The effect of the surrounding media on the distribution of electromagnetic fields in a conductor

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The Effect of the Surrounding Media On the
Distribution of Electromagnetic Fields In a Conductor

by

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Submitted for Degree of Doctor of Philosophy

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SUMMARY

The distribution of the current-density, and the associated electric and magnetic fields within long, straight conductors of circular and rectangular cross-section, is well known. Much of the associated analytical and experimental work was stimulated by the development of high-frequency line communications.

The theoretical work was naturally based on the differential equation form of Maxwell's field equations. The concepts of field impedance and equivalent field networks were evolved to aid the understanding of line, waveguide and indeed, free-space transmission systems.

The work described in this thesis has arisen out of an attempt to exploit the field impedance and equivalent field network concepts to explain the current-density and field distributions in inductor and transformer-like structures.

The physical models, which were investigated both analytically and experimentally, were designed such that their behaviour could, hopefully, be explained in terms of one-dimensional theory.

During the course of the work, each new configuration presented problems because the early experimental results failed to support the theory. In each case the sources of the major discrepancies were explained by identifying the likely causes of error and modifying the physical model accordingly. The alternative approach of elaborating the mathematical model to explain the experimental results would, very likely, lead to much tedious and unrewarding work.
The equivalent field network, Fig. 2.2-2, representing the distribution of fields in the energizing strip, is believed to represent an original contribution. It was important, therefore, to obtain experimental support for this mathematical model before proceeding to more sophisticated structures. This work is described in Sections 6 and 7.

Section 8 is concerned with the propagation of "waves" through the walls of a copper "box", simulating the secondary winding of a transformer; a short-circuited single turn, Fig. 8.1-1. Whilst interesting in its own right, this investigation was a necessary preliminary to Section 9, in which the transformer secondary turn is half copper and half zinc. The need to understand the behaviour of a model of this kind arose in connection with some Helicon-wave investigations at Harwell.

In Section 10, the model of Section 8 is extended to incorporate an external load into which power is fed. Arising out of this work is an unconventional explanation of transformer action, which is interesting because it embodies circuit and field concepts in one simple idealised model.
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External Adjustable Load. All Four Plates of Fig. 10.1-1 Assembled and Wired.

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LIST OF PRINCIPAL SYMBOLS

S.I. units are used throughout the analytical work, but some experimental results are displayed in derived units; Eg: - Electric field in \( \mu \text{V/cm/amp} \).

- \( E \) = Electric field
- \( E_s \) = Sending-end field
- \( E_r \) = Receiving-end field
- \( H \) = Magnetic field
- \( \phi \) = Magnetic flux
- \( J \) = Current density
- \( I \) = Current
- \( V \) = Potential difference
- \( \sigma \) = Conductivity
- \( \rho \) = Resistivity
- \( \mu \) = Permeability
- \( \varepsilon \) = Permittivity
- \( L \) = Inductance
- \( C \) = Capacity
- \( R \) = Resistance
- \( G \) = Conductance

per meter when applied to transmission lines

(vii)
\[ Z = \text{Impedance, both field impedance and circuit impedance.} \]

\[ Z_0 = \text{Characteristic field impedance of medium} \]

\[ \Gamma = \text{Propagation coefficient of medium} \]

\[ S_1, S_2, S_3, S_4 = \text{Metal plates - strips - of either Cu or } Z_n \]

\[ a = \text{Thickness of metal plates} \]

\[ b = \text{Width of energizing plates} \]

\[ c, g = \text{Distances between plates} \]

\[ l, L = \text{Length of plates - there is unlikely to be confusion between this and "inductance".} \]

\[ x, y, z = \text{Cartesian Co-ordinates} \]

\[ r = \text{Radius or resistance} \]
The Effect of the Surrounding Media On the
Distribution of Electromagnetic Fields In a Conductor

Section 1. Introduction

The distributions of $E$, $H$ and $J$ as functions of space and frequency - or time - in very long, isolated conductors of either circular or rectangular - flat strip - cross-sections are well known. A total current $I_0 e^{j\omega t}$ is usually assumed to flow along the conductor. The return path for this current is frequently neglected in order to avoid considerable and often unwarranted complications. The phenomena of "skin effect" are satisfactorily explained this way.

Consider now the theory of the coaxial transmission line as an example in which the return path is clearly defined. This is usually developed on the assumption that the conductors are of infinite conductivity, so that a straightforward application of a one-dimensional wave equation, including dielectric losses, leads to a satisfactory explanation of the longitudinal wave propagation and field distributions within the dielectric.

In the practical case where the dielectric walls are bounded by surfaces of finite conductivity, a rigorous analysis would be most complicated. The isolated conductor solution is not directly applicable, but there is a generally accepted artifice for overcoming this difficulty. Fig. 1.1 (a) is the equivalent network for an elemental length of coaxial line with perfect conductors, and Fig. 1.1 (b) its counterpart with imperfect conductors.
The value of the series resistance, $R$ per meter, is derived by assuming radial propagation due to a longitudinal electric field, resulting from current flow in the conductors, in conjunction with the existing circular magnetic field. This radial propagation into the metallic walls is a diffusion process rather than a wave one, but the important point is that Fig. 1.1 (b) is the key, presumably intuitive, to combining the results of the two transport processes, one longitudinal and the other radial; $R$ turns out to be frequency dependent.

These ideas are common-place in the technology of line and waveguide transmission where the differential and integral forms of the field equations are applied with equal facility.

Another common "circuit" in which the conduction current path is clearly defined is the loop of wire or flat strip Fig. 1.2 which may be regarded as an idealised inductor.
A practical conducting loop will have finite conductivity, in which case an equivalent network of resistance $R$ in series with inductance $L$ is frequently an adequate representation. The resistance is usually regarded as a property of the strip itself but that can be a gross oversimplification, because the current density distribution over the cross section of the strip depends upon the media on either side of, i.e. outside, the strip. The analysis of this (commonly used) structure does seem to have been rather neglected.

The first part of this thesis (Section 2) describes the development of an equivalent field network or mathematical/pictorial model which represents the behaviour of a conducting loop but which is simplified to the extent that only one space variable, radial distance, is taken into account. As with the transmission line, some intuitive thinking is involved.

From Section 3 onwards the thesis describes experimental and theoretical work arising out of the basic model of Section 2. The experiments have been designed in each case to support a one-dimensional theory, so that the physical models chosen are somewhat idealised. Nevertheless, the results are relevant to many technological problems to which reference is made in the appropriate sections.
FIG. 2.1-1 A SIMPLE INDUCTOR EXPRESSED IN TERMS OF FIELD IMPEDANCES
Section 2. A One-Dimensional Field Network Model of an Inductor

2.1 Strip of Infinite Conductivity

The conducting loop of Fig. 1.2 is formed from a flat infinitely thin strip of infinite conductivity. At low frequencies it behaves as an inductance of \( \frac{\mu_0}{2} \frac{r_c^2}{L} \) henries, if we neglect end-effects by, say, assuming that the cylinder is part of a toroid. This result is normally derived from the integral form of the field equations, the approach, with the assumption that the field \( H^+ \) outside the loop, i.e. at \( r > r_c \), is zero. The magnetic \( H^- \) field inside the loop is uniform and equal to \( I_0 \frac{r_c}{L} \) amps/m, so that \( \Phi = \mu_0 \pi r_c^2 I_0 \frac{r_c}{L} \).

The same result can be obtained as a low-frequency approximation to a more general solution derived from an application of the differential form of the field equations\(^1\) including the concept of field impedance.\(^3,4\). This latter we shall use as a major analytical tool in the following pages.

In Fig. 2.1-1 \( Z^+ \) represents the ratio \( \frac{E^+}{H^+} \) of the radially outward wave propagation derived from a straightforward application of the field equations in one-dimensional circular co-ordinates.\(^1\) See also Fig.1.2.

At radius \( r_c \), \( Z^+ \) turns out to be:

\[
Z^+(r_c) = Z_1 \frac{K_1(\pi r_c)}{K_0(\pi r_c)}
\]

Where
\[
Z_1 = \frac{\gamma_0 \mu_0}{\gamma \sigma + j \omega \varepsilon_1}
\]

the intrinsic or characteristic field impedance of the medium outside the loop.
is the intrinsic propagation coefficient of the medium outside the loop.

For free space, \( Z_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \) and \( \Gamma_1 = j\omega\sqrt{\mu_0\varepsilon_0} \), and by using the small argument approximations to the \( K_n \) Bessel functions, \( Z_1^+ \) is found to approach infinity at low frequencies so that \( H^+ \) approaches zero since \( \frac{E^+}{H^+} = Z^+ \).

In Fig. 2.1-1 the radial propagation inwards from \( \gamma = \gamma_0 \) towards the centre, is represented by:

\[
Z^-(r_0) = Z_2 \frac{I_n \left( \frac{r_0}{r_0} \right)}{I_0 \left( \frac{r_0}{r_0} \right)}
\]

For a free-space medium and at low frequencies where the \( I_n \) Bessel functions can be suitably approximated:

\[
Z^-(r_0) = \frac{j\omega\mu_0r_0}{2}
\]

Looking now at Fig. 2.1-1, we see that \( E = H^+ \frac{j\omega\mu_0r_0}{2} \) and because \( Z^+ \) is infinite, \( H^- = H_0 = \frac{I_0}{r_0} \).

Hence

\[
E = \frac{I_0}{r_0} \frac{j\omega\mu_0r_0}{2}
\]
So that \( V_0 \), which is the line integral of \( E \), becomes

\[
V_0 = \frac{I_0}{b} j \omega \mu_0 \frac{2 \pi r_0}{2} = I_0 \frac{j \omega \mu_0 \pi r_0}{b} \quad \text{--- \text{2.1-6}}
\]

This approach is, of course, unnecessarily complicated for solving the simple circuit Fig. 2.1-1, but it comes into its own when the field impedances on either side of the energizing strip are comparable in magnitude; an iron core inside, and a conducting secondary load on the outside of Fig. 1.2 for example.

2.2 Strip of Finite Conductivity

So far we have ignored the properties of the energizing strip itself, viz. its thickness, conductivity and permeability. When these factors are taken into account, we can guess from the isolated strip-conductor analysis that at very low frequencies the current will spread itself uniformly over the cross-section of the strip, and at very high frequencies it will concentrate on the surface because of the skin effect. But how much will flow on the outer surface compared with that on the inner surface? We must, in fact, be able to calculate the distribution of the fields within the strip in order to understand and compute the contribution which the energizing strip makes to the total impedance of the system, i.e. to evaluate the so called "copper losses" of the inductor or transformer. In other words, the effective series resistance of the energizing winding depends upon the media on either side into which power is flowing; the winding impedance can in fact vary by as much as two to one due to "proximity effects".
The propagation of plane "waves", through a conducting medium is in fact a diffusion process and can be expressed in terms of an elemental line\(^2\), Fig. 2.2-1(a) somewhat similar to Fig. 1.1(a).

**FIG. 2.2-1 EQUIVALENT FIELD NETWORKS FOR A CONDUCTING MEDIUM**

Fig. 2.2-1(b) is the equivalent field network for a strip of thickness "a".

\[
Z_o = \sqrt{\frac{j\omega \mu}{\sigma}} \quad \quad \quad 2.2-1
\]

\[
\Gamma = \sqrt{\frac{j\omega \mu \sigma}{\epsilon}} \quad \quad \quad 2.2-2
\]

These two parameters \(Z_o\) and \(\Gamma\) are similar to equations 2.1-2 and 2.1-3 but with zero permittivity.

This well known model can be obtained by a straightforward application of the one dimensional wave equations \(\nabla^2 \vec{E}\) and \(\nabla^2 \vec{H}\) in cartesian co-ordinates.\(^6\)
The great virtue of Fig. 2.2-1(b) - there is in fact a " \Pi " or delta equivalent - is that it completely expresses the characteristics of the one-dimensional medium devoid of boundary conditions which can so easily obscure the basic nature of the transport process. The value of this pictorial-mathematical model approach for dealing with linear transport processes of all kinds can hardly be over-estimated. It can be applied quite easily in two dimensions and with certain restrictions in three dimensions\(^5,6\). In electrical engineering terms, the boundary conditions of linear transport processes of all kinds are expressions of terminating "impedances", i.e. the ratio of the two dependent variables involved in the process. It is for this reason that the impedance concept is such a powerful tool in the application of field theory to many disciplines.

Accepting now, that the energizing strip can be represented by Fig. 2.2-1(b), and that the media on either side of the strip can be represented by the field impedances \( Z^+ \) and \( Z^- \) as in Fig. 2.1-1, how should the two transport processes be combined? This, like the transmission line problem, appears to demand an intuitive step, and Fig. 2.2-2(b) seems a plausible way of taking it. The article\(^2\), in which this was published, gave rise to no correspondence, so there has been no indication of the correctness, worthwhileness or originality of this representation.

Section 3. The Design of an Experiment to Check the Mathematical Model of Fig. 2.2-2(b).

3.1 An Idealised Physical Model in Cylindrical Geometry

The object of the first investigation was to devise suitable experiments to see to what extent the influence of the external fields on the current distribution in the energizing strip can be explained by the model of Fig. 2.2-2(b). In other words, to determine by calculation and experimentally, the effect of the surrounding media on the distribution of the fields within the energizing conductor.
A comparable technique for developing equivalent circuits, usually referred to as the transverse resonance method (ref. Collin), is used in relation to waveguide propagation. This method is readily applicable to closed structures, but difficulties arise with open structures such as those discussed in this thesis. The essential point in developing equivalent circuits is to arrange the interconnections in such a way that the field boundary conditions are satisfied. A modification of this approach can be used to provide a more rigorous derivation of the equivalent circuits used in this thesis. Some differences in the interpretation of the equivalence between the field and circuitry problems arise but these do not affect the electric field values which are investigated experimentally.
Fig. 2.2–2. Equivalent field network including energizing strip.
Briefly then, we need to devise an experiment in which the relative magnitudes of $Z^-$ and $Z^+$ can be controlled and calculated, and in which the electric fields $E_1$, $E_2$, $E_3$, $E_4$ and $E_T$ can be both calculated and measured; see Fig. 2.2-2(b).

A cylindrical air cored inductor, energized by a strip of thickness "a", will, at relatively low frequencies, look like Fig. 3.1-1(a).

At rather higher frequencies it will look like Fig. 3.1-1(b) since $\coth \Gamma a$ approaches unity as $\Gamma a$ approaches infinity.

If now a tightly fitting metallic tube, "d" metres thick, is placed over the energizing strip, but insulated from it by an infinitely thin layer of insulating tape, the equivalent field network will look like Fig. 3.1-2(a). At rather higher frequencies the hyperbolic approximations simplify to produce Fig. 3.1-2(b). In this instance the energizing strip and outer tube are of the same material.
OUTER TUBE: INSULATION

\[ Z_0 \left( \cosh \Gamma d - 1 \right) \frac{1}{\sinh \Gamma d} \]

\[ H_0 = \frac{I_0}{b} \]

(a) OPEN CIRCUIT

Z\to\infty \quad \omega \to 0

FOR FREE SPACE

ENERGIZING STRIP

H_0

(b) "HIGH-FREQUENCY" APPROXIMATION OF (a).

FIG. 3.1-2. AIR-CORED INDUCTOR WITH OUTER CONDUCTING TUBE.
Fig. 3.1-1 and Fig. 3.1-2 show clearly that the presence of the outer tube will influence the distribution of the fields and the current densities over the inner and outer surfaces of the energizing strip and hence throughout its cross-section.

Fig. 3.1-3 shows how, in theory, the various electric fields can be measured. The energizing strip and the outer tube have been well separated to simplify the diagram. Likewise, the field measuring leads are shown separated from the surfaces along which they should lie, insulated, but as closely as is physically possible.

The measured voltages are, of course, the line integrals of the associated electric fields. We note that - Fig. 3.1-3(b) - in general \( V_3 \) will not be equal to \( V_4 \), and yet they might appear to be measures of potential differences between the same two points a and b in Fig. 3.1-3(a). In fact, the difference between \( V_3 \) and \( V_4 \) is counterbalanced by the time rate of change of flux \( \frac{d\phi}{dt} \), due to the field actually inside the energizing strip. Considering this in rather more detail we return for a moment to Fig. 2.2-2(b) and work out \((E_3 - E_4)\) in terms of two generalised impedances \( Z^- \) and \( Z^+ \), calculate \( H \) throughout the strip and from it determine, by integration, the total flux in the strip, and hence \( \frac{d\phi}{dt} \).

We then see that the line integral of \( E \) is the same as \( \frac{d\phi}{dt} \).

This, of course, is an application of the integral form of Faraday's Law:

\[
\oint E \, dl = -\frac{d\phi}{dt} = -\iint \mu H \, ds.
\]
(a) 

ENERGIZING STRIP

OUTER TUBE

(b) 

$H_0 = \frac{I_0}{b}$

$V_2$ & $V_3$ MUST LIE ALONG INNER SURFACE & $V_4$ ALONG OUTER SURFACE OF ENERGIZING STRIP. $V_1$ ALONG INNER SURFACE OF TUBE.

FIG. 3.1-3. ARRANGEMENT OF ELECTRIC FIELD-MEASURING LEADS.
The distances along which the electric fields are measured and integrated in Fig. 3.1-3 are unimportant, but the higher the voltage the easier it is to measure because we shall be in the micro-volt region. We note also that the lead could follow a complete revolution, in which case there need be no electrical connection between it and the inner surface of the tube.

The construction of an experimental set-up in circular geometry is likely to be quite expensive as well as being difficult to assemble and to modify compared with a parallel plate equivalent or near-equivalent. The design of a suitable parallel-plate inductor assembly is described in 3.2.

3.2 A Physical Model in Rectangular Geometry

The essential geometry is detailed in Fig.3.2-1(a), in which an infinite conductivity energizing strip is assumed.

The \( \frac{d\phi}{dt} \) approach produces an inductance of \( \mu_0 \frac{L}{G} \). Considering the loop as a parallel plate transmission line, with \( E_z \times H_\infty \) propagating along the \( OY \) direction, the field-impedance \( Z_\infty \) looking into the sending end of the line is:

\[
Z_\infty = Z_0 \tan \h R \Gamma L - - - - - - - - - - 3.2-1
\]

\[
Z_0 = \frac{\mu_0}{N \xi}
\]

\[
\Gamma = j \omega \sqrt{\mu \xi}
\]

since we are assuming an air core.

At frequencies such that the line is but a small fraction of a wavelength long, equation 3.2-1 reduces to:

\[
Z_\infty = j \omega \mu_0 L
\]
FIG. 3.2-1 THE PARALLEL-PLATE AIR-CORED INDUCCTOR
Assuming again that the outward looking field-impedance is infinite then

\[ H = \frac{I}{\nu} \]

So that

\[ E_s = \frac{I_0}{\nu} Z_s \]

\[ = I_0 \frac{j \omega \mu_0 L}{\nu} \]

and

\[ V_s = I_0 \frac{j \omega \mu_0 L C}{\nu} \]

It will be more relevant to our subsequent analysis, however, if we work in terms of \( E_y \times H_x \) and, hence, consider propagation along the \( OZ \) axis. Fig. 3.2-1(a) (long-wavelength)

Low-frequency assumptions allow both \( E_y \) and \( H_x \) to be regarded as uniform in the \( OX \) and \( OY \) directions, varying only along the \( OZ \) direction. The low-frequency equivalent field network is shown in Fig. 3.2-1(b).

We must remember that this network simply represents two infinitely long strips carrying currents in opposite directions; each one is radiating and reflecting plane electromagnetic waves of wavelength very long compared with the distance "c" between them. We are not, therefore, in this particular picture, at liberty to join the ends together or to consider fields other than \( H_x \) and \( E_y \).
$E_{su}$ is the electric field developed along the upper strip, and $E_{sl}$ that along the lower strip. The total voltage around the loop is $L(E_{su} + E_{sl})$, i.e. $\frac{\text{I}_0}{\text{L}} j\omega \mu_0 C$, which agrees with equation 3.2-2.

We note that the symmetry of Fig. 3.2-1(b) is such that there is no electric field across the shunt capacitance, so that each half of the system can be considered separately, as in Fig. 3.2-1(c), to which we have added a finite outward-looking impedance to complete the picture.

Incorporating now the finite thickness and conductivity of the energizing strip, we arrive at Fig. 3.2-2 as the equivalent field network for one half of the system. This is identical with Fig. 3.1-1 for the circular inductor but with "C" replacing "$Y_0$", so that the discussion in Section 2 is directly applicable to the half-section of the parallel-plate inductor.

---

**FIG. 3.2-2. EQUIVALENT FIELD NETWORK FOR ONE HALF OF PARALLEL-PLATE INDUCTOR.**
Section 4. The Inductor Assembly

A non-ferrous material is desirable to avoid uncertainties in the value of permeability, so that copper is a fairly obvious choice, at least initially. Soldered connections are easily made, and the conductivity of commercial copper seems to be well established at $6 \times 10^7$ mho/metre.

The physical dimensions must be related to the frequency spectrum to be explored, and since we shall be measuring micro-volts, a fair degree of frequency selectivity is necessary. The Marconi Wave Analyser TF 2330 covering the frequency range from about 20 Hz to 50 kHz with a band-width of some 5 Hz, has proved to be very suitable.

The attenuation $\alpha l = \frac{1}{2} \log f$ of copper to plane waves is about 14 dB per mm at 10 kHz, and it turns out that the surface impedance is within 5% of $Z_0 = \sqrt{\frac{\mu_0}{\omega}}$ at 10 kHz and above for 3 mm thickness. This assumes energization from both surfaces; the worst case. This thickness will enable the simplified conditions of Fig. 3.1-2(b) to be established experimentally at and above 10 kHz.

Even at 50 kHz the free space wavelength of plane-waves is so long that the low-frequency approximations associated with dimensions $L$, $\mu$ and $C$ of Fig. 3.2-1(a) are readily satisfied. There is, however, another factor, in that we have so far used no more than a one-dimensional analysis to represent what is, in fact, a three-dimensional system. On this score, it is reasonable that $C$ and $\alpha$ should be very small compared with $L$ and $\mu$. Another reason for keeping $C$ very small is that $\frac{j \omega \mu C}{2}$ should be of the same order of magnitude as $Z_0$, see Fig. 3.1-2(b), so that these terms can be measured to similar orders of accuracy.
\[ |Z_o| \text{ for copper at } 8 \text{ kHz is about } 3 \times 10^{-5} \Omega, \text{ and } \frac{j \omega \mu \epsilon}{2} \]

with \( \epsilon = 2 \text{ mm at } 8 \text{ kHz is } 6 \times 10^{-5} \Omega \).

SRBP sheet of 1 mm thickness is readily obtained, and we decided to make \( \epsilon = 2 \text{ mm at least for the initial experiments.} \)

The ratio of \( L \) to \( \varphi \) should be large to obtain a uniform distribution of current in the XY plane, see Fig.3.2-1(a). Some analytical work was undertaken to determine the connections between \( L, \varphi \) and the current feed conditions. This is described in Appendix I and the plate construction is shown in Fig. 4.1

---

**FIG. 4.1. INDUCTOR PLATE CONSTRUCTION.**

- Field-measuring leads soldered at points 1 cm in from each end.
- \( L = 10 \text{ cms.} \)
- \( b = 5 \text{ cms.} \)
- \( a = 3 \text{ mm.} \)
On the assumption that each of the ten feed wires carry equal currents, the longitudinal field distribution should be uniform to within 1% at 0.5 cm in from the ends. Because of measurement difficulties, no great effort was spent on trying to measure the detailed field distributions; they were found to be uniform to within ±10% at DC using a potentiometric method. Instead, we concentrated on measuring the fields on both sides of the plate over the 8 cm lengths from about 100 Hz to 20 kHz, and this brought to light the first major practical difficulty associated with the experimental work. It became clear that it was necessary to place the field measuring leads extremely close to the metallic surface along which the field is to be measured. Various lead arrangements have been tried with encouraging results, and they will be described later.

Fig. 4.2 shows the inductor assembly, current-feed arrangements and measuring system. The measuring system will be described in Section 5.

The reactance of the inductor, \( C = 2 \text{ mm} \) is about \( j2 \times 10^{-3} \Omega \), and the impedance of the plates about \( 4 \times 10^{-4} \Omega \) at 50 kHz. The resistive feeds add up to 0.02 \( \Omega \), so the drive transformer was designed to feed 1 amp into 0.03 \( \Omega \) from a Marconi TF 885A/1 oscillator. On test it was found that 12.0 volts were required from the oscillator at 100 Hz rising fairly steadily to about 20V at 50 kHz. Below 200 Hz the waveform was poor due to a slight underestimate of the shunt-inductance of the feed transformer, so it was decided to operate the system with 0.5 amp feed-current to avoid any unforeseen errors which might arise due to non-linearities at the low frequencies.
The double electric screen in the transformer could be useful if at any time it is necessary to estimate the magnitude of undesired earth currents which might be flowing around the system. The earthing arrangements of the system have been carefully organised and appear to be quite satisfactory.

The field-measuring leads, Fig. 4.2, are designated $E_{S_1}$ meaning the sending end of the strip in number one position, i.e. $S_1$. $E_{R_1}$ means the receiving end of the strip in position $S_1$. The terms sending-end and receiving-end are meaningful in relation to the equivalent field network, Fig. 3.2-2.

Initially the feed current was set by measuring the drop across one of the 0.1 $\Omega$ feed resistors, but later on during the experiments this arrangement was found to be introducing an error of perhaps 10% at higher frequencies. A suitable current shunt was designed, and calibrated against a known resistance at 400 $\text{Hz}$. The wavemeter was checked to ensure that its range-switch was scaling correctly, since the measuring system is based upon the measurement of voltage ratios; an absolute measurement of voltage is not required. The frequency response of the current-feed "shunt" was uniform to within $\pm 1\%$ up to 20 $\text{kHz}$. For this latter measurement the shunt was fed with current from a constant voltage source via a 1 $\Omega$ carbon rod resistor; all voltage measurements were made using the Wave Analyser.
Input XFMR 1:20 turns

WAVE ANALYSER

100KΩ

Co-ax Lead

CO-AX LEAD

SYSTEM EARTH LEAD

W.A. EARTH

20 S.W.G. Insulated & Twisted Together

20 S.W.G. Insulated & Twisted Together

0.1Ω x10

0.1Ω x10

E1

E2

E3

ES1

X 10

3.0mV for 0.5 AMP
ACTUAL CALIB: 60mV VIA XFMR FOR 0.5 AMP.

*“HF SHUNT. 30 PAR: INSULATED EUREKA WIRES 30 S.W.G EA: 2cms. LONG BENT INTO "U" SHAPE TO MINIMISE INDUCTANCE. RESISTANCE 0.006 Ω*.

FIG. 4.2. SYSTEM ASSEMBLY.
Section 5. The Measuring System

From Fig. 4.2 we see that the constant-current feed arrangement places even the earthy end of \( S_1 \) at 10 mV above earth with 1 amp flowing. So we need to measure potential differences of a few micro-volts along the strip, which is itself 10 mV above earth. A transformer with a carefully screened primary winding was a likely solution, being much more stable in operation than active devices balanced about earth.

Since the input-impedance of the Analyser is 100 k\( \Omega \) and the impedance of the strips of the order of \( 10^{-3} \Omega \), a step-up transformer is very worthwhile. Actually, the limiting factor is the resistance of the field measuring leads, which, as it turns out, begin to introduce an error at about 200 Hz and below if the leads are no larger than 49 SWG.

The input impedance of the loaded transformer is about:

- 5 \( \Omega \) at 50 Hz
- 6 \( \Omega \) at 100 Hz
- 10 \( \Omega \) at 200 Hz
- 20 \( \Omega \) at 400 Hz

Its frequency response with an earthed constant-voltage input of 100 mV was measured using the Wave Analyser with the following results:
Input 100 mV.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Voltage as measured at secondary terminals with Wave Analyser</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 Hz</td>
<td>1.5 volts</td>
</tr>
<tr>
<td>200 Hz</td>
<td>1.8 volts</td>
</tr>
<tr>
<td>400 Hz</td>
<td>1.9 volts</td>
</tr>
<tr>
<td>1 kHz</td>
<td>1.9 volts</td>
</tr>
<tr>
<td>4 kHz</td>
<td>1.9 volts</td>
</tr>
<tr>
<td>8 kHz</td>
<td>1.95 volts</td>
</tr>
<tr>
<td>16 kHz</td>
<td>2.0 volts</td>
</tr>
<tr>
<td>30 kHz</td>
<td>2.1 volts</td>
</tr>
<tr>
<td>50 kHz</td>
<td>2.2 volts</td>
</tr>
</tbody>
</table>

The fall at 100 Hz is probably due to the DC resistance of the transformer primary winding. The errors in frequency response have not been applied as corrections to the results recorded in this memo; all curves are records of Wave Analyser measurements. It is important to note that the field measurements will not be in error to the extent recorded in the above table, because the transformer is used when setting up the main current at $V_0$, Fig. 4.2. The only error, though fairly small, is due to the extra resistance inserted by the field measuring leads, as mentioned above, and this is not a simple correction; no correction would be needed beyond say 0.4 kHz. The rising frequency response of the transformer beyond
20 kHz, is probably due to series resonance between the leakage-inductance and self-capacity of the secondary winding; the setting-up procedure again removes this error.

The operation of the transformer when measuring 1 μV at 10 mV with respect to earth was checked using the circuit arrangement of Fig. 5.1.

---

**FIG. 5.1 CIRCUIT FOR CHECKING OPERATION OF INPUT TRANSFORMER**
The voltage $V_2$ was within $5\%$ of $V_1$ at all frequencies up to $20$ kHz provided "A" was connected to terminal 4 and "L" to terminal 5. On reversing these connections $V_2$ appeared to be:

- $5\%$ above $V_1$ at $10$ kHz
- $15\%$ above $V_1$ at $16$ kHz
- $25\%$ above $V_1$ at $20$ kHz

Because of these errors we arranged to connect terminal 5, i.e. the outer of the coaxial measuring lead, see Fig. 4.2, to the earthy end of the system.

It is perhaps worth noting how small the direct capacitance between the primary winding and the high potential end of the secondary must be to avoid serious errors. For the $10$ mV to be reduced to say $0.1 \mu V$ across the primary, vis. $2 \mu V$ across the secondary, we need a voltage reduction of some $10^4$ times. If the coupling is due to capacitance $C$ in series with $100 \, k\Omega$, the input impedance of the Analyser, then:

$$\frac{1}{\omega C} \gg 10^9 \Omega$$

At $20$ kHz

$$\frac{1}{C} \gg 10^9 \times 10^5$$

$$C \ll 10^{-14} \text{ farads direct capacitance}$$
WAVE-ANALYSER & INPUT TRANSFORMER

OSCILLATOR & OUTPUT TRANSFORMER

MEASURING EQUIPMENT, FIG. 4.2, WITH 4-PLATE
SYSTEM OF FIG. 7.1-1 IN POSITION
Section 6. **Calculations and Measurements on the Two-Plate System of Fig. 4.2.**

6.1 **Calculations**

The equivalent field network is Fig. 3.2-2 which immediately introduces the difficulty of deciding upon the value of $Z^+$. In one-dimensional circular co-ordinates $Z^+$ tends to infinity at lower frequencies — see Section 2 — and the curves of Fig. 6.1-1 see also Fig. 4.2, have been calculated on this assumption because any other mathematical model would introduce considerable and probably needless analytical and computational difficulties.
The curve (b) of Fig. 6.2-1 is calculated on the assumption that the field measuring lead runs a distance \( d_0 = 0.05 \text{ mm} \) from the inner surface of \( S_1 \), so that in addition to \( E_{S_1} \) there is an induced voltage due to the rate of change of flux in the free space; relative phases have been taken into account.

6.2 Measurements and Comment

Experimental curve (d) was amongst the first to be taken and it employed relatively heavy gauge leads such that \( d_0 \) was around 0.5 mm.
Various forms of field measuring leads were tried; they were designated as follows:

- **Plate PX** - Heavy gauge with \( d_o \approx 0.5 \text{ mm and centre lead-out.} \)
- **Plate PY** - 49 SWG with centre lead-out.
- **Plate PA and PB** - 1 thou (0.025 mm) x 1 mm strip with centre lead-out 46 SWG.
- **Plate PC** - 46 SWG with clamped connections and end lead-out.

The best results at the higher frequencies were obtained from Plate PC Curve (c) in Fig. 6.2-1. The fall-off at low frequencies is because the resistance of the 46 SWG lead cannot be neglected compared with the impedance looking into the measuring transformer Fig. 4.2. The flat strip lead construction of Plates PA and PB overcomes that defect, and it was decided that future plates would be like PA and PB but with end lead-out.

The curves in Fig. 6.2-1 suggest that the \( Z^+ \) infinity assumption is adequate so far as \( E_{S1} \) calculation is concerned.

The \( E_{R1} \) results, Fig. 6.2-2 suggest that the magnetic field actually outside the strip is not zero, possibly because the one-dimensional presumably conditions are not satisfied, and at 10 kHz the term is of the same order as \( E_{R1} \). Above this frequency \( \frac{d\phi}{dt} \) will increase at 6dB per octave, whereas the \( E_{R1} \), transmitted through the plate, will fall at an even greater rate.

Further work on \( E_{R1} \) with the two-plate configuration was likely to be unprofitable, because it would involve a complicated and tedious two-dimensional analysis, so it was decided to proceed to a four-plate configuration in which the \( E_{R1} \) conditions were likely to be realised on a one-dimensional basis.
Section 7. Calculations and Measurements on a Four-Plate System

7.1 Calculations

The four-plate configuration, Fig. 7.1-1 (a) and (b), is the cartesian counterpart of Fig. 3.1-2 and Fig. 3.1-3, but in which a clearly defined separation "S" 9 mm is made between the energizing strip and the outer conducting medium. This experiment was designed to check the validity of Fig. 2.2-2.

No attempt has been made to measure $E_{R3}$; see comments at the end of Section 6.
The equivalent field network for one half-system is shown in Fig. 7.1-2(a), with the low-frequency approximation in Fig. 7.1-2(b), and the high-frequency approximation in Fig. 7.1-2(c).
\[ H_0 = \frac{I_0}{b} \]

\[ A = Z_0 \left[ \frac{\cosh \Gamma a - 1}{\sinh \Gamma a} \right] \]

\[ B = \frac{Z_0}{\sinh \Gamma a} \]

(a) COMPLETE SYSTEM

(b) LOW-FREQUENCY APPROXIMATION

(c) HIGH-FREQUENCY APPROXIMATION

FIG. 7.1-2 EQUIVALENT FIELD NETWORKS OF 4-PLATE SYSTEM (ACTUALLY FOR ONE HALF OF SYSTEM)
The complete solution of Fig. 7.1-2(a) is straightforward but tedious, and preliminary sums suggest that the low-frequency approximation might be adequate up to 500 Hz and the high-frequency approximation might hold above 4 kHz. The result is displayed in Fig. 7.1-3, which suggests that the complete solution needs extending at both ends.
For the time being the junctions have been smoothed artistically in Fig. 7.1-4 which we are calling "first calculation" of Fig. 7.1-1, because it might be necessary to improve on this later; the model or the sums or both could be faulty.
In Fig. 7.1-5 we repeat the curves of Fig. 7.1-4 and superimpose the calculated $E_{S1}$ and $E_{R1}$ curves of Fig. 6.1-1 which is for the two-plate system.

This clearly displays the effect we are looking for above about $2 \text{ kHz}$. The presence of the outer loop, i.e. reduced outward looking field impedance, has radically increased $E_{R1}$ the current-density on the outside of the energizing strip. It has, of course, lowered $E_{S1}$ the current-density on the inside surface of the energizing strip. We see that by the time $30 \text{ kHz}$ is reached, the

$$*\mathbf{j} = \sigma \mathbf{E}.$$
sum of $E_{S1}$ and $E_{R1}$ in the four-plate system, is equal to the $E_{S1}$ of the two-plate system. A glance at Fig. 7.1-2(c) indicates that this equality should be approached as the frequency tends to infinity.

7.2 Measurement and Discussion

The measured results are shown in Fig. 7.2-1 along with the "first calculations" of Fig. 7.1-4. The trends are encouraging and suggest that the mathematical model at least provides a useful foundation for further work.
Whilst the agreement between theory and measurement above say 4 kHz is satisfactory, there are two quite major discrepancies between theory and experiment which need explaining. One is the $E_{R1}$ response in the 1 to 2 kHz region, the other is the $E_{S3}$ curve at frequencies below say 1 kHz.

One would hardly suspect the measuring technique when the high-frequency results are so encouraging, but there were other avenues to be explored:

(a) Faults in the calculations.

(b) The one-dimensional limitations of the present mathematical model at low frequencies.

(c) The external plates do not form a continuous homogeneous loop because they are connected by wires. The "effective conductivity" of the external loop might, therefore, be different from that calculated assuming a continuous homogeneous loop.

Item (c) was investigated in the following manner. The resistance of the end connections of the outer plates, Fig. 7.1-1, could not readily be reduced, but it could be increased. Accordingly, the wires joining $S_3$ and $S_4$ were increased from about 1 cm length to 10 cm length, firstly at one end and then at both ends, with results shown in Fig. 7.2-2.
There is little doubt that the resistance of the end connecting wires (20 SWG Cu) is responsible for the major discrepancies in both $E_{R_1}$ and $E_{S_2}$ at the lower frequencies.
In Fig. 7.1-1(a) each end connecting wire joining $S_3$ to $S_4$ is approximately 1 cm long, i.e. approximately $2 \times 10^{-4} \Omega$. There are ten wires in parallel, i.e. $2 \times 10^{-5} \Omega$, at each end, but the DC resistance of each plate is only $1.1 \times 10^{-5} \Omega$. This doubtless explains the source of the errors between the simple theory and experiment at low frequencies. Above some 4 kHz the surface impedance of the plates rises faster than that of the connecting wires, so that the errors decrease progressively with increase in frequency.

In an attempt to reduce the low frequency errors, a new set of plates were constructed as follows:

**Drive Plates**
- 2 x PF: 30 cms x 5 cms.
- Voltage Probes 1 Thou x 1 mm Cu
- with 46 SWG end lead-out.
- 10 x 20 SWG feed wires.

**Outer Plates**
- 2 x PG: 30 cms x 5 cms.
- Voltage Probes as above.
- End connecting leads 14 SWG Cu.

Each plate resistance is now $3.3 \times 10^{-5} \Omega$ and each end connection approximately $5 \times 10^{-6} \Omega$, giving almost an order of magnitude improvement on the original plates at DC.

The experimental results of Fig. 7.2-3 are judged adequate to support the theory. A measurement error (low reading) of approximately 10% is present at 100 Hz, due to the 46 SWG measuring leads feeding the input transformer impedance of $(4 + j4)$. The error is 6% at 200 Hz and negligible above 400 Hz. The curves have not been corrected. See also note in Section 5. It seems likely that the remaining discrepancies are due to imperfect end connections combined with the inadequacy of a one-dimensional mathematical model.
The calculated curves of Fig. 7.1-4 show pronounced weaves characteristic of wave-like systems. The phase change through copper is about 0.5 rads/mm at 1 kHz, i.e. about $\pi/2$ for 3 mm. At 4 kHz the phase change will be $\pi$ rads for 3 mm. It may be that the experimental curves show little tendency to weave because of the "damping" introduced by the imperfect short-circuits on $S_3$, $S_4$. 
THE 4-PLATE SYSTEM OF FIG. 1-1
Section 8. Secondary Loop Enclosed Within the Primary Energizing Loop

8.1 Introductory

From the point of view of investigating further the agreement between the one-dimensional mathematical model of Fig. 2.2-2 and physical experimentation, we now see that the geometry of Fig. 7.1-1 has two rather serious limitations. One is that the field impedance terminating the outer strip can be taken as infinite in calculating $E_{S3}$ but the same is not true of $E_{R3}$ and hence of propagation within the $S_3$ strip. The other limitation is the physical problem of providing good short circuits at the ends of $S_3$ and $S_4$ because the current energizing leads to $S_1$ and $S_2$ must pass through to the generator. These limitations can be overcome by enclosing the secondary loop within the energizing loop as in Fig. 8.1-1.

Admittedly such an arrangement will not enable the current distribution within the energizing strips to be diverted as in Fig. 7.1-1 et seq., but it will enable propagation into and through the metal strips $S_1$ and $S_2$, Fig. 8.1-1 to be investigated experimentally.

From a practical point of view, such an investigation could lead to a better understanding of the current distribution and "copper losses" in the secondary windings of transformers. It should also provide a useful demonstration of electromagnetic screening, since the free space within the "secondary" loop is screened by $S_1$ and $S_2$ from the plane waves arriving from the energizing strips $S_3$ and $S_4$. 

- 44 -
a = 3 mm, \( g_1 = g_2 = 1 \text{ mm} \), \( c = 2 \text{ mm} \).

*SHORT-CIRCUIT PROVIDED BY 14 S.W.G. Cu x 10 LEADS OR BY Cu 'FILLETS' AT EACH END.*

SCHEMATIC OF CONNECTIONS & FIELD MEASURING LEADS.

**IF SYSTEM IS SYMMETRICAL ABOUT \( c/l \), ONLY \( E_{51} \), \( E_{53} \) & \( E_{RI} \) ARE NEEDED.**

FIG. 8·1·1. FOUR-PLATE SYSTEM, SECONDARY INSIDE.
8.2 The One-Dimensional Model

Relying once again on symmetry as in Section 3.2 and Fig. 3.2-2, the equivalent field network for one-half of the system is shown in Fig. 8.2-1(a) with the low-frequency approximation Fig. 8.2-1(b) via asymptotic values of the hyperbolic functions for vanishingly small values of the arguments. The series inductor $\mu g$ is the low-frequency one-dimensional approximation to the space between $S_3$ and $S_1$.

It is interesting to note that the low-frequency network corresponds to the generally accepted equivalent network for a one-to-one transformer. The leakage-inductance is $\mu g$ and the shunt-inductance is $\frac{\mu \cdot c}{2 \cdot \delta a}$ may be regarded either as the load resistance or the shorted-circuited secondary winding resistance. These values all apply to the "half-system", of course.
LOW-FREQUENCY APPROXIMATION

FIG. 8-2-1 EQUIVALENT FIELD NETWORK OF FIG. 8.1-1
8.3 Calculations

The three electric fields, $E_{s1}$, $E_{r1}$, and $E_{s3}$ were calculated for $l_r = 5$ cm, $c = 2$ mm, $g = 1$ mm, $\alpha = 3$ mm, $\sigma = 6 \times 10^{-7}$ mho/m, and $\mu_0 = 4 \pi \times 10^{-7}$ H/m, using the complete expressions Fig. 8.2-1(a); a twenty-inch slide rule was used and the results plotted in Fig. 8.3-1. As a check on these calculations, curves were plotted of the low-frequency and high-frequency approximations. There was good agreement over the appropriate parts of the spectrum.

8.4 Experimental

Many experiments were conducted at intervals of two or more months, because a new physical model was needed with each modification.

A description of each experiment in chronological order would be far tedious, so we shall describe the first, which turned out to be from the theoretical model and one of the last which is very close to the model. Some of the intermediate experiments will then fall more conveniently into place.

The first experiment used drive plates PF as described in Section 7.2, in positions $S_3$ and $S_4$, of Fig. 8.1, with secondary plates PG in positions $S_1$, $S_2$.

The results are shown in Fig. 8.4-1, in which the most interesting curve $E_{r1}$, is sadly at variance with theory.
CALCULATIONS OF FIG. 8.2-1(a)

AS DETAILED IN SECTION 8.3

FIG. 8.3 - 1
One of the last experiments conducted in this particular series used the mechanical configuration of Fig. 8.4-2 with results Fig. 8.4-3.
SEGMENTED DRIVE PLATES SPA & SPB IN POSITIONS S3 & S4 OF FIG. 8.1-1.

INNER SECONDARY LOOP. IN POSITIONS S1 & S2.

PLATE S1 - LEADS ON BOTH SIDES.

FIG. 8.4-2. CONSTRUCTIONAL DETAIL.
There is good agreement between the calculated and experimental results, particularly above 4 kHz. At lower frequencies the $E_{R1}$ curve departs from theory to about the same extent as does the $E_{S1}$ curve. It is that in terms of transmission through the $S_1$ strip, theory and experiment are in good agreement at all frequencies; the
error is at the $E_{S_1}$ face. We shall return to this question towards the end of this section when discussing the significance of the various mechanical features of the experimental model of Fig. 8.4-2.

The driving plates SPA and SPB are segmented to force a uniform lateral current distribution all along the 30 cm length, but the width of these plates remains at 5 cms. The secondary plates $S_1$ and $S_2$ are, however, 10 cms wide, but the current cannot be spreading much beyond the 5 cms "shadow" region because spreading would reduce $E_{R_1}$.

Some theoretical support for this effect is given in Appendix II which incidentally must be applicable to all diffusion transport processes. The 10 cm wide plates were used with a view to reducing the external field leaking in via the exposed edges. The copper screening tube considerably reduced pick-up in the exposed short length of twisted pair $E_{R_1}$ measuring leads at high frequencies where the signal level is at its lowest.

It became evident during the experiments that the size of the short-circuiting fillets and the method of securing them to the $S_1$, $S_2$ plates are very important factors. The arrangement shown in Fig. 8.4-2 was the most satisfactory of many which were tried.

For the final experiment the 10 cms $S_1$, $S_2$ plates were sawn down to 5 cms width, but leaving the screening tube and field measuring leads undisturbed. The results of Fig. 8.4-4 demonstrate the considerable improvement due to the 10 cms width, particularly at the higher frequencies, as one might expect.
One of the more interesting intermediate experiments was an attempt to measure the lateral field spreading at the surfaces of the 10 cms plate - by placing additional field measuring leads $E_{sia}$ and...
$E_{RIq}$, 3.5 cm off the centre line, i.e. 1 cm beyond the shadow results of $S_3$, with as in Fig. 8.4-5.

---

**Drive Plates SPA, SPB.**
**Secondary Plates similar to Fig. 8.4-2 but no $E_{RI}$ shield.** 10 cm wide with extra field meas'g leads 3.5 cm from o/l, $E_{S1}$ & $E_{RIq}$

**Fig 8.4-5**
The off-centre $E_{s1a}$ field is less than 10% of $E_{s1}$ - the centre-line field - at higher frequencies, but at the lower frequencies $E_{s1a}$ is rather more than 10% of $E_{s1}$. This increase of lateral spreading which presumably takes place in the 1 mm "g" air gap doubtless accounts for the departure of $E_{s1}$ and hence of $E_{rs}$ from theory at the lower frequencies. The $E_{rs}$ curve is fairly well below $E_{rs}$ at lower frequencies, but this is by no means so at higher frequencies. On the other hand, the absolute value of $E_{rs}$ is so low that little importance can be attached to it. In any case we have no discrepancies to account for in connection with transmission through $S_1$ plate.

Finally we need to explain the reason for the segmented driving plates, SPA and SPF.

Returning to the first experiment of the series Fig. 8.4-1, we note that at the lower frequencies even $E_{s3}$ departs from theory; an effect which was not present with the secondary loop outside, as in Section 7. The significant difference turned out to be due to the spacing of the drive plates. With the secondary loop inside, the inner surfaces of the drive plates are separated by 10 mm instead of 2 mm. Off-centre field measuring leads were placed along the $E_{s3}$ surface and the centre-line field was found to be some 10% lower than that, only 1.5 cms off-centre at 400 Hz. The segmented plates eliminated this effect, but it is interesting to note that with the SPA/SPB plates, air spaced by 2 cm, the measured $E_{s3}$ field at higher frequencies was significantly below the calculated $E_{s3}$. Below 1 cm spacing there is good agreement between theory and experiment, see Fig. 8.4-6. In all cases the lateral field distribution with the segmented plates proved to be uniform.
The segmented drive plates SPA and SPB were used in all but the earlier experiments of Section 8, and in all experiments in Sections 9 and 10.

The effect of the "9" air-space upon $E_{S1}$ was briefly investigated; see Fig. 8.1-1 and Fig. 8.4-3. The dimensions of $9_1$ and $9_2$ were reduced from 1 mm to 0.1 mm, resulting in an increase in $E_{S1}$ of about 10% at 100 Hz and correspondingly less increase as the frequency was increased to 2 kHz. This suggests that the air-space fields in $9_1$ and $9_2$ tend to spread rapidly.
Section 9. **Bi-metal Secondary Loop Enclosed Within a Primary Loop**

9.1 **Introductory**

The geometry we have in mind is the same as Fig. 8.1-1, with the $S_1$ plate of copper but the $S_2$ plate of some other metal. In circular geometry the arrangement would look like Fig. 9.1-1(a).

**FIG. 9.1-1(a). BI-METAL SECONDARY LOOP ENCLOSED WITHIN PRIMARY LOOP.**

$S_1$ - METAL $M_1$ LENGTH $L_1$ THICKNESS $Q_1$

$S_2$ - METAL $M_2$ LENGTH $L_2$ THICKNESS $Q_2$

**FIG. 9.1-1(b). ESSENTIAL FEATURES OF HELICON-WAVE EXPERIMENT.**

$M_2$ - CROSS SECTION OF "COMMERCIAL" Cu BOX

$M_1$ - CROSS SECTION OF PURE Cu LID
The need for a fairly detailed understanding of this kind of problem arose in connection with some work at Harwell concerned with the propagation of so-called Helicon Waves through a copper plate. The particular Helicon Wave experiment was devised to demonstrate the feasibility of propagating Electromagnetic Waves through a highly conducting thin copper plate, viz. pure copper at a very low temperature, in the presence of a steady magnetic field, of predetermined strength, applied in the direction of wave propagation. The plate formed the lid of a very thick-walled box made of commercial copper and intended to exclude all fields other than that transmitted through the thin lid. The essential elements from the point of view of this Section 9 investigation are shown in Fig. 9.1-1(b), where the electric fields $E_{R1}$ and $E_{R2}$, transmitted through the two media, can be directly related from Fig. 9.1-1(b) through Fig. 9.1-1(a) to Fig. 8.1-1 but with $S_1$ different from $S_2$. This arrangement is a simplified version of Fig. 9.1-1(a) to the extent that $L_1 = L_2$ because plates $S_1$ and $S_2$ are of equal lengths.

The $S_1$ plate could naturally be the same as in Fig. 8.4-2 and zinc seemed a suitable choice of metal for $S_2$; its resistivity is $\frac{5.75}{1.6} = 3.6$ times that of copper, which is a significant difference. It is inexpensive, easily machined and soldered. Brass and Bronze were rejected because their resistivities are so variable and ferrous metals have uncertain permeabilities.

9.2 The One-Dimensional Model

There is, of course, no difficulty in formulating the one-dimensional wave equation for the separate plates in the system. The problem, as always, is one of deciding upon the correct boundary conditions. This was done by writing down a plausible form of equivalent field network, Fig. 9.2-1, and examining its validity under certain significant conditions.
\( \mu_0 g_1, \mu_0 g_2 \) & \( \mu_0 c \) ARE APPROXIMATIONS TO THE WAVE NETWORKS FOR FREE-SPACES \( g_1, g_2 \) & \( c \).

\[
A_1 = \frac{Z_c (\cosh \Gamma_c a_c - 1)}{\sinh \Gamma_c a_c} \quad B_1 = \frac{Z_c}{\sinh \Gamma_c a_c} \quad Cu
\]

\[
A_2 = \frac{Z_z (\cosh \Gamma_z a_z - 1)}{\sinh \Gamma_z a_z} \quad B_2 = \frac{Z_z}{\sinh \Gamma_z a_z} \quad Zn
\]

\[
Z_c = \frac{j \omega \mu_0}{\sigma_c} \quad \Gamma_c = \sqrt{j \omega \mu_0 \sigma_c} \quad Cu
\]

\[
Z_z = \frac{j \omega \mu_0}{\sigma_z} \quad \Gamma_z = \sqrt{j \omega \mu_0 \sigma_z} \quad Zn
\]

\( a_c = \) THICKNESS OF Cu PLATE \quad \( a_z = \) THICKNESS OF Zn PLATE

FIG.9.2-1 MATHEMATICAL/PICTORIAL MODEL OF BI-METAL SECONDARY
If \( S_1 \) and \( S_2 \) are identical in every respect there is symmetry about the dotted line, and one half of Fig. 9.2-1 is the same as Fig. 8.2-1, as one might expect. But it was the experimental work of Section 8 resulting in increased confidence in the model of Fig. 8.2-1 which led to Fig. 9.2-1.

The most uncertain feature of the Fig. 9.2-1 representation is the tacit assumption that the magnetic fields along the outward looking surfaces of \( S_1 \) and \( S_2 \) are identical and equal to \( H_0 \). Accepting this, it follows that the \( H_R \) field will be the same along the inner surfaces of \( S_1 \) and \( S_2 \).

This must be so because of Amperes Law:
\[
\oint H \, dl = \oint J \, ds
\]
and the fact that the total current flowing along \( S_1 \) will be the same as the total current flowing along \( S_2 \) to satisfy continuity of the current flow around the \( S_1/S_2 \) loop.

Simply from inspection of Fig. 9.2-1 we can see that at high frequencies, say above 4 kHz;

\[
E_{S3} = E_{S4} = E_{S1} = H_0 Z_c
\]

Where \( Z_c \) is the characteristic impedance of copper
\[
Z_c = \sqrt{\frac{\omega \mu_0}{\sigma_c}}
\]

Whilst
\[
E_{S2} = H_0 Z_Z \text{, } Z_Z \text{ is characteristic impedance of zinc}
\]

\[
= H_0 Z_c \sqrt{3.6} \text{ impedance of zinc}
\]

\[
= 1.99 E_{S1}
\]
At very low frequencies, perhaps well below 100 Hz, \( H_R = H_0 \)
i.e. the magnetic field will be uniform throughout the structure
because the shunting effect of \( B_1 \) and \( B_2 \) viz.
\[
\left[ \frac{1}{\sigma_z} a_c + \frac{1}{\sigma_z} a_z \right]
\]
will become negligible. These asymptotic deductions lend support to the
field network model.

9.3 The Calculations

In order to minimise the calculating work, the copper strip \( S_1 \)
remained at \( a_c = 3 \text{ mm} \) whilst two specially selected values of
zinc thickness \( a_z \) were chosen. One was \( a_z = 5.7 \text{ mm} \) in order
that
\[
\frac{1}{\gamma_1} a_c = \frac{1}{\gamma_2} a_z
\]
\[
\frac{\sqrt{3 \omega \mu \sigma_c}}{a_c} = \frac{\sqrt{3 \omega \mu \sigma_z}}{a_z}
\]
\[
\frac{a_z}{a_c} = \frac{\sqrt{\sigma_z}}{\sqrt{\sigma_c}}
\]
\[
= \sqrt{\frac{\varepsilon_z}{\varepsilon_c}}
\]
\[
= \frac{5.7}{3}.
\]

For the other plate \( a_z = 2.9 \text{ mm} \) in the hope that the
hyperbolic expressions for \( \Gamma a/2 \) in terms of \( \Gamma a \) would
again reduce the calculating effort; in the event this was a vain hope.
Referring now to Fig. 9.2-1, the calculations were organised simply by working out:

\[
\begin{align*}
H_R &= \frac{H_0 (B_1 + B_2)}{A_1 + A_2 + B_1 + B_2 + Z_R} \quad \text{--- 9.3-1} \\
E_{R1} &= H_0 B_1 - H_R (A_1 + B_1) \quad \text{--- 9.3-2} \\
E_{R2} &= H_0 B_2 - H_R (A_2 + B_2) \quad \text{--- 9.3-3} \\
E_{S1} &= H_0 (A_1 + B_1) - H_R B_1 \quad \text{--- 9.3-4} \\
E_{S2} &= H_0 (A_2 + B_2) - H_R B_2 \quad \text{--- 9.3-5}
\end{align*}
\]

and in which

\[
\begin{align*}
(A_1 + B_1) &= Z_c \cot \theta \Gamma_c a_c \quad \text{--- copper} \\
(A_2 + B_2) &= Z_2 \cot \theta \Gamma_2 a_2 \quad \text{--- zinc} \\
B_1 &= Z_c \sqrt{\frac{\rho}{\Delta_m}} \Gamma_c a_c \\
B_2 &= Z_2 \sqrt{\frac{\rho}{\Delta_m}} \Gamma_2 a_2
\end{align*}
\]

If now \( \Gamma_c a_c = \Gamma_2 a_2 \) as it does in the case of the 5.7 mm zinc plate experiment, then Characteristic Impedance is a common factor in equations 2 to 5 above, and it follows that at all frequencies:

\[
\begin{align*}
E_{R2} &= \frac{Z_2}{Z_c} E_{R1} \\
&= 1.9 E_{R1} \\
E_{S2} &= 1.9 E_{S1}
\end{align*}
\]
No such simplified relationship exists in the case of the calculation for the 2.9 mm zinc plate experiments, but there are some useful approximations which are recorded in Section 9.4.

The drive plates were SPA and SPR so that the $E_{53}$ as in Section 8 and of course the $E_{54}$ calculations are the same as in Section 8. The calculations for $E_{51}$, $E_{52}$, and $E_{53}/E_{54}$ are displayed in Fig. 9.3-1, and those for $E_{R1}$ and $E_{R2}$ in Fig. 9.3-2. In each case solid lines are used for calculations of 5.7 mm zinc/3 mm Cu system and dotted lines for the 2.9 mm zinc/3 mm Cu system.
CALCULATIONS FIG 9.2-1 \( E_s1, E_s2, E_s3/E_s4 \)

FIG 9.3-1
CALCULATIONS FIG. 9.2-1 $E_{R1}, E_{R2}$

**Fig. 9.3-2**
9.4 Some Interesting Approximations to the Model of Fig. 9.2-1

9.4.1 High-Frequency Approximations

Included in Fig. 9.3-2 is the \( E_{R1} \) curve for an internal loop of Cu 3 mm/Cu 3 mm as in Fig. 8.3-1, and we note that this curve coincides with 5.7 mm Zn/3 mm Cu, \( E_{R1} \) at high frequencies.

Using the HF approximations to equations 9.3-1 and 2, with

\[
\Gamma_s a_c = \Gamma_s a_z \quad \text{we find that:}
\]

\[
E_{R1} = \frac{-\Gamma_s a}{2 H_0 Z_c \epsilon Z_R} \frac{Z_R}{Z_c + Z_z + Z_R} \quad \text{--- 9.3-7}
\]

The corresponding expression for Fig. 8.3-1 is

\[
E_{R1} = \frac{-\Gamma_s a}{2 H_0 Z_c \epsilon Z_R} \frac{Z_R}{Z_c + Z_z + Z_R} \quad \text{--- 9.3-8}
\]

The only difference is in the denominator, and since \( \frac{Z_R}{\sqrt{\omega}} \) varies as "\( \omega \)" and \( (Z_c + Z_z) \) varies as \( \sqrt{\omega} \), then equation 9.3-7 approaches equation 9.3-8 as \( \omega \rightarrow \infty \).

At 8 kHz:

\[
\begin{align*}
Z_a &= 0.128 \times 10^5 \Omega \\
|Z_c| &= 3.24 \times 10^5 \Omega \\
|Z_z| &= 6.15 \times 10^5 \Omega
\end{align*}
\]

So we can expect the two curves to coincide above about 8 kHz.
These results could have been deduced directly from the approximate field network in Fig. 9.4.1-1 relying on Thevenin’s Theorem.

FIG. 9.4.1-1. AN ALTERNATIVE VERSION OF FIG.9.2-1 FOR CALCULATING E_{R1} AND E_{R2} AND HIGH-FREQUENCY APPROXIMATIONS TO E_{R1} AND E_{R2}.

\[ E_C = \frac{H_0 Z_C}{\sinh \alpha_c} z_2 H_0 Z_C e^{-\alpha_c z} \]

\[ E_Z = \frac{H_0 Z_Z}{\sinh \alpha_z} z_2 H_0 Z_Z e^{-\alpha_z z} \]

\( E_C \) and \( E_Z \) are the open-circuit fields at the ends of the two transmission lines, and \( Z_C \) and \( Z_Z \) are the field impedances (characteristic) looking back into the lines at higher frequencies where the attenuation is considerable.

Using this network we can now consider the relationship between \( E_{R1} \) and \( E_{R2} \) at high frequencies for the 2.9 mm Zn/3 mm Cu system, Fig. 9.2-1.
The only significant signal is:

\[ E_z = \frac{H_0 Z_z}{\sin \phi \Gamma_z q_z} \quad \text{because} \quad \Gamma_z q_z = \frac{1}{2} \Gamma_z q_z \]

\[ H_R = \frac{E_z}{Z_c + Z_z + Z_R} \]

\[ E_{R2} = E_z - H_R Z_z \]

\[ = E_z \left( 1 - \frac{Z_z}{Z_c + Z_z + Z_R} \right) \]

\[ E_{R1} = -H_R Z_c \]

\[ \therefore \quad \left| \frac{E_{R2}}{E_{R1}} \right| = \frac{Z_c + Z_R}{Z_c} \]

\[ = \frac{4.6 + j4.6 + j51.2}{6.5} \approx 32 \text{ kHz} \]

\[ = 8.6 \]
Finally we shall estimate the high-frequency value of $E_{R1}$ for the 2.9 mm Zn/3 mm Cu system:

$$E_{R1} = \frac{H_0 Z_z}{\tan \delta Z_2} \frac{Z_c}{Z_c + Z_z + Z_R}$$

At 32 kHz:

$$Z_z = 8.7 + j 8.7 \ \Omega$$
$$Z_c = 4.6 + j 4.6 \ \Omega$$
$$Z_R = j 51.2 \ \Omega$$

$$\frac{Z_z}{\tan \delta Z_2} = 0.4$$

$$\therefore E_{R1} = 0.08 \mu V/cm/\delta$$

which agrees very well with the curve in Fig. 9.3-2.

9.4.2 Low-Frequency Approximations

The appropriate network is Fig. 9.4.2-1 and this was used as a check on the curves of Fig. 9.3-2 at 100 Hz with the following results:

- 3 mm Cu/3 mm Cu: $E_{R1} = E_{R2} = 0.148 \mu V/cm/\delta$
- 3 mm Cu/2.9 mm Zn: $E_{R1} = 0.062$
  $E_{R2} = 0.45$
- 3 mm Cu/5.7 mm Zn: $E_{R1} = 0.11$
  $E_{R2} = 0.19$
9.5 Experimental

It will simplify the presentation of the experimental results if the 3 mm Cu/2.9 mm Zn experiment is described first because it worked according to plan, whereas the 5.7 mm Zn system presented difficulties. Four physical models were constructed before satisfactory results were obtained.

9.5.1 The 3 mm Cu/2.9 mm Zn System

The physical layout and mechanical construction were similar to Figures 8.1-1 and 8.4-2. Plate $S_1$ being of Cu and plate $S_2$ of Zn; there was no screening tube.
The $E_{S1}$, $E_{S2}$, $E_{S3}$ and $E_{S4}$ experimental results are compared with calculations in Fig. 9.5.1-1.

The only disagreement is at the lower frequencies and this is presumably due to lateral field spreading in the 1 mm "g" space. See curve $E_{S1}$ of Fig. 8.4-3 and associated comments in Section 8.4.
The $E_{R_1}$ and $E_{R_2}$ experimental results are compared with theory in Fig. 9.5.1-2, which together with Fig. 9.5.1-1 lend encouraging support to the mathematical model of Fig. 9.2-1.
9.5.2 The 3 mm Cu/5.7 mm Zn System

Three physical models were made, as described below, and the measured results recorded in Fig. 9.5.2-1 for the $E_{S1}$, $E_{S2}$, $E_{S3}$, $E_{S4}$ fields, and in Fig. 9.5.2-2 for the $E_{R1}$ and $E_{R2}$ fields.

**Fig 9.5.2-1.**
Apart from the error which we have to accept at lower frequencies, the sending end fields $E_{S_1}$ etc. agree well with calculations for all three experiments. It is the $E_{R_1}$ and $E_{R_2}$ fields which are sadly in disagreement with theory, and it was this fact which led to much cross-checking of the $E_R$ calculations, on the one hand, and the production of three physical models, on the other.

The construction of all three models was similar to Fig. 8.4-2, but with slight variations as follows:

The first model was 10 cm wide, but the short-circuiting fillets for $S_1$ and $S_2$ were each 1 cm wide and soldered to the Zn plate. The Cu plate was screwed down with one row of five brass screws. Although the $E_{R_2}$ curve agrees with theory the $E_{R_1}$ curve is wildly out, so that instead of being separated by a factor of 1.9 times they are separated by some 3.5 times in the middle of the spectrum.

The short-circuiting arrangements of the second model were identical with Fig. 8.4-2 but there was no screening tube, whilst the third model was made by sawing the second model down to a width of 5 cm.

Whilst the second and third models gave almost identical results, they still depart considerably from theory, although the ratio of $E_{R_2}$ to $E_{R_1}$ is much more in accord with theory.

Assuming that the source of the errors lies in the $S_1/S_2$ short-circuitry technique, it is necessary to explain why the 2.9 mm Zn/3 mm Cu experiment was so satisfactory.
A possible explanation is that the thickness of the 5.7 mm zinc plate resists the forces of the screw-heads and thereby inhibits a uniform contact with the copper short-circuiting fillets. The elementary theory of loaded beams and cantilevers indicates that the deflection for a given load is inversely proportional to the moment-of-inertia of the cross-section, and hence inversely proportional to the beam thickness cubed. In the case of the zinc plates, a factor of eight times is involved.

A fourth physical model was constructed in which the copper-zinc loop was 76 cms long and 10 cms wide. The short-circuiting fillets were each 3 cms wide instead of the 2 cms of the earlier models. In addition to the two rows of screws, Fig. 8.4-2, the plates and fillets were forced into contact by a row of five bolts and nuts acting through steel plates, at each end. The longer plates were intended to reduce the effect of the end connections to some, admittedly unknown, extent. The energizing current was fed along four segmented plates, two above and two below the copper-zinc secondary loop. Two spare energizing plates, SPA1 and SPB1, were available as standbys for the earlier experiments.

The results, which are recorded in Fig. 9.5.2-3, are encouraging because they suggest that the departure from the theoretical model of Fig. 9.2-1 can be explained by the relatively poor end connections.
Fig 9.5.2-3.
THE "FOURTH MODEL"

FIG 9.5.2 - 2

Cu 3mm  Zn 5.7mm  10cm x 76cm.
Section 10. Primary Loop Encircling the Secondary Loop which Feeds Power to an External Load of \( T \) Ohms.

10.1 Introductory

The physical arrangement shown in Fig. 10.1-1 has much in common with Fig. 8.1-1, the \( S_1, S_2 \) plates being short circuited at one end with copper fillets but terminated at the other in 14 SWG Cu wires arranged to provide an adjustable external load. The DC resistance of 14 SWG Cu wire is \( 5 \times 10^{-5} \Omega \) per cm.

Short circuited fillets were also provided at the load end so that measurements could be made with \( \gamma = 0 \) and compared with Fig. 8.4-3.

FIG. 10.1-1 PHYSICAL MODEL FOR EXTERNAL LOAD EXPERIMENT.
An investigation of this arrangement should provide a quantitative assessment of the importance of the end connections which have been the major source of uncertainty in the earlier experiments. It might also provide some interesting insight into the distribution of current in transformer windings and hence the associated losses. From a purely academic point of view this exercise is interesting because, unlike the earlier problems, its solution seems to demand a combined wave-like and approach, viz an application of the field equations in both the differential and integral forms.

10.2 The One-Dimensional Model

When the external load \( y = 0 \) the model must be the same as Fig. 8.2-1 and accordingly the only fields inside the "C" space are those transmitted through the \( S_1, S_2 \) plates.

For finite values of \( y \) the fields within the "C" space, \( H_R \) for example, are compounded of two fields. One arrives through the \( S_1, S_2 \) plates and the other presumably arrives through the resistance-loaded gap at the end of the plates. When \( y = \infty \) we expect to find \( H_R = H_0 \) because then the total current flowing along the \( S_1, S_2 \) strips is zero, and we have \( \oint H \, dl = 0 \) where the line integral is taken around the plate surface. There are surface currents, of course, on \( S_1 \) and \( S_2 \) but they flow in opposite directions on the two faces.

At high frequencies the field cannot propagate through the strips, so it must propagate into the "C" space via the open end. It seems unlikely that a wave-like analysis would explain the propagation mechanism of the resistance-loaded gap and so an integral - circuit equation - approach seems appropriate.
The proposed model is shown in Fig. 10.2-1.

\[ A = Z_0 \frac{\cosh \Gamma a - 1}{\sinh \Gamma a} \quad Z_0 = \sqrt{\frac{j \omega \mu}{\sigma}} \]

\[ B = \frac{Z_0}{\sinh \Gamma a} \quad \Gamma' = \sqrt{-j \omega \mu \sigma} \]

\[ \mu = 4 \pi \cdot 10^{-7} \text{H/m} \quad \sigma = 6 \cdot 10^{-7} \text{mho/m} \]

**Fig. 10.2-1.** Composite Field/Circuit Mathematical Model of One Half of External Load System.
One way of determining $H_R$ is to equate the voltage-drops around the secondary loop as follows:

$$E_o = j\omega \mu \frac{c}{2} H_R \quad 10.2-1$$

and

$$V_o = E_o l \quad \text{where} \quad l \quad \text{is the length of the plates.} \quad 10.2-2$$

Now

$$I_1 = \frac{1}{2} [H_o - H_R] \quad 10.2-3$$

This is Amperes Law:

$$\oint H \, dl = \int \int J \, ds \quad \text{applied to the} \quad S_1 \quad \text{and} \quad S_2 \quad \text{plates.}$$

$$V_r = I_1 \frac{\gamma}{2}$$

$$= \frac{r l r}{2} \left[ H_o - H_R \right] \quad 10.2-4$$

and finally:

$$V_{R1} = l E_{R1} \quad 10.2-5$$

Equating voltages around the secondary loop:

$$V_o = V_{R1} + V_r$$

Hence

$$E_{R1} = j\omega \mu \frac{c}{2} H_R - \frac{r l r}{2} \left[ H_o - H_R \right] \quad 10.2-6$$

From Fig. 10.2-1

$$E_{\varepsilon_1} = H_o B - H_R [A+B] \quad 10.2-7$$
and from equations 6 and 7:

\[
H=R = \frac{H_0 \left[ Z_0 l + \frac{rb}{2} \sinh Ta \right]}{\left( \frac{j \omega cl + \frac{rb}{2} }{2} \right) \sinh Ta + lZ_0 \cosh Ta} \tag{10.2-8}
\]

and

\[
E_{R_1} = \frac{H_0 Z_0 \left[ j \omega cl + \frac{rb}{2} - \frac{rb}{2} \cosh Ta \right]}{\left( \frac{j \omega cl + \frac{rb}{2} }{2} \right) \sinh Ta + Z_0 l \cosh Ta} \tag{10.2-9}
\]

From equation 4, the voltage across the load \( \gamma \) will be:

\[
\mathcal{E}_\gamma = rb \left( H_0 - H_R \right) \]

\[
= rb H_0 \left[ \frac{j \omega cl \sinh Ta + Z_0 l ( \cosh Ta - 1) }{\left( \frac{j \omega cl + \frac{rb}{2} }{2} \right) \sinh Ta + Z_0 l \cosh Ta} \right] \tag{10.2-10}
\]

Before plunging into detailed calculations of \( E_{R_1} \) and \( \mathcal{E}_\gamma \), it will be instructive to inspect some asymptotic approximations.
10.3 Field Distributions when $\gamma = \infty$.

The field network is shown in Fig. 10.3-1.

\[
H_0 = \frac{I_0}{b}
\]

\[
E_{SI} = E_{RI} = H_0 A - \text{SEE FIG. 10.2-1.}
\]

\[
= H_0 Z_0 \frac{\cosh \Gamma a - 1}{\sinh \Gamma a} \quad \text{10.3-1}
\]

EXTERNAL LOAD $r = \infty$

FIG. 10.3-1.

This also agrees with equation 10.2-9 with $\gamma \rightarrow \infty$.  

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Calculated results are shown in Fig. 10.3-2.
10.4 High-Frequency Approximations and Finite Values of $\gamma$.

These results can, of course, be obtained directly from equations 10.2-8 and 10.2-9 with $\eta_0 \to \infty$, but it is instructive to deduce them from the equivalent network, Fig. 10.4-1.

The analysis is the same as before up to equation 10.2-6, and then instead of equation 10.2-7 we have

$$E_{R1} = -H_R Z_0$$ \hspace{1cm} 10.4-1

and hence

$$H_R = \frac{H_0 \frac{\gamma l}{2}}{j\omega C \frac{\gamma l}{2} + \frac{\gamma l}{2} + Z_0 l}$$ \hspace{1cm} 10.4-2

$$E_{R1} = -H_0 \frac{\gamma l}{2} Z_0 \frac{\gamma l}{2} + j\omega C \frac{\gamma l}{2} + Z_0 l$$ \hspace{1cm} 10.4-3
\[ V_r = \frac{V}{2} \left[ H_0 - H_R \right] \quad \text{see eqn. 10.2-4} \]

\[ = \frac{V}{2} H_0 \left[ \frac{j \omega \mu l}{2} + Z_0 l \right] \quad \text{--- 10.4-4} \]

\[ \frac{j \omega \mu l}{2} + \frac{V}{2} + Z_0 l \]

The voltage across the complete load is:

\[ 2V_r = \frac{V}{2} H_0 \left[ \frac{j \omega \mu l}{2} + Z_0 l \right] \quad \text{--- 10.4-5} \]

\[ \left[ \frac{j \omega \mu l}{2} + \frac{V}{2} + Z_0 l \right] \]

\[ = \frac{1}{2} \omega l \left[ \frac{j \omega \mu l}{l} + \frac{2 Z_0 l}{l} \right] \quad \text{--- 10.4-6} \]

\[ \left[ \frac{j \omega \mu l}{l} + \frac{1}{2} + \frac{2 Z_0 l}{l} \right] \]

The equivalent circuit (voltage/current) of equation 10.4-6 is shown in Fig. 10.4-2.
The inductance $L$ is the inductance of the "C" space, and the impedance $2Z_0 l/l_b$ is the total surface impedance of the inner faces of the $S_1$ and $S_2$ strips in series.

It might seem a little odd that a portion of the secondary winding self-impedance turns up in series with the inductance rather than in series with the load, but this follows from the model of Fig. 10.4-1 and also, therefore, from Fig. 10.2-1.
LOW-FREQUENCY APPROXIMATION TO FIG. 10.2-1.

FIELD-NETWORK BRIDGE BALANCED WHEN:—

$$\frac{a}{c} = \frac{1}{\sigma} \frac{2l}{rb}$$

FIG. 10.5-1.
10.5 Low-Frequency Approximations

The equivalent network is shown in Fig. 10.5-1, which gives results as in equation 10.2-9 provided $\sinh \Gamma a$ is replaced by $\Gamma a$ and $\cosh \Gamma a$ is replaced by $1 + (\Gamma a)^2$.

Then:

$$E_{R1} = H_0 \left[ \frac{cl}{2} - \frac{\sigma a^2 yl}{4} \right] \frac{c + a}{\sigma a} \frac{1}{j\omega \mu} \left( \frac{y\sigma a + l}{2} \right)$$

The most revealing feature of this analysis is that there is a particular value of load resistance $y$ which enables $E_{R1}$ to become zero.

If, in Fig. 10.5-1, we write the load resistance as $\frac{yl}{2l}$ instead of $\frac{y}{2}$, then from inspection of the redrawn "bridge" in Fig. 10.5-1 we see that $E_{R1}$ becomes zero when:

$$\frac{a}{c} = \frac{1}{\sigma a} \cdot \frac{2l}{yl}$$

which agrees with the null of equation 10.5-1.

It seems that $\frac{yl}{2l}$ can be regarded as a "normalised" load which can be used in place of $\frac{y}{2}$ in Fig. 10.2-1 for example. Under these circumstances the analysis can remain in "field form", and there is then no need to introduce voltage and current concepts.
The null point occurs at $\gamma = 4.5 \times 10^{-5}$ $\Omega$ with the following parameters:

$C = 2$ mm, $\alpha = 3$ mm, $l = 5$ cms, $l = 30$ cms

$\gamma = 6 \times 10^7$ $\text{mho/m}$

It is convenient to express the load resistor $\gamma$ in terms of length of 14 SWG wire run $X$ cms from the secondary plates. There are ten wires fed from each plate so that

$$\gamma = 2X \times 5.10^{-5} \Omega$$

$$= X.10^{-5} \Omega$$

Where "X" is length in cms from the end of the plates to the short circuit clamps.

So the null in $E_{R1}$ can be expected to occur at 4.5 cm and clearly measurements of $E_{R1}$ with this order of load will be very sensitive to changes of $\gamma$. Early on in the experiments this point was not appreciated and erratic results were obtained when measuring around $X = 4$ cm; the low-frequency approximation had not been investigated and the complete expression of equation 10.2-9 is too complicated to sort out in detail.

The "null" point was subsequently determined experimentally at 3.5 cm at 200 Hz.
The low-frequency approximations to the voltage across the load is:

\[ 2V_r = \frac{r \text{H}_0 \cdot j \omega \mu (a+c) l \cdot \frac{l}{v}}{j \omega \mu (a+c) l + (\text{rL} + \frac{2l}{\sigma a})} \]

which can be written as:

\[ 2V_r = \frac{r \text{I}_0 \cdot j \omega \mu (a+c) l/v}{j \omega \mu (a+c) l/v + \text{rL} + \frac{2l}{\sigma a}} \]

This can be drawn as an equivalent voltage/current circuit as in Fig. 10.5-2. This is an accepted equivalent circuit for the secondary side of a transformer.

FIG. 10·5·2. EQUIVALENT CIRCUIT NETWORK OF EQUATION 10·5·3.
10.6 Experiments with $\gamma = \infty$

The majority of the experiments described in Sections 8 and 9 used secondary plates $S_1$ and $S_2$ which were 10 cms wide with the primary current fed along the 5 cms wide SPA/SPB plates. The first experiment was with this arrangement, firstly with one end of $S_1/S_2$ loop open-circuit and then with both ends open-circuit; there was in fact no measurable difference between these two conditions.

The results in Fig. 10.6-1 show that above 1 kHz the field $E_{R1}$ and hence $H_R$ in the "C" space is half the expected value. Apparently the field spreads out to fill the "C" space, an interesting fact but one we will not pursue at this stage.

![Diagram showing measurements of $E_{R1}$ and $E_5$ versus frequency.](image)
Fig. 10.6-2 shows results of a similar experiment on 5 cms plates throughout; they are in good agreement with theory. In spite of the short-comings of the 5 cms plates, which became evident during the experiments of Sections 8 and 9, it is clear that the external load experiment could not succeed with other than 5 cms - equal width - plates throughout. The inner plates on the Fig. 10.6-2 experiment were actually a spare set of segmented plates like SPA and SPB.
Fig. 10.6-3 records the final experiment on the physical model 5 cms wide, Fig. 10.1-1, with the adjustable external load. It is interesting that $E_{S1}$ is nearer to the calculated values than Fig. 10.6-2, probably because of the reduced value of 0.1 mm compared with 1 mm. See note at the end of Section 8.
10.7 Experiments with Finite External Load of $\gamma$ Ohms.

For no very good reasons the first experiments were made with $\gamma$ around $3.10^{-5}$ $\Omega$. The results were so far removed from calculations using equation 10.2-9, that they are not worth recording. When the null conditions using equation 10.5-1 were appreciated, measurements were made well away from $4.10^{-5}$ $\Omega$ – see Section 10.5 – actually at

$\gamma = 15.10^{-5}$ $\Omega$.

The results of $E_{r_1}$ were compared with the L.F. and H.F. approximate equations, because these can be evaluated so quickly compared with equation 10.2-9. Below 1 kHz the agreement was good but the errors increased progressively as the frequency increased. Since the DC resistance of the load wires, $15.10^{-5}$ $\Omega$, was used in these early high frequency calculations some discrepancy was inevitable. The change in $\gamma$ from DC to 16 kHz was estimated as follows:

The resistance of isolated 14 SWG Cu wire at DC is $5 \times 10^{-5}$ $\Omega$/cm.
The radius of the wire is approximately 1 mm.
The attenuation of copper is around 16 dB/mm at 16 kHz, so that the surface field impedance is

$$|Z_0| = 4.6 \times 10^{-5} \Omega$$

Infinite line conditions.

Now

$$H_0 = \frac{I_0}{2\pi R}$$

$R$ is radius of the wire in metres.

$$E_0 = H_0 Z_0$$

$$= \frac{I_0 Z_0}{2\pi R}$$

$$= \frac{Z_0}{2\pi R}$$

$$= \frac{4.6 \times 10^{-5}}{2\pi 10^{-3}} = 7.3 \times 10^{-5} \text{V/cm}$$

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The load wires run in pairs with no more than a few times 0.001 inch spacing between the wires of a pair. Accordingly the current will tend to flow on the inner surfaces of the pairs as in Section 6. We might, therefore, expect the impedance at 16 kHz to rise to say \( 2 \times 7.3 \times 10^{-5} \Omega \) which is some three times the D.C. resistance. Using this figure for \( \gamma \) in equation 10.4-4 at 16 kHz, gave an answer approaching the experimental figure.

The feasibility of constructing a load resistor, less frequency dependent or more calculable in performance, was considered using perhaps multiply stranded wire. The increased bulk might well increase the self inductance and the proximity effect would be almost impossible to calculate.

Instead, the modulus of the impedance of one loop was measured using the working rig Fig. 4.2 which is after all, designed to measure very low mutual impedances.

| Measured \( |Z| \) of closed loop 14 SWG Cu 15 cms. length i.e. 30 cms of wire: |
|-------|-------------------|
| \( f \) | \( |Z| \) |
| 100 Hz | 17.75 \( 10^{-5} \) |
| 200 Hz | 18.1 \( 10^{-5} \) |
| 400 Hz | 17.1 \( 10^{-5} \) |
| 800 Hz | 17.75 \( 10^{-5} \) |
| 1 kHz  | 18.8 \( 10^{-5} \) |
| 2 kHz  | 19.5 \( 10^{-5} \) (13.8 + j 13.8) \( 10^{-5} \) |
| 4 kHz  | 24.7 \( 10^{-5} \) (17.5 + j 17.5) \( 10^{-5} \) |
| 6 kHz  | 30.2 \( 10^{-5} \) (21.4 + j 21.4) \( 10^{-5} \) |
| 8 kHz  | 37.4 \( 10^{-5} \) (26.5 + j 26.5) \( 10^{-5} \) |
| 10 kHz | 43.5 \( 10^{-5} \) (30.8 + j 30.8) \( 10^{-5} \) |
| 16 kHz | 61.6 \( 10^{-5} \) (43.5 + j 43.5) \( 10^{-5} \) |
| 20 kHz | 72.6 \( 10^{-5} \) (51.5 + j 51.5) \( 10^{-5} \) |
The figures on R.H.S. of the table are based on the assumption that E.M. propagation into the copper behaves according to a semi-infinite diffusion process. This is a valid assumption above say 8 kHz, but becomes increasingly inaccurate below this frequency. The table also assumes that at 1 kHz and below, the impedance is purely resistive. One could arrive at a more accurate estimate of the phase angle of $\gamma$ by working in cylindrical co-ordinates and hence Bessel functions but even then, the proximity effect would at best be but a crude approximation. There is, however, another and probably more important factor to take into account at the higher frequencies. The practical load Fig. 10.1-1 is inductive to the extent that the 14 SWG wires are separated by rather more than 2 mm for a distance of some 2 cms where they enter the $S_1$ and $S_2$ plates. This theoretically represents an inductance of $10^{-9}$ H with a reactance of $j6.10^{-5}$ $\Omega$ at 10 kHz.

From 4 kHz upwards this factor was added to the imaginary parts of the impedance, already recorded above, to arrive at complex values for $\gamma$ to use in evaluating equations 10.4-3 and 10.4-5 at the high-frequency end of the spectrum. The values of $\gamma$ in the table at 1 kHz and below, were used to evaluate equations 10.5-1 and 10.5-2 in the low-frequency region.

The solid curve of Fig. 10.7-1 was calculated from equation 10.5-1 below 1 kHz and from equation 10.4-3 above 1 kHz, using values of $\gamma$ as detailed above. The agreement with experiment is surprisingly good; it may, however, be misleading as explained later. Nevertheless, the experiment lends considerable support to the model of Fig. 10.2-1.
Fig. 10.7-2 shows corresponding information for $2V_0$ and again the approximate theory and measurements are in surprisingly good agreement. There is considerable disagreement, however, between the complete expression and the approximate expressions over an unexpectedly large part of the spectrum, but the results go some way towards establishing the validity of the model, Fig. 10.2-1.
CALCULATIONS & MEASUREMENTS OF 2Vx

FIG 10.7-2
It may well be that similar discrepancies are inherent in Fig. 10.7-1 but the calculations are very tedious and of doubtful value in terms of increased overall accuracy, because the values of $\gamma(\omega)$ are, as already explained, likely to be most in error over the central part of the spectrum, where the fields penetrate the wire.

10.8 Estimation of the Influence of Imperfect Short Circuits on Field

This discussion is concerned with the system of Fig. 8.4-2 which behaves according to Fig. 8.4-3. On the assumption that even at frequencies as low as 100 Hz the current flow in $S_1 / S_2$ is confined more or less to a track 5 cm wide, the low-frequency resistance of a 30 cm loop would be around $6.6 \times 10^{-5} \Omega$.

Assume $\gamma$, the combined short-circuit resistance, is say:

$$\gamma = 4 \times 10^{-6} \Omega$$

$$\frac{\gamma \ell}{2} = 10^{-7}.$$

Using the low-frequency approximation 10.5-1

$$E_{R1} = H_0 \left[ \frac{\ell \ell}{2} - \frac{\sigma a^2 \gamma \ell}{4} \right]$$

$$\frac{\ell \ell}{2} = 3 \times 10^{-4} \quad \frac{\sigma a^2 \gamma \ell}{4} = 2.7 \times 10^{-5}$$

$$\frac{\ell \ell}{2} - \frac{\sigma a^2 \gamma \ell}{4} = (3 - 0.27) \times 10^{-4}.$$
\[
\frac{vl\sigma a}{2} = 0.018, \quad l = 0.3
\]

\[
\frac{v\sigma l (c+a)}{2} = 135
\]

\[
\frac{1}{j\omega \mu} \text{ at } 100 \text{ Hz} = -j1250
\]

\[
\text{at } 100 \text{ Hz} =
\]

\[
E_{R1} = H_0 \left[ 3 - 0.27 \right] 10^{-4}
\]

\[
\frac{135 - j1250(0.3 + 0.02)}{}
\]

So that the \( Y \) term in numerator reduces \( E_{R1} \), below

\( Y = 0 \) value, by about 10%.

The \( Y \) term in the denominator causes a further reduction of
say 10% at 100 Hz and becomes negligible at about 1 kHz.

Hence, \( Y = 4 \times 10^{-6} \) reducing \( E_{R1} \) below the perfect
system ( \( Y = 0 \)), by about 20% at 100 Hz and by about 10% at 1 kHz.

Now consider the high-frequency approximation; equation 10.4-3.

\[
E_{R1} = H_0 \frac{vl}{2} \frac{Z_0}{j\omega \mu cl + \frac{vl}{2} + Z_0 l}
\]
At 20 kHZ

\[ j \omega \mu c l \frac{1}{\varepsilon} = j 4 \cdot 8 \times 10^{-5}; \quad \frac{\gamma}{\varepsilon} = 10^{-7} \]

\[ Z_0 l = (1 + j \cdot 1) \times 10^{-5}; \quad |Z_0| = 5.7 \times 10^{-5} \]

The \( \frac{\gamma}{\varepsilon} \) in denominator can be neglected.

\[ E_{R_1} = 0.02 \mu V/cm/\alpha \]

This happens to be equal to the \( E_{R_1} \) propagated through the copper strip at 20 kHz, see Fig. 8.3-1. So that the contribution to \( E_{R_1} \) via "\( \gamma \)" is significant.

We note, therefore, that the effect of D.C. \( \gamma = 4 \times 10^{-6} \Omega \) on a loop of D.C. \( 6.6 \times 10^{-5} \Omega \) is significant, whereas

\( \gamma = 4 \times 10^{-7} \Omega \), i.e. a hundredth of the loop D.C. resistance would probably go undetected.

Equations 10.4-2 and 10.4-3 might be useful in estimating the effectiveness of practical "screens" which usually contain joins of some description.

10.9 A Direct Field Approach to the External Load System Fig. 10.1-1 and Fig. 10.2-1 but with Infinitely Conducting \( S_1, S_2 \) Plates.

In this sub-section we describe a somewhat unusual but interesting and perhaps revealing way of considering transformer operation. It arises directly out of the earlier work in this Section 10.
Fig. 10.9-1 represents two parallel plates of infinite conductivity \( c \) meters apart, air-spaced, \( w \) meters wide, short-circuited at one end and terminated in a resistor \( \gamma \) ohms at the other. To simplify the explanation, \( \gamma \) is chosen such that:

\[
\gamma = \frac{Z_a c}{l} \text{ ohms} \quad \ldots \quad 10.9-1
\]

\( Z_a \) is the intrinsic (characteristic) field impedance of free space \( \sqrt{\frac{\mu_0}{\varepsilon_0}} \), so that \( \gamma \) is the characteristic circuit—impedance of the parallel-plate line.

Current is fed into the system by insulated feed wires \( l \) meters from the short-circuit end, carrying a total of \( I_o \) amps and spread over a distance \( w \) meters.
Hence the parallel plate system is energized by a magnetic field:

\[ H_0 = \frac{I_0}{l} \text{ amps/m.} \]

The field impedance at the energizing position is:

\[ Z_T = \frac{Z_a \cdot j \cdot Z_a \tan \beta l}{Z_a + j \cdot Z_a \tan \beta l} \]

resulting in an electric field:

\[ E_o = H_o \cdot Z_T \]

and hence a corresponding voltage:

\[ 2V_f = H_o \cdot Z_T \cdot c \]

\[ = \frac{H_o \cdot c \cdot j \cdot Z_a \beta l}{1 + j \cdot \beta l} \]

Provided the "l" line is very much shorter than \( \frac{\lambda}{4} \)

\[ 2V_f = \frac{H_o \cdot c \cdot j \cdot \omega \cdot \sqrt{\frac{\mu}{\varepsilon}} \cdot \sqrt{\mu \varepsilon}}{1 + j \cdot \omega \cdot \sqrt{\mu \varepsilon} \cdot l} \]

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from 10.9-1:  

\[ r = \frac{\mu}{N \varepsilon} \cdot \frac{C}{u} \]

\[ \sqrt{\varepsilon} = \sqrt{\mu} \cdot \frac{C}{r u} \]

\[ 2V_t = \frac{H_o e^{j\omega l t}}{1 + j\omega \mu l / v r} \]

This agrees with equation 10.2-10 when \( \sigma \rightarrow \infty \) so that \( Z_o \rightarrow 0 \) and \( \sinh \Gamma a \) and \( \cosh \Gamma a \rightarrow \infty \).

Fig. 10.5-2 is the corresponding equivalent circuit.

Returning to Fig. 10.9-1, the feed wires could lie insulated but otherwise in contact with the infinitely conducting surfaces of the "l" line, in which case the impedance seen by the primary system is no more than \( Z_T \cdot \frac{C}{l} \cdot R \) assuming all the wires are fed in parallel as in the experiments described in this section. If the wires are fed in series, then the usual impedance transformation follows. The self-impedance of the feed wires has, of course, been neglected in this discussion.
EXTERNAL ADJUSTABLE LOAD

PLATES S₁ & S₂ OF FIG 10.1-1

SHOWING

E₅₁ FIELD MEASURING LEAD; 1MM Cu FOIL
EXTERNAL ADJUSTABLE LOAD

ALL FOUR PLATES OF FIG. 10.1-1 ASSEMBLED & WIRED
APPENDIX I

The Current Distribution in a Flat Strip Conductor at D.C.

The strip of length "L" and breadth "b" is energized by a current density \( J_s \), uniformly over "d" at one end and uniformly over "b" at the other, as in Fig. A1. We are interested in \( J_x(x,y) \) and \( J_y(x,y) \) as \( d/L \) is reduced to quite small values, so that we might regard \( d \) as a piece of wire feeding a strip

![Diagram of current distribution in a flat strip conductor](image)

Because \( H = 0 \), \( \nabla \times E = 0 \)

\[ \therefore E = -\nabla \phi \]

\[ J = \sigma E = -\sigma \nabla \phi \quad (1) \]
\[ \nabla \cdot J = 0 \quad \cdots \quad (2) \]

Hence
\[ J_x = -\sigma \frac{\partial \phi}{\partial x} \quad \cdots \quad (3) \]
\[ J_y = -\sigma \frac{\partial \phi}{\partial y} \quad \cdots \quad (4) \]
\[ 0 = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} \quad \cdots \quad (5) \]

Hence
\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \cdots \quad (6) \]
\[ \therefore \phi = (A \cos kx + B \sin kx)(C \cos ky + D \sin ky) \quad \cdots \quad (7) \]
\[ J_y = -\sigma k (A \cos kx + B \sin kx)(-C \sin ky + D \cos ky) \quad \cdots \quad (8) \]
\[ J_x = \sigma k (A \sin kx + B \cos kx)(C \cos ky + D \sin ky) \quad \cdots \quad (9) \]

Boundary conditions:
\[ J_y \left( x, \frac{1}{2} \right) = J_y \left( x, -\frac{1}{2} \right) = 0 \]
\[ \therefore -C \sin k\frac{1}{2} + D \cos k\frac{1}{2} = 0 \quad \cdots \quad (10) \]
and
\[ C \sin k\frac{1}{2} + D \cos k\frac{1}{2} = 0 \quad \cdots \quad (11) \]

If \( J_x \) is energized symmetrically about \( OX \), then

from equation (9)
\[ I = 0 \quad \cdots \quad (12) \]
In equations (10) and (11), since \( \alpha \) cannot be zero then
\[
\sin \frac{kL}{2} \quad \text{must be zero.}
\]

\[
\therefore \quad \frac{kL}{2} = 0, \pi, 2\pi \text{ etc}
\]

\[
k = \frac{g\pi}{L} ; \quad g = 0, 2, 4 \text{ etc} \quad \cdots \quad (13)
\]

We can now write equation (7) as:-

\[
\psi = \left( N \cosh kx + M \sinh kx \right) \cos ky
\]

Now

\[
\psi(L, y) = 0 \quad \text{see Fig. A1.}
\]

\[
\therefore \quad 0 = N \cos kL + M \sin kL
\]

\[
M = -N \cosh kL
\]

\[
\Phi = N \left( \cosh kx - \cosh kl \cos kL \right) \cos ky \quad \cdots \quad (14)
\]

\[
J_x = -ok N k \left( \sinh kx - \cosh kl \cos kL \right) \cos ky
\]

We have used the symbol \( N_k \) to denote that for each
value of \( k \) there is a corresponding value of \( N \).

Hence from equation (14)

\[
J_x(0, y) = \sum_n \alpha k N k \cosh kL \cos ky
\]

\[
= \sum_n \alpha N k \frac{\pi g}{L} \cos \left( \frac{\pi g y}{L} \right) \cos \left( \frac{\pi g y}{L} \right) \quad \cdots \quad (15)
\]

\[
g = 0, 2, 4, \quad \cdots
\]
The zero-order mode term in equation (15) is:

\[ J_x(0, y) = \frac{\sigma N_0}{L} \quad (15a) \]

Looking at Fig. A1 we can express \( J_x(0, y) \), the energizing current distribution, as a Fourier Series, so:

\[ J_x(0, y) = J_s \frac{d}{l} + \frac{4J_s}{\pi} \sum_{g=2, 4, \ldots}^{\infty} \frac{1}{g} \sin\left(\frac{\pi gd}{2l}\right) \cos\left(\frac{\pi gy}{l}\right) \quad (16) \]

To determine the various values of \( N_k \), we equate equations (15) and (16) term by term.

For \( g = 0 \):

\[ \frac{0}{L} = \frac{J_s d}{l} \]

\[ \therefore N_0 = \frac{J_s d}{l} \]

For the other values of \( g \):

\[ \frac{4J_s}{\pi g} \sin\left(\frac{\pi gd}{2l}\right) = \sigma N_k \frac{\pi g}{l} \cos\left(\frac{\pi gL}{l}\right) \]

\[ \therefore N_k = \frac{4J_s}{\pi g} \sin\left(\frac{\pi gd}{2l}\right) \frac{l}{\sigma \pi g} \tanh\left(\frac{\pi gL}{l}\right) \]
Inserting these values for $N_0$ and $N_k$ into equation (14) we get:

$$J_x = \frac{dJ_s}{d^2} \left[ \sum_{g=2,4,...} \frac{4J_s}{\pi g} \sin \left( \frac{\pi g d}{2L} \right) \left[ \tanh \left( \frac{\pi g l}{L} \right) \sin \left( \frac{\pi g x}{L} \right) \right. \right.$$

$$\left. - \frac{\cosh \frac{\pi g x}{L}}{\sin \left( \frac{\pi g x}{L} \right)} \right] \cos \left( \frac{\pi g y}{L} \right) \right] - - - (17)$$

We now look for a simplified approximation to equation (17), which is likely to be good enough for our purpose.

If $L > L - \tanh \left( \frac{\pi g L}{L} \right) \approx 1$

and

$$\sin \left( \frac{\pi g x}{L} \right) - \cosh \left( \frac{\pi g x}{L} \right) \approx -e^{-\frac{\pi g x}{L}}$$

and equation (17) becomes:

$$J_x = \frac{dJ_s}{d^2} + \frac{4J_s}{\pi} \left[ \sum_{g=2,4,...} \frac{1}{g} \sin \left( \frac{\pi g d}{2L} \right) e^{-\frac{\pi g x}{L}} \cos \left( \frac{\pi g y}{L} \right) \right] - - - (18)$$
The exponential term decays so rapidly that only the \( g = 2 \) term is significant at \( x = 1 \)

\[
\frac{-2\pi x}{l} \approx 0.002 \quad \text{at} \quad x = 1
\]

\[
\therefore J_x(l, y) = J_s \frac{d}{l} + \frac{4J_s}{2\pi} s\hat{w} \left( \frac{2\pi d}{2l} \right) 0.002 \cos \left( \frac{2\pi y}{l} \right)
\]

If now \( \frac{d}{l} < \frac{1}{4} \) say, we can write \( \frac{\pi d}{l} \) instead of \( s\hat{w} \left( \frac{\pi d}{l} \right) \), and then:

\[
J_x(l, y) = J_s \frac{d}{l} + 2J_s \frac{d}{l} 0.002 \cos \left( \frac{2\pi y}{l} \right) - - - (19)
\]

Hence

\[
J_x(l, 0) = J_s \frac{d}{l} \left( 1 + 0.004 \right) - - - - (20)
\]

\[
J_x(l, \frac{L}{2}) = J_s \frac{d}{l} \left( 1 - 0.004 \right) - - - - (21)
\]

and so, when \( L > l \) \{ both conditions in which we are interested \}

\[
\text{and } \frac{d}{l} < \frac{1}{4}
\]

then at \( x = 1 \) and beyond, the fluctuation of \( J_x \) across the strip is rather less than 1\%, and the fluctuation is independent of the value \( \frac{d}{l} \).
Our strip is 5 cms wide, so 10 uniformly spaced feeds should give $J_x$ with less than 1% fluctuation 0.5 cms from the ends, since such an arrangement may be regarded as 10 independent strips each 0.5 cms wide, i.e. $\nu = 0.5$ cms

and if $\frac{d}{\nu} < \frac{1}{4}$, $d < 1$ mm.

20 SWG Wire is 0.9 mm diameter.
APPENDIX II

A Two-Dimensional Analysis of Electromagnetic Propagation into a Conducting Medium

The geometry is shown in Fig. AII-1, in which the energizing strip of width "d" meters carries a current of $I_0 e^{jot}$ amps. This corresponds to strip $S_3$ of Fig. 8.1-1. Power flows into the conducting medium, in the $+\infty$ direction in accordance with

$$P_x = E_y x H_z.$$  

This corresponds to strip $S_1$ of Fig. 8.1-1.

The one-dimensional analysis is concerned with field variations in the $\infty$ direction only.

![Diagram](image)
We now wish to consider propagation in both \( \infty \) and \( Z \) directions. The field \( H_\infty \) will permit propagation in the \( Z \) direction, since:

\[
P_2 = E_y \times H_x
\]

If the field spreads in the conducting medium, this will be the underlying mechanism.

The appropriate field equations are therefore:

\[
\frac{\partial E_y}{\partial z} = j \omega \mu H_x \quad (1)
\]

\[
\frac{\partial E_y}{\partial x} = -j \omega \mu H_z \quad \ldots \ldots \quad (2)
\]

\[
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \sigma E_y \quad \ldots \ldots \quad (3)
\]

and hence

\[
\nabla^2 H_z = \Gamma_0^2 H_z \quad (4)
\]

\[
\Gamma_0^2 = j \omega \mu \sigma^*.
\]

It is convenient to express the diffusion equation in \( H_z \) rather than \( E_y \) or \( H_x \), because the medium is energized in terms of \( H_z \); Eqn(4).
A solution of Eqn. 4 is:

\[ H_z = (A \sin \alpha + B \cos \alpha)(C \sin k_z + D \cosh k_z) \]

where \[ n^2 + k^2 = -\Gamma_o^2 \]

or \[ n = \sqrt[4]{k^2 + \Gamma_o^2} \]

Hence \[ H_z = \left( P \sin \frac{\pi}{N} \sqrt{k^2 + \Gamma_o^2} \alpha + Q \cos \frac{\pi}{N} \sqrt{k^2 + \Gamma_o^2} \alpha \right) 
\left( C \sin k_z + D \cosh k_z \right) \ldots \ldots (5) \]

To simplify the analysis we shall assume that the conducting medium is sufficiently deep to behave like an infinite line in the \( \alpha \) direction, so that equation 5 becomes:

\[ H_z = e^{-\sqrt{k^2 + \Gamma_o^2} \alpha} \left( C \sin k_z + D \cosh k_z \right) \ldots \ldots (6) \]

If the medium is energized symmetrically about the \( OY \) axis, the constant \( C \) will be zero giving:

\[ H_z = D e^{-\sqrt{k^2 + \Gamma_o^2} \alpha} \cos k_z \ldots \ldots (7) \]

where \( D \) and \( k \) are constants to be determined from additional boundary conditions.
From equations 1, 2 and 7

\[ H_x = -k \frac{D e^{-\sqrt{k^2 + \gamma_0^2} z}}{\sqrt{k^2 + \gamma_0^2}} \sin k z \quad - - - \quad (8) \]

If now we assume that for all values of \( \infty \), \( H_x \) is zero where \( z = \pm \frac{L}{2} \) because there will be total reflection of power at the metal free-space boundary, then:

\[ \sin \frac{kL}{2} = 0 \]

\[ \therefore \quad kL = 0, \pi, 2\pi \ldots \]

or

\[ k = g \frac{\pi}{L} \quad \text{where} \quad g = 0, 2, 4 \ldots \quad - - - \quad (9) \]

Hence:

\[ H_z(x, z) = \sum D_k e^{-\sqrt{k^2 + \gamma_0^2} x} \cos k z \quad - - - \quad (10) \]

and along the energizing surface at \( x = 0 \):

\[ H_z(0, z) = \sum D_k \cos k z \quad - - - \quad (11) \]

or

\[ H_z(0, z) = \sum D_g \cos g \frac{\pi z}{L} \quad - - - \quad (12) \]
The pattern of energization displayed in Fig. AII-1(b) can be represented by the series:

\[ H_0(z) = H_0 \frac{d}{r} + \frac{4H_0}{\pi} \sum_{g=2}^{\infty} \frac{1}{g} \sin \left( \frac{\pi gd}{2L} \right) \cos \frac{\pi g z}{L} \]  

The values of \( D_g \), equation(12) can now be determined by identifying equation(12) with equation(13), term by term.

\[ D_0 = \frac{H_0 d}{r} \]

and for the other "g" terms

\[ D_g = \frac{4H_0}{\pi} \frac{1}{g} \sin \left( \frac{\pi gd}{2L} \right) \]

Hence

\[ H_2(z, z) = H_0 \frac{d}{r} e^{-\Gamma_0 x} + 4 \frac{H_0}{\pi} \sum_{g=2}^{\infty} \frac{1}{g} \sin \left( \frac{\pi gd}{2L} \right) e^{-\sqrt{\left( \frac{g^2 \pi^2}{L^2} + \Gamma_0^2 \right)} x} \cos \frac{\pi g z}{L} \]  

\[ \ldots \] (14)
Equation (14) evaluated for $v = a$ becomes:

$$H_z (x; z) = H_0 e^{-\frac{\Gamma_0 x}{2}}$$

because the sine term is zero for all even values of $9$. This condition corresponds to one-dimensional propagation, simply because the energization contains no mode other than zero order.

Now consider the case of $d = l/2$, with $l = 10$ cm, and confining attention to $Z = 0$. The zero order mode, i.e. first term, in equation (14) is:

$$0.5 H_0 e^{-\frac{\Gamma_0 x}{2}}$$

It is the term $(g \pi v_r)^2$ in the exponent of equation (14) which distorts the pattern of the fields as they diffuse into the conducting medium.

The magnitude of the sum of the "$9$" terms up to $g = 12$ is $0.552 H_0$, neglecting, for the moment, the exponential factor. This $0.552 H_0$, added to the zero order of $0.5 H_0$, gives $1.05 H_0$ or $\%$ above $H_0$, the one-dimensional value. In fact beyond $g = 12$ the fluctuation in $H(x, 0)$ becomes progressively less than $\%$ of $H_0$, the one-dimensional value. Now consider the exponential term:

$$e^{-\sqrt{(g \pi v_r)^2 + \Gamma_0^2} x}$$

At $g = 12$ $(g \pi v_r)^2 = 1.4 \times 10^5$

At 250 Hz $\Gamma_0^2 = 3.1 \times 10^5$ for Copper.
Hence at frequencies above say 250 Hz, the \( (\frac{g \pi L}{v})^2 \) term can be neglected compared with \( \frac{12}{v} \) for all values of \( g \) less than 12.

Under these circumstances:

\[
H_z(0, x) = 0.5 H_0 e^{-\frac{12}{\pi} x} \\
+ (0.5 \pm 0.05) H_0 e^{-\frac{x}{2} x}
\]

Since the value of \( H_z(x) \) along \( z = 0 \) agrees reasonably well with the one-dimensional result, it is not unreasonable to suppose that, with \( d = 5 \) cm and \( l = 10 \) cm, the pattern of energization will be preserved throughout the conducting medium at frequencies above some 250 Hz.

If we now consider the case of two-dimensional wave propagation in free-space, such as the "\( g \)" space of Fig. 8.1-1, the situation is very different. Instead of \( \Gamma_o^2 = j \omega \mu \sigma \) in the conductor medium, we have \( \Gamma_o^2 = j \omega \mu_0 \sigma \) in the free-space, which even at 20 kHz is only \( 10^{-11} \). The mode term \( k^2 \) is undoubtedly important compared with the \( \Gamma_o^2 \) term, and no simple picture emerges.
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