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FOR REFERENCE ONLY
THE MEASUREMENT OF THE BULK MODULUS LOSS FACTOR
OF SMALL SOLID SPECIMENS

by

GEOFFREY LEONARD WILSON

A Doctoral Thesis

Submitted in partial fulfilment of the requirements
for the award of the degree of
Doctor of Philosophy
of the Loughborough University of Technology

February 1975

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THE MEASUREMENT OF THE BULK MODULUS LOSS FACTOR OF SMALL SOLID SPECIMENS

By Geoffrey L. Wilson

The bulk modulus loss factor of small, relatively loss-free solid specimens is one of the more elusive elastic constants to measure, as it is essential that the losses in the apparatus be reduced below those in the sample. A procedure described by Tamm in 1942 involves the insertion of the sample at a pressure maximum in a column of liquid resonating in a longitudinal mode; the loss factor was determined from the change in damping of this resonance. Essentially the same procedure was followed by Niemic in 1972, who reports a Q of 1300 as typical for the resonator without a sample.

Simon in 1965 attempted to eliminate viscous loss in the boundary layer at the tube wall by the use of purely radial modes in spherical vessels such as have commonly been used for the measurement of the attenuation in liquids. He suspended the flask in a vacuum jacket to reduce radiation loading, and used a common transducer at the centre for the initial drive and the measurement of decay. He obtained a Q of over 20 000, though the excitation of wall resonances and the asymmetry at the neck caused considerable difficulty.

The present method makes use of a cylindrical vessel in order to produce a more practical system. Initially it was hoped that viscous losses could be reduced by using a very thin, resilient wall and a predominantly radial mode of vibration. This approach proved unsuccessful due to losses in the material of the wall itself. A vessel with a thick wall and thick bottom is now used; the resonance used is a longitudinal 3\lambda/4 mode, and the sample is placed on the bottom. The driving transducer is a ceramic button cemented to the underside of the bottom, and the receiving transducer is a small crystal microphone suspended over the air-water interface. The vessel may be suspended by three strings, or supported on nails inside an external pressure container. By this means a Q of 5000 is obtainable at a frequency of 6 kHz.

Results are given for typical polymeric materials, root vegetables, and samples of animal tissue.
PREFACE

Most of the work described herein was performed in the Physics Department of the Pennsylvania State University under the supervision of Dr. E. J. Skudrzyk, Professor of Physics, and of Dr. M. T. Pigott, Professor of Engineering Research and Assistant Director of the Applied Research Laboratory, where I am a staff member. I wish to thank Dr. R. H. Good, Professor of Physics and Head of the Physics Department, and Dr. D. H. Rank, Professor Emeritus, his predecessor, for the use of the facilities of that department, and Dr. J. C. Johnson, Professor of Engineering Research and Director of the Applied Research Laboratory, and many of my colleagues at A.R.L., for their advice and support. I also wish to acknowledge the encouragement given by Professor J. W. R. Griffiths, Head of the Department of Electronic and Electrical Engineering at the Loughborough University of Technology, and the many helpful discussions and suggestions by Dr. J. Szilard, Lecturer in that Department.

Except where otherwise referenced, this thesis describes the original work of the author, and no part of it has been submitted to any other university.

Geoffrey L. Wilson
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LIST OF SYMBOLS

a  radius of tube
b  velocity of sound
c  velocity of sound in liquid of resonator
d  differential operator
e  base of natural logarithm
f  frequency
g  mode factor
h  length of tube (depth of liquid)
i  running index
j  \( \sqrt{-1} \); running index
k  wave number (= \( \omega/c \)); Boltzman's constant; running index
kr  radial component of wave number
kz  longitudinal component of wave number
k\( \phi \)  circumferential component of wave number
l  running index
m  order of Bessel function
n  direction of normal to surface; integer numeral (order of sinusoidal variation)
p  instantaneous pressure
po  pressure amplitude
pm  local pressure amplitude
q  mode factor; magnitude of electric charge
r  radial coordinate
List of Symbols (continued)

t  
time variable; thickness of wall

u  
instantaneous particle velocity

u₀  
particle velocity amplitude

uₘ  
local particle velocity amplitude

ur  
radial component of particle velocity

uᵦ  
circumferential component of particle velocity

uz  
longitudinal component of particle velocity;
particle velocity amplitude as function of z

z  
longitudinal coordinate

A  
area

E  
energy

Eₖ  
mean kinetic energy

E₀  
energy in resonator

Eₛ  
energy in sample

Eₗ  
rate of energy loss

Eₗₜ  
energy loss from viscous damping at wall

Eₗₜₜ  
energy loss from viscous damping at bottom

Eₗₚ  
energy loss to wall material

Eₛ  
energy loss in sample

F  
force; rate of energy loss per unit volume

I  
current

I₂  
quadratic strain invariant

Jₘ(x)  
Bessel function of order m

K  
compliance

K'  
mode compliance

M  
mass

M'  
mode mass

Q  
quality factor

Re  
real component of
List of Symbols (continued)

$S$ surface area
$S_{ij}$ strain tensor
$S_{,ij}$ strain rate

$T$ time; kinetic energy density; transmission factor
$T_{ij}$ stress tensor

$V$ volume; voltage
$V_0$ volume of resonator
$V_s$ volume of sample

$W$ potential energy density

$\alpha$ attenuation in a liquid (water)

$\beta$ Korteweg-Kuhl frequency reduction factor

$\gamma_{m,v}$ root of Bessel function $J_m(\gamma_{m,v}) = 0$
$\gamma'_{m,v}$ root of Bessel function $J'_m(\gamma'_{m,v}) = 0$

$\delta$ differential operator; decay rate
$\delta_0$ decay rate of resonator
$\delta_s$ decay rate component attributable to sample
$\delta_{ij}$ unit tensor

$\zeta$ acoustic impedance ratio

$\eta$ loss factor
$\eta_E$ Young's modulus loss factor
$\eta_K$ bulk modulus loss factor (of sample)

$\lambda$ wavelength; elastic modulus; Lamé's "dilatational" constant
$\lambda_E$ Young's modulus
$\lambda_G$ shear modulus
$\lambda_K$ bulk modulus
$\lambda_w$ bulk modulus of liquid (water)
$\lambda_v$ Lamé's volume viscosity coefficient
$\lambda'$ complex modulus
List of Symbols (continued)

\(\mu\) Lamé constant (shear modulus)

\(\mu_1\) Lamé's shear viscosity coefficient

\(\nu\) order of root of Bessel function

\(\rho\) density

\(\rho_0\) density of liquid of resonator

\(\sigma\) Poisson's ratio

\(\tau\) fractional volume change (volume displacement)

\(\phi\) angular coordinate

\(\psi\) velocity potential

\(\omega\) angular frequency

\(\Delta\) differential operator

\(\dot{\Delta}\) dilatational rate
The two elastic constants usually used to describe the properties of a linearly elastic homogeneous and isotropic medium are the Lamé constants $\lambda$ and $\mu$ (described originally as "dilatational" and "shear" respectively, though the former term is a misnomer). From these all other constants of interest may be derived. The shear modulus can be obtained directly from torsional measurements, but there is no motion that is controlled exclusively by Lamé's dilational modulus, and its value must therefore be obtained from a measurement of one of the derived constants. In most of these the shear is the dominant factor, and the effects of inaccuracies in its determination can swamp the dilational term. The constant with the least component of shear is the bulk modulus $\lambda + 2 \mu/3$. Apart from its intrinsic importance therefore, this constant is of particular interest, as its evaluation is likely to lead to the most accurate determination of Lamé's "dilatational" modulus $\lambda$.

Internal losses can be taken into account by describing the vibration by complex variables and permitting the constants to assume complex values. In fact, it is convenient to express the complex bulk modulus as $\lambda_k = \lambda_k (1 + j\eta_k)$ where $\lambda_k$ is the real component and $\eta_k$ is the bulk modulus loss factor.

A knowledge of this loss factor is useful in the evaluation of polymers such as the acoustic "windows" or "domes" surrounding sonar transducers, where a low value is desired, or the materials used in the design of shock and vibration mountings, for which a high value may be preferred. More esoteric applications may be formed in agriculture, in the evaluation of the properties of various food materials such as grain, fruit and root vegetables, and perhaps in
medicine, in the examination of bone and tissue for possible defects. However, the measurement is difficult, especially on small, low-loss samples, as it is necessary to excite a purely volumetric vibration in the material in such a way that the inherent material losses are not exceeded by those of the driving system.

Several methods are described in the literature for the measurement of the dynamical properties of elastic materials. Nolle describes methods for the measurement of Young's modulus in the frequency range 0.1 Hz to 0.1 MHz using a resonance decay method. Kuhl and Meyer also describe such an apparatus for the frequency range 20 to 200 Hz, and for the attenuation of longitudinal waves at higher frequencies up to 15 kHz. Marvin, Aldrich and Sack describe measurements of the modulus governing propagation of longitudinal waves in a frequency range 0.7 to 7 MHz. Both of these suffer from the disadvantage alluded to above that the losses are small and that the shear losses dominate the dilatational losses, rendering the determination of the latter inaccurate. Using a vibration tester, Philippoff and Brodnyan made direct measurements of dynamic compressibilities at very low frequencies ($10^{-5}$ to 10 Hz); they attempted to determine the associated loss factor from the phase angle between stress and strain, but the results were somewhat indeterminate. Sharma and McCarty describe somewhat similar measurements in the range $10^{-4}$ to $10^{-1}$ Hz. McKinney, Edelman and Marvin made measurements in the frequency range 50 Hz to 10 kHz, using the pressure changes in a liquid filled cavity excited by a barium titanate disc. Heydemann used a similar apparatus in the range 10 Hz to 60 kHz. Wada, Hirose, Umebayashi and Otomo describe measurements of a sound velocity and attenuation in a suspension of powders at frequencies of 1/3, 1 and 3 MHz.
Meyer and Tamm\textsuperscript{9} used a method based upon acoustic measurements in a resonant fluid column. Both the frequency and the bandwidth of the resonance of the column were measured, followed by a similar measurement with the sample introduced and suspended at a node, enabling the properties of the sample to be derived. This was further developed by Sandler,\textsuperscript{10,11} by Cramer and Silver,\textsuperscript{12} and by Niemic.\textsuperscript{13} He makes use of the second longitudinal resonance (at about 1551 Hz) in a water filled steel tube about 8.2 mm thick, 11.4 cm inside diameter and 92 cm long; with this he has achieved a half-power bandwidth of 1.2 Hz, corresponding to a $Q$ of about 1300 (in the absence of a sample).

As an alternative to the measurement of bandwidth the decay of the resonance may be measured when the excitation is removed. Simon\textsuperscript{14} adopted this approach, using a standing wave in a U-tube so that both ends of the liquid column might be free. The driver was above one end, the receiver above the other end; the test piece was placed at the bottom of the U-tube. Because of the symmetry, the test piece was always at a pressure maximum, and the particle velocity at the test piece was zero. This method worked well, but it did not eliminate the main problem, which was the friction at the wall.

For the measurement of sound absorption in liquids Kurtze and Tamm\textsuperscript{15} and Karpovich\textsuperscript{16} used the decay of a purely radial mode of resonance in a spherical vessel in order to avoid viscous losses due to friction at the walls. Simon\textsuperscript{14}, in his later experiments, used a spherical vessel suspended in a vacuum jacket to minimize radiation losses. The arrangement is illustrated diagramatically in Figure 1. The main problem in this method was that of resonances of the walls. These resonances are always coupled with the resonances of the fluid;
Figure 1. Simon's Spherical Vessel Suspended in a Vacuum Jar (from ref. 14).
for some of the resonances most of the energy is located in the water, for others in the walls, and it is very difficult to decide what type of resonance is excited. Figure 2 shows the resonances Simon obtained when the driver was at the wall and the receiver at the centre; none of the resonances could be identified as being a specific mode in the water. To reduce the response to resonances in which the energy was mainly in the wall vibration Simon then used the same transducer at the centre both for driving and for monitoring the decay; figure 3 shows the improvement obtained, though some of these remaining resonances may still have been wall resonances, especially in view of the inevitable asymmetries because of the existence of the neck. With another vessel Simon was able to achieve a $Q$ of about 25 000, in the absence of a sample, and was able to obtain a figure for the loss factor of a sample introduced at the centre (with the transducer moved to a point above it). However, the difficulty of searching for the resonance when the same transducer had to be used for reception as for excitation (with a send/receive key), and of deciding that it was indeed a volume and not a wall resonance, rendered the method impractical. It was finally concluded that the wall modes can only be suppressed by making the walls either very thick or very thin. A solution in between was not possible.

Reinhardt$^{17}$ repeated Simon's measurements with a slightly different arrangement and reported that $Q$'s of the order of 80 000 were attainable. He then attempted to dispense with the vacuum jacket by using a thick-walled vessel, which was not itself expected to be excited, and was able to achieve a $Q$ of 50 000 in this way. However, asymmetries due to the neck of the container and the weld at its equator remained, and still introduced a great number of resonances which tended to obscure the radial resonance in the liquid.
Figure 2. $Q$ as a function of frequency for the resonances found in water in a sphere when driven at the wall and monitored at the centre. (From Simon, ref. 14)
Figure 3. $Q$ as a function of frequency for the high amplitude resonances in water in a sphere when driven and monitored by a single transducer at the centre. (From Simon, ref. 14)
He concluded that the complexity of the experiment was not justified, and to avoid these problems he made a few exploratory measurements using cylindrical vessels, which, however, were not successful. A much more thorough investigation was clearly desirable.
II. OBJECTIVES AND PRACTICAL CONSIDERATIONS

The intent of the work now to be described was to develop a practical method that is simple and reliable, and which could be used by a reasonably skilled technician in any industrial or medical laboratory. We were no longer necessarily interested in the attainment of such high Q's as 80 000 as Simon and Reinhardt had achieved. The vacuum jacket was to be dispensed with; spherical containers were considered impractical, for the reasons discussed above. The work was therefore confined to vessels of cylindrical form, and with the objective of finding the best length to diameter ratio and the optimum wall thickness.

The essence of the experiment is the measurement of the change in damping when a sample is introduced into a volume of liquid excited into resonance. For this to be an effective procedure it is essential that the experiment be designed so as to minimize the losses inherent in the apparatus and that any resonance which is to be used be discrete, easily identified and separable from all other resonances in the liquid and surrounding structure.

The density of resonances in a cylindrical tube increases greatly above the frequency of the first resonance with a radial component. This "cut-off" frequency for a long rigid walled tube is given by

\[ f = 0.293 \frac{c}{a} \]

where \( a \) is the radius and \( c \) the sound velocity. [The resonance modes are discussed in section 3.8. The cut-off frequency is obtained from equation 3.8.6 for the lowest non-zero root of \( J_m'(\gamma) = 0 \), setting \( k_z = 0 \).] The highest frequency that is practical for decay
measurements is of the order of magnitude of this frequency (for
which the diameter \( 2a = 0.58\lambda \)), perhaps two or three times this
frequency if the tube is short and the damping is low. The lowest
frequency is determined by the length of the tube; at this fre-
quency a tube open at both ends is half a wavelength long.

For a tube with thin (resilient) walls there can be no purely
longitudinal mode of vibration; the lowest possible resonance fre-
quency for a long tube with a resilient wall is

\[
f = \frac{0.384}{c/r}
\]

for which \( d = 2a = 0.77\lambda \), and again the density of resonances in-
creases rapidly above this frequency.

It is clear, therefore, that any particular piece of apparatus
can only be of use in a specific frequency band, and furthermore
that the range of frequencies in which this technique is at all
useful is quite limited.

The principal losses were expected to be losses in the material
of which the vessel is made, acoustic radiation from the outside of
the vessel, and viscous losses in the boundary layer of the liquid
near the walls of the vessel due to the relative motion between the
wall and the main body of the liquid. The minimization of these
gives rise to conflicting requirements, and a "trade-off" becomes
necessary.

Viscous losses will be minimized if the wall is very thin and
highly resilient, so that there is little relative velocity between
the liquid and the wall material. However, this "thin wall" approach
implies that the wall material itself is in motion, and the losses
in the wall material itself become intolerably great since the loss
factor in metals is always much greater than that in liquids.
Furthermore, the resonances in the wall are strongly coupled to the liquid, rendering the whole pattern of resonances more confusing.

If, on the other hand, the wall were completely rigid there would be no problem with losses in the wall material or by acoustic radiation, nor would support of the vessel create any difficulty. There would, however, be relative motion in the boundary layer, and viscous losses would therefore be significant. The main problem with this "thick wall" approach is that the acoustic impedance of solid materials from which a vessel might be made is only of the order of 10 to 30 times that of liquids, so that a considerable proportion of the acoustic energy is transmitted into the wall and dissipated there. A crude estimate is made in Appendix A of the energy loss from a liquid-filled cavity in an infinite solid medium, and it is seen that only a very low Q, about 6, can be expected. Thus the wall thickness must be finite; there should be an optimum value for this thickness for each size of tube and each type of wall material.

It was decided to attempt to evaluate both approaches, experimenting both with thick and with thin walled tubes. Three types of vessel were constructed, one with both thin walls and thin bottom, one with a thin wall but a relatively thick bottom, and lastly a type with both thick walls and thick bottom. In each case the top surface of the liquid was free, though a cover did in fact close off the air space above.

There may be additional losses in the driving and monitoring transducers; it was felt that this could be minimized by using relatively loose coupling to them, the resulting loss of sensitivity not being significant.
Losses in the suspension of the vessel (and the effects of resonances in that suspension) must be minimized, particularly in the thin-walled case, for example by tuning with additional components to form a mechanical low-pass filter. Losses in the body of the liquid due to its viscosity and to thermal relaxation were expected to be negligible in comparison with other unavoidable losses (the attenuation of sound in water is only of the order of 1 dB/km at 100 kHz).

Water is certainly the most convenient liquid to choose for the experiments, though as will be seen later considerable care is required in its preparation. Simon reports one experiment in his U-tube apparatus using DC 200 65 centistoke silicone fluid (from Dow Corning Company, Midland, Michigan); with a reported specific gravity of 0.761 this has a viscosity in SI units of 0.0005 Pa.s, compared with 0.001 Pa.s for water. This low viscosity fluid is extremely volatile, and the rapid evaporation caused a drift in resonant frequency, rendering the experiment difficult (and expensive). Though there is considerable scatter on his experimental data, he did achieve a small reduction in loss factor of the apparatus, around 25%, in conformance with his theoretical predictions. The principal advantage to the choice of a different liquid would seem to be in the measurement of materials which float in water; however, in view of the practical difficulties involved in cleaning the apparatus after a change of liquid, water was used throughout the experiments to be described.

The most important cause of loss, and the one which is likely give rise to the most variability in the results, is from air bubbles adhering to the walls of the vessel or to the test sample. This problem is discussed in more detail in another section. Here it will
suffice to say that results of any measurements are bound to be sus-
pect if there is a possibility of air bubbles being present, and in
fact this particular point was the cause of considerable difficulty
and delay in the experimental programme.
III. THEORETICAL CONSIDERATIONS

3.1 Outline of Approach

We are especially interested in two types of vessel. We expect that the effect of wall vibrations coupled to those of the fluid will be minimized with either

a) thick-walled vessels, that are stiff compared to the elastic impedance of the water

or b) very thin-walled vessels, in which the losses should be small because of their low acoustic impedance.

The first method has the advantage of eliminating losses due to acoustic radiation in the surrounding air; the second could be expected to eliminate much of the viscous friction at the layer near the wall, because the tangential velocity of the fluid vanishes at a pressure-release boundary.

The tool for investigating both types of arrangement is the calculation of the normal mode structure. Accordingly, in this chapter we will first discuss the use of two independent (Lamé) constants in the classical theory of elasticity, and the extension to the use of complex constants to take losses into account. We will then show how the decay in a resonance when the excitation is removed is related to these losses, and discuss the effect on the decay rate produced by inserting a lossy specimen into the system. We will continue with a discussion of the effect of the shape of the vessel on the absorption in the liquid, and then proceed to a detailed calculation of the normal modes in the liquid contained in a cylindrical vessel with completely resilient or absolutely rigid walls and their tabulation for specific dimensions. We will calculate the kinetic energy contained
in the radially symmetric modes and use the perturbation method to estimate the losses in the material of a thin-walled (nearly resilient) vessel and the frictional losses at the walls of a thick (nearly rigid) vessel. These calculations will enable us to select the dimensions and the mode with the smallest loss factor (or highest Q).

In practice no wall can be either completely resilient or absolutely rigid. A tube of finite thickness approximates more closely to the rigid case, and Korteweg\textsuperscript{18} has derived a correction factor to the velocity of propagation in a longitudinal mode to take the thickness and elasticity of the wall into account. Field\textsuperscript{19} and others tried to improve upon the simple Korteweg formula by taking radial resonances of the tube into account also. Kuhl\textsuperscript{20}, however, has shown that radial resonances are never excited in liquid-filled thick walled tubes at low frequencies, and that the Korteweg formula accurately describes sound propagation in liquid filled tubes up to frequencies that are considerably higher than the first radial mode of the side wall. Only in extremely thin-walled tubes do radial modes influence the sound propagation; in others the bending stiffness of the tubes seems to counteract the excitation of any modes which are neither purely longitudinal nor purely radial.
3.2' The Stress-Strain Relationship in the Classical Theory of Elasticity

In the classical theory of elasticity a linear relationship is assured between the components of the stress tensor $T_{ij}$, and the strain tensor $S_{kl}$, expressed by the generalized form of Hook's Law,

$$T_{ij} = \lambda_{ijkl} S_{kl}$$  \hspace{1cm} (3.2.1)

The assumption of linearity enables the principle of superposition to be applied to successive strains.

In principle there are 81 ($3^4$) values for the elastic constants $\lambda_{ijkl}$. However, considerations of symmetry reduce these to 21 values. In the case of an isotropic material with no elastic after effects it can be shown that of the 81 coefficients only 15 are non-zero, and furthermore that these fifteen coefficients have the values $\lambda$, $\mu$, or $(\lambda + 2\mu)$; the two independent constants $\lambda$ and $\mu$ are known as the Lamé' constants.\textsuperscript{21,22}

It can be shown that the Lamé constant $\mu$ is the same as the shear modulus (or modulus of rigidity) $\frac{G}{2}$, and thus a direct method of measurement of its value may be devised. Every other commonly used modulus of elasticity involves both $\lambda$ and $\mu$, for example Young's Modulus $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$, the bulk modulus or modulus of compression $K = \lambda + 2\mu/3$, and Poisson's ratio $\sigma = \lambda/2 (\lambda + \mu)$. Thus there is no practical method for the direct measurement of the value of $\lambda$, which must be obtained indirectly using measured values for $\mu$ and for another constant; since the bulk modulus contains the smallest component of shear, the determination of this modulus is of particular interest.
3.3 The Definition of a Complex Bulk Modulus

The static bulk modulus is defined as

\[ \lambda_K = \frac{\Delta p}{\Delta V/V} \]  \hspace{1cm} (3.3.1)

where \( \Delta p \) is the change in pressure, and \( \Delta V/V \) the fractional change in volume.

The same definition can be extended to the dynamic modulus. Suppose there is a cyclic variation of pressure of angular frequency \( \omega \) and pressure amplitude \( p_m \), written \( p = p_m \cos \omega t \), or in complex notation \( p = p_m e^{j\omega t} \), and an associated fractional change in volume \( \tau = \tau_m e^{j(\omega t - \phi)} \), where \( \phi \) is the phase difference between pressure and volume change; then a complex modulus may be defined as

\[ \overline{\lambda}_K = \frac{p_m e^{j\omega t}}{\tau_m e^{j(\omega t - \phi)}} = \frac{p_m}{\tau_m} e^{j\phi} \]  \hspace{1cm} (3.3.2)

It is usual to write \( \overline{\lambda}_K = \lambda_K e^{j\eta_K} \), where \( \eta_K \) is called the loss factor, so that \( \lambda_K = p_m/\tau_m \) and \( \phi = \eta_K \). If the losses are relatively small, as is usually the case, we may expand \( e^{j\eta_K} = (1 + j\eta_K + \ldots) \), and ignoring the higher order terms write

\[ \overline{\lambda}_K = \lambda_K (1 + j\eta_K) \]  \hspace{1cm} (3.3.3)

The relationship of various elastic constants to each other and to the Lamé constants has already been discussed. These relationships can be extended for the case of complex constants. For example, it can be shown that the following relationship should hold between
the "imaginary", loss factor, components:

\[ \eta_K = \frac{3m \eta_E - 2(m + 1) \eta_G}{m^2 - 2} \]  \hspace{1cm} (3.3.4)

where \( m \) is the reciprocal of the Poisson's ratio,
\( \eta_E \) is the loss factor of the Young's modulus,
\( \eta_G \) is the loss factor of the Shear modulus
and \( \eta_K \) is the loss factor of the Bulk modulus.

3.4 Resonance Decay in a system Containing a Lossy Specimen

The decay rate of any damped resonant system of loss factor \( \eta \) and angular frequency \( \omega \) may be written as

\[ \delta = \omega \eta / 2 = \dot{E} / 2E \text{ nepers/unit time} \]  \hspace{1cm} (3.4.1)

We note also that \( \delta = \omega / 2Q \), where \( Q \) is the quality factor of the system.

The power dissipated in a specimen of volume \( V_s \) is the work done per unit time, i.e.

\[ E_s = \frac{V_s}{T} \int_0^T p_m \cos \omega t \cos (\omega t - \eta_K) \, dt \]  \hspace{1cm} (3.4.2)

Here \( p_m \) and \( \tau_m \) are the local maxima of the pressure and of the volume change respectively. Remembering that when the integration is carried out over a time that is long compared with the period of oscillation, \( \frac{1}{T} \int_0^T \cos^2 \omega t \, dt = \frac{1}{2} \) and \( \frac{1}{T} \int_0^T \cos \omega t \sin \omega t \, dt = 0 \), we
obtain

\[
E_s = \frac{1}{2} V_s p_m \tau_m \omega \sin \eta_K = \frac{1}{2} V_s p_m \tau_m \omega \eta_K
\]

\[
= \frac{1}{2} V_s p_m^2 \omega \eta_K / \lambda_K
\] (3.4.3)

It is our purpose to determine this energy loss \( \dot{E}_s \) from the change in decay rate when it is inserted into a measuring vessel. Using the suffix \( \circ \) to indicate properties of resonant system in the absence of the sample and the suffix \( s \) for those of the sample alone, we may write the decay rate in the presence of the sample as

\[
\delta_{s+\circ} = (\dot{E}_s + \dot{E}_\circ)/(E_s + E_\circ)
\]

We assume that the specimen is small, so that its presence perturbs the resonance only slightly and that its energy \( E_s \) in the denominator may be neglected. We may then write

\[
\delta_s = \delta_{s+\circ} - \delta_\circ = \dot{E}_s / E_\circ
\]

and we merely need to calculate the energy contained by the fluid in the vessel. Though we shall estimate the contribution of some individual items to the total system loss, they are all in fact taken care of by the single measurement of the decay rate \( \delta_\circ \) in the absence of the specimen.

The principle may best be illustrated in terms of a simple case, that of a purely longitudinal mode of resonance of a body of liquid contained by a vessel with either completely free or completely rigid end boundaries. For this we may write the local velocity amplitude

\[
u_z = u_0 \sin (kz), \text{ where } z \text{ is a longitudinal dimension, } u_0 \text{ is the peak velocity amplitude, and } k = \omega/c \text{ is the wave number. The kinetic energy over an elemental length } dz \text{ and cross-section } A \text{ is then } \frac{\rho A dz}{2} \frac{u_z^2}{2} \text{ (the additional factor } \frac{1}{2} \text{ being necessary to obtain the mean square value over a period of oscillation). If the resonator is of total length } h, \ V = Ah \text{ and } kh = n\pi/2, n \text{ being an integer.
which is even when both ends are similar (i.e., both open or both closed) and is odd when they are dissimilar. Thus, the mean kinetic energy

\[ E_K = \int_0^{z_0} \frac{\rho A u^2}{4} dz \]

\[ = \frac{\rho A u^2}{4} \int_0^{z_0} \sin^2(kz) dz \]

\[ = \rho A h u_0^2 / 8 \]

\[ = p_o^2 V o / 8 \rho c_s^4. \quad (3.4.4) \]

For other than longitudinal modes of resonance the energy is different; it is convenient to replace the factor \( \frac{1}{2} \) which arises from the integral in equation (3.4.4) by a "mode factor" \( q \), defined as the ratio of the space average of the square of the velocity to the square of the peak velocity; this of course has the value unity for a simple "lumped-constant" case, and will have other values for other modes of resonance (it is calculated for radial-longitudinal modes in a cylinder in section 3.9).

Since at resonance the mean potential energy equals the mean kinetic energy, we may write the total energy as

\[ E_o = q p_o^2 V o / 4 \rho c^2 = q p_o^2 V o / 4 \lambda_w, \quad (3.4.5) \]

where \( \lambda_w = \rho c^2 \) is the bulk modulus of the liquid.

Using equations (3.4.3) and (3.4.5) we may now write

\[ \delta_s = \frac{\rho A u^2}{q V o} \frac{\lambda}{\lambda_w} \eta_K, \quad (3.4.6) \]
from which

\[ \eta_K = \frac{q}{\omega} \frac{V_0}{V_s} \frac{\lambda_K}{\lambda_w} \delta_s \]  

(3.4.7)

3.5 Change in Frequency due to Insertion of a Specimen

It is evident from the foregoing section that in order to determine the bulk modulus loss factor \( \eta_K \) of the sample we must know the ratio of its bulk modulus \( \lambda_K \) to that of the fluid \( \lambda_w \). This we may measure from the change in resonant frequency when the sample is inserted. There are two effects which must be considered, firstly that due to the difference in modulus between the specimen and the water it displaces, which we can calculate using the mode mass and mode compliance, and secondly that due to the added volume of the resonator.

For the purely longitudinal resonator, the mode mass is defined as \( M' = qM \), where \( M \) is the total mass of the fluid and \( q \) is the mode factor, defined above as the ratio of the space average of the square of the velocity to the square of the peak velocity. The mode compliance is defined as \( K' = 1/\omega^2 M' \), where \( \omega \) is the angular resonant frequency. It must be noted that this definition is in terms of the resonant frequency and not of the static compliance \( K \), though they are proportional to each other. We may also write a "distributed" compliance \( K'' = gK \), where \( g \) is defined in an analogous manner as the ratio of the space average of the square of the (volume) displacement to the square of the peak displacement, which for longitudinal modes has the same value \( \left(\frac{1}{2}\right) \) as does \( q \).
Thus for the resonator in the absence of the sample we may write the distributed compliance as

\[ K''_0 = \frac{V}{2\lambda_w} \]  \hspace{1cm} (3.5.1)

When a small sample is introduced at the pressure maximum (velocity node) it can have no effect on the mode mass, since it contains no kinetic energy; however, its effect on the compliance is not subject to averaging. Thus, provided it is small and its modulus is not too different from that of the fluid we may write the perturbed effective compliance as

\[ K'' = \frac{1}{2} \frac{V}{\lambda_w} + \frac{V_s}{\lambda_s} - \frac{V}{\lambda_w} \]  \hspace{1cm} (3.5.2)

If the new angular frequency is \( \omega' \), we may then write

\[ \frac{\omega^2 M'}{\omega_0^2 M} = \frac{K'}{K''} \hspace{1cm} \frac{\omega^2 M'}{\omega_0^2 M} = \frac{K'}{K''} \]

\[ \frac{1}{2} \frac{V}{\lambda_w} + \frac{V_s}{\lambda_s} - \frac{V}{\lambda_w} \]

or, by Taylor expansion,

\[ \omega_1 = \omega_0 \left\{ 1 - \frac{V_s}{V_0} \left( \frac{\lambda_w}{\lambda_s} - 1 \right) \right\} \] \hspace{1cm} (3.5.4)

The change due to the added volume is simply computed by logarithmic differentiation of the frequency relation \( kh = n\pi/2 \) or \( \omega h = n\pi c/2 \). This gives \( \frac{\delta \omega}{\omega} = -\frac{\delta h}{h} = -\frac{V_s}{V_0} \). We may thus write
the perturbed resonant frequency \( \omega'' \) due to the added volume as

\[
\omega'' = \omega' \left(1 - \frac{V_s}{V_0}\right) .
\] 

(3.5.5)

Combining this with (3.5.4) we obtain for the new frequency

\[
\omega_s = \omega_0 \left[1 - \frac{V_s}{V_0} \left(\frac{\lambda_w}{\lambda_k} - 1\right)\right] \left(1 - \frac{V_s}{V_0}\right)
\]

\[
= \omega_0 \left[1 - \frac{V_s}{V_0} \frac{\lambda_w}{\lambda_k}\right] .
\] 

(3.5.6)

We can thus write the proportional change in frequency as

\[
\frac{\Delta f}{f} = \frac{\Delta \omega}{\omega} = -\frac{V_s}{V_0} \frac{\lambda_w}{\lambda_k} ,
\] 

(3.5.7)

the negative sign indicating a decrease in frequency. It is implicit in this analysis that the loss factors \( \eta_k \) and \( \eta_w \) associated with the appropriate bulk modulus are themselves small.

Meyer and Tamm\(^9\) obtained the same result using a transmission line approach. The advantage of the present method is that it is in principle applicable to other than purely longitudinal modes of vibration, though the actual expressions become more complicated.

[The factors \( \frac{1}{2} \) in equations (3.5.2) and (3.5.3) have to be replaced by the appropriate factor \( g \), and equation (3.5.5) must be replaced by a new expression based upon the appropriate frequency equation; for the radial-longitudinal modes discussed in sections 3.7 and 3.8 it can easily be shown that \( g = q \), and that the term \( V_s/V_0 \) in equation (3.5.5) should be multiplied by the factor \( k_z^2/k^2 \).]
For the purely longitudinal modes we can combine equation (3.4.7) with (3.5.7) and obtain

\[
\lambda_K = -\lambda_w \frac{V_b}{V_0} \frac{f}{\Delta f} \tag{3.5.8}
\]

and

\[
\eta_K = \frac{1}{4\pi f} \frac{V_b}{V_s} \frac{\lambda_K}{\lambda_w} \frac{\delta_s}{\delta_s} \tag{3.5.9a}
\]

\[
= \frac{1}{4\pi} \frac{\delta_{s+f} - \delta_s}{\Delta f} \tag{3.5.9b}
\]

(An additional factor of 0.115 may be inserted in the numerator of equation (3.5.9) if the decays are to be measured in decibels rather than in nepers.)

This is the pair of equations we shall use to calculate \( \lambda_K \) and \( \eta_K \) from experimental results. Two points become evident. The first is that the losses in the system govern a lower limit to the value of the loss factor \( \eta_K \) which can be measured for a given size of specimen, and that for some relatively loss-free materials a fairly large volume will be necessary. The second was less obvious, that the size of the sample required is determined largely from the accuracy with which the change in frequency can be measured, but that the actual volume does not need to be known if merely the loss factor is required, a convenience when the samples may be of irregular shape. These points will be discussed further in section 4.6.
3.6 Energy in the Vibrating Fluid - General Discussion

The energy contained in the vibrating fluid consists of two parts, potential and kinetic. It is usual to assume that these are equal at resonance. However, this assumption warrants further discussion, especially in the case where the boundary conditions are not ideal. Greenspan has investigated this situation in a critique of the use of the resonator-decay method for the measurement of absorption of sound in liquids. Following his analysis, we may write the strain rate at a particular point in the fluid as

\[ \dot{S}_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) = -\psi,_{ij} \quad (i = 1, 2, 3) \quad (3.6.1) \]

It is convenient to use the summation convention. The \( i \)'s and \( j \)'s denote the three coordinates in the chosen coordinate system (cylindrical in our application). Here \( u_i \) is the velocity component, given by

\[ u_i = -\psi,i \quad (3.6.2) \]

\( \psi \) being the velocity potential and the comma in the suffix indicating differentiation. The dilatation rate is

\[ \dot{\lambda} = \ddot{S}_{ii} = -\psi,_{ii} = -\nabla^2 \psi = k^2 \psi \quad (3.6.3) \]

(using the wave equation \( \nabla^2 \psi = k^2 \psi \), where \( k = \omega/c \) is the wave number).

Following Greenspan's derivation, the stress tensor is given by

\[ T_{ij} = (-p + \lambda \dot{\lambda}) \delta_{ij} + 2\mu \dot{S}_{ij} \quad (3.6.4) \]

in which \( T_{ij} \) is the stress, \( \lambda \), and \( \mu \), are the two viscosity coefficients, and \( \delta_{ij} \) is the unit tensor.

The strain (or potential) energy density is given by
In this equation $S_{ij}$ is in quadrature with both $\dot{S}_{ij}$ and with $\dot{\lambda}$; thus these two product terms are zero and we have

$$2W = p\delta_{ij}S_{ij}$$

$$= p\Delta$$

$$= p\Delta/j\omega$$

$$= \rho k^2 \psi^2$$  \hspace{1cm} (3.6.6)

(from equation (3.6.3) above).

The kinetic energy density is

$$T = \rho u_i u_i/2 = \rho \psi_i \psi_i/2.$$  \hspace{1cm} (3.6.7)

Using one step in the derivation of Green's theorem

$$\int_V \psi_i \psi_i dV + \int_V \psi V^2 \psi dV = -\int_S \psi \frac{d\psi}{dn} dS,$$  \hspace{1cm} (3.6.8)

we obtain for the steady state condition

$$\int_V (T - W) dV = -\frac{\rho}{2} \int_S \psi \frac{d\psi}{dn} dS$$

$$= Re \left( \frac{\rho}{2j\omega p} \int_S p \bar{u}_r dS \right)$$  \hspace{1cm} (3.6.9)

(from equation (3.6.2)).

Thus the total energy in the fluid is

$$E = \int_V (T + W) dV = \int_V [2W + (T - W)] dV$$

$$= \int_V \rho k^2 \psi^2 dV - \frac{\rho}{2} \int_S \frac{d\psi}{dn} dS$$  \hspace{1cm} (3.6.10)

The difference term in equations (3.6.9) and (3.6.10) represents the interaction between the fluid and the walls. In the special cases...
of resonance in a completely rigid or perfectly resilient vessel such as we have been considering this difference term is zero, since in the one case the normal velocity, proportional to $\frac{d\psi}{dn}$, is zero and in the other the pressure (proportional to $\psi$) is zero, so we can write without approximation

$$E = \int \rho k^2 \psi^2 dV \quad (3.6.11)$$

In all other cases interaction occurs between the fluid and the walls. However, we can justify the use of equation (3.6.11) on the basis that we are really interested in the energy contained in the total system when we apply equation (3.4.6), not merely that in the fluid alone. At resonance there is always a cyclic transfer between kinetic energy and potential energy. A thick, nearly rigid, wall will carry relatively little kinetic energy due to the low velocity, and a thin, almost perfectly resilient, wall will carry little because of its low mass. Thus for practical systems we will assume that although the potential energy may be shared between the wall material and the fluid, the kinetic energy resides mostly in the fluid, and that to a good approximation equation (3.5.11) applies.

3.7 Energy Losses in the Fluid - General Discussion

The loss factor of the fluid describes its average absorption over a distance of exactly one wave length, or over many wave lengths. If the losses are viscous, maximum energy is dissipated in the regions of the fluid where the velocity gradients are greatest. If the medium has the properties of a solid, hysteretic losses will usually dominate and most of the energy dissipation will take place in regions of the strain maxima.

If the boundary is non rigid, the relative distribution of
velocity maxima and minima will depend on the impedance of the boundary. For a rigid boundary, the velocity is zero near it; for a pressure release boundary it is a maximum. Thus the energy dissipation is a function of the vibration pattern in the medium and the shape of the container. It is well known that for an infinite sound wave, the energy dissipation is described by the two viscosities $\lambda_1$, $\mu_1$ in the combination $\lambda_1 + 2\mu_1$, whereas for a small particle it is given by $\lambda_1 + 2/3\mu_1$. To investigate this situation, we have to start out with the Stokes Navier equations or with equivalent equations on the basis of the energy principle.

Accordingly we write the dissipation function for the fluid, i.e. the rate of energy loss per unit volume, as

$$2F = T_{ij} \dot{S}_{ij} = (-p + \lambda \dot{\omega}) \delta_{ij} \dot{S}_{ij} + 2\mu \dot{S}_{ij} \dot{S}_{ij} \quad (3.7.1)$$

We note that $p$ is in quadrature with $S_{ij}$. Expanding the remaining terms we obtain

$$2F = \lambda \dot{\omega} S_{ij} + 4\mu (\dot{S}_{12}^2 + \dot{S}_{23}^2 + \dot{S}_{31}^2 - \dot{S}_{11} \dot{S}_{22} - \dot{S}_{22} \dot{S}_{33} - \dot{S}_{33} \dot{S}_{11}). \quad (3.7.2)$$

From equation (3.6.6) this can be written as

$$2F = (\lambda + 2\mu) k^2 \psi^2 + 4\mu I_2 \quad (3.7.3)$$

Here,

$$I_2 = \dot{S}_{12}^2 + \dot{S}_{23}^2 + \dot{S}_{31}^2 - \dot{S}_{11} \dot{S}_{22} - \dot{S}_{22} \dot{S}_{33} - \dot{S}_{33} \dot{S}_{11}. \quad (3.7.4)$$

is the quadratic strain invariant.

The attenuation in a travelling wave may be written

$$\alpha = \dot{E}/2cE \text{ nepers/unit distance}.$$
Using (3.6.6)

\[ \alpha = \frac{\int f dV}{c \int f dV} = \frac{\lambda + 2\mu}{2\rho c^3} + \frac{2\mu \int f I dV}{\rho c^2 \int f^2 dV} \]  \hspace{1cm} (3.7.5)

Two special cases can be noted immediately. Firstly, in the case of a plane wave in the \( x_1 \) direction, say, only \( S_{11} \) in (3.7.4) is non-zero. Then \( I_2 = 0 \), so that (3.7.5) reduces to

\[ \alpha = (\lambda + 2\mu)\omega^2/2\rho c^3 \]  \hspace{1cm} (3.7.6)

the known result for attenuation in a plane wave. It can thus be seen that the condition for the resonator-decay experiment to yield the plane wave result is that \( \int I_2 dV = 0 \).

Secondly, in the case of a uniform expansion the shear rates \( S_{12} \), etc., in (3.7.4) are zero and \( \dot{S}_{11} = \dot{S}_{22} = \dot{S}_{33} = \dot{S} \), say. Then the dilational rate \( \dot{\lambda} = 3\dot{S} \), and \( I_2 = -1/3 \dot{\lambda}^2 \). Using (3.6.3), equation (3.7.5) becomes

\[ \alpha = (\lambda + 2\mu/3)\omega^2/2\rho c^3 \]  \hspace{1cm} (3.7.7)

as would be expected, \( \lambda + 2\mu/3 \) being the compressional viscosity.

Greenspan examines the situation for spherical, cylindrical and rectangular vessels, all of them on the assumption that equation (3.6.11) is true. For the spherical vessel he shows that \( I_2 = 0 \) when the vessel has rigid walls, so that equation (3.7.6) applies and the experiment yields the plane-wave value for the attenuation in the liquid. However, for "pressure-release" walls, the only other case that can easily be analysed, and for which equation (3.6.11) applies rigorously, the second term in the right-hand side of (3.7.5) results in a difference between the measured and plane-wave values; he estimates that the measurement is 20\% lower for the first order mode, but that the difference decreases to less than 1\% for modes of order greater than 5.
The analysis of the radial-axial modes in a cylindrical vessel is more complicated, as there are two sets of boundary conditions to be considered. These are discussed in more detail in section 3.7 which follows. On the basis of these boundary conditions, Greenspan concluded that a measurement in a cylindrical tube with rigid walls and each end either rigid or pressure release will yield the true plane-wave or free-space value for the attenuation. However, when the wall is resilient or pressure release, there is an error, which again is least for high order modes. The worst case arises in the case of the lowest order mode in a relatively long tube with a resilient wall, for which on the basis of his calculation we can estimate that the measurement will yield a result that is 17% too low.

It is clear from the foregoing that if we are to avoid awkward computations that require a knowledge of the exact behaviour of the walls we should use either very thick or very thin walls, and this is the approach which has been followed.

These calculations relate only to the viscous losses in the body of the fluid. Their purpose is to demonstrate that the use of the plane wave value for the attenuation in the liquid will result in an over-estimate rather than an under-estimate of the losses due to that cause. They do not take into account the viscous loss in the boundary layer adjacent to a rigid surface, nor the losses in the wall when this is not completely rigid, which will be discussed in the following sections. Here it will suffice to say that such losses will substantially outweigh losses in the body of the liquid for the low order modes which will be used in the experimental procedures in our particular application.
3.8 Normal Modes in the Liquid in a Cylindrical Vessel

We can now turn our attention in more detail to the case of a cylindrical vessel. We will assume that the boundary conditions are nearly ideal, that is that the walls are either very rigid or very resilient.

The solution of the wave equation may be written in terms of cylindrical coordinates $r$, $\phi$, and $z$ as

$$p = \sum_m p_m J_m(k r) \cos(k z + \xi_m) \cos(m\phi + \alpha_m) e^{jwt},$$

where

$$\omega^2/c^2 = k^2 = k_z^2 + k_r^2;$$

(3.8.1)

here $k$ is the wave number and $J_m$ the Bessel function of the first kind of order $m$. The second solution for the wave equation involves the Neumann function $N_m(k r)$ and is eliminated by the condition that the pressure remains finite at $r = 0$. For continuity around the walls of the cylinder $m$ must be zero or a positive integer.

Since we will be dealing with individual modes, we will, in what follows, take the sinusoidal time dependence $e^{jwt}$ as understood, and without loss of generality can omit the phase terms $\xi_m$ and $\alpha_m$ if we regard the origins of the $z$ and $\phi$ coordinates as arbitrary. We can thereby write the equation for a single mode somewhat more simply as

$$p = p_0 J_m(k r) \cos(k z) \cos(m\phi).$$

(3.8.2)

In a vessel with an open surface and a thin (resilient) bottom, the pressure is zero at these points, so that

$$k_z h = m\pi,$$

(3.8.3)
where $h$ is the height of the liquid and $n$ is an integer.

If on the other hand the vessel has a thick (rigid) bottom, the normal velocity, and therefore the pressure gradient, must be zero, at the surface, so that

$$k_{z}h = (2n-1) \pi/2 .$$  \hspace{1cm} (3.8.4)

At the walls of the vessel similar considerations apply. For a thin-walled vessel of radius $a$ the boundary condition is

$$J_{m}(k_{r}a) = 0 ,$$

or

$$k_{r}a = \gamma_{m,v} ,$$  \hspace{1cm} (3.8.5)

where the values of $\gamma_{m,v}$ are the roots of $J_{m}(\gamma) = 0$.

For a thick-walled vessel the condition is

$$u_{r} = \frac{i}{kpc} \frac{\partial p}{\partial r} = 0 ,$$

or

$$k_{r}a = \gamma'_{m,v} ,$$  \hspace{1cm} (3.8.6)

where $\gamma'_{m,v}$ are the roots of $J'_{m}(\gamma) = 0$.

By the use of the above relations and of tabulations for the values for $\gamma$ and $\gamma'$ such as can be found in standard texts, the frequencies of the normal modes for any particular arrangement can readily be calculated.

In actual experiments we must be sure of the relative pressure amplitudes at the location of the sample. Thus we are restricted to the use of the radially symmetric modes for which $m = 0$. For the higher order modes, there is a pressure null on the axis, and even if it were possible to locate the position of the pressure peaks they
would almost certainly move when a sample were introduced. It is important therefore that every resonant frequency found experimentally be properly identified and associated with a particular mode, or rejected as spurious (since the walls of the vessel cannot in practice be either ideally rigid or perfect "pressure-release" surfaces some coupling of wall resonances is to be expected). Though slide rule accuracy is adequate for the calculations, it was found convenient to make the tabulations by means of a computer programme, described in Appendix C.

3.9 Energy in a Normal Mode

We can rewrite equation (3.8.2) as

$$\psi = \psi_0 J_m(kr) \cos(kz) \cos(m\phi) \quad (3.9.1)$$

where $\psi$ is the velocity potential, related to the pressure by the Euler equation

$$p = \rho \frac{d\psi}{dt} ,$$

or, for the mode of frequency $\omega$,

$$\psi_0 = p_0 / j\omega$$

(3.9.2)

the $j$ indicating the quadrature phase relationship.

Thus we may write the three components of peak velocity as

$$u_z = \frac{\partial \psi}{\partial x} = k \cdot \psi_0 \cdot J_m(kr) \sin(kz) \cos(m\phi) \quad (3.9.3)$$

$$u_\phi = \frac{1}{r} \frac{\partial \psi}{\partial \phi} = \frac{m}{r} \psi_0 \cdot J_m(kr) \cos(kz) \sin(m\phi) \quad (3.9.4)$$

and

$$u_r = \frac{\partial \psi}{\partial r} = -k \cdot \psi_0 \cdot J'_m(kr) \cos(kz) \cos(m\phi) \quad (3.9.5)$$
The kinetic energy in the element of volume $\delta z \delta r \delta \phi$ is

$$E = \frac{1}{2} \rho \frac{u^2}{2} \delta r \delta z \delta \phi,$$

(3.9.6)

where $u^2/2 = \frac{1}{2}(u_z^2 + u_\phi^2 + u_r^2)$ is the mean square velocity.

For the ideal resonators which we are considering the potential energy is equal to the kinetic energy; the total is double this amount. Thus the total energy is

$$E = \frac{1}{2} \int_0^a \int_0^h \int_0^{2\pi} \rho u^2 \, dr \, d\phi \, dz.$$

(3.9.7)

Only the radially symmetrical modes are useful for experiment. For these $m = 0$ and $u_\phi = 0$. Equation (3.9.7) then may be written

$$E = \frac{1}{2} \rho \psi^2 \int_0^a \int_0^h \int_0^{2\pi} [k_z^2 J_0(k_r) \sin(k_z z) + k_r^2 J_1(k_r) \cos(k_z z)] \, dr \, d\phi \, dz.$$

(3.9.8)

We note that because of the symmetry produced by the boundary conditions the integrals over $z$ for each term merely yield $h/2$, both for the rigid wall and the resilient wall case. The integral over $\phi$ merely gives $2\pi$, so we may simplify the expression to

$$E = (\rho \pi h/2) \psi^2 \int_0^a [k_z^2 r J_0^2(k_r) + k_r^2 r J_1^2(k_r)] \, dr.$$

(3.9.9)

Using the solution for this type of integral given by McLachlan, this becomes

$$E = (\rho \pi a^2/4) \psi^2 [k_z^2 [J_0^2(k_a) + J_1^2(k_a)] + k_r^2 [J_1^2(k_a) - J_0(k_a) J_2(k_a)]]$$

(3.9.10)
For the thin-walled vessel $J_0(k_\tau a) = 0$ and we have

\[ E_{\text{thin}} = \left(\rho \pi a^2/4\right) \psi_0^2 k^2 J_1^2(\gamma) \]
\[ = \left(p_0^2/4c^2\right) V_0 J_1^2(\gamma) , \]  

(3.9.11)

since $k^2 = k_z^2 + k_\tau^2$ and $J_1(x) = J_0'(x)$, and using equation (3.9.2) to relate the velocity potential to the pressure.

For the thick-walled vessel $J_1(k_\tau a) = 0$ and using the recurrence relation $28$

\[ \frac{2m}{z} J_m(z) = J_{m+1}(z) + J_{m-1}(z) , \]

from which $J_1(\gamma') = -J_0(\gamma')$, we have

\[ E_{\text{thick}} = \left(\rho \pi a^2/4\right) \psi_0^2 k^2 J_0^2(\gamma') \]
\[ = \left(p_0^2/4c^2\right) V_0 J_1^2(\gamma') . \]  

(3.9.12)

Thus we see that the "mode-factor" $q$ defined in section 3.4 takes on the values $q_{\text{thin}} = J_1^2(\gamma)$ and $q_{\text{thick}} = J_0^2(\gamma')$ respectively.

It is interesting to compare the actual quantities in the two cases. For a long thick tube at the first radial resonance $ka = \gamma' = 3.83$ and the mode factor is 0.162. For a long thin tube at its first radial resonance $ka = \gamma = 2.41$ and the mode factor is 0.269. Both these values are considerable lower than the value 0.5 for a purely longitudinal mode$^1$, and modes of higher order have an even lower mode factor.
3.10 Viscous Losses at Thick Container Boundaries

A derivation according to Stokes of the losses due to viscosity of a fluid in motion relative to a rigid body is given by Rayleigh\(^{29}\); an alternative derivation is given in Appendix B.

The loss per unit area is

\[
\dot{E}/A = -\sqrt{\omega \mu/8} \ u_m^2 ,
\]

(3.10.1)

where \( \rho \) is the density, \( \mu \) the viscosity, and \( u_m \) the local peak relative velocity.

A thin boundary is assumed to move with the liquid, so that the viscous losses are zero. However, in this section we calculate the viscous losses in the type of vessel which has a thin wall but a thick bottom, as well as those in a thick-walled vessel.

At the thick bottom of the container, the vertical component of velocity is zero, as is the circumferential component for the radially symmetric modes. Hence

\[
\dot{E}_b = -\sqrt{\omega \mu/8} \int_0^{2\pi} \int_0^a u_r^2 r \ dr \ d\phi
\]

\[
= -\sqrt{\omega \mu/8} \ (2\pi) \ \psi^2 (2) \ J_0^1(k_r a) \ dr . \quad (3.10.2)
\]

This integral is similar to the integral in section 3.9. The solution is

\[
\dot{E}_b = -\sqrt{\omega \mu/8} \ \pi \ a^2 k_r^2 \ \psi^2 \ J_0^1(\gamma)
\]

(3.10.3)

for a vessel with a thin side wall (and a thick bottom), or

\[
\dot{E}_b = -\sqrt{\omega \mu/8} \ \pi \ a^2 k_r^2 \ \psi^2 \ J_0^2(\gamma')
\]

(3.10.4)
for a vessel with a thick side wall and thick bottom.

The kinetic energy is given by equation (3.9.11) or (3.9.12). Thus the contribution to the damping factor from friction at the bottom is

\[ \eta_b = \frac{1}{2} \frac{E}{\omega} = \frac{\sqrt{2\mu/\rho \omega}}{(k_r^2/k^2 h)} \]  

(3.10.5)

independent of whether the side wall is thin or thick.

At a thick cylindrical wall the radial component of velocity is zero, as is the circumferential component for the radially symmetric mode. Hence

\[ \dot{E}_w = -\frac{\omega \rho \mu}{8} \int_0^{2\pi} \int_0^2 u_r^2 r \, dz \, d\phi \]

\[ = -\frac{\omega \rho \mu}{8} \pi h a \frac{k_z^2 \psi_0^2 \chi^2(y')} \]

(3.10.6)

and the contribution to the damping factor is

\[ \eta_w = \frac{1}{2} \frac{E}{\omega} = \frac{\sqrt{2\mu/\rho \omega}}{(k_z^2/k^2 a)} \]  

(3.10.7)

3.11 Losses in the Material of a Thin Cylindrical Wall

We will assume that the cylinder is sufficiently resilient that the pressure at the wall is small and can be neglected compared with the internal pressures in the vessel, so that our assumption of the boundary condition is not upset. With this assumption we can calculate the magnitude of this pressure from the velocity of the liquid at the wall.

The tangential stress in a thin walled tube produced by an internal pressure \( p \) is \( \sigma = p \rho / t \), where \( t \) is the wall thickness and \( a \) is the radius. If \( \lambda_E \) is the Young's modulus, the tangential
strain is \( \frac{ap}{t \lambda_E} \), so that the change in radius \( \Delta r = a^2 p / t \lambda_E \).

Conversely, therefore, the component of pressure at frequency \( \omega \) at the wall is

\[
p = t \lambda_E \Delta r / a^2
\]

\[
= t \lambda_E (1 + j \eta_E) u_a / j \omega a^2
\]

(3.11.1)

where \( u_a \) is the radial component of velocity at the wall, and \( \lambda_E = \lambda_E (1 + j \eta_E) \), \( \eta_E \) being the Young's modulus loss factor of the wall material.

The power loss per unit area is the product of the wall velocity and the component of pressure in phase with it, and is thus

\[
\dot{E}/A = t \lambda_E \eta_E u_a^2 / 2 \omega a^2
\]

(3.11.2)

(the factor \( \frac{1}{2} \) being included to provide the rms value).

Thus the total power loss (from this cause) is

\[
\dot{E}_m = (t \lambda_E \eta_E / 2 \omega a^2) \int_0^{2\pi} \int_0^h u_a^2 \, dz \, d\phi
\]

(3.11.3)

Making use of equation (3.9.5) and inserting the boundary conditions we obtain for the radially symmetric mode

\[
\dot{E}_m = \pi h t \eta_E \lambda_E \kappa \psi^2 J^2(\gamma) / 2 \omega a
\]

(3.11.4)

Making use of the energy given in equation (3.9.11) we obtain for the contribution of the wall material to the damping factor in the thin walled case

\[
\eta_m = \dot{E}_m / E\omega = (2t \eta_E \lambda_E / \omega^2 a^3 \rho) k^2 / k^2
\]

(3.11.5)

The viscous losses at the wall are assumed to be zero because both the longitudinal and tangential components of velocity vanish at a resilient wall.
3.12 Losses in the Material of a Thick Cylindrical Wall

In order to estimate the losses in a thick wall we make use of the formula derived by Korteweg\(^{18}\) for the change in velocity of propagation in a tube due to the interaction with the wall, which was modified by Kuhl\(^{20}\) for the thick wall case and demonstrated experimentally to apply with reasonable accuracy. The vibration losses can be deduced from the formula by assuming a complex modulus of elasticity for the material of the wall.

Kuhl's formulation for a thick wall may be written

\[
\frac{c'}{c_0} = 1 + \frac{\beta}{1 + 2\beta} \approx 1 - \beta + O(\beta^2) \tag{3.12.1}
\]

where

\[
\beta = \frac{\lambda_0}{\lambda_E} \left( \frac{a_2}{a_1} \right)^2 + \frac{1}{1 - \frac{t}{a_1}} = \frac{\lambda_0}{\lambda_E} \left( \frac{t/a + 1}{t/a} \right) \left( \frac{a_2/a_1}{2 + t/a} \right) \tag{3.12.2}
\]

where \(\lambda_0 = \rho_0 c_0^2\) is the bulk modulus of the liquid, \(\lambda_E\) is the Young's modulus of the wall material, \(a_2\) and \(a_1\) are the outer and inner radii of the tube, and \(t = a_2 - a_1\) is the wall thickness.

For a progressive wave the amplitude is proportional to \(e^{jwx/c}\) or \(e^{jux/c}\). Inserting the corrected sound velocity this becomes \(e^{jwx/c_0} = e^{\frac{j\omega}{c} c_0} = e^{j\kappa(1 + \beta)}\). Assuming a small loss factor \(\eta_E\) for the wall material, we can replace the modulus \(\lambda_E\) by the complex modulus \(\lambda_E = \lambda_E(1 + j\eta_E)\), and therefore \(\beta\) by \(\beta(1 - j\eta_E)\).

Thus the damping due to wall material losses in a progressive wave, or in a standing wave in the case of a resonant system, is, to a first approximation, \(k\beta\eta_E\) per unit length, and we may write

\[
\eta_m = \omega \beta \eta_E \frac{h}{c} \tag{3.12.3}
\]
This result is quite different from the thin walled case, equation (3.11.5), because a thick wall has only a small effect on the particle velocity due to its stiffness, which thus reduces the effect of losses in the wall material.

3.13 Numerical Results and Discussion

The equations of sections 3.9 through 3.12 enable the frequencies of the normal modes to be found and the principal types of loss to be estimated for any given geometry. Though great accuracy is not required the use of a digital computer enabled the calculations to be performed and the results tabulated quickly and conveniently. The FORTRAN computer program is described in detail in Appendix C.

For the appraisal of the different types of vessel in the different modes the following values were assumed:

Sound velocity \( c = 1.5 \times 10^8 \text{ m/sec} \)
Density (water) \( \rho = 10^{-3} \text{ kg/m}^3 \)
Viscosity (water) \( \mu = 0.001 \text{ Pa.s} \)
Young's modulus (steel) \( \lambda_E = 2.0 \times 10^{11} \text{ N/m}^2 \)
Young's modulus loss factor \( \eta_E = 10^{-4} \)

In Figures 4, 5 and 6 are plotted the frequencies for some of the lower order modes in a vessel of 50 mm radius as a function of water depth for the three cases considered in section 3.8. The annotation of the modes is in the sequence \( m, n, \lambda \), where \( m \) is the order of the Bessel function, controlling the number of nodal lines along diameters of the vessel, \( \nu \) is the order of the Bessel function zero controlling the number of nodal loops, and \( n \) is the order of the longitudinal (sinusoidal) component. The figures illustrate the
Figure 4. Frequencies of the lower order modes in a thin-walled, thin-bottomed vessel of 100 mm diameter as a function of water depth.
Figure 5. Frequencies of the lower order modes in a thin-walled, thick-bottomed vessel of 100 mm diameter as a function of water depth.
Figure 6. Frequencies of the lower order modes in a thick-walled, thick-bottomed vessel of 100 mm diameter as a function of water depth.
grouping of modes at the extremes of very shallow and very deep vessels (with the exception of the purely longitudinal modes in deep thick-walled vessels); in fact far more modes converge to these groupings than can conveniently be shown in graphical form. It is obvious that these extremes should be avoided so that a discrete resonance may be found and employed, otherwise the frequency shift resulting from the insertion of a sample might lead to a different mode being measured. In the intervening range only low order modes are of practical use; high order modes in any case will have a lower mode factor, leading to a reduced sensitivity in the experiment as can be seen from equation (3.4.7). For these reasons the discussion which follows is directed mainly at the lower order symmetrical modes in depths from a half to double the vessel diameter. In interpreting the results it must be remembered that while figures for the viscosity of water and the Young's modulus of metals are well established, the figure of $10^{-4}$ for the Young's modulus loss factor is an assumed figure; the actual loss is likely to vary considerably from sample to sample as a result of small variations in composition, heat treatment and other processing.

Computed losses in the wall material for the thin walled thin bottom case are shown in Figure 7. It can be seen that for the parameters assumed above the contribution to the damping factor is less than $10^{-4}$ for all modes, corresponding to a $Q$ in excess of 10 000 if this were the only cause of loss. However, such a vessel is hard to produce in actual practice without solder, cement or welded joints which would introduce loss, and of a strength adequate to support the weight of the liquid without deformation. There is also the problem of radiation loading from the outer surface, which can only partly be reduced by enclosing a outer tube unless that tube were evacuated. As
Figure 7. Estimated Losses in a Thin-Walled, Thin-Bottomed Vessel of 100 mm diameter.
will be seen, any attempts to use this principle were unsuccessful.

If a vessel has a thick bottom the side walls might be made thin without excessive loss of rigidity. Figure 8 shows losses in this case. As can be seen, however, viscous losses at the bottom then became commensurate with the wall material losses, and in fact are likely to predominate for shallow vessels.

Figure 6 shows the computed losses in longitudinal modes in a thick-walled vessel of thickness 10 mm, one tenth of the diameter. There are, of course, no viscous losses at the bottom since there is no radial or circumferential component of motion in the longitudinal modes. It can be seen that viscous losses at the wall dominate over the wall material losses, so that any increase in thickness (and therefore weight) in order to increase the wall stiffness would not be justified.

Though some radial-axial modes are also shown in Figure 9, they do not seem useful for experiment. It can be seen that for these the viscous losses at the bottom tend to outweigh those at the wall. Estimates of material losses are also given; they were computed on the basis of equation (3.12.3), though it has not been shown that this formula should apply to other than purely longitudinal modes, but they indicate that wall material losses are likely to be commensurate with the viscous losses. All these modes are in a higher frequency range than the longitudinal modes, and indeed there are asymmetrical modes of lower frequency.
Figure 8. Estimated Losses in a Thin-Walled, Thick-Bottomed Vessel of 100 mm diameter.
Figure 9. Estimated Losses in a Thick-Walled, Thick-Bottomed Vessel of 100 mm diameter
IV. EXPERIMENTAL DETAILS

4.1 The Measurement of Decay - Electrical Apparatus

The basic electrical block diagram is given in Figure 10. The principle is to search for resonances by sweeping the signal frequency by hand, observing the response on an oscilloscope. When a resonance is found, the signal is switched off for the measurement of its decay.

The decay is normally measured on a Brüel and Kjaer Model 2305 recorder (though the later model 2307 would be preferable if it were available). The input to this recorder is applied to its 50 dB logarithmic potentiometer at signal frequency; the attenuated signal from the slide wire is passed to a square law detector, and the rectified signal compared with a reference source within the instrument, the difference being applied to a servo motor driving the slide wire and recorder pen in a null balance arrangement. The procedure is to switch the recorder paper on immediately before switching off the excitation, and ideally a straight line decay curve results. A paper speed of 100 mm/sec is used, with a maximum writing speed of 2000 mm/sec corresponding to a minimum decay time measurement of 1000 dB/sec.

The decay envelope may also be observed directly on the oscilloscope (Tektronix Model 564 B) in storage mode, the time base being set for single sweep operation triggered by a relay contact on the send-receive key (Figure 11). This display of the exponential decay envelope is a little less convenient for actual measurement, requiring additional computation, but is valuable for preliminary assessments, especially when a sample of unexpectedly high loss is encountered.

In Appendix D a circuit is described which produces an output
Figure 10. Electrical Block Diagram of Apparatus for the Measurement of Decay.
Figure 11. Detail of Transmit Circuitry.
proportional to the logarithm of the instantaneous voltage. It was thought that such a circuit would be of use in providing a linear decay envelope which would be more easily measured, and a breadboard model was built and tested. Though it functioned properly in quasi-static bench tests, the decay envelope when a rapidly decaying signal was applied was completely asymmetrical. The cause was not fully determined, but it was suspected that a small switching surge or an initial overload might have produced an offset in the centre of the sinusoidal waveform which would effectively be amplified by the open-loop gain of the feedback amplifier. Since the square law detector of the recorder eliminates the effect of any such offsets, its use was preferred for actual measurements.

A Brüel and Kjaer model 1014 Beat Frequency Oscillator is generally used for the signal source (the current transistorized model is designated 1022). This has both coarse and fine frequency controls and excellent stability, factors of importance when high Q resonances are to be examined. It also has a very convenient selection of output impedance ranges, including a high impedance 120 volt range which was very suitable for driving the ceramic transducers normally employed. A crystal controlled decade counter, counting over a 10 second period, is generally used to monitor the frequency.

For a few of the measurements a General Radio frequency synthesizer was used. Its facility for substituting a continuously adjustable decade in place of any switched decade enabled progressively finer tuning to be obtained, with a direct read-out of frequency to 0.01 Hz. The synthesizer was followed by a power amplifier for driving the transducer, but the counter was unnecessary.

The preamplifier was an integrated circuit amplifier with 20 dB gain controlled by feedback, followed by a simple RC filter to produce
a roll-off below 1 KHz, though the use of a more elaborate filter might have been helpful to cut down the noise level; it was contained with its battery in a shielded box to avoid hum problems. Two of these amplifiers could be operated in cascade if desired.

The relay circuitry of Figure 11 is normally used to switch the signal, as it provides a trigger to the oscilloscope. The momentary switches which control the relays are brought on a cord to a convenient location, enabling the wiring carrying the actual transmit signal to be physically apart from the receive wiring. It is sometimes found more convenient to use the "oscillator stop" button on the B & K instrument; this requires the precaution of noting the frequency beforehand, as there might be a slight frequency shift when the instrument resumes oscillation.

For tuning purposes the transmit signal is on continuously and is displayed on channel 1 of the oscilloscope, with the time base locked to it; the received signal is displayed on channel 2. By displaying both waveforms overloading and frequency-doubling effects are avoided. Also it is found that the sideways movement of the received waveform due to the relative phase change as the frequency passes through resonance facilitates the precise location of the resonance peak.

A typical recorder trace from the B. & K. recorder is reproduced in Figure 12. At the bottom of the trace can be seen some fluctuation due to background noise. This is mostly noise in the room which is being picked up by the microphone. Figure 13 is a tracing of a typical exponential decay envelope as displayed on the oscilloscope, from which an estimate of decay can be made, though to less accuracy.
Figure 12. Typical Decay Trace on B. & K. Recorder.

Figure 13. Typical Oscilloscope Trace of Exponential Decay Envelope.
4.2 Cleaning and Degassing - The Problem of Air Bubbles

An air bubble is a tuned vibrator, and an almost ideal sound absorber, particularly when the frequency approaches the resonant range. For example, one tiny bubble of volume \(10^{-3}\) \(\text{mm}^3\) per litre of water leads to a sound absorption of 100 dB per metre. Even if the frequency differs from the bubble's resonant frequency, the sound absorption is significant. Microscopic layers of adsorbed gases that tend to adhere to all surfaces that have been exposed to air are almost as harmful. Thus degassing of the liquid and thorough cleaning of all surfaces is essential. The development of a good working procedure is of sufficient importance to warrant its narration in detail.

The cleaning procedure was dependent upon the material from which the vessel was made. For those of stainless steel or of brass a chromic-sulphuric acid dip was used, followed by a rinse with distilled water. Occasionally a scouring action seemed necessary, using cotton swabs and a solvent such as methyl-ethyl-ketone, followed by an alcohol rinse. The final treatment consisted of an overnight soak in a dilute (2 to 3%) citric acid solution, followed by a rinse with distilled water and then with alcohol. Care was always taken to avoid subsequent contamination by finger-prints.

Vessels made of aluminium alloy require a different treatment. Aluminium is highly reactive, and oxides gradually form on surfaces exposed to the air. It therefore needs frequent repolishing if left unprotected. This can be done quite quickly in a lathe using 400 grit emery paper, followed by polishing with a swab soaked in alcohol to remove loose particles, and a final alcohol rinse. An effective protective coating was sometimes applied by giving the vessel a hot alkali dip, a water rinse, a nitric acid dip and a water
rinse, followed by a sodium dichromate soak and another water rinse, and then treatment by a chromate conversion process. (The particular coating used was "Alodyne 1200" from Amchem Products Inc., Ambler, Pennsylvania, marketed in Britain by the Paints Division of Imperial Chemical Industries Ltd. under the trade name "Alochrom"). Again this was followed by a distilled water and alcohol rinse. Anodizing would provide a more durable coating, but requires specialized equipment, and its effect has not been tried.

Distilled water from the laboratory supply was used for the first experiments. The classic method of boiling to drive out dissolved air, and then cooling, was used; this reduced the gas content to about 9 parts per million as measured on a Van Slyke Manometric Blood Gas Apparatus, compared with a saturation value of about 18 ppm at room temperature. This had been thought to be adequate to ensure that trapped air would go into solution, but minute bubbles were seen adhering to the surfaces of the vessel. Accordingly, water was siphoned into the vessel under vacuum using a fine nozzle so as to form a spray; by doing this over a period of about four hours the gas content could be reduced to about 2 ppm, but this failed to give any lasting improvement; bubbles were still observed to form despite frequent recleaning of the vessel. For some time the difficulty was attributed to the possibility of nuclear contamination in the basement laboratory in which the experiments were being conducted, but shielding of the test vessel and the eventual move of the whole experiment into a newly constructed building that was free of contamination yielded no improvement. Eventually the water itself had to be held responsible.

The clue came with the publication of an article describing the difficulty of obtaining pure sterile water for clinical purposes.
The campus water supply is heavily chlorinated, and it was found that the ordinary distillation process does not remove all the chlorine ions. Consequently electrolysis may be set up in all the metal containers between neighboring domains of slightly differing electrochemical potential due to surface dislocations, cracks and crystallites. This caused the generation of gas bubbles which adhered tenaciously to the surface, since the water nearby soon became saturated, and they could not be driven into solution even by the application of an over pressure.

The solution to the problem was to use water which had been carefully demineralized by passing through two ion exchange demineralizers (the particular units used are made by the Barnstead Company, W. Roxbury, Massachusetts, model DO 809). The resistivity is monitored between the first and second cartridges to exceed 2MΩ·cm; the second cartridge, which is always the newer one, increases the purity further so that the resistivity is too high to be measurable. The freshly demineralized water is boiled in a glass flask which is stoppered while cooling. It is then poured gently down the side of the tilted test vessel, preferably while still warm. This gives a gas content of about 9 ppm, but in the absence of the possibility of electrolysis gas bubbles do not form, and the water may remain usable for several days. The present demineralizer unfortunately does not remove organic material, and there is a tendency for algae to grow and form a viscous (and lossy) scum if the water is not boiled soon after treatment or is left exposed to the air for too long.
4.3 Experiments with Thin-Walled Vessels

Figure 14 is an exploded view of the basic experimental arrangement used for thin-walled vessels. The vessel itself is suspended by three cords (a waxed nylon braid used as lacing cord in the electronics industry) inside a steel alloy outer container of 158 mm inside diameter and 9 mm wall thickness. A lead-titanate-zirconate disc 12.5 mm diameter and 6.2 mm thick, whose natural resonant frequency is about 300 kHz, is cemented on to the underside of the vessel with a silver bearing epoxy cement so as to provide an electrical connection to one electrode. A light twisted pair lead as used in cheap gramophone pick-up arms is used, the black (earth) lead being cemented alongside the disc and the red (live) lead to the exposed electrode.

The outer container has a loose fitting lid of transparent acrylic sheet, with clearance holes for the suspension cords and the transducer lead. A cheap crystal lapel microphone is attached to the lid for use as a receiver; by using loose coupling through the air space above the liquid the introduction of additional losses is avoided. For a few measurements a small loudspeaker was mounted in place of the microphone and the drive voltage applied to it, the resonance being monitored by the ceramic disc instead; though the effect should be similar, there were impedance matching and shielding problems with this arrangement, and the microphone was normally used.

The screw jack is on the floor directly below the support plate which is clamped to a work bench so as to overhang the edge. With the jack in the raised position the test vessel rests on the support ring, so that the cords may easily be threaded through the holes in the lid and the slots in the support plate, wound round the capstan posts and secured with the screws. On lowering the jack the vessel
Figure 14. Apparatus used with Thin-Walled Vessels.
is left hanging by the cords; if necessary the jack may be raised again and the cords re-adjusted so that the vessel will hang level.

The purpose of the outer container is to shield the vessel from draughts and from electrical disturbances, and to prevent the loss of energy by radiation of sound from the vessel.

For the first experiments a vessel was constructed from stainless steel tubing 155 mm long, of 0.9 mm (nominally 1/32 inch) wall thickness and 126 mm inside diameter. The bottom was a disk 9.5 mm (3/8 inch) thick brass into which a groove was machined to receive the end of the tube, which was secured by epoxy cement. In order to reduce the wall material losses as much as possible the tubing was turned down carefully in a lathe to a thickness of 0.4 mm.

The frequencies of the low order normal modes for the water in this vessel when nearly full and the estimated losses are tabulated in Appendix C, Table C1, using a value for the sound velocity of $1.49 \times 10^3$ m/sec, calculated for room temperature of 24°C from a standard formula. It can easily be seen that the viscous losses at the thick bottom were likely to outweigh the wall material losses, a point that was overlooked when the vessel was constructed.

The results of a search for resonances in this vessel are shown in Figure 15. A considerable number of minor resonances were found, all of low $Q$. On the figure are flagged the calculated frequencies for the low order modes. It is impossible to identify any of the radially-symmetrical modes of resonance which had been expected. Such modes seem to have been completely damped out, though $Q$'s of several thousand had been expected. With the possible exception of two resonances which might be the asymmetrical 101 and 102 modes all the resonances are believed to be in the structure. The reasons for the heavy damping can only be surmised. Perhaps the machining of the
Figure 15. $Q$ as a function of frequency for the high amplitude resonances in water in the thin-walled cylindrical vessel with a brass bottom.
wall introduced internal stresses or a lack of uniformity, causing 
wall material damping. Probably there was damping introduced by the 
epoxy glue joint at the bottom. It is, however, noteworthy that on 
removal of the outer jacket the Q's of those resonances of moderate 
Q were reduced to about half their previous value, indicating that 
the jacket does indeed serve to reduce losses by acoustic radiation.

Because of the suspicion that the glue joint might be responsible 
for the high losses a search was made for vessels made in one piece, 
without any welding, solder or glue joints. The least unsuccessful 
of these was a 1800 ml stainless steel beaker made by the Vollrath 
Corp., Sheboygan, Wisconsin; its internal diameter was 123 mm and 
the wall thickness was 0.8 mm. The frequencies and estimated losses 
for this vessel are tabulated in Appendix C, Table C2 for a water 
depth of 159 mm. The Q for the measured resonances are shown in 
Figure 16, on which the calculated mode frequencies are also shown. 
Again the measured values for Q are disappointingly low. Annealing 
the vessel in a hydrogen atmosphere at 860°C and air quenching made 
no significant improvement (Figure 16 is in fact the results for the 
annealed condition). It can only be assumed that the loss factor for 
metals in the spun, drawn or pressed condition is considerably higher 
than the figure of $10^{-4}$ assumed for the purposes of calculation.

Several other vessels were tried, with no success. One was a 
one piece stainless steel developing tank, whose performance was 
similar in character to that of the beaker. Another was a plastic 
beaker, in which no resonances could be discerned at all.

In an attempt to obtain low losses at a really thin wall, a 
thin plastic bag was by siphoning it in under vacuum filled with water 
and was suspended inside the outer vessel. The driving transducer was 
a lead metaniobate cube of 6.3 mm (1/4 inch) side, approximately
Figure 16. $Q$ as a function of frequency for the high amplitude resonances in a 1800 ml stainless steel beaker.

Calculated mode frequencies

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (kHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>001</td>
<td>18</td>
</tr>
<tr>
<td>002</td>
<td>14</td>
</tr>
<tr>
<td>101</td>
<td>10</td>
</tr>
<tr>
<td>003</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
in the middle of the bag. A pair of fine wires were soldered to the
two electrodes and threaded up a length of stainless steel capillary
tubing, the bottom end of which was sealed with epoxy. The whole
cube was dipped in varnish for water proofing. Since no resonances
could be found, this somewhat crude preliminary attempt was not pursued
further.

It was suggested that a non-wetting coating on the inside of the
vessel might produce the effect of a resilient boundary. To investi­
gate this possibility a check was made with an ultrasonic probe
operating at 2 MHz connected to an "Ultrasonoscope". The reflection
pulse from a metal surface was first examined qualitatively. The
reflection from a Teflon coated surface (a "non-stick" fry pan) was
found to be similar, as was the reflection from a Teflon rod. How­
ever the reflection from an air surface had the opposite phase. This
demonstrated that the Teflon water-repellent coating does not result
in a pressure relief or resilient surface. Measurements were not
therefore attempted on a Teflon coated vessel.

In view of all the experiments described above, the thin wall
approach was abandoned as unprofitable.
4.4 Experiments with Thick-Walled Vessels

The fluid column measurements mentioned in the Introduction used a relatively long thick-walled tube in a longitudinal mode. The tube was closed at the bottom by a thin membrane which was assumed to form a pressure node. Simon's U-tube experiments were an attempt to eliminate losses in such a membrane, the driving transducer and in the joint to the tube, by having both ends of the resonant column completely free.

The opposite approach is described here, that of utilizing a thick, nearly rigid, membrane. This rigidity results in loose coupling between the fluid and the transducer, so that losses in the transducer do not greatly affect the decay of the resonance in the fluid. An advantage of this approach is that the sample may merely be placed on the bottom of the vessel, since this is a region of maximum pressure. It was decided that the vessel should be machined from a solid billet of metal, so as to avoid any losses that might be introduced due to welding, soldering or even cementing pieces together.

It can be seen from equations (3.12.2) and (3.12.3) that the wall losses are dependent upon the stiffness of the walls and hence the Young's modulus of the material. It would therefore seem desirable to choose a material with a high value for this modulus. Steel has a modulus of about $20 \times 10^{11}$ Pa.s, compared with about $10 \times 10^{11}$ for brass and $7 \times 10^{11}$ Pa.s for aluminium. Mild steel, of course, is unsuitable as it would rust. Stainless steel is heavy, and relatively expensive and hard to machine. It was therefore decided to make the initial measurements with a softer metal.

Figure 17 illustrates the general arrangement. The vessel illustrated is approximately 112 mm inside diameter, 188 mm inside height,
Figure 17. Pressure Apparatus used with Thick-Walled Vessels.
with a 25 mm thick wall, and was machined from a solid billet of 6061 T 6 aluminium alloy. The outer container measures 200 mm inside diameter and has a wall thickness of 6.3 mm. The lid enables an excess pressure to be applied (using an automobile tyre pump and a Schrőder valve), with the idea that any gas bubbles which might form on the walls of the container will be driven into solution. It was designed so that the vessel might rest on four springs set into recesses in a lead block so as to provide vibration isolation; however, it was found that merely balancing it on three nail points inserted in holes drilled up into the vessel's wall was adequate and did not increase the damping. When pressure is not required, the vessel may be supported by strings in a manner similar to the thin walled vessel, the outer container serving merely as a shield. The microphone is attached to an acrylic disc of slightly greater diameter than the outside diameter of the vessel, so that it could be wedged above it between the strings without actually touching it.

The frequencies of the normal modes and estimated losses for this vessel were computed and are given in Appendix C, Table C3. The tabulated frequencies do not include the Korteweg-Kuhl correction, whose numerical value is given in the heading of the Table. For the wall loss calculation a loss factor for the metal of $10^{-4}$ was again assumed. As can be seen, the unavoidable viscous losses exceed the estimated wall material losses only at the lower frequencies, due to the modulus of aluminium being only about one third that of steel.

In order to confirm the estimate of $10^{-4}$ for the loss factor a microphone was hung inside the empty vessel, and a search made for resonances. Among the several found was one particularly high $Q$ resonance at approximately 7535 Hz with a $Q$ of 11 000, corresponding
to a loss factor of $0.9 \times 10^{-4}$. Considering that losses other than those in the metal must be present to some degree, it was felt that the initial estimate of $10^{-4}$ was an appropriately conservative one.

The measured resonances and the associated $Q$ values are displayed in Figure 18, together with the computed frequencies, with the Korteweg-Kuhl correction applied to the purely longitudinal modes only. It can be seen that the 002 resonance, the second-order longitudinal mode, is quite prominent, and the agreement of frequency is very close. The $Q$ value of about 6000 for this resonance is higher than had been expected, and indeed this value has only been noted once; the $Q$ usually measures about 4000, more in keeping with the calculations.

It seemed desirable to obtain additional confirmation of the mode structure before making measurements on actual specimens. This was done rather crudely by exciting the resonance continuously and then moving a probe transducer around in the water, watching the response on the oscilloscope. The probe used was the lead-metaniobate cube described in the preceding section. No attempt was made at accurate positioning, nor was the excitation frequency adjusted. The $3\lambda/4$ and $5\lambda/4$ structure of the longitudinal 002 and 003 modes could be clearly identified, with approximately uniform levels in the diametral planes. (The lack of an exact harmonic relationship between these modes is attributed to end effects at the bottom.) Other resonances such as the 011 resonance could also be identified.

It is interesting to note that the $\lambda/4$ (001) mode is not strongly excited. The 002 mode was normally used for measurements on samples, though the 003 mode might be of use if a higher frequency measurement were needed, though at the full water depth there are other resonances close by.
Figure 18. $Q$ as a function of frequency for the high amplitude resonances in water in the thick-walled aluminium cylindrical vessel.
4.5 Thick-Walled Vessel with Conical Tapered Section

A vessel tried in a few experiments is shown in Figure 19. It was machined from a solid bar of brass. The main part is cylindrical, but it is tapered near the bottom to a smaller cylindrical section in which the sample may be placed. The recess underneath the bottom cavity accommodates a lead-zirconate-titanate driver, which is coupled through a relatively thin (4.8 mm) membrane to the liquid in the cavity. A radial groove along the under side accommodates the leads while enabling the vessel to sit on a flat surface, though during measurements it was found necessary to hang the vessel by strings.

Only the first longitudinal resonance is used in this apparatus. The intention was that tapering the vessel to a smaller cross-section at the closed end would cause amplification of the relative pressure amplitude by a corresponding ratio, without a corresponding increase in the viscous loss at the open end. However, this is not the case when the length over which the tapering is done is less than a quarter of a wavelength. In fact the whole vessel behaves as a Helmholz resonator, the frequency being controlled by the total volume. The effect of the conical tapered section and the small bottom section may be closely approximated by an equivalent cylindrical section of the same volume and of the same cross section as the upper portion.

A more detailed frequency calculation, based upon Rayleigh's solution for a conical pipe, is given in Appendix E. It confirms that the pressure amplification is negligible, and that the calculation based upon the equivalent volume is entirely adequate.

The major resonances found with this pot are shown in Figure 20, with the frequencies of the longitudinal modes, corrected by the
Figure 19. Brass Vessel with Conical Tapered Section.
Figure 20. $Q$ as a function of frequency for the high amplitude resonances in water in the brass vessel with conical tapered section.
Korteweg-Kuhl formula, the correction factor being 0.98. The results are not as good as had been achieved with the aluminium cylinder. The complicated shape gave no advantage, so the aluminium cylinder was preferred for measurement on samples.

4.6 Sample Size, Preparation and Measurement Procedures

It was decided that only longitudinal modes should be used for the final measurements on samples, so that equations (3.5.8) and (3.5.9) apply without modification. It should be noted that because of the way in which these equations were derived (the sample is substituted for an equivalent volume of liquid) the true value must be taken for \( \lambda = \rho c^2 \); the Korteweg-Kuhl reduction of sound velocity does not apply.

To obtain the greatest accuracy the size of the sample has to be selected in accordance with its loss factor and bulk modulus, so that the losses in it are commensurate with those in the rest of the system. With the thick-walled aluminium vessel a \( Q \) of about 5000, i.e. a loss of \( 2 \times 10^{-4} \), is obtained, corresponding to a 40 dB decay in 1.22 seconds or 122 mm on the recorder, which may easily be measured to within 1%; the frequency is about 6 kHz for a depth of 165 mm, i.e. a volume of \( 1.65 \times 10^{-3} \) m\(^3\). To obtain a similar loss it can be seen from equation (3.5.9a) and (3.4.1) that the product of the volume ratio and the sample loss factor should be about 5 \( \times 10^{-5} \) if the bulk modulus is similar to that of water. Thus for a material with loss factor \( 10^{-3} \) the sample volume should be about 5% of the volume of the resonator. If the product is much lower than this inaccuracies in the measurement of the two decay times \( \delta_{s+0} \) and \( \delta_0 \) are magnified when the difference is taken. However, the
method still yields an approximate value, or an upper bound to the loss factor corresponding to a volume-ratio loss-factor product of about $10^{-5}$, for which the difference in decay time ought certainly to be discernible, and this information in itself can be useful.

It is convenient to use the frequency shift given in equation (3.5.8) to obtain the bulk modulus itself, and thus the form (3.5.9b) of the equation for the loss factor. This procedure works well for most substances, but gives rise to difficulty with very small specimens and with very lossy materials, for which the accuracy of the frequency shift measurement is limited.

For the resonance of the resonator by itself, the bandwidth to the $-3$ dB points is 1.2 Hz; it is relatively straightforward to tune the signal frequency to the actual peak to within about 0.2 Hz, and to measure the signal frequency to within 0.1 Hz on a decade counter over a 10 second period. For the low loss sample considered above, with a volume ratio of 5%, the frequency shift is also 5% or 300 Hz and this can be measured without any difficulty. If the loss factor were only $10^{-2}$ the volume ratio should ideally be 0.5% and the frequency shift can still be measured to an accuracy of about 0.6 Hz or 2%. However, if the loss factor were as high as 0.1 the ideal volume ratio of 0.05% would result in too small a frequency shift to be measurable with reasonable accuracy. The sample may of course be increased in size, but this damps the resonance further and makes precise tuning more uncertain, so the accuracy cannot be increased in that way. Consequently when the loss factor of the sample is relatively high or the sample is very small the form (3.5.9a) for the equation for the loss factor is preferred, requiring a knowledge of the volume and of the bulk modulus obtained from other methods.
The problem with air bubbles has already been mentioned. It is important that no air bubbles be introduced and remain attached to a specimen. It is appropriate to make measurements on several samples of a material when this is a possibility, and to use the lowest value for the loss factor. Solid samples were usually left in another vessel of demineralized and boiled water so that any air bubbles would dissolve out in advance of their insertion into the measurement vessel. The actual measurement of decay took place after several minutes, and after gentle stirring, again so that bubbles would dissolve. The decay measurement was then repeated after removal of the sample. (A cork extractor, designed for the removal of corks from the inside of wine bottles, was found to be a convenient tool.) Care was taken to disturb the water as little as possible, but to shake back as many droplets as possible, as their removal would cause a small frequency shift and give rise to errors. If the sample would otherwise float a piece of tinned copper wire was wound round it to weight it down. (A check on a piece of wire showed that it had a negligible effect on the resonance.)

Samples of root vegetables were cut and trimmed under water in a dish, and the sample remained immersed in a small beaker of water while being transferred to the measuring vessel, so that there would be no chance of new bubbles adhering. When meat samples were handled in this way the water in the measuring vessel rapidly became murky, as some blood was inevitably contained in the water surrounding the sample in the transfer beaker. This made it hard to see to the bottom of the vessel to get the sample out, though it seemed to have no effect on the losses once the sample was removed. The sample was therefore transferred via a rinsing bath of clear water before being put into the measuring vessel. As an alternative, the
sample might be deposited in a wire cage, which could be pulled out with a piece of nylon fishing line; it is not believed that the cage should increase the losses unduly, but this point should obviously be checked upon as part of the regular procedure.
V. DISCUSSION AND CONCLUSIONS

The theoretical and experimental results have confirmed the conclusion drawn by previous authors that the resonator-decay method (or the equivalent bandwidth method) of measuring the losses in a sample are limited in the case of a thick-walled tube primarily by the viscous loss in the boundary layer at the wall. While no similar limitation was predicted for thin-walled vessels on theoretical grounds, it was found that in practice no improvement could be obtained by their use. The only way in which an improvement seems to be obtainable is with the spherical vessels used by Simon and Reinhard, with all the complications that these involve. Once a cylindrical shape is decided upon, there seems to be a significant practical advantage in making this a thick-bottomed vessel; it must, however, be machined out of a solid billet. Though the use of stainless steel is likely to result in the lowest losses in the wall material, for lightness of weight and ease of construction an aluminium alloy may be used at frequencies up to about 10 kHz; however, it can be seen from a comparison of equations (3.12.3) and (3.10.7) that the wall material losses are likely to become dominant at higher frequencies so that stainless steel ought to be employed. By extrapolation, it seems likely that the wall material losses will become intolerable above about 50 kHz.

The problem of air bubbles can hardly be over-emphasized. Even with the utmost precautions, one can never be quite sure that bubbles adhering to a sample are not causing loss, and precautions must be taken to eliminate this possibility as much as possible.

Some results for typical specimens of visco-elastic polymers are given in Table I. The variation between the polyurethane samples is
particularly interesting, as it was not until after the sample with
high loss was measured that it was examined thoroughly and the air
bubbles were noticed.

Some results for root vegetables and meat products are given in
Table II. The carrot sample was a slice about 16 mm diameter and
about 1 mm thick. The potato sample was a little larger. An attempt
was made to prepare a sample of celery, but on cutting under water
with a razor blade bubbles were seen to appear, and even a 1 mm thin
slice produced so much damping that the resonance could not be dis-
cerned.

Loss factors for animal tissue were replotted by Skudrzyk\textsuperscript{36} from
the results of measurements reported by Hueter and Meyer\textsuperscript{35} in the 0.4
to 4.2 MHz range. The curves show that the damping is due to hyster-
esis effects in a wide frequency range; they are plotted, together
with the present results, in Figure 21. It had been suspected that
animal tissue might exhibit a strong structural relaxation phenomenon
at lower frequencies and consequently should be more absorptive. The
present results, on samples obtained from local butchers, tend to
confirm this hypothesis, though it is suspected that the unusually
high damping in kidney is due to minute cavities or to gases given
off within the structure. A more definitive study of these and other
types of tissue from freshly slaughtered animals would confirm
whether cavities were responsible for the high loss in kidney and
would yield information on the effect of aging and decomposition,
perhaps leading to a method for testing the freshness of meat at the
point of sale. A comparison between healthy and cancerous tissue
might be particularly interesting, but such a study is beyond the
scope of this report.
<table>
<thead>
<tr>
<th>Material</th>
<th>Loss Factor</th>
<th>Bulk Modulus (N/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neoprene <em>(Goodrich 35003)</em></td>
<td>0.003</td>
<td>2.8 x 10⁹</td>
</tr>
<tr>
<td>ABS <em>(Acrylonitrile-Butadiene-Styrene)</em></td>
<td>0.002</td>
<td>3.3 x 10⁹</td>
</tr>
<tr>
<td>(Goodrich Absonic A)*</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nylon</td>
<td>0.003</td>
<td>5.0 x 10⁹</td>
</tr>
<tr>
<td>Polyurethane <em>(CONAP EN-6, poured by CONAP)</em></td>
<td>0.004</td>
<td>3.0 x 10⁹</td>
</tr>
<tr>
<td>Polyurethane <em>(EN-6, poured at PSU)</em>*</td>
<td>0.003</td>
<td>3.1 x 10⁹</td>
</tr>
<tr>
<td>Polyurethane with visible 2 mm air bubble and smaller bubble</td>
<td>0.04</td>
<td>3.1 x 10⁹</td>
</tr>
<tr>
<td>Teflon</td>
<td>0.007</td>
<td>3.3 x 10⁹</td>
</tr>
</tbody>
</table>

* B. F. Goodrich Aerospace and Defense Products, Akron, Ohio 44311

** CONAP, Inc., Allegany, New York 14706
<table>
<thead>
<tr>
<th>Typical Results for Root Vegetable and Animal Tissue</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Loss Factor</strong></td>
</tr>
<tr>
<td>Carrot</td>
</tr>
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<td>Potato</td>
</tr>
<tr>
<td>Steak</td>
</tr>
<tr>
<td>Tongue</td>
</tr>
<tr>
<td>Liver</td>
</tr>
<tr>
<td>Kidney</td>
</tr>
</tbody>
</table>
Figure 21. Loss Factor of Animal Tissue as a Function of Frequency.
APPENDIX A

LOSSES FROM A CAVITY IN AN INFINITE MEDIUM

Consider a cylindrical cavity of radius \( a \) in an infinite solid medium containing a liquid in which there exists a radial progressive wave of pressure amplitude \( P_p \) at the walls.

If \( \rho_m \) and \( c_m \) are the density and sound velocity in the solid material, and \( \rho_o \) and \( c_o \) those in the liquid, the transmission factor

\[
T = \frac{2\zeta}{(\zeta + 1)} \quad (A1)
\]

where \( \zeta = \frac{\rho_m c_m}{(\rho_o c_o)} \), and the power (or rate of energy absorption) in the metal, for a height \( h \), is

\[
\dot{E} = \frac{dE}{dt} = \frac{1}{2} \left( \frac{2\zeta}{\zeta + 1} \right)^2 \frac{P_p}{\rho_m c_m} 2\pi ah \quad (A2)
\]

If the pressure amplitude were uniform in the cavity, the energy contained in the liquid would be

\[
E = \frac{1}{2} \frac{P_p^2}{\rho_o c_o^2} \pi a^2 h \quad (A3)
\]

But \( \dot{E}/E = \omega\eta = \omega/Q \).

Hence

\[
Q = \omega\dot{E}/E
\]

\[
= \omega \frac{\rho_m c_m}{\rho_o c_o^2} \left( \frac{\zeta + 1}{2\zeta} \right)^2 \frac{a}{2}
\]

\[
= \frac{\omega (\zeta + 1)^2 a}{8\zeta c_o}
\]

\[
\approx \frac{ka\zeta}{8} \quad (A4)
\]
Typically $\zeta = 33$ (for a steel/water interface) and for a radius $a = 60$ mm and a frequency of 6 kHz we obtain a Q of only 6.
Consider one element of the fluid of unit area and thickness $\delta x$ adjacent to a plate, where the relative velocity is $u = u_0 e^{j \omega t}$. The frictional force at the plate is

$$ F = \mu \frac{\partial u}{\partial x}, \quad \text{(B1)} $$

where $\mu$ is the dynamic viscosity, and the net force acting on the volume is

$$ F - (F + \frac{\partial F}{\partial x} \delta x) = - \mu \frac{\partial^2 u}{\partial x^2} \delta x. \quad \text{(B2)} $$

The equation of motion becomes

$$ \rho \frac{\partial^2 u}{\partial x^2} \delta x = - \mu \frac{\partial^2 u}{\partial x^2} \delta x $$

or

$$ \rho \frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial^2 u}{\partial x^2} = 0. \quad \text{(B3)} $$

Assume a solution of the form $u = u_0 e^{j(\omega t + kx)}$ [this satisfies the boundary condition at $x = 0$]. Then $\frac{\partial u}{\partial t} = j \omega u$ and $\frac{\partial^2 u}{\partial x^2} = - k^2 u$, so that $\rho j \omega - uk^2 = 0$, from which

$$ k = \pm \sqrt{\frac{\rho \omega}{\mu}} = \pm \sqrt{\frac{\rho \omega}{2\mu}} (1+j) \quad \text{(B4)} $$

and

$$ u = u_0 e^{\left( \pm \sqrt{\frac{\rho \omega}{2\mu}} (1-j)x + j \omega t \right)}. \quad \text{(B5)} $$

The negative sign must be taken to satisfy the boundary condition for a plate in an infinite medium that $u = 0$ at $x = \infty$. 


so that the force at the boundary is

$$\mu \frac{\partial u}{\partial x}_{x=0} = -\mu \frac{\sqrt{\omega \mu}}{2\mu^2} (1-j) u_0 e^{j\omega t}. \quad (B6)$$

The power loss (per unit area) is the product of the velocity and the component of force in phase with it, i.e.,

$$\frac{\dot{E}}{A} = -\sqrt{\omega \mu/8} u_0^2 \quad (B7)$$

(a factor 1/2 being included to give the rms value).
Appendix C

Computer Programme

For the identification of resonances found in any particular experiment a simple slide rule calculation using the relations in section 3.8 is quite adequate. However, for convenience a FORTRAN programme was written to calculate and tabulate these frequencies. The programme incorporates the calculations necessary for estimating the losses in the radially-symmetric modes using the equations in sections 3.9 through 3.12, and the losses are added so as to enable a calculation to be made of the $Q$ and the decay rate.

The data to be provided by the user includes a key to determine the type of vessel, the number of modes to be computed, the dimensions of the vessel, the sound velocity, density and dynamic viscosity of the liquid, and the wall thickness and Young's modulus, including the associated loss factor. The output is deliberately utilitarian in format so that any self-consistent set of units may be used; though the numbers resulting from the calculations are printed out to several figures, it must be remembered that these results are no more accurate than the input data and the assumptions underlying the basic equations. In particular, while the values of parameters governing the viscous losses are reasonably well known, one can usually only guess at the Young's modulus loss factor, so that the wall material losses can be in considerable error.

The programme listing follows, together with the input data cards used for the three sets of example calculations. The results for these calculations, which are in fact for three vessels which have been used experimentally, are given in Tables C1, C2 and C3.
CXX ZEROS OF THE 'J' BESSEL FUNCTIONS.

C IDENTIFY WIT'T STEGEN P. 409:
C ZEROS OF HIGHER ORDERS ASSOCIATED WITH ZERO ARGUMENT ARE
C OMITTED.)

CXX NOTE: 'N' IS INCREASED BY 1 IF COMPUTE PURPOSES.
DATA GPI(1,1), GPI(1,2), GPI(1,3), GPI(1,4), GPI(1,5), GPI(1,6), GI, 1
A/ 2.31201, 5.52901, 6.54371, 11.79153, 14.93091, 18.100601, 21.215641
B/ 3.21711, 7.01555, 10.17347, 13.32369, 16.47085, 19.15962, 22.160081
C/ 4.13562, 6.40721, 11.14904, 14.70959, 17.95992, 21.17004, 27.207111
D/ 5.16416, 7.54307, 11.01750, 14.52797, 19.43042, 23.94573, 29.541721
E/ 7.58044, 11.06711, 14.72541, 17.91970, 20.82649, 24.11027, 27.199041

READ(1,2) DAY

C O R E 2 

2 FCPWAT (3044)
1 READ(1,5) CDE
5 FPCWAT (2044)
1 IF(CDE(204), EC, EC) GO TO 999
WRITE(IOUT,11) DAY, CDE
10 FORMAT(1E14.6, 1E14.6)
READ(1,20) KEY, WMAX, NUMAX, WMAX, UNUMAX, RADMAX, HEIGH, C
INH, VISC, THCKAS, YLCLSS
20 FCPWAT (31170, 44)
25 IF((RADIUS LT 1.0E-1) OR (HEIGHT LE 1.0E-10 OR (C.LE.1E-10))
1 GO TO 99
IF( (KEY .EQ. 0) .OR. (KEY .EQ. 0) ) CC TO 990
IF( (KEY .EQ. 1) ) WRITE(IOUT,11)
IF( (KEY .EQ. 2) ) WRITE(IOUT,32)
IF( (KEY .EQ. 2) ) WRITE(IOUT,33)
31 FORMAT (10D14.6, 10D14.6) THIN-WEIRED CYLINDRICAL VESSEL
32 FORMAT (10D14.6, 10D14.6) THICK-WEIRED CYLINDRICAL VESSEL
33 FORMAT (10D14.6, 10D14.6) THIN-WEIRED GLASS CYLINDRICAL VESSEL
WRITE(IOUT,40) RADIUS, HEIGHT, C, YLCLSS
40 FORMAT (10D14.6, 10D14.6, 10D14.6, 10D14.6, 10D14.6, 10D14.6, 10D14.6)
WRITE(IOUT,40) RADIUS, HEIGHT, C, YLCLSS
45 FORMAT (10D14.6, 10D14.6, 10D14.6, 10D14.6, 10D14.6, 10D14.6, 10D14.6)
110M Y-LOSS FACTOR = 1.0G1/18
BETA = 0.
TRA = THCKAS/RADMAX
IPF(1). = IF( (KEY .EQ. 3) .OR. (TRA .LE. 1E-12) ) GO TO 47
C MODIFIED KERWEIC FORMULA
BETA = (1.5*C/(1.014**211711/TRA(2) +TRA(3) -47)
BETA = 1.5*C/(1.014**211711/TRA(2) +TRA(3) -47)
WRITE (ICUT,46) PTEAK
46 FORMAT (39H KORESP FREQUENCY REDUCTION FACTOR =,F8.5)
47 THMC = (RHIC,EF,LOF-10)
48 TVISC = (VISC,LE,LOF-10),ANL.(KEY.EQ.3)
49 TLOSS = (VLLSS,LE,LOF-10),ANL.(KEY.EQ.11)
50 TLOSS = TRHD,OP,TIVSC,OR,TY pos,

C**
USp TO SUPPRESS CALCULATION AND PRINTING OF LOSS FACTORS WHEN
DATA Is NOT SUPPLIED.
 IF (TLOSS) GO TO 55
WRITE (ICUT,50)
50 FORMAT (39H,5X,12HARMONIC MODES,37X,12H LOSS FACTORS/2X,THM NU N,2X,
10H FREQUENCY,10X,6HBRDFOM,4X,THWALL(1),4X,THWALL(1M),6X,3HETC,9X,10H
2X,4X,5XDELT A)
FACT1 = SQRT(2.*VISC/RHIC)
FACT2 = 4.*THICKS*VYLESS/(RHOS*ACLIUS**3)
FACT3 = HEIGHT*VYLESS*ETH/C
CC TO 57
55 WRITE (ICUT,56)
56 FORMAT (39H,5X,12HARMONIC MODES /
10H FREQUENCY/)
57 ANH = PI/HEIGHT
58 DO 90 K=1,MAX
41 = M-1
DO 90 NU=1,MAX
NU = NU-1
IF (KEY.EQ.1) GAMMA = GPR(NU)
IF (KEY.EQ.3) GAMMA = GPM(NU)
DO 90 K=1,MAX
IF (GAMMA.EQ.0.0) NUAMAX = 0
IF (GAMMA.GT.MANUMP) GC TO 90
AKH = GAMMA/FACTLS
AKRS2 = AK#AKP
FN = FLCAT(N)
IF (KEY.EQ.11) FA = FN-.5
AKL = AKH*FA
AKSQ = AKL*AKL
AKSC = AKPSQ + AKSQ
AK = SQRT(AKSC)
SIGMA = AK*AK
FREQ = CASE/PHASE(PI/2.3)
IF (M,NE.11) FRACTLS) GC TO 65
RTECH = 0.
WALLY = 0.
SRTRM = SORT(OMEGA)
IF (KEY.EQ.3) WALL=FACT2*AKPSQ/|AKSC|*OMEGAVS2)
IF (KEY.EQ.3) WALL = FACT1*AKPSQ/|AKSC|*RADIUS*XRTOM)
IF (KEY.EQ.3) WALL = FACT1*OMEGA
IF (KEY.EQ.11) RCTCHM = FACT1*AKSC/|AKSC|*HEIGHT*XRTOM)
ETA = WALL + WALLY + RCTCHM
DELTA = ETA*CASE/2.
WRITE (ICUT,60) M,NU,NUM,FREQ,RCTCHM,WALLY,Wall,ETA,DELTA
60 FORMAT (39H,11.2X,11.2X,11.2X,6G12.6,4X,6G11.4)
GO TO 70
65 WRITE (ICUT,66) M,NU1,A,FREC
66 FORMAT (39H,11.2X,11.2X,11.2X,6G11.4)
70 CONTINUE
90 WRITE (ICUT,100)
170 FORMAT (39H)
GO TO 1
990 WRITE (ICUT,991)
991 FORMAT (4H01,EM RUN TERMINATED DUE TO ERROR IN DATA CARD//)
999 WRITE (ICUT,998)
998 FORMAT (100H)
100 STOP
END
//DATA INPUT NO *
8 JAN 1975
THIN STEEL WALL, BRASS PICTOP VESSEL
2335
613.0 9-0314.0 0-0314.4 E 031.0 E 030.001 0.4 0-032.0 0 E 111.0 0-04
1800 ML REAVER
1335
613.0 9-0315.0 0-031.4 E 031.0 E 030.001 0.8 0-032.0 0 E 111.0 0-04
ALUMINIUM ALLOY THICK-WALLED VESSEL
3335
4.5 9-03170.6 0-031.44 E 031.0 E 030.001 13.0 0-037.1 0 E 101.0 0-04
*/
TABLE Cl.

8 JAN 1975
THIN STEEL WALL, BRASS POTION VESSEL
NORMAL MODES, WALL MATERIAL LOSSES AND VISCOSITY LOSSES
IN A THIN-WALLED, THICK-BOTTOMED CYLINDRICAL VESSEL
RADIUS = 0.63006-01
HEIGHT = 0.1430
SOUND VELOCITY = 1490.
FLUID DENSITY = 1000.
VISCOSITY = 0.1000E-02
WALL THICKNESS = 0.4000E-03
YOUNG'S MODULUS = 0.7000E 12
Y LOSS FACTOR = 0.1000E-03

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TABLE C2.

8 JAN 1975

1000 PL BEAKER

NORMAL MODES AND WALL MATERIAL LOSSES IN A THIN-WALLED, THIN-WALLED CYLINDRICAL VESSEL

RADIUS = 0.6150E-01
HEIGHT = 0.1530

SOUND VELOCITY = 1490,
FLUID DENSITY = 1000.
VISCOSITY = 0.1000E-02

WALL THICKNESS = 0.8000E-03
YOUNG'S MODULUS = 0.2000E 12
Y LOSS FACTOR = 0.1000E-03

| MODAL M 
| FREQUENCY |
|---|---|---|---|---|
| 0 0 1 | 10471.6 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 0 2 | 13447.2 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 0 3 | 17302.5 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 0 4 | 21571.8 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 0 5 | 26952.5 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 1 1 | 21835.9 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 1 2 | 23407.2 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 1 3 | 25915.6 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 1 4 | 28851.6 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 1 5 | 32740.9 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 2 1 | 33721.7 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 2 2 | 34760.4 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 2 3 | 36425.7 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 2 4 | 38166.8 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 2 5 | 41406.1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 3 1 | 45566.6 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 3 2 | 47076.7 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 3 3 | 49777.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 3 4 | 52447.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 3 5 | 55118.8 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 4 1 | 57748.5 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 4 2 | 60375.1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 4 3 | 63043.9 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 4 4 | 65734.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0 4 5 | 68444.1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 1 1 | 39529.4 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 1 2 | 40419.1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 1 3 | 41399.9 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 1 4 | 43797.2 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 2 1 | 20397.5 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 2 2 | 22067.8 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 2 3 | 24607.6 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 2 4 | 27716.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 2 5 | 31383.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 3 1 | 32815.7 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 3 2 | 33686.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 3 3 | 35592.3 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 3 4 | 37852.1 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 4 1 | 45065.2 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 4 2 | 45851.6 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1 4 3 | 47126.7 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
TABLE C3.

8 JAN 1975
ALUMINIUM ALLOY THICK-WALLED VESSEL
NORMAL MODES, WALL MATERIAL LOSSES AND VISCOSITY LOSSES
IN A THICK-WALLED, THICK-BOTTOMED CYLINDRICAL VESSEL
RADIUS = 0.6150E-01
HEIGHT = 0.1706
SOUND VELOCITY = 14900.
FLUID DENSITY = 100 C.
VISCOITY = 0.1000E-02
WALL THICKNESS = 0.1100E-01
YOUNG'S MODULUS = 0.7100E 11
Y LOSS FACTOR = 0.86707
KORTEG FRICTION REDUCTION FACTOR = 0.86707

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APPENDIX D

A LOGARITHMIC AMPLIFIER

When a permanent record was not needed, and for the shorter decay times for which the Brüel and Kjaer recorder was not suitable, it was the practice to observe the exponential decay of the envelope of the received signal directly on an oscilloscope. It was thought that a logarithmic amplifier would be of use, as the decay curve displayed would then be linear. Accordingly the circuit of Figure D1 was built in breadboard form and tested.

The essence of this circuit is the negative feedback applied to a high-gain integrated circuit amplifier by means of silicon diode junctions. The relationship between voltage V and current I of such a junction is

\[ I = I_s \left[ e^{qV/kT} - 1 \right] \]

where \( q \) is the magnitude of the electronic charge,
\( k \) is Boltzman's constant
and \( T \) is the absolute temperature.

Here \( I_s \) is the asymptote of the diode reverse current \( D1 \).

The equation may be rewritten

\[ V = \frac{(kT/q)\log_e(I/I_s + 1)}{\log_e(I/I_s + 1)} \]

The amplifier operates so as to bring the input current and the feedback current into balance with its input voltage near zero. Thus the output of the whole device is proportional to the logarithm of the instantaneous input voltage, the positive and negative parts of the waveform being controlled by the appropriate diode chain, except for
Figure D1. Logarithmic Amplifier Schematic Diagram.
a small region around zero input voltage, which is subjected to the full open-loop gain of the amplifier.

The performance of the breadboard amplifier is shown graphically in Figure D2. It is seen that a logarithmic relationship exists over a 50 dB dynamic range, and it is believed that with more care this could easily be extended. (From the slope of the curve it is concluded that each 1N914 diode unit contains two junctions in series.)

However, in experimental use the behaviour was disappointing. The amplifier seemed to be highly susceptible to initial overload due to switching surges (even though it was operated from its own batteries) and any power line ripple superimposed additively on the desired signal distorted the waveform, so that it was found impossible to get meaningful results. Indeed, the resulting waveform seemed quite variable from one occasion to another. Figure D3 is a tracing of one oscilloscope display, taken just after the display in Figure 13.

While it is probable that these difficulties can be overcome it was felt that further effort in this direction was not immediately justified, and either the recorder or the exponential display was used in all the actual measurements.
Figure D2. Response of Logarithmic Amplifier to Slow Changes of Input Level.

Figure D3. Typical Response Envelope of Logarithmic Amplifier to Exponential Decay.
APPENDIX E

RESONANT FREQUENCY FOR THICK-WALLED VESSEL
WITH CONICAL TAPERED SECTION

The general case is illustrated in Figure El. It is convenient to choose three different coordinate origins for the three sections as by so doing the equations can be written in a more simple form. The relationship between the coordinate systems is evident from the diagram.

For the cylindrical section OA an origin of coordinate $x$ at the bottom 0 is chosen. Assuming that the wave front is plane, and since there is a node at this point, we can write the pressure

$$p = p_0 A \cos(kx),$$

(E1)

where $A$ is some multiplying factor.
The velocity is

\[ u = \frac{1}{k \rho c} \frac{dp}{dx} \]

\[ = -u_o Ak \sin(kx) \quad \text{where} \quad u_o = \frac{ip_o}{k \rho c} \quad (E2) \]

For the conical section AB it is more convenient to use an origin at the apex of the cone, and following Rayleigh, we can write

\[ p = p_o \frac{B \cos(kx + \phi)}{x} \quad (E3) \]

and

\[ u = -u_o B \left[ \frac{k \sin(kx + \phi)}{x} + \frac{\cos(kx + \phi)}{x^2} \right] \quad (E4) \]

where \( B \) and \( \phi \) are amplitude and phase constants to be found.

Implicit in these equations is an assumption of spherical spreading of the wave front in the conical section.

For the open-ended cylindrical section a third coordinate \( y \), measured from the open end, is used, and we have

\[ p = p_o C \sin(ky) \quad (E5) \]

and

\[ u = -u_o Ck \cos(ky) \quad (E6) \]

(a negative sign being introduced for the velocity because of the change in direction for measurement of coordinate \( y \)).

Assuming coincidence of the plane wave in the section OA and the spherical wave in the section AB, we have for continuity of pressure at this boundary:

\[ A \cos(ka) = (B/d) \cos(kd + \phi) \quad (E7) \]
and for continuity of velocity

\[ A_k \sin(ka) = B \left[ (k/d) \sin(kd + \phi) + \frac{1}{d^2} \cos(kd + \phi) \right], \quad (E8) \]

whence we can write an admittance equation

\[ \tan(ka) = \tan(kd + \phi) + \frac{1}{kd}, \quad (E9) \]

Using the standard expansion for \( \tan(kd + \phi) \) the last equation may be rewritten

\[ \tan(kd) + \tan \phi = \left[ 1 - \tan(kd) \tan \phi \right] \left[ \tan(ka) - \frac{1}{kd} \right], \]

or

\[ \tan \phi \left[ 1 + \tan(kd) \left[ \tan(ka) - \frac{1}{kd} \right] \right] = \tan(ka) - \tan(kd) - \frac{1}{kd}, \quad (E10) \]

Similarly at the boundary \( B \)

\[ (B/b) \cos(kb + \phi) = C \sin(kc) \quad (E11) \]

and

\[ B \left[ (k/b) \sin(kb + \phi) + \frac{1}{b^2} \cos(kb + \phi) \right] = Ck \cos(kc), \quad (E12) \]

whence

\[ \tan(kb + \phi) + \frac{1}{kb} = \cot(kc) = \tan\left(\frac{\pi}{2} - kc\right) \]

or

\[ \tan \phi \left[ 1 + \tan(kb) \left[ \tan\left(\frac{\pi}{2} - kc\right) - \frac{1}{kb} \right] \right] = \tan\left(\frac{\pi}{2} - kc\right) - \tan(kb) - \frac{1}{kb} \quad (E13) \]
From these we obtain the frequency equation:

\[\frac{\tan(ka) - \tan(kd) - 1/kd}{\{1 + \tan(kb) \left[\tan\left(\frac{\pi}{2} - kc\right) - 1/kb\right]\}} - \left[\tan\left(\frac{\pi}{2} - kc\right) - \tan(kb) - 1/kb\right] \cdot \left\{1 + \tan(kd) \left[\tan(ka) - 1/kd\right]\right\} = 0. \quad (E14)\]

This equation may conveniently be solved for \( k \) to any desired accuracy for a particular case by an iterative procedure using a digital computer, and values for \( \phi, B/A \) and \( C/A \) may then be obtained. For the vessel of Figure 19, filled to within 10 mm of the top, for which \( a = 25 \text{ mm}, b = 50 \text{ mm}, c = 130 \text{ mm}, d = 25 \text{ mm}, k \) was found to be 10.3796 m\(^{-1}\), to a computational accuracy far exceeding the accuracy implied by the input data and the assumptions made. The pressure magnification \( A/C \) is found to be 1.07. The tapered section and the small end have a volume equivalent to that of a length of 18 mm of the full diameter, and thus the equivalent depth for a cylindrical vessel would be 148 mm, for which \( k = 10.6 \text{ m}^{-1} \). The 2.2% discrepancy in frequency is well within the accuracy implied by the input data and the assumptions regarding the boundary conditions.

The kinetic energy may be obtained by integration over the volumes of the three sections in a manner analogous to section 3.9.

For the section B to C

\[E = \frac{1}{2} \pi r^2 \rho u_o^2 C^2 k^2 \int_0^C \cos^2(ky) \, dy\]

\[= \frac{1}{8} \pi r^2 \rho u_o^2 C^2 k \left[2kC + \sin(2kC)\right]\]

\[= \frac{1}{8kpc^2} \pi r^2 \rho_o^2 C^2 \left[2kC + \sin(2kC)\right] \quad (E15)\]
For the section 0 to A the radius is \(rd/b\), and

\[
E = \frac{1}{8k_0c^2} \pi r^2 \rho A^2 (d/b)^2 [2ka - \sin(2ka)]
\]

(E16)

For the section A to B, of varying radius \(rx/b\),

\[
E = \frac{1}{8k_0c^2} \pi r^2 B^2 (1/b)^2 4k^2 \\
\int_0^b \left[k \sin(kx + \phi) + (1/x) \cos(kx + \phi)\right]^2 dx
\]

(E17)

When this is expanded, all the terms lead to tabulated functions.

Experiment has shown that the shape of vessel does not lead to any significant improvement in \(Q\) over a straight cylinder, and since there is no pressure amplification the complication of the tapered section serves no useful purpose, merely restricting the volume of a sample that can be measured. It was therefore decided to abandon this approach, and the calculation has not proceeded further.
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