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Extending a Theorem of Bergweiler and Langley Concerning Nonvanishing Derivatives

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Abstract
We consider the differential operator $\Lambda_k$ defined by
$$\Lambda_k(y) = \Psi_k(y) + a_{k-1} \Psi_{k-1}(y) + \ldots + a_1 \Psi_1(y) + a_0,$$
where $a_0, \ldots, a_{k-1}$ are analytic functions of restricted growth and $\Psi_k(y)$ is a differential operator defined by $\Psi_1(y) = y$ and $\Psi_{k+1}(y) = y \Psi_k(y) + (\Psi_k(y))'$ for $k \in \mathbb{N}$. We suppose that $k \geq 3$, that $F$ is a meromorphic function on an annulus $A(r_0)$, and that $\Lambda_k(F)$ has all its zeros on a set $E$ such that $E$ has no limit point in $A(r_0)$. We suppose also that all simple poles $a$ of $F$ in $A(r_0) \setminus E$ have $\text{Res}(F, a) \not\in \{1, \ldots, k-1\}$. We then deduce that $F$ is a function of restricted growth in the Nevanlinna sense. This extends a theorem of Bergweiler and Langley [1]. We show also that this result does not hold for $a_0, \ldots, a_{k-1}$ meromorphic functions.

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1 Introduction
In [1], Bergweiler and Langley define a differential operator $\Psi_k(F)$ for $k \in \mathbb{N}$ by
$$\Psi_1(F) = F, \quad \Psi_{k+1}(F) = F \Psi_k(F) + (\Psi_k(F))',$$
for which we have the following lemma.

Lemma 1.1 ([1]) Let $f$ be meromorphic on a domain $\Omega$ and let $F = f'/f$. Then for each $k \in \mathbb{N}$ we have $\Psi_k(F) = f^{(k)}/f$.

Bergweiler and Langley then prove the following theorem. For background material regarding Nevanlinna theory, the reader is referred to [5].

Theorem A ([1]) Let $k \geq 3$ be an integer, and let $F$ be a meromorphic and non-constant function in the plane that satisfies both of the following conditions:

(i) $\Psi_k(F)$ has no zeros.

(ii) if $a$ is a simple pole of $F$ then $\text{Res}(F, a) \not\in \{1, \ldots, k-1\}$.

Then $F$ has the form
$$F(z) = \frac{(k-1)z + \alpha}{z^2 + \beta z + \gamma},$$
(2)
or
$$F(z) = \frac{1}{az + \beta}.$$
(3)

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Here $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha \neq 0$ in (3).
Conversely, if $F$ has the form (2) or (3), and if (ii) holds, then $\Psi_k(F)$ has no zeros. If $F$ has the form (2) or (3), but (ii) does not hold, then $\Psi_k(F) \equiv 0$.

We note that the conclusion of this theorem can be summarised as saying that $F$ is a rational function, of a special form, and hence,

$$T(r, F) = O(\log r) \quad \text{as } r \to \infty.$$  

Defining an annulus $A(r_0)$ by

$$A(r_0) = \{z : r_0 \leq |z| < \infty\}$$  

we extend Theorem 1 in two ways. First, we let $F$ be meromorphic and non-constant on $A(r_0)$, by which we mean that $F$ is meromorphic in a domain containing $A(r_0)$. Second, we weaken condition (i) as follows.

We let $a_0, \ldots, a_{k-1}$ be analytic functions of restricted growth as $z \to \infty$, and define $\Lambda_k(F)$ by

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0.$$  

We then assume that $\Lambda_k(F) = 0$ only on a set $E$ such that $E$ has no limit point in the annulus $A(r_0)$. This implies that $\Lambda_k(F) = 0$ only on a countable set $E$. The new conclusion is that $F$ is a function of restricted growth in the Nevanlinna sense. We state the extended theorem as follows, denoting by $S(r, F)$ any quantity satisfying

$$S(r, F) = O(\log r + \log^+ T(r, F)),$$

as $r \to \infty$ outside a set of finite measure, not necessarily the same set at each occurrence.

**Theorem 1.2** Let $k \geq 3$ be an integer and let $F$ be meromorphic and non-constant in an annulus $A(r_0)$, as defined by (4). Suppose $a_0, \ldots, a_{k-1}$ are analytic functions on $A(r_0)$, with

$$a_j(z) = O(|z|^{(\lambda-1)(k-j)}) \quad \text{as } z \to \infty,$$

for some fixed $\lambda \geq 0$. Let $f_1, \ldots, f_k$ be solutions of $L(w) = 0$ where $L$ is defined by

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_0f,$$

in $A(r_0) \setminus \mathbb{R}^-$. Let $\Lambda_k(F)$ be defined as in (5) by

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0.$$  

Suppose there exists a set $E$, such that $E$ has no limit point in $A(r_0)$, and such that $\Lambda_k(F)$ has all its zeros in $E$. Suppose further that all simple poles $a$ of $F$ in $A(r_0) \setminus E$ have $\text{Res}(F,a) \notin \{1, \ldots, k-1\}$. Set

$$N_E(r) = \int_{t=a}^{t=b} \frac{n_E(t)}{t} dt$$

where $n_E(t)$ is the number of points in $E \cap \{z : r_0 \leq |z| \leq t\}$.

Then either:

(i) $T(r, F) \leq cN_E(r) + S(r, F)$, as $r \to \infty$, where $c$ is a constant depending only on $k$,

or

(ii) $F$ is a rational function of the $f_j$ and their derivatives, in which case

$$T(r, F) = O(r^\lambda + \log r), \quad \text{as } r \to \infty.$$  

We note that when $\lambda = 0$, it follows from (ii) that $\lim_{z \to \infty} F(z)$ exists.

The following corollaries are deduced from Theorem 1.2.
Corollary 1.3 Let \( k \geq 3 \) be an integer and let \( F \) be meromorphic and non-constant in an annulus \( A(r_0) \), as defined by (4). Suppose there exists a set \( E \), such that \( E \) has no limit point in \( A(r_0) \), and such that \( \Psi_k(F) \) has all its zeros in \( E \). Suppose further that all simple poles \( a \) of \( F \), such that \( a \notin E \), have \( \text{Res}(F, a) \notin \{1, \ldots, k-1\} \). Then either:

(i) \( T(r, F) \leq cN_E(r) + O(\log r + \log^+ T(r, F)) \) \( \text{(n.e.)} \), where \( c \) is a constant depending only on \( k \), or

(ii) \( \lim_{z \to \infty} F(z) \) exists.

Corollary 1.4 Let \( k \geq 3 \) be an integer, and let \( F \) be meromorphic and non-constant in an annulus \( A(r_0) \), as defined by (4). Suppose \( F \) satisfies both of the following conditions:

(i) \( \Psi_k(F) \) has no zeros.

(ii) if \( a \) is a simple pole of \( F \) then \( \text{Res}(F, a) \notin \{1, \ldots, k-1\} \).

Then \( \lim_{z \to \infty} F(z) \) exists.

Corollary 1.5 Let \( F \) be meromorphic on \( \mathbb{C} \) and satisfy both of the following conditions:

(i) \( \Psi_k(F) \) has finitely many zeros.

(ii) for all but finitely many simple poles of \( F \) we have \( \text{Res}(F, a) \notin \{1, \ldots, k-1\} \).

Then \( F \) is a rational function.

The following example shows that we cannot extend Theorem 1.2 to the case where \( a_0, \ldots, a_{k-1} \) are meromorphic functions.

**Example** Let \( k \geq 3 \) and let \( F = f'/f \) where \( f \) is a meromorphic function which is nonvanishing, that is, has no zeros. Define \( a_0, \ldots, a_{k-1} \) by

\[
a_j = \begin{cases} 
-\frac{f^{(k)}}{f'} & \text{if } j = 0, \\
\frac{f^{(j)}}{f'} & \text{if } j = 1, \ldots, k-1.
\end{cases}
\]

Then \( a_0, \ldots, a_{k-1} \) are meromorphic functions and we have, using Lemma 1.1, that

\[
\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0 = \frac{f^{(k)}}{f} + \frac{f}{f^{(k-1)}}f' + \ldots + \frac{f}{f'}f^{(k)} - \frac{f^{(k)}}{f} = k - 1
\]

Thus \( \Lambda_k(F) \) has no zeros, and \( E = \emptyset \). And so, all the hypotheses of Theorem 1.2 are satisfied except for the analyticity and growth of the \( a_j \). However, since \( f \) may be any nonvanishing meromorphic function, no conclusions may be drawn about the growth of \( F \).

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Some preliminaries for the proof of Theorem 1.2 are in § 2, and the main part of the proof is in § 3.
2 Proof of Theorem 1.2 (Preliminaries)

First we include some background material about Wronskians, referring the reader to [7] for further material. Let \( f_1, \ldots, f_k \) be meromorphic functions. We define the Wronskian \( W(f_1, \ldots, f_k) \) as follows

\[
W(f_1, \ldots, f_k) = \begin{vmatrix}
    f_1 & \cdots & f_k \\
    f'_1 & \cdots & f'_k \\
    \vdots & \ddots & \vdots \\
    f^{(k-1)}_1 & \cdots & f^{(k-1)}_k
\end{vmatrix}.
\]

Following the notation in [7], we denote by \( W_q(f_1, \ldots, f_k) \), for \( q = 0, \ldots, k-1 \), the determinant which is obtained from \( W(f_1, \ldots, f_k) \) by replacing the row \((f^{(q)}_1, \ldots, f^{(q)}_k)\) by \((f^{(k)}_1, \ldots, f^{(k)}_k)\). We include two useful lemmas regarding Wronskians, the second of which combines results from [4] and [7].

**Lemma 2.1** ([7]) Let \( f_1, \ldots, f_k \) be meromorphic functions in a domain \( \Omega \). Then \( W(f_1, \ldots, f_k) \) vanishes identically on \( \Omega \), if and only if \( f_1, \ldots, f_k \) are linearly dependent on \( \Omega \).

**Lemma 2.2** ([4], [7]) Let \( k \geq 1 \) and let \( f_1, \ldots, f_k \) be linearly independent meromorphic functions in a domain \( \Omega \), that satisfy the homogeneous linear differential equation \( L(w) = 0 \), where \( L \) is defined by (7), and in which the \( a_q \) are meromorphic on \( \Omega \). Then the \( a_q \) can be written in the form

\[
a_q = -\frac{W_q(f_1, \ldots, f_k)}{W(f_1, \ldots, f_k)}.
\]

for \( q = 0, \ldots, k-1 \) and, in particular,

\[
a_{k-1} = -\frac{W(f_1, \ldots, f_k)}{W(f_1, \ldots, f_k)}.
\]

The poles of \( a_q \) in \( \Omega \) have multiplicity \( \leq k-q \) and can only arise among the poles of \( f_1, \ldots, f_k \) and the zeros of \( W(f_1, \ldots, f_k) \). Furthermore, if \( f \) is meromorphic on \( \Omega \) then

\[
W(f_1, \ldots, f_k, f) = W(f_1, \ldots, f_k)L(f).
\]

Several lemmas are needed for the proof of Theorem 1.2. We state them here, providing proofs where necessary. The first assertion in the following lemma is proved in [1], and the second is an extension which follows immediately.

**Lemma 2.3** ([11]) Let \( k \geq 2 \) be an integer. Let \( y \) be meromorphic on a domain \( \Omega \), such that if \( a \) is a simple pole of \( y \) then \( \text{Res}(y, a) \notin \{1, \ldots, k-1\} \). Let \( n \in \mathbb{N} \) be such that \( n \leq k \). If \( y \) has a pole at \( a \) of multiplicity \( m \) then \( \Psi_n(y) \) has a pole at \( a \) of multiplicity \( nm \), and \( \Lambda_n(y) \) has a pole at \( a \) of multiplicity \( nm \), where \( \Lambda_n(y) \) is defined as in (5) by

\[
\Lambda_n(y) = \Psi_n(y) + a_{n-1}\Psi_{n-1}(y) + \ldots + a_1\Psi_1(y) + a_0
\]

where \( a_0, \ldots, a_{n-1} \) are analytic functions on \( \Omega \).

The following summarises some results from Nevanlinna theory which are used in Lemma 2.8. For background material about Nevanlinna theory in an annulus, we refer the reader to [2] or [4].

**Lemma 2.4** Let \( \gamma \in \mathbb{N} \) and let \( B \) be a simply connected domain in the annulus \( A(r_0) \), as defined by (4). Suppose that \( w \) is meromorphic on \( B \) such that \( W = w^\gamma \) is meromorphic on \( A(r_0) \). Then \( w^{(k)}/w \) is meromorphic on \( A(r_0) \), for each \( k \in \mathbb{N} \), with poles of multiplicity at most \( k \). Further, we have

\[
T(r, w^{(k)}/w) = O(T(r, W)), \quad (n.e.),
\]

and

\[
m(r, w^{(k)}/w) = S(r, W), \quad (n.e.),
\]

as \( r \to \infty \).
Suppose that $F \in \mathcal{A}(r_0)$, as defined by (4), and satisfy $\log^+ \log^+ |f_j(z)| = O(|\log |z| |)$ for $z$ in $S = \{z : |z| > r_0, |\arg z| < 2\pi\}$. Suppose that $F$ is meromorphic in $\mathcal{A}(r_0)$. Suppose further that, for some non-negative integer $Q$, each of the functions $h_1, \ldots, h_k$ on $S$ is a polynomial in the $f_j^{(m)}, F^{(m)}, 1 \leq j \leq k, 0 \leq m \leq Q$. Suppose finally that $h_1, \ldots, h_k$ are linearly independent solutions in $S$ of an equation

$$w^{(k)} + \sum_{j=0}^{k-1} B_j w^{(j)} = 0$$

in which the $B_j$ are meromorphic in $\mathcal{A}(r_0)$. Then we have, for $j = 0, \ldots, k-1$,

$$m(r, B_j) = S(r, F).$$

The following lemma is proved in [4] for $a_0, \ldots, a_{k-1}$ rational functions, and $F$ meromorphic in the plane. The proof extends without modification to the case where $a_0, \ldots, a_{k-1}$ are analytic functions and $F$ is meromorphic in $\mathcal{A}(r_0)$. This gives the following lemma.

**Lemma 2.6** Let $\lambda > 0$ and $k \geq 2$, and let $a_0, \ldots, a_{k-1}$ be analytic functions satisfying $a_j(z) = O(|z|^{|\lambda-1|(k-j)})$ as $z \to \infty$. Suppose that $F$ is meromorphic in the annulus $\mathcal{A}(r_0)$, as defined by (4), and has in some domain $B$ a representation as a rational function in solutions $f_j$ of the equation $L(w) = 0$ and their derivatives, where $L$ is defined by (7). If $\lambda > 0$ then $T(r, F) = O(r^k)$ as $r \to \infty$. If $\lambda = 0$ then $\lim_{r \to \infty} F(z)$ exists.

The following lemma is found in [8].

**Lemma 2.7** Let $k \geq 1$ and that $a_0, \ldots, a_{k-1}$ are analytic in an annulus $\mathcal{A}(r_0)$, as defined by (4), such that, for some $\lambda \geq 0$,

$$a_j(z) = O(|z|^{(\lambda-1)(k-j)}), \quad as \quad z \to \infty.$$  

(11)

Let $f_j(z)$ be a solution of $L(w) = 0$, where $L$ is defined by (7), in a sectorial region

$$S = \{z : |z| > r_0, \alpha < \arg z < \alpha + 2\pi\},$$

where $\alpha$ is real. Then as $z \to \infty$ in $S$,

$$\log^+ |f_j(z)| = O(|z|^\lambda + \log |z|).$$

(12)

Suppose now that $k, F, \mathcal{A}(r_0)$ and $a_0, \ldots, a_{k-1}$ are as in the statement of Theorem 1.2. In particular, the $a_j$ satisfy (11). We may define linearly independent analytic solutions $f_1, \ldots, f_k$ of (7) in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$. These are analytic in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$ and since the $a_j$ are analytic in $\mathcal{A}(r_0)$, the $f_j$ admit unrestricted analytic continuation in $\mathcal{A}(r_0)$. By (12), $\log^+ |f_j(z)| = O(|z|^\lambda + \log |z|)$ for $j = 1, \ldots, k$ and thus the continuations satisfy $\log^+ \log^+ |f_j(z)| = O(\log |z|)$ for $|z| > r_0, |\arg z| < 2\pi$. Then $f_1, \ldots, f_k$ satisfy the conditions of Lemma 2.5.

Choose a simply connected domain $B \subseteq \mathcal{A}(r_0)$, on which $F$ has no poles and $\Lambda_k(F)$ has no zeros. Define functions $f, g, \Lambda_k(F)$ by $f^\prime / f = F, \quad \Lambda_k(F) = g^{-k}, \quad h = -F g$.

(13)

Then $f, g$ and $h$ are analytic on $B$.

We note that by Lemma 1.1, we have in $\mathcal{B}$ that

$$L(f)/f = f^{(k)}/f + \sum_{j=0}^{k-1} a_j f^{(j)}/f = \Psi_k(F) + \sum_{j=1}^{k-1} a_j \Psi_j(F) + a_0 = \Lambda_k(F).$$

(14)

The following lemma is fundamental to our proof of Theorem 1.2 and uses ideas such as analytic continuation and Wronskians. Background material on these topics can be found in [3] and [7].
Lemma 2.8 Define on \( \mathcal{B} \) functions \( w_j \) and \( h_j \), \( j = 1, \ldots, k \), by
\[
 w_j(z) = f_j(z)g(z) + f_j(z)h(z), \quad h_j(z) = -f_j'(z) + f_j(z)F(z).
\] (15)

Then the \( w_j \) form a fundamental solution set on \( \mathcal{B} \) of the differential equation
\[
 w^{(k)} + \sum_{j=0}^{k-1} A_j w^{(j)} = 0,
\] (16)
in which the \( A_j \) are meromorphic functions on \( \mathcal{A}(r_0) \) with
\[
 T(r, A_q) \leq cN_E(r) + S(r, F), \quad r \to \infty,
\] (17)
for \( j = 0, \ldots, k-1 \), where \( c \) is a constant depending only on \( k \).

**Proof** We divide this proof into a number of steps.

(i) The \( w_j \) are linearly independent solutions of an equation (16) on \( \mathcal{B} \).

We note first that \( w_j = fg(f_j/f)' \) on \( \mathcal{B} \) by (13), and by (10) and the properties of Wronskians we have
\[
 W(w_1, \ldots, w_k) = f^{-1}g(-1)^kW(f_1, \ldots, f_k, f) = f^{-1}g(-1)^kL(f)W(f_1, \ldots, f_k).
\] (18)

Then since \( L(f)/f = \Lambda_k(F) \) by (14), and \( g^k = (\Lambda_k(F))^{-1} \) by (13), we have from (18) that,
\[
 W(w_1, \ldots, w_k) = (-1)^kW(f_1, \ldots, f_k).
\] (19)

By Lemma 2.1, the right-hand side is not identically zero, since the \( f_j \) form a linearly independent solution set of (7). Thus \( w_1, \ldots, w_k \) form a linearly independent solution set for a differential equation (16). Also, by (9), we have
\[
 A_{k-1} = -\frac{W(w_1, \ldots, w_k)'}{W(w_1, \ldots, w_k)} = \frac{(-1)^kW(f_1, \ldots, f_k)'}{(-1)^kW(f_1, \ldots, f_k)} = a_{k-1}.
\] (20)

(ii) The \( h_j \) are linearly independent solutions of a differential equation,
\[
 w^{(k)} + \sum_{j=0}^{k-1} B_j w^{(j)} = 0,
\] (21)
with coefficients \( B_j \) that are meromorphic on \( \mathcal{A}(r_0) \setminus \mathbb{R}^- \).

We have by (13) and (15) that \( w_j = -h_jg \), and so
\[
 W(w_1, \ldots, w_k) = (-1)^k g^kW(h_1, \ldots, h_k) = (-1)^k\Lambda_k(F)^{-1}W(h_1, \ldots, h_k).
\]

Then by (19), we have
\[
 W(h_1, \ldots, h_k) = \Lambda_k(F)W(f_1, \ldots, f_k).
\]

The right-hand side is not identically zero on \( \mathcal{B} \) since \( \Lambda_k(F) \neq 0 \) on \( \mathcal{B} \) and since \( f_1, \ldots, f_k \) form a linearly independent solution set to (7) on \( \mathcal{A}(r_0) \setminus \mathbb{R}^- \). Thus, \( h_1, \ldots, h_k \) form a linearly independent solution set to a differential equation (21) on \( \mathcal{B} \). By (15), we have \( h_j = -f_j' + f_jF \) and thus the \( h_1, \ldots, h_k \) are meromorphic on \( \mathcal{A}(r_0) \setminus \mathbb{R}^- \) since the \( f_j \) are analytic there, and \( F \) is meromorphic on \( \mathcal{A}(r_0) \). Hence the coefficients \( B_j \) are meromorphic on \( \mathcal{A}(r_0) \setminus \mathbb{R}^- \).

(iii) The \( B_j \) extend to be meromorphic on \( \mathcal{A}(r_0) \) and have poles of multiplicity \( \leq k - j \) on \( \mathcal{A}(r_0) \).

By analytic continuation of the \( f_j \), the \( B_j \) extend to be meromorphic on \( \mathcal{A}(r_0) \). Furthermore, (21) has \( k \) linearly independent solutions on a neighbourhood of each point of \( \mathcal{A}(r_0) \), namely \( h_1, \ldots, h_k \). Hence, by Lemma 2.2, the \( B_j \) have poles of multiplicity \( \leq k - j \) on \( \mathcal{A}(r_0) \).
(iv) Estimate for \( m(r, A_j) \).
First we recall by (15) that \( h_j = -f_j' + f_j F \), and note that all the conditions of Lemma 2.5 are satisfied, and thus we have that
\[
m(r, B_j) = S(r, F) \tag{22}
\]
for \( j = 0, \ldots, k - 1 \). Next expressing \( A_j \) in terms of \( B_j \) and \( g^{(p)}/g \), we have for \( j = 0, \ldots, k - 1 \),
\[
A_j = B_j - \binom{k}{j} \frac{g^{(k-j)}}{g} - \sum_{q=j+1}^{k-1} A_q \binom{q}{j} \frac{g^{(q-j)}}{g}, \tag{23}
\]
which is initialised by
\[
A_{k-1} = B_{k-1} - \binom{k}{k-1} \frac{g'}{g}.
\]
We now note by Lemma 2.4 that \( g^{(p)}/g \) is meromorphic in \( A(r_0) \) for all \( p \in \mathbb{N} \), and that \( m(r, g^{(p)}/g) = S(r, F) \) for \( p = 1, \ldots, k \). Thus we have by (22) and by induction on (23), that
\[
m(r, A_j) = S(r, F)
\]
for \( j = 0, \ldots, k - 1 \).

(v) Estimate for \( N(r, A_j) \).
We show first that the poles of \( A_j \) can only arise on \( E \), the set containing all points where \( \Lambda_k(F) = 0 \).

We know by Lemma 2.2, that the poles of \( A_j \) can only arise among the zeros of the continuation of \( W(w_1, \ldots, w_k) \) and the poles of the continuations of \( w_1, \ldots, w_k \). By (20), \( a_k-1 = -W(w_1, \ldots, w_k)'/W(w_1, \ldots, w_k) \) and since \( a_k-1 \) is analytic on \( A(r_0) \), we have that \( W(w_1, \ldots, w_k) \) continues without zeros. We recall from (13) and 15) that \( w_j = (f_j - f_j F)/((\Lambda_k(F))^{1/k} \). Thus the poles of \( A_j \) only arise at poles of \( F \) and zeros of \( \Lambda_k(F) \).

Now let \( z_0 \in A(r_0) \setminus E \) and suppose that a pole of \( A_j \) arises at \( z_0 \). Since \( \Lambda_k(F) \neq 0 \) on \( A(r_0) \setminus E \) we must have that \( z_0 \) is a pole of \( F \), of multiplicity \( m \) say, and if \( m = 1 \) then \( \text{Res}(F, z_0) \notin \{1, \ldots, k - 1\} \) since \( F \) satisfies the hypotheses of Theorem 1.2. Then by Lemma 2.3, \( \Lambda_k(F) \) has a pole at \( z_0 \) of multiplicity \( mk \), and so \( g \) can be analytically continued to a neighbourhood of \( z_0 \) and has a zero of multiplicity \( m \) there. Thus \( h = -F g \) can be analytically continued to \( z_0 \) and since the \( f_j \) can be continued analytically in \( A(r_0) \), we have that \( w_j = f_j g + f_j h \) can be analytically continued to \( z_0 \). We therefore deduce that the \( A_j \) are analytic at \( z_0 \). This contradicts our hypothesis, and so the poles of \( A_j \) can only arise in \( E \).

We recall from (iv) that the poles of \( B_j \) have multiplicity \( \leq k - j \) on \( A(r_0) \). We note also by Lemma 2.4, that \( g^{(p)}/g \) have poles of multiplicity \( \leq p \) there. Thus we have by (23) that the poles of \( A_j \) must have multiplicity \( \leq c \) where \( c \) is a constant depending only on \( k \). Therefore we have, for \( j = 0, \ldots, k - 1 \),
\[
N(r, A_j) \leq c N_E(r).
\]

(vi) Conclusion.

We have since \( T(r, A_j) = N(r, A_j) + m(r, A_j) \) that
\[
T(r, A_j) \leq c N_E(r) + S(r, F), \quad \text{as } r \to \infty,
\]
for \( j = 0, \ldots, k - 1 \), where \( c \) is a constant depending only on \( k \). This completes the proof of Lemma 2.8.

The following lemma appears in [4] and is the final lemma needed for the main part of the proof of Theorem 1.2.

**Lemma 2.9 ([4])** Let \( k \geq 1 \) be an integer, and let \( f_1, \ldots, f_k, G, H \) and \( a_0, \ldots, a_{k-1} \) and \( A_0, \ldots, A_{k-1} \) be meromorphic in a domain \( \Omega \). Suppose that \( f_1, \ldots, f_k \) are linearly independent solutions in \( \Omega \) of \( L(w) = 0 \), where \( L \) is defined as in (7). Then the functions \( f_1 g + f_1 h, \ldots, f_k g + f_k h \) are solutions in \( \Omega \) of the equation (16) if and only if, setting \( A_k = 1 \) and \( A_{-1} = a_{-1} = 0 \) and, for \( 0 \leq q \leq k \),
\[
M_{k,q}(w) = \sum_{m=q}^{k} \binom{m}{q} A_m w^{(m-q)}, \quad M_{k,-1}(w) = 0,
\]
we have, for $0 \leq q \leq k - 1$,

$$M_{k,q}(h) - a_q h = -M_{k,q-1}(g) + a_q M_{k,k-1}(g) - (a_q a_{k-1} - a'_q - a_{q-1}) g.$$  \hspace{1cm} (24)

### 3 Proof of Theorem 1.2 (Main Part)

We are now in a position to complete the proof of Theorem 1.2. We note that we use methods found in [4, Theorem 3].

**Proof of Theorem 1.2** We apply Lemma 2.9 to equation (16) and to $g$ and $h$ in $\mathcal{B}$. The $k$ equations (24) can be written in the form

$$T_q(g) = S_q(h) = \sum_{j=0}^{k-q} c_{j,q} h^{(j)} , \quad 0 \leq q \leq k - 1,$$  \hspace{1cm} (25)

in which $T_q$ and $S_q$ are homogeneous linear differential operators with coefficients $\lambda_\nu$, which are rational functions in the $a_j$, $A_j$ and their derivatives. Then by (17) we have

$$T(r, \lambda_\nu) \leq c N_E(r) + S(r, F), \quad \text{as } r \to \infty,$$  \hspace{1cm} (26)

where $c$ is a constant depending only on $k$.

We have in particular that $q = k - 1$ gives

$$M_{k,k-1}(h) - a_{k-1} h = -M_{k,k-2}(g) + a_{k-1} M_{k,k-1}(g) - (a_{k-1} a_{k-1} - a'_{k-1} - a_{k-2}) g.$$ 

Then since

$$M_{k,k-1}(h) = A_{k-1} h + k h',
M_{k,k-1}(g) = A_{k-1} g + k g',
M_{k,k-2}(g) = A_{k-2} g + (k-1) A_{k-1} g' + k(k-1) g'/2,$$

we now have that

$$A_{k-1} h + k h' - a_{k-1} h = -A_{k-2} g - (k-1) A_{k-1} g' - k(k-1) g'/2 + a_{k-1} A_{k-1} g + k a_{k-1} g' - a_{k-1} a_{k-1} - a'_{k-1} - a_{k-2} g$$

which gives

$$h' = U(g) = -(k-1) g''/2 + a_{k-1} g'/k + (a'_{k-1} + a_{k-2} - A_{k-2}) g/k$$  \hspace{1cm} (27)

since $a_{k-1} = A_{k-1}$ on $\mathcal{B}$ by (20). We note that we can then write (25) in the form

$$T_q(g) = c_{0,q} h + \sum_{j=1}^{k-q} c_{j,q} \frac{d^{j-1}}{dz^{j-1}}(U(g)).$$  \hspace{1cm} (28)

We distinguish two cases here.

**Case 1.** We assume that the coefficient of $h$ in at least one $S_q$ in (25) is not identically zero.

Let $\nu$ be the largest integer, $0 \leq \nu \leq k - 1$, such that $c_{0,\nu} \neq 0$. Then since $h = -Fg$ by (13), equations (25) and (28) give

$$h = -Fg = (c_{0,\nu})^{-1} \left( T_{\nu}(g) - \sum_{j=1}^{k-\nu} c_{j,\nu} \frac{d^{j-1}}{dz^{j-1}}(U(g)) \right) = V(g).$$  \hspace{1cm} (29)

Then by (25), (27) and (29) we know that $g$ solves the system of equations

$$U(g) = \frac{d}{dz}(V(g)), \quad S_q(V(g)) = T_q(g), \quad 0 \leq q \leq k - 2.$$  \hspace{1cm} (30)
Here we distinguish two sub-cases.

**Case 1A.** We assume that the solution space of (30) has dimension 1. That is, every common solution of the equations (30) is a constant multiple of \( g \).

Then (26) and a standard reduction procedure, see [6, p.126], give a first order equation

\[ p_1 g' + p_2 g = 0, \quad p_1 \neq 0, \]

where the \( p_j \) are rational functions in the \( \lambda_v \) and their derivatives. It follows by (26) that

\[ T(r, g'/g) \leq cN_E(r) + S(r, F), \quad \text{as } r \to \infty, \]

where \( c \) is a constant depending only on \( k \). Hence, since \( F = -h/g \) and using (26) and (29),

\[ T(r, F) \leq cN_E(r) + S(r, F), \quad \text{as } r \to \infty. \]

Hence we have conclusion (i) of the theorem.

**Case 1B.** We assume that there is a solution \( G \) for the system (30) such that \( G/g \) is non-constant. In particular, we note that this will be the case if the system (30) is trivial.

Define \( H \) by \( H = V(G) \). Then, by (30),

\[ H' = U(G), \quad S_q(H) = T_q(G), \quad 0 \leq q \leq k - 2. \]

In particular, the equations (25) hold with \( g \) and \( h \) replaced by \( G \) and \( H \) respectively. And so, by Lemma 2.9, the functions \( f_j, H + f_j'G \) are solutions of (16) and so are linear combinations of \( w_1, \ldots, w_k \). Hence, there are solutions \( g_j \) of \( L(f) = 0 \) where \( L \) is defined by (7) such that

\[ f_j H + f_j' G - g_j h - g'_j g = 0, \quad 1 \leq j \leq k. \]

(31)

We regard the equations in (31) as a system of \( k \) equations in \( H, G, h, g, g' \), over the field \( \mathbb{F} \) of functions meromorphic in \( B \), with coefficients \( f_j, f'_j, g_j, g'_j \).

Next, we note that the rank of the coefficient matrix is \( \leq 3 \), since there is a non-trivial solution for the system. Using the method found in [4, Theorem 3], we have that the rank of the system (31) is precisely 3.

We can then solve for \(-F = h/g\) as a quotient of determinants in \( f_j, f'_j, g_j, g'_j \). Thus \( F \) is a rational function of the \( f_j \) and their derivatives. By Lemma 2.6

\[ T(r, F) = O(r^\lambda + \log r), \quad \text{as } r \to \infty, \]

and so we have conclusion (ii) of the theorem.

**Case 2.** We assume that \( c_{0,q} \equiv 0 \) for \( 0 \leq q \leq k - 1 \) in (25).

We then have that the equations (25) are satisfied when \( g \) and \( h \) are replaced by 0 and 1 respectively and thus so are the equations (24). Then, by Lemma 2.9, the \( f_j \) are solutions of (16). Thus the equations \( L(f) = 0 \) and (16) are the same, where \( L \) is defined by (7), and for \( 1 \leq q \leq k \) we may write

\[ f_j h + f'_j g = g_j, \]

(32)

in which each \( g_j \) is a solution of \( L(f) = 0 \). Then since \( f_1 \) and \( f_2 \) are linearly independent, we have \( f_1 f'_2 - f'_1 f_2 \neq 0 \) and so

\[ F = -h/g = (f'_1 g_2 - f'_2 g_1)/(f_1 g_2 - f_2 g_1), \]

which gives \( F \) as a quotient of determinants in \( f_1, f'_1, g_j, g'_j \). Then by Lemma 2.6 we have that

\[ T(r, F) = O(r^\lambda + \log r), \quad \text{as } r \to \infty, \]

and so we have conclusion (ii) of the theorem.

This completes the proof of Theorem 1.2.
References


