The stresses in granular material due to applied vibration

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<th>EASTHAM, J.</th>
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THE STRESSES IN GRANULAR MATERIAL
DUE TO APPLIED VIBRATION

Ian E. Eastham

Supervised by
B. Scarlett

Director of Research
Professor D.C. Freshwater

Submitted for the Doctor of Philosophy degree
of Loughborough University of Technology.

October 1969
ACKNOWLEDGEMENTS

I wish to thank the directors of Morganite Research and Development Limited for financial support during this work.

Also I wish to thank Professor D.C. Freshwater for the excellent research facilities, B. Scarlett for his patient supervision and all the staff of the Derby road site for their tolerance during the past three years.

My wife, Stephanie, and son have made many sacrifices whilst I completed this research, most of all whilst Stephanie learnt to type on the manuscript. I am eternally grateful to them both.
SUMMARY

The dissipation and spread of stresses in a granular material due to an applied disturbance is investigated.

A single impulse was applied to a granular material to help understand its behaviour when disturbed by an oscillating vertical force. Measurements were made of the effect of the impulse at the surface of the beds of sand of various heights. The spread of the disturbance in a direction normal to that of application was measured by a radial traverse of the surface of each bed.

The behaviour of a granular material when acted upon by a force is comprehensively discussed in the literature survey. The effect on the structure throughout the material of a disturbance applied at a distant point is then assessed.

A theory is presented to predict the profile of force at the surface of granular material disturbed internally by a vertical impulse. A second theory is presented which extends established methods of estimating the velocity of propagation of a disturbance. In the course of the development of the velocity of propagation an expression is derived for Poisson's ratio of a granular material.

Comparison of theory with experiment shows an accurate prediction of Poisson's ratio and an inaccurate prediction of velocity of propagation due to simplified evaluation of the functions involved. The prediction of profile of force varies in the correct manner but the values are low. One constant
proposed in the equation for force profile cannot yet be theoretically evaluated.
# CONTENTS

| ACKNOWLEDGEMENTS | SUMMARY | ii |
| INTRODUCTION | | iii |
| CHAPTER ONE | PRELIMINARY EXPERIMENTAL WORK | 5 |
| 1.2 Discussion of results | | 16 |
| 1.3 Conclusions | | 30 |
| 1.4 Effect of these experiments on the major experimental work | | 31 |
| CHAPTER TWO | THE LITERATURE SURVEY | 33 |
| 2.1 Introduction to the survey | | 34 |
| 2.2 The effect of vibration on a granular material | | 48 |
| 2.3 The physical state of a granular mass | | 60 |
| 2.4 The approach of experiment to theoretical conditions of porosity | | 64 |
| 2.5 Theory of the effect of vibration | | 70 |
| 2.6 Mechanics of granular matter | | 78 |
| 2.7 Phenomena of propagation of stress | | 91 |
| 2.8 Statistical models | | 97 |
| 2.9 Particle characterisation | | 99 |
| 2.10 Conclusions from the literature | | 106 |
| CHAPTER THREE | THEORY | 106 |
| 3.1 Introduction to the theory | | 106 |
| 3.2 The distance between two contacts on an irregular particle | | 106 |
3.3 Theory of propagation of an applied force through a cohesionless material.
The distribution of force from a point source.
The distribution of force from a source of finite area.

3.4 Velocity of propagation.
Properties of a contact between two particles.
Deformation of an element of granular material.
Poisson's ratio for a granular material deformed by a static load.
The decrease in volume of a granular material due to a small increase in load.

3.5 Distribution of variables.

CHAPTER FOUR EXPERIMENTAL WORK
4.1 The apparatus
4.2 Calibration
4.3 Experimental procedure

CHAPTER FIVE RESULTS AND DISCUSSION
5.1 The experimental conditions
5.2 Results
Measurement of profile of force at the surface.
Measurement of velocity of propagation
5.3 Theoretical predictions
5.4 Further work

CHAPTER SIX CONCLUSIONS

APPENDIX A

Rules of procedure of the Contact Theory
developed by R.D. Mindlin and H. Deresiewicz. 237

APPENDIX B

Volume, area and length fractions of a
disperse component in space. 241

APPENDIX C

A correction for the finite area of the
receiving disc. 246

APPENDIX D

Tables of experimental results. 249

APPENDIX E

Nomenclature 284

BIBLIOGRAPHY 289
INTRODUCTION

The purpose of this work is to predict and observe effects of interest to engineers which occur when a force, particularly due to an oscillating source is applied to a granular material.

The macroscopic effect of greatest interest is the production of non-recoverable deformation of the granular mass, be it desired as in the problem of powder flow of the process engineer or undesired as in the settlement problem of civil engineers. A complete understanding of this deformation will only be possible when the relationship between the powder behaviour and the fundamental properties is known. This relationship will only be unique and simple mathematically if the correct fundamental properties are measured. This issue has not yet been fully resolved.

The technique at present is to measure a macroscopic property of strength and treat the material as a plastic continuum. The rules of procedure are all based on the Coulumb equation and have been developed extensively. Terzaghi, himself a world authority on soil mechanics, has suggested that the failing of the basic principles of his science is that continuum is considered when in reality soils are composed of many discrete grains. The approach here will be to consider the individual grains and thereby predict macroscopic properties which can then be used to predict behaviour in some ways already established.
The sequence of properties to aid a complete understanding of the propagation of stress in granular material is:-

(1) Primary properties e.g. particle size distribution

(2) Physical state e.g. porosity and particle orientation

(3) Macroscopic properties e.g. bulk modulus, shear strength

(4) Behaviour e.g. velocity of propagation of a disturbance. Dissipation and spread of applied force.

The step from (3) to (4) is established. Theories presented here attempt to relate all four sections.

The relationships developed between the primary properties of the particles, the physical state and the macroscopic properties rely heavily on the theories of behaviour of interparticle contacts. Present theory treats the contact behaviour as being close to elastic and in consequence cannot predict the degree of non-recoverable deformation of the bulk material. However the present modified elastic theory can be used to predict the onset of non-recoverable deformation. This will be done by predicting the shear occurring within the material which will be related to the properties of shear strength, to define the onset of deformation.
Non-recoverable deformation of a granular material can be of two types.

(1) The disordered process of interparticle contact slide which allows the particles to move relative to one another but generally in the direction of applied force such that they try to relieve the force. This process causes an overall reduction in porosity and it will be called compaction.

\[ \text{applied force} \]

Force applied to open void. Contact slides and void closes

Applied force causing compaction by contact slide.

(2) The more ordered process of bulk movement where many particles move in the direction of the applied force with or without relative interparticle movement. This bulk movement is bounded by zones of shear and only occurs if the shear strength of the material is exceeded. This process will be called bulk flow.

The thesis will consider the history of both types of non-recoverable deformation, particularly caused by vibration, and will present experiment and theory to describe the local
conditions of stress in a granular material in order to predict the onset of deformation. Since interparticle contact slide is not considered the non-recoverable deformation predicted will be bulk flow. Later discussion suggests how compaction can be predicted.
CHAPTER ONE

THE PRELIMINARY EXPERIMENTAL WORK

An early look at the literature suggested no obvious approach to this study of vibration of granular material. Some problems that were being faced by industry created whilst using vibration in chemical engineering were forthcoming but no common solution was apparent. It was decided then to conduct some simple visual experiments to observe reported phenomena and look at the problems involved.

A small vibrating table was bought, which was manufactured as a commercial packer. The vibrating table was made of a steel plate which easily allowed attachment of apparatus to contain the material to be vibrated. Vibrational energy was generated electromagnetically at 50 cycles per second. The visual experiments involved vibrating glass beads or sand in a measuring cylinder which was clamped to the table, Fig. 1. The capacity of the measuring cylinder, the height of the granular material and the amplitude were varied and bulk circulation, packing and expansion of the bed were observed.

Similar experiments formed the basis of an undergraduate project conducted at the time and their results are included as they represent the phenomena that was observed.
Fig.1, The commercial packer and measuring cylinder.
The experimental technique was to vary the setting of the vibrator controller on a linear scale and thus obtain different maximum accelerations. The acceleration was measured using a cathetometer focused on a knife edge attached to the vibrating table. The experiments were to find the minimum porosity obtainable by varying the acceleration, to measure the variation of the rate of packing with time, and to measure the effect of increasing a surface load on the powder.

<table>
<thead>
<tr>
<th>Acceleration of vibration</th>
<th>Fine sand &lt; 100μ</th>
<th>Fine sand &lt; 600μ</th>
<th>Coarse sand &lt; 600μ</th>
<th>Chalk &lt; 45μ</th>
<th>Rubber chips = 2000μ</th>
</tr>
</thead>
<tbody>
<tr>
<td>.00g</td>
<td>.420</td>
<td>.441</td>
<td>.383</td>
<td>.450</td>
<td>.450</td>
</tr>
<tr>
<td>.15g</td>
<td>.396</td>
<td>.400</td>
<td>.355</td>
<td>.411</td>
<td>.440</td>
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<td>4.43g</td>
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<td>.338</td>
<td>.321</td>
<td>.341</td>
<td>.403</td>
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<td>8.35g</td>
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<td>.325</td>
<td>.306</td>
<td>.341</td>
<td>.379</td>
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<td>9.66g</td>
<td>.350</td>
<td>.330</td>
<td>.303</td>
<td>.356</td>
<td>.370</td>
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<td>.346</td>
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<td>12.13g</td>
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<td>.303</td>
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</tr>
<tr>
<td>12.80g</td>
<td>.350</td>
<td>.338</td>
<td>.306</td>
<td>.364</td>
<td>.364</td>
</tr>
</tbody>
</table>

Table 1 The effect of vibrational acceleration on the porosity of granular material.
Fig. 2. The steady state porosity of granular materials at a fixed vibrational acceleration.
The effect of vibrational acceleration on the porosity of granular material is shown in Table 1 and Fig. 2. The experiment was conducted with the vibration continuing at fixed maximum acceleration of $N$ times that of gravity ($N_g$) until no further decrease in porosity resulted. The rate of compaction results at vibration of 8.35g, are shown in Table 2 and Fig. 3. The first three porosity columns were measured with the vibrator off and time continued between measurements. The fourth column of results was taken with the vibrator off but the sample was loosened to the initial porosity before the vibration was restarted. In this case time was taken for compaction from the initial porosity to that at which the reading was taken.

<table>
<thead>
<tr>
<th>Vibration time (sec)</th>
<th>Run 1</th>
<th>Run 2</th>
<th>Run 3</th>
<th>Run 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.401</td>
<td>0.406</td>
<td>0.410</td>
<td>0.400</td>
</tr>
<tr>
<td>0.5</td>
<td>0.365</td>
<td>0.344</td>
<td>0.348</td>
<td>0.345</td>
</tr>
<tr>
<td>10</td>
<td>0.341</td>
<td>0.341</td>
<td>0.339</td>
<td>0.341</td>
</tr>
<tr>
<td>15</td>
<td>0.333</td>
<td>0.337</td>
<td>0.333</td>
<td>0.331</td>
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<td>20</td>
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<td>30</td>
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<td>0.330</td>
<td>0.323</td>
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<tr>
<td>35</td>
<td>0.326</td>
<td>0.326</td>
<td>0.326</td>
<td>0.322</td>
</tr>
</tbody>
</table>

Table 2 The rate of compaction of sand at a peak vibrational acceleration of 8.35g.
Fig. 3. The rate of compaction of sand whilst being vibrated at an acceleration of 8.35g.
The effect of a surface load on the packing of the granular material was measured; these results are shown in Table 3 and Fig. 4. The vibrator was stopped for each reading and time continued as the vibrator was restarted. (The results in Table 2 show no difference due to an interrupted vibration time.) The load was applied to the surface by a bottle loosely fitting into the measuring cylinder.

<table>
<thead>
<tr>
<th>Load on surface gm.</th>
<th>0</th>
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<th>149</th>
<th>199</th>
<th>299</th>
<th>399</th>
</tr>
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<td></td>
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<td></td>
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<td>.325</td>
<td>.325</td>
<td>.322</td>
<td>.322</td>
<td>.322</td>
</tr>
</tbody>
</table>

Table 3  The effect of surface load on the packing of fine sand of size < 600 μ.
Fig. 4. The effect of surface load on the rate of compaction by vibrational acceleration of 8.35g.
During the course of these experiments reproducibility measurements were made and standard deviations on final porosities calculated. The results of these calculations are shown in Table 4, below:

<table>
<thead>
<tr>
<th>Material</th>
<th>Sand</th>
<th>Sand</th>
<th>P.V.C</th>
<th>P.V.C</th>
<th>Sand</th>
<th>Sand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>&lt;600μ</td>
<td>&lt;600μ</td>
<td>&gt;3000μ</td>
<td>&gt;1500μ</td>
<td>&lt;600</td>
<td>&lt;600</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of results taken</th>
<th>17</th>
<th>17</th>
<th>17</th>
<th>17*</th>
<th>17†</th>
<th>17‡‡</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacity of vessel</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>500</td>
</tr>
<tr>
<td>Initial volume</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>500</td>
</tr>
<tr>
<td>Mean final volume</td>
<td>886.3</td>
<td>883.2</td>
<td>908.4</td>
<td>894.1</td>
<td>902.4</td>
<td>449.7</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>6.6</td>
<td>5.0</td>
<td>6.0</td>
<td>5.2</td>
<td>7.9</td>
<td>4.9</td>
</tr>
<tr>
<td>Initial porosity</td>
<td>.403</td>
<td>.385</td>
<td>.423</td>
<td>.419</td>
<td>.410</td>
<td>.411</td>
</tr>
<tr>
<td>Mean fuel porosity</td>
<td>.326</td>
<td>.303</td>
<td>.364</td>
<td>.350</td>
<td>.346</td>
<td>.344</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>.005</td>
<td>.004</td>
<td>.004</td>
<td>.003</td>
<td>.006</td>
<td>.008</td>
</tr>
</tbody>
</table>

* Vibrated for one minute only.
† 500 cc. of sand in 1000 cc. measuring cylinder vibrated for one minute.
‡‡ 500 cc. of sand in 500 cc. measuring cylinder vibrated for one minute.
Following visual observation some understanding of possible mechanisms of propagation of vibration and microscopic behaviour of the material was necessary to plan the major experimental part of the project. A high speed camera was used to observe this behaviour, being able to take several photographs during each cycle of vibration. Such a technique was limited by the subject being close to a wall of the container but since it was not intended to take precise measurements from the resulting films the experiment was considered of value.

The camera used was a 'Fastax' capable of recording up to 8,000 frames per second, i.e. 160 photographs per cycle of 50 Hertz vibration. The first sequences of film were of a vertical monolayer of glass beads contained between two perspex sheets. The beads and the air between them were perfectly dry. Illumination from behind the subject gave the best photographic contrast, films being taken before, as and after the vibrator was started. The second series produced films of the monolayer of beads totally immersed in air free water subjected to the same vibration. The third series was of the monolayer of beads drained of water but held by the remaining pendular moisture. Photographs produced from the films are shown in Figs. 5 and 6.

Experiment could proceed little further without a more versatile vibrator. A variation of frequency and amplitude was required as well as a higher power output than had
previously been possible. A rapid response time of the vibrating system was also essential since changes in the vibration wave form likely necessary to investigate propagation phenomena had to be closely followed by the vibrator. An electromagnetic vibrator capable of following a wave-form generator output was considered closest to our specification.

The vibrator chosen was manufactured by Derritron, England being their V.P. 25 model driven by their 1000 watt low frequency amplifier. The system was limited by the first resonance of the vibrator itself at 6,000 Hertz but this allowed us to work up to 4,000 Hertz with linear response. The maximum power output of the amplifier produced a vector force of 140 kilograms at the vibrator table, i.e. with a load of L kilograms (including the vibrating table mass) a maximum acceleration of \( \frac{140}{L} \) times that of gravity was possible.

After installation of the new vibrator (described fully in Chapter 4) it was possible to extend our photographic experiments to look at material circulation in a vibrated bed. Fig. 7. The materials used were glass beads and later sand of approximately the same sieve size. Alyanak¹ and Barkan⁸ have observed a circulatory motion away from the source of vibration which we hoped to observe using alternate layers of black and white beads, Fig. 8. A flow of material down the walls soon masked the behaviour of the black and white layers so a second experiment was set up using a vertical core of
black beads in a bed of white ones. For the black core to
appear flowing down the sides of the box a flow up the centre
away from the vibration source must be present. Following the
circulation experiments the container was filled with sand
and subjected to the same vibration, Fig.9.

1.2 Discussion of results

The dependence of packed porosity on the acceleration
of vibration reported by Youd and Barkan was verified
with the exception of the chalk sample Fig.2. At the highest
amplitude in this case the circulation caused during the
vibration entrained air which could not percolate during the
vibration cycle (Yoshida and Kousaka). Each of the other
materials would exhibit a similar rise in porosity as the
acceleration increases. The difference in porosity of the
chalk sample between accelerations of 8 and 10g is
significant considering the range of standard deviations in
Table 4.

The rate of compaction experiments Fig.3 suggest that
the shorter compaction times quoted in the literature
are accurate although a higher vibrational acceleration will
cause faster compacting. The difference between the curves is
not significant at the minimum porosity according to data in
Table 4.

The effect of a surface load is not clear from our
experiments. The difference in final porosities is not
significant since the extreme values are not for the lowest
and highest surface load. Literature which records an advantage of surface load 16, 10, 11, & 12 reports loads above the range we investigated but porosities not less than .3 can be attained with natural size ranges of sand without crushing the particles.

The high speed photographic experiments yielded motion film, stills from which are shown in Figs. 5, 6, 8 & 9. The vertical monolayer of dry glass spheres is shown before vibration, Fig. 5a and after 2.110 seconds of vibration of 50 Hertz and with a maximum acceleration 8 times that of gravity Fig. 5b. It can be seen that the porosity of the vibrated layer is lower than the original state and that the layer is more ordered after vibration.

The sequence of six photographs during motion are .005 seconds apart, Fig. 5b-5g, and show the effect of a ring of six spheres transmitting higher force than between most other particles. This higher force occurs on impact and is due to the ring being constrained laterally and bridging a region of high porosity (shown ++ Fig. 5b). The ring is also at the apex of a triangle of hexagonally packed spheres which will not rearrange during the cycle thus quickly transmitting the effect of the high force through the bed. The result is that the triangle of regularly packed particles moves downwards relative to the container and the particles above move upwards. A zone of high porosity results around the ring which upon collapsing during the next vibration cycle caused lesser impacts which could be seen in later frames of the
Figure 5 A dry, vertical monolayer of glass spheres.
(e) after 2.125 seconds vibration

(f) after 2.130 seconds vibration

(g) after 2.135 seconds vibration

Figure 5 (cont.) A dry vertical monolayer of glass spheres.
For the triangle of regular particles to move downwards shear must take place at its boundary. This is visible in Fig. 5c along with the expansion to allow slide over the bounding layers termed dilation. During the four seconds of vibration that was filmed this effect occurred three times with subsequent minor impacts. In a three dimensional bed of spheres being vibrated the triangle of regularly packed particles which moves relatively downwards would be a cone. High forces would be generated at abnormalities in the packing, occurring more frequently the more regular the system. This would limit the porosity and not allow a final ordered packing to be achieved. A system of spheres manufactured to high tolerance reduces this effect as has been observed by other workers. Similar disturbances can be imagined in a three dimensional bed of irregular particles but in that case many minor disturbances would occur. The effect would cause a more turbulent bed than for spheres; compare Figs. 8b to g of spheres and Fig. 9b to g of similarly sized sand subjected to the same vibrational energy.

Film of the monolayer immersed in air free water showed a more damped system when subjected to the same vibrational acceleration. Areas of regular packing were visible earlier, only six cycles from the start of vibration. The water acts as a lubricant at the contacts, and delays the approach of two particles as the fluid flows from the voids.

After the water had been drained from the container the
Figure 6 A vertical monolayer of glass spheres bounded by pendular moisture.
(e) after 1.925 seconds  (f) after 1.930 seconds

(g) after 1.935 seconds

Figure 6 (cont.) A vertical monolayer of glass spheres bounded by pendular moisture.
Vertically suspended mirror.

Vibrator

Trunnion

Sample box

plinth

Fields of view

'Fastax' camera.

Oscilloscope screen

Fig. 7. Plan of the 'Derritron' vibrator, the sample box, the 'Fastax' high speed camera and the oscilloscope.
spheres were connected by water menisci known as pendular moisture. The bed was again subjected to the same vibrational acceleration but no particle rearrangement was visible; the moisture firmly holding the spheres in place. The only motion visible, but not detectable in the photographs shown in Fig. 6, was two separate particles held by menisci equally spaced above and below the sphere. The vibration caused the spheres to oscillate as if suspended elastically. No menisci from these particles to either face of the container was evident. Menisci from other spheres to the container would undoubtably hold the bed in place but such bonding would not be as strong as if to adjacent layers of spheres.

The photographic investigation of bulk circulation gave Fig. 8b to f of glass spheres vibrated at 30 Hertz and 17g. as the clearest evidence. The flow of material down the walls of the vessel is affected by the friction between the walls and the spheres, since flow down the corners of the container was fastest, but it was felt that this was not the only cause. Scant evidence suggested that the vibration cycle propagated most strongly up the centre of the vessel thus exhibiting higher amplitude there. Since recovery of the bed before the next cycle is due to gravity alone and recovery time is nearly constant across a horizontal section of the bed the central spheres will not fall the same distance as they were projected. The net result would be a flow upwards at the centre of vibration. Clear evidence of this phenomena has been demonstrated only recently by Takahashi, Suzuki and
(a) before vibration  (b) during vibration

(c) during vibration  (d) during vibration

Figure 8 Glass beads vibrated at 30 Hertz and 17g.
Figure 8 (cont.) Glass beads vibrated at 30 Hertz and 17g.

(e) during vibration  (f) during vibration
Tanaka. Their figure four of a 'Vee' shaped container shows flow downward towards the apex of the 'Vee' and upwards from the sides where most of the contained material receives the oscillating force. Friction between the particles and wall of the type we observed cannot be present in Tanaka's experiment.

Vibration of a bed of sand yielded Figs. 9b to g. Vibration conditions were identical to those of the layered glass bead experiment, however conditions in the bed were markedly different. From the beginning of vibration, circulation was very turbulent and erratic. The maximum acceleration was 17g, causing a cavity at the bottom of the bed 8 millimetres high. This cavity had to fill and empty with air on each cycle thus causing an appreciable flow of air through the bed. Air flow and local high stress caused the turbulence. The holes formed in the bed of sand are of great interest. They were stable through the vibration cycle and stationary as long as the vibration was applied. Granular material was seen to fall through the holes suggesting an upward flow elsewhere. The overall appearance of the hole was like a bubble in a fluidised bed. A recent paper by Yoshida and Kousaka points out the importance of air circulation in a vibrated bed and expresses equations which equate air velocity with minimum fluidisation velocity. This of course would only apply for part of the cycle but the air may not be able to flow through the bed in the cycle time.

A point of interest is that the holes only occur in the
Figure 9  Sand vibrated at 30 Hertz and 17g.
(e) during vibration  
(f) during vibration

(g) during vibration

Figure 9 (cont.) Sand vibrated at 30 Hertz and 17g.
corner of the vessel. The edges of the container were not gas tight and it is possible that the hole contains air drawn in to replace that forced out of the container from the cavity between the bed and the vessel base. The air drawn in is trapped there unable to flow away.

The particle turbulence around the holes is high. It appears as though further investigation may lead to a technique of contacting fluid and particles without the fluid flow rate being subjected to the demands of fluidisation.

1.3 Conclusions

1. The final porosity of a vibrated powder depends on the acceleration of vibration.

2. The majority of the packing due to vibration occurs within ten seconds of vibration at an acceleration of 8.35 times that of gravity and 50 Hertz.

3. Surface load does not significantly affect the rate of compaction in the range investigated.

4. Areas of high stress in the granular material cause turbulence and temporary uncompaction.

5. Shear occurs in the bed.

6. Immersion in fluid damps the particle motion

7. Pendula moisture strongly bonds glass spheres and inhibits interparticle movement.

8. A bulk flow of granular material occurs during vibration at a peak acceleration of 17g.

9. The vibration of an irregular material causes more
turbulence than the vibration of spheres at the same energy input.

1.4 The effect of these experiments on the major experimental work.

The preliminary experimental work allowed observation of bulk phenomena and also suggested possible particle behaviour. It became apparent during the experiments that all effects were in some way due to the mechanism of force transmission through the bed. It was decided that the major experiments would investigate force transmission.

A problem being faced by industry during vibratory powder moulding suggested the approach our experiments would take. Zones of lower and higher porosity could be found in the compact which would cause areas of weakness in the finished product. It was thought that these could be caused by zones of vibrational inactivity due to the interaction of the primary wave from the source of vibration and a reflected wave from inside the mould. A divergence of the effect of an applied force was suspected since it would be unreasonable to assume that only an area projected from the input in the direction of the force was affected. The divergence possibly subjected some parts of the granular material to higher forces than others; this would also be investigated.

In planning the experiments the form of the force propagation appeared complex, and to study the phenomena with an oscillating force would entail attempting to sort the
summation of primary waves from reflections. It was decided to investigate propagation by applying a pulsed force to the bed of granular material.
CHAPTER TWO

THE LITERATURE SURVEY

2.1 Introduction to the survey

The intention of this survey is to discuss all aspects of the stresses in granular material due to applied vibration. The first section will consider the variables an engineer has at his disposal and the different effects in a granular system caused by vibration. The stresses due to vibration produce all the effects but a detailed knowledge of these is not necessary to cause compaction, bulk circulation, etc.

The second section will consider the physical state of material one could wish to vibrate and the physical laws governing the arrangement of the particles in a static state. Also considered will be the way they behave during disturbance. Then the approach by means of vibration to these theoretical physical states will be discussed.

The history of understanding the local stresses within a granular material will be followed via sections entitled, 'Mechanics of granular material' and 'Phenomena of propagation of stress'. Within these titles, observations and theories of explanations will be discussed critically whilst the important points relevant to the Thesis will be emphasized.

Since the approach we make to understanding the distribution of stresses within a bed of material draws on
statistical theory and a new form of particle size measurement, the histories of relevant aspects of these will be described in sections entitled, 'Statistical models' and 'Particle characterisation'.

Conclusions from the literature are discussed in a final section.

2.2 The effect of vibration on a granular material.

A simple vibration apparatus was described by Westman and Hugill in their paper on the packing of particles in 1930, the first recorded data known to the author of the use of vibration. Commercial vibrators were available at that time but little published data is available. Scientific interest was increased in the early nineteen fifties when the peaceful utilisation of atomic energy was being studied. Vibration was used to compact nuclear fuel into cylindrical containers for reactor feed and consequently empirical data became available. (A bibliography by Jones lists references relevant to nuclear work up to 1967).

Bell, who was amongst the first researchers, produced much data which appeared in publications on the subject of ceramics. He pointed out that little basic research had been done. His work was empirical. He concluded that particles vibrated with slight restraint sought the closest packing, the smaller fitting between the larger particles. In a powder of uniform size but different shape, orientation produced closer packing. Lubricants to reduce contact friction were used but may have produced undesired effects.
in subsequent processing. The variables of the system according to Bell included inertia and resonance, and the interparticle friction which he said affected the cohesion of the particles. (In the light of recent work it would be more correct to consider friction at a contact and cohesion quite separate). The particle size distribution was another variable. He achieved more dense compacts when there was the correct proportion of fines to fill the voids of the larger particles. Vibratory compaction wears the dies and lowers size segregation compared with pressure techniques.

The references of his later work included many similarly empirical papers using pneumatic vibrators acting on both bottom and top surfaces of the compacting material. The materials investigated include: - Alumina, Alumina-Chromium 'Cermet', Titanium carbide-Nickel 'Cermet', Uranium oxide etc., but little basic information for general application.

Likhtman et. al. 76 summerised Russian work before 1960 apparently conducted in a similarly empirical way as in America. He stated that 10 seconds was a long enough vibration time for low porosity and that 100th of the static pressing pressures are required. A more uniform product exhibiting no stress cracks was formed by vibration.

The frequency of vibration in experiments before 1960 ranged from 10 Hertz to 30 Kilohertz although with most success in the lower sonic frequencies. The ultrasonic experimentation 11 was an attempt to utilise a technique which had been applied to many processes ranging from mist
agglomeration to detecting cracks in metals. Webb\textsuperscript{113} used 20 - 48 Hertz to pack a three component system and reported that a little water or alcohol (2 - 5\%) increased the packing rate and stopped the finer material passing right through the coarse matrix. Hauth\textsuperscript{19} conducted experiments in the same vibration range but obtained similar results to Webb only when the container was vibrated as it was filled.

An important factor reported in all these papers is the use of a load on the surface of the granular material. Bell suggested slight constraint but the other workers produced conflicting results. More recent research has indicated a detrimental effect on high surface load. However in 1964 Brackpool and Phelps\textsuperscript{16} produced their lowest porosity with a surface load of 1470 pounds per square inch, which they applied only after unloaded compaction was completed. Their work was with spherical copper powder, produced commercially as a raw material for sintering. They concluded that lowest porosity resulted from a two part process; firstly by impacting, when the surface load was less than the vibrator force, and then by densification when a high surface load deformed the particles. The latter part was only effective on ductile materials. Their experiments on the ceramic alumina suggested no lower density after initial impaction was complete. A test on fine (5 \(\mu\)) tungsten powder did not compact under any conditions. Increasing surface load increased the density of product at low frequencies (100 Hertz), vice versa at higher frequencies (300 Hertz).
At the same time Evans and Millman had conducted their research on bronze powder, litharge, alumina and Bakelite with an electromagnetic vibrator amplified by a resonant beam. They quoted an equation of power consumption in vibrating granular systems:

\[ W = K m a^2 v^3 \]

where \( W \) is the power consumption, \( K \) is a constant, \( m \) is the seismic mass, \( a \) is the amplitude and \( v \) the frequency.

Once again the first few seconds caused the majority of settlement. The most effective frequency for packing was investigated and two peaks with each material were found which were thought to be resonances. The best compacts from experiments of frequency variation were obtained by cycling the frequency around a resonance as had been reported by Hauth. The authors concluded that particle size is the most important factor in producing low porosity compacts but point out that the best blends of size distribution for one material are not necessarily the best for others. The hardness of the material compacted had no significant effect on its behaviour.

Selig experimented with sand on an electromagnetic vibrator and produced contours of final product density on axes of maximum acceleration and frequency. Whilst the separate effect of amplitude and frequency had been previously reported the presentation technique Fig. 10 allows easy determination of the optimum conditions with a controlling parameter.
Fig. 10. Selig's contours of constant product density, as a function of frequency and maximum vibrational acceleration.
Ivashchenko et al.\textsuperscript{63,64} worked on the formation of metalloceramic (cermet) filter elements used in engineering. Their powders were spherical and showed a dependency of rate of packing on the frequency, which suggested the number of cycles experienced by the material is a parameter. The process, investigated with two components, entailed percolating the finer fraction through a coarse matrix, a common practice. However in their case they were able to produce a uniform packing or a specific porosity, varying along the height of the material; an important factor in filter design. They also observed that accelerations of vibration above that which gave the required density, disrupted the upper layers of coarse material which upset the filtering of the fines.

Kroll\textsuperscript{72} reported a bulk circulation in a vibrated bed of granules in a paper which discussed qualitatively the compression and expansion during vibration. Alyanak\textsuperscript{1} similarly observed this and produced photographs of the motion of traced sand in a cylindrical container. However the motion he deduced from the photographs is not what is usually observed. The rings of tracer, the visible edge of trace layers, normally descend the wall parallel to each other, demonstrating circulation in a uniformly vibrated vessel only by the fact they descend. The clamping technique in Alyanak's experiment gave only two points of connection to the vibrator effectively producing two vibration sources. Hence the distorted traces show fastest descent at a position displaced
90° horizontally from the two clamping positions. Alyanak explained circulation by wall friction.

Takahashi et al.\textsuperscript{10} attempted correlation of the circulation effect. They reported that circulation could only occur when the maximum acceleration of vibration exceeded that of gravity (also reported by Alyanak) i.e. a finite relative amplitude between the vessel and some of the particles existed. Takahashi's experiments were with a model hopper with the two inclined sides acting as independent vibration sources. The circulation they observed was upwards from the vibration sources and consequential downward flow in the area of least vibration. They also presented data of the variation of particle velocity along the length of each vibration source Fig. 11 expressed as height in the vessel. It is of interest that the point on the vibration source producing the highest particle velocity is also the point of connection of the stand inducing the vibration in the source. However Takahashi's results show a circulation without a wall friction of the kind Alyanak used to explain the behaviour.

We now have two threads of evidence that a propagation phenomena is involved i.e. that a line in the direction of the vibratory displacement throughout the centre of the vibration source experiences a relative upward motion during each cycle of vibration. The preliminary experiments conducted in this work showed a flow up the centre of a container, the base of which experiences uniform vibratory displacement. In that case the flow was uniform about a
Fig. 11. Takahashi's graph of particle velocity in relation to position on the vibrating plate.
vertical axis.

The work of Takahashi was directed towards the understanding of flow through slits at the bottom of a hopper when induced by vibration. His co-author Suzuki continued the work on hopper flow. In obtaining data to verify a theory he showed that the rate of discharge increased as the acceleration of a base plate increased, through a hole in which the material was flowing. Suzuki concluded that if a slit was blocked under the effect of gravity, a vibration with an acceleration greater than that of gravity would make it possible for particles to flow out. The application of vibration to the blocked hopper must be useful. This statement may be a little misleading since a vibration causing flow at the outlet could compact the bulk of material not at that time involved in the flow at the outlet.

The effect of air flow through an enclosed granular system during vibration was discussed by Yoshida. When the bed was vibrated at an acceleration greater than that of gravity, a cavity between the vessel base and the material can be formed as the bed and container separate. A rarefaction and compression took place in an air filled system which induced the flow of air through the granular system. When the cavity was compressed at a rate such that the air velocity through the granular material exceeded the minimum fluidisation velocity the bed could no longer be impacted and began to expand from the minimum porosity state. They concluded that even before the minimum fluidisation velocity
the drag of air on the particles had a significant effect in counteracting the compaction effect of the vibration.

Segregation of different sized fractions of particles had been mentioned by Brackpool and Phelps. They concluded that mixtures of powders of widely differing densities and size showed little or no segregation after vibratory compaction. Segregation on a vibrating plane was the major topic of research by Williams and Shields. They thought the mechanism of segregation was twofold, that the fines filtered through the coarse material and the large particles had more inertia during the vibration cycle and could move relative to the fines. They experimented on fertiliser granules, separating the upper and lower layers from material flowing down a vibrating plate inclined at 11 degrees. The segregation caused was worse when the vibration was directed at between 25 and 30 degrees to the horizontal. It was not uniquely controlled by the acceleration either vertically or in the direction of vibration.

Lawrence and Beddow also studied segregation. Their work was concerned with filling dies with lead particles of different sizes. They presented data of the zones of different particle behaviour during vibration, in order of increasing acceleration, Stability, Churning and Bouncing. When their fine particles were poured into the matrix of coarse particles a cone of fines formed within the coarse grains. This was levelled when vibration was applied, thus
evening the radial distribution of fine to coarse. Segregation was low during the stable period (packing), high when the bed was churning and low when the bed was bouncing. They concluded that the fines tended to filter through the coarse during vibration.

Civil engineers have long been interested in the vibration of soil. Converse\textsuperscript{26 25} studied the vibration of unbounded sands in 1952. He used a compactor to vibrate a sandy beach at its resonant frequency and measured the behaviour of the sand mass for different vibration conditions. He presented a theory for the unbounded system assuming that a truncated cone of sand with the vibrator at the apex was affected. A spring modulus for the cone was estimated which he defined as the reaction of the sand per unit displacement of the vibrator. The mass of the cone was estimated and also the reaction of the surrounding soil in a horizontal plane. His experiments showed it to be an accurate model, however inertia of the sand mass must be included. The angle of the cone he used in the model cannot be separated from the theory.

The shear strength of a granular material is affected by vibration. Mogami and Kubo\textsuperscript{90} vibrated various soils in a direction normal to the shear plane. The soil was constrained by static load on the surface and data of shear force against acceleration of vibration was produced for different surface loads. Their results showed that the shear strength was
markedly reduced with increasing vibrational acceleration. An acceleration twice that of gravity reduced the shear strength with each normal load to that value for an unloaded static test at the same depth. Kutzner\textsuperscript{73} carried out similar tests on glass beads.

Alyanak\textsuperscript{1} calculated the shear strength by considering the intergranular pressure. He concluded that that alone caused the reduction of strength.

Barkan\textsuperscript{8} conducted experiments with vibration in the direction of shear on sand. Shear strength varied linearly with normal load both in the dynamic case and the static case. Moist sand exhibited a lesser reduction in shear strength during vibration due to cohesion between the granules. The expansion of the bed due to shear for materials denser than the critical porosity (see section 2.3) was less when the material was vibrated. As the load on the surface was increased, the change in porosity due to shear decreased.

The Phd. thesis of Youd\textsuperscript{123} provided the most comprehensive explanation of shear strength and porosity variation due to vibration yet produced. The work of Roscoe was used as an example of the behaviour of granular material during static shear and in the critical porosity state. In a vibrated bed which was subjected to shear a critical porosity existed. Youd suggested and showed experimentally that the 'equilibrium porosity'\textsuperscript{*} (maximum porosity attainable at fixed

\* See note at the foot of the following page.
vibrational acceleration) was the critical porosity for shear occurring during vibration.

The 'equilibrium porosity' curve was of the type obtained in the preliminary experimental work Fig. 2 and by Barkan, and represents the conditions that would be observed for the compaction process and for shear during vibration. To compact a material at initial porosity \( \varepsilon_0 \), the vibrational acceleration must be that on the curve corresponding to \( \varepsilon_0 \) before compaction would occur. The fact that a compaction of a granular material would not begin until a certain acceleration was achieved has been reported by most workers on compaction. It has been commonly termed 'the threshold of compaction'. The only state of porosity of the material which would not exhibit a threshold of compaction was at point 0, the loosest possible stable state of the material, any disturbance causing a compaction.

Thus from an experimental determination of the relationship between porosity and vibrational acceleration for the material, and a shear box test at no normal load, the behaviour of a granular material of any initial porosity can be predicted during the compaction process and during shear.

* The term porosity has been substituted for the more common civil engineering term of 'voids ratio'. The relation between the two is: voids ratio = \( \frac{\varepsilon}{1-\varepsilon} \), where \( \varepsilon \) is the porosity. Porosity is the percentage voids in the bed of granular material.
In conclusion, the parameters which increased the packed density were:

(a) Vibration whilst filling
(b) A slight surface load such that the surface force was less than the peak vibrator force.
(c) A proportion of fine to coarse grains such that the fines filled the voids of the coarse.
(d) A lubricant (water or alcohol 2-5% by weight) reduced contact friction.
(e) Ten seconds vibration caused the majority of packing.
(f) Forty to two hundred Hertz the most effective frequency range.
(g) Cycling the frequency around the resonant frequency.

Other technically important effects of vibration which were reported were:

(h) Bulk circulation occurs when the vibrational acceleration is in excess of that of gravity.
(i) Flow from a hopper can be initiated when the vibrational acceleration is in excess of that of gravity.
(j) Particle size segregation is low when the acceleration is below that of gravity but increases with increasing acceleration.
(k) Vibration both in axial and normal directions reduces shear strength.
2.3 **The physical state of a granular mass.**

Kolbuszewski\(^6\) studied the fundamental factors controlling the loose packing of sand. He attempted to predict the porosity of sand depending on the way it was deposited. His experiments entailed varying the intensity of deposition and measuring the profile of particle intensity normal to the direction of fall. A dependence of packed porosity on the intensity of deposition was found but similar experiments conducted underwater did not fit the data. A velocity of fall effect was suspected and verified by experiments in water, ether, and air.

The conclusions were that a low velocity of fall produced a high porosity, and a high velocity of fall produced a low porosity at a low intensity of deposition. Increasing the intensity of deposition increased the porosity at a fixed velocity of fall until the intensity was that of a falling mass of sand. Then the porosity produced was that minimum determined by the tilting test. High intensity of deposition locked the grains before they could achieve their stable position. High velocity of deposition provided enough energy for rearrangement to take place.

McCrae and Gray\(^8\) developed the work of Kolbusewski and included the effect of material properties. Their experiments were with steel, phosphor bronze, and lead spheres and two types of glass spheres. They looked closely at the packing zone, where the descending particles rearranged and formed the packed bed. Four regions were observed, in order as:
height increased; (a) The packed bed, (b) the region of bed with bulk flow, (c) a region of individual particle movement, and (d) the region of falling and rebounding spheres. Observation showed that densest packing occurred when region (b) was most active. (The activity of (b) is affected by the kinetic energy of the falling particles, the resilience (fraction of energy transferred on impact) and the rigidity of the stable bed. The zone will be at the critical porosity.) Their conclusions were that to produce an ordered packing the energy of each falling particle must be above a critical level and intensity of deposition below another critical level. The dependence of packing performance on the resilience of the particles falling onto the surface of the packing is shown in Fig. 12.

A tabular form of the effect of variables on the packing of particles due to McCrae and Gray is extended to include more recent work and shown as Table 5.

Conditions which produce an ordered packing with spheres are likely to be those which produce the densest packing with irregular particles but not the same porosity. The work of McCrae and Gray and that of Youd suggests that energy levels of individual particles are important. The regular packings produced by McCrae and Gray are of low potential energy since consecutive layers nest into the previous layers. With a material of irregular granules one
can envisage a similar nesting when an upper particle reorientates until its centre of gravity is as low as possible. This will occur when it is in the space between lower particles. A dense packing would result.

Fig. 12 The effect of resilience of falling particles on the porosity of the bed they produce.

The study of particle packing was intense around 1930. Smith, Foote and Busang\textsuperscript{192} studied the actual packing of shot. They observed that for certain statistical purposes the arrangement could be treated as a mixture of close hexagonal
## Table 5  Evidence in the literature concerning variables considered to be important in the packing of particles.

(after McCrae and Gray)
and simple cubic pilings required to yield the observed porosity. One such purpose for the assumption was to estimate the average number of contacts per sphere which agreed with the experiment. The distribution function of numbers of spheres having a certain co-ordination number was also included which showed the distribution to be Gaussian for the higher porosities.

Westman and Hugill\textsuperscript{115} compacted many materials of different size and particle shape. They observed that single sized spheres packed to the same porosity and that around sand was limited to a higher porosity the more fine the sample. A calculation of porosity of spheres showed that the packed porosity of 39.5\% can be predicted if the packing is hexagonal in one plane and cubic in the other. McGeary showed that vertical force tended to compact the vertical plane producing hexagonal packing whilst cubic packing predominated in the horizontal plane. Westman and Hugill observed a minimum porosity with a binary system of two sizes of particles and both tertiary and quaternary mixtures produced a minimum.

Furnas\textsuperscript{38,39} was interested in the flow of gases through broken solids and produced mathematical relations for a system of minimum porosity. Experimentally he showed this to be accurate down to a size difference of 5 to 1. The relationship of volumes occupied by consecutive sizes was:-
\[
V_1:V_2:V_3 \ldots \ldots \ V_n = 1:\varepsilon:\varepsilon^2 \ldots \ldots \varepsilon^{n-1}
\]

where \(\varepsilon\) is the porosity.

Anderegg\textsuperscript{2} presented the paper following that by Furnas\textsuperscript{19} having worked in co-operation with him. Anderegg applied the mathematical relations to cement products and concrete. Experiments showed that the relations developed by Furnas gave good results. On one or two occasions the aggregate mixture proposed was hard to mix. However the density results obtained were such that techniques were developed to handle the aggregate.

Graton and Fraser\textsuperscript{42} extensively discussed the packing of spheres, being convinced that an understanding would answer the problems of packing of irregular particles. They discussed observable physical phenomena and their ideas of probable arrangements of spheres and frequency of arrangements. An important observation was that a perfect packing could only be achieved if the container had exactly the right side length and corner angles. They also observed that zones of hexagonal packing occurred within initially randomly packed bed of spheres. During the work for this thesis observation of packing of spheres in two dimensional beds show areas of hexagonal packing.

The porosity of regular packings of spheres were calculated by White and Walton\textsuperscript{16}; Cubic packing 47.64%, Single stagger (orthorhombic) 39.55%, Double stagger (tetragonal) 30.20%, Pyramidal 25.95% and Tetrahedral 25.95%.
Secondary, tertiary, quaternary and quinary spheres, each fitting the voids of the larger spheres, could be arranged to a porosity of 14.9%. A very fine filler could then be included to reduce it to 3.9%. Elliptical particles produced no less porosity but a marked reduction was calculated with cylinders. The authors discussed methods of producing cylindrical particles with ceramic materials. In trying to approach the calculated porosity with flint and clay in the correct proportions a porosity of 33% was produced. 3.9% was calculated.

Carman discussed the porosities of regularly packed spheres and reproduced data of porosity for packings of co-ordination number (number of contacts per sphere) from 3 to 12. In a dense random packing of spheres the porosity was 38.5%. Smith, Foote and Busang produced co-ordination number against porosity curves for random packings of spheres which for the higher porosities were normal distributions. At lower porosities this was not so, probably because the shape of the spheres introduce a regular effect which overrides the random effects. Carman pointed out that the size of spheres affects packed porosity only below .1 millimetre. Surface area has an effect on the packing below this range.

Brown and Hawksley stated that the study of regularly packed spherical systems was not likely to help in the study of irregular particle packing. Their reasons were that equal spheres poured into a box take up a disordered arrangement with a porosity of between .45 and .37 and irregular
particles pack in an entirely different manner. Although the packing is disordered, the frequency distribution of number of contacts per particle is different; compare the results with spheres of Smith, Foote and Busang with those of Bennett and Brown\textsuperscript{13} from irregular starch particles.

Brown and Hawksley suggested there were two opposing effects in the packing of irregular particles. Spiers quoted porosities of .45 for 1 inch coal to .54 for .0625 inch coal while the authors obtained .43 for 1 inch coal to .47 for .125 inch coal by experiment in a 10 inch cube. They say that these two effects may cancel each other in certain cases suggesting a linear relationship by experiment. Dilatancy discovered by Reynolds\textsuperscript{93} and investigated by Jenkin\textsuperscript{65}, among others, not only accounted for non-uniformity in the transmission of force in a granular material but also non-uniformity in the distribution of local porosity. Variation in local porosity was, in practice, likely to be more significant than the overall porosity. For these reasons the study of the geometry of ordered arrangements of spheres was of little assistance.

Hudson\textsuperscript{54,55} investigated the interstices between spheres. He found that those of cubic and hexagonal close packing are the same shape but of different distributions. The void was bounded by four or six convex surfaces which were connected by a labyrinth through which balls exceeding \((2/\sqrt{3})-1\) times the radius of the spheres could not pass. The increment of porosity reduction caused by introducing a
further sized sphere into the voids of the previous was reported. He showed that only a few spheres in the wrong place could disrupt the whole system.

The wall effect on a packing of particles was discussed by several workers. Scott allowed for container edge effects on a packing of equal spheres in a spherical container by assuming the effect is a function of surface to volume ratio, or the reciprocal of the radius of the spheres. The radius is a linear function of $N^{-1/3}$ (where $N$ is the number of balls contained.) He plotted the packing density (one minus the porosity) against $N^{-1/3}$ and obtained a straight line. Extrapolation to $N^{-1/3}$ tending to zero gives a value of packing density for an infinite number of spheres. A similar plot was suggested for spheres in a cylindrical container.

McCrae and Gray found that wall effect extended through two particles. However when the zone, in which the packing of the bed was formed, was very turbulent, the wall effect extended through five particles.

A mathematical approach to the porosity of different states of packing has been made by Supnick. He considered spheres in a cylinder such that two adjacent spheres can jam across the diameter, a state he refers to as incompressible. The prediction of the order of sphere diameters for the loosest and densest incompressible packings is of little relevance to packing in bulk. However a development of the
theory to packing in a matrix of other spheres may prove useful.

Wise\textsuperscript{120} considered a dense random packing of unequal spheres and made extensive use of statistical theory. Dense random packing was defined in a new way. A density function $w$ of tetrahedra with apexes at the centres of four spheres such that each sphere touches the three others was described. Thus $w$ is defined by four radii. Boundary effects were ignored and general equations for $w$ were deduced. A specific application of development of the theory for $w$ being a log normal distribution was developed to predict physical properties of a real heap of spheres.

Hogendijk\textsuperscript{52} extended Wise's work by considering a log-normal distribution of 1, 2, 3, 4 and 5 different sizes of spheres. The radius distribution of the largest possible interstitial sphere was calculated and the inclusion of these interstitial spheres had a calculated effect on the density. The fact that this theory concerned tetrahedra of centres of spheres mutually in contact restricted the application of the work to the real case. However, an estimate of the number of tetrahedra with one side between spheres not in contact could be made, which would extend the utility of the theory.

A similar approach to the geometry of centres was made by Kallstenius and Bergau\textsuperscript{68}. They considered the tetrahedra between sphere centres but did not restrict it to spheres in mutual contact. Experiments were conducted to measure the distribution of the tetrahedra in a cylinder of spheres. The
bed investigated was formed in a cylinder with a flat base by free fall. It was found that the heights of tetrahedra were not consistent with isotropy. They concluded that examining granular matter by statistical analysis and experiment was a useful attempt to understand the behaviour of granular materials.

Hvorslev had described a continuous function of criterion of failure for a soil. His tests had shown a good agreement with:

\[ J_f = u_0 \sigma'_f + \nu \exp \left( -B e_f \right) \]

where \( J \) is the shear stress, \( \sigma \) is the normal stress, \( \nu \) is the voids ratio and subscript \( f \) denotes the failure condition. \( u_0, \nu \) and \( B \) are constants for the system. A diagram of the surface of failure for a cohesionless granular material shown in Fig. 13. Any point above the surface in the direction of shear stress cannot exist since the failure condition has been exceeded.

Roscoe extended Hvorslev's failure surface study to include the condition of critical porosity. Continuous shear failure of a granular material occurred at a fixed porosity, the critical porosity, dependent on the normal load applied. The porosity at failure is termed the critical porosity. The critical porosity for each failure condition lay on the Hvorslev failure surface and represented the condition of continuous shear. Failure could begin anywhere on the plane but tended toward the critical porosity line.
critical porosity line on which steady state shear occurs

critical porosity line on zero shear force plane

Change of porosity with increase in shear force at constant normal force for a sample initially below the critical porosity.

Change in porosity with increase in shear force at constant normal force for a sample initially above the critical porosity.

Fig. 13. Hvorslev's failure surface for a cohesionless granular material, including the critical porosity line of Roscoe on the surface.
Packing of granular material has been described by means of spherical models almost without exception. The step from a spherical system to one of irregular particles is proving difficult to make. However the study of packing of spheres has given a qualitative idea of granular behaviour.

The definition of the failure condition described refers to failure in which many particles move in one direction relative to the stationary bulk. The failure surface predicted for a cohesionless soil is an example of such a surface. The diagram can be constructed theoretically, although not yet absolutely, and once found completely describes bulk flow failure conditions.

2.4 The approach of experiment to theoretical conditions of porosity.

Initial experimental work considered the effect of variation of size distribution of the constituent grains and also often included a qualitative description of the shape of the particles. Westman and Hugill experimented with spheres and achieved a porosity of 37% with lead and 39.2% with steel. The porosity calculated for hexagonal packing in one plane and cubic in the other was 39.5%. Particles of various shapes, tending to be round showed no trend due to shape, but the porosities produced were close to those for spheres. Round, washed sand had a constant porosity of about 38% for three monosize experiments with approximate sizes of 2000\(\mu\), 500\(\mu\), and 250\(\mu\). A sample of the same sand of size
about 50μ packed to 42.5%. They investigated the porosity of binary mixtures of a coarse (≈ 4000μ), medium (300 - 400μ) and fine (75 - 100μ) round washed sand and obtained minimum porosities in all cases at 70% of the coarser fraction, 30% of the finer. The minimum porosities were 18.5% for coarse - fine, 26.7% for medium - fine and 27% for coarse - medium mixtures. A graph of the porosities of mixtures of 70% coarse to 30% fine of various materials of different coarse to fine diameter ratios is shown in Fig. 14.

Their investigation of mixtures of the three sizes of sand produced a minimum porosity of 15.5% at 70% of ≈ 4000μ, 10% of 300 - 400μ and 20% of 75 - 100μ.

Carman suggested that low porosities could be obtained if the ratio 4 to 1 was maintained between adjacent sizes. He measured a porosity of 38.5% for stackings of spheres of lead and glass but obtained 15% porosity for a three sized system. He thought that surface effects were dominant with particles below 30 microns in size.

Bell found that the best ratio of coarse to fine in a binary mixture was 60% coarse, 40% fine which produced a porosity of 15%. He used vibration for compaction whereas the previous authors had used a tapping technique. The difference in the techniques, according to Yoshida and Kousaka, is the number of cycles applied if the geometry of the systems was the same; Bell likely having applied far more cycles than either Westman and Hugill or Carman.

Webb quoted Dalla Valle as obtaining 14.6% porosity
mixing five sizes. However Dayton and Brown obtained only 17% with a mixture of three of the same sizes. Hauth claimed to have achieved 5% without giving details but obtained 8% with a three sized mixture.

![Graph showing porosity percentage vs. diameter ratio](image)

**Fig. 14** The porosity of 70% coarse fraction in a binary mixture of various materials.

Other workers using spheres consistently obtained a porosity of between 38% and 40%. Scott experimentally arrived at this figure but extrapolation to an infinite container porosity of between 36 and 38%. McGeary measured 37.5% for three separate sizes of steel spheres but a slightly lower porosity for 30 micron tungsten and aluminium spheres. His
experiments with binary mixtures of spheres predicted 86% of coarse size to 14% of an infinitely fine filler for minimum porosity. The results he extrapolated were of the packing of spheres where the proportion of coarse to fine varied from 60% to 40% for a diameter ratio of 20:1. He also obtained a porosity of 4.9% with a mixture of four sizes of spheres of diameter ratio 3:1:6:7:1.

Ayer and Soppett agreed with the figure of 14% fines as the best binary mixture of spheres. Extrapolation of their experimental results for binary mixtures predicted a minimum porosity for tertiary systems of 4.9% which compares well with the experimental value claimed by Hauth. Ayer and Soppett similarly extrapolated their results with irregular particles and obtained 9.7% which compares with the data presented by Hauth.

Brackpool and Phelps experimented primarily with copper powder and achieved 25% porosity with an 80% mix of ~100μ coarse material and 60μ fines. They also observed that the highest porosities were obtained with 50μ alumina, again suggesting the effect of surface area.

Evans and Millman achieved 22% porosity with an 'as received' litharge. They also experimented with the same sized fractions of four different materials as a component mixture. They achieved porosities of 14.1% for a spherical bronze, 23.8% for litharge, 25.4% for alumina and 47.8 for Bakelite.

Ivashchenko measured a minimum porosity of 17% with a
mixture of 72% coarse and 28% fines of spherical copper.

A wall effect predicted statically to extend two or three particles into the bulk of granular material has been measured by several authors. Ayer and Soppett detected a levelling off of a packing efficiency curve at a sphere to container diameter ratio of twelve; that being a measure of wall effect. McGeary only detected the levelling out at a diameter ratio of 40 to 1. McCrae and Gray measured the wall effect as extending inward 5 particle diameters for a system which packed due to falling grains.

2.5 Theory of effect of vibration

An attempt at explanation of observed phenomena both qualitatively and quantitatively is usually made in a technical publication and the study of the effect of vibration is no exception. Qualitative explanation in those papers so far reviewed (they are thought to have the most bearing on process engineering applications) invariably entails the discussion of the behaviour of individual particles. On the other hand two types of quantitative theory are apparent; that which considers the individual particles and that which considers the granular material as a continuum. The author believes the particle approach is finally to be most fruitful. Some excellent work has been done particularly by civil engineers in predicting behaviour by the continuum approach, however it is discussed here in a minority.
Brackpool and Phelps explained the onset of impacting in their mould of granular material as that point where the force upward due to vibration exceeded the downward force due to the surface load. The resultant force moved the granular material upward relative to the container and the consequential movement toward each other caused the impact. When the surface force exceeded the vibratory force only granular rearrangement took place.

The reduction of force between grains during the vibration cycle is explained by Bažant and Dvořák by a propagation phenomena. Vibration propagated from the underlying to the overlying grains. The lower layers rose first whilst the upper layers did not move (due to a finite propagation velocity) and a compression took place. Then all the layers rose in succession. The base layers were the first to fall whilst the upper layers were still rising. Consequently dilation took place. For low vibrational accelerations the compression and dilation was negligible and stability was retained. (The design condition for machinery foundations.) For higher accelerations compaction occured and for even higher accelerations impaction occured with a more severe compaction on impact, and with a bigger dilation lasting longer. (The design conditions for compaction equipment.)

Yoshida and Kousaka described the compaction process when not considering air flow in two parts, impaction when the bed moved relative to the vessel and compaction due to
acceleration fluctuation. The latter case they considered minor as was found experimentally by Brackpool and Phelps. When air flow was considered the mechanism was slightly altered. Instead of the granular material impacting in free fall with the rising vessel it impacted at a reduced velocity due to the drag of air flowing from the cavity between the material and the containing vessel. The compaction due to impaction was reduced as the air velocity increased until the minimum fluidisation velocity was reached. The granular material was then deposited gently on the container bottom during each cycle. They also concluded that the application of vibration had exactly the same effect as tapping if the following variables were the same; the vessel, the particles, the number of impacts and the impact velocity.

A model proposed by Likin did not consider particulate behaviour. He included the effect of the material to be compacted as if it were a continuum. His model was of a mass of granular material with a compressive stiffness $K_2$, vibrated against a shock absorber of stiffness $K_1$. The Lagrangian equation of motion was:

$$m \frac{d^2x}{dt^2} + (K_1 + K_2)x = F \sin \omega t + P_0$$

where $m$ is the mass being compacted, $x$ is the displacement from the rest position, $F \sin \omega t$ is the vibratory force and $P_0$ is the static compressive force. By substituting an equation of plastic deformation due to Hencky and Bridgeman
the equation of vibratory pressing was:

\[ \log_{10} \epsilon = \log_{10} \epsilon_0 + 0.65 \frac{P}{\left( \frac{\sigma d}{2} + K_1 \right) x - \frac{P w_0^2 \sin \omega t}{(w_0^2 - w^2)}} \]

where \( \epsilon \) and \( \epsilon_0 \) are the porosity at pressure \( P \) and the initial porosity respectively, \( \sigma \) is the shear modulus of the material being compacted and \( d \) is the diameter of the mould, \( w_0 \) is the natural frequency of compaction.

Good agreement was found below a porosity of 35% but above this figure (higher porosity) the prediction was not so accurate. Likin thought the disagreement was due to the assumption of plasticity.

The mechanism of segregation of sizes in a vibrated mass proposed by Williams and Shields was in two parts. The fines tended to percolate through the matrix of coarse material. The second mechanism proposed was that the larger particles received more kinetic energy during the impaction part of a cycle since they were pressed by inertia against the packed bed. Their kinetic energy caused more inertia which permitted relative motion of the coarse particles through the fines at the limit of upward displacement of the fines.

Packing behaviour was predicted by Suzuki et.al.. They considered the relationship of relative velocity at which that particle was projected from the plate. Their expression
of packed density was:

\[(1-\varepsilon) = (1-\varepsilon_0) + K \left( \frac{AV}{V_0} \right)\]

where \(\varepsilon\) and \(\varepsilon_0\) are the porosities at that condition and initial porosities respectively, \(K\) is a constant and \(\frac{AV}{V_0}\) is the velocity function previously described.

The equation for the term \(\frac{AV}{V_0}\) does not contain any particle parameters and in view of the effect of shape observed by many authors and of resilience observed by McCrae and Gray, the closeness of theoretical prediction to experiment is something of a coincidence.

The discharge rate of particles from a hopper was also predicted by Suzuki et.al. They used their term \(\frac{AV}{V_0}\) in the form of relative particle acceleration which was substituted for \(\Delta P\) in the equation of flow rate of a fluid; \(Q = c \sqrt{\Delta P}\), where \(c\) is a constant, \(Q\) is the flow rate and \(\Delta P\) the pressure drop. There was fairly good agreement with practice.

Youd\(^{123}\) included several explanations of the effect of vibration on density. Converse stated that "The basic requirements for compaction of soil is that the shearing resistance or friction between the particles of soil be reduced to a point where the superimposed loads can press the particles closer together". Winkerhorn stated "The mechanism of vibrodensification is one of loosening the contact of a particle with its neighbours for a sufficiently long period for it to assume, under the influence of gravity and normal pressures, positions of lower potential energy". Johnson and
Sallberg stated "that particles in a granular system do not have equal contact pressures between particles. When a normal force is applied the particles are forced into the adjoining void spaces. As the force is released more deformations take place. Vibration consists of alternate loading and unloading". Simply stated they say "Adequate vibration meets those requirements of having sufficient force (deadweight and dynamic force), acting through the required distance (amplitude) for a sufficient length of time (frequency) for movement of grains to take place".

The theory of Youd described a system unable to compact further. The movement of particles to a lower potential energy state was prohibited by the frictional and interlocking forces on the particles. These forces formed an energy barrier above which the particle must pass in order to attain a lower energy state. Kinetic energy was provided by the variation which can allow the energy barrier to be crossed when the particle is in a position to achieve a lower state of potential energy.

With the aid of a graph of porosity plotted against vibrational acceleration, he described the packing and shear force behaviour of the material during vibration. The shear force is reduced to the no-load static value at the critical porosity which is the equilibrium porosity for that acceleration of vibration. If the material was not at the equilibrium porosity, the critical porosity must still be achieved during shear. This was accomplished by an expansion
of the bed to the equilibrium porosity for that vibrational acceleration. The force required to expand the bed in the shear zone could be estimated, hence the shear force could be deduced.

2.6 Mechanics of granular material

In order to reduce the uncertainty of a theoretical model of any system, the development of that model must be directed towards a complete understanding of the system. The continuum approach to the stresses in a granular material has to be extended with little reference to the actual processes involved and could never accurately predict the behaviour of one particle. The most advanced state of knowledge of particle behaviour is due to Mindlin, Deresiewicz and others, who considered the behaviour of contacts between like spheres. By similar reasoning to that of Brown and Hawksley, the treatment of spheres is far from the behaviour of materials encountered in practice. The contact of two irregular grains can hardly be likened to the contact of two spheres on account of dissimilar radii of curvature alone. What is the radius of curvature of point on an irregular particle?

However the theory of Mindlin et.al. is the most advanced available and has approached prediction of the behaviour of like spheres. Unfortunately a mathematical description of like spheres is somewhat different from an engineer's description, no set of spheres could be made without a finite tolerance between their radii. In
consequence deviation from theory has resulted. Reducing the tolerance of spheres used for experiment has produced results closer to theory. However it appears that some account of the variation in size and shape of the best set of spheres man can make must be made for theory to predict practice.

The contact theory of Mindlin* et al. developed the Hertz theory of the contact of two spheres to include a tangential stress. Cattaneo²³ was the first to include tangential stress but his work only became known to Mindlin after parallel development to the same stage. Mindlin further developed the theory to include the effect of both normal and tangential stress oscillations and tortional oscillation.

The Hertz theory predicted the radius of contact, the displacement of the sphere centres normal to the common tangent, the rate of change of displacement due to an increasing normal load, and the radial profile of local stress across the contact**.

* This work was the result of effort by a team of workers and there is no intention to credit it all to Mindlin.

** Throughout this description of contact theory the direction of the common tangent along the contact face is termed the tangential direction. The normal to it is termed the normal direction.
Non-linearity of these changes with changing normal load suggested difficulties when considering a dynamic case. It was also decided by Mindlin that the initial normal stress affected the role of the elastic constants. The Hertz theory predicted the velocity of a wave travelling normally across the contact as proportional to the one sixth root of the initial isotropic pressure and proportional to the one third root of the shear modulus. This was in poor agreement with fact.

A tangential force applied to a contact experiencing normal force produced infinite tangential traction* at the edge of the contact if it was assumed that there was no relative displacement. It was therefore assumed that such a displacement occurred and because of symmetry it took place on an annulus inward from the contact radius. The displacement of the two contact surfaces within that annulus was termed slip, which was distinct from the whole contact displacement which was termed slide. Slip occurred when Coulumb's law of friction at a point was exceeded for a constant coefficient of friction. Equations of annulus limits and tangential compliance† were obtained, and Johnson 66 confirmed these experimentally.

* Traction is the force in the direction described which tends to move the point at which it acts, relative to the sphere's centre.

† See note at the foot of the following page.
When the tangential force was released the displacement returned, but not to zero. If the tangential force cycle continued to oscillate, one complete cycle formed a hysteresis loop. For that case an energy loss, the area of the hysteresis loop, occurred which was described by an equation. The prediction of energy loss agreed well with Johnson's experiments at large amplitudes, varying with the cube of tangential force. At lower amplitudes the variation of energy loss was to the square of tangential force. A dynamic effect was suspected of being responsible for the error.

Johnson found in his energy dissipation measurements that a velocity dependent mechanism was involved which overshadowed the static effect at small amplitudes. At the centre of the amplitude range there appeared to be a geometrical factor missing since Johnson observed a dependence of energy dissipation on both sphere diameter and normal load.

When both normal and tangential forces varied (as was the case in a vibrated assemblage of spheres) the inelastic character of the relation between the tangential load and displacement introduced a complication. The force-displacement relation depended on the entire past history of

† Compliance is the movement of a point relative to the sphere's centre in the direction described. It is caused by the traction at the point.
normal and tangential loading. The factors which affected subsequent behaviour were:

(a) Tangential or normal force constant, each or both.
(b) Both forces increased or decreased, or both varied oppositely.
(c) Whether the rate of change of the tangential force to normal force exceeded the coefficient of friction.
(d) Whether the disturbance was in the same or opposite direction to the loading history.

Little development of the basic theory was made after 1954, however applications of the theory were put to test. Mindlin who had explained the theories described the conditions of stress in a regular array of spheres. The array was hexagonally packed and fully consolidated. The approach was to consider the spheres under a state of initial stress and calculate the deformation due to an arbitrary increase in stress. The deformation of the spheres was calculated by adding the deformations of each contact. Not enough was known of the system to allow complete analysis but the case of small amplitude vibrations was considered when the assembly of spheres was confined by a high load. In that case the change in contact compliance could be assumed constant. An experiment showed good agreement with the predicted natural frequency with high tolerance spheres. The agreement improved with both high and low tolerance balls as the pressure increased.

Deresiewicz considered a cubic array of elastic
spheres deforming up to the failure condition. That condition was where contact slide occurred, i.e. when the inner radius of the annulus of slip was zero. The relationship between normal and tangential forces and annulus dimension was:

\[(\frac{c}{a})^3 = 1 - \frac{T}{\mu N}\]

where \(c\) is the annulus inner radius, \(a\) is the contact radius, \(\mu\) is the coefficient of sliding friction and \(T\) and \(N\) are Tangential and Normal forces respectively. When \(c = 0\), \(T = \mu N\) which was the overall condition of slide according to Coulomb's law.

Deresiewicz\(^3\) produced his excellent review of 'mechanics of granular matter' in 1958. He discussed the theory of packing of particles and the latest advances in their contact theory which included only new applications to that described previously. The response of granular materials to a disturbance is extensively described, with little emphasis on the continuum approach. Propagation of stress by a collection of granules was reviewed, all models of which consider spheres. His suggestions for further work described the information lacking for the extension of contact theory for spheres and the development to apply to irregular particles. Specific points which were needed were:

(a) An exact solution to the Hertz theory.

(b) The direction of application of oblique forces could change. It had not at that time been considered (or
(c) Stress strain relations and criteria of failure of a randomly packed array of arbitrary sized and shaped particles, probably described by empirical distribution functions.

The restrictions of usage and the equations of the behaviour of a contact between two spheres due to Mindlin et al. is given in Appendix A.

Deresiewicz mentioned the discrete particle approach to understanding the behaviour of granular materials. The work mentioned included the discovery of dilatancy by Reynolds. Reynold's approach was somewhat philosophical, directed towards an understanding of the nature of ether which supposedly comprised space at the time. His argument has since had an important bearing on the understanding of the behaviour of granular material.

"The difference between granular material and solids or liquids was its property of dilatancy," stated Reynolds. Dilatancy was the definite change in the volume due to a change in overall shape or a distortional strain. In liquids and solids, dilation and distortion were essentially distinct but in granular media distortion always produced dilation. When a granular material was deformed each grain's position was controlled by those around it since it could not move through them without a major disturbance. Hence during distortion each grain would stay in the vicinity of its
neighbours.

When the structure of grains of a granular material cannot expand, the mass cannot be distorted. He described his famous balloon experiment. The balloon contained water and solid spheres. Water was drawn into the balloon when it was distorted suggesting an expansion of the spheres. A more inelastic bag which could not expand was sealed so that the water with the spheres could not flow in or out. Distortion of the bag was impossible.

Before discussing the ether, Reynolds described the well known example of the dilatancy of sand. The pressure of a footstep on the sand caused the particles to dilate as they deformed under the load. The expansion drew water from around the footstep and gave the sand a white appearance. When the pressure was released the assembly of grains contracted again leaving an excess of water on the surface.

Jenkin applied the principle of dilatancy spheres and sand in hoppers and bunkers. He calculated the pressures exerted on a horizontal base and vertical walls. He verified the calculation and concluded that the angle between the force acting on a wall and the normal to the wall was the angle of friction. The pressure was not transmitted uniformly and did not vary linearly with height above the base. The results of the experiments depended on the pattern of packing and on grain shape.

Andrade and Fox experimented on the mechanism of dilatancy. They described observations of overall effects
when dilatancy takes place. No effect of either the rates of loading from 600 to 16 seconds or varying the pore water pressure was detected. Graphs of load against displacement showed consecutive slip and shear which caused dilatancy. They produce a theory of propagation of a slowly applied load for a system of cylinders and it is developed for carbon shot and sand. The carbon shot photographed during distortion clearly showed a 90° inverted cone below the base of the piston applying the load. This cone was stable while the neighbouring shot dilated and moved. A similar 90° cone was observed under the piston compressing sand Fig. 15.

An interesting discussion of the study of elastic deformations is presented by Holubec. He concluded that such study was not of use to all branches of soil mechanics, but he studied elastic behaviour since it was necessary in the work of preconsolidation of large areas before civil engineering work took place. His definition of elastic behaviour is an extension of a strict meaning. A deformation is considered elastic if the width of the hysteresis loop is small compared with the magnitude of reversible strain. Much data on the stress-strain relations of soil was included.

2.7 Phenomena of propagation of stress

Geophysical interest in the propagation of disturbances in granular material was established at the turn of the century. The economics of oil exploration provided much
Spheres from the same original line

(a) Carbon shot.

(b) Sand.

Fig.15. A diagram of the effect of slowly loading the surface of a granular material. (a) Carbon shot, and (b) Sand, as observed by Andrade and Fox.
impetus to empirical research with the result that changes in strata in the earth's crust could be located by observing reflections from explosive disturbances. However present research gains no information from that early work.

The carbon microphone, used in telephone mouthpieces for thirty years before, was the subject of the research of Hara$^5$. He presented a model of the carbon granules in order to predict the propagation phenomena of sound waves in the microphone. His model comprised a cubic lattice of spheres connected in the direction of propagation by springs. The spring was to model the elasticity of the contacts between the granules, their behaviour being given by the Hertz theory. The contacts which were orientated normally to the direction of propagation were assumed to play no part. Hara's model predicted that the elastic wave velocity varied with one sixth power of the confining pressure.

Ishimoto and Iida$^6$ began their work on the elastic properties of soils by making an investigation using vibration. They varied the geometry of their sample and the moisture content and evaluated the elastic constant at the resonant frequency. Iida$^6$ used the resonant column technique to investigate soil samples from many regions of Japan. He experimentally established the dependence of the velocity of propagation on one sixth of the confining pressure. He also observed the decrease in velocity with increasing porosity. A slight increase in the velocity was observed with increasing grain size. Nasu$^7$ used the same
technique in the field.

The Japanese investigations were directed towards the understanding of the propagation of stress which caused earthquakes from a source deep within the earth's crust. Takahashi and Sato\textsuperscript{106} modelled the propagation similarly to Hara. In one section they produced a parallel calculation to Hara and then allowed the orientation of the contacts of the constituent grains of one shape and size to be in all directions. The Hertz theory was again used to calculate the contact strain. Then average values of strain and contact elasticities were found. These were applied to a unit volume of the model granules to allow the velocity of propagation to be calculated as if a continuum.

Gassman\textsuperscript{98} treated the earth's crust as if made of an array of similar spheres in hexagonal close packing. The contact strain was calculated by the Hertz theory and the grains compressed under their own weight. The wave velocity was again dependent on one sixth root of the confining pressure (in this case the pressure was due to the overlying granular material.)

Chambre\textsuperscript{29} derived the velocity of a compression wave in a suspension of porous medium which obeyed a simple mixture of densities law. Hysteresis and unloading phenomena, after the passage of the compression cycle, were ignored. The whole system was assumed to have no rigidity hence the resulting equation was independent of confining pressure. The analysis showed the velocity varied with the composition of the medium.
and shows a minimum between the values for either component. (Iida's dependence of velocity on the porosity.) He derived the same equation two ways; one way by considering a slice of the medium normal to the direction of propagation and of infinitely small depth. The structure of the medium remained unchanged during compression.

An extension of Chambré's work was developed by Brandt. He attempted to make some allowance for the shape and arrangement of the fundamental particles in rocks. He stated that the derivation of the speed of sound in the aggregate of non-spherical particles followed closely the derivation with spherical particles, provided certain average characteristics of the shape of particles are known.

Brandt's model aggregate comprised four sizes of spheres, each size stacked randomly in the interstices of the larger. The mass of spheres was deformed and contact response was again calculated by the Hertz theory. A bulk volume-confining pressure relationship was evolved which he used in a modified continuum equation for the velocity of propagation.

The work was extended to be valid for irregular particles. However, they were restricted to those which could be described by an average radius of curvature to validate the Hertz theory of contact of spheres. Brandt stated that the velocity was then dependent on the following particle characteristics: the number of contacts between the grains, the porosity, and the relation between the average radius of curvature and volume. These were all related factors so could
be grouped into one velocity measurement. The velocity predicted was inevitably proportional to the one sixth power of the confining pressure since the Hertz theory was the basis of deformation.

Brandt's theory predicted the velocity data of Nasu, and decrease in volume due to compaction by static pressures. The experimental values for velocity of Hughes et al. \(^5\) \(^6\) \(^7\) in saturated sandstone were predicted at low pressures but deviation occurred at higher values. The discrepancy was likely due to the limit of the Hertz theory. Fatt\(^3\) \(^7\) verified the prediction of a slight decrease in porosity with pressure and the predicted relationship between the bulk volume and pressure for spheres.

Matsukawa and Hunter\(^8\) \(^3\) experimented with a laboratory sample of sand. They pulsed a continuous oscillation applied to the bed and measured the variation of sound velocity with pressure. Gassman's predictions of velocity variation with confining pressure were verified, but a sensitivity of velocity to grain size was apparent.

Casagrande and Shannon\(^2\) \(^2\) applied transient loads to soils. The loading rates were restricted to those which exhibited no time delays in the small cylindrical samples they used. However with a clay they observed as much as 220% increase in the compressive strength due to a rapid loading in .01 seconds. Only a 10 - 15% increase in strength was recorded with sand at the same loading rate. The results for
sand included much scatter and were only considered approximate.

Casagrande's results suggested that a previously unknown mechanism was involved when granular materials were stressed rapidly. Whitman investigated these phenomena further at Massachusetts Institute of Technology from 1954. He measured a similar 10% increase in strength of sand when he conducted his highest rate of loading test in 0.005 seconds. However, tests were conducted with less control on the rate of strain down 0.001 seconds duration where Whitman found two factors interfered with his experiments. The first was that a measurable time was elapsing for the wave to travel from the impact end of the sample to the other. The second factor he termed the lateral inertia effect. Lateral strains had to occur before failure could take place. In very rapid tests inertia delayed the development of radial strains, thus it was possible to develop, during short periods of time, stresses far in excess of the true peak strength. Both effects increased with increasing sample size and were found for their samples of about 4 centimetres in diameter 11 centimetres long to begin to interfere with a 0.001 second loading.

Shannon et al. conducted rapidly loaded triaxial tests as he had with Casagrande. The purpose of the experiments was to find the dynamic modulus of elasticity from a modified 'triaxial' apparatus. From the dynamic moduli wave velocities were calculated which fitted well with
velocities measured by other authors, Fig. 16.

The phenomena of lateral inertia which appears probable when one considers the structure of granular material, was disputed by Parkin92. He predicted the impact wave form by means of continuum theory without consideration of lateral inertia. The curves predicted were as close as any other theory had been which caused Parkin to firmly dispute the existence of lateral inertia.

A granular model to aid prediction of wave propagation, particularly with geophysics in mind, was presented by Brutsaert. He extended Brandt's theory to apply to three phase systems, solid, liquid and gas. The elastic constants for an aggregate of his model particles were calculated by a modified technique of Brandt. Brutsaert concluded that Brandt's reasoning for extension of the spherical particle model to irregular materials was correct. In consequence he included the same constants. Dissipation was a result of the two interstitial fluids moving relative to the solid.

The propagation equations without dissipation resulting from the work were the same as Brandt's for a saturated granular medium. Equations of a form already predicted were produced when including dissipation. An important conclusion is that at low frequencies of vibration there is only one type of compressional wave. At high frequencies the dissipation effect is the same as for interstitial fluid with no viscosity.
Fig. 16. The velocity of propagation of longitudinal waves from a disturbance in a granular medium, based on data from Hardin and Richart.
The text book on foundation and soil dynamics by Barkan included discussion of propagation phenomena. He concluded that since the amplitude of an oscillation decreased with the distance from the source, its effect decreased. Applying this to the problem of vibratory compaction, he wrote that since a threshold of vibratory compaction existed (see section 2.2) a three dimensional surface at a certain distance from the source would be the limit of the compaction effect. The distance depended on the amplitude of the source and the transmission behaviour of the material through which the wave propagated. Beyond the limit the structure of the material would not change and an elastic wave would be propagated.

The fraction of energy absorbed by a granular material was found by Barkan to be independent of the rate of loading but the fraction absorbed was considerably affected by the grain size of the material. It increased as the grain size increased.

The aforementioned phenomena are examples of the limitations of considering the granular material as a continuum. Barkan was aware that continuum theory was an analogy but he described what it predicted since it was at that time the best available. When one considered a granular material as an isotropic-homogeneous elastic body two forms of body waves must be present. The longtitudinal, with displacements in the direction of propagation, and the transverse with displacements perpendicular to the direction of propagation. Only if the appearance of the waves were such
that the components of displacements at the initial moment corresponded only to change in volume the transverse waves would not appear.

Kolbuszewski suggested that porosity is not a sufficient parameter for relating the propagation properties of different sands. It was also necessary to account for particle shape, diameter, size distribution and packing. He indicated that it was possible to obtain the same relative porosity many times with the same sand yet obtain different response. Assuming that his procedures were identical, the difference of results was explained by differences in the arrangements of the grains which were possible for the same numerical value of porosity.

The experimental technique of Iida in 1939 using resonant columns of Japanese soil to find elastic wave propagation phenomena was adopted by Hall, Hardin and Richart. Their sequence of experiments was designed to isolate the effect of more parameters on fewer materials. The variables considered were moisture content, porosity, grain characteristics and vibration conditions. The porosity was the most significant variable, the wave velocity varied almost linearly with it. Preloading was found to have only a small effect on the velocity of the sand, cycling from 16 - 50 psi produced only 1 - 4% variation in velocity.

Water content appeared only to affect the velocity
indirectly. It changed the consolidation characteristics i.e. porosity, and the dynamic behaviour was changed. Grain size and shape which affected the packed porosity similarly, indirectly affected the propagation characteristics.

The amplitude of vibration affected the velocity such that an increase in amplitude decreased the velocity. The order of decrease in the range of amplitude investigated was 10 to 15%. The effect of the porosity was to increase velocity 10 to 15% between maximum to minimum porosity.

The variation in velocity due to amplitude change was higher for a lighter confining pressure at any porosity. This suggested that the velocity is more affected by amplitude when the contacts between two particles approached a condition of slide, i.e. not subjected to such high normal contact pressures. It is also interesting that a system of glass beads has a high velocity change due to water content whereas sand is little affected. This phenomena suggested that the coefficient of contact friction was reduced by the water, and glass beads exhibited the reduction more than sand because glass had a smoother surface.

Hall and Richart obtained spurious results when they vibrated a fine, compacted quartz. It is not clear what amplitudes were used in this experiment but it appears as though the amplitudes investigated were of the same order of the particle size. Specific test conditions were not reported but their range of frequencies for all experiments was 138 -165 Hertz, and maximum amplitude of about 0.5 x 10^{-3}
inches for all tests. A calculation shows that for the minimum frequency investigated the acceleration of vibration was about 1.5 times that of gravity. Such an acceleration would undoubtably cause particle rearrangement and may have caused packing. Kolbuszewski observed different propagation phenomena at the same porosities. Further investigation by Hall and Richart suggested that the velocity was 'time of loading' dependent but the increment of increase of velocity due to various times of load application was approximately uniform. This suggests that the system is in fact susceptible to the number of tests carried out on it, not the time of load application.

Hardin and Richart produced a graph of velocity of propagation data by many workers. A modified form of the graph is shown in Fig. 16.

An alternative method of investigating propagation phenomena to the resonant column is by local stress and time of arrival measurements. Selig and Vey experimented with horizontal sand columns laterally loaded. A highly developed gauge system was embedded in the column intervals. It was observed that the wave velocity varied linearly with bulk density. Also observed was the peak pressure attenuation decreased as the density increased. It also decreased with increased confining pressure.

Selig and Vey concluded that velocities measured by a time of arrival technique compared well with those predicted by resonant columns. This is likely to be so if there is no
particle rearrangement in the resonant column. There was least change in wave shape along the specimen with the highest density, highest confining pressure and smallest shock pressure. There was always a reflection from the opposite end of the specimen to the input. The peak stress attenuation increased as the input stress increased and as the peak stress duration decreased. There was no peak stress attenuation for a step pulse but the wave front deformed.

Baker\textsuperscript{7} carried out resonance experiments with samples similar to Selig's. None of his observations conflicted with Selig but Baker also noted that wave velocity decreased with increasing bar diameter to wavelength ratio. He concluded that the amplitude of vibration only slightly influenced velocity of propagation (in elastic propagation region.) Wave velocity varied linearly with porosity.

The spread of disturbance in a granular medium has not recently been investigated. Pokrovski (Goldstein, Misumski and Lapidus\textsuperscript{41}) and Smoltozyk\textsuperscript{103} assumed that the spread would form a normal distribution, but no experiments observing propagation in a semi-infinite system have been made.

2.8 \textbf{Statistical models}

In 1929 Smith, Poote and Busang observed that a Gaussian distribution described the number distribution of the number of contacts per particle in an assembly of spheres. The study of the geometry of a packing of spheres hardly
suggests a completely random system. However all the small effects dictating the co-ordination number are random by definition of a Gaussian distribution. It is reasonable to assume as an extension of this principle that the packing of an assembly of irregular particles of different size is controlled by many more small effects which add together to form similar Gaussian distributions.

Wise applied statistics to the understanding of packing phenomena. He considered an assembly of spheres with a distribution of radii described by a mathematical function, chosen for convenience to illustrate his argument. (Whilst all size analyses produce a number distribution of one form or another, they are rarely completely described by a mathematical equation. The distributions yield sufficient data to characterise the particles and moments, means and other statistical tools can be readily calculated from them using a digital computer.) He defined and calculated a probability and density function (see section 2.3). He defined and calculated a probability density function describing the dense packing of the spheres from the size distribution. The probability density function was used to calculate density functions of geometrical behaviour of the packing.

An important aspect of Wise's paper was the technique he used to suggest a distribution function which at that time was not known. He introduced a function of a form which did not contradict the laws of the system he was studying in
order that he may demonstrate his theory. For example, Wise discussed the probability of obtaining the tetrahedra he used to describe the packing (see section 2.3) throughout a real heap of spheres. He thought that a tetrahedra with one side between the centre of two balls not touching may be more feasible. In that case five variables (instead of the four constituent sphere radii) were necessary to define the tetrahedra, the four radii and the gap between the two spheres on the odd side. The radii distribution was assumed or measured but he would still have had to have guessed at the distribution of the gap width. However he felt that many of the interesting geometrical properties of a mass of spheres could still be estimated even if the guess was not too good.

Other authors who have used statistics in the study of packing phenomena are Hogendijk, who extended Wise's work, Kallstenius and Bergau and Higuti.

Herdan's book first published in 1953, described and correlated many aspects of particle behaviour which was dependent on the size analysis. Extension of the size analysis by number to area and volume analyses was also covered. Wise also worked on conversion of size analyses. Weibel and Tomkeieff described techniques of sampling particles and prediction of distribution from them (see section 2.9.)

An extension of the use of statistics from the
understanding of packing geometries is made by Litwiniszyn. He considered the probability of displacement of a particle forward and sideways when moving towards a hole in the base of the box. His theory produced Gaussian and Hyperbolic flow profiles. The profiles were close to the shape of coloured trace layers embedded in sand which was allowed to flow through a hole in the base of its container.

A further investigation was made by Smoltczyk. He considered the propagation of stress in a soil. In his reasoning for modifying the Boussinesq theory of elastic 'potentials' within the soil, he stated, "Elastic strain is a minor and sometimes insignificant part of soil deformation". Smoltczyk proposed that there are three forms of particle behaviour being acted on by a force:

(a) The particle held its position relative to the adjoining ones.
(b) The particle left its position but returned to it when the disturbance stopped.
(c) The particle left its position and did not return. Elastic deformation could not account for the third displacement.

Smoltczyk discussed the propagation of stress. The impulse transmitted to any particle decreased with growing distance from the source and with its lateral location denoted by the angle its direction from the source made with the vertical. The degree of scattering of the impulse
depended on physical qualities of the soil such as porosity, coarseness, size distribution. This qualitative reasoning was sufficient information for him to write the distribution of normal and shear stress on a horizontal plane due to vertical force as a normal distribution of the force on the plane. The force on the plane was the total integral of the stress at all points on the plane. For a horizontal stress acting at a distant point the distribution of stress on a horizontal plane was given by the simplest assymetrical distribution function. Sketches of the distributions of stress on a plane from a point normal load and parallel load are shown in Fig. 17. The prediction is a distribution of static load. He applied it to the stresses under footings.

Goldstein, Misumski and Lapidus stated that the probability approach to problems of soil mechanics was first suggested in U.S.S.R. by Pokrovski in 1933. His fundamental principles were:-

(1) The stress distribution in soil is controlled by the law of probability (usually normal.)

(2) The settlement is a process of transition of a foundation to a more stable state. Boltzman's theorem as to the proportionality of entropy change to logarithm of probability is applicable to a structure - foundation system.

(3) The strength of soil is determined by the maximum local stress coinciding with the weakest points of the material and statistically distributed over the whole volume of soil.
Fig. 17. The force profiles predicted by Smolczyk.
(a) From a normal force,
(b) From a shear force, applied at a distant point.
Takahashi and Sato used the statistical definition of the mean value of the energy loss per contact in a model of many similar particles to obtain the energy loss per unit volume of material. Their model was composed of many small particles of one kind with the points of contact arranged in all directions. The energy loss per unit volume is used to calculate the velocity of propagation of an elastic wave in the granular model.

2.9 Particle characterisation

For many processes occurring within a packed bed of granular material a conventional size analysis cannot be used with accuracy. For example the distribution of diameter given by a sedimentation technique of size analysis is only an indication of the particle size and would be inaccurate if used to calculate surface area of the particles in the bed.

This thesis is concerned with the way a force propagates through a bed of particles. Propagation is clearly via particle to particle contacts which lie in various positions around any one particle. It can be seen that it is important to have an accurate measurement of the distance from one contact to another across a particle.

Attempts at correlating particle shape have been published regularly. Wadell (1935) presented his results of estimating the sphericity of a quartz sand. His results were from a large number of particles and the technique much too tedious for practical adoption. Powers and Rittenhouse
proposed similar estimates of shape. Lees discussed the short-comings of sphericity and presented a non-empirical method for estimating quantitatively the degree of angularity of the particles. He attempted particularly to estimate shape of crushed material. The work of Powers, Rittenhouse and Lees is mentioned by Selig.

None of the methods were sufficiently advanced or practical and since a sizing technique which accounts for particle shape appeared to be more developed and closer to our needs it was adopted with modifications.

The particle sizing technique has been used for many years by metallurgists and pathologists to obtain size distribution data from a section and is now being used by particle technologists. Wiebel applied the technique to the study of the human lung. His sample was two dimensional slice of three dimensional tissue from which was measured the distribution of the lengths of solid chords on many straight lines drawn across the section. The principles used to convert the data obtained from the one dimensional sampling technique to apply to three dimensions were by Delesse and Rosiwal. Delesse established a principle that the area of solid on a sample section was the same ratio to the total area as the solid volume to the total volume. Rosiwal extended the principle to the fraction of a line in a random direction cutting a particle to that part cutting the void, being the same as the solids volume of the original. (A mathematical proof of the principles of Delesse and Rosiwal
presented by Weibel appears as Appendix B.)

Tomkeieff and Campbell developed a suggestion of Moran that the surface area of the solids and their volume could be easily calculated from the chord measuring technique described above. They dispelled the premise that the solids needed to be the same shape as one other and that they needed to be entirely enclosed in a unit volume. The only rider attached to the analysis that the sections should be randomly disposed to the sampling line.

The measurement of the distance between the intercepts of the sampling line with a particle (which will be called the 'Random Chord size analysis') can provide much more data than is used by the metallurgists or pathologists. The length of solid intercepts can be plotted to produce a random chord distribution which is related to any other size analysis, but in a complicated manner. However a random chord has itself a physical significance and can be used to predict the distances between contacts on a particle, since it is a distribution of the distance in random orientation across the particles.

2.10 Conclusions from the literature

A disturbance applied to a bed of granular material does not propagate solely in the direction of application. A spread of effect about the line of application is found. The spread has been described as a normal distribution with the mean value on the line of application. The total area...
under the distribution is the force at that level. No experimental verification of this has come to light even though it has important bearing on what effect of a disturbance is found in which position in a bed of material.

In view of the divergence of the effect of a disturbance the attenuation along the length of a cylindrical sample of granular material measured by many workers may not bear a simple relationship to the attenuation measured within an infinitely wide bed. The effect of the boundary wall is not yet understood. It must affect lateral strain during axial loading and in consequence could cause a similar effect of increasing compressive strength, and the force propagating, as is attributed to lateral inertia. A close boundary wall would not affect the velocity of propagation of a disturbance measured by timing the beginning of a shock wave at various distances along the sample. Selig found that the impulse measuring technique produces the same velocity values as the resonant technique. Hence the velocity of propagation is the same in a cylindrical sample as in an infinitely wide bed of material.

The divergence of a disturbance must be understood to enable the prediction of local conditions in a disturbed bed of granular material to be made. One application which will benefit is the vibration of a hopper to produce discharge conditions. It is commonly known that either a flow condition or packing can occur but the physical limits of either effect is not known. Youd presented enough knowledge to allow the
porosity and its relation to critical porosity to be predicted, given local vibration conditions. Mogami and Kubo and Youd presented values of the shear strength of sands at specific vibration conditions, and the latter author suggested how to extend the values to all energies of vibration.

The 'threshold of vibratory compaction' explained by Youd is seen by Barkan to be a limit in an infinite system of granular material. Within the limit, vibration conditions will be of sufficient energy to cause compaction and outside no compaction would occur. In effect Barkan said that the amplitude of vibration decays as one gets further from the source. This decay has been measured by Selig and Richart along the length of a cylindrical sample but an increase was noticed as reflection occurred from the boundary forming the end opposite to the source. The decay has also been reported by authors compacting granules by vibration. A region of higher porosity was noticed near the surface of the material.

The constraining force acting there due to the weight of the bed is lower which tends to allow the vibration to exhibit a higher acceleration there. However the porosity-acceleration of vibration curves presented by Youd and Barkan suggest that a higher acceleration produces a lower porosity. For there to be a region of high porosity near the surface, the acceleration of vibration must be lower there. For this to be so, the acceleration must decay as the
distance from the source increases. The addition of the constraining surface load reduced the porosity of the upper region by being a source of reflection.

The mechanism of propagation suggested by Bázant and Dvořák best describes the processes involved. The effect of amplitude and frequency can also be included in their explanation and the mechanism of packing and bulk flow can be superimposed. The disturbance is propagating from the bottom of the vessel. It passes via particle to particle contact from underlying to overlying grains. The lower layers rise first whilst the upper layers do not move (finite time of disturbance progression) and compression takes place. Successively all the layers rise. The base layers next begin to fall before the upper layers, causing dilation and reduction of contact pressures.

For low acceleration of vibration (less than that of gravity) compression and dilation is negligible and rearrangement of particles can only occur when the contact failure condition described by Deresiewicz occurs, i.e. the tangential force at the contact exceeds the product of the coefficient of sliding fraction and the normal force. (Both forces oscillate due to the vibration.)

For acceleration of vibration little above that of gravity, dilation occurs, allowing the particles to separate and reorientate discretely. During dilation the resistance to external shear force is lowered and flow occurs more readily.
For vibrational acceleration very much higher than that of gravity dilation, impact and compression take place. In this condition bulk circulation is observed. The divergence of the disturbance as it passes through the material can explain bulk circulation. The profile of stress at any level has a maximum on the axis of the disturbance. If this is the case during vibration, maximum stress produces a maximum amplitude on the axis of vibration. Since the recovery of any layer of particles projected by vibration (as they are when the acceleration is greatly in excess of gravity) is due only to gravity, those particles exhibiting the largest amplitude would require a longer time to recover to their original positions. The time of recovery is essentially constant at each horizontal level in the bed as can be seen from figure 8 in the Preliminary Experimental work. The particles experiencing the largest amplitude do not recover completely and in consequence, experience a resultant, upward motion relative to the neighbouring particles. The resulting flow away from the source of vibration is shown by Takahashi and the consequential downward flow is in the region of least disturbance.

The equilibrium porosity of a granular material during vibration at a fixed acceleration is the same as the critical porosity for shear at those conditions. If the granular material is circulating in the vessel it must be at its critical porosity for that local high energy vibration
condition. Since the critical porosity is the lowest porosity at those conditions and the equilibrium porosity always becomes lower as acceleration of vibration increases⁸, no higher porosity results when the bed exhibits bulk circulation. However segregation of sizes of particles⁷ and air flow¹² may be caused which would increase the porosity.

Propagation of a disturbance is via particle to particle contact and the theory which can best predict contact behaviour at this time is due to Mindlin and Deresiewicz⁸⁸. Their theory has been applied to arrays of spheres with success. Brandt¹⁷ applied the Hertz theory of contact behaviour to a sphere model of a granular material to produce the most widely quoted propagation theory. Mindlin's contact theory is an extension of the work of Hertz and could be applied to a randomly packed bed of particles (within the limitations of the assumption of the contact being between two spherical surfaces of the same radii.) The technique of Brandt is to generate a term \( v \frac{dp}{dv} \) (where \( v \) is volume and \( p \) is pressure) considering the contacts in an assembly of granules which is used as a compression modulus in an equation for the velocity of propagation of the disturbance. The technique could be followed incorporating Mindlin's contact theory.

Since the approach of this work is towards application of vibration to granular materials in process industries,
propagation of vibration to produce discrete or bulk flow is the main interest. It can be seen from the literature that a common propagation investigation technique is by means of the resonant column. In such experiments the structure of the granular material must not change. A system at resonance must be strong enough and stable enough to behave in the same way, at the same time during each cycle of vibration. When the structure is not stable, unrepeatable results occur as reported by Hall and Richart in their experiments with Novaculite. Since the behaviour of geometrically unstable material is of interest, the measurement of propagation by applying and observing pulses was considered more versatile.

It is noticeable from the application of statistical analysis to the problem of mechanics of assemblies of irregular granules, that the normal distribution describes some properties. The problems of packing irregular materials has been seen to be extremely complex with many factors involved in a small way. Smith, Poote and Busang found close to normal distributions for certain conditions in the packing of spheres. In view of the normal distributions being observed in the packing of regular particles it is reasonable to assume that the same distribution will apply to many irregular material properties, especially as a first approximation.
3.1 Introduction to the theory

The divergence of the effect of an applied force in a granular material has been prophesied by Pokrovski$^{1}$ and predicted by Smoltczyk$^{103}$. The latter made no account of the properties of the particles, the geometry and orientation of which is totally responsible for the divergence.

A size analysis technique which takes account of the geometry of irregular particles has been described. Data from the size analysis will be used to predict the profile of force due to divergence.

In order to predict the time at which a disturbance affects a point distant from the source, the velocity of propagation of the disturbance must be known. The established technique by Brandt$^{17}$ to calculate a compression modulus for an assembly of particles from particle to particle, contact behaviour is extended to include the effect of the tangential force at a contact. The modified contact theory used here is by Mindlin and Deresiewicz$^{88}$.

3.2 The distance between two contacts on an irregular particle.

A random chord is defined as that part of a line drawn in a random direction that lies within a particle, or, that
part of a line in a fixed direction that lies within a randomly orientated particle. A Random Chord size distribution is the distribution of chords defined above. Each particle in a granular mass has an infinite number of random chords varying between the maximum dimension of the particle and zero. The distribution of the the random chords for one particle will be a function of particle shape. If for all the particles in a bed of granular material, individual particle random chord distributions were evaluated, a technique for completely characterising the shape of the particles would be achieved. Such an analysis would be as tedious as that to evaluate shape by Wadell\textsuperscript{112} and of little practical value.

The Random Chord size distribution formed by adding together the individual particle Random Chord distributions will be a collective measure of the shape of the particles. The overall distribution can be estimated by the technique of scanning a two dimensional section in a random plane of the particles by lines in random directions outlined in section 2.9. The number of random chords measured to form the exact distribution should be infinite. However, sampling theory (Lloyd\textsuperscript{114}) allows an estimate of known accuracy from a finite number of random chords. Data shown by Lloyd suggests that as many particles as is practically possible should be sampled.

A Contact Chord is defined as the length of line between two contacts on the same particle. Since there are
.5 k (k-1) contact chords in a particle, where k is the number of contacts on that particle, and their lengths can vary from the longest dimension of the particle to zero, they form a distribution of number against length. The limits of the distribution for one particle are the same as for the individual particle random chord size distribution. Similarly the overall limits of length of contact chord are the same as the limits of the overall random chord size distribution. The mean value of Random Chord for all the particles will now be shown to be related to the mean value of the contact chord.

Fig. 18  The routes through a bed of granular material via random chords and contact chords.
Consider a particle within which there is a contact chord separated by angle theta from a random chord through one of the contacts Fig. 18. Consider also the second, third, \ldots Nth particles on the line of the random chord, until a contact again coincides with the line. N particles have been traversed by the line. The shortest route via the particles and contacts (i.e. crossing no voids) is through M contact chords each inclined at an angle $\theta_i$ to the line which cuts the random chords. The sum of the projections of each contact chord on the random chord line is the length of the random chord line. If the length of an individual random chord is $r_i$ and of an individual contact chord $c_i$ then:

$$\sum_{i=0}^{i=M} c_i \cos \theta_i = \frac{1}{1-\varepsilon} \sum_{i=0}^{i=N} r_i$$

since the ratio of random chords to voids is porosity (Rosiwal)\(^9\)

It is assumed that $N=M$, i.e. the same number of particles are intersected through the contact chords as on the straight line, then:

$$c \cos \Theta = \frac{\overline{r}}{1-\varepsilon}$$

The distributions of $c$ and $\Theta$ are independent, therefore,

$$c \cos \Theta = \overline{c} \cdot \cos \Theta$$

The mean value of $\cos \Theta$ is calculated from the
distribution of $\theta$. In three dimensions the number distribution of $\theta$ is given by:

\[ p(\theta) = \sin \theta \]

(A proof of this is given later.)

The relationship of the number distribution of $\cos \theta$ to $\theta$ is:

\[ p(\cos \theta) \cdot d \cos \theta = p(\theta) \cdot d\theta = \sin \theta \cdot d\theta \]

The mean value of $\cos \theta$ is given by:

\[
\begin{align*}
\cos \theta &= \frac{\int_0^{\pi/2} p(\cos \theta) \cdot \cos \theta \cdot d\cos \theta}{\int_0^{\pi/2} p(\cos \theta) \cdot d\cos \theta} \\
&= \frac{\int_0^{\pi/2} \sin \theta \cdot \cos \theta \cdot d\theta}{\int_0^{\pi/2} \sin \theta \cdot d\theta} \\
&= \frac{\int_0^{\pi/2} \sin \theta \cdot d\theta}{\int_0^{\pi/2} \sin \theta \cdot d\theta} \\
&= \frac{1}{2}
\end{align*}
\]

Substituting $\cos \theta$ in $c \cos \theta = \frac{\gamma}{(1-\varepsilon)}$

\[ \bar{c} = \frac{2 \gamma}{(1-\varepsilon)} \quad \text{(1)} \]
The mean lengths of the random chords is calculable from the random chord size distribution and hence the mean length of the line within a particle between two inter-particle contacts is found (the contact chord.)

3.3 Theory of propagation of an applied force through a cohesionless material.

Consider several irregular shaped solid particles grouped three dimensionally around a particle A, shown diagramatically in Fig. 19. Consider force (F) is applied to A in any direction as shown. It is opposed by reaction on the opposite side of the particle A at contacts b, c, d, and e. The increased contact pressures there, are in turn, opposed by reaction on the opposite sides of particles B, C, D, E at contacts f to p. Action and reaction continue in this way through a network of contacts so that two particles distant from A experience an increase in common contact pressure. The increase depends on the magnitude of the original force F and the distance from particle A.

It can be seen from Fig. 19 and imagined in three dimensions that a force F on a particle A causes increased contact pressure on particles B, C, D, and E in contact with A and increase also on particles F, G, I, J, L, M, O, and P. Thus the effect of an applied force on one particle is met by reaction at an increasing number of contacts as they are
further from the original force. The reaction is spread as the effect of force propagates through the bed.

Fig. 19 The reaction of neighbouring particles to a force applied to one particle.

Consider now a static bed of granular material of finite height but of infinite width. If a vertical force is applied upward to one particle in the bottom of this bed (a point source) a profile of force is formed in a horizontal plane a distance $H$ above the source. For a bed which has uniform porosity and is not segregated in particle size, the force profile is symmetrical about the vertical axis through
the source, and at an increasing horizontal displacement (D) from the centre of the system the force tends to zero. If the profiles at each level in the bed were known, the shear and compressive force between two distant points could be found. If these forces exceeded the strength for the bulk material, failure would occur.

In this theory an attempt will be made to predict the force profile at any height above a vertical applied force from a point source, and later from an input of finite area. The packing of the bed will be considered static whilst propagating the force and it will remain stable until shear or compressive forces exceed the failure condition.

A point of contact in a cohesionless granular material exists only if the normal force at the contact is greater than, or equal to zero. A negative force would tend to separate the two particles and since there is no cohesion there, the contact would fail to exist; i.e., a contact in a cohesionless granular material cannot resist a tensile force. Thus when the effect of a force applied to a cohesionless granular material passes through the points of contact only compressive forces are involved.

Consider the particle A in Fig. 19. The force F compresses those contacts which lie above its point of action and tends to tense those below its point of action. A compression of a contact below this point is possible if the force F exerts a turning moment on the particle. However such
a compression will be neglected since it is opposed by an
equal compression above the point of action of F. Hence only
those contacts above the point of action of the force on a
particle will be considered.

If the effect of an applied force reaches the surface
of the bed of the granular material, it must have described
many routes through particles and contacts in its travel from
the source. This part of the theory will describe the
probable routes to a point on the surface and thereby
produce a distribution of arrival points. The routes through
the bed will be described by many lines within the particles
between two contacts on each particle, the contact chord, and
the inclination of this line to the vertical, angle theta
(θ) Fig. 20. The mean length of the contact chords has been
described for the particles and a number of distribution of
theta can be deduced from the state of isotropy of the bed.
If a statistically large number of particles are affected by
the force, their contact chords and inclinations, theta will
form their respective number distributions.

Along each of these routes a packet of force will
travel. The concept of a quantity of force can be explained
by considering a pile of bricks in the form of an inverted
cone, Fig. 21.

If a force of magnitude 1 is applied to the lowest
brick its effect will divide equally between the two above it.
Similarly the force acting on underlying bricks will combine
and divide throughout the pile in passing to overlying bricks.
Fig. 20. Contact chords ($c$), the angle theta ($\Theta$), and the angle lambda ($\lambda$).
To avoid the complication of dividing the applied force, it will be considered as many small packets of force which add together to give the force acting on any brick. If the packets are considered separately their density at any point in the pile is the force acting. This idea is easily extended to a bed of particles.

![Diagram](image)

Fig. 21 A diagram of the effect of force.

The position at which each packet arrives at the surface is determined by the route through the bed via particles and contacts. The number distribution of arrival points on the surface will also be the number distribution of packet of force arrivals. This in turn is the distribution of force on the surface.

At each point of contact in the bed, the reaction of the upper particle can equal the propagating force until the particle slides. In this theory the contacts in the bed are
assumed stable until the bulk strength of the material is exceeded. The profile of force predicted by assuming the interparticle contacts are stable is compared with the bulk strength measured for the material. If the profile predicted does not exceed the bulk strength, a higher force is applied and the calculations reported until the material fails. The condition of failure is of importance to the process engineer.

The non recoverable strain at a contact due to the effect of an oscillating tangential force is considered not to affect the structure of the bed. Thus, until failure occurs due to the bulk strength being exceeded, each contact reacts equally and oppositely against the propagating force and all force applied to the bed is detectable at the surface.

The distribution of force from a point source.

Consider a three dimensional bed of granular material containing many particles randomly orientated such that a line through the bed in any direction cuts the same proportion of voids to solids. Consider a contact on a particle is contact chord \( c \) from a second contact on the same particle, and this contact chord is inclined at an angle \( \theta \) from the vertical, Fig. 22. The horizontal displacement \( d \) of the two contacts is \( c \sin \theta \) and the vertical displacement \( h \) is \( c \cos \theta \). Since we are considering that propagation occurs by means of contact compression and only via those contacts above that one which first receives the force, the angle \( \theta \) varies between
- \pi/2 and +\pi/2. Hence \(d\) varies between \(-c\) and \(+c\), and \(h\) between 0 and \(c\).

To account for the direction of the horizontal displacement \(d\) a further angle must be defined. This will be called lambda (\(\lambda\)) and is the deflection in a horizontal plane of displacement \(d\) from a vertical, fixed reference plane, Fig. 21.

The total horizontal displacement \(D\) after \(N\) displacements (\(d\))

\[
D = \sum_{i=1}^{N} d_i \cos \lambda_i = \sum_{i=1}^{N} c_i \sin \Theta_i \cos \lambda_i
\]

Subscript \(i\) refers to one set of conditions.

The number of displacements (\(N\)) is given by the shortest distance of travel of the force divided by the projection of the mean contact chord onto that line. The number of displacements to a point, a horizontal displacement \(D\), and at a height \(H\) from the source is given by:

\[
N = \frac{\sqrt{H^2 + D^2}}{c \cos \Theta}
\]

The number distributions of the length of contact chord and theta are independent of each other, hence:

\[
\overline{c \cos \Theta} = \overline{c} \cos \overline{\Theta}
\]

For the force to have arrived at a horizontal displacement \(D\) at a height \(H\) above the source, the mean angle
theta for the route is given by:-

\[ \theta_{HD} = \arctan \frac{D}{H} \]

Since an initial assumption was that a line through the bed in any direction cuts the same fraction of voids to solids as the porosity, the line also cuts random chords. This is only so if we assume the particles are randomly orientated. Then the line at angle \( \arctan \frac{D}{H} \) to the vertical cuts random chords in the particles and has the same properties as the line cutting random chords described in section 3.2. The angle of inclination of each contact chord to the line, can be considered the same as theta and has the same distribution. In consequence the mean value of \( \cos \theta \) is:

\[ \cos \theta = \frac{1}{2} \]

Substituting \( \cos \theta \) in equation 3, the number of steps to reach a point on the surface is:

\[ N = \frac{2}{\cos \theta} \left( \frac{H^2 + D^2}{\sigma^2} \right) \]

(4)

If a large number of packets of force pass through the bed of material, taking a large number of steps \( N \) the distribution of force on the surface will be Normal (Gaussian) (Smoltczyk\textsuperscript{103} and Goldstein, et.al.\textsuperscript{141}). To describe the normal distribution its standard deviation (\( \sigma_D \)) must be
Fig. 22. The vertical and horizontal displacement of one step along a contact chord of length \( c \).
described which can be found from the standard deviation of one displacement ($\sigma_D$). The normal distribution of force is:

$$g(D) = \frac{1}{\sqrt{2\pi \sigma_D^2}} \exp\left(-\frac{D^2}{2\sigma_D^2}\right)$$  \hspace{1cm} (5)

The variance of displacement for one step is given by:

$$\sigma_d^2 = \int_{d_{\text{min}}}^{d_{\text{max}}} (\bar{d} - d)^2 \cdot p(d) \cdot \text{dd}. \hspace{1cm} (6)$$

where $\bar{d}$ is the mean value of horizontal displacement and $p(d)$ is the probability of displacement $d$.

For one displacement, $\bar{d} = 0$. Therefore the variance is given by:

$$\sigma_d^2 = \int_{d_{\text{min}}}^{d_{\text{max}}} d^2 \cdot p(d) \cdot \text{dd}. \hspace{1cm} (7)$$

The probability of displacement $d$ is the probability of $c \sin \theta$. Since $c$ and $\theta$ are independent, consider one value of $c (c_j)$. Then:

$$p(d) \cdot \text{dd} = p(\theta) \cdot d\theta$$

The distribution of theta $p(\theta)$ is found by considering the three dimensional space above the contact (Q) which first receives the force, Fig. 23. Consider a contact chord from
that contact of a fixed length ($c_j$). If it were allowed to pivot freely about $Q$ in all directions above $Q$ the opposite end of the chord to $Q,(S)$ would describe the surface of a hemisphere of radius $c_j$ with the centre at $Q$. Let the chance of $S$ being at any position on the surface of the hemisphere be random. Then the probability of theta is given by the area subtended by an infinitely small angle $d\theta$ as it revolves about the vertical.

![Diagram](image)

Fig. 23 The hemisphere described by $S$ and the area subtended by revolving $d\theta$ about the vertical.

The probability of $\theta$ is:-
\[ \frac{2\pi c_j \sin \theta \cdot c_j d\theta}{2 \pi c_j^2} \]

\[ p(\theta) d\theta = \sin \theta d\theta. \quad \text{(8)} \]

Substituting equation (8) in (7) and \( d = c \sin \theta \) in (7),

\[ \sigma_d^2 = c^2 \int_{-\pi/2}^{\pi/2} \sin^2 \theta \cdot \sin \theta \cdot d\theta \]

\[ = 2c^2 \left[ \frac{\cos^3 \theta}{3} - \cos \theta \right]_0^{\pi/2} \]

\[ \sigma_d^2 = \frac{4c^2}{3} \quad \text{(9)} \]

Consider now all possible values of \( c \). The equation for variance becomes:

\[ \sigma_d^2 = \frac{4}{3} \int_{c_{\text{min}}}^{c_{\text{max}}} c^2 \cdot p(c) \cdot dc \]

A technique has not yet been established to find the number distribution of the contact chords. Only the mean contact chord has been calculated. It is desirable to derive
the contact chord distribution from a random chord analysis because of the convenience of the analysis, however it is not yet possible. In consequence an assumption will be made that the second moment of the contact chord distribution is related to the second moment of the random chord distribution. The assumption is primarily to demonstrate the theory.

The first moment of the contact chord distribution is related to the random chord by equation (1). It is assumed that the ratio of the second moments is the square of that given in equation (1). The equation for the variance of one step now becomes:-

$$\sigma_d^2 = \frac{4 \cdot 4}{3(1-\varepsilon)^2} \int_{r_{\min}}^{r_{\max}} r^2 \cdot p(r) \cdot dr.$$  

The integral term is the second moment of the random chord distribution and has a constant value for a granular material. It will be subsequently written as $M_{2,0}$ in accordance with the nomenclature established at the Institute Verfarhenstechnic of Kalsruhe University. Then the variance on one step is given by:-

$$\sigma_d^2 = \frac{16}{3(1-\varepsilon)^2} \cdot M_{2,0}.  \tag{10}$$
The displacement for one step was considered in three dimensions. In consequence the variance of the horizontal displacement cannot simply be added together to form the variance of the normal distribution on the surface. Allowance has to be made for the angle that the vertical plane, through the horizontal displacement, makes with a fixed reference vertical plane. This angle is lambda (λ) (Fig. 20) and its number distribution will be considered linear since the bed is considered isotropic. Then the variance on each step $\sigma_d^2$ must be projected onto the vertical reference plane which for convenience includes both the point source and the point of arrival at the surface.

The projected variance is given by:-

$$\sigma_p^2 = \sigma_d^2 \cdot \cos \lambda$$  \hspace{1cm} (11)

The mean value of $\cos \lambda$ is given by:-

$$\int_0^{\pi/2} p(\cos \lambda) \cdot \cos \lambda \cdot d \cos \lambda$$

$$\int_0^{\pi/2} p(\cos \lambda) \cdot d\lambda$$

Now $p(\cos \lambda) \cdot d \cos \lambda = p(\lambda) \cdot d\lambda$

$p(\lambda)$ is assumed linear $= \frac{2}{\pi}$

Then $\cos \lambda$ is given by:-
\[
\int_{0}^{\pi/2} \cos \lambda \, d\lambda = \frac{2}{\pi}
\]

Substituting equation 12 in 11

\[
\sigma_p^2 = \frac{2}{\pi} \cdot \sigma_d^2.
\]

Then the variance of the normal distribution of force on a plane, height \(H'\) above the source is:

\[
\sigma_D^2 = N \cdot \frac{2N}{\pi} \cdot \sigma_d^2
\]

N is given by equation 14 and \(\sigma_d^2\) by equation 10. Substituting these in equation 13:

\[
\sigma_D^2 = \frac{64}{3\pi} \frac{\sqrt{H'^2 + D^2}}{c} \frac{M_2,0}{(1-\varepsilon)^2}
\]

The variance for N steps is now substituted in the normal distribution equation (5) to give the distribution of force at the surface.

The standard deviation for N steps is :-
\[
\sigma_D = \frac{8^{\frac{1}{4}} \sqrt{H^2 + D^2}}{(1-\epsilon) \sqrt[4]{3\pi c}} \sqrt{M_{2,o}}
\]  

(14)

Then the surface distribution from a point source is:

\[
g(D) = \frac{(1-\epsilon) \sqrt{3\pi c}}{8 \sqrt{2\pi M_{2,o}}} \frac{\sqrt{H^2 + D^2}}{\sqrt{H^2 + D^2}} \exp \left[ -\frac{3\pi \, c \, D^2 \, (1-\epsilon)^2}{128 \cdot M_{2,o} \cdot \sqrt{H^2 + D^2}} \right]
\]

(15)
The distribution of force from a source of finite area.

The distribution of force on a horizontal plane a height \( H \) above a point source is given by equation 15. It is a function of the porosity of the bed of granular material \( (e) \), the size and shape of the grains \( (c) \), the height of the surface being considered \( (H) \), and the radial position \( (D) \).

In order to calculate the force received at a height \( H \) in the bed from an input source of finite area, it is necessary to integrate the contribution of all points of the input area.

Consider an input disc with radius \( R' \), and the force from it which reaches a point \( T \) on the surface. The vertical displacement of \( T \) above the surface is \( H \) and the horizontal displacement from the centre of the input is \( D \), Fig. 24. The projection of \( T \) vertically downward onto the plane of the input disc is the point \( P \). The contribution of all points on the input disc a horizontal displacement \( x \) from \( P \) will be integrated. Each point on the input disc contributes a force \( g(x) \cdot dA \), where \( g(x) \) is the distribution given by equation 15 and \( dA \) is the area of the point being considered.

The assumption here is that the effect of a point on an input of finite area reaching a position on the surface, is governed by the same equation as the effect received by a point on the surface from a point source input, i.e.
Fig. 24. The geometry of the finite input disc system.

Fig. 25. Case 1, when $P$ is within the input disc.

Fig. 26. Case 2, when $P$ is outside the input disc.
Each point on the input disc is assumed to apply the same force to the bed of material. Therefore the element of area of the input disc, a horizontal displacement \( x \) from \( P \), must be calculated.

The element of area is given by the product of the length of arc of the circle radius \( x \) which coincides with the input disc and the width of the element \( dx \). The length of arc coinciding is given by the product of \( x \) and the angle \( (2\phi) \) subtended by the arc. The angle \( \phi \) is given by the cosine rule on the triangle formed by the horizontal displacement \( D \) of the point \( P \) from the centre of the input disc \( R \), and the radius \( (x) \) of the arc being considered, Fig. 25.

If the point \( P \) is within the input disc some of the arcs of radius \( x \) will be full circles, i.e. \( \phi = 2\pi \). The condition of this is \( D < R \) or = \( R \). If \( D > R \) the arcs of radius \( x \) intercepting the input disc will be incomplete, Fig. 26.

The general form of the contribution of each element of area is:

\[
g(x) \cdot x \cdot 2\phi \cdot dx
\]

(16)

The distribution of force on the surface \( f(p) \) is given by integrating equation 16 between the limits of the nearest and furthest point of the input disc to the point \( T \). There are two cases:

Case 1: When the projection \( (P) \) of point \( T \) lies within
the input disc, i.e. \( D < R^i \) when the distribution is given by:

\[
\begin{align*}
    f(D) &= \int_0^{R-D} g(x) \cdot 2\phi \cdot x \cdot dx \\
         &+ \int_{R^i-D}^{R^i+D} g(x) \cdot 2 \cdot \text{arccos} \left[ \frac{x^2 + D^2 - R^2}{2 \cdot x \cdot D} \right] \cdot x \cdot dx.
\end{align*}
\]

(17)

**Case 2:** When the projection (P) of point T lies outside the input disc, i.e. \( D > R^i \), the distribution is given by:

\[
\begin{align*}
    f(D) &= \int_{D-R^i}^{D+R^i} g(x) \cdot 2 \cdot \text{arccos} \left[ \frac{x^2 + D^2 - R^2}{2 \cdot x \cdot D} \right] \cdot x \cdot dx
\end{align*}
\]

(18)
3.4 Velocity of propagation

The foregoing theory calculated the profile of force at the surface at a steady state condition. The force arriving at any point on the surface is the sum of force arriving via many contact routes of various lengths through the bed. Because of the finite time of progression the effect of applied force will take longer to reach the surface the longer the route.

Consider an element of a bed of granular material containing M contacts such that contact properties in the element form the same distribution as contact properties for the bed, (Lloyd 124). Consider that in the element the vertical force due to the weight of the bed above is constant and that the whole element when disturbed by a vertical upward force is affected uniformly. Within the element the contact pressures will increase depending on the force of the static load, the magnitude of the disturbing force, and the angles these make with the normal to their common tangent. Tangential and normal compliance (rate of change of displacement of points in the particles due to applied force) will occur. The effect of the compliance will be a reduction in element volume. Calculation of the reduction will yield a function \( \frac{vdp}{dv} \), where \( v \) is the element volume and \( p \) is disturbing force per unit area. The ratio of element height reduction to the reduction in width will yield Poisson's ratio (\( \nu \)). The equation used by Brandt 17 for the velocity of
a longitudinal wave in an isotropic medium of infinite extent:–

\[ S = \left[ \frac{3 \rho}{d} \left( \frac{v \, dp}{dv} \right) \right]^{\frac{1}{2}} (18) \]

will be used with the functions calculated for our element to calculate the velocity of propagation.

Properties of the contact between two particles

The mathematical theory of deformation of material composed of elastic grains due to Cattaneo, Mindlin, Deresiewicz\textsuperscript{23, 36-38, 46-48} and other workers is the best available at this time. The limitations they impose in dealing with spheres must be imposed on the system here considered\textsuperscript{†}. Whilst it is desirable to treat irregular particles, the geometry of each contact must be considered as that between two spheres; i.e. the contact surfaces are circular and flat, share a common tangent and have a profile of normal stress given by the Hertz theory. The consideration of particles with these properties will limit this theory but contact orientation, relative position, stress due to load

\textsuperscript{†} The system treated by Mindlin and Deresiewicz is presented in Appendix A.
and those stresses due to the applied force are given by consideration of a random assembly of irregular particles.

The deformation of an element of granular material.

Consider a contact between two particles within a bed of granular material. The contact can be fully described for our purpose by two parameters. (Their range of values for all the particles in the bed will form the number distribution discussed later.) The orientation is described by angle $\beta$ which is that between the vertical and the normal to the common tangent through the contact. The vertical force $W$, due to the weight of the bed and any load on the surface, is used to describe the static history of the contact by resolution of $W$ in the direction of the normal and tangent, Fig. 27. In the case where $\beta$ exceeds the angle of sliding friction at the contact, tangential stress due to other contacts on the particle must be present to maintain stability. The minimum value can be readily calculated.

It is unlikely that tangential stress is zero at any contact in a static bed. It is also unlikely that the maximum static tangential stress is in the same direction as the tangential stress due to an applied force (considering the third dimension.) Since the contact is stable the component of static tangential stress in the direction of the applied tangential stress does not achieve the sliding condition. The direction of the applied force is assumed vertical since along whatever direction it acts, the vertical component must be that fitting the overall condition
of applied force per unit area (P).

The diagram of forces on a contact is shown in Fig. 27.

Fig. 27  A diagram of forces on a contact between two particles.

The static normal stress is:-

\[ N' = W \cos \beta \]  \hspace{1cm} (19)

The static tangential stress for \( \beta < \tan^{-1} \mu \) is:-

\[ T = W \sin \beta \]  \hspace{1cm} (20)

The static tangential stress for \( \beta > \tan^{-1} \mu \) is limited to:-

\[ T = \mu W \cos \beta \]  \hspace{1cm} (21)
For the condition where $\beta > \tan^{-1} \mu$, the normal and tangential stresses will not resolve to give the vertical force $W$. Since $T > \mu N$, the contact has achieved the slide condition but cannot move due to the effect of the other contacts on the particle. The bed is assumed stable. Therefore such a contact cannot react to any further tangential stress unless the normal stress increases.

The element of the bed being considered here is only isotropic under no-load conditions, the initial state. The deformation from the initial state due to the static load, is given by equations $A_i$ to $A_v$ in Appendix A. The volume of the element $(v)$ considered must be that deformed due to the weight of the bed and surface load.

The isotropic volume of the element is the horizontal area of the element, unity, times the height. The height that will be considered is the mean random chord, this distance being measured between some point in one particle to a point in another; thus always considering two rows of particles separated by the contacts between the rows, Fig. 28. The isotropic volume of the element will be considered as $\Omega$.

The tangential displacement due to static load is given by equations $A_v$ and $A_{iii}$:

$$\delta = \frac{3(2-\nu)\mu N'}{8 \sigma a} \left[ 1 - \left( 1 - \frac{T}{\mu N} \right)^{\frac{2}{3}} \right]$$

(22)
Fig. 28. The undeformed element of granular material.
where \( N \) is the normal force and \( T \) is the tangential force, \( \mu \) is the coefficient of friction at the contact and \( a \) the radius of the contact. \( v \) is Poisson's ratio for the material of the particles and \( s \) is the shear modulus, and the normal displacement is:

\[
\alpha = \frac{1}{2} \left[ \frac{3(1-v)N}{S R^{1/2}} \right]^{2/3}
\]

(23)

where \( R \) is the radius of the spheres.

For one contact substituting equations 19, 20 and \( A_i \) in equation 22 for \( \beta < \tan^{-1} \mu \), the tangential displacement is

\[
\delta_i = \frac{3(2-v) \mu}{4} \sqrt[3]{\frac{(W \cos \beta)^2}{3(1-v) R s^2}} \left[ 1 - \left( 1 - \frac{\tan \beta}{\mu} \right)^2 \right]^{2/3}
\]

(24a)

For \( \beta > \tan^{-1} \mu \)

\[
\delta_j = \frac{3(2-v) \mu}{4} \sqrt[3]{\frac{(W \cos \beta)^2}{3(1-v) R s^2}}
\]

(24b)

Substituting equation 19 in 23 the normal displacement is:

\[
\alpha = \frac{1}{2} \left[ \frac{3(1-v) W \cos \beta}{S R^{1/2}} \right]^{2/3}
\]

(25)

The reduction in size of the element due to a static load is found by projecting \( \delta \) and \( \alpha \) into the vertical and
horizontal directions and finding the mean of the projections over all values of \( W, \beta, R, \) and \( \lambda. \) \( R \) is the radius of curvature at the contact and \( \lambda \) the angle of deflection of the contact normal in the horizontal plane.

The vertical deflection is:

\[
V_i = \alpha \cos \beta + \delta \sin \beta
\]  
(26)

Combining equations 24a, 25 and 26

\[
V_i = \sqrt[3]{\frac{(W \cos \beta)^2}{R}} \left[ \frac{(1-\nu)^3}{s} \right]^{\frac{2}{3}} \cos \beta + \frac{3(2-\nu)\mu}{4}
\]

(27)

The horizontal deflection is:

\[
H_i = (\alpha \sin \beta - \delta \cos \beta) \cos \lambda
\]  
(28)

Combining equations 24a, 25 and 28

\[
H_i = \sqrt[3]{\frac{(W \cos \beta)^2}{R}} \left[ \frac{(3(1-\nu)^3}{s} \right]^{\frac{2}{3}} \sin \beta - \frac{3(2-\nu)\mu}{4}
\]

(29)

This equation is for the case where \( \beta < \tan^{-1} \frac{1}{\mu} \)

To ease manipulation of these equations let:

\[
A = \frac{1}{2} \left[ \frac{(1-\nu)^3}{s} \right]^{\frac{2}{3}}
\]
and \( B = \frac{3(2-v)\mu}{4} \left[ \frac{2}{3(1-v)} \right] \)

so that

\[
A, B, = \frac{3(2-v)\mu}{4} \left[ \frac{1}{S^2(1-v)3} \right]^{\frac{1}{3}}
\]

For the case where \( \beta > \tan^{-1} \mu \), the horizontal deflection is:

\[
H_j = \frac{3(W \cos \beta)^2}{R} \left[ A \sin \beta - A, B, \cos \beta \right] \cos \lambda.
\]

(30)

and similarly the vertical deflection is:

\[
V_j = \frac{3(W \cos \beta)^2}{R} \left[ A \cos \beta + A, B, \sin \beta \right]
\]

The number distributions of \( B, W, R \) and \( \lambda \) are necessary to determine mean deflections \( \overline{H} \) and \( \overline{V} \). These distributions are expressed as \( p(\beta), p(W), p(R) \), and \( p(\lambda) \). Probability of deflections \( V_i, V_j, H_i, H_j \), can now be written:

\[
p(H_i) \, dH_i = p(W) \, dW \cdot p(\beta) \, d\beta \cdot p(R) \, dR \cdot p(\lambda) \, d\lambda.
\]

and \( p(V_i) \, dV_i = p(W) \, dW \cdot p(\beta) \, d\beta \cdot p(R) \, dR \).

Let the probability of \( \beta \) be linear†, therefore \( p(\beta) = 1/\pi \)

where \( \beta \) varies between \(-\pi/2\) and \(+\pi/2\).

† See footnote on next page.
Let the probability of $W$ be linear, therefore $p(W) = \frac{1}{2L}$ where $W$ varies between 0 and $2L$, $L$ is the vertical load $P$ on the element, divided by the number of contacts $N^2$.

The probability of radius of curvature $p(R)$ can be derived from a random chord size analysis using the relationship derived by Campbell\(^2\) and Tomkeieff\(^3\). They stated that the average random chord is four times the volume of the particle divided by the surface area. The mean radius of curvature is the volume of the particle divided by the surface area. Then for each particle the mean radius of curvature $\bar{R}$ is the mean random chord $\bar{r}$ divided by four/three. $\bar{R} = \frac{\bar{r}}{4/3}$. Since this is true for each particle the distribution of radius of curvature is the distribution of random chords i.e. $p(R) = p(r)$ where $R = \frac{3r}{4}$.

The probability of $\lambda$ is linear due to the isotropy of the bed, therefore $p(\lambda) = \frac{1}{\pi}$, where $\lambda$ varies between $-\pi/2$ and $+\pi/2$.

Then:

\[
p(H_i) \; dH_i = \frac{1}{2\pi^2L} \; p(R) \; dR. \; dW. \; d\beta. \; d\lambda
\]

and

\[
p(V_i) \; dV_i = \frac{1}{2\pi L} \; p(R) \; dR. \; dW. \; d\beta.
\]

The mean values of $H$ and $V$ are given by equations of the form:

\[
\text{Discussion of the distributions assumed for } \beta \text{ and } W \text{ is to be found under the appropriate heading in the section "Distributions of Variables."}
\]
\[ \bar{H} = \frac{\int_{H_1}^{H_{1\text{ max}}} H_1 \cdot p(H_1) \cdot dH_1 + \int_{H_2}^{H_{2\text{ max}}} H_2 \cdot p(H_2) \cdot dH_2}{\int_{H_1}^{H_{1\text{ min}}} p(H_1) \cdot dH_1 + \int_{H_2}^{H_{2\text{ min}}} p(H_2) \cdot dH_2} \]

(H and V are synonymous) where the numerator is the total deflection and the denominator is the number of deflections. \( V_j, H_j \), refer to those values of \( H_i \) and \( V_i \) where \( \beta \) exceeds \( \tan^{-1} \mu \).

Now \( p(H_i) \cdot dH_i = p(H_j) \cdot dH_j \)

Therefore:

\[ \bar{H} = \frac{\int_{H_1}^{H_{1\text{ max}}} H_1 \cdot p(H_1) \cdot dH_1 + \int_{H_2}^{H_{2\text{ max}}} H_2 \cdot p(H_2) \cdot dH_2}{\int_{H_1}^{H_{1\text{ min}}} p(H_1) \cdot dH_1} \]

(32)

The denominator of the expression evaluates to one by choice of the original distribution functions. Including the limits for the separate integrations \( \bar{H} \) is

\[ \bar{H} = \int_{-\pi/2}^{\pi/2} \int_{R_{\text{ min}}}^{R_{\text{ max}}} \int_{0}^{2L} \tan^{-1} \mu \int_{0}^{\tan^{-1} \mu} H_1 \frac{d\beta \cdot dW \cdot dR \cdot d\lambda}{\pi^2 L} \]

\[ + \int_{R_{\text{ min}}}^{R_{\text{ max}}} \int_{0}^{2L} \tan^{-1} \mu \int_{0}^{\tan^{-1} \mu} H_j \frac{d\beta \cdot dW \cdot dR}{\pi^2 L} \]

(33)

after substitution of equations 31 and 32.
Expansion of equation 33 yields several functions containing \( \cos^{5/3} \beta \) which cannot be integrated analytically. However by combining the two parts of equation 33 over their common range of values only one function is not integrable. Then substituting equations 29 and 30, in 33 and integrating with respect to \( W \) and \( \lambda \).

\[
\bar{H} = \frac{12(2L)^{5/3}}{5\pi^2L} \int_{R \text{ min}}^{R \text{ max}} \frac{P(R)}{R^{1/3}} dR \left[ \tan^{-1} \mu \cos^{5/3} \beta \left[ 1 - \frac{\tan \beta}{\mu} \right]^{2/3} d\beta \right]
\]

Integrating by \( d \beta \) those parts which can be evaluated analytically \( \bar{H} \) is given by:

\[
\bar{H} = \frac{12(2L)^{5/3}}{5\pi^2L} \int_{R \text{ min}}^{R \text{ max}} \frac{P(R)}{R^{1/3}} dR \left[ \frac{3A}{5} - \tan^{-1} \mu \cos^{5/3} \beta \left[ 1 - \frac{\tan \beta}{\mu} \right]^{2/3} d\beta \right]
\]

(34)

The expression for the mean vertical deflection \( \bar{V} \) is of the same form as that for the mean horizontal deflection with the exclusion of \( \lambda \) terms. \( \bar{V}_i \) and \( \bar{V}_j \) are given by equation 27. Evaluation of \( \bar{V} \) requires the same rules as for \( \bar{H} \).
and after integrating by \( W \) and where possible by \( \beta \), \( \bar{V} \) reduces to:

\[
\bar{V} = \frac{6(2L)^{5/3}}{5\pi L} \left[ \int_{R_{\min}}^{R_{\max}} \frac{P(R)}{R^{1/3}} \, dR \right] \left[ \int_{0}^{\pi/2} \cos^{5/3} \beta \, d\beta + \frac{3A.B.}{5} \tan^{-1} \mu \right. \\
\left. - A.B. \left\{ \int_{0}^{\pi/2} \sin \beta \cos^{2/3} \beta \left[ 1 - \frac{\tan \beta}{\mu} \right]^{2/3} \, d\beta \right\} \right]
\]

Those integrals not soluble analytically were evaluated by digital computer. Their values are shown in Table 8.

The volume of the element deformed by vertical static load \( P \) is given by:

\[
\text{Vol.} = (r - \bar{V}) (1 - NH)^2
\]

where \( N \) is the number of contacts per unit length in the element.

Poisson's ratio for a granular material deformed by a static load.

To evaluate Poisson's ratio for the granular material compressed due to a static vertical load, consider a cube of material of unit volume. There are \( g \) contacts per unit length in the horizontal direction (isotropy) and \( f \) contacts per unit length in the vertical direction. As a first approximation let us assume that the number of contacts in
\[ \int_0^{\pi/2} \cos^{5/3} \beta \, d\beta = 0.841309 \]

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<th>[ \tan^{-1} \mu \int_0^{\pi/2} \cos^{5/3} \beta \left( 1 - \frac{\tan \beta}{\mu} \right)^{2/3} , d\beta ]</th>
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Table 8 Computed values of integrals.
each axis direction of the cube remain constant and only the axis lengths deform due to the vertical load on the cube L.

The mean deflections per contact $H$ and $V$ calculated in equations 34 and 35 are deflections from the no-load, isotropic condition where $f = g$. Then Poisson's ratio for the bulk material due to a vertical static load is:

$$\nu = \frac{gH}{fV} = \frac{H}{V}$$  \hspace{1cm} (37)

Substituting equation 34 and 35 in equations 36

$$\nu = 2 \left[ \frac{\pi/2 - B}{5} \int_{0}^{\pi/3} \cos^{5/3} \beta \, d\beta - \int_{0}^{\pi/3} \cos^{5/3} \beta \left[ 1 - \tan^{-1} \varphi \right]^{2/3} \, d\beta \right]$$

$$\nu = \frac{\pi}{\tan^{-1} \varphi} \left[ \int_{0}^{\pi/3} \cos^{5/3} \beta \, d\beta + \frac{3B}{5} - B \int_{0}^{\pi/3} \sin \beta \cos^{2/3} \beta \left[ 1 - \tan^{-1} \varphi \right]^{2/3} \, d\beta \right]$$  \hspace{1cm} (38)

Values of Poisson's ratio for the bulk material calculated by computer for different values of the grain material Poisson's ratio and coefficient of sliding friction are given in Table 9. The method of evaluation was to fit an eighth order equation to the values of those integrals evaluated between zero and $\tan^{-1} \varphi$. 
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Table 9 Values of Poisson's Ratio (ν) for granular material calculated from the solid material values of Poisson's Ratio (ν) and coefficient of sliding friction (μ).
The decrease in volume of a granular material due to a small increase of load.

Evaluation of the rate of decrease in element volume with applied pressure is achieved by evaluation of the vertical and tangential compliance of each contact. Addition of these compliances in the vertical and horizontal directions will give a reduction in height and width due to each increment of applied pressure. The volume change due to the applied pressure will then be calculated.

Increase in load is assumed to act vertically upward on the lower face of the element of granular material. The direction of the resulting stress at each contact is then the same as that due to the static load. For one contact in the element where the normal to the common tangent is inclined at angle \( \beta \) to the vertical the tangential deflection \( \Delta \delta \) due to applied tangential stress \( \Delta T \) is given by:

\[
\Delta \delta = \left[ \frac{2-\nu}{4\nu a} \right] \Delta T
\]

For \( \beta < \tan^{-1} \mu \).

No further deflection occurs in the case where \( \beta > \tan^{-1} \mu \), due to the small increase in load. All possible deflection in this case was taken up by the static load on the contact. However the number of contacts where \( \beta > \tan^{-1} \mu \) must be considered in order to obtain the mean deflection at a contact.

The normal deflection \( \Delta \alpha \) due to applied normal stress \( \Delta N \) is given by:
$\Delta \alpha = \left[ \frac{1-v}{2sa} \right] \Delta N'$  \hfill (40)

Equations 39 and 40 are found in the paper by Mindlin and Deresiewicz.\textsuperscript{83}

The contact in question is acted upon by increase of stress $\Delta W$ above the value $W$ for the static case. Then $\Delta N' = \Delta W \cos \beta$ \hfill (41)

and $\Delta T = \Delta W \sin \beta$ \hfill (42)

Then equations 39 and 40 become:-

$\Delta \delta = \left[ \frac{1-v}{4sa} \right] \Delta W \sin \beta$, for $\beta \leq \tan^{-1} \mu$ \hfill (43)

$\Delta \alpha = \left[ \frac{1-v}{2sa} \right] \Delta W \cos \beta$ \hfill (44)

To obtain vertical and horizontal deflections due to increase in stress $\Delta W$, equations 26 and 28 will be used in the form:-

$\Delta V = \Delta \alpha \cos \beta + \Delta \delta \sin \beta$ \hfill (45)

$\Delta H = (\Delta \alpha \sin \beta - \Delta \delta \cos \beta) \cos \lambda$ \hfill (46)

Substituting equations 42 and 44 in 45:-

$\Delta V = \left[ \frac{1-v}{2sa} \right] \Delta W \cos^2 \beta + \left[ \frac{2-v}{4sa} \right] \Delta W \sin^2 \beta$ \hfill (47)

similarly:-

$\Delta H = \Delta W \cos \lambda \left[ \frac{1-v}{2sa} \right] \cos \beta \sin \beta - \left[ \frac{2-v}{4sa} \right] \cos \beta \sin \beta$ \hfill (48)
The mean horizontal ($\bar{H}$) and vertical ($\bar{V}$) deflections are required to calculate the change in volume. They can be found by equation 32 where $\Delta H$ substitutes $H$, and in a similar equation where $\Delta V$ substitutes $V$.

To ease manipulation of the equation let

$$C = \frac{1-\nu}{2s} \left[ \frac{8s}{3(1-\nu)} \right]^{1/3}$$

and

$$D' = \frac{2-\nu}{4s} \left[ \frac{8s}{3(1-\nu)} \right]^{1/3}$$

Using equation Ai in the form:-

$$a = \left[ \frac{3(1-\nu)(W \cos \beta + \Delta W \cos \beta)}{8s} R \right]^{1/3}$$

and equation 31:-

$$p(\Delta H) \, d\Delta H = \frac{1}{2\pi^2 L} \int p(R) \, dR, \, d\beta, \, dW, \, d\lambda.$$
The limits $\Delta H$ max and 0 transform into $-\pi/2$ to $+\pi/2$ for integration by $\lambda$; $R$ max to 0 for integration by $R$; $2L$ to 0 for integration by $W$; and $-\tan^{-1}\mu$ to $+\tan^{-1}\mu$ for integration by $S$.

In the limit $\Delta W \to 0$ the form of the integral becomes:

$$
\frac{dH}{dL} = \frac{2}{\pi^2 L} \int_0^{\pi/2} \cos \lambda \, d\lambda \cdot \int_0^{R \text{ max}} \frac{p(R)}{3\sqrt{R}} \, dR \cdot \int_0^{2L} \frac{dW}{3\sqrt{W}}
$$

In the form:

$$(C-D') \int_0^{\tan^{-1}\mu} \cos^{2/3} \beta \cdot \sin \beta \cdot d\beta.$$

Hence:

$$
\frac{dH}{dL} = \frac{3(2L)^{2/3}}{\pi^2 L} \cdot (C-D') \left(1-\sqrt{1-\mu^2}\right) \int_0^{R \text{ max}} \frac{p(R)}{3\sqrt{R}} \, dR.
$$

Mean vertical deflection $\overline{\Delta V}$ is found by combining equations 47, 49, 50 and 51 with 31a in the form:

$$p(\Delta V) \, d\Delta V = \frac{1}{2\pi L} \cdot p(R) \, dR \cdot d\beta \cdot dW.$$

to give $\overline{\Delta V}$:

$$\overline{\Delta V} = \int_0^{\Delta V \text{ max}} \frac{\Delta W}{2\pi L} \left[ \frac{3}{\sqrt{R \cos \beta (W + \Delta W)}} C \cos^2 \beta + \frac{3}{\sqrt{R \cos \beta (W + \Delta W)}} D' \sin^2 \beta \right] \, p(R) \, dR \cdot d\beta \cdot dW.$$
Transforming the limits as for $\Delta H$ and in the limit of $\Delta W \to 0$, the form of the integral becomes:

\[
\frac{dV}{dL} = \frac{1}{\pi L} \int_0^{R_{\text{max}}} \frac{p(R)}{\sqrt[3]{R}} \, dR \int_0^{2L} \frac{dW}{\sqrt[3]{W}}
\]

\[
\tan^{-1} \mu \int_0^{\tan^{-1} \mu} \left( C \cos^{5/3} \beta + D \sin^{5/3} \beta \tan^{1/3} \beta \right) d\beta
\]

Integrating by $dW$:

\[
\frac{dV}{dL} = \frac{3(2L)^{2/3}}{2\pi L} \int_0^{R_{\text{max}}} \frac{p(R)}{\sqrt[3]{R}} \, dR \left[ \tan^{-1} \mu \int_0^{\tan^{-1} \mu} C \cos^{5/3} \beta \, d\beta + \tan^{-1} \mu \int_0^{\tan^{-1} \mu} D \sin^{5/3} \beta \tan^{1/3} \beta \, d\beta \right]
\]

(53)

The term $\sin^{5/3} \beta \tan^{1/3} \beta$ cannot be integrated analytically. Its computed values are shown in Table 10.

The change in volume $\Delta Vol$ due to an incremental increase in Pressure $\Delta P$ is given by:

\[
\frac{\Delta Vol}{\Delta P} = \left[ (r - \bar{V}) - \frac{dV}{dL} \right] \left[ (1 - NH) - \frac{MdH}{dL} \right]^2 - (r - \bar{V})(1 - NH)^2.
\]

where $r$ is the undeformed element height.
The term $\frac{v dp}{dv}$ (where $v$ is a volume) is given by the volume deformed by static pressure $P$ on the element multiplied by the inverse of $\frac{\Delta \text{Vol}}{\Delta P}$ in the limit $\Delta P \to 0$.

Therefore:

$$
\frac{v dp}{dv} = \frac{(r-\bar{V})(1-N\bar{H})^2}{(r-\bar{V}) - \frac{dv}{dL} \left[ (1-N\bar{H}) - \frac{Nd\bar{H}}{dL} \right]^2 - (r-\bar{V})(1-N\bar{H})^2}
$$

$$
= \frac{(r-\bar{V})(1-N\bar{H})^2}{2N(r-\bar{V})(1-N\bar{H}) \frac{dH}{dL} - (r-\bar{V}) N^2 \frac{dH}{dL}^2 - \frac{dv}{dL} (1-N\bar{H})^2}
$$

$$
- \frac{dV}{dL} \frac{N dH}{dL}^2 \neq 2N \frac{dH}{1-N\bar{H}} \frac{dv}{dL}
$$

(54)
$H, V, \Delta H$ and $\Delta V$ are given by equations 34, 35, 52 and 53 respectively. They are all functions of the coefficient of sliding friction of the solid ($\mu$), Poisson's ratio for the solid which forms the grains ($v$) and the mean radius of curvature of the grains $R$. The pressure on the element $P$ appears in each equation as a term of $L$. For a particular granular material $\mu, v$ and $R$ are constant and consequently the behaviour of $-\frac{v \, dp}{dv}$ is governed by $L$.

The evaluation of equation 54 will be made in terms of functions of $L$, thus:

\begin{align*}
\Delta H &= h \cdot \left( \frac{L^2}{3} \right) \\
\Delta V &= i \cdot \left( \frac{L^2}{3} \right) \\
\Delta H &= g \cdot \left( \frac{L^{-1}}{3} \right) \\
\Delta V &= j \cdot \left( \frac{L^{-1}}{3} \right)
\end{align*}

where $h, i, g$ and $j$ are constant for a particular material.

Equation 54 becomes, in terms of functions of $L$:

\[ -\frac{v \, dp}{dv} = \frac{f_1(L^2) + f_2(L^{4/3}) + f_3(L^{2/3}) + c_1}{f_4(L) + f_5(L^{1/3}) + f_6(L^{-1/3}) + f_7(L^{-2/3}) + c_2} \]

where $c_1$ and $c_2$ are constants.

When expanded the equation becomes:

\[ -\frac{v \, dp}{dv} = f_8(L) + f_9(L^{1/3}) + c_3 + f_{10}(L^{-1/3}) + f_{11}(L^{-2/3}) + f_{12}(L^{-1}) \ldots \ldots \]  \tag{55}

After the term $f_8(L)$ the subsequent terms play an increasingly smaller part in the behaviour of $-\frac{v \, dp}{dv}$. 

Whilst the $f_s(L^{1/3})$ term and the constant term $c$, may be significant only the first term $f_s(L)$ will be evaluated.

$$f_s(L) \text{ is given:} = \frac{N^2 \cdot \bar{H}^2 \cdot \bar{V}}{dL \cdot 2N^2 \cdot \bar{H} \cdot \bar{V} + dV \cdot N^2 \bar{H}^2}$$

$$= \frac{\bar{H} \cdot \bar{V}}{2dH + dV \cdot dL}$$

Since $\frac{\bar{H}}{\bar{V}}$ is the bulk material Poisson's ratio ($\nu$) the term becomes:

$$= \frac{\bar{H}}{2dH + \nu \cdot dV \cdot dL}$$

Substituting the values of $\bar{H}$, $dH$ and $dV$ from equations 34, 52 and 53 and simplifying the simplified equation for $- \frac{v dp}{dv}$ becomes:

$$-\frac{dp}{dv} = \frac{\frac{HL}{5} \left[ \frac{3A}{5} - B.A \right] \int_{0}^{\pi/2} \cos^2 \beta d\beta - \int_{0}^{\pi/2} \cos^2 \beta \left[ 1 - \frac{\tan \beta}{\mu} \right]^2 d\beta}{(C-D)(1 - \sqrt{1-\mu^2}) + \frac{\pi \nu}{4} \int_{0}^{\pi/2} (C \cos^2 \beta d\beta + D \sin^2 \beta \tan^3 \beta) d\beta}$$

$$L$$ is the mean vertical load on a contact and is related to the pressure on an element by the equation:

$$L = \frac{P}{N^2}$$

where $N^2$ is the number of contacts in an element of unit
horizontal area and height of a mean random chord. The physical value of N for evaluation of the velocity of propagation will be taken as the number of contacts between the particles intercepted by a line of unit length through the bed.

\[ N = \frac{1}{r} \]

It is to be noticed that the major term of \( \frac{v \, dp}{dv} \) is dependent on the number of interparticle contacts per unit volume. It is also a function of only the static pressure on the granular material and properties of the solid which constitutes the granules.

The velocity of propagation can now be written, substituting the functions calculated in equation 18. \( -\frac{v \, dp}{dv} \) is given by equation 56 and \( \tilde{v} \) is given by equation 38.

\[
S = \left[ \frac{3g}{\rho(1-\varepsilon)} \left( - \frac{v \, dp}{dv} \right) \frac{(1-\tilde{v})}{(1+\tilde{v})} \right]^{\frac{1}{2}}
\]

(18)

Hence the velocity of propagation is a function of the confining pressure to the half power, the reciprocal of the density of the granular material \( \rho \) to the half power, the reciprocal of number of contacts between particles on a straight line through the material, and of the properties of the solid which constitutes the granules. Computer evaluation of \( -\frac{v \, dp}{dv} \frac{(1-\tilde{v})}{(1+\tilde{v})} \), excluding the terms of \( P \) and \( p \), yields Table 11.
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<td>0.50</td>
<td>1.4602</td>
<td>0.9516</td>
<td>0.5732</td>
<td>0.3030</td>
<td>0.0618</td>
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</tbody>
</table>

Table 11 Velocity of propagation constant K, in equation for velocity \[ \left[ \frac{K}{3 \cdot g \cdot P \cdot \frac{R}{r}} \right]^{1/2} \]
3.5 Distributions of variables

The distributions of several variables included in the theory have never been measured and cannot yet be derived. The task would be complex and lengthy and may never have to be done. It was thought sufficient at this stage of presentation of these theories that reasonable assumptions of distributions would demonstrate the trends of the theory.

The choice of the distributions in all cases depends on the isotropy (the same in all directions) in the bed of granular material. A material deviates from isotropy as soon as it is acted upon by an external force. Such a force is the confining pressure due to the weight of the material above the point being considered. The grains tend to reorientate to present their shortest axis to the force. For reorientation to occur, the contacts on the particles must slide and form new contacts. It is feasible that light confining pressures are not sufficient to cause contact slide and the material will remain isotropic. A condition of both theories is that no contact slide occurs so the assumption of isotropy is reasonable if the material was initially isotropic.

The velocity of propagation theory makes allowance for the change of geometry of the granules due to confining pressure but slide is not considered.
The angle ($\theta$) between a contact chord and a fixed direction.

If the bed is isotropic (the same in all directions) theta ($\theta$) can only be affected by the positions of the two contacts at each end of the contact chord. The case where the contacts are at any position on the particle surface is considered in the text. This cannot be strictly so since two contacts cannot be separated by an infinitely small distance. However it is reasonable to assume that the distribution of theta will be uniform about zero and will be random between the nearest possible contacts to the one where theta is measured.

The distribution of the angle ($\beta$) between the normal to the common tangent at a contact and the vertical.

If the granular material is considered isotropic it is implied that the particles are randomly orientated. The normal at a contact will tend to point to a hypothetical centre of a particle. If the contacts are randomly positioned on the particle surface then $\beta$ will be randomly orientated to the vertical or any fixed direction.

The distribution of the value of the vertical force ($W$) acting on each contact.

The propagation of force in a granular material has been seen to be governed by many random effects causing the distribution of force to be normal, when distant from the source. The lattice of particles and contacts through which
the pressure is transmitted in a granular material is the main cause of the random effects. Consequently $W$ is assumed to be random.

The increment of increased load $\Delta W$ acts through the same lattice of particles and contacts as $W$ and is consequently added directly to $W$.

The distribution of the angle $\lambda$ in a horizontal plane.

The direction of a contact chord to a fixed horizontal direction is described by lambda $(\lambda)$. The distribution is assumed random since lambda $(\lambda)$ is a special case of the angle theta $(\theta)$. 
4.1 The Apparatus

The apparatus was designed to permit the study of attenuation and propagation velocity of a vertical impulse travelling upward in a packed bed of granular material. The containing vessel was made of such a diameter that the angle of the limit of propagation of the impulse could be investigated and intensity measurements could be made before the vessel wall interfered with the propagated force. The general arrangement of the apparatus is shown in Figs. 30 & 31.

An electromagnetic vibrator was mounted on a heavy concrete plinth which passed through the floor and was built integrally with a concrete foundation Fig. 32. This structure provided the vibrator with a total reaction load of 4,000 kilograms. To reduce the effect of neighbouring vibrating machinery the plinth was insulated from the floor of the laboratory. The vibrator was bolted into six steel rods 30cm. long 2.5cm. in diameter grouted into tapering holes in the plinth. This structure was considered rigid during all measurements.

The granular material under investigation was contained in a glass vessel 45cm. in diameter and 100cm. tall with the ends sealed by a base and lid of ground steel. Through the
Central hole for filling and depth probe

Top disc

Output transducer

Bed of material

Input disc

Glass vessel

Input transducer

Support frame

Bottom disc

Vacuum gland

Piston

Vibrator and trunnion

Insulated, heavy base

Fig 30 The propagation apparatus
Fig. 31a A general view of the apparatus.
Fig. 31b The glass vessel, supporting framework and vibrator and trunnion.
Fig. 32 The vibrator, plinth and foundation.
base passed a piston which was connected to the vibrator and on which was mounted an interchangeable rigid input disc. The piston passed through two glands, Fig. 33. The lower gland sealed the vessel so that experiments could be carried out at reduced gas pressure and the upper gland was to protect the vacuum seal from the material in the vessel.

The vibrator was driven by a power amplifier to which an electrical signal was fed from a wave form generator. The impulse was detected as it passed into the bed of material by a piezo-electric force transducer mounted between the piston and the input disc, Fig. 33.

The impulse was detected at the surface by a second force transducer mounted between an aluminium receiving disc and a reaction load Fig. 34. The receiving disc had an area of 10.00 sq. cm. and rested on the material surface where it could be moved to any position. The output of the two transducers was recorded on an ultra violet oscillograph via separate charge amplifiers. Also recorded was the electrical signal to the power amplifier and a timing oscillation was superimposed.

The principle of operation of the chart recorder was the reflection of ultra violet light from mirror galvanometers onto sensitive paper. The five galvanometers used had response frequencies at least twice that of the signal each recorded. The chart speed was approximately 2.5 metres per second. A diagram of the electrical apparatus is shown in Fig. 35.
The input piston and sealing glands.

The surface force receiver.

Fig 33 The mountings of the transducers
Fig. 34 The input disc (right) showing the powder gland and tension protection cage. The receiver (left) is resting on the lower sealing disc of the glass vessel.
Fig. 35 A diagram of the electrical apparatus.
The method of detecting the impulse at the surface was chosen primarily for simplicity. It was felt that the effect of a surface receiver on the system being measured could be estimated with greater ease than one embedded in the material.

An embedded receiver would have a minimum size so that it measured the effect over a representative sample of the material, but would have to be as small as possible to minimise the disturbance of the system. These parameters are conflicting and since any measurement beyond an embedded gauge would be affected by that gauge a measurement on the surface only was considered more accurate.

The reaction load system was chosen, again for simplicity. It was essential that the receiver was mobile both horizontally to traverse the surface and vertically to allow the glass vessel to be further filled. An alternative would have been to have had the transducer mounted on an adjustable frame attached to the vessel upper sealing disc. In that case adjusting the position of the receiving disc vertically to coincide with the material surface would have been extremely difficult. The horizontal traverse problem would have been easier to solve. A support system for the transducer which would have allowed mobility would have been a vertical rod. Such a support would have induced complex reflections of the disturbance in the transducer and the bed of material. The reaction load system was thus considered the best to use.
The choice of transducers was made by considering how they would affect the system being measured. The quartz piezo-electric transducer had the lowest strain, i.e., highest Youngs modulus of those considered. Since a dynamic system was to be measured and one particular measurement was to be the velocity of propagation, the quartz transducer was that which would introduce the least time delay in the measuring system.

The oscillograph was chosen in preference to an oscilloscope since it could accommodate the five channels we wished to record. The fast chart speed allowed clear recording of the shortest time of pulse we could produce and would also record the decay of the pulse in the bed. It would have been impossible to record both of these accurately on an oscilloscope as the ratio of pulse time to time length of decay was about 1 to 50. A continuous record could be made on a camera but 'Polaroid' film was not available to produce a convenient record.

The oscillograph produced a 15.2cm wide record which developed on exposure to artificial light emitting little in ultra violet range. A permanent record could be made by chemical development and chemical fixing of the trace but this was carried out on only a few occasions since the record decayed extremely slowly. Consequently the undeveloped records were stored in rolls in a dark cupboard.
Modification of the apparatus was necessary for two propagation experiments; those to measure the rate and decay of an impulse travelling vertically downward through the bed and those to measure the progress of a negative impulse travelling vertically upward. Traces of the three types of input are shown in Fig. 36.

The change of the apparatus for experiments with the downward travelling impulse entailed removing the piston from the bottom sealing disc and replacing it by the force transducer rigidly mounted on the base. The receiving transducer was modified to measure the input in such a way that it withstood collision with a 2.54 cm diameter falling steel ball. The ball was released from an electromagnet suspended above the new input transducer. The general arrangement of this experimental apparatus is shown in Fig. 37. The height of release of the steel ball was important only for reproducibility since the transducer system on the upper surface of the sand measured the impulse reaching the bed.

The second modification for upwardly propagating negative impulse entailed making a 'cage' to protect the force transducer in tension Fig. 38. Whilst the maximum compressive force measurable is 2000 kilograms the maximum tensile force acceptable is 100 kilograms. (This is due to the ultimate tensile strength of the phosphor bronze sheath which preloads the quartz discs in the transducers). Further preload was provided by the 'cage' which also protected the
Compressive force
(a) Force applied by the input piston.

Compressive force
(b) Force applied by a falling steel ball.

(c) Force due to the weight of material reduced by the input disc.

Fig. 36. Tracings of the three types of applied pulse.
Fig.37 Apparatus for an impulse propagating downward
Tightening sequence for preloading screws

Fig. 38. The tension protection cage for the input transducer.
transducer from rupture. The body contained the transducer and the lid preloaded it by tension in the set screws. Care was taken when mounting the lid to keep the load normal to the upper face of the transducer. Inaccuracy in reading and local areas of stress on the quartz discs of the transducer above the maximum permissible could be formed by non-normal force. The threads of the set screws were made loose fitting so that the torque on each screw could be measured. The tightening sequence is also shown in Fig. 38. The preload force on the transducer was measured from its output on an oscilloscope.

The system for evacuating the granular material and subsequently filling the interstices with carbon dioxide is shown in Fig. 39. The vacuum pump was connected via the pressure gauges to the vessel. Between the pump and gauges were two inlet pipes and valves; the first to inlet air and the second was connected to a cylinder of carbon dioxide via a pressure reducing valve. Two gauges were included, one indicating the order of magnitude of the vessel pressure and another reading from 100 to 0 millimetres of mercury. Since it was expected that the large diameter top and bottom discs would be difficult to seal on the vessel, two vacuum pumps were used in parallel.

The problem of filling the experiment vessel with granular material was overcome at the expense of radial
Vacuum pumps in parallel.

Wide and narrow range vacuum gauges.

Glass vessel

Receiving transducer

Fig 39 The gas system and force transducer arrangement.
segregation. Any two horizontal sections of the vessel had to have identical properties so a continuous flow of material into the vessel was necessary. A pneumatic conveying system was used since it also overcame the problem of lifting the sand from the storage bins on the laboratory floor through a height of 3 metres into the vessel, Fig. 40.

The conveying apparatus was designed so that sand entered at the terminal velocity of the fraction being filled through a pipe in the centre of the partially evacuated vessel. It then arrived at the sand surface at the same velocity wherever the surface level in the vessel. The work of Kolbuszewski suggests that both uniform settling velocity and intensity of deposition give uniform packing. To achieve uniform intensity of deposition a self feeding device was made for the pneumatic conveyor, Fig. 40. The conveying air was controlled by a valve which when admitting more air into the conveyor caused less sand to be drawn into the pipe. Since a specific porosity was not needed, merely uniformity, we achieved a uniform feed rate at that conveying air velocity depositing the material into the vessel at its terminal velocity.

The radial segregation was due to the coarse particles having higher momentum than the fine. A greater proportion of coarse particles were deposited in the centre than at the edge of the vessel. The sand tended to form a deposition cone but the conveying air, in flowing over the surface of the sand, carried some particles radially outwards as the air
Emptying of the test vessel achieved by partially evacuating a receiving bin and immersing the feed end of the conveyor in the test vessel.

Fig. 40. The pneumatic conveyor.
moved to the outlet and the evacuating fan. The effect of this flow was to move centrally deposited sand to the outside thus tending to even the distribution.

The depth of the bed of granular material between the input disc and the surface was measured by a depth probe from the upper sealing disc, Fig. 4.1. The probe was made of a length of screwed rod with a British Association size '0' thread. The pitch of this thread is nominally one millimetre, and by measuring the number of threads exposed to a datum on the vessel upper sealing disc the depth of granular material could be found. The probe was made more accurate by clamping a boss holding a micrometer a measured distance from one end of the rod and measuring back to the datum with the micrometer. The depth probe is shown in Fig. 4.1.

4.2 Calibration

The apparatus as shown in Fig. 3.1 was set up with the knowledge gained during the preliminary experiments. No literature was found describing any similar experimental rig although help was to be found on propagation testing of cylindrical samples. Therefore the equipment was calibrated for sufficient accuracy for the experiments envisaged. The calibration covered two separate sections:

(1) Mechanical performance
(2) Transducer performance
Micrometer mounted on a screwed boss

Marks on each tenth thread

Clamping screws

Hole to locate micrometer when adjusting the probe

Micrometer datum

Lid of test vessel

Cross-section of the screwed rod

Plumb bob

Fig. 41. The depth probe.
Mechanical performance

The vacuum gland between the input piston and the lower sealing disc was made with the piston diameter 25 - 30 microns smaller than the gland body. Such clearance on a hole 10cm long presented an alignment problem which we could only overcome by means of the non rigid mountings of the containing vessel. Alignment was checked with the vibrator driving the input piston sinusoidally and the receiving transducer measured the induced vibration in the vessel structure. The best alignment was that which induced a minimum vibration in the lower sealing disc. This was checked by looking for minimum interference on the sine wave trace from the transducer mounted on the input piston.

The screwed thread of the depth probe was calibrated to give correct readings of the depth of granular material. Clear marks had been made every ten threads on the flattened side of the screwed rod. The boss holding the micrometer was clamped to one of the marks. The datum boss, which screws into the experimental vessel for measurements to be taken, was sleeved right up to the micrometer boss. The micrometer was mounted such that it gave a zero reading when both bosses were in this position. The datum boss was then turned exactly 10 revolutions to screw it away from the micrometer. Then the distances between both bosses was measured with the micrometer. The micrometer boss was then moved down to the next mark and the procedure repeated. In this way the
distance between every ten threads on the screwed rod was measured to .001 centimetre. The rod was found to be accurate to 1 millimetre on its one metre length at 20°C.

Transducer performance

All propagation measurements were to be made from the charts produced by the oscillograph and since event frequencies close to the limit of accuracy of the oscillograph galvanometers were to be recorded the recording circuits had to be critically damped. The damping circuits affected the current through the galvanometers and charge amplifiers so the whole transducer, charge amplifier, oscillograph system was calibrated by static loads on the transducers.

A known mass was mounted on the transducer and the deflection on the oscillograph measured at a known charge amplifier setting. A charge is released from the quartz plates as load is applied and this charge is replaced from the charge amplifier as the load is released. After earthing the charge amplifier following measurement of the oscillograph displacement a negative displacement of the same amplitude resulted upon removing the load. This mass addition and removal procedure was continued to check linearity of both transducers and amplifiers whilst providing an overall calibration of force against oscillograph displacement. The curves are shown in Fig. 42. The frequency response of the whole system was limited by that of the galvanometers so
Fig. 42 Calibration curve of load against oscillograph displacement.
static calibration was valid for work in the range of the galvanometer response frequency.

A correction of the output signal from the surface receiver was necessary, since the receiver accelerated when acted on by the force it was measuring. The mass of the transducer was a significant part of the receiver mass and a correction for the reduction of force at the quartz measuring plates due to the transducer mass was made.

Figure 4.3 shows a sectional diagram of the surface receiver. Consider the mass of the transducer and receiving disc is \( m \) and the load on the receiver mass is \( M \). The actual force at the surface is \( F_a \) and the measured force is \( F_m \). Then at the surface of the granular material the force equation is:

\[
F_a = (M + m) \cdot a
\]

where \( a \) is the acceleration of all the receiver

At the quartz plates the force equation is :

\[
F_m = M \cdot A
\]

where \( A \) is the acceleration of the receiver load.

Since the transducers used were those available with the lowest strain the system can be assumed rigid. i.e. The acceleration of the receiver load is the same as the receiving disc on the material surface, \( A = a \). Then:

\[
F_a = \frac{F_m (M + m)}{M}
\]

The receiving disc area was 10.0 square centimetres. It was thought that a correction was necessary to account for
Fig. 43. The forces on the sectioned force receiver.
the variation of local force on the disc. The correction equation is developed in appendix C, and its significance is discussed.

The oscillograph maximum chart speed allowed two events separated in time by .0004 seconds to be recorded 1 millimetre apart. In order that the traces could be used with confidence some measure of the speed stability of the system was necessary. A close to 2,000 Hertz vibration was monitored for deviation from set frequency by a frequency counter and recorded simultaneously on the oscillograph. The trace produced was measured by a cathetometer over ten sets of ten adjacent trace peaks, the ten sets being 100 milliseconds apart. The standard deviation of the distance between the peaks in each set was between 11.5 and 12.5 microseconds, this being a measure of oscillograph speed stability and measurement accuracy. The frequency response of the oscillograph galvanometers which recorded the transducer output, limits the accuracy of the system to events lasting 100 microseconds or more. The accuracy of oscillograph paper speed and trace measurement was therefore insignificant.

4.3 Experimental procedure

The main objects of the experiments were to measure the velocity of propagation, the limiting angle of propagation and the decay of an applied impulse in a granular material. For this the glass vessel was filled with sand to a measured
depth and an impulse was applied to the sand. The characteristics of the propagating impulse were measured at the surface.

The electrical equipment was checked before the vessel was filled since difficulty was frequently experienced with the connections to the piezo-electric transducers. The insulation resistance of the connections had to be of the same order as the resistance in the charge amplifier circuit. However since the connections were embedded in the sand it was extremely difficult to keep the connecting surfaces dust free. The pneumatic filling process invariably filled the vessel with extremely fine quartz particles which, when on the connection surface, much reduced the insulation resistance. If the oscillograph traces remained stationary (no drift) before filling the vessel an experiment was conducted. No maintenance to the input transducer was possible when the vessel was full, in consequence some experiments gave spurious results.

The vessel was filled with the pneumatic conveyor. The evacuation fan was connected to the sealed vessel and the conveyor pipe immersed in the sand. The conveying air valve was adjusted such that the sand entering the vessel was close to its terminal velocity, and such that conveying was continuous. Care was taken to ensure that the feed inlet to the conveyor was immersed in sand until the required depth
was achieved.

A small pile always formed in the centre of the surface of the sand during filling. The shape was approximately conical, being constantly 5cm high and 30cm in diameter. The cone was levelled to provide a flat surface by rotating a horizontal straight edge across the diameter of the vessel. The level of the prepared surface was checked with a spirit level in two horizontal directions at right angles to each other. The height of sand at the surface which was caused by levelling the conical pile was 7 millimetres. Since it was not known how this levelled volume of sand would affect propagation of force, no correction could be made. Such a zone was formed at each level in the bed.

After filling and levelling, the receiver was carefully placed at the centre of the surface ensuring that it rested horizontally. The depth of the bed of material was measured with the depth probe Fig. 41. The probe was screwed down through the datum boss attached to the top sealing disc. When the pointed end just touched the top of the receiver, the micrometer boss was clamped to one of the calibrated marks on the probe screwed rods. The distance of the calibration mark from the datum was measured with the micrometer. A similar procedure had been carried out when the vessel was empty to locate the input piston. The depth of material in the vessel was then calculated by subtracting the measurement to the receiver from the empty vessel measurement. Account was taken of the receiver thickness and the deflection of the input
disc due to the weight of the bed. The deflection was measured with a cathetometer focussed on a knife edge on the vibrator table.

The peak force and duration of the input impulse to be applied to the granular material during the propagation measurements were determined by two series of experiments. The receiver remained in the centre of the surface of the same bed of material for both series. The first investigated the effect of the pulse width at a fixed peak force (Run 7.)

After the measurements to characterise the bed of granular material an impulse was applied to the vibrator. This was done by switching a positive half-sine wave from the signal generator into the vibrator power amplifier. The peak force on this occasion was chosen for convenience as that which gave an output at the receiver which was clearly detectable above the receiver noise. (The noise was mainly due to incomplete rectification of the power to the field coil of the vibrator and mains frequency picked up in the input stages of the power amplifier.) The width of the applied pulse was varied from 1 millisecond to 10 milliseconds where it became difficult to identify the limits of the pulses on the oscillograph record. The results of this experiment are shown in Fig. 45.

It had been reported in the literature that higher stresses could pass through a granular material at high rates of loading. The peak of propagated force at short pulse
widths shown in figure 45 suggested this to be the case with sand. The use of vibration to cause packing and shear force reduction requires that the propagation of the force from the source should be as efficient as possible. It was assumed that the half-sine wave pulse which we applied behaved in a similar fashion to a half cycle of continuous vibration. The evidence of similar propagation phenomena for a shock wave and vibration at resonant frequency of Selig and Vey suggested it to be so. Thus it was thought that the investigation of propagation phenomena in the region where less impulse was dissipated was technically of more interest. It was therefore decided to conduct the experiments at a pulse width of 1.25 milliseconds.

The second series of pulse characteristic measurements entailed varying the peak height of the pulse at the width already established (Run 8). The pulse height was varied from $10^4$ dynes per square centimetre which caused an impulse at the receiver undetectable above the noise level, to $10^5$ dynes per square centimetre which was just below the failure condition of the input transducer in tension. A graph of the results of this experiment is shown in Fig. 46. The impulse ratio and peak height ratio was constant. It was felt therefore that that input peak force most convenient for individual experimental conditions was the best to use.

The effect of the number of impulses applied to the bed was investigated (Run 11). The vessel, initially empty, was
filled and the depth of sand measured with the receiver carefully placed in the centre of the surface. An impulse propagation measurement was taken with the first series of pulses. A known number of unmeasured pulses were applied to the bed followed by a further measurement. This was continued until five measurements had been taken and two hundred pulses had been passed. The sand was then compacted by inverting the commercial packer, used in the preliminary experiments, on the surface whilst vibrating the input piston. After 30 minutes compaction with both vibration sources vibrating at 50 Hertz a further measurement of pulse propagation was made. Two more measurements were taken after a total of one hour's compaction and after three hours.

The effect of the number of pulses applied to the bed of material on subsequent pulses is shown in figure 47. It was found that reproducible results were only obtained after packing and in consequence continuous vibration was applied to the sand before a series of experiments were conducted at each bed depth.

Subsequent propagation experiments were performed at the conditions then established. A summary of all experiments is shown in Table 6, and a summary of important experimental conditions in Table 7.

The procedure for a propagation experiment was to fill the vessel to the smallest bed height of the series. The input disc was vibrated at 50 Hertz for 15 minutes. The bed
<table>
<thead>
<tr>
<th>RUN NUMBER</th>
<th>GRANULAR MATERIAL</th>
<th>PURPOSE</th>
<th>RESULTS TABLE</th>
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</thead>
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<tr>
<td>1 and 2</td>
<td>Fine sand</td>
<td>Calibration</td>
<td></td>
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<td>3 and 4</td>
<td>Fine sand</td>
<td>Propagation investigation</td>
<td>D3 and 4</td>
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<td>Fine sand</td>
<td>Evacuated voids</td>
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<td>Fine sand</td>
<td>Carbon dioxide in voids</td>
<td>D6</td>
</tr>
<tr>
<td>7</td>
<td>Fine sand</td>
<td>Effect of pulse width</td>
<td>D7 and 8</td>
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<td>8</td>
<td>Fine sand</td>
<td>Effect of peak force</td>
<td>D9</td>
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<td>9</td>
<td>Fine sand</td>
<td>Effect of receiver load</td>
<td>D10</td>
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<td>10</td>
<td>Fine sand</td>
<td>Obtained conditions for 11</td>
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<td>11</td>
<td>Fine sand</td>
<td>Effect of pulse history</td>
<td>D11</td>
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<td>18 &amp; 19</td>
<td>Various</td>
<td>Eliminating electrical fault</td>
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<td>Propagation in thin bed</td>
<td>D16</td>
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<td>Coarse sand</td>
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<td>54 &amp; 55</td>
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<td>Propagation of negative pulse upward</td>
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Table 6  A summary of the experiments.
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<th>OUTPUT DISC DIAMETER CM.</th>
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<td>D12 to 15</td>
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<td>3.56</td>
<td>458.1</td>
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</table>

Table 7  A summary of important experimental conditions.
was then levelled and the depth measured. The receiver was located at the centre of the surface by means of an aluminium rule which closely fitted the diameter of the vessel. The rule was calibrated with the centre of the vessel and with radial distance in one millimetre divisions. An impulse was applied with the receiver in the centre of the surface. For each experimental condition four separate impulses were applied when the bed showed no trace of the previous pulse. The experiment was only considered valid if there was no difference between the traces produced by the four pulses.

The receiver was then moved radially outward to the next position, usually in 2 centimetre steps. A further set of impulses was applied before the receiver was again moved to a new radial position so that a radial traverse was made. The traverse ended when either the wall was reached or no impulse was detected at the surface. The receiver was returned to a central position at the end of a traverse and an impulse measured. This checked that the bed had remained stable during the traverse.

The receiver was next removed from the vessel and the top sealing disc replaced. More sand was conveyed into the vessel to the next height to be investigated. The new bed was vibrated for fifteen minutes before a traverse of impulse measurements was made. The system of filling and taking results was continued until all the sand was in the vessel.

The vessel was emptied pneumatically. A receiving vessel was evacuated by the conveyor fan and the conveyor
inlet end immersed in the sand. The vessel was always emptied completely.

The measurement procedure was repeated for a bed of granular material evacuated of the interstitial air. The bed which was evacuated had previously been investigated with air in the interstices. Each time the receiving transducer was moved to a new radial position the vessel had to be re-evacuated since the top sealing disc of the containing vessel had been removed. The pumping operation took five minutes with the aid of a second vacuum pump. Evacuation caused the input piston to be slightly drawn into the vessel but a cathetometer measurement of the knife edge position on the vibrator again located the input disc precisely; thus allowing an accurate bed depth measurement.

After the measurements in the evacuated vessel were completed, carbon dioxide was substituted for air as the interstitial gas. The object of this was to produce a third set of results to allow extrapolation of effect of the medium in the interstices. To fill the vessel with carbon dioxide it was initially evacuated of air, then flooded with the gas before re-evacuation. A second flooding of carbon dioxide provided the interstitial gas for the experiment. The gas replacement procedure was repeated after each relocation of the receiving transducer. The bed of material investigated was the same as that evacuated and had had air filled pores for one series of experiments.
The Random Chord size analysis described in section 2.9 is being developed at Loughborough. The latest technique was to set a representative sample of the granular material in a resin glue, taking care to avoid segregation. Once hardened the resin was sectioned on a diamond wheel and the surfaces of the section polished. A high contrast photograph was produced of the polished surface which was analysed by either a computer scanning technique or a visual scanning method.

The computer technique entailed scanning the photographic image with a photocell. The lines of scan were straight and the co-ordinates of their intercepts with the particles were recorded. The random chord distribution was then computed from the co-ordinates. The manual method entailed measuring the lengths of chords on many lines drawn across the photograph and plotting their number distribution.

Both techniques rely on scanning along a line in a fixed direction. For the lines on the solid to be random chords by definition, the particles must be randomly orientated in space. To achieve this, the particles must be set in the liquid resin without sedimentation.

The shear strength of the granular material used in our experiment was measured. The shear cell used was a modified form of the Jenické cell and comprised upper and lower rings. The material was consolidated by three different normal loads and measurement was made at normal loads below that used for consolidation.
The technique was to fill the cell with granular material with the two similar rings fastened together one above the other. A close fitting piston was placed inside the upper ring and the material consolidated by a normal load above that at which the tests were to be carried out. After consolidation, the cell was placed in a retaining ring such that the lower ring of the cell was located laterally. The normal load at which the test was to be made was applied to the upper surface of the material. The upper ring was then forced laterally until the material failed. The shear force divided by the area of the sheared face was plotted against the normal load.

The measurements were carried out three times; the material consolidated at three different loads. The procedure was repeated with the second material.
5.1 The experimental conditions

The increase in propagated impulse at high rates of loading reported in the literature was found for sand in our experiments. Figure 44 shows the ratio of output/input impulse per unit area for increasing width of applied pulse. It can be seen that the output/input ratio increases as the pulse width decreases, the increase beginning at the upper limit of loading rate investigated by Whitman and Healy. It became increasingly difficult to detect points of interest on recordings of pulses of duration in excess of 15 milliseconds and the level portion to the right hand side of the curve can only be assumed to extend to much longer pulses. Since no further data was obtained it was assumed that the value of output/input ratio for the level portion of the curve was that found with infinitely slow ratios of loading and, in consequence, will be referred to as the static case. The peak value of output/input ratio is in the order of 300% higher than the static case, an increase of the same order as that found by Casagrande and Shannon for clay.

Experiment could proceed with few shorter pulse widths beyond the peak value due to the limitation of the equipment. The trend suggested by the shortest pulse width experiment is that the output/input ratio of impulse begins to decrease
beyond the peak but evidence is very scant.

It is of technical importance that such a peak of output impulse exists since a much higher impulse can be propagated into a granular material at that pulse width. It is a matter for further work whether the same phenomena exists for continuous vibration at the same half cycle width.

The experiment to detect whether there was any difference in propagation behaviour due to increasing peak applied force showed no change, Figure 45. It was of interest whether a change from propagation via purely elastic particle deformation to propagation via non-recoverable deformation was detectable. The lower limit of applied force was that which produced a measurable trace from the output transducer. No technique was known which could detect whether, at that level of force, propagation was purely by elastic forces. However increasing peak force to a level which threatened to rupture the input transducer in tension, showed no deviation from a straight line. Peak force conditions could be subsequently chosen for convenience.

The compaction effect of the applied pulses and their effect on propagation was investigated, Figure 46. It was found that the number of pulses applied to a newly filled bed of sand changed propagation behaviour. The larger the number of pulses applied, the more strongly the force propagated. This was to be expected since applied pulses were
of sufficient force to reorientate the particles whilst not reducing the overall porosity of the sand. Kolbuszewski had observed similar phenomena. The vibration compaction of the sand produced no further change in the propagation behaviour of the mass. Vibration was subsequently applied to each bed of granular material investigated to stabilise the propagation behaviour. The sand was not visibly compacted during the vibration, however more uniform results were obtained.

The random chord size of the two fractions of sand investigated is shown in Figure 47. The best technique available when the samples were analysed was that of measuring the lengths of projected chords on a microscope slide of a representative sample of the material. The disadvantages of the method are that the particles are randomly orientated on the slide and that not enough measurements could be taken for accurate distribution. However more recent techniques of analysis have not been available due to incomplete development work. It was felt that Figure 47 gives an idea of the random chord size distribution of our materials, although a better analysis is required for strict application to the theories developed in the Thesis.

Segregation in the glass vessel is shown in Figure 48 and the mechanism which caused it is discussed on page 178. Insufficient evidence of the dependence of propagation
phenomena on the size of the material has been obtained to suggest what part the segregation has to play in the propagation observed. It was felt, however, that uniform conditions existed on each horizontal plane within the bed.

The shear strength characteristics of the two gradings of sand used in the experimental work are shown in Figure 49. It was intended to relate the measured shear strength to the predicted profile of force within the granular material, however, not enough is yet known about the confining pressure normal to the predicted force to be able to define a shear strength at any point in the bed. It can be seen however that once the lateral pressure is known, due to both static and dynamic force, the shear strength of the material can be predicted at each point. Failure of the granular material will then occur if the force profile exceeds the local shear strength of the granular material.

5.2 Results
Measurement of force profile at the surface
The traces produced during the radial traverses of three bed surfaces to form the profile of force there, are shown in Fig. 50. The figure is of tracings of the actual record reduced to about one quarter of full size and is intended only to be illustrative. Measurement was taken from the actual record. Only that output trace which corresponded
Fig. 50 (a) Oscillograph records produced whilst measuring the force profile at the surface of a bed of sand. The depth of sand is 12 centimetres.
Fig. 50 (b) Oscillograph records produced whilst measuring the force profile at the surface of a bed of sand. The depth of sand is 24 centimetres.
Fig. 50 (c) Oscillograph records produced whilst measuring the force profile at the surface of a bed of sand. The depth of sand is 38 centimetres.
Fig. 44 The variation of the ratio output/input impulse with the width of the applied pulse.
Fig. 45 The variation of the ratio output/input force for increasing applied force.
Fig. 46 The variation of the ratio output/input force with the number of pulses applied.
Fig 47  Random chord size of sands used
Fig. 48 The segregation of particle size due to filling.

Fig. 49 Variation of shear strength with normal load for the sands used.
to the initial input was considered. The measurement of the areas of the pulses and their height being taken for calculation of impulse ratio and force ratio respectively. The output trace which continues after the initial peak likely holds information on the reflection behaviour of the system, however, no attempt was made to extract information from that part of the trace. Figure 50 shows the output peak decreasing as the receiver was traversed radially outward from the centre of the surface until it is no longer detectable above the noise level.

The velocity of propagation of the applied force in the granular material was measured by counting the superimposed standard frequency oscillations. Measurements were taken from the traces produced during the traverse of two beds to decide between which points the velocity should be calculated. Measurements were made between the start of the input and output peaks, the midway points on the rise of each, the peaks, the midway points on the fall and the point where each peak reached or recrossed the base line. There was no significance between any of the points measured and in consequence the distance between the beginning of the traces was measured. When this was not clear the peak-peak distance was compared with the probable distance between the beginning of the trace and if the two appeared of the same order the peak-peak distance was recorded.

The force profiles at the surface which were recorded are shown in Figures 51 to 56 inclusive. It can be
Fig. 51 Force profiles on the surface of beds of finer sand, large input disc.
Fig. 52 Force profiles on the surface of a bed of coarser sand, large input disc.
Fig. 53 Force profiles on the surface of shallower beds of coarser sand, small input disc.
Fig. 54 Force profiles on the surface of shallower beds of finer sand, small input disc.
Fig. 55 Force profiles at the bottom of a bed of finer sand disturbed at the surface.
Fig. 56 Force profiles at the bottom of a bed of coarser sand disturbed at the surface.
immediately seen that the majority of the curves have the form of the Gaussian (Normal) distribution as predicted by Smoltczyk. The behaviour of the curves is to broaden and flatten as the bed depth increases and the peak value of output/input force ratio is in all cases on the central axis of the applied force.

The peak value for output/input ratio of force per unit area for the shallowest bed of each investigation is in excess of one. This is of great interest since it suggests that higher forces per unit area than one applied can be produced within a bed of granular material. Before attempting to explain how this could be so, a possible measurement or calculation error will be discussed.

The ratio of output/input force or impulse per unit area is calculated by: \[ R = \frac{A_1\mathcal{O}}{A_I} \] where \( \mathcal{O} \) is the output, \( I \) is the input, \( A_1, A_0 \) are the input and output areas respectively. Whereas the actual output \( \mathcal{O} \) cannot exceed the input \( I \), it is clear that if the input area is larger than the output area, \( R \) can exceed one. If the medium between the output and input were a fluid then \( R \) would always be one, however, if the medium between were a solid \( \mathcal{O} \) would equal \( I \) and \( R \) would be the ratio of the areas of the input/output. A granular material can be expected to behave in an intermediate way, with \( \mathcal{O} \leq I \) and consequently \( R \) varying from \( \mathcal{O} \) to \( A_1/A_0 \).

However a granular material can cause an error if the value of \( A_0 \) is used in the calculation.

Consider a load \( W \) acting on a circular area \( A_0 \) on the
surface of a granular material. Let $W$ be a value such that the surface sinks under $A_o$. In sinking, the surface compresses a frustrum of a cone of sand with the loaded area at the apex. Sinking would continue until the larger circular area at the base of the frustrum can spread the load of $W$, and the uncompressed sand below the compressed region can support the load.

Such is the situation below the loaded receiver used in the experiments. The cone of compressed sand is more rigid than the surrounding material and consequently forms a mechanical coupling with the receiver. The base of the compressed zone reacts to forces over a larger area than on the surface and, if there are no losses in the compressed region, couples the higher force to the receiver. The output force $O$ would be higher than that measured by an unloaded surface receiver. In order to correct for the enlarged value of $O$ the actual area over which the receiver measures would need to be included. However no attempt was made at the correction in view of the complex problems involved.

It can be seen that even with the correction for receiving area a value of output/input force ratio greater than one could exist. For this to be so the granular material between the input and output must behave like a solid which could happen in two ways. It has been seen that the experiments were conducted at very high loading rates, where lateral inertia effects were apparent. Lateral strain cannot develop as fast as is necessary because of the inertia
of the grains. Consequently the material behaves more rigidly than if loaded more slowly. The granular material is also made more rigid as it is compressed by the applied force, in a similar way to the compression below the receiving disc.

Both lateral inertia and compression tend to make the granular material more rigid and in consequence force can be propagated as if through a solid. Hence the value of R can exceed one.

The experiments designed to detect the effect on propagation of the interstitial gas showed no effect. The measured force profiles of each experiment were within the limits of experimental accuracy to each other. Whilst all care was taken to ensure reproducibility of the results, an operation such as evacuating and reflooding the interstices with gas likely affects the orientation of some of the particles. Consequently small deviations in reaction to an applied force occurred which masked any small change due to different interstitial gases.

**Measurement of velocity of propagation**

The accuracy of velocity of propagation measurement depends very much on the intensity of the output force. In all cases a noise oscillation was detectable on the output trace and unless the peak due to the received force was much higher than the noise level, difficulty was experienced in measuring between the beginnings of the peaks. Variation in
velocity was always found as the distance from the centre of the surface increased. The values at the centre and at 2 and 4 centimetres radius were usually used to calculate the velocity for that bed height.

The velocity of propagation at the central axis of the bed of two sizes of sand is shown in Figure 58. No significant difference is detectable between the data for the coarse and fine materials. The best straight line is drawn through the results and the prediction of the present theory is also shown.

The line through the results has a slope differing from the one sixth power of the confining pressure, a dependence commonly predicted and occasionally measured. The results have a steeper slope and fit well with those of many workers shown in Figure 60.

The variation of velocity with radial distance is shown in Figure 59. It was found that the results of the downward propagation of force experiment showed much less scatter than those for upward propagation. These results clearly show the trend of velocity increase with increasing depth of material and also a decrease with radial distance. The increased distance between the measuring point and the source accounts for the radial decrease.

The downward propagation of force into the containing vessel tends to compress the granular material. The structure of the material is stabilised by the pre-experiment vibration and consequently good reproducibility was achieved. In the
Fig. 58 The measured velocity of propagation plotted against bed depth.
Fig. 59 The variation of velocity of propagation with radial position of the receiver.
Fig. 60 The predicted velocity of propagation together with the results of this work and those of others.
case of upward propagation of force the applied disturbance lifts the material and then allows it to fall back, thus increasing the possibility of disturbance of the structure of the material. In that case the reproducibility was not as good.

The two experiments conducted with a negative pulse propagating upward (i.e. the source in the bottom of the granular material was retracted away from the material) are also shown in Figure 59. The force profiles showed little variation with radius and the velocity curves shown there are inconclusive. It is clear that the velocity of propagation is much lower than that in compression experiments but that is likely due to the lower rate of acceleration of particles due to gravity alone.

The intention in conducting the negative pulse experiments was to observe the behaviour of the other half cycle to the compression pulse in a continuous sinusoidal vibration. It had been suspected that the recovery part of the cycle (negative pulse) was a much slower process, primarily since the driving force involved is much lower than the compression. The result suggests that it is reasonable to assume that continuous vibration is a succession of positive pulses as long as there is sufficient time for recovery to take place. More work is needed to verify the application of this work to continuous vibration.

It was noticed in the results that the width of the output pulse varied with the experimental conditions,
Figure 61 shows the variation with bed depth and Figure 62 shows the variation with radial distance. In both cases the ratio of out/input width decreases with distance from the source.

A possible explanation of the decrease in pulse width ratio is due to the compression of the bed due to the applied force. The bed compresses as the force is applied to state of more rigidity. The more rigid state then transmits the remainder of the pulse relatively unchanged in shape or form. Upon unloading the output follows the input until the bed begins to expand. Thus only the upper part of the applied force pulse reaches the surfaces.

5.3 Theoretical predictions

The predicted velocities from the theory are shown on Figure 60. The values of constants for the granular material are only approximate, the predicted velocity is too low, and the curve has too steep a slope. The slope is due to the velocity being predicted to the half power of the confining pressure which is markedly different from the common prediction of variation of velocity to the one sixth power of pressure. Such a prediction is obtained when the Hertz theory of contact deformation is used. The theoretical prediction here is in the right direction since the experiments of many workers show a greater dependence on pressure than the Hertz theory suggests.

It can be seen in the theory that a simplification was
Fig. 61 The variation of the ratio output/input pulse width with bed depth.
Fig. 62 The variation of the ratio output/input pulse width with radial distance.
made to facilitate the computation of a value of velocity. The simplification entailed neglecting terms of $P^{1/3}$ and a constant term, as well as other functions of $p$, in the equation $-\nu \frac{dp}{dv}$. The evaluation of these terms may improve the prediction, in fact inclusion of the $P^{1/3}$ term, if negative, will bring the predicted dependence of velocity on the pressure to a slope parallel to the best line through the results. Evaluation of all the constants will show the change.

It is significant that the inclusion of tangential compliance at an interparticle contact has changed the form of the dependence of velocity on pressure. It has previously been theoretically impossible to break from the one sixth power dependence. It is to be hoped that complete evaluation of the proposed theory will produce more accurate predictions than at present. It is felt that the above evidence suggests such a step would be valuable.

The prediction of Poisson's ratio of a granular material as an interim step in the calculation of velocity of propagation can be checked against that value for sand. The figure for sand is commonly accepted to vary between .2 and .3. The predicted value, taking solid material Poisson's ratio as .16 and the coefficient of sliding friction at a contact as .2, is .29 which is an acceptable prediction. Much more experimental evidence is necessary for the theory to be accepted.
The predicted force profile is shown in Figure 63. The values are much lower than measured although the curves broaden and flatten as the depth of the bed increases. The curves were computed for the same bed depths as the series of results shown in Figure 53. In evaluating the equation for profile of force it was necessary to make an assumption of the value of the second moment of the contact chord distribution. The value assumed had little physical significance and may possibly account for the low predicted values. An earlier equation for force profile (reference 126) showed better agreement with experiment when the total force on the surface was equated to the area under the predicted profile. The present equation varies with bed height and particle size in the same way but differs by the inclusion of porosity and a more complex dependence on the radial distance on the surface (D).

If the constant terms are evaluated by one experiment the theory follows practice as well as the prediction in reference 126. However the intention was to predict the profile from basic properties of the constituent grains but this cannot be done until the actual value of the second moment of the contact chord distribution is found as well as a more accurate value for the mean random chord.

Figure 64 shows the dependence of peak measured force on the depth of bed. The separation of the two sets of curves is due to the different loads on the receiver. This is verified by the continuation of the curves of runs 37 to 41 and 42 to 46 by the similarly loaded runs 51 to 53 and 48 to
Fig. 63 The theoretically predicted force profiles.
Fig. 64 The dependence of peak measured force on the depth of a bed of granular material.
50 respectively. It can be seen that the fine material results and those for the coarse have a dependence on depth close to the inverse of the square root of bed depth. Such a relationship is predicted for the profile at the centre of the surface where \( D = 0 \).

5.4 Further work

It has been shown that a random chord size analysis has physical significance and that it can be used to evaluate the behaviour of granular material. However since the second moment of contact chord cannot yet be derived from the random chord analysis the accuracy of prediction cannot be completely assessed. It is felt that the present results suggest that evaluation of the second moment of contact chord would be valuable.

Throughout the thesis only single pulses have been considered and the validity of assuming continuous vibration is a succession of applied pulses must be evaluated. When such an assumption is valid the profile of effect of vibration in a granular material will be known. Sufficient information is available in the literature to define powder behaviour at a point given local vibration conditions.

It is clear that a more fundamental approach is necessary in the study of the contact of irregular particles. Particular problems are:

(1) What is the shape of the surface of a contact between two points of different radius of curvature?
(2) What is the distribution of normal stress?
(3) If the surface of contact is not flat, as it is unlikely to be, how does the fitting of one particle into the other affect the coefficient of friction there?
(4) How does a complex contact surface shape affect the distribution of tangential stress?

If these points can ever be evaluated an approach to relating the bulk powder properties to the properties of the basic grains has been demonstrated.

The theory presented here considers only interparticle contacts which do not slide. Such an approach limits the prediction of flow behaviour to only the onset of non-recoverable deformation. A study of the compressional behaviour of granular material when contact slide is considered will predict the magnitude of non-recoverable deformation. Such a study would also be of value for the prediction of the onset of compaction. The present study can only predict the onset of bulk flow with the aid of bulk strength data of the material.
CHAPTER SIX

CONCLUSIONS

1. The distribution of force on a plane normal to the direction of propagation of force has the form of a Normal (Gaussian) distribution.

2. The greatest force is propagated along the central axis of the direction of the applied force.

3. The greatest force on the central axis decreases with increasing distance from the source. The decrease is approximately a function of the inverse of the square root of the distance from source.

4. Sand transmits a higher impulse at high rates of loading. A 300% increase was recorded.

5. The ratio of output/input force on the central axis is linear with applied force in the range investigated.

6. The velocity of propagation of a disturbance is the same whether propagated vertically upward or downward in a bed of sand.

7. The velocity of propagation of a disturbance of downward displacement, propagating vertically upward is lower
than the velocity of propagation of a compressive disturbance.

8 The velocity of propagation to radially displaced points on the surface is lower but the decrease is accounted for by the increased distance of travel.

9 The velocity of propagation measured in this apparatus with distant boundaries is comparable with the velocity measured by the resonant column method and by measuring the time of arrival of a pulse in a small diameter cylindrical sample.

10 A discrete particle approach has been used to predict the variation of force profile but not the absolute value of force.

11 A discrete particle approach has been used to predict the values of bulk granular material Poisson's ratio.

12 A discrete particle approach has been used to attempt to predict the velocity of propagation in a granular material.

13 The inclusion of the tangential stress in the dynamic behaviour of an interparticle contact has resulted in an equation for the velocity of propagation which is not dependent on the one sixth power of confining pressure. The one sixth power result is inevitable when the Hertz theory of contact deformation is used.
APPENDIX A

Rules of procedure of the Contact Theory developed by R. D. Mindlin and H. Deresiewicz.

Rule 1 The radius of (a) of the contact surface and the distribution of the normal component of stress (n) on it are given by the Hertz formulae;

\[
a = \left[ \frac{3(1-\nu) N' R}{8s} \right]^{\frac{1}{3}} \quad \text{Ai}
\]

\[
n = \frac{3N'}{2\pi a^3} \left( a^2 - b^2 \right)^{\frac{1}{2}} \quad \text{Aii}
\]

Rule 2 With every application or change of tangential force (T), slip will be initiated whenever, in the absence of slip, the local component of tangential stress (t) at any point, exceeds the product of a constant coefficient of friction (\(\mu\)) and the normal component of stress (n) at that point, Fig. Ai.

Rule 3 Slip, in the direction of the force causing it, progresses concentrically, radially inward from the boundary of the contact surface, forming an "annulus of slip".

Rule 4 The small lateral component of relative tangential displacement which accompanies the major slip in
Fig. A1 Distribution of normal $(n)$ and tangential $(t)$ stress on the contact surface of two spheres.
the direction of the applied force produces a lateral tangential stress, which is neglected.

**Rule 5** At any point on a contact surface, the magnitude of the tangential component of stress is at most equal to the product of a constant coefficient of friction and the normal component of stress at that point. The equality necessarily holds at a point at which slip has just occurred, in which case the stress has the same sense as the slip.

**Rule 6** The adhered portion of the contact surface, i.e. the portion encircled by an annulus on which slip occurs, is subjected to a change of tangential stress and undergoes a rigid-body tangential displacement. The radius of the adhered portion, the distribution of the stress and the magnitude of the displacement ($\delta$) are obtained from Cattaneo's and Mindlin's Formulae;

\[
c' = a \left[ 1 - \frac{T}{\mu N} \right]^{\frac{1}{3}} \quad \text{Aiii}
\]

\[
t = \frac{3\mu N'}{2\pi a^3} (a^2 - b^2) \quad : \quad c' < b < a
\]

\[
t = \frac{3\mu N}{2\pi a^3} \left( a^2 - b^2 \right)^{\frac{1}{2}} - \left( c'^2 - b^2 \right)^{\frac{1}{2}} \quad : \quad b < c'
\]
\[ \delta = \frac{3(2-v)\mu N'}{8sa} \left[ 1 - \frac{d^2}{a^2} \right] \]

\[
\frac{d\delta}{dT} = \frac{(2-v)}{4sa} \left[ 1 - \frac{T}{\mu N} \right]^{\frac{1}{2}}
\]

Rule 7: Beginning with an equilibrium position, for which the displacement and the distribution of stress have been established in accordance with the preceding rules, the effects of a change in the state of loading are obtained by advancing to the desired state through a sequence of equilibrium positions, each of which is obtained from its predecessor by applying Rules 1 to 6.
Volume, area and length fractions of a dispersed component in space.

The problem of sampling a three dimensional system by means of a two dimensional microscope slide led E.R. Weibel to demonstrate the principles developed by Delesse in 1842 and Rosiwal in 1892.

Delesse stated that a planar section through space containing a dispersed component, cut a fraction of area of the component equal to the fraction of volume it occupied. Rosiwal extended this principle to a line through space. He stated that the fraction of line passing through a randomly dispersed component in space is approximately equal to the fraction of volume occupied by the component. Weibel extends Rosiwal's principle to an exact equality when accounting for the sample size.

The demonstration of these principles is as follows.

Delesse:

Suppose that a cube (Fig. B1) with volume
\[ V = L^3 \]  \hspace{1cm} \text{(B1)}
contains granules of any shape and size which together have a volume
\[ V = \gamma V. \]  \hspace{1cm} \text{(B2)}

Consider, now, a thin slice of this cube of thickness \( dx \) parallel to the \((z,y)\) plane having a volume...
Fig. B1 A planar section of a space containing particles.

Fig. B2 Variation of the fraction of solids on a planar section.

Fig. B3 Random lines across a planar section of a space containing particles.

Fig. B4 Variation of the fraction of solids on a random line.
\[ dV = L^2 \cdot dx. \quad (B3) \]

In this slice a volume
\[ dv = n(x) \cdot dV = n(x) \cdot L^2 \cdot dx \quad (B4) \]
will contain segments of the granules. \( L \) and \( dx \) are constant wherever we place the slice but \( n(x) \) will vary with \( x \), as is indicated in Fig. Bii. If \( dx \to 0 \) the total volume of the granules \( v \) is
\[ v = \int_0^L dv = L^2, \quad \int_0^L n(x) \, dx = \gamma \cdot L^3. \quad (B5) \]

But
\[ \frac{1}{L} \int_0^L n(x) \, dx = \bar{n} \quad (B6) \]
is the average value of the coefficient \( n(x) \) between 0 and \( L \) so that it follows from (3.5) and (3.6) that
\[ \bar{n} \cdot L^3 = \gamma \cdot L^3 \]
or
\[ \bar{n} = \gamma. \quad (B7) \]

Relation (B7) means that the volumetric frequency \( \gamma \) of a given component is reflected on sections of the component in occupying a corresponding fraction \( n \) of the section area. In more practical terms, an average fractional coefficient \( \bar{n} \) determined on sections through a volume represents an estimator of the volumetric frequency \( \gamma \) of the component.
under investigation. From this follows that "A section is a quantitatively representative two-dimensional sample of a three dimensional system of randomly distributed structures."

Rosiwal:-

Let a square area $S = L^2$ be covered by spots (transsections of the structure $G$ under investigation) as illustrated in Fig. Biii over an area

$$s_g = \eta_g \cdot S$$

(B8)

and consider a strip of width $dx$ and area

$$dS = L \, dx.$$  

(B9)

A fraction

$$ds_g = \psi_g(x) \, dS = \psi_g(x) \, L \, dx$$

(B10)

of this strip will thus be covered by spots. We shall now find that $\psi_g(x)$ again varies with $x$ (Fig. Biv). If $dx \to 0$

$$s_g = L \int_0^L dx \cdot \psi_g(x) = \eta_g \cdot L^2.$$  

(B11)

But since

$$\frac{1}{L} \int_0^L dx \cdot \psi_g(x) = \overline{\psi}_g$$

(B12)

is the average value of $\psi_g(x)$ between 0 and $L$, it follows that

$$\overline{\psi}_g = \eta_g.$$  

(B13)
In deriving Delesse's principle, it was shown that average value of \( n_G \) is an estimator of the volumetric fraction \( \gamma_G \) occupied by the structures under investigation. We, therefore, find that

\[
\gamma_G = \bar{n}_G = \bar{\psi}_G. \tag{B14}
\]

Rosiwal also demonstrated that the line along which \( \psi_G \) is determined need not be straight but may have any shape, so long as its course is not biased by the underlying array of transsections.
A correction for the finite area of the receiving disc.

Since the lower limit of response to applied force of the transducers dictated the area of the receiving disc, the disc could not be made as small as was desired. The weight of the transducer and reaction load had to be supported on a definite area to stop the receiver sinking into the sand. The area of receiving disc close to these conditions was ten square centimetres. Such an area did not measure the profile of force at a point.

The actual profile of force on the surface was considered as concentric rings of width dS a distance S from the centre of the surface, Fig. Cl. The vertical stress detectable at a ring is G(S), where G(S) is a function of S. The measured force G(Z) is the sum of all the stress acting on the receiving disc, i.e.

\[ G(Z) = \Sigma G(S) \, dA \]  \hspace{1cm} (Ci)

\[ dA \] is the area over which stress G(S) acts.

The area dA is given by the length of arc a distance S from the centre of the surface. The length of arc is given by the product of the angle subtended at the centre and S.

\[ dA = S \cdot \theta \cdot dS \]  \hspace{1cm} (Cii)

\[ \theta \] is given by the cosine rule on the triangle of sides Y, S, Z shown in Fig. Cl.
Fig. C1. The relationship between the actual profile of force $G(S)$, and the measured profile $G(Z)$. 
Y is the radius of the receiving disc and Z is the distance between the centres of the surface and the receiver.

\[ \theta = \cos^{-1} \left[ \frac{Z^2 + S^2 - Y^2}{2ZS} \right] \]  

(Ciii)

Substituting equations Cii and Ciii in equation Ci the relationship between the measured profile \( G(Z) \) and the actual profile \( G(S) \) is:–

\[ G(Z) = \int_{Z-Y}^{Z+Y} S \cdot 2 \cdot \cos^{-1} \left[ \frac{Z^2 + S^2 - Y^2}{2ZS} \right] G(S) \, dS. \]  

(Civ)

This equation cannot be integrated either analytically or numerically since \( G(S) \) is unknown. Only by assuming the form of the function of \( G(S) \) can equation (Civ) be evaluated.

It was felt at this stage that an attempt at solving Civ would lead to more inaccuracy than assuming that \( G(S) \) was linear within the limits of the receiving disc; i.e. that the measured profile \( G(Z) \) was the same as the actual profile \( G(S) \). Consequently the correction was abandoned and all the profiles measured are assumed to be the actual profiles.
APPENDIX D

TABLES OF EXPERIMENTAL RESULTS

Some sets of experimental results for a particular bed depth of granular material had to be discarded because of indifferent performance of the transducers. Those included here are all considered to be accurate. However, occasionally, one column of results on a table is spurious, e.g., the velocity of propagation in Table D21 is low. In such a case if the other columns offered apparently valuable data the table was included, although it is known that calibration had drifted for one column of results.

In those columns where one set of data appears above another, the upper figure is the data and the lower figure the accuracy, e.g., The value for impulse ratio on the top line of Table D1 is 2.13 which has a measurement accuracy of plus or minus .11.
Table D1 To measure the radial profile of force at the surface. The first bed of series 16, 17, 3 and 4.

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### BED DEPTH 24.7 CM. OF FINE SAND.

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Table D2  To measure the radial profile of force at the surface. The second of the series 16, 17, 3 and 4.
Table D3  To measure the radial profile of force at the surface. The third bed of material in the series 16, 17, 3 & 4.
Table D4  

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To measure the radial profile of force at the surface. The deepest bed of material in the series 16,17,3 & 4
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Table D6 To measure the effect of carbon dioxide in the pore space. Compare with runs 4 and 5.
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</table>

Table D7 To measure the effect of variation of pulse width.
Bed depth 57.7 cm. of fine sand.

<table>
<thead>
<tr>
<th>RUN</th>
<th>Radial Position CM.</th>
<th>Radial Impulse Ratio</th>
<th>Radial Height Ratio</th>
<th>Radial Width Ratio</th>
<th>Radial Velocity M/SECS</th>
<th>Pulse Position Ratio</th>
<th>Pulse Width MSEC</th>
</tr>
</thead>
<tbody>
<tr>
<td>7a</td>
<td>0.0</td>
<td>0.06</td>
<td>0.15</td>
<td>0.41</td>
<td>77</td>
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</tr>
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<td>0.06</td>
<td>0.15</td>
<td>0.37</td>
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<td>5.0</td>
</tr>
<tr>
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<td>0.05</td>
<td>0.16</td>
<td>0.34</td>
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<td>0.13</td>
<td>0.29</td>
<td>69</td>
<td>0.0</td>
<td>6.0</td>
</tr>
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<td>7e</td>
<td>0.0</td>
<td>0.03</td>
<td>0.15</td>
<td>0.24</td>
<td>68</td>
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</tr>
<tr>
<td>7f</td>
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<td>0.04</td>
<td>0.14</td>
<td>0.27</td>
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</table>

Table D8 To measure the effect of variation of pulse width.
BED DEPTH 57.7 CM. OF FINE SAND.

<table>
<thead>
<tr>
<th>RUN</th>
<th>RADIAL POSITION CM.</th>
<th>IMPULSE RATIO</th>
<th>HEIGHT RATIO</th>
<th>WIDTH RATIO</th>
<th>VELOCITY M/SECS</th>
<th>PULSE HEIGHT % OF MAX.</th>
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<tbody>
<tr>
<td>8D</td>
<td>0.0</td>
<td>0.15</td>
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<td>1.09</td>
<td>160.</td>
<td>14.6</td>
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<tr>
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<td>0.02</td>
<td>0.04</td>
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<td></td>
</tr>
<tr>
<td>8E</td>
<td>0.0</td>
<td>0.12</td>
<td>0.09</td>
<td>1.25</td>
<td>160.</td>
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</tr>
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<td>0.01</td>
<td>0.04</td>
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<td></td>
</tr>
<tr>
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<td>0.0</td>
<td>0.11</td>
<td>0.12</td>
<td>0.94</td>
<td>144.</td>
<td>29.8</td>
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<td>0.01</td>
<td>0.03</td>
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<td></td>
</tr>
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<td>8G</td>
<td>0.0</td>
<td>0.12</td>
<td>0.11</td>
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<td>167.</td>
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<td>0.01</td>
<td>0.03</td>
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</tr>
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<td>0.02</td>
<td>0.04</td>
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<tr>
<td>8J</td>
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<td>1.23</td>
<td>141.</td>
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<td>0.01</td>
<td>0.03</td>
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</tr>
<tr>
<td>8K</td>
<td>0.0</td>
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<td>0.01</td>
<td>0.03</td>
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</tr>
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<td>138.</td>
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<td>0.01</td>
<td>0.03</td>
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<td>0.01</td>
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</tr>
</tbody>
</table>

Table D9  To measure the effect of input force at a pulse width of 1.25 milliseconds.
<table>
<thead>
<tr>
<th>RUN</th>
<th>RADIAL POSITION CM.</th>
<th>IMPULSE RATIO</th>
<th>HEIGHT RATIO</th>
<th>WIDTH RATIO</th>
<th>VELOCITY M/SECS</th>
<th>LOAD ON RECEIVING TRANSDUCER</th>
</tr>
</thead>
<tbody>
<tr>
<td>9A</td>
<td>0.0</td>
<td>0.02</td>
<td>0.08</td>
<td>0.29</td>
<td>165.</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>.01</td>
<td>.02</td>
<td>.02</td>
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<td></td>
</tr>
<tr>
<td>9B</td>
<td>0.0</td>
<td>0.01</td>
<td>0.06</td>
<td>0.27</td>
<td>156.</td>
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</tr>
<tr>
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<td>.02</td>
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<td></td>
</tr>
<tr>
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<td>0.04</td>
<td>0.31</td>
<td>144.</td>
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</tr>
<tr>
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<td>.01</td>
<td>.02</td>
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<tr>
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</tr>
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<td>.02</td>
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<td>0.08</td>
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<td>.02</td>
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<tr>
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<td>0.05</td>
<td>0.08</td>
<td>0.59</td>
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<td>.02</td>
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<td>.03</td>
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<tr>
<td>9L</td>
<td>0.0</td>
<td>0.05</td>
<td>0.06</td>
<td>0.93</td>
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Table D10 To measure the effect of load on the surface receiver.
# Bed Depth 59.2 cm. of Fine Sand

<table>
<thead>
<tr>
<th>RUN</th>
<th>Radial Position cm.</th>
<th>Impulse Ratio</th>
<th>Height Ratio</th>
<th>Width Ratio</th>
<th>Velocity M/Secs</th>
<th>Number of Pulses Applied</th>
</tr>
</thead>
<tbody>
<tr>
<td>11A</td>
<td>0.0</td>
<td>0.05</td>
<td>0.08</td>
<td>0.67</td>
<td>141.</td>
<td>3</td>
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<td>0.01</td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11B</td>
<td>0.0</td>
<td>0.08</td>
<td>0.17</td>
<td>0.51</td>
<td>119.</td>
<td>16</td>
</tr>
<tr>
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<td>0.00</td>
<td>0.01</td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11C</td>
<td>0.0</td>
<td>0.07</td>
<td>0.15</td>
<td>0.51</td>
<td>119.</td>
<td>27</td>
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<tr>
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<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td></td>
<td></td>
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<tr>
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<td>0.17</td>
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<td>0.01</td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11E</td>
<td>0.0</td>
<td>0.10</td>
<td>0.21</td>
<td>0.49</td>
<td>96.</td>
<td>199</td>
</tr>
<tr>
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<td>0.01</td>
<td>0.02</td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11F</td>
<td>0.0</td>
<td>0.10</td>
<td>0.17</td>
<td>0.59</td>
<td>198.</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td></td>
<td>30 min, 50 Hertz vibration</td>
</tr>
<tr>
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<td>0.0</td>
<td>0.09</td>
<td>0.15</td>
<td>0.58</td>
<td>198.</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td></td>
<td>30 min, 50 Hertz vibration</td>
</tr>
<tr>
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<td>0.09</td>
<td>0.15</td>
<td>0.55</td>
<td>198.</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td></td>
<td>2 hours, 50 Hertz vibration</td>
</tr>
</tbody>
</table>

Table D11: To measure the compaction of the number of pulses applied.
**BED DEPTH 12.4 CM. OF COARSE SAND.**

<table>
<thead>
<tr>
<th>RUN</th>
<th>RADIAL POSITION CM.</th>
<th>IMPULSE RATIO</th>
<th>HEIGHT RATIO</th>
<th>WIDTH RATIO</th>
<th>VELOCITY M/SEC</th>
<th>PROPAGATION ANGLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>12C</td>
<td>0.0</td>
<td>1.23</td>
<td>1.32</td>
<td>0.93</td>
<td>131.</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>.07</td>
<td>.12</td>
<td>.04</td>
<td>17.</td>
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</tr>
<tr>
<td>12D</td>
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<td>0.93</td>
<td>146.</td>
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<tr>
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<td>.04</td>
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</tr>
<tr>
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<td>.11</td>
<td>.03</td>
<td>17.</td>
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<tr>
<td>12F</td>
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<td>0.77</td>
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<tr>
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<td>.05</td>
<td>.11</td>
<td>.03</td>
<td>15.</td>
<td></td>
</tr>
<tr>
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<td>.07</td>
<td>.03</td>
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<tr>
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<td>0.96</td>
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<td>.04</td>
<td>.04</td>
<td>11.</td>
<td></td>
</tr>
<tr>
<td>12I</td>
<td>12.0</td>
<td>0.27</td>
<td>0.30</td>
<td>0.91</td>
<td>98.</td>
<td>20.0</td>
</tr>
<tr>
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<td>.02</td>
<td>.04</td>
<td>.03</td>
<td>9.</td>
<td></td>
</tr>
<tr>
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<td>.02</td>
<td>.03</td>
<td>7.</td>
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</tr>
<tr>
<td>12K</td>
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<td>0.03</td>
<td>0.05</td>
<td>0.60</td>
<td>76.</td>
<td>34.5</td>
</tr>
<tr>
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<td></td>
<td>.01</td>
<td>.02</td>
<td>.03</td>
<td>5.</td>
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<tr>
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<td>0.03</td>
<td>0.52</td>
<td>76.</td>
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<td>.01</td>
<td>.03</td>
<td>5.</td>
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Table D12 To measure the profile of force at the surface. The first of the series 12,13,14,and 15.
BED DEPTH 23.6 OF COARSE SAND.

<table>
<thead>
<tr>
<th>RUN</th>
<th>RADIAL POSITION CM.</th>
<th>IMPULSE RATIO</th>
<th>HEIGHT RATIO</th>
<th>WIDTH RATIO</th>
<th>VELOCITY M/SECS</th>
<th>PROPAGATION ANGLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>13B</td>
<td>0.0</td>
<td>0.42</td>
<td>0.72</td>
<td>0.58</td>
<td>157.</td>
<td>0.0</td>
</tr>
<tr>
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<td>.07</td>
<td>.02</td>
<td>12.</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.37</td>
<td>0.65</td>
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Table D13: To measure the profile of force at the surface. The second of the series 12, 13, 14, and 15.
**BED DEPTH 37.5 CM. OF COARSE SAND.**

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Table D14 To measure the profile of force at the surface. The third of the series 12,13,14 and 15.
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<th>PROPAGATION ANGLE</th>
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Table D15: To measure the profile of force at the surface. The fourth of the series 12, 13, 14 and 15.
BED DEPTH 2.0 CM. OF FINE SAND.

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Table D16 To measure the profile of force at the surface of a shallow bed.
Table D17 To measure the profile of force at the surface. The series 37 to 41 is of shallow beds.

BED DEPTH 2.2 CM. OF COARSE SAND.

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Table D 18  
To measure the profile of force at the surface. The series 37 to 41 is of shallow beds.
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Table D19: To measure the profile of force at the surface. The series 37 to 41 is of shallow beds.
Table D20 To measure the profile of force at the surface. The series 37 to 41 is of shallow beds.
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Table D21 To measure the profile of force at the surface. The series 37 to 41 is of shallow beds.
Table D23  To measure the profile of force at the surface. The series 42 to 46 of shallow beds.

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<th>WIDTH RATIO</th>
<th>VELOCITY M/SECS</th>
<th>PROPAGATION ANGLE</th>
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Table D24 To measure the profile of force at the surface. The series 42 to 46 of shallow beds.
Table D25: To measure the profile of force at the surface, the series 42 to 46 of shallow beds.

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Table D26  To measure the profile of force at the surface. The series 42 to 46 of shallow beds.
Table D27

To measure the profile of force at the surface. The series 42 to 46 is of shallow beds.

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<th>IMPULSE HEIGHT RATIO</th>
<th>WIDTH RATIO</th>
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**Bed depth 17.3 cm, of fine sand.**

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Table D28 To measure propagation phenomena of an impulse travelling downward,
### Table D29

To measure propagation phenomena of an impulse travelling downward.

**BED DEPTH 31.1 CM. OF FINE SAND.**

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Table D30 To measure propagation phenomena of an impulse travelling downward.
**BED DEPTH 26.3 CM. OF COURSE SAND.**

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<th>VELOCITY (M/SECS)</th>
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Table D31 To measure propagation phenomena of an impulse travelling downward.
## Table D32

To measure propagation phenomena of an impulse travelling downward.

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<th>WIDTH RATIO</th>
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Table D33  

To measure propagation phenomena of an impulse travelling downward.

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**BED DEPTH 48.6 CM. OF COARSE SAND.**
BET DEPTH 16.8 CM.

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Table D35 To measure propagation phenomena of an upward propagating negative pulse.
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Table D34 To measure propagation phenomena of an upward propagating negative pulse.
APPENDIX E

NOMENCLATURE

The symbols used in the Thesis are divided into two sections.

(1) Those which are used in the present theories and are maintained throughout.

(2) Those which were derived by other authors and are maintained for clarity.

1 2

A Constant to ease manipulation.

a Interparticle contact radius.

a (p.37) Amplitude of vibration.

B Constant to ease manipulation.

B (p.58) Constant for that granular material.

b The radius at which local normal stress \( n \) is calculated.

C Constant to ease manipulation.

c Length of a contact chord.

c' Inner radius of annulus of slip at an inter-particle contact.

D Horizontal displacement at a point on the surface from the centre of the system.

D' Constant to ease manipulation.
The horizontal displacement of two contacts across one particle.

\( d \) (p.67) Diameter of the mould.

\( F \) Force.

\( F \sin wt \) Vibratory force.

\( G(S) \) Vertical stress at radius (S) on the surface of granular material.

\( G(Z) \) Measured profile of vertical stress at radius (Z) on the surface of granular material.

\( g \) The acceleration of gravity.

\( g(D) \) The distribution of force on a horizontal plane at radial distance (D).

\( g(x) \) The distribution of force on a horizontal plane at radial distance (x).

\( H' \) The height of a bed of granular material.

\( H \) Horizontal displacement of an interparticle contact due to vertical force (W).

\( I \) Input force.

\( i \) (subscript) One particular value of.....

\( J \) Shear stress.

\( j \) (subscript) For the case where \( \beta > \tan^{-1} \mu \).

\( k \) The number of contacts on one particle.

\( L \) The mean value of vertical force on one contact.

\( L \) (p.15) Load on the vibrating table.

\( L \) (pp.241-244) Length of side of cube of volume V, and of side of square of area S.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>Number of particles traversed via contacts in order to follow a line which intercepts (N ) particles.</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>The second moment of the random chord distribution.</td>
</tr>
<tr>
<td>( m )</td>
<td>(p.37) Seismic mass.</td>
</tr>
<tr>
<td>( m )</td>
<td>(p.66) Mass of powder being compacted.</td>
</tr>
<tr>
<td>( N )</td>
<td>The number of particles cut by a line in a random direction.</td>
</tr>
<tr>
<td>( N' )</td>
<td>Normal force at an interparticle contact.</td>
</tr>
<tr>
<td>( N )</td>
<td>(p.56) Number of balls contained in an experiment.</td>
</tr>
<tr>
<td>( n )</td>
<td>Local normal stress on an interparticle contact surface.</td>
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<tr>
<td>( O )</td>
<td>Output force.</td>
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<td>( P )</td>
<td>The vertical load on an element of granular material of unit horizontal area.</td>
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<tr>
<td>( P_0 )</td>
<td>(p.66) Static compressive force.</td>
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<td>Pressure.</td>
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<td>( p(d) )</td>
<td>The probability of ( \ldots )(d).</td>
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<td>( R )</td>
<td>Mean radius of curvature of an irregular particle.</td>
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<tr>
<td>( R' )</td>
<td>The radius of the input disc.</td>
</tr>
<tr>
<td>( r )</td>
<td>Length of a random chord.</td>
</tr>
<tr>
<td>( S )</td>
<td>Area of a section of granules and voids.</td>
</tr>
<tr>
<td>( s )</td>
<td>Shear modulus of the material of the particles.</td>
</tr>
<tr>
<td>( S_G )</td>
<td>Area of the granules.</td>
</tr>
<tr>
<td>( T )</td>
<td>Tangential force at an interparticle contact.</td>
</tr>
</tbody>
</table>
Vertical displacement of a contact.

Volume of a cube of granules and voids.

Volume of an element of granular material.

Volume of granules.

Vertical force on an interparticle contact.

Power consumption.

Displacement, length.

Radius of receiving disc.

Horizontal distance between the centres of the input and receiving disc.

Normal displacement of an interparticle contact.

The angle between the normal to the common tangent at an interparticle contact and the vertical direction.

Fraction of volume V occupied by solid granules.

Small value of the following symbol.

Suzuki's velocity function.

Tangential displacement of an interparticle contact.

Porosity, fraction of voids in a granular material.

Variation of the fraction of solids in an elemental slice of granular material.

Mean value of \( \eta(x) \).
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_G$</td>
<td>Fraction of granules in section of granules and voids.</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Angle between a contact chord and a fixed direction.</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Angle subtended by the arc of radius $S$ within the receiving disc.</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Horizontal angle between a fixed vertical plane and a contact chord.</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Following a number, microns, $10^{-6}$ metres.</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Coefficient of Limiting Friction.</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson's ratio for the material of the particles.</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Poisson's ratio for a granular material.</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density of material of the particles.</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Standard deviation.</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>(p.67) Shear modulus.</td>
</tr>
<tr>
<td>$\sigma_d^2$</td>
<td>Variance of $d$.</td>
</tr>
<tr>
<td>$\sigma_D^2$</td>
<td>Variance of $D$.</td>
</tr>
<tr>
<td>$\sigma_F^2$</td>
<td>Projected variance.</td>
</tr>
<tr>
<td>$2\phi$</td>
<td>Angle subtended by the arc of radius $x$ coinciding with the input disc.</td>
</tr>
<tr>
<td>$\psi_G(x)$</td>
<td>Variation of the fraction of solids in an elemental strip of granular material.</td>
</tr>
<tr>
<td>$\psi_G$</td>
<td>Mean value of $\psi_G(x)$.</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Natural frequency.</td>
</tr>
</tbody>
</table>
BIBLIOGRAPHY

ALYANAK I.
'Vibration of sands'
1 Phd. Thesis. Univ. of Birmingham 1962

ANDEREGG F.O.
'Grading aggregates - Part 2, The application
of mathematical formulae to mortars.'
2 Ind. and Eng. Chem.
Vol. 23 pp 1058-1064 1931

ANDRADE E.N.da C. and Fox J.W.
'The mechanism of dilatancy'
Vol. 62 (8B) pp 483-500 1949

AUERBACH F.
'Die Gleichgewichtsfiguren pulverförmiger
Massen'
Vol. 310 (4.Folge 5) 170-219 1901

AYER J. and Soppett F.
'Vibratory compaction - Part 1, Compaction
of spherical shapes.'
5 J. Am. Ceram. Soc.
Vol. 48(4) pp 180-183 1965

AYER J. and Soppett F.
'Vibratory compaction - Part 2, Compaction
of angular shapes.'
Vol. 49(4) pp 207-210 1966

BAKER W.J.
'Wave propagation studies in confined sands'
7 Phd. Thesis Univ. of New Mexico 1966

BARKAN D.
'Dynamics of bases and foundations'

BAZANT Z. and Dvořák A.
'Effects of vibrations on sand and the
measurement of dynamic properties.'
9 6th Int. Conf. on Soil Mech. and Found. Eng.
Vol. 1 pp 161-164 1965
BELL W.C. and others
'Vibratory compacting of metal and ceramic powders'
Vol. 38(11) pp 396-404
1955

BELL W.C.
'Ceramic fabrication processes'
ed. Kingery, published by Wiley
pp 74-77
1957

BELL W.C.
'Vibratory compaction of metal and ceramic powders'
W.A.O.D.C. Tech. Rep. 53-193
Part 1
Part 2
Part 3
1953
1954
1956

BENNETT J.G. and Brown R.L.
J. Inst. Fuel
Vol. 15 p 232
1940

BERNAL J.D. and Mason J.
'Co-ordination of randomly packed spheres'
Nature
Vol. 188 pp 910-911
1960

BOERDIJK A.H.
'Some remarks concerning close-packing of equal spheres'
Vol. 7 pp 303-313
1952

BRACKPOOL J.L. and Phelps L.A.
'Vibratory compacting of metal powders'
Powder metallurgy
Vol. 7(14) p 213-227
1964

BRANDT H.
'A study of the speed of sound in porous granular material.'
Vol. 22 p 479-486
1955

BROWN R.L. and Hawksley P.G.W.
'Packing of regular (spherical) and irregular particles.'
Nature
Vol. 156 pp 421-422
1945
BRUTSAERT W.
'The propagation of elastic waves in unconsolidated unsaturated granular mediums'

CAMBELL H. and Tomkeieff S.I.
'Calculation of internal surface of a lung'
Nature Vol. 170 p 117

CARMAN P.C.
'The fundamental principles of industrial filtration'

CASAGRANDE A. and Shannon W.L.
'Research on stress deformation and strength characteristics of soils and soft rocks under transient loading.'
Harvard University, Soil mech. series No. 31 Publication 447

CATTANE0 C.
'Sul contatto di due corpi elastici'
Rend. R. Acad. dei. Lincei. (Ser.6) Vol. 27 pp 342-348, 434-436, 474-478

CHAMBRE P.L.
'Speed of a plane wave in a gross mixture'
J. Accoust. Soc. of America Vol. 26(3) pp 329-331

CONVERSE F.J.
'Vibration compaction of sand at resonant frequency'

CONVERSE F.J.
'Foundations subject to dynamic forces'

CONVERSE F.J.
'Vibration compaction of sand'
California Inst. of Tech. Research project sponsored by the U.S. Bureau of Yards and Docks.
DAYTON J.J. and Brown T.
28 (TID-5985) 1960

DELESSE M.A.
'Procédé mécanique pour déterminer la
composition des roches'
C.R. Acad. Sci. (Paris)
Vol. 25 pp 544-545 1847

DERESIEWICZ H.
'Mechanics of granular matter'
Advances in Applied Mechanics
Vol. 5 pp 233-306 1958

DERESIEWICZ H.
'Oblique contact of non-spherical elastic
bodies'
Vol. 24 1957

DERESIEWICZ H.
'Contact of elastic spheres under an
oscillating torsional couple'
Vol. 21 pp 52-56 1954

DERESIEWICZ H.
'Stress strain relations of a simple cubic
array of elastic spheres'
Paper No. 57-A-90 1957

DUFFY J. and Mindlin R.D.
'Stress strain relations and vibrations of a
granular medium'
Paper No. 57-APM-39 1957

ELAIS H. (editor)
'Stereology'
35 Springer Verlag, New York 1967

EVANS P.E. and Millman R.S.
'The vibratory compacting of powders'
Powder Metallurgy
Vol. 7 No. 13 p 50-63 1964

FATT I.
'Compressibility of a sphere pack,
Comparison of theory and experiment'
Vol. 24 pp 148-149 1957
FURNAS C.C.

38 'Flow of gases through beds of broken solids'
Bureau of Mines Bulletin
U.S. Dept. of Commerce
Vol. 307 pp 74-83
1929

39 'Mathematical relations for beds of broken solids of maximum density'
Ind. and Eng. Chem.
Vol. 23 pp 1052-1058
1931

GASSMANN F.

40 'Elastic waves through a packing of spheres'
Geophysics
Vol. 16 pp 673-685
1951
Vol. 18 p 268
1953

GOLDSTEIN M.N., Misumski V.A. and Lapidus L.S.

41 'Theory of probability and statistics in relation to the Rheology of soils'
Proc. 5th Int. Conf. on Soil Mech. and Found. Eng.
Vol. 1 p 123
1961

GRATON L.C. and Fraser H.J.

42 'Systematic packing of spheres with particular relation to porosity and permeability'
J. of Geology
Vol. 43 pp 785-909
1935

HALL J.R.

43 'Effect of amplitude on damping and wave propagation in granular material'
1962

HALL J.R. and Richart F.E.

44 'Dissipation of elastic wave energy in granular soils'
Vol. 89 pp 27-56
1963

HARA G.

45 'Theory of acoustic vibrations propagating in granular material and experimental investigation on carbon powder' (in German)
Elektrische Nachrichten-Technik
Vol. 12 pp 191-200
1935
HARDIN B.O.  
'Study of elastic wave propagation and damping in saturated granular materials'  

HARDIN B.O. and Richart F.E.  
'Elastic wave velocities in granular soils'  
Vol. 89  
p 33-65  
1963

HARRIS C.C. and Morrow N.R.  
'Pendular moisture in packings of equal spheres'  
Nature  
Vol. 203  
pp 706-708  
1964

HAUTH  
'Vibratory compacted fuel elements'  
Nucleonics  
Vol. 20 (9)  
pp 50-54  
1962

HENNIG A.  
'A critical survey of volume and surface measurements in microscopy'  
Zeiss Werkzeitschrift  
No. 30  
1959

HERDAN G.  
'Small particle statistics'  
Butterworths London, 2nd edition  
1960

HOGENDIJK M.J.  
'Random dense packing of spheres with a discrete distribution of radii'  
Philip's Res. Reps.  
Vol. 18  
pp 109-126  
1963

HORSFIELD H.T.  
'The strength of asphalt mixtures'  
Vol. 53  
pp 107T-115T  
1934

HUDSON D.R.  
'Close-clustering of spheres round a kernel'  
Proc. Leeds Phil. Lit. Soc. (Sci. Sec.)  
Vol. 5  
pp 65-74  
1947

HUDSON D.R.  
'Density and packing in an aggregate of mixed spheres'  
J. Appl. Physics.  
Vol. 20  
pp 154-162  
1949
HUGHES D.S. and Jones H.J.
'Variation of elastic modulii of igneous rocks with pressure and temperature'
Vol. 61 pp 843-856 1950

HUGHES D.S. and Cross H.J.
'Elastic wave velocities in rocks at high pressures and temperatures'
Geophysics
Vol. 16 pp 577-593 1951

HUGHES D.S. and Kelly J.L.
'Variation of elastic wave velocity with saturation in sandstone'
Geophysics
Vol. 17(4) pp 739-752 1952

HVORSLEV M.J.
'Uber die Festigkeitseigenschaflem gestorter bindiger Dosen'
Ingeniuruidenskals Skitter
Vol. A,45 p 155 1937

IIDA K.
'The velocity of elastic waves in sand'
Bull. Earthquake Res. Inst. (Tokyo Univ.)
Vol. 16 pp 131-144 1938

IIDA K.
'Velocity of elastic waves in a granular substance'
Bull. Earthquake Res. Inst. (Tokyo Univ.)
Vol. 17 pp 783-808 1939

ISHIMOTO and Iida
'Determination of elastic constants of soils by means of vibration methods'
Vol. 14 pp 632-656 1936
Vol. 15 pp 67-85 1937

IVASHCHENCO V.V., Tartakowskii I.P. and Golubev T.M.
'Vibration packing of spherical powders'
Translated from 'Poroshkovaya Metallurgiya'
No. 8(32) pp35-39 1965

IVASHCHENCO V.V., Tartakowskii I.P. and Golubev T.M.
'The investigation of the vibration compaction of two-component systems of spherical powders'
Translated from 'Poroshkovaya Metallurgiya'
No. 9(33) pp 40-44 1965
JENKIN C.F.
65 'The pressure exerted by granular material - an application of the principle of dilatancy'
Vol. 131 pp 53-89 1931

JOHNSON K.L.
66 'Surface interaction between elastically loaded bodies under tangential forces'
Proc. Roy. Soc. (Series A)
Vol. 230 pp 531-548 1955

JONES P.J.
67 'Vibratory compaction of metal and ceramic powders - Bibliography'
A.E.R.E. Bib 153 1967

KALSTENIUS T. and Bergau W.
68 'Research on texture of granular masses'
Proc. 5th Int. Conf. on soil Mechanics and Found. Eng.
Vol. 1 Part 1 pp 165-170 1961

KOLBUSZEWSKI J.
69 'General investigation of the fundamental factors controlling the loose packing of sand'
Proc. 2nd. Int. Conf. on soil Mech.
Vol. 7 1948

KOLBUSZEWSKI J.
70 'Fundamental approach to the basic factors controlling the behaviour of sands'
Vol. 4 pp 9-18 1961

KOLBUSZEWSKI J., A1yanak I.
71 'Effect of vibrations on shear strength and porosity of sands'
The Surveyor
30th May p 23 1964
6th June p 31 1964

KROLL W.
72 'Fliesserscheinungen an Haufwerken in schwingenden Gefässen'
Vol. 27 p 33-38 1955
KUTZNER C.H.
'Über die Vorgänge in Körnigen Schüttingen bei der Rüttelverdichtung'
Institute für Boden mechanik und Grundbau der Technische Hochschule Fridericiana, Karlsruhe 1962

LAWRENCE L.R. and Beddow J.K.
'Some effects of powder segregation during die filling'
Powder Technology Vol. 2 pp 125-130 1968

L'HERMITE R. and Tournon G.
'La vibration du béton frais'
Annales de l'Institute Technique du Bâtiments et des Travaux Publques, Paris No. 11 Nouvelle Série 1965

LIKHTMAN V.I., Gorbunov, Shatalova, Rebinder.
'Vibrational compacting in powder metallurgy'

LIKIN
"Equation of vibration pressing of powdered materials'
Translated from 'Poroshkovaya Metallurgiya' No. 1(37) pp 1-4 1966

LITWINISZYN J.
'An application of the random walk argument to the mechanics of granular media'
I.U.T.A.M. Symposium, Grenoble 1964

LITWINISZYN J.
'Statistical methods in the mechanics of granular bodies'

LUBKIN J.L.
'The torsion of elastic spheres in contact'

MACRAE J.C. and Gray W.A.
'Significance of the properties of materials in the packing of real spherical particles'
MARVIN J.W.
'The shape of compressed lead shot and its relation to cell shape'
Am. J. Botany.
Vol. 26 pp 280-288 1939

MATSUMARAWA E. and Hunter A.N.
The variation of sound velocity with stress in sand'
Vol. 69 pp 847-848 1956

MATZKE E.B.
'Volume shape relationships in lead shot and their bearing on cell shapes'
Am. J. Botany
Vol. 26 pp 288-295 1939

McGEARY
'Mechanical packing of spherical particles'
Vol. 44 pp 513-522 1961

MINDLIN R.D.
'Compliance of elastic bodies in contact'
Vol. 16 pp 259-268 1949

MINDLIN R.D., Mason, Osmer, and Deresiewicz.
'Effects on an oscillating tangential force on the contact surfaces of elastic spheres'
pp 203-208 1951

MINDLIN R.D. and Deresiewicz H.
'Elastic spheres in contact under varying oblique forces'
Vol. 20 pp 327-344 1953

MINDLIN R.D.
'Mechanics of granular media'
Ann Arbor
pp 13-20 1954

MOGAMI T. and Kubo K.
The behaviour of soil during vibration'
3rd Int. Conf. on Soil Mech. and Found. Eng.
Vol. 1 pp 152-155 1953
NASU N.

'Studies of the propagation of an artificial earthquake wave through superficial soil or sand layers, and the elasticity of soil and sand' (In Japanese)
Bull. Earthquake Res. Inst. (Tokyo)
Vol. 18 pp 289-304 1940
Vol. 27 pp 101-106 1949

PARKIN B.R.

'Impact waves in sand; Theory compared with experiment on sand columns'
Vol. 87 1961

REYNOLDS O.

'On the dilatancy of media composed of rigid granules in contact'
Phil. Mag. (5. Series) Vol. 20 pp 469-481 1885

ROSCOE K.H., Schofield A.N., and Wroth C.P.

'On the yielding of soils'
Geotechnique Vol. 8 N.1 pp 22-52 1958

ROSIWAL A.

'Uber geometrische Gesteimanalysen. Ein einfacher Weg zur ziffermäßigen Feststellung des Quantit ätsverhältnisses der Mineralbestandteile gemengter Gesteine'
Verk. K.K. Geol. Reichsamt. Wein. p 143 1898

SCOTT G.D.

'Packing of equal spheres'
Nature Vol. 188 pp 908-909 1960

SELEIG E.T.

'Effect of vibration on density of sand'

SELEIG E.T.

'Shock induced stress wave propagation in sand'
Illinois Institute of Technology Phd. Thesis 1964
SELG E.T. and Vey E.E.  
'Shock induced stress wave propagation in sand'  
Vol. 91 p 19-49  
1965

SHANNON W.L., Yamane G., Dietrich R.J.  
'Dynamic triaxial tests on sand'  
Vol. 1 pp 473-488  
1959

SHINOHARA K., Suzuki A., Tanaka T.  
'Gravity and vibration effects on flow of cohesive materials from a hopper'  
A.S.M.E. Paper 68-MH-33  
1968

SMITH W.O., Foote P.D., Busang P.F.  
'Packing of homogeneous spheres'  
Physical review Vol. 34 pp 1271-1274  
1929

SMOLTZYK H.U.  
'Stress computation in soil media'  
Vol. 93  
1967

SUPNICK F.  
'On the dense packing of spheres'  
1949

SUZUKI A., Takahashi H., Tanaka T.  
'Behaviour of a particle bed in the field of vibration'  
'Flow of particles through slits in the bottom of a vibrating vessel'  
Powder Technology Vol. 2 pp 72-77  
1968

Takahashi T. and Sato Y.  
'On the theory of elastic waves in granular substance'  
Bull. Earthquake Res. Inst. Vol. 27 pp 11-16  
Vol. 28 pp 37-43  
1949 1950
Takahashi H., Suzuki A., Tanaka T.
Behaviour of a particle bed in the field of vibration.
'Analysis of particle motion in a vibrating vessel'
Powder Technology
Vol. 2 pp 65-71
1968

Thurston C.W. and Deresiewicz H.
'Analysis of a compression test of a model of a granular medium'
Vol. 26 pp 251-258
1959

Timoshenko S. and Goodier J.N.
'Theory of elasticity' (2nd ed.)
McGraw Hill, N.Y.
pp 372-382
1951

Tomkeieff S.I.
'Linear intercepts, Areas and Volumes'
Nature
Vol. 155 p 24
1945

Vibratory Packing
U.S.A.E.C. Reports NYO 2571-2575
111

Wadeell H.
'Volume, shape, and roundness of quartz particles'
J. of Geology.
Vol. 43 pp 250-280
1935

Webb
'Vibratory compaction'
113

Weibel E.R.
'Morphometry of the human lung'
1963

Westman A.E.R. and Hugill H.R.
'The packing of particles'
Vol. 13 pp 767-779
1930

White H.E. and Walton S.F.
'Particle packing and particle shape'
Vol. 20 pp 155-166
1937
WHITMAN R.V.  
"The behaviour of soils under transient loadings"  
Vol. 1  p 207  
1957

WHITMAN R.V. and Healy  
"Shear strengths of sands during rapid loadings"  
Vol. 88  pp 99-132  
1962

WILLIAMS J.C. and Shields G.  
"Segregation of granules in a vibrated bed"  
Powder Technology.  
Vol. 1  pp 134-142  
1967

WISE M.E.  
"Dense random packing of unequal spheres"  
No. 7  pp 321-343  
1952

WISE M.E.  
"Converting a number distribution of particle size into one for volume and surface area"  
Vol. 9  pp 231-237  
1954

YOSHIDA T. and Kousaka Y.  
"Mechanism of vibratory compaction of granular solids"  
Kagaku-Kōgaku In English (abridged addition)  
Vol. 5 No. 1  
1967

YOUD T.L.  
"The engineering properties of cohesionless materials during vibration"  
Phd. Thesis Iowa State University  
1967
Bibliography of work being carried out and work privately presented at Loughborough.

The subjects referred to in these papers have been supplemented by private communication.

LLOYD P.J.
'Sampling from bulk powders and Fluids'
124 Particle Characteristics Conference, New Jersey, U.S.A.
Published by Department of Chemical Engineering, Loughborough in conjunction with Coulter electronics.

LLOYD P.J.
'The mathematics of particle size distributions'
125 (Based on a series of lectures presented by Dr. Ing. K. Leschonski at Loughborough during 1969)
Ibid.

SCARLETT B. and Eastham I.E.
'The stresses in granular materials due to applied vibration'
126 Tripartite Chemical Engineering Conference Montreal.
Session 14
1968

SCARLETT B. and Todd A.C.
'The critical porosity of free flowing solids'
1969