of the oscillator amplitude to its steady state value after a small disturbance. A real oscillator will not correspond to either of the oscillator models we have used but it will have similar properties if instantaneous amplitude limiting is used, so that equation 3.45 is still valid. The value of \( \mu \) for a real oscillator might have to be found experimentally and \( f_n(t) \) might be due to a combination of thermal and shot noise. The relaxation of the ideal hard limiting oscillator can be used to estimate the relaxation time of a real oscillator. For a real oscillator \( \frac{1}{\mu} \) is always smaller than the value of \( \frac{1}{\mu} \) for the ideal hard limiting oscillator, because no real oscillator can have harder limiting than the ideal. On the other hand \( \frac{1}{\mu} \) cannot be made very much larger than the value for the ideal hard limiting oscillator because the oscillator would then become unreliable. With a very large value for \( \frac{1}{\mu} \) the oscillator has soft limiting and a loop gain only just greater than unity.

We will consider the oscillator to be of the hard limiting type so that equation 3.45 becomes:

\[
\theta(t) = \int_0^t e^{-\frac{\omega_0}{2\alpha}(t-t_i)} \left[ \frac{i_n(t)}{C} \right] \sin(\omega_0 t + \varphi) \, dt_i
\]
Following the method used for the phase this is solved in a mean and mean squared sense.

We have

\[ \langle i_n(t) \rangle = 0 \quad \& \quad \langle i_n(t_1)i_n(t_2) \rangle = \frac{N_o}{2} \delta(t_1-t_2) \]

Therefore

\[ \langle a(t) \rangle = 0 \]

Also

\[ \langle a(t)a(t-\tau) \rangle = R_a(t,t-\tau) \]

\[
= \int_0^t e^{-\mu(t-t_1)} e^{-\mu(t-t_2)} \langle i_n(t_1)i_n(t_2) \rangle \sin(\omega_0 t_1 + \varphi) \sin(\omega_0 t_2 + \varphi) \, dt_1 dt_2
\]

\[
= \frac{N_o}{4C^2} \left[ \frac{e^{-\mu \tau} - e^{-2\mu t + \mu \tau}}{2\mu} - \frac{e^{-\mu \tau} \sin 2(\omega_0 t - \omega_0 \tau + \varphi)}{2\omega_0} + e^{-2\mu t + \mu \tau} \frac{\sin 2\varphi}{2\omega_0} \right]
\]

This autocorrelation function is correct if the oscillator is started at time \( t = -\infty \) but without any noise acting until after time \( t = 0 \). In order to let the oscillator reach a steady state we must let \( t \to \infty \).

\[
R_a(\tau) = \frac{N_o}{4C^2} \left[ \frac{e^{-\mu \tau}}{2\mu} - \frac{e^{-\mu \tau}}{2\omega_0} \sin 2(\omega_0 t - \omega_0 \tau + \varphi) \right] \quad (3.46)
\]
The second term in the auto-correlation function is periodic and much smaller than the first, because the relaxation time of the oscillator $1/\mu$ is much greater than the period of oscillation $2\pi/\omega_0$. It is convenient to simplify the equation by ignoring the small periodic part.

Its effect on the amplitude spectrum is to add a small image of the main spectrum shifted up to a frequency of $2\omega_0/\pi$. It is difficult to place any useful practical interpretation on this result so the small periodic term in the auto-correlation function is best neglected.

Thus we write the autocorrelation function as:

$$ R_a(\tau) = \frac{D_o A_o^2}{2\mu} e^{-\mu \tau} \quad \ldots \quad (3.47) $$

where

$$ D_o = \frac{N_o}{4A_o^2C^2} $$

It is also instructive to find the mean squared value of the amplitude fluctuations by making $\tau = 0$. Then from 3.46 we get:

$$ \langle a^2(t) \rangle = R_o(0) = \frac{N_o}{4C^2} \left[ \frac{1}{2\mu} - \frac{\sin(\omega_0 t + \varphi)}{2\omega_0} \right] \quad \ldots \quad (3.48) $$

Compare this with equation 3.41. The mean squared amplitude error reaches a limiting value about which it moves in a periodic manner, once the oscillator has reached a steady state. However, the mean squared phase error is continually increasing along a periodically varying path as shown by Figure 3.5.
By using equation 3.47 and applying the Wiener-Khitchine theorem the amplitude spectrum can be found.

\[ S_a(f) = 4 \int_{-\infty}^{\infty} R_a(\tau) \cos \omega \tau d\tau \]
\[ = \frac{2 D_0 A_0}{\mu} \int_{-\infty}^{\infty} e^{-\mu \tau} \cos \omega \tau d\tau \]
\[ S_a(f) = \frac{2 D_0 A_0}{\mu^2 + \omega^2} \] \hspace{1cm} \ldots (341)

3.3.3 Complete Oscillator Signal

Finally we shall find the RF spectrum of the oscillator.

\[ \nu(t) = [A_o + a(t)] \sin [\omega_o t + \varphi + \phi(t)] \]

\[ R_v(\tau) = \left< [A_o + a(t)][A_o + a(t-\tau)] \sin [\omega_o t + \varphi + \phi(t)] \sin [\omega_o t + \varphi - \omega_o \tau + \phi(t-\tau)] \right> \]

For narrow band signals (3.9) we can write the auto-correlation as:

\[ R_v(\tau) = \left< \left[ A_o^2 + A_o a(t) + A_o a(t-\tau) + a(t-\tau)a(t) \right] \frac{\cos [\omega_o \tau + \phi(t-\tau) - \phi(t)]}{2} \right> \]

\[ \ldots \ldots \ldots (3.50) \]

Since \( [\phi(t-\tau) - \phi(t)] \) is a random variable with a symmetrical distribution about zero and because sine is an odd function then \( \left< \sin [\phi(t-\tau) - \phi(t)] \right> \)
is zero. The phase and amplitude can be considered to be statistically independent. The same noise acts on both phase and amplitude but the expression for the phase contains a cosine weighting function and the expression for amplitude contains a sine weighting function. The phase perturbations are mainly determined by the action of the noise when the cosine function is large. During this time the noise has little effect on the amplitude and we would expect the phase and amplitude to be statistically independent. This can be proved by calculating \( \langle a(t) \phi(t) \rangle \) from equations 3.45 and 3.43. The result is not quite zero but gives a small oscillatory term. This term is of the same order of magnitude as the oscillatory terms found in the expressions for \( \langle a(t) a(t-\tau) \rangle \) and \( \langle \phi^2(t) \rangle \). These oscillatory terms have been considered negligible and we will also assume that \( \langle a(t) \phi(t) \rangle \) is zero.

Now we make use of the following properties of \( \phi(t) \) and \( a(t) \)

\[
\langle a(t) \rangle = 0 \quad \langle \phi(t) \rangle = 0 \quad \langle a(t) \phi(t) \rangle = 0 \\
\& \quad \langle \sin [\phi(t-\tau) - \phi(\tau)] \rangle = 0
\]

Equation 3.50 gives:–

\[
R_v(\tau) = \left[ A_o^2 + \langle a(t)a(t-\tau) \rangle \right] \frac{C}{2} \langle \cos [\phi(t-\tau) - \phi(\tau)] \rangle \quad (3.51)
\]

From 3.47

\[
\langle a(t)a(t-\tau) \rangle = R_a(\tau) = \frac{D_o A_o^2}{2\mu} e^{-\mu \tau}
\]
Also $[\phi(t-\tau) - \phi(t)]$ is the phase error accumulated in the time $\tau$ and equation 3.42 gives the mean squared value of this.

$$\langle [\phi(t-\tau) - \phi(t)]^2 \rangle = D_0 \tau$$

Therefore

$$\langle \cos[\phi(t-\tau) - \phi(t)] \rangle = \int \cos \phi p(\phi) d\phi$$

Where

$$p(\phi) = \frac{e^{-\frac{\phi^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} \quad \& \quad \sigma = D_0 \tau$$

$$\langle \cos \phi \rangle = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos \phi e^{-\frac{\phi^2}{2\sigma^2}} d\phi = e^{-\frac{\sigma^2}{2}}$$

Therefore

$$R_V(\tau) = \left[ A_o^2 + \frac{D_o A_o^2 e^{-\mu\tau}}{2\mu} \right] \left[ \frac{\cos \omega_0 \tau}{2} \cdot e^{-\frac{D_o \tau}{2}} \right]$$

By using the Wiener Khitchine theorem we can find the power spectrum of the oscillator signal.

$$S_V(f) = 4 \int_0^{\infty} R_V(\tau) \cos \omega \tau d\tau$$

For $|\omega - \omega_0| \ll \omega_0$ and $\alpha$ we find :-
Figure 3.6: The RF spectrum of an oscillator
\[
\frac{S_r(f)}{A_0^2/2} = \frac{D_0}{\frac{1}{4} D_0^2 + \Delta \omega^2} + \frac{D_0}{\mu^2 + \Delta \omega^2}
\]

Where \(\Delta \omega = \omega_o - \omega\)

The RF spectrum is divided by the carrier power since it is the relative magnitude of the noise sidebands which are of interest. The RF spectrum consists of two terms which are plotted in Figure 3.6.

The first term in 3.52 is due to phase noise and the second due to amplitude noise. For any oscillator with a white noise current generator of spectral density \(N_o\) in parallel with the tank circuit \(D_0\) is given by :-

\[
D_0 = \frac{N_o}{4A_o^2 C^2}
\]

If the thermal noise is predominant \(N_o = 4kT/R\) so that :-

\[
D_0 = \frac{kT}{RA_o^2 C^2}
\]
Figure 3.7: Delayed amplitude limiting oscillator

\[ g_m = \frac{rC}{M} + G(A_0 - Z) \]
For an oscillator with a series loss resistance $r$ and thermal noise predominant then:

$$D_o = \frac{kT r \omega_o^2}{A_o^2} \quad \ldots (355)$$

This can also be written in terms of $Q$.

$$D_o = \frac{kT \omega_o}{A_o^2 C Q} \quad \ldots (356)$$

### 3.4 Delayed Amplitude Limiting Oscillator

The effect of white noise in a delayed amplitude limiting oscillator has been investigated by Golay (3.10) and others (3.11, 3.12, 3.13). Here we will use a method of analysis similar to Golay's but modified so that a direct comparison can be made with our results for the Van der Pol and hard limiting oscillators. The oscillator shown in Figure 3.7 is similar to the Van der Pol oscillator in Figure 3.1, but with a detector, low-pass filter and servo amplifier added. The valve has also been modified so its gain is linear but can be varied by the servo amplifier. This can easily be done by using a variable $\mu$ valve so that the bias voltage controls the gain. The level of oscillations must not be so great as to drive the valve into limiting.
The equations which govern the operation of this oscillator are:

\[ g_m = \frac{rC}{M} + G[A_o - Z(t)] \quad \ldots (3.60) \]

\[ Z(t) + T \frac{dz(t)}{dt} = A(t) \quad \ldots (3.61) \]

\[ \frac{d^2v}{dt^2} + \frac{dv}{dt} (rC - Mg_m) \omega_o^2 + \omega_o^2 v = \omega_o^2 e_n(t) \quad \ldots (3.62) \]

The delayed amplitude limiting oscillator we have chosen to study has a linear amplitude detector and a first order low-pass filter. A square law amplitude detector could equally well have been chosen. This would not make any difference to the analysis because the equation for amplitude disturbances due to noise must be linearised even with a linear detector. It would also have been possible to choose a more complicated type of low-pass filter. This would have made the analysis more difficult so we will only consider the oscillator with a first order low-pass filter. As with the Van der Pol oscillator we make the following substitutions in equation 3.62.
\[ x = v \quad \& \quad y = \frac{1}{\omega_0} \frac{dv}{dt} \]

Therefore

\[ \dot{x} = \omega_0 y \]

\[ \dot{y} = -\omega_0 x - (rC - Mg_m)\omega_0^2 y + \omega_0 e_n(t) \]

Also let \( x = A \cos \theta \) and \( y = A \sin \theta \)

\[ \frac{d\theta}{dt} = -\omega_0 - (rC - Mg_m)\omega_0^2 \sin \theta \cos \theta + \frac{\omega_0 \cos \theta}{A} e_n(t) \]

\[ \frac{dA}{dt} = -(rC - Mg_m)\omega_0^2 A \sin^2 \theta + \omega_0 \sin \theta e_n(t) \]

Finally let \( A = A_0 + a(t) \) and \( \Theta = -\omega_0 t + \phi(t) \)

\[ \dot{\phi} = - (rC - Mg_m)\omega_0^2 \frac{\sin 2\theta}{2} + \frac{\omega_0 \cos \theta}{A} e_n(t) \]

\[ \dot{\alpha} = -(rC - Mg_m)\omega_0^2 A \left( \frac{1 - \cos 2\theta}{2} \right) + \omega_0 \sin \theta e_n(t) \]

These equations can be "smoothed" in the same way as those for the Van der Pol oscillator.
\[ \dot{\phi} = \frac{\omega_0 \cos \theta}{A} e_n(t) \quad \cdots \quad (3.63) \]

\[ \dot{a} = - (rC - Mg_m) \frac{\omega_0^2 A}{2} + \omega_0 e_n(t) \sin \theta \quad \cdots \quad (3.64) \]

Equation 3.63 for the phase is identical to equation 3.14a for the phase of the Van der Pol oscillator. The amplitude equation requires further consideration before we can make any comparisons.

We have \( A = A_o + a(t) \) Also let \( Z = A_o + z(t) \)

Then from equation 3.64 using 3.60 we get:

\[ \dot{a} = - \frac{MG \omega_0^2 A_o}{2} z(t) - \frac{MG \omega_0^2}{2} a(t) z(t) + \omega_0 e_n(t) \sin \theta \]

Since \( A_o \gg a(t) \)

\[ \dot{a} = z(t) \frac{MG \omega_0^2 A_o}{2} + \omega_0 e_n(t) \sin \theta \quad \cdots \quad (3.65) \]

Equation 3.61 gives:

\[ z + T \dot{z} = a \quad \cdots \quad (3.66) \]
The simultaneous differential equations 3.65 and 3.66 define the random motion of the amplitude under the influence of noise. If they are compared with equation 3.14b it is clear that the effect of noise on the amplitude in a delayed amplitude limiting oscillator is more complicated than in an instantaneous limiting oscillator. The complicated nature of the amplitude makes it impossible to use the same methods as in Section 3.3 for finding the RF spectrum of the oscillator. However, we can by using Laplace transforms find the amplitude spectrum.

Let \[ \tau = \frac{2}{MG\omega_n^2 A_o} \] \& \[ f(t) = \omega_n e_n(t) \sin \Theta \]

Then equations 3.66 and 3.65 give:

\[ pa'(p) = -\frac{1}{\tau} g'(p) + f'(p) \]
\[ a'(p) = pTg'(p) + g'(p) \]

Therefore

\[ a'(p) = \frac{\tau (1 + Tp)}{(p^2 T\tau + p\tau + 1)} f'(p) \]

\[ S_a(f) = S_f(f) \left| \frac{\tau (1 + Tp)}{(p^2 T\tau + p\tau + 1)} \right|^2 \]
Figure 3.8: The amplitude power density spectra of an instantaneous and delayed amplitude limiting oscillator.
If \( e_n(t) \) is due to the thermal noise in \( r \) then:

\[
S_p(f) = \omega_0^2 2kT_r
\]

In order to make a comparison with the instantaneous limiting oscillator we introduce \( D_0 \) as defined by equation 3.55.

\[
S_p(f) = 2D_0A_0^2
\]

\[
S_a(f) = 2D_0A_0^2 \left| \frac{(1+j\omega T)\gamma}{(1+j\omega \gamma - \omega^2 T\gamma)} \right|^2
\]

This can now be compared with equation 3.49 for the instantaneous limiting oscillator which is:

\[
S_a(f)_{\text{inst.}} = \frac{2D_0A_0^2}{\omega^2 + \mu^2}
\]

Equation 3.67 with \( T \gg \gamma \) is plotted in Figure 3.8 together with equation 3.49 for comparison. Note that the damping coefficient in equation 3.67 is \( \frac{1}{2} \sqrt{\frac{\gamma}{T}} \).

If \( T \ll \gamma \) then

\[
S_a(f) = \frac{2D_0A_0^2}{\omega^2 + \frac{1}{\gamma^2}}
\]
With $T \ll \gamma$ the amplitude spectrum is of the same form as for the
instantaneous limiting oscillator. The numerical values of a typical
60 MHz oscillator are given below and they will be used to estimate
the magnitudes of $T$, $\gamma$ and $\mu$.

\[ C = 10 \text{pF} \quad Q = 100 \quad r = 2.6 \Omega \]
\[ A_0 = 1 \text{V} \quad L = 0.705 \mu \text{H} \quad \omega_o = 3.77 \times 10^8 \]

If this oscillator has hard instantaneous limiting then from page 65 we
get:

\[ \frac{1}{\mu} = \frac{2Q}{\omega_o} = 5.33 \times 10^{-7} \]

Let the delayed amplitude limiting oscillator use a field effect transistor
having the following characteristics.

\[ I_D = I_{DSS} \left(1 - \frac{V_{GS}}{V_p}\right)^2 \]
\[ I_{DSS} = 16 \text{mA} \quad V_p = 4 \text{ volts} \]

Let the gate of the FET be biased with a voltage $V_{GS}$ of 2 volts then
$g_m = 4 \text{ mA/volt}$.

Therefore
\[ \frac{dg_m}{dV_{GS}} = \frac{2I_{DSS}}{V_p^2} = 2 \times 10^{-3} \]
Figure 3.9: The RF power density spectra of an instantaneous and delayed amplitude limiting oscillator.
If the servo amplifier is a wide band voltage amplifier of gain 10 which is connected to the gate of the FET then:

\[ G = 10 \times \frac{d g_m}{d V_{cs}} = 2 \times 10^{-2} \]

The value of \( M \) which gives a steady state amplitude with a \( g_m \) of 4mA/volt is given by equation 3.60 with \( Z \) equal to \( A_0 \).

\[ M = \frac{rC}{g_m} = 0.0065 \mu H \]

We can now calculate \( \gamma \)

\[ \gamma = \frac{2}{MG \omega^2 A_0} = 1.85 \times 10^{-7} \]

It will be assumed that the low-pass filter cut off frequency is 100 kHz so that \( T = 1.6 \times 10^{-6} \).

The numerical values we have obtained for \( \mu \), \( T \) and \( \gamma \) correspond to the situation \( T > \mu > \gamma \) shown in the sketch of \( S_a(t) \) in Figure 3.8.

In Figure 3.6 the RF spectrum of an instantaneous limiting oscillator is shown to consist of two parts. The part due to phase noise is dominant except for frequencies above \( \frac{\mu}{2\pi} \) where the amplitude noise and phase noise contributions are equal. The situation is obviously very different in a delayed amplitude limiting oscillator with \( T \gg \gamma \). As shown in Figure 3.9 the RF spectrum will have a peak at a frequency \( \frac{1}{2\pi \sqrt{T\gamma}} \) due
to the peak in the amplitude spectrum.

In some applications this peak might be undesirable so that it is best to have $T \ll \tau$ so that the peak in the RF spectrum does not occur.

This might be done by raising the cut-off frequency of the low-pass filter. Unfortunately the model we have used for our analysis becomes unrealistic if this is done. We have neglected to include some additional phase lags in the model to allow for the limited bandwidth of the servo amplifier and amplitude detector. In a real oscillator these additional phase shifts will make the feedback system unstable if the low-pass filter time constant is made too small. Amplitude detectors cannot be made to follow changes in level quickly without producing a large amount of radio frequency signal at their output. Thus if $T$ is to be made small and the phase lag in the amplitude detector is also small, then some of the oscillator signal will be fed through the servo amplifier back into the oscillator. This unwanted feedback might interfere with the proper operation of the oscillator if it is too large.

It would seem that delayed amplitude limiting can increase the noise sidebands of an oscillator, unless care is taken to ensure that the feedback loop is stable and well damped. When this is done the noise sidebands are about the same as for an instantaneous limiting oscillator.

The amplitude controlling mechanism in an instantaneous limiting oscillator
involves only one time constant $\frac{1}{\mu}$ and is unconditionally stable.

In a delayed amplitude limiting oscillator at least two time constants $T$ and $\tau$ are involved. With a simple signal lag filter the damping of the system is given by $\frac{1}{2} \sqrt{\frac{T}{\tau}}$. This constraint, together with the practical limitations of the amplitude detector and servo amplifier, makes the delayed amplitude limiting oscillator as noisy as the instantaneous limiting oscillator. If the servo amplifier adds noise to the oscillator then the amplitude performance of the delayed amplitude limiting oscillator might be considerably worse than with instantaneous limiting. The extra complexity involved in using delayed amplitude limiting is thus unlikely to result in a lower noise oscillator.
References for Chapter 3


\[ v(t) = A_0 \sin \omega_0 t + n(t) \]

**Figure 4.1**: Noise added to the output of an oscillator.

**Figure 4.2**: Representation of noise by a series of discrete sine waves.
CHAPTER 4

THE EFFECT OF OTHER TYPES OF OSCILLATOR NOISE

4.1 Noise Added to the Output of an Oscillator

Consider now the effect of noise which is added to the output of an oscillator. This noise is normally referred to as amplifier noise since it just adds to the oscillator's output signal. However, its effect when added to the oscillator signal is to destroy its perfect periodicity and effectively add noise modulation sidebands. Thus it is important to establish the amplitude and phase spectra of the oscillator signal after the noise has been added. A knowledge of these is essential if the noise performance of a system such as radio receivers is to be found.

Consider the situation shown in Figure 4.1 where a noise free oscillator signal is fed into a noisy amplifier with a band-pass transfer function $H(\omega)$ centred on the oscillator frequency $\frac{\omega_0}{2\pi}$. The noise in the amplifier is represented by an equivalent noise voltage generator $n'(t)$ at the input, which at radio frequencies can be considered to be white noise of spectral density $N_0$.

The spectrum of the output noise is given by the familiar expression shown below:

$$S_n(f) = \left|H(\omega)\right|^2 S_{n'}(f) = \left|H(\omega)\right|^2 N_0$$
Having established the RF spectrum of the oscillator signal we now wish to find its phase and amplitude spectra. Bennett (4.1) and Rice (4.2) have shown that a Gaussian process can be approached as the limiting form of the sum of a large number of sinusoidal distributions. This is the basis for the method of analysis which is almost invariably used to find the phase and amplitude spectra of a sinusoidal signal with narrow band noise. (4.3) (4.4). The noise is represented by a series of discrete sine waves spaced $\Delta f$ Hz apart as shown in Figure 4.2. We then have:

$$n(t) = \sum_{l=-\infty}^{\infty} a_l \sin [(\omega_0 + 2\pi l \Delta f) t + \Theta_l] \quad \ldots (4.1)$$

We equate power in one member of the series to power in the continuous spectrum over a bandwidth $\Delta f$.

$$S_n (f_0 + l \Delta f) = \frac{a_l^2}{2} \quad \ldots (4.2)$$

Now we want to obtain $n(t)$ in the following form.

$$n(t) = a(t) \sin \omega_0 t + b(t) \cos \omega_0 t \quad \ldots (4.3)$$

This can be done by expanding 4.1 as follows

$$n(t) = \sum_{l=-\infty}^{\infty} a_l \left[ \sin \omega_0 t \cos (l \Delta \omega t + \Theta_l) + \cos \omega_0 t \sin (l \Delta \omega t + \Theta_l) \right]$$
Hence

\[ a(t) = \sum_{\ell=-\infty}^{\infty} a_{\ell} \cos (\ell \Delta \omega t + \Theta_{\ell}) \]

\[ b(t) = \sum_{\ell=-\infty}^{\infty} a_{\ell} \sin (\ell \Delta \omega t + \Theta_{\ell}) \]

Consider now the discrete spectrum of \( a(t) \) and equate this to its continuous power spectrum. When this was done for \( n(t) \) there was only one term but for \( a(t) \) there are two terms corresponding to the same frequency.

\[
S_a(\ell \Delta f) \Delta f = \left[ a_{\ell} \cos (\ell \Delta \omega t + \Theta_{\ell}) + a_{-\ell} \cos (-\ell \Delta \omega t + \Theta_{\ell}) \right]^2
\]

Since \( n(t) \) is a Gaussian random variable then \( \Theta_1 \) must be a uniformly distributed random variable so the +\( \ell \) and -\( \ell \) terms are independent.

Hence

\[
S_a(\ell \Delta f) \Delta f = \frac{a_{\ell}^2}{2} + \frac{a_{-\ell}^2}{2} \quad \ldots \quad (4.4)
\]

Comparing 4.2 with 4.4 and letting \( \ell \Delta f = f \) we get:

\[
S_a(f) = S_n(f_0 + f) + S_n(f_0 - f)
\]

Similarly for \( b(t) \) we find that:

\[
S_b(f) = S_n(f_0 + f) + S_n(f_0 - f)
\]

Having found the spectra of \( a(t) \) and \( b(t) \) we now write the output of the noisy amplifier \( v(t) \) using equation 4.3.
\[ v(t) = A_0 \sin \omega_0 t + a(t) \sin \omega_0 t + b(t) \cos \omega_0 t \]

\[ v(t) = \sqrt{[A_0 + a(t)]^2 + b(t)^2} \sin [\omega_0 t + \tan^{-1}\left(\frac{b(t)}{A_0 + a(t)}\right)] \]

Since \( A_0 \gg a(t) \) and \( A_0 \gg b(t) \)

We have

\[ v(t) = (A_0 + a(t)) \sin [\omega_0 t + \frac{b(t)}{A_0}] \]

Thus the amplitude spectrum is \( S_a(f) \) and the phase spectrum \( S_\phi(f) \) is \( S_a(f)/A_0 \).

\[ S_a(f) = S_n(f_0 + f) + S_n(f_0 - f) \quad \ldots \quad 4.5 \]

\[ S_\phi(f) = \frac{S_n(f_0 + f) + S_n(f_0 - f)}{A_0^2} \quad \ldots \quad 4.6 \]

When the noise spectrum \( S_n(f) \) is symmetrical about \( f_0 \) these expressions simplify

\[ S_a(f) = 2S_n(f_0 - f) \quad \ldots \quad 4.7 \]

\[ S_\phi(f) = \frac{2}{A_0^2} S_n(f_0 - f) \quad \ldots \quad 4.8 \]
\[ i_a = \alpha v_i^2 - \beta v_i^3 - \gamma v_i \]

**Figure 4.3**: Flicker noise in a Van der Pol oscillator

\[ i_o = 2\alpha v_{in} - 2\gamma v_{in}^3 \quad \text{when} \quad v_n = 0 \]

**Figure 4.4**: A symmetrical oscillator with flicker modulation
These are the results of this section and give the amplitude and phase spectra of a noise free oscillator signal when low level narrow band noise is added to it.

4.2 The Effect of Flicker Noise

4.2.1 Amplitude Modulation

In this section the effects of flicker noise will be examined. It might be supposed that low frequency noise is unimportant because it is far below the oscillator frequency. This is not so because of the non-linear nature of an oscillator. We will first consider the Van der Pol oscillator shown in Figure 4.3 and see how flicker noise modulates the amplitude.

It is assumed that the flicker noise has a power spectrum $S_{en}(f)$ and that it can be represented by a voltage generator at the grid of the valve. It would be difficult to rigorously justify this model in terms of the properties of a real oscillator. The flicker noise is generated at the cathode due to fluctuations in the emission current. An equivalent voltage generator at the grid may be used to represent flicker noise in a linear amplifier but it is not clear whether this is valid in a non-linear situation. For the moment let us assume that the model agrees with a real oscillator in a qualitative manner even if not quantitively.
We have 

\[ i_a = \alpha' v_j - \beta' v_j^2 - \gamma' v_j^3 \]

and 

\[ v_j = v + e_n \]

Since 

\[ e_n \ll v \]

Then 

\[ i_a = (\alpha' + 2e_n\beta')v - (\beta' + 3e_n\gamma')v^2 - \gamma v^3 + \alpha' e_n \]

Since \( e_n \) is a slowly varying voltage compared with \( v \), we may consider that its effect is to change \( \alpha' \) to \( \alpha' + 2e_n \). Using the results from page 34 we have:

\[ A_o = 2\sqrt{\frac{\alpha'}{\delta}} = 2\sqrt{\frac{rC - M\alpha'}{3M\delta}} \]

When the effect of \( e_n \) is included the amplitude will deviate from its steady state value and it will be given by:

\[ A = A_o + a(t) = 2\sqrt{\frac{rC - M\alpha' - 2e_nM\beta'}{3M\delta}} \]

With 

\[ A \gg a(t) \quad a(t) = -\frac{4e_n\beta'}{3A_o\delta} \]

Since the power density spectrum of \( e_n \) is \( S_{e_n}(f) \)

Then 

\[ S_a(f) = \frac{16\beta'}{qA_o^2\delta^2} S_{e_n}(f) \quad \ldots \quad \text{(4.10)} \]
Equation 3.38 shows that the oscillator's amplitude is modulated by the flicker noise due to the presence $\beta$ of the coefficient of $\nu^2$. Whereas $\alpha$ and $\gamma$ are essential for proper amplitude limiting, $\beta'$ is undesirable because it mixes in low frequency noise and creates additional harmonics. If the non-linear characteristics of the valve could be made symmetrical then $\beta'$ and all other even coefficients would be zero. By adjusting the bias voltage a valve can be set to an inflection point on its characteristic so that $\beta'$ is at a minimum.

Some doubt was cast on the validity of representing the noise as an equivalent input voltage. The analysis shows that if $\beta'$ is zero noise at the input is not mixed in but one wonders whether this is also true of noise introduced within the valve. It is easy to devise a model of a non-linear valve which is perfectly symmetrical (for its input output characteristics), but which gives $1/f$ noise modulation due to internal noise. Consider the representation of a non-linear device shown in Figure 4.4.

The two non-symmetrical parts together form a symmetrical characteristic so that $1/f$ noise applied with the input voltage $V_{1N}$ has no effect. Noise such as $V_n$ which occurs internally will cause amplitude modulation despite the overall symmetry of the device.

Without a full understanding of the physical causes of low frequency noise and non-linear properties of valves and transistors it is impossible
to deduce exactly how the amplitude modulation can be minimised, except by choosing a device with as little low frequency noise as possible.

4.2.2 Frequency Modulation

We will now consider how flicker noise can affect the frequency of an oscillator. The capacitance and inductance of the tank circuit determine the frequency of oscillation. The input capacitance of the active device forms a small part of the tank circuit capacitance. Any fluctuations in this input capacitance will cause fluctuations in frequency. Flicker noise in the active device or its associated resistors can cause the input capacitance to vary and give rise to a $1/f$ frequency modulation spectrum. Flicker noise is produced by carbon resistors which carry a current and can be largely eliminated by using metal film resistors. The effects of flicker noise due to the active device are much more difficult to deal with.

Flicker noise in a valve is caused by fluctuations in the emission current at the cathode. This is normally observed as a noise current at the anode but it also creates a fluctuating space charge. The grid cathode capacitance is dependent on the space charge so that the fluctuating cathode emission causes the input capacitance to vary. The fluctuating capacitance cannot be simply related to the noise current at the anode and it is not easily measured. This fluctuating capacitance only becomes apparent if the valve is used to make an
oscillator and the frequency modulation of the oscillator is measured.

In a bipolar transistor the flicker noise is due to fluctuations in the recombination current in the emitter-base space charge region. This cannot be observed directly although the fluctuating collector current can. The emitter-base diffusion capacitance is proportional to the emitter current so that the fluctuating emitter injection gives rise directly to a fluctuating diffusion capacitance. The field effect transistor has a reverse biased junction and so it is the transition capacitance which is of importance. This is varied by the flicker noise but there is no simple relation between these variations in capacitance and the noise current observed at the drain electrode.

In order to estimate the effects of flicker noise on the frequency of an oscillator, we will assume that the equivalent input noise voltage modulates the input capacitance of a field effect transistor. The low frequency equivalent circuit for flicker noise is shown in Figure 4.5.

![Figure 4.5: Equivalent circuit for flicker noise](image)
The relationship between the junction capacitance of a reverse biased semiconductor junction and the voltage across the junction depends on the type of junction. When the junction is very abrupt so that the P-type semiconductor changes suddenly to an N-type, then the capacitance is inversely proportional to the square root of the voltage. If we assume that the field effect transistor has such a junction then:

\[ C_{IN} = \frac{K}{\sqrt{V}} \]

Therefore

\[ \frac{dC_{IN}}{dV} = -\frac{C_{IN}}{2\sqrt{V}} \]

If the input capacitance appears in parallel with the tank circuit capacitance and the frequency of oscillation is \( \nu(t) \) then:

\[ \nu(t) = \frac{1}{2\pi\sqrt{LC}} \quad \& \quad \frac{d\nu}{dC} = -\frac{\nu_o}{2C} \]

where \( \nu_o \) is the average frequency of oscillation.

Using 4.11 we get

\[ \frac{d\nu}{dV} = \frac{d\nu_o}{dC} \times \frac{dC_{IN}}{dV} = \frac{C_{IN}\nu_o}{4C\nu} \]

The noise voltage \( e_n \) is a small change in \( \nu \) so that:

\[ \Delta\nu = \frac{C_{IN}\nu_o}{4C\nu} e_n \]
Figure 4.6: The phase and amplitude spectra of an oscillator showing three types of oscillator noise.
Let the power density spectrum of the noise voltage $e_n$ be $S_{e_n}(f)$.

Then

$$S_{\Delta v}(f) = \left(\frac{C_{IN}V_o}{4CV}\right)^2 S_{e_n}(f) \quad \text{...4:12}$$

Equation 4.12 gives the frequency modulation spectrum in terms of the flicker noise spectrum. The effect of flicker noise on the frequency of the oscillator can be reduced by choosing a large value for C and a small value for L. This makes the capacitance of the tank circuit $C$ large compared with $C_{IN}$ so reducing the frequency modulation by flicker noise.

4.3 The Combined Effect of Different Types of Oscillator Noise

We have seen how electrical noise in an oscillator produces three types of oscillator noise. In a real oscillator all three of these may be important so we must consider what their combined effect will be on the oscillator signal. Normally the noise sources responsible for each type of oscillator noise are independent, so that the power density spectra for the oscillator can be found by adding up the power density spectra due to each type of oscillator noise. Figure 4.6 shows the amplitude and phase spectra of an oscillator in which all three types of noise are significant. The amplitude spectrum is drawn for an instantaneous limiting oscillator but the dotted line shows how a delayed amplitude limiting oscillator might behave.

At high frequencies noise added to the output of the oscillator is dominant. If the output signal is not filtered, then the noise added to
the output is flat and gives a flat amplitude and phase spectrum at high frequencies. In the middle frequency range white noise in the tank circuit is dominant, so that the phase spectrum slopes at 6 dB/octave. The amplitude spectrum for the instantaneous limiting oscillator slopes at 6 dB/octave above the frequency \( \frac{\mu}{2\pi} \) and is flat below it. At low frequencies flicker noise is dominant so that the amplitude spectrum slopes at 3 dB/octave. The low frequency deviation spectrum given by Equation 4.12 slopes at 3 dB/octave so that the phase spectrum will slope at 9 dB/octave. In some devices flicker noise does not slope at exactly 3 dB per octave so that the phase and amplitude spectra at low frequencies can vary from that shown in Figure 4.6.
REFERENCES FOR CHAPTER 4


CHAPTER 5
CHARACTERISATION AND MEASUREMENT OF OSCILLATOR NOISE

5.1 Characterisation of Oscillator Noise

An oscillator is a device which produces a simple periodic time waveform such as a sine or square wave. The output waveform of the oscillator is, however, never perfectly periodic because of the effects of noise, although it can be considered to be periodic for many purposes. Here we will consider only sinusoidal oscillators or oscillators which are used in such a way that only their fundamental frequency is of importance. Most radio frequency oscillators produce sine waves and many of the systems with which they are used have narrow band filters at their inputs. Thus for many purposes it is sufficient to use a sinusoidal representation of an oscillator signal. It will also be assumed that noise in an oscillator does not produce any power in the oscillator signal at frequencies far away from the fundamental frequency. Again this restriction is fulfilled by many oscillators. Also because of the narrow band filters used in many radio frequency systems, one can usually consider an oscillator signal with broad band noise as a narrow band signal.

Having limited our consideration to narrow band oscillator signals we can use the following representation for an oscillator signal \( v(t) \).

\[
v(t) = [A_o + o(t)] \sin[\omega_o t + \phi(t)] \quad \ldots (5.1)
\]
Here $a(t)$ and $\phi(t)$ are randomly varying due to the effect of noise. It is assumed that

$$\frac{a(t)}{A_0} \ll 1 \quad \& \quad \frac{\dot{\phi}(t)}{\omega} \ll 1$$

If this oscillator signal is to be characterised in the frequency domain then there are four power spectral densities that might be used.

(i) $S_v(f)$ the spectral density of the whole oscillator signal which has the units of volts$^2$/Hertz. This can be thought of as consisting of a carrier with noise modulation sidebands. It is often called the RF spectrum.

(ii) $S_a(f)$ the spectral density of the amplitude fluctuations $a(t)$ which has the units of volts$^2$/Hertz. This can be found by passing the signal into an amplitude detector and analysing its output.

(iii) $S_p(f)$ the spectral density of the phase fluctuations $d(t)$ which has the units of radians$^2$/Hertz. This can be found by passing the signal into a phase detector and analysing its output.

(iv) $S_{\Delta\nu}(f)$ the spectral density of the frequency deviation $\Delta\nu(t) = \dot{\phi}/2\pi$ which has the units of Hertz$^2$/Hertz. This can be found by passing the signal into a frequency discriminator and analysing its output.

The phase spectrum $S_{p}(f)$ and the frequency deviation spectrum $S_{\Delta\nu}(f)$ can easily be related because differentiation in the time domain is equivalent to multiplication by $j\omega$ in the frequency domain.
Therefore

\[ S_\varphi(f) = \omega^2 S_\theta(f) \]

\[ S_{\Delta v}(f) = \frac{1}{4\pi^2} S_\theta(f) = f^2 S_\theta(f) \]

The phase and amplitude spectrum of an oscillator are usually independent so that one cannot be found from the other. In order to completely characterise the oscillator noise performance (as far as second moment measures are concerned) we need to know the spectra of both \( a(t) \) and \( \phi(t) \).

The RF spectrum is of great importance in many applications and is frequently used to characterise the noise performance of an oscillator. However, it does not completely characterise the oscillator noise since it is not possible to find \( S_a(f) \) and \( S_\phi(f) \) from it, although the reverse is possible. For an oscillator in which either the amplitude or phase noise is predominant the RF spectrum can be very simply related to the noise modulation spectrum. When both amplitude and phase noise are important then the relationship between \( S_a(f) \), \( S_\phi(f) \) and \( S_v(f) \) is more complicated, particularly when \( a(t) \) and \( \phi(t) \) are correlated.

When the phase noise is predominant and the modulation index low, then the RF spectrum is given by the following expression as shown in appendix 8.1.

\[ \frac{S_v(f)}{A_0^2/2} = \frac{1}{2} S_\phi(f-f_0) + S(f-f_0) \]
This expression is an important and widely used result, because in all high quality oscillators the phase noise is predominant and the modulation index low. It states that the phase noise spectrum is twice the single sideband noise spectrum over the carrier power. In the unlikely event of the amplitude noise being predominant then a similar expression can be obtained relating the RF spectrum to the amplitude spectrum.

\[
\frac{S_v(f)}{A_o^2/2} = \frac{S_o(f-f_o)}{2A_o^2} + S(f-f_o) \quad \ldots \quad (5.4)
\]

The effects of noise can also be characterised in the time domain. Although there are many ways this may be done, only the variance of the fractional frequency deviation is of any practical use. This can be measured by a high resolution counter by taking a large number of readings of the frequency of a signal source. From these readings the mean and variance of the frequency measured over the averaging time of the counter can be found (5.1). This method of characterising the output of an oscillator gives a measure of the frequency stability but not the amplitude stability. The variance of the fractional frequency deviation can be found from the phase spectrum but the reverse is in general more difficult. It is thus more useful if oscillator noise is characterised in the frequency domain than in the time domain by measuring fractional frequency deviation. Correlation functions are not used to characterise oscillator noise because they are more difficult to measure. They are, however, useful for theoretical work and can be found by taking the Fourier transform of the power density spectrum.
Figure 5.1: Measurement of the RF spectrum

Figure 5.2: The effect of automatic frequency control on the RF spectrum of an oscillator
At present only second moment measures are used to characterise oscillator noise. No real use has been found for distribution functions so these are not considered.

5.2 Methods of Measuring Oscillator Noise

5.2.1 The RF Spectrum

Although the RF spectrum does not completely characterise an oscillator it is often measured when the phase and amplitude spectra are not required. We will consider the practical problems involved in measuring the RF spectrum. Commercial spectrum analysers have insufficient resolution and dynamic range to measure the effects of oscillator noise from all but very noisiest oscillators. A specially built test set as shown in Figure 5.1 can be used to measure oscillator noise (5.2).

The signal from the oscillator being tested is shifted down to a low intermediate frequency, by a local oscillator, so it can be measured by a low frequency wave analyser. The oscillator being tested must be tuned exactly 25 kHz from the local oscillator frequency so that a 25 kHz IF frequency is produced. The noise sideband amplitude 1 kHz from the carrier can then be measured by tuning the wave analyser to 24 kHz. In order to prevent the wave analyser being overloaded by the carrier, while the low level noise sidebands are measured, a tunable IF amplifier must be used. By this means a dynamic range of 130 dB can be achieved. Noise from the local oscillator will be added to the noise which is being measured. Fortunately it is easy to make a crystal local oscillator which has much less noise than any variable frequency oscillator. It is thus possible to
Figure 5.3: Amplitude and frequency noise modulation measurement

Figure 5.4: Measurement of phase noise modulation
neglect local oscillator noise but measurements can only be made at the fixed frequencies for which crystals are available.

Frequency drift of the oscillator while it is being tested must be avoided if accurate readings of the noise spectrum close to the carrier are to be obtained. If the oscillator drifts relatively slowly then it can be retuned with the aid of a frequency counter before each reading is taken. When the oscillator drifts too rapidly to be corrected manually, then an automatic frequency control system is required. This can only be used with oscillators which can be tuned by an externally applied voltage. Automatic frequency control can be achieved by the action of a discriminator as shown in Figure 5.1. The measured noise spectrum can be modified by the automatic frequency control unless its bandwidth is sufficiently small. Figure 5.2 shows the type of effect which can be obtained. Frequency drift of an oscillator can be considered to be a type of very low frequency oscillator noise. The drift is not usually entirely random like true noise, because part of it is usually due to deterministic changes such as variations in temperature. The automatic frequency control system tries to prevent changes in frequency and cannot distinguish between drift and low frequency noise. If automatic frequency control is used then its time constant should be adjusted so that a true noise spectrum is measured.

5.2.2 Amplitude and Frequency Noise Modulation Measurement

The amplitude and frequency noise modulation spectra can be found by passing the oscillator signal into an amplitude detector and frequency discriminator and measuring the outputs. This can be done much more easily than the measurement of the RF spectrum. Frequency drift is not a problem
Figure 5.5: 60MHz crystal oscillator

Figure 5.6: 1.5 MHz A.M. detector
and the carrier can easily be removed from the noise modulation by simple low-pass filtering. It is difficult to make a discriminator suitable for use over a wide range of radio frequencies but this can be avoided by using a local oscillator to bring the signal to a fixed IF for detection as shown in Figure 5.3.

The noise produced by the local oscillator, mixer IF amplifier and detectors limits the performance of this system. However, it is no worse than the RF spectrum method in this respect. The internal noise produced by this method is not so large as to prevent measurements being made on the best variable frequency oscillators but the method is not good enough for measurements on a crystal oscillator.

5.2.3 Measurement of Phase Noise Modulation

A phase detector may be used instead of a frequency discriminator to measure noise phase modulation rather than noise frequency modulation (5.3). With this method the oscillator being tested must be phase locked, so it is at the same frequency as the local oscillator. This is a disadvantage because only oscillators with electrical tuning can be tested by this method. However, it is capable of a much greater dynamic range because the detection is carried out very early in the system as shown in Figure 5.4.

With this phase detector method, noise from a crystal oscillator can be measured even when the noise from the local oscillator is of the same order of magnitude. This is done by using a third crystal oscillator and taking three sets of readings, one for each pair out of the three. With these results three simultaneous equations can be written and solved to give the
Figure 5.7: Mixer circuit
individual noise from each oscillator. As with automatic frequency control, phase locking can modify the phase spectrum of the oscillator being tested. Care must be taken to ensure that the time constant of the feedback path is sufficiently large to avoid this.

5.3 A Practical Test Set for Measuring Oscillator Noise

The work done for this thesis was intended to be mainly applicable to variable frequency VHF oscillators. Thus oscillator noise produced by this type of oscillator had to be measured. For this purpose it was decided that a test set consisting of an amplitude detector and frequency discriminator was most suitable, because this method does not require the use of phase locking or automatic frequency control. A test set was built corresponding to the block diagram shown in Figure 5.3.

The crystal local oscillator used to bring the signal down to the intermediate frequency of 1.5 MHz is shown in Figure 5.5. A second crystal oscillator is needed to check the noise performance of the test set and prove that it is negligible compared with the noise from the oscillator being tested. Figure 5.7 shows how the mixer, which is a balanced ring modulator, is constructed. Hot carrier diodes are used rather than point contact diodes because they produce less noise. The transformers have a transmission line form to give a wide bandwidth (5.4). In order to ensure linear operation of the mixer the local oscillator signal must be about $\frac{1}{2}$ volt and the signal from the oscillator being tested must be attenuated to about 100 mV.

When amplitude noise is to be measured the mixer intermediate frequency output is connected to the AM detector shown in Figure 5.6. The
Figure 5.8: 1.5 MHz limiting amplifier and discriminator
circuit consists of a voltage amplifier, diode peak detector and low-pass filter. A transformer input is used so that the source impedance seen by the transistor amplifier can be adjusted for the lowest noise figure. Normally, in a low noise amplifier transistors are run at very low emitter currents to give a low noise performance. Unfortunately in an AM detector large signals must be handled which require large emitter currents. The choice of emitter current for the first transistor in the AM detector is thus a compromise. Series feedback is used to stabilize the voltage gain of the amplifier. Shunt feedback is used to lower the input impedance of the amplifier, so the input impedance of the detector is 50 Ω and matches the output of the mixer. A 50 Ω input impedance could have been obtained by connecting a 50 Ω resistor across the input. This would worsen the noise performance of the amplifier whereas the 50 Ω input impedance created by negative feedback does not. The AM detector is designed to operate with an input signal of about 50 mV which gives a DC output of 5 volts. The DC output voltage is proportional to the carrier amplitude $A_o$ and the AC output voltage is proportional to the modulation $a(t)$. Thus the RMS value of $a(t)/A_o$ is found by dividing the RMS value of the AC voltage by the DC voltage.

When frequency noise modulation is to be measured then the mixer is connected to the limiting amplifier and frequency discriminator shown in Figure 5.8. The limiter consists of two integrated circuits which employ a matched pair of emitter coupled transistors to give symmetrical current limiting at the output. The discriminator uses a double tuned transformer with diodes rather than one of the more modern pulse circuits because it was found to give less noise. The design of the discriminator is based on
Figure 5.9: Detail of transformer TR1

\[ L_1 = \left( N_1^2 + N_2^2 \right) A_L \]

\[ L_2 = 4N_3^2 A_L \]

\[ M = 2N_2N_3 A_L \]
K. R. Sturley's paper on the phase discriminator (5.5). The linear part of the discriminator characteristic is 80 kHz wide so the oscillator being tested can drift several kHz while measurements are made. The construction of the double tuned transformer is shown in Figure 5.9. By using ferrite ring cores it is very easy to obtain exactly the required amount of coupling between the two circuits and very good symmetry for the centre tapped secondary.

The sensitivity of the discriminator is found by measuring the output voltage when a FM signal generator with a known amount of frequency deviation is connected to the input. The discriminator sensitivity is expressed in 'volts per hertz' where the volts and hertz are both either RMS, peak or average. Since all signal generators are calibrated for peak deviation, then a peak reading voltmeter must be used to find the sensitivity of the discriminator. When the noise modulation is being measured then mean squared values are required so that a RMS voltmeter is used.

Before the oscillator noise modulation is measured the level and frequency of the oscillator must be adjusted so that the AM detector and discriminator are working properly. They both require the same conditions, so that the frequency can be set by adjusting the oscillator until the discriminator output is zero. The correct level is obtained when the AM detector output is 5 volts. When this has been done the output of the detector being used is taken to the 60 dB gain low noise amplifier shown in Figure 5.10. This is a conventional amplifier which has a low noise figure when used with high source impedances.
Figure 5.10: 60 dB gain low noise amplifier
The spectrum at the output of the low noise amplifier is measured with a TF 2330 Marconi Instruments Wave Analyser. This wave analyser has a constant analysis bandwidth of 6.5 Hz and a frequency range from 20 Hz to 50 kHz. It is not entirely suitable for measuring noise because the meter is a full-wave rectified average type scaled to read RMS for sine waves and the meter time constant is too short for accurate readings of noise. Taking into account the average responding meter and assuming that Gaussian noise is to be measured, then the noise equivalent bandwidth is 5.0 Hz. In order to improve the accuracy an external meter is used with a time constant of 10 seconds. With this only 5% of the readings will be in error by greater than 1 dB (5.6).

Figures 5.11 and 5.12 show the phase and amplitude spectra of an oscillator obtained from measurements made with the test set. The noise produced by the test set is also shown and it can be seen that it is well below the oscillator noise. The oscillator used was a Marconi Instruments Signal Generator type TF 144H/4 which by comparison with other signal generators has a small amount of noise modulation. Figures 5.13 and 5.14 give the tabulated results and show how the readings taken with the test set are used to find the phase and amplitude spectra. The results show in addition to the noise modulation that the mains frequency and its harmonics cause both AM and FM modulation. The plotting of a continuous and discrete spectrum together presents some difficulty because the continuous spectrum is a power density while the discrete spectrum is just power. This is overcome by plotting the discrete spectrum as a power density with a line width of 1 Hz.
The power spectrum and power density spectrum of a discrete signal are then numerically the same and the problem of plotting dirac delta functions of infinite height is avoided.

The spectra in Figures 5.11 and 5.12 were measured in a quiet environment so that any additional modulation due to microphony was negligible. The mains frequency modulation and microphony are not the main concern of this thesis but it is important to take account of these effects in order to obtain true measurements of the effects of electrical noise. Mains frequency and low frequency noise modulation occur in a similar way but the mains frequency modulation can be reduced to any desired level by filtering the power supplies. Microphony is also more straightforward compared with the effects of electrical noise. It is caused by sound waves which set the mechanical parts of the tank circuit vibrating. This mainly affects the resonant frequency of the tank circuit so that microphony has more effect on the phase spectrum than the amplitude spectrum.

The phase and amplitude spectra in Figures 5.11 and 5.12 are typical of a high quality valve signal generator. The noise modulation is predominantly phase modulation so that the RF spectrum divided by the carrier power can be found by subtracting 3 dB from the phase spectrum. The noise modulation is less than in most modern transistor signal generators but the microphony and mains frequency modulation is worse. The phase noise spectrum falls at about 9 dB per octave with frequency and the amplitude noise spectrum initially falls at 3 dB per octave but flattens out at around 20 kHz.
Figure 5.12: Amplitude noise spectrum

-60
-80
-100
-120
-140
-160

$f (Hz)$

$S_a(f)/A_o^2 (dB)$

- MAINS FREQUENCY & HARMONICS
- OSCILLATOR NOISE
- NOISE FROM TEST SET

(f (Hz) LINE WIDTH)
Figures 5.11 and 5.12 can be compared with the theoretical amplitude and phase spectra shown in Figure 4.6 of Section 4.3. It can be seen that the phase noise spectrum in Figure 5.11 is due to flicker noise because it slopes at about 9 dB per octave. The test set is not able to measure the phase spectrum at sufficiently high frequencies to see the effects of white noise in the tank circuit or additive noise. The amplitude spectrum in Figure 5.12 is due to flicker noise where it slopes at 3 dB per octave and white noise in the tank circuit where it is flat. Again the test set is not able to measure sufficiently high in frequency to see the effect of noise added to the output of the oscillator.
### Oscillator Noise (TF144/H)

<table>
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<th>$f$ (Hz)</th>
<th>$V$ (Volts RMS)</th>
<th>$\Delta V$</th>
<th>$\Delta V$</th>
<th>$S_{aw}(f)$</th>
<th>$S_{aw}(f)/f^2$</th>
<th>$S_{aw}(f)$</th>
<th>$S_{aw}(f)/f^2$</th>
<th>$S_{aw}(f)$</th>
<th>$S_{aw}(f)/f^2$</th>
<th>$S_{aw}(f)$</th>
<th>$S_{aw}(f)/f^2$</th>
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<td>0.461</td>
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<td>0.55 $\mu$V</td>
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<td>0.0036</td>
<td>2.6 $\times 10^{-6}$</td>
<td>6.5 $\times 10^{-15}$</td>
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<td>70 $\mu$V</td>
<td>8.4</td>
<td>79          *</td>
<td>3.16 $\times 10^{-2}$</td>
<td>-150</td>
<td>35 $\mu$V</td>
<td>0.42</td>
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<td>7.04 $\times 10^{-5}$</td>
<td>-41.8</td>
<td></td>
</tr>
<tr>
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<td>-47.5</td>
<td>6.5 $\mu$V</td>
<td>0.0779</td>
<td>0.0068</td>
<td>6.08 $\times 10^{-7}$</td>
<td>-62.2</td>
<td></td>
</tr>
<tr>
<td>1500</td>
<td>75 $\mu$V</td>
<td>0.90</td>
<td>0.81        *</td>
<td>3.58 $\times 10^{-5}$</td>
<td>-44.5</td>
<td>8.5 $\mu$V</td>
<td>0.102</td>
<td>0.0104</td>
<td>4.62 $\times 10^{-8}$</td>
<td>-73.4</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>40 $\mu$V</td>
<td>0.48</td>
<td>0.23        *</td>
<td>5.52 $\times 10^{-6}$</td>
<td>-52.6</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td></td>
</tr>
</tbody>
</table>

* Discrete components are not corrected for 5 Hz Filter  
† Disc. 0.012 Hz/$\mu$V
| \( f \) (Hz) | \( V \) (volts r.m.s.) | \( \frac{a}{A_o} \) | \( \frac{S_a(f)}{A_o^2} \) | \( \frac{a^2}{5A_o^2} \) | \( dB \) | \( V \) (volts r.m.s.) | \( \frac{a}{A_o} \) | \( \frac{S_a(f)}{A_o^2} \) | \( \frac{a^2}{5A_o^2} \) | \( dB \) |
|-----|---------|--------|-------------|----------------|---|---------|--------|-------------|----------------|---|-------|
| 25 | 14.5 | 2.8\times10^{-6} | 1.57\times10^{-12} | -118.0 | 2.3 | 0.46\times10^{-6} | 0.0422\times10^{-12} | -133.7 |
| 75 | 7.5 | 1.5 \times 10^{-6} | 0.45 \times 10^{-12} | -123.5 | 1.4 | 0.28 \times 10^{-6} | 0.0157 \times 10^{-12} | -138.0 |
| 225 | 4 | 0.9 \times 10^{-6} | 0.128 \times 10^{-12} | -128.9 | 0.8 | 0.16 \times 10^{-6} | 0.00512 \times 10^{-12} | -142.9 |
| 625 | 2 | 0.4 \times 10^{-6} | 0.032 \times 10^{-12} | -135.0 | 0.45 | 0.08 \times 10^{-6} | 0.00162 \times 10^{-12} | -147.9 |
| 2k | 1.3 | 0.26 \times 10^{-6} | 0.0185 \times 10^{-12} | -138.7 | 0.33 | 0.066 \times 10^{-6} | 0.00087 \times 10^{-12} | -150.4 |
| 6k | 0.75 | 0.15 \times 10^{-6} | 0.0045 \times 10^{-12} | -143.5 | 0.22 | 0.044 \times 10^{-6} | 0.000388 \times 10^{-12} | -154.1 |
| 20k | 0.65 | 0.13 \times 10^{-6} | 0.0033 \times 10^{-12} | -144.7 | 0.20 | 0.04 \times 10^{-6} | 0.00032 \times 10^{-12} | -154.9 |
| 50 | 130 | 2.6 \times 10^{-6} | 6.77 \times 10^{-12} | -91.7 | - | - | - | - |
| 100 | 65 | 1.3 \times 10^{-6} | 1.69 \times 10^{-12} | -97.7 | 12 | 2.4 \times 10^{-6} | 5.77 \times 10^{-12} | -112.4 |
| 150 | 19 | 3.8 \times 10^{-6} | 1.4 \times 10^{-12} | -108.4 | - | - | - | - |
| 200 | 25 | 5.0 \times 10^{-6} | 2.5 \times 10^{-12} | -106.0 | 5.5 | 1.1 \times 10^{-6} | 1.21 \times 10^{-12} | -119.2 |

* DISCRETE COMPONENTS ARE NOT CORRECTED FOR 5TH FILTER
† D.C. OUT OF DETECTOR 5 VOLTS
References for Chapter 5


CHAPTER 6

COMPARISON BETWEEN THEORETICAL AND EXPERIMENTAL RESULTS

6.1 Philosophy of Experimental Work

This chapter describes the experimental work which was undertaken to confirm that the theoretical results derived in Chapters 3 and 4 are correct. Only the effect of flicker and white noise inside the oscillator is considered. The effect of white noise which is added to the output of an oscillator is well known and does not require experimental confirmation. This is because it is so frequently encountered when a radio receiver is receiving a weak signal. Thermal noise from the aerial and receiver noise, accompany the radio signal, so that the output of the amplitude detector or discriminator of the receiver, contains noise modulation.

In order to check the theoretical results for oscillator noise, the location and magnitude of all of the sources of electrical noise in an oscillator must be known. The magnitude of some of the noise sources can be calculated and the magnitude of others measured but it is rarely possible to find the magnitude of every source of noise. This difficulty can be overcome by using a noise generator to inject noise into an oscillator. If the oscillator noise spectra is made much bigger by the externally generated noise, then it alone can be considered responsible for the resulting noise spectra. Since only one source of noise of known location and magnitude need be considered, then the theoretical results
Figure 6.1: 50.5 Hz instantaneous limiter oscillator
can be checked more easily and with greater certainty.

The preceding theoretical work predicted that hard limiting oscillators have less amplitude noise than soft limiting oscillators. If the limiting could be changed from hard to soft in a real oscillator, then one might be able to observe a change in the amplitude noise. Normally in an instantaneous limiting oscillator the limiting and amplification are provided by the same active device. With this arrangement it is not possible to change the hardness of limiting without also changing the amplitude of oscillation and the noise generated by the active device. Thus when the limiting is altered, any changes in the oscillator noise cannot be attributed to it alone.

6.2 Experimental Instantaneous Limiting Oscillator

6.2.1 Design of Oscillator

A special experimental instantaneous limiting oscillator has been built to observe the effects of oscillator noise. The circuit of this oscillator is shown in Figure 6.1. The limiting and amplification in this are performed by separate parts of the circuit. It is anticipated that with this arrangement, the limiting can be changed without affecting the amplitude of oscillation or the magnitude of the noise sources in the oscillator. The diodes D1 and D2, and the variable resistors VR1 and VR2 control the limiting action. With VR2 set for maximum resistance and VR1 at a minimum, the diodes perform symmetrical clipping of the oscillator signal and the oscillator is set for hard limiting.
If then, VR1 is set for maximum resistance, the level of oscillation in the tank circuit doubles showing that non-linearities in the field effect transistors, limit the amplitude of oscillation at a higher level than the limiting circuit. When the oscillator is to be changed to soft limiting, VR1 is set at a maximum and VR2 is slowly reduced until the original amplitude of oscillation is obtained.

The limiting circuit is placed at a low impedance point between the pair of cascode connected field effect transistors. This tends to prevent changes in the limiting from affecting the loading on the tank circuit. The output is also taken from this point in the circuit, so that a measure of the hardness of limiting can be obtained from the distortion in the output signal. An output taken from the tank circuit always appears sinusoidal even with hard limiting because of the high Q of the tank circuit. Less noise would be added to the output of the oscillator signal if the output was taken from the tank circuit. Since this type of oscillator noise is not to be investigated and it is only important at frequencies well away from the carrier, then a large amount of additive noise in the output can be tolerated.

Noise from an external source can be added into the tank circuit through a small coupling coil placed near to the tank circuit. Only loose coupling is permissible between the tank circuit and coupling coil, so that the tank circuit is not appreciably loaded when a noise source is connected to the oscillator. Unfortunately this means that only a
fraction of the noise power applied to the oscillator is induced into the tank circuit.

The experimental oscillator was designed to have small noise sidebands close to the carrier. This makes the small amount of external noise which can be added to the tank circuit predominate over the internal noise. Equation 3.56 indicates how the effect of thermal noise in the tank circuit can be minimised. The capacitance of the tank circuit, the amplitude of oscillation and the Q of the tank circuit must all be as large as possible. There are practical limitations to the values which can be used for the experimental oscillator. The capacitance, for example, must not be made too large or the inductance of the tank circuit becomes comparable with the inductance of component leads. The tap on the tank circuit coil is close to the end of the coil connected to the drain of the FET. Thus the tank circuit appears at the input of the active device rather than the output. Section 2.2.2 showed that this minimises the effect of noise from within the active device.

Field effect transistors were used for the experimental oscillator for several reasons. Valves and FET oscillators have generally been found to produce less oscillator noise than bipolar transistors. They have a high input impedance which usually enables a higher Q to be obtained for the tank circuit. They tend to produce less flicker noise modulation. It is convenient to use an FET rather than a valve because then only a low voltage power supply is required.
The cascode connection of common source and common gate field effect transistors is used to reduce miller effect capacitance. Since this is due to the gate-drain junction capacitance, flicker noise causes the miller effect capacitance to fluctuate and generate noise frequency modulation.

6.2.2 Internal Noise in Experimental Oscillator

The noise performance of the experimental oscillator will now be considered. The phase and amplitude noise modulation were measured with the test set described in Section 5.3 and the results are shown in Figures 6.2. The amplitude noise was found to be comparable with the noise from the test set and is not plotted. The mains frequency modulation sidebands are not shown since it is the noise modulation which is of interest. The noise performance of the experimental oscillator is about the same as for the Marconi Instruments Signal Generator TF 144H which is shown in Figures 5.11 and 5.12.

The results given in Figure 6.2 for the experimental oscillator were for hard limiting. When set for soft limiting the phase noise at low frequencies is about 3 dB less. There should be no difference in the phase noise for hard and soft limiting, provided the limiting mechanism does not alter the noise inside the oscillator. It will become apparent later on, when the results obtained with external noise injected into the oscillator tank circuit are seen, that this discrepancy must be due to the clipping diodes generating more noise when set to
give hard limiting. This illustrates how easy it is to draw the wrong conclusions from experimental results, unless it is known for certain which source of electrical noise in an oscillator is producing the oscillator noise being measured.

Before investigating the effect of externally injected noise it is worth considering what types of noise are responsible for the phase spectrum in Figure 6.2. A straight line drawn through all the points for the phase noise spectrum would slope at 27 dB per decade. The theoretical results predict that flicker noise gives a 30 dB per decade and white noise in the tank circuit a 20 dB slope to the phase spectrum. Thus in Figure 6.2 a straight line sloping at 30 dB per decade has been drawn through the low and middle frequency points and a 20 dB per decade line drawn through the upper frequency points. At the point where the phase noise spectra due to flicker and white noise are equal, the combined spectrum is of course 3 dB higher than the straight line approximation. The theoretical results suggest that most of the phase noise spectrum plotted is due to flicker noise. Only at frequencies above 10 kHz does white noise become predominant.

6.3 White Noise in the Instantaneous Limiting Oscillator

6.3.1 Oscillator Noise Due to Noise Generator

The experimental oscillator described in the previous section has been used to check the theoretical results given in Section 3.3. This was achieved by injecting white noise into the tank circuit from
Figure 6.3: Noise Generator

3 dB BANDWIDTH 12-100 MHz AND OUTPUT E.M.F. 1mV IN 300 kHz
a specially designed noise generator. With this method the oscillator noise spectra can be attributed, with some certainty, to the one source of electrical noise.

The circuit for the noise generator is shown in Figure 6.3. The 12 V Zener diodes produce noise which is amplified by a current feedback transistor amplifier. The noise generator has an output EMF of 1 mV measured in a 300 kHz bandwidth. Its source impedance is 50 Ω and the 3 dB bandwidth of the noise spectrum is from 12 MHz to 100 MHz.

With the noise generator connected to the experimental oscillator, the phase and amplitude spectra shown in Figure 6.4 were obtained. The phase spectrum is independent of the limiting and slopes at 20 dB per decade. The amplitude spectrum is flat and is reduced by 11 dB by changing the oscillator from soft to hard limiting. This agrees exactly with the theoretical result for an instantaneous limiting oscillator, when the only source of noise is white noise in the tank circuit. These results are encouraging but it is important to see if the theoretical results also give the correct magnitude for the spectra. It is not very easy to do this because some of the oscillator parameters which are required are difficult to measure.

We need to know the amplitude of oscillation in the tank circuit. This is difficult to measure accurately because any connection made to the tank circuit tends to stop the oscillator working. The voltage in the tank circuit can be found approximately by using two RF voltmeters
Figure 6.4: Noise modulation due to externally added noise
with high impedance probes. One voltmeter can be used to measure the voltage in the tank circuit but this reading is low by about 50% because of the loading effect of the voltmeter probe. The second voltmeter is used to measure the voltage at the noise input socket which has a 50 Ω load connected to it. This is a relatively low impedance point so that the voltmeter here does not load the tank circuit. The fall in reading of this second voltmeter, when the first is connected to the tank circuit gives a measure of the error in measurement of the tank circuit voltage.

The magnitude of the equivalent noise current generator across the tank circuit must also be found. This can be done by observing the magnitude of the oscillator signal voltage which appears at the noise input socket and then applying the reciprocity theorem. Consider the circuit shown below in Figure 6.5a) where a voltmeter is connected to the noise input socket to measure the oscillator signal voltage there. Let the total parallel loss resistance of the tank circuit be $R'$. This includes any loading effect due to $R_N$.

Then we have:

$$I = \frac{V_t}{R'}$$
a) Oscillator signal in tank circuit

\[ v_t = A_0 \sin \omega t \quad V_t = \frac{A_0}{\sqrt{2}} \quad \omega_0 = \frac{1}{\sqrt{LC}} \]

b) Noise signal in tank circuit

Figure 6.5: Calculation of equivalent noise current generator in the tank by the reciprocity theorem
Let $Z$ be the transfer impedance for the voltmeter reading and current generator $I$.

$$Z = \frac{V_o}{I} = \frac{V_o R'}{V_t}$$

Now exchange the current generator and ideal voltmeter and look at Figure 6.5b in which the noise signal is considered. By the reciprocity theorem we have:

$$V_N = Z I_N = \left(\frac{V_o R'}{V_t}\right)\left(\frac{E_N}{R_N}\right)$$

Here $E_N$ is the EMF of the noise generator in volts per root hertz.

Thus the power density spectrum of the equivalent noise current generator across the tank circuit $N_o$ is given by:

$$N_o = \left(\frac{V_N}{R'}\right)^2 = \left(\frac{E_N V_o}{R_N V_t}\right)^2 \text{ AMPS}^2/\text{Hz}$$

The measured values for the oscillator parameters which we require are:

- $A_o = 2.5 \text{ V}$ (V$_t = 1.77 \text{ V RMS}$)
- $V_o = 21 \text{ mV}$
- $R_N = 50 \text{ }\Omega$
- $E_N = 10^{-3} \sqrt{300,000}$
- $C = 13 \text{ pF}$

Substituting in these values gives $N_o = 1.9 \times 10^{-19} \text{ AMPS}^2/\text{Hz}$.
Now we can use equations 3.53 and 3.44 to calculate the theoretical phase noise spectrum $S_\phi(f)$.

$$S_\phi(f) = \frac{2D_0}{\omega^2}, \quad \ldots \quad (3.44)$$

$$D_0 = \frac{N_0}{4A_0^2C^2}, \quad \ldots \quad (3.53)$$

$$D_0 = 4.5$$

The phase noise spectrum at 1 kHz may be calculated

$$S_\phi(1000) = 2.29 \times 10^{-6}$$

$$10 \log_{10} S_\phi(1000) = -56.4 \, \text{dB}$$

From Figure 6.4 it can be seen that the measured phase noise spectrum at 1 kHz is $-65$ dB. The measured and theoretical phase noise spectra of this 60 MHz oscillator differ by 9 dB. This is almost certainly due to measurement errors which tend to be large at high frequencies. A difference of only 1.5 dB between theoretical and measured phase noise has been observed for a 1.5 MHz oscillator. Details of this are given on the additional pages 142c - 142f which have been added overleaf.

One of the most critical measurements required for the 60 MHz oscillator was the value of $A_0$. Since the phase noise is
Figure 6.16: Circuit of 1.5 MHz Oscillator

L₁, L₂ & L₃ wound on LA1377 POT CORE
L₁ = 57.80 µH (36 turns), M₁₃ = 4.87 µH
L₃ = 0.63 µH (4 turns), L₂ is 9 turns
Additional Experimental Results

These additional experimental results, which were obtained with a 1.5 MHz oscillator, are in close agreement with the theoretical phase noise. Usually oscillator noise is unimportant in such a low frequency oscillator because it is smaller and more difficult to measure than at higher frequencies. By injecting noise into the oscillator, the resulting oscillator noise can be made much larger than it would normally be. This enables us to use the test set described in section 5.3 whose inherent noise is as great as that of a good 1.5 MHz oscillator. The test set has an I.F frequency of 1.5 MHz so that the oscillator can be connected directly to the frequency discriminator without the need for a local oscillator or mixer.

The circuit of the 1.5 MHz instantaneous limiting oscillator is shown in figure 6.16. The amplitude of oscillation is determined by the impedance seen by the collector of TR4 and the 5mA constant current source which feeds the emitters of TR3 and TR4. The impedance seen by TR4 was chosen for a relatively small amplitude of oscillation. This was done so that the input voltage rating of TR3 and TR4 is not exceeded and so that TR4 is not saturated. This ensures that the limiting mechanism is hard and symmetrical.

The noise is injected into the tank circuit by a resistive network which prevents the noise generator from loading the tank circuit and correctly terminates the 50Ω output of the noise generator. The noise generator's e.m.f. was set to 20 mV and its bandwidth was 20 MHz, but when the e.m.f. was checked it was
Figure 6.17: Equivalent Circuit for Calculating $N_0$

![Equivalent Circuit](image)

Figure 6.18: Table of Results for 1.5 MHz Oscillator

<table>
<thead>
<tr>
<th>$f$ (Hz)</th>
<th>$V^2$ (Volts RMS)</th>
<th>$\Delta V$ (Hz RMS)</th>
<th>$S_{\Delta V}(f)$</th>
<th>$S_{\phi}(f)$</th>
<th>$S_{\phi}(f)$</th>
<th>$S_{\phi}(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>14.85</td>
<td>0.1893</td>
<td>0.007167</td>
<td>1.147 x 10^-5</td>
<td>-49.4</td>
<td></td>
</tr>
<tr>
<td>75</td>
<td>14.52</td>
<td>0.1851</td>
<td>0.006852</td>
<td>1.218 x 10^-6</td>
<td>-59.1</td>
<td></td>
</tr>
<tr>
<td>225</td>
<td>14.46</td>
<td>0.1844</td>
<td>0.006801</td>
<td>1.343 x 10^-7</td>
<td>-68.7</td>
<td></td>
</tr>
<tr>
<td>625</td>
<td>14.30</td>
<td>0.1823</td>
<td>0.006647</td>
<td>1.702 x 10^-8</td>
<td>-77.7</td>
<td></td>
</tr>
<tr>
<td>2k</td>
<td>15.36</td>
<td>0.1958</td>
<td>0.007668</td>
<td>1.917 x 10^-9</td>
<td>-87.2</td>
<td></td>
</tr>
<tr>
<td>6k</td>
<td>15.10</td>
<td>0.1925</td>
<td>0.007913</td>
<td>2.198 x 10^-10</td>
<td>-96.5</td>
<td></td>
</tr>
<tr>
<td>20k</td>
<td>16.85</td>
<td>0.2149</td>
<td>0.00924</td>
<td>2.310 x 10^-11</td>
<td>-106.4</td>
<td></td>
</tr>
</tbody>
</table>

* DISC. 0.01275 Hz/μV
found to be 18.84 mV.

The magnitude of the equivalent noise current generator \( N_0 \) across the tank circuit can be found by using the reciprocity theorem as described earlier.

\[
N_0 = \left( \frac{E_n}{R_n} \frac{V_o}{V_t} \right)^2 \text{ Amps}^2/\text{Hz} \quad \text{(6.10)}
\]

The measured values for the 1.5 MHz oscillator were:

\[
A_o = 1.98\text{V} \quad (V_t = 1.40\text{V rms}) \quad C = 200 \text{ pF}
\]

\[
V_o = 5.6\text{mV} \quad R_n = 50 \quad E_n = 18.84/2 \times 10^7 \text{ mV/ Hz}
\]

Substituting these values in 6.10 gives

\[
N_0 = 1.136 \times 10^{-19} \text{ Amps}^2/\text{Hz}
\]

This can be checked by analysing the circuit shown in figure 6.17, because all of the component values are known. The coil is wound on a pot core and all the self and mutual inductances were measured with a Q meter at 1.5 MHz. This was impractical for the 60 MHz oscillators because the inductances were very much smaller. The following expression can be found for \( N_0 \).

\[
N_0 = \left| \frac{E_n/2}{\rho M_{13} + \rho^2 R_1 C_1 M_{13} + R_3 + \frac{L_3}{\rho^2 M_{13} C_1} + \frac{1}{\rho M_{13} C_1}} \right|^2
\]

Substituting the appropriate component values into this equation gives

\[
N_0 = 1.348 \times 10^{-19} \text{ Amps}^2/\text{Hz}
\]

These two values are fairly close and we will take the average value of \( 1.242 \times 10^{-19} \text{ Amps}^2/\text{Hz} \) as being correct. We can now use equations 3.43 and 3.53 to calculate the frequency deviation power spectrum \( S_{\Delta \nu}(f) \).

\[
D_0 = \frac{N_0}{4A_o^2 C^2} \quad \text{(3.53)} \quad S_{\Delta \nu}(f) = \frac{D_0}{2\pi^2} \quad \text{(3.43)}
\]
Figure 6.19: Phase Noise Power Spectrum of 1.5 MHz Oscillator

\[ S_\phi(f) \]

(dB)

\[ f \ (\text{Hz}) \]
The measured phase and frequency deviation power spectra are tabulated in figure 6.18 and the phase noise spectrum is plotted in figure 6.19. The results show that $S_{\phi}(f)$ remains almost constant at 0.007 Hz²/Hz from 25 Hz to 2 kHz and rises to 0.009 Hz²/Hz at 20 kHz. Thus most of the results are within 1.5 dB of the value given by the simplified theory which has been presented in this thesis. The rise in $S_{\phi}(f)$ at 20 kHz is to be expected because the noise added into the tank circuit not only perturbs the oscillator signal but also appears added to the signal at the output. The effect of noise which is added to the output was considered in section 4.1 and equation 4.8 enable us to calculate its effect. The amplitude of the oscillator signal at the point 'A' is 340 mV peak to peak and the magnitude of the noise which adds to it is 1 mV. The phase noise due to noise added to the output signal is $3.5 \times 10^{-12}$ Rad²/Hz. This additional phase noise becomes significant at about 20 kHz and is responsible for the slight increase in phase noise shown by the results.

$$D_o = 0.198 \quad \text{and} \quad S_{\phi}(f) = 0.01003 \text{ Hz}^2/\text{Hz}$$
inversely proportional to the fourth power of $A_0$, because $A_0$ is used to find $N_0$ as well as appearing in the expression for $D_0$, then a small error in $A_0$ has a large effect on the phase noise.

The theoretical results of Section 3.3 are only strictly applicable to the simple type of oscillators such as the Van der Pol oscillator. Our experimental oscillator is a Hartley oscillator which according to the analysis in Appendix 8.2 will have slightly different coefficients. A more accurate expression for $D_0$ is obtained if the term $(-a_k/c)$, which appears in the theory of Section 3.2, is replaced by $(-a_k/c)(L_2 + M)/L$. This gives:

$$D_0 = \frac{N_0}{4A_0^2C^2}\left(\frac{L_2 + M}{L}\right)^2$$

In the experimental oscillator $L_2$ consists of four turns of wire and $L_1$ one turn, so we expect $(L_2 + M)$ to be almost as large as $L$. We will not try to use the more accurate expression since it would be very difficult to find the values for $L_1$, $L_2$ and $M$. However, this could contribute to the 9 dB difference between the measured and theoretical results.

6.3.2 Effect of Thermal Noise in Tank Circuit

By using an external source of noise we have shown that the phase noise spectrum, due to white noise in the tank circuit slopes at 20 dB per decade and is independent of the hardness limiting. However, the
theoretical magnitude and measured magnitude of the phase noise spectra were found to differ by 9 dB. Thus we look to see what other checks on the theoretical magnitude of the phase noise spectrum can be made. Consider Figure 6.2 which shows the phase noise spectrum of the experimental oscillator due to the oscillators internal noise. The slope of the phase noise spectrum changes at around 10 kHz suggesting that thermal noise in the tank circuit is responsible for the spectrum above that frequency. Let us see if the theoretical results give the correct magnitude for the spectrum at 20 kHz.

In addition to the oscillator parameters which have already been found, we need to know the Q of the tank. The loaded Q, rather than the Q of the coil on its own, can be found by using the oscillator as an amplifier. The oscillator circuit is broken at point A and the two FET's separately biased.

Then the common gate FET can be used as an input amplifier and the common source FET as an output stage. The effective Q is found by measuring the 3 dB bandwidth of the amplifier. The Q was found to be 87 so that:

\[
D_o = \frac{kT\omega_0}{A_o^2CQ} \quad \ldots (3.56)
\]

\[
D_o = 2.16 \times 10^{-4}
\]
The measured phase noise spectrum at 2.0 kHz is at \(-134 \text{ dB}\) so that the agreement here is slightly better than that obtained with externally added noise. However, it is not certain that the measured phase noise spectrum is due entirely to thermal noise in the tank circuit.

6.3.3 Relaxation Time of Oscillator

The phase and amplitude spectra of the experimental oscillator shown in Figure 6.4 will be used with the theoretical results of Section 3.3 to find the relaxation time of the oscillator. This will then be compared to the relaxation time found by direct measurements made on the oscillator.
Section 3.3 gives the following theoretical equations for the phase and amplitude spectra:

\[
S_\phi(f) = \frac{2D_o}{\omega^2} \quad \ldots (3.44)
\]

\[
\frac{S_a(f)}{A_o^2} = \frac{2D_o}{\omega^2 + \mu^2} \quad \ldots (3.49)
\]

Hence:

\[
\mu = \sqrt{\frac{S_\phi(f)}{S_a(f)/A_o^2}} \omega \quad (\omega \ll \mu)
\]

Figure 6.4 gives \(S_\phi(100) = -45\) dB and \(S_a(100)/A_o^2 = -127.5\) dB for hard limiting. Therefore the hard limiting relaxation time \(1/\mu_H\) is found to be:

\[
\frac{1}{\mu_H} = 0.120 \quad \mu\text{Sec}
\]

Similarly the soft limiting relaxation time \(1/\mu_s\) is found to be:

\[
\frac{1}{\mu_s} = 0.412 \quad \mu\text{Sec}
\]
Chapter 3 gives a theoretical value for the relaxation time of the Van der Pol oscillator and of the ideal hard limiting oscillator. Of these two only the expression for the ideal hard limiting oscillator is easy to find numerically. Section 3.3.2 gives:

\[ \mu = \frac{\omega_0}{2Q} \]

This can be used as a rough check on the value we obtained for the relaxation time of the experimental oscillator with hard limiting. We have

\[ \omega_0 = 2\pi \times 58.5 \times 10^5 \]

\[ Q = 87 \]

Therefore \( \frac{1}{\mu} = 0.47 \mu \text{Sec.} \)

This result for the ideal hard limiting oscillator should be rather less than the value of 0.120 \( \mu \) sec obtained for the experimental oscillator. The experimental oscillator cannot limit the amplitude of oscillations as perfectly as the ideal hard limiting oscillator, so the ideal hard limiting oscillator must have the smallest possible relaxation time for a given value of \( Q \) and \( \omega_0 \). Again we have quite a large discrepancy. This could be due to the \( Q \) of the tank circuit being much lower for normal operating conditions than when measured under small signal conditions. The gate of the FET connected to the tank circuit might be brought near to forward conduction by the peaks of the oscillator signal and so damp the tank circuit.

A method of direct measurement for the relaxation time of experimental oscillator has been devised. In principle all one need
Figure 6.6: Measurement of relaxation time of an oscillator

Figure 6.7: Avalanche transistor
do is to disturb the amplitude of oscillation by a small amount and observe the time taken for the amplitude to return to its steady state. However, the disturbing signal must be much shorter than the period of oscillation. It should be timed to occur when the oscillator signal reaches a maximum so that it disturbs the amplitude and not the phase of oscillation. The disturbance should preferably be periodic so that it is easily observed but with sufficient time between disturbances for the oscillator to return to its steady state.

The arrangement shown in Figure 6.6 was devised to fulfil these requirements. The output of the experimental oscillator is taken to a unit which divides the frequency by a hundred. This unit is intended to be used as a range extender for a low frequency counter and gives a square wave output. The divider is connected to trigger the avalanche transistor circuit shown in Figure 6.7. This generates a pulse of about 3 nanoseconds width and 10 volts amplitude which is injected into the oscillator tank circuit by using the noise input socket. Thus every hundredth period of oscillation is disturbed by a short pulse. The timing of the pulse is controlled by adjusting the voltage applied to the avalanche transistor. (6.1)

The amplitude disturbance of the oscillator can be observed by using a sampling oscilloscope triggered by the output of the frequency divider. The gain of a sampling oscilloscope can be increased so that
Figure 6.8: Sampling oscilloscope display of oscillator amplitude transient
only a part of the signal applied to the oscilloscope appears on the screen. The display is valid when this is done because a sampling oscilloscope does not overload when the trace is deflected off the top or bottom of the cathode ray tube. Figure 6.8 shows the type of display of the oscillator amplitude which can be obtained. The actual displays obtained for disturbances to the amplitude of the experimental oscillator were of rather poor quality. Pick up of mains frequency signals, jitter in triggering the oscilloscope and noise made the display blurred.

For hard limiting the relaxation time of the experimental oscillator was found to be about 0.2 \( \mu \) sec and for soft limiting about 1.5 \( \mu \) sec. It is difficult to find the time constant of an exponential fall in amplitude from an oscilloscope display. One has to imagine the envelope of the waveform and then measure the time taken for it to fall by 0.707 of its initial value. Thus the measured values of 0.2 \( \mu \) sec and 1.5 \( \mu \) sec agree reasonably well with the values of 0.12 \( \mu \) sec and 0.412 \( \mu \) sec obtained from the noise spectra.

6.4 The Delayed Amplitude Limiting Oscillator

In this section we will look at the effect of noise on the amplitude spectrum of an experimental delayed amplitude limiting oscillator. The theoretical analysis of this type of oscillator in Section 3.4, dealt only with the effect of white noise in the tank circuit. In the experimental
Figure 6.9: 60 MHz A.M. detector circuit and frequency response when used with 60 dB low noise amplifier.
oscillator many types of noise affect the amplitude spectrum but the characteristic response of the control system can still be seen.

The parameters of the oscillator were chosen so that the control system is under-damped and produces a peak in the amplitude spectrum. This type of amplitude spectrum is distinctive and quite unlike that produced by an instantaneous limiting oscillator. The theory cannot be properly checked by the experimental results because the amplitude spectrum is caused by many types of noise and so it does not correspond exactly to the theoretically predicted shape. At low frequencies it is modified by flicker noise and at high frequencies by another source of noise. However, over a central part of the amplitude spectrum, the peak in the spectrum can be clearly seen. From this the theoretical results are used to find the time constant of the loop filter. This is in agreement with the measured time constant of the loop filter.

The test set which has been used for all the previous measurements of oscillator noise can only be used to measure noise modulation up to a frequency of 50 kHz. The amplitude noise spectrum of a delayed amplitude limiting oscillator can rise to its peak at a frequency greater than 50 kHz. In order to observe this effect properly a new amplitude detector was built. This works at the oscillator frequency and does not need a local oscillator or mixer. The modulation frequency bandwidth can be made much greater because the radio frequencies to be filtered out are at 60 MHz instead of 1.5 MHz. It would not be easy to make a frequency discriminator to work properly at 60 MHz and thus only the
Figure 6.10: Delayed amplitude limiting oscillator
amplitude spectrum is measured. Fortunately it is only the effect of noise on the amplitude of the oscillator which we wish to investigate in the experimental instantaneous limiting oscillator. The circuit for the new amplitude detector is shown in Figure 6.9 together with the frequency response of the detector for an amplitude modulation signal when it is used with a 60 dB gain low noise amplifier. The readings of noise modulation at frequencies above 50 kHz were corrected to take account of the fall off in gain of the amplitude detector and low noise amplifier.

The circuit diagram for the experimental delayed amplitude limiting oscillator is shown in Figure 6.10. First let us identify its main parts. The field effect transistor TR1 is used to make an oscillator in which the loop gain can be varied by altering the voltage applied to the gate. The transistors TR2 and TR3 form an amplifier which take a small signal from the oscillator without loading it and provides a large signal for both the amplitude detector and output transistor TR4. The amplitude detector consisting of D1 and D2 is not connected directly to the tank circuit as in the theoretical model because it loads the tank circuit. The functions of the servo amplifier and low-pass filter are performed by the integrated circuit operational amplifier IC1. Figure 6.11 shows the frequency response of this amplifier.
Figure 6.11: Servo amplifier frequency response
For normal operation as a delayed amplitude limiting oscillator the switch is set to A and the amplitude of oscillation is controlled by VR2. The limiting can be changed to instantaneous by setting the switch to B and then VR1 is used to adjust the amplitude of oscillation. Thus the amplitude noise spectrum of the instantaneous limiting oscillator can be measured for comparison with that for the delayed amplitude limiting oscillator.

Section 3.4 showed that the amplitude spectrum of a delayed amplitude limiting oscillator depends on the time constants $T$ and $\tau'$. The low-pass filter time constant is $T$ and $\tau'$ is given by the following equation.

$$\tau' = \frac{2}{MG\omega_0^2 A_0} \ldots (6.4)$$

In Section 3.4 $A_0$ is the amplitude of oscillation in the tank circuit which is the same as the voltage applied to the amplitude detector. In our experimental oscillator these are not the same. It can be seen from the analysis in Section 3.4 that the voltage applied to the amplitude detector should be used for $A_0$ in equation 6.4.

The theoretical results are most convincingly demonstrated by making $T \gg \tau'$ so that the amplitude spectrum has a distinctive peak at a frequency of $\frac{1}{(2\pi \sqrt{\tau'T})}$ as shown in Figure 3.8. The parameters of the experimental oscillator were chosen so that this distinctive peak was produced. The 60 kHz low-pass filter determines the
Figure 6.12: Amplitude noise spectra of instantaneous and delayed oscillators.

\[ S_o(f)/A_o^2 \] (dB)

\[ f \quad (Hz) \rightarrow \]
time constant $T$. The time constant $\tau$ can only be roughly estimated because $M$ cannot easily be measured. However, $\tau$ is proportional to the gain of the operational amplifier so that the effect of changing $\tau$ by a known factor can be observed.

Figure 6.12 shows the amplitude spectrum of the delayed amplitude limiting oscillator for two values of $\tau$ which are in the ratio of 4:1. The spectrum of the experimental oscillator when it is set for hard and soft instantaneous limiting is also shown. We will consider the delayed amplitude limiting first and compare these results with Figure 3.8. The measured amplitude spectra in Figure 6.12 appears similar in shape to the theoretical results over the frequency range of 10 kHz to 200 kHz. At frequencies greater than 200 kHz the measured spectra appear to be levelling off at about 138 dB rather than falling with increasing frequency like the theoretical spectra. This could be due to high frequency noise in the operational amplifier adding to the oscillator signal in the tank circuit. For frequencies below 10 kHz flicker noise, either in the oscillator field effect transistor or some part of the amplitude control loop becomes predominant. The amplitude spectrum between 10 kHz and 200 kHz is produced by white noise within the oscillator or amplitude control system. It is not necessarily due to white noise in the tank circuit since any source of white noise within the amplitude control loop will produce the same response.

From Figure 6.12 we can obtain approximate values for the time constants $T$ and $\tau$. From equation 3.67 the following approximate
expressions can be obtained assuming that $T \gg \gamma$.

\[ \frac{f_{\text{peak}}}{S_a(f)_{\text{peak}}} \quad \frac{S_a(f)_{\text{peak}}}{S_a(f)_{\text{flat}}} = \left( \frac{T}{\gamma} \right)^2 \]

For the spectrum marked $\gamma'$, we note that the peak occurs at 130 kHz and is 18 dB above the flat part of the spectrum. Using the above expressions $T$ and $\gamma$ can be found.

\[ T = 3.7 \times 10^{-6}, \quad \gamma' = 0.41 \times 10^{-6} \]

Similarly for the spectrum marked $4\gamma'$

\[ T = 2.99 \times 10^{-6}, \quad 4\gamma' = 0.86 \times 10^{-6} \]

The correct value of $T$ is $2.65 \times 10^{-6}$ since the servo amplifier $3$ dB bandwidth is $60$ kHz. Also the values for $\gamma'$ should be in the ratio $1:4$. The results are reasonably close to the correct values considering that the theory deals only with white noise in the tank circuit and that the experimental oscillator has other sources of noise which modify its spectrum.

Finally, consider the amplitude spectrum in Figure 6.12 for the instantaneous limiting oscillator. With the amplitude of oscillation of the instantaneous limiting oscillator set the same as for the delayed amplitude limiting oscillator, the spectrum marked soft, was obtained.
Figure 6.13: Measurement of flicker noise in FET's
The amplitude of oscillation for the delayed amplitude limiting oscillator was made small so that the instantaneous non-linearities had little effect. The instantaneous limiting oscillator when set to the same amplitude is only marginally stable and requires frequent adjustments. Its amplitude spectrum is similar to the theoretical spectrum sketched in Figure 4.6. The limiting is soft because the instantaneous non-linearities have only a small effect. The relaxation time can be found from the knee in the spectrum at 15 kHz. This is normally not noticed in a hard limiting oscillator because noise added to the output of the oscillator obscures it. With the amplitude of oscillation increased slightly the spectrum marked hard was obtained. This amplitude setting did not require careful adjustment and gives the spectrum of a typical hard limiting oscillator.

6.5 Effect of Flicker Noise

We will now consider the theoretical work of Section 4.2 which deals with the effect of flicker noise inside an oscillator. The analysis presented in Section 4.2 is relatively simple and hardly worth checking experimentally. It is the initial assumption upon which the analysis is based, which requires experimental verification because it has no theoretical basis. It is assumed that flicker noise can be represented by an equivalent noise voltage generator at the input to the active device. This is then assumed to act on the voltage dependent input capacitance of the device as though it was an externally applied signal. Thus we cannot prove anything useful by trying to simulate the effect of the
Figure 6.14: Pickeer noise in different FMs.

\[ V \propto f^{-n} \]

- 2N5248
- BF244B
- IDEAL I/F SPECTRUM
- FA 1300
flicker noise with an external noise generator. This would only check the simple analysis which is unlikely to be wrong and tell us little about the important assumptions we have made about the internal effect of the flicker noise on the input capacitance.

In order to investigate the effect of flicker noise we must observe an oscillator in which the flicker noise is predominant. The experimental instantaneous limiting oscillator is suitable for consideration if we use only the phase noise spectrum below 10 kHz. Section 6.3.2 shows that the effect of white noise is not significant in determining the phase noise spectrum below 10 kHz. Only the phase noise spectrum can be considered because the amplitude spectrum was too small to measure.

The flicker noise produced by a sample specimen of three types of FET's is shown in Figure 6.14. The method of measuring the flicker noise is shown in Figure 6.13. The low noise amplifier and wave analyser used as part of the test set for measuring oscillator noise were also used to measure the flicker noise. Flicker noise in the E300 and BF 244B FET's has a power spectrum which is approximately of a $1/f$ shape. The 2N5248 type of FET has a different shaped power spectrum which is probably due to some kind of burst noise raising its spectrum level around 100 Hz. Considerable variation in the flicker noise was found between different devices of the same type and the results shown in Figure 6.14 are not necessarily
Figure 6.15: Noise frequency modulation due to flicker noise.

\[ \Delta \nu = \frac{1}{f} \] (in Hz)

\[ f(\text{Hz}) \]

- Measured noise modulation
- Calculated noise modulation due to transistor I/R noise
typical of their type. These results show that the phase spectrum of an oscillator in which flicker noise is predominant need not have a perfect 30 dB per decade slope and can be quite irregular.

Consider now the experimental instantaneous limiting oscillator which uses E300 field effect transistors. The flicker noise spectrum given in Figure 6.14 will be used to find the noise frequency modulation which it produces. It is more convenient to deal with noise frequency modulation measured in a 6 Hz bandwidth than with the phase spectrum used earlier. The oscillator noise of the experimental instantaneous limiting oscillator is plotted in this new form in Figure 6.15.

A small voltage was applied to the gate of the input FET of the experimental oscillator and its change in frequency was noted. This gives the frequency voltage dependence constant K which can be used to find the noise frequency modulation due to the flicker noise.

\[ K = \frac{\Delta f}{\Delta V} = 0.25 \times 10^6 \text{ Hz/volt} \]

The predicted noise frequency modulation is shown in Figure 6.15.

The measured and predicted noise frequency modulation graphs have a similar shape but the measured results are about five times greater than the predicted results. This might be due partly to experimental errors. It is also possible that the fluctuation in capacitance caused by the flicker noise is bigger than that which would
result from the equivalent noise input voltage acting across the input junction. Our knowledge of flicker noise is inadequate because it is normally only observed as a fluctuating output current from the device. Unfortunately fluctuations in input capacitance are only apparent when the device is used as an oscillator.

Reference for Chapter 6
CHAPTER 7

CONCLUSIONS

In this chapter we see what conclusions can be drawn from this study of oscillator noise. The theoretical results which have been derived provide a reasonable explanation of the noise performance of the two oscillators which have been built. Further theoretical work is required in order to establish the physical mechanism which causes flicker noise modulation. Also some additional experimental confirmation of the theoretical results is desirable. However, despite this, the theoretical results enable many aspects of the noise performance of an oscillator to be calculated with reasonable confidence.

7.1 How to Make a Low Noise Oscillator

The noise performance of many transistor oscillators used in electronic instruments has been found inadequate. Thus it would be useful if one could determine how to design a low noise oscillator. First one must decide what aspects of the oscillator's noise performance are important. An oscillator designed to produce as little noise as possible in one situation is not necessarily a low noise oscillator in another situation.

Consider the local oscillator used in a radio receiver to convert the radio frequency signal to an intermediate frequency. The local oscillator usually feeds a large signal into the receivers mixer so that the conversion gain of the mixer is largely independent of small changes in the
amplitude of the local oscillator signal. Thus, the effect of amplitude noise in the local oscillator is suppressed by the mixer and does not have much effect on the intermediate frequency signal. The local oscillator’s phase noise is however, transferred to the intermediate frequency signal because its frequency is determined by the local oscillators frequency.

It is only the phase noise spectrum of a local oscillator which is important and thus local oscillator noise is not important in an AM receiver. For an FM receiver used for receiving speech, the audio bandwidth is usually from 300 Hz to 3 kHz. Local oscillator phase noise in such a receiver need only be minimised in this frequency range and it is likely that only flicker noise need be considered.

In the introduction the use of signal generators for checking adjacent channel rejection was mentioned. A low noise oscillator for such a signal generator has very different requirements to one used as a local oscillator in a radio receiver. For checking adjacent channel rejection the oscillator noise sidebands at a frequency of 12.5 kHz from the carrier must be as low as possible. Thus both phase and amplitude noise is important. Noise frequency modulation at 12.5 kHz is usually due to the effects of white noise in the oscillator tank circuit. Some care though must be taken in choosing a suitable oscillator circuit so that the effect of the other two types of noise are negligible. The oscillator signal should preferably be taken directly from the tank circuit, so that noise is not added to
It by an amplifier.

It would be unwise to use a delayed amplitude limiting oscillator for checking adjacent channel rejection. Unless a great deal of care is taken with the stability of the feedback loop the amplitude noise tends to rise to a peak either side of the carrier signal as shown in Figure 4.6. This peak in the amplitude noise sidebands may well occur at a frequency near to 12.5 kHz. The noise sidebands at 12.5 kHz will then be larger than for an instantaneous limiting oscillator in which only the phase noise sidebands are significant.

The preceding two examples show how important it is to specify which aspect of the noise performance of an oscillator is important. It is not sufficient to state that a low noise oscillator is required. Considerations other than the noise performance of the oscillator also effect the choice of the type of oscillator. If a signal with low harmonic distortion is needed then delayed amplitude limiting must be used in the oscillator and the resulting degradation of the noise performance must be accepted.

Having established which types of oscillator noise are likely to be important in a particular application, one can refer to the appropriate theoretical results. These are summarised in the following three sections together with some general comments. These results when used together with a suitable method of measuring oscillator noise (Section 5.2) should facilitate the design of low noise oscillators.
Each type of oscillator noise imposes its own characteristic shape on the phase and amplitude spectra of an oscillator. Usually one can see which type of oscillator noise is predominant from the shape of these spectra. (Figure 4.6). If the noise performance of an oscillator is unsatisfactory, then by inspecting its noise spectra one can identify the type of oscillator noise which requires attention. With this knowledge it is easier to improve the noise performance of the oscillator.

7.2 White Noise Inside An Oscillator

The phase and amplitude noise spectra of an instantaneous limiting oscillator perturbed by white noise inside the tank circuit is given by equations 3.44 and 3.48.

\[
S_\phi (f) = \frac{2D_o}{\omega} \quad \ldots (3.44)
\]

\[
S_\alpha (f) = \frac{2D_o A_o}{\mu^2 + \omega^2} \quad \ldots (3.49)
\]

In order to use these equations to find the noise performance of an oscillator the values of the constants \( \mu \), \( D_0 \) and \( A_0 \) must be found. It is difficult to do this accurately so that results derived from equations 3.44 and 3.48 may be only approximate. A good theoretical estimate of the relaxation time \( \frac{1}{\mu} \) of an oscillator may be found by assuming
it to be slightly greater than that of the ideal hard limiting oscillator.

This follows from Section 3.3.2 where the expression for the relaxation time of a hard limiting oscillator is given as:

\[ \frac{1}{\mu} = \frac{2Q}{\omega_0} \]

Here \( \omega_0 \) is the oscillator's angular frequency and \( Q \) is the Q of the tank circuit. The phase diffusion constant \( D_0 \) is given by equation 3.53.

\[ D_0 = \frac{N_0}{4A_o^2C^2} \]

In this expression \( A_0 \) is the amplitude of oscillation in the tank circuit, \( C \) is the capacitance of the tank circuit and \( N_0 \) is the spectral density of the equivalent noise current generator in the tank circuit. Section 2.2.2 showed that if the tank circuit is connected to the input of the active device used in the oscillator rather than its output, then usually \( N_0 \) is due only to the thermal noise in the tank circuit. Then we get:

\[ N_0 = 4kT/R \]

The loss resistance \( R \) can easily be found from the Q of the tank circuit. When only thermal noise in the tank circuit need be considered then only the values \( \mu, A_0, Q \) and \( C \) are required to find the phase and amplitude spectra.
If the effect of white noise in the tank circuit of an oscillator are to be minimised then clearly $A_0$, $Q$ and $C$ must be made as large as possible. The maximum value for $A_0$ depends on the breakdown voltage of the active device used in the oscillator. Most radio frequency transistors have a relatively low breakdown voltage compared to radio frequency valves which is probably one of the factors which favour the use of valves for low noise oscillators. The $Q$ of the tank circuit depends on the size and construction of the tank circuit. If the best materials and construction technique is used then the $Q$ of the tank circuit can only be increased by increasing its size. Thus the effects of the noise can be reduced at the expense of the size, weight and cost of the oscillator. For any desired frequency of oscillation one can reduce the inductance of the tank circuit so that $C$ can be increased to improve the noise performance. However a point is reached when the inductance of the tank circuit becomes comparable with the inductance of component leads and the oscillator becomes unreliable. Oscillations at the wanted frequency can be suppressed by oscillations at some high unwanted frequency.

If the oscillator uses delayed amplitude limiting then the expression for the amplitude noise spectrum is given by equation 3.67.

\[
S_a(f) = 2D_0 A_o^2 \left| \frac{(1 + j\omega T)\tau}{1 + j\omega \tau - \omega^2 T \tau} \right|^2
\]  \[3.67\]
Here \( T \) is the time constant of the loop filter and \( C \) is a time constant which is dependant on the loop gain of the feedback system. The practical limitations imposed by the performance of real devices usually means that delayed amplitude limiting oscillators have larger noise sidebands than instantaneous limiting oscillators.

The phase noise spectrum due to white noise in the tank circuit is the same for the instantaneous and delayed limiting oscillators. However, additional phase noise is usually found in the delayed amplitude limiting oscillator due to noise in the servo amplifier and amplitude detector.

7.3 Noise Added to Oscillator Output

Additive noise can be dealt with far more easily than the other types of oscillator noise. Semiconductor manufacturers usually give the noise figure for transistors intended for use as RF amplifiers. Thus the spectral density of the white noise which is added to an oscillator signal by an amplifier can be easily found. The amplitude spectrum \( S_a(f) \) and phase spectrum \( S_\phi(f) \) due to additive noise of spectral density \( S_n(f) \) is given by equations 4.7 and 4.8.

\[
S_a(f) = S_n(f_0+f) + S_n(f_0-f) \quad (4.7)
\]

\[
S_\phi(f) = \frac{S_n(f_0+f) + S_n(f_0-f)}{A_o^2} \quad (4.8)
\]
7.4 Flicker Noise

The analysis of flicker noise in the thesis is based on a plausible assumption which seems to be approximately correct but does not have a proper theoretical basis. It is assumed that the effect of flicker noise in an oscillator can be found by placing an equivalent noise voltage generator across the input of the active device. This model has been used to show how flicker noise can produce phase and amplitude noise modulation. The results of this analysis are of a qualitative nature and cannot be used to accurately predict the noise spectra of an oscillator. The analysis however is useful because it shows how the effects of flicker noise in an oscillator can be reduced.

Flicker noise modulation is predominant over other types of oscillator noise at low modulation frequencies. The phase noise modulation spectrum at low frequencies is almost always much greater than the amplitude noise modulation spectrum so that it is usually the effect of flicker noise on the frequency of an oscillator which is important. Flicker noise causes phase modulation by varying the input capacitance of the active device used in the oscillator. The effect of flicker noise can be reduced by making the input capacitance of the active device as small a part of the tank circuit capacitance as possible. This can be done by choosing a device with a small input capacitance and making the tank circuit Q high so the active device need only be loosely coupled to the tank circuit. Active devices with low flicker noise and an input capacitance with low voltage dependence tend to give less flicker noise modulation.
Further work is required in order to find a proper theory for flicker noise modulation. The importance of flicker noise in oscillators was not appreciated when the theoretical work for this thesis was started. The importance of flicker noise only became apparent when the oscillator noise produced by an experimental oscillator was compared with the theoretical results. In this thesis an oscillator is considered to be a system containing several known sources of noise and it is thus purely a mathematical problem to find the noise performance of the oscillator. This approach is satisfactory for shot or thermal noise because they are well understood and their equivalent noise generators can be included in the oscillator model. However, flicker noise is not well understood as it cannot be properly represented in an oscillator model. Further work on oscillator noise should approach the problem by a study of solid state physics rather than signal processing mathematics.
APPENDIX

8.1 SPECTRA OF SINEWAVE AND NOISE

Consider a narrow band signal $v(t)$ with a fluctuating amplitude $a(t)$ and fluctuating phase $\phi(t)$.

$$v(t) = [A_0 + a(t)] \sin[\omega_0 t + \phi(t)]$$

The autocorrelation function (equation 3.51, page 69) is then:

$$R_v(\tau) = [A_0^2 + \langle a(t)a(t-\tau) \rangle] \frac{\cos \omega_0 \tau}{2} \langle \cos[\phi(t-\tau) - \phi(t)] \rangle$$

The spectrum of the signal will first be found with the fluctuating phase made zero. Then we have:

$$R_v(\tau) = \frac{A_0^2}{4} e^{j\omega \tau} + \frac{A_0^2}{4} e^{-j\omega \tau} + \frac{R_a(\tau)}{4} e^{j\omega \tau} + \frac{R_a(\tau)}{4} e^{-j\omega \tau}$$

Throughout the main body of this thesis only the one sided power density spectrum $S(f)$ has been used. Now it will be necessary to use the two sided power density spectrum $S^*(f)$.

$$\text{TOTAL POWER} = \int_{-\infty}^{\infty} S^*(f) df$$

The Weiner Khintchine theorem gives

$$S^*(f) = \int_{-\infty}^{+\infty} R(\tau) e^{-j\omega \tau} d\tau$$

The following identity is also required

$$S(x-x_0) = \int_{-\infty}^{+\infty} e^{\pm 2\pi j(x-x_0)t} dt$$
Figure 8.1
From 8.2 and letting $2\pi f_0 = \omega_0$ we get:

$$S_\nu^*(f) = \frac{A_0^2}{4} \delta(f-f_0) + \frac{A_0^2}{4} \delta(f+f_0) + \frac{1}{4} S_\phi^*(f-f_0) + \frac{1}{4} S_\phi^*(f+f_0)$$

Therefore

$$S_\nu(f) = \frac{A_0^2}{2} \delta(f-f_0) + \frac{1}{4} S_\phi(f-f_0) \quad \ldots (8.3)$$

This result can best be explained by drawing the relevant power spectra.

Let the amplitude spectrum at low frequencies be $N_0$. Figure 8.1 shows how the carrier signal shifts the two sided amplitude power spectrum up to a frequency $f_0$.

Now we let the amplitude noise be zero so that the RF spectrum can be related to the phase spectrum. From 8.1 we get:

$$R_\nu(\tau) = \frac{A_0^2}{2} \cos \omega_0 \tau \left\langle \cos \left[ \phi(t-\tau) - \phi(t) \right] \right\rangle$$

If the phase angle $[\phi(t-\tau) - \phi(t)]$ is small then:

$$R_\nu(\tau) = \frac{A_0^2}{2} \cos \omega_0 \tau \left(1 - \left[ \frac{\phi(t-\tau) - \phi(t)}{2} \right]^2 \right)$$

This is equivalent to stipulating that the modulation index is low so that the narrow band phase modulation is obtained.

$$R_\nu(\tau) = \frac{A_0^2}{2} \cos \omega_0 \tau \left[1 - \langle \phi^2 \rangle + R_\phi(\tau) \right]$$

Therefore if $1 \gg \langle \phi^2 \rangle$

$$S_\nu(f) = \frac{A_0^2}{2} \delta(f-f_0) + \frac{A_0^2}{4} S_\phi(f-f_0) \quad \ldots (8.4)$$
Figure 8.2

Figure 8.3

L = L₁ + L₂ + 2M
8.2 The Hard Limiting Model of a Hartley Oscillator

Here we will show how the hard limiting oscillator model used in 3.2 can be applied to a Hartley oscillator. Consider the circuit of a Hartley oscillator shown in Figure 8.2.

For the AC analysis of this oscillator the choke, coupling capacitor $C_1$ and cathode biasing components $R_c$ and $C_c$ can be neglected. If the valve is represented by an ideal limiter as in Section 3.2.1, then the equivalent circuit shown in Figure 8.4 can be obtained.

The losses in the tank circuit are represented by the resistor $R$ and the accompanying noise generator by $I_n$. Let the oscillator be started by the limiter current changing from $-I_o$ to $+I_o$ at time $t = t_o$.

The resulting oscillations in the tank circuit up to the first zero crossing of the tank circuit voltage $v$ can be easily found by linear analysis. In order to include the effect of noise, one impulse is allowed to occur at time $t = t_k$.

Thus we have:

$$i = 2I_o H(t - t_o) \quad \ldots \quad 8.10$$

$$i_n = a_k \delta(t - t_k) \quad \ldots \quad 8.11$$

The analysis of the oscillator can be done most easily by transforming the circuit from the time domain into the complex frequencies domain and then using mesh analysis, and is shown in Figure 8.5.

Mesh 1 gives:

$$O = I_1 \left( R + \frac{1}{pC} \right) - I_2 \left( \frac{1}{pC} \right) - I_n (R)$$

$$I_1 = \frac{I_2 \left( \frac{1}{pC} \right) + I_n (R)}{\left( R + \frac{1}{pC} \right)} \quad \ldots \quad 8.12$$
Figure 8.4
Mesh 2 gives:

\[ O = I_2 \left( pL + \frac{1}{pC} \right) - I_1 \left( \frac{1}{pC} \right) + I(pL_1 + pM) \]

where \( L = L_1 + L_2 + 2M \)

Substituting for \( I_1 \) from 8.12 gives:

\[ O = I_2 \left( pL + \frac{1}{pC} \right)(R + \frac{1}{pC}) - I_2 \left( \frac{1}{pC} \right)^2 - \frac{1}{I_2} \left( \frac{R}{pC} \right) + I(pL_1 + pM)(R + \frac{1}{pC}) \]

\[ \therefore \quad I_2 = \frac{I_n \left( \frac{1}{lC} \right) - IP(L_1 + M) \left( \frac{P}{L} + \frac{1}{LRC} \right)}{\left( P^2 + P \frac{1}{RC} + \frac{1}{LC} \right)} \] \[ \therefore \quad I_2 = \frac{I_n \left( \frac{1}{lC} \right) - IP(L_1 + M) \left( \frac{P}{L} + \frac{1}{LRC} \right)}{\left( P^2 + P \frac{1}{RC} + \frac{1}{LC} \right)} \] ... \[ 8.13 \]

\[ V = I_2 P (L_2 + M) + IP M \]

Substituting \( I_2 \) from 8.13 gives:

\[ V = pI_n \left( \frac{1}{lC} \right)(L_2 + M) - p^2 I(L_1 + M)(L_2 + M) \left( \frac{1}{lRC} \right) - \frac{1}{3} I(L_1 + M)(L_2 + M) \left( \frac{1}{lC} \right) \]

\[ + \quad pMI \left( P^2 + P \frac{1}{RC} + \frac{1}{LC} \right) \]

From 8.10 and 8.11 we have:

\[ I = \frac{2I_0 e^{-\rho t_o}}{P} \]

\[ I_n = a_k e^{-\rho t_k} \]
Thus:

\[ V(p) = \frac{p \left( \frac{a_k}{c} \right) \left( \frac{L_2 + M}{L} \right) e^{-pt_k}}{(p^2 + \frac{1}{RC} + \frac{1}{LC})} + 2I_o Me^{-pt_0} \]

\[ + 2I_o (L_1 + M) \left( \frac{L_2 + M}{L} \right) e^{-pt_0} \left( -\frac{1}{RC} - \frac{p}{p^2 + \frac{1}{RC} + \frac{1}{LC}} \right) \]

This expression can be written as:

\[ V(p) = \frac{p \left( \frac{a_k}{c} \right) \left( \frac{L_2 + M}{L} \right) e^{-pt_k}}{(p^2 + \frac{1}{RC} + \frac{1}{LC})} + \frac{2I_o \left( \frac{M^2 - L_1 L_2}{L} \right) e^{-pt_0}}{L} \]

\[ + 2I_o (L_1 + M) \left( \frac{L_2 + M}{L} \right) \left( \frac{1}{LC} \right) e^{-pt_0} \left( p^2 + \frac{1}{RC} + \frac{1}{LC} \right) \]

Taking the inverse Laplace transform and letting \( t_0 \to 0 \) we get:

\[ v(t) = 2\omega_b I_o \left( \frac{L_1 + M}{L} \right) \left( \frac{L_2 + M}{L} \right) e^{-\frac{\omega_b t}{2Q}} \sin \omega_b t + 2I_o \left( \frac{M^2 - L_1 L_2}{L} \right) S(t-t_0) \]

\[ + \left( -\frac{a_k}{c} \right) \left( \frac{L_2 + M}{L} \right) H(t-t_k) e^{-\frac{\omega_b}{2Q} (t-t_k)} \left[ \cos \omega_b (t-t_k) - \frac{1}{2Q} \sin \omega_b (t-t_k) \right] \]

\[ \ldots 8.14 \]
where \( \omega_0^2 = \frac{1}{LC} \) and \( \frac{1}{RC} = \frac{\omega_0}{Q} \) and for \( Q \gg 1 \).

This result can now be compared to that given by equation 3.28 which was obtained for the simple tuned grid oscillator in Section 3.2.1. Equation 3.28 is written here again for convenience.

\[
\nu(t) = 2\omega_0 IM e^{-\frac{\omega_0 t}{2Q}} \sin \omega_0 t + H(t-t_k) e^{-\frac{\omega_0 (t-t_k)}{2Q}} \left[ \cos \omega_0 (t-t_k) \right] - \frac{1}{2Q} \sin \omega_0 (t-t_k)
\]

Comparing the first term of 3.28 with the first term of 8.14 it can be seen that the only difference is that the \( M \) in 3.28 becomes \((L_1 + M)/(L_2 + M)/L\) in 8.14. The second term in 8.14 does not occur in 3.28. This term can be ignored because it does not affect the noise performance of the oscillator. It does not alter the position of the zero crossings or the response of the tuned circuit after the limiter has changed state. For coils which are closely coupled this term is small and for a unity coupled coil \((M^2 = L_1 L_2)\) it is zero. The terms due to noise in 3.28 and 8.14 have coefficients of \((-a_k/c)\) and \((-a_k/c) (L_2 + M)/L\) respectively.
8.3 LIST OF PRINCIPAL SYMBOLS

\( \alpha, \beta, \delta, \epsilon \) = constants in power series representation of nonlinearity

\( \Theta \) = argument of trig. function

\( \lambda \) = shot noise constant

\( \frac{1}{\mu} \) = relaxation time of oscillator

\( \nu \) = instantaneous frequency

\( \sigma \) = standard deviation

\( \varphi \) = accumulated phase error

\( \phi \) = phase angle

\( \omega \) = angular frequency

\( A(t) \) = amplitude of signal

\( A_s \) = steady state value of \( A(t) \)

\( a(t) \) = departure of \( A(t) \) from its steady state value

\( a_k \) = amplitude of noise pulse

\( A_2 \) = transfer impedance

\( b(t) \) = quadriture noise component of oscillator signal

\( C \) = capacitance

\( D_0 \) = phase diffusion constant

\( E_n, e_n \) = noise voltage generator EMF

\( f \) = frequency

\( g_n \) = mutual conductance

\( G \) = gain of servo. amp.

\( H(\omega) \) = transfer function

\( I_a, i_a \) = anode current

\( I_s, i_s \) = shot current

\( I_n, i_n \) = noise current

\( I_p \) = drain current in FET

\( I_{oss} \) = drain current with zero gate source voltage

\( j = \sqrt{-1} \)

\( k \) = Boltzmann's constant

\( L \) = self inductance

\( M \) = mutual inductance

\( N_n \) = spectral density of noise

\( p \) = complex frequency

\( p(\phi) \) = probability density function

\( Q \) = tank circuit quality factor

\( R(\gamma) \) = autocorrelation function

\( R, r \) = resistance

\( S(f) \) = single sided power density spectrum

\( t_o, t_k \) = time constant

\( T, t, \tau \) = time

\( T \) = absolute temperature

\( V_p \) = pinch off voltage of FET

\( V_s, v \) = signal voltage

\( V_g \) = grid voltage

\( x, y \) = variables

\( Z(t), z(t) \) = voltage input to servo. amp.