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Control of Sampled-data Systems with Variable Sampling Rate

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Abstract

This paper addresses stability and performance of sampled-data systems with variable sampling rate, where the change between sampling rates is decided by a scheduler. A motivational example is presented, where a stable continuous time system is controlled with two sampling rates. It is shown that the resulting system could be unstable when the sampling changes between these two rates, although each individual closed-loop system is stable under the designed controller that minimizes the same continuous loss function. Two solutions are presented in this paper. The first solution is to impose restrictions on switching sequences such that only stable sequences are chosen. The second solution presented is more general, where a piecewise constant state feedback control law is designed which guarantees stability for all possible variations of sampling rate. Furthermore, the performance defined by a continuous time quadratic cost function for the sampled-data system with variable sampling rate can be optimised using the proposed synthesis method.

Keywords: Sampled-data systems; hybrid systems; stability; performance

1 Introduction

Sampled-data systems with varying sampling rate arise for different reasons. The first reason is the optimal usage of central processing unit (CPU) resources (Eker, 1999; Cervin, 2000). In the area of embedded systems which is of broad interest, several tasks including computing control effort, management, data processing and fault diagnosis are carried out on the same CPU. When enough computational resources are available, the control law is computed more frequently than when the resources are used for other computations, management or data processing. This leads to variations in sampling rate. Secondly, it also arises in the situations where sampling rate depends on certain variables; for example, in brushless DC motor control, a few hall sensors are used to determine the position of the rotor and the speed measurement frequency is velocity dependent (Yen *et al.*, 2002). The third reason is to use sampling rate as an extra control variable; for instance, a wide range of sample interval adaption schemes for stablising a single-input-single-output (SISO) system were proposed in (Owens, 1996). Previously, variations in sampling rate were often neglected. In other cases, it was assumed that designing a piecewise continuous controller consisting of controllers which are optimal for the current sampling rate would lead to reasonable results. This paper shows that such assumptions are not justified and such a control strategy does not guarantee stability.

A motivational example is first given in this paper, where a stable continuous-time system is sampled at two different sampling rates. Two controllers are designed by minimizing the same continuous quadratic loss function and each individual controlled system is stable at a fixed sampling rate. However, it is shown that the resulting closed-loop system might be unstable when the sampling changes between these two rates. It is then pointed out that a sampled-data system with variable sampling rate is a kind of hybrid system which attracts considerable attention recently. The stability of this kind of system not only depends on the continuous control and dynamics but also the discrete dynamics (switching strategies between different sampling rates) (Branicky, 1998; Ye *et al.*, 1998; Chen and Ballance, 2002).

To avoid instability of this kind of system as in the motivational example, two solutions are suggested in this paper. The first solution shows how restrictions on switching sequences can be imposed such that only stable sequences are chosen. This can be achieved by identifying all possible unstable switching sequences. However, in engineering, not only stability but also performance are of concern. Moreover, in some cases, it is impossible to impose restrictions on the scheduling strategies. Therefore, the second solution presents an optimal controller design where the bound on the cost for all possible switching sequences is minimised. This results in a piecewise constant state feedback control law and guarantees stability regardless of switching sequences. The controller synthesis is cast into an LMI, which conveniently solves the synthesis problem. To illustrate the procedure, the introduction example is revisited using the proposed LMI synthesis method and a piecewise constant control law is given, which is stable for all switching sequences while minimising the bound of the cost.

2 A motivational example

As an example of instability for sampled-data systems with variable sampling rate, the real-time control of a linear continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

is considered, where

$$A = \begin{bmatrix} 0 & 1 \\ -10000 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{2}$$

are the system, input and output matrices. The continuous-time system is stable with poles in the left hand side of the complex plane, $p_{1,2} = -0.05 \pm 100i$.

The continuous-time system is discretized with two different zero order hold circuits, where the sampling times are $h_1 = 0.002s$ and $h_2 = 0.0312s$, respectively. The two discretizations, i.e., discrete-time systems, are represented by

$$x(k+1) = \Phi_q x(k) + \Gamma_q u(k) \tag{3}$$

$$q \in \{1, 2\}$$

where

$$\Phi_q = e^{Ah_q}, \qquad \Gamma_q = \int_0^{h_q} e^{A(h_q - s)} B ds \tag{4}$$

and q denotes the discretized system obtained with sampling time h_q. For the sake of simplicity, $x(k)$ and $u(k)$ denote the state at the kth sampling instant with sampling time either h_1 or h_2. With the data, it can be calculated that

$$\Phi_1 = \begin{bmatrix} 0.98007 & 0.0019865 \\ -19.8649 & 0.97987 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0.0000 \\ 0.0019865 \end{bmatrix}, \tag{5}$$

$$\Phi_2 = \begin{bmatrix} -0.9982 & 0.00021558 \\ -2.1558 & -0.99822 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0.0001998 \\ 0.0002125 \end{bmatrix} \tag{6}$$

Both discretisations lead to stable discrete systems with the spectral radius $\rho(\Phi_1) < 1$ and $\rho(\Phi_2) < 1$, respectively, where $\rho(\Phi_q)$ denotes the largest eigenvalue of Φ_q.

A discrete linear quadratic optimal controller is designed for both discretizations by minimizing the continuous loss function

$$J = \int_0^{\infty} (x(t)^T Q_c x(t) + u(t)^T R u(t)) dt \tag{7}$$

subject to system's dynamics (1) sampled at h_1 and h_2, respectively, where

$$Q_c = \begin{bmatrix} 20000 & 0 \\ 0 & 20000 \end{bmatrix} \quad R = 50$$

The discretized performance index at the sampling time h_q is given by (Åström and Wittenmark, 1997)

$$J = \sum_{i=1}^{\infty} x(i)^T Q_{1,q} x(i) + 2x(i)^T Q_{12,q} u(i) + u(i)^T Q_{2,q} u(i) \tag{8}$$

where

$$Q_{1,q} = \int_0^{h_q} (\Phi_s^T Q_c \Phi_s) ds \tag{9}$$

$$Q_{12,q} = \int_0^{h_q} (\Phi_s^T Q_c \Gamma_s) ds \tag{10}$$

$$Q_{2,q} = \int_0^{h_q} (\Gamma_s^T Q_c \Gamma_s + R) ds \tag{11}$$

$$\Phi_s = e^{As} \tag{12}$$

and

$$\Gamma_s = \int_0^{s} e^{A(s-s_1)} B ds_1. \tag{13}$$

Solving the discrete algebraic Riccati equation

$$P_q = \Phi_q^T P_q \Phi_q + Q_{1,q} - (\Phi_q^T P_q \Gamma_q + Q_{12,q})(\Gamma_q^T P_q \Gamma_q + Q_{2,q})^{-1}(\Gamma_q^T P_q \Phi_q + Q_{12,q}^T)$$

gives the state feedback law $u = K_q x$ where

$$K_q = -(\Gamma_q^T P_q \Gamma_q + Q_{2,q})^{-1}(\Gamma_q^T P_q \Phi_q + Q_{12,q}^T)$$

With the data, the feedback control gains for sampling rate h_1 and h_2 are given by

$$K_1 = \begin{bmatrix} 195.401 & -19.4121 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1313.1 & 10.284 \end{bmatrix}, \tag{14}$$

respectively.

For both discretizations, it can be shown that the closed-loop systems are stable. However, as shown in figure 1, when the system is sampled with h_1 once and then the system is sampled with h_2 twice repeatedly, and in each sampling rate, the corresponding optimal controller (14) is applied, it is found that the closed-loop system is unstable. Figure 1 shows 240 sampling points of the continuous trajectory of this unstable system in the phase-plane. The system (1) is sampled with h_1 once, i.e., small distance between initial and first sample, and twice with h_2, i.e., larger distance between first, second and third sample. It can be seen that the trajectory gets further away from the origin as time goes. The instability of the closed-loop system with variable sampling rate is confirmed by checking the spectral radius of the resulting system $\rho((\Phi_2 + \Gamma_2 K_2)^2(\Phi_1 + \Gamma_1 K_1)^1) > 1$, while the spectral radiuses of the closed-loop system under the optimal control at a fixed sampling rate are $\rho(\Phi_1 + \Gamma_1 K_1) < 1$ and $\rho(\Phi_2 + \Gamma_2 K_2) < 1$, respectively. The spectral radius of the resulting system is obtained by writing the solution for sampling at h_1 once as $x_{h_1} = (\Phi_1 + \Gamma_1 K_1)x_0$ and sampling at h_2 twice as $x_{2h_2+h_1} = (\Phi_2 + \Gamma_2 K_2)^2 x_{h_1}$. Substituting the former into the latter gives $x_{2h_2+h_1} = (\Phi_2+\Gamma_2 K_2)^2(\Phi_1+\Gamma_1 K_1)x_0$. Since this is done repeatedly, it can be considered as a new system with the spectral radius larger than one, which implies that the resulting system is unstable.

It turns out that this is not the only sequence between these two sampling rates which destabilises the system. Table 1 gives other sequences for which the resulting system is unstable.

Table 1: Unstable sequences

	$\rho((\Phi_2 + \Gamma_2 K_2)^{mh_2}(\Phi_1 + \Gamma_1 K_1)^{nh_1}) > 1$								
$n \cdot h_1$	$1 \cdot h_1$	$1 \cdot h_1$	$1 \cdot h_1$	$2 \cdot h_1$	$2 \cdot h_1$	$2 \cdot h_1$	$2 \cdot h_1$	$2 \cdot h_1$	$2 \cdot h_1$
$m \cdot h_2$	$1 \cdot h_2$	$3 \cdot h_2$	$4 \cdot h_2$	$2 \cdot h_2$	$3 \cdot h_2$	$4 \cdot h_2$	$5 \cdot h_2$	$6 \cdot h_2$	$7 \cdot h_2$

3 Stable scheduling strategies

It should be noticed that sampled-data system with varying sampling rate can be represented as a hybrid system. In this setting, the same continu-

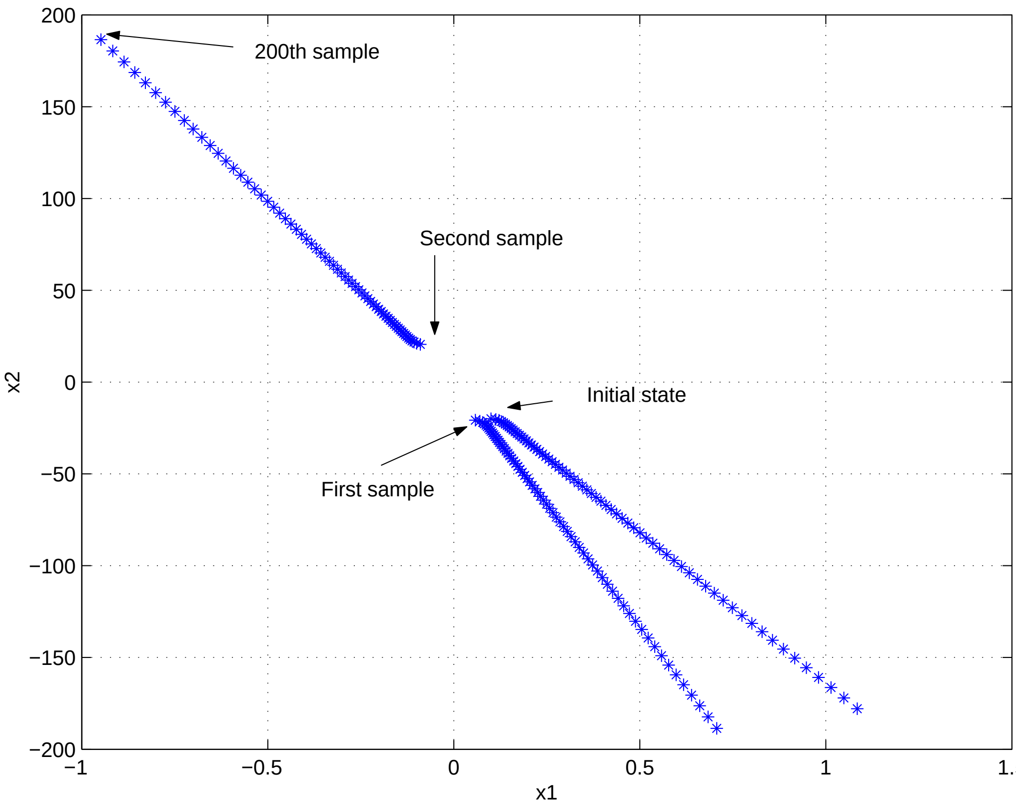

Figure 1: Unstable sequence

ous time system (1) is discretized at different sampling rates into different discrete-time systems (3). After the controllers are designed based on each discrete time model using LQR method, the closed-loop systems under the designed controllers are different although the same continuous time performance index (7) is optimized. The closed-loop system at a fixed sampling rate can be regarded as a subsystem. When the sampling rate changes, the controller is switched between the corresponding controllers for different subsystems. Therefore, the variation of the sampling rate can be considered as switching between different subsystems.

One immediately interesting question arising from this example is that when the controller is switched between two sampling rates, how many switching sequences lead to unstable scheduled systems. Theorem 1 states that the number of the possible switching sequences leading to unstable closed-loop systems, as shown in Table 1, is limited.

Theorem 1: Consider a continuous time system controlled with two sampling rates and the closed-loop system under each controller with a fixed sampling rate is exponentially stable. When the controller is switched between these two stablising controllers, depending on the corresponding sampling rate, the number of possible unstable switching sequences used repeatedly, which result in the new closed loop matrix $(\Phi_2+\Gamma_2 K_2)^i(\Phi_1+\Gamma_1 K_1)^l$, is upper bounded by $p=(m-1)\cdot(n-1)$, where m and n are sufficiently large positive integers satisfying

$$((\Phi_1+\Gamma_1 K_1)^n)^T P_1(\Phi_1+\Gamma_1 K_1)^n-\frac{1}{a}P_2<0 \tag{15}$$

$$((\Phi_2+\Gamma_2 K_2)^m)^T P_1(\Phi_2+\Gamma_2 K_2)^m-\frac{1}{a}P_2<0 \tag{16}$$

$P_1,\ P_2>0$ and $a\in\mathbf{R}^+$ is a positive scalar such that $aP_1>P_2$.

Proof: To show that a switching sequence between these two sampling rates is stable, it is sufficient to find a Lyapunov function candidate for the resultant closed-loop system. Since for a fixed sampling rate, each discrete-time closed loop system is exponentially stable, there exists a Lyapunov function for each system satisfying

$$(\Phi_q+\Gamma_q K_q)^T P_q(\Phi_q+\Gamma_q K_q)-P_q<0 \quad P_q=P_q^T>0 \tag{17}$$
$$q\in\{1,2\}$$

where $P_q > 0$. Since $P_1 > 0$ and $P_2 > 0$, there exists a scalar $a \in \mathbf{R}^+$ such that $aP_1 > P_2$. We can take a piecewise quadratic Lyapunov function $V(x)$ as

$$V(x) = \begin{cases} x^T a P_1 x & \text{at subsystem 1} \\ x^T P_2 x & \text{at subsystem 2} \end{cases} \tag{18}$$

The Lyapunov function decreases while staying at one subsystem. However, when switching from one subsystem to another one, the Lyapunov function might increase. Therefore, to guarantee the overall decrease of the Lyapunov function, the system shall stay sufficiently long either with subsystem 1 before switching to subsystem 2 or with subsystem 2 before switching to subsystem 1. Since $aP_1 > P_2$, this implies that the Lyapunov function $V(x)$ decreases when switching from subsystem 1 to subsystem 2, whereas the Lyapunov function increase when switching from subsystem 2 to 1, which causes concern. Condition (15) implies that after staying subsystem 1 for n sampling intervals, the associated Lyapunov function is less than that at the subsystem 2 before it switches to the subsystem 1; condition (16) means that after staying the subsystem 2 for m sampling intervals, the decrease of the Lyapunov function is larger than the increase of the associated Lyapunov function due to switch from subsystem 2 to 1. In both cases, the decrease of the overall piecewise Lyapunov function is ensured and hence the stability. In other words, all sequences where subsystem 1 is active for at least n cycles or subsystem 2 is active for at least m cycles are stable. Therefore, unstable sequences can only consist of the remaining $p = (m-1) \cdot (n-1)$ combinations.

□

Theorem 1 indicates that the number of switching sequences that possibly destabilise the sampled-data systems with two sampling rates are upper bounded by $p = (m-1) \cdot (n-1)$. Hence, we need to check the spectral radius of the p combinations $\rho((\Phi_2 + \Gamma_2 K_2)^i (\Phi_1 + \Gamma_1 K_1)^l) > 1$, $i \in \{1, \ldots, m-1\}$, $l \in \{1, \ldots, n-1\}$ to find all switching sequences that are unstable.

As pointed out earlier, a sampled-data systems with variable sampling rate can be considered as a kind of hybrid system. In many cases, control of a hybrid system can be implemented by not only continuous control, but also discrete dynamics (for example, switching sequences). One might choose a performance index that penalizes continuous and discrete dynamics. In particular, discrete mode changes need to be penalized to avoid Zeno executions (Johansson *et al.*, 1999*b*; Johansson *et al.*, 1999*a*).

Unfortunately, our application does not allow the choice of the discrete dynamics freely since the change of allowable computational resource needs to be taken into account. That is, the system should be able to switch from fast to slow sampling at any sampling time if necessary. However, the opposite is of course not required, i.e., the system can stay with the slow sampling rate as long as the scheduler wants, although it is desirable, for the sake of good performance, that the system shall switch back to fast sampling as soon as possible.

This fact is exploited by imposing sensible restrictions on the scheduling strategies. We proceed with computing a minimum dwelling time for slow sampling required for guaranteeing the stability of the system, i.e., the allowable time interval between switching from fast to slow sampling and switching back to fast sampling again if computational resources allow it. It will be shown that if such a scheduling strategy for sampling rate is applied, the scheduled system is stable.

It follows from the proof of Theorem 1 that if the system stay in the slow sampling for a certain time, then the closed-loop system with variable sampling rate should be always stable regardless of how long the system stays in fast sampling. Suppose that h_2 is slow sampling. Then it follows from (16) that the minimum dwelling time should be $m \cdot h_2$ where m is an integer such that condition (16) is satisfied.

This approach can be further generalised to system with several sampling rates. Suppose that the sampling periods are given by h_q, $q \in \{1, 2, \dots, N\}$. Let P_1 be associated with h_1 which is the fastest sampling time. Then, the minimum dwelling time can be calculated as follows: pick an $a \in \mathbf{R}^+$ such that $aP_1 \geq P_q$ for all $q \in \{1, 2, \dots, N\}$, and then solve iteratively for each m_q which satisfies

$$((\Phi_q + \Gamma_q K_q)^{m_q})^T P_1 (\Phi_q + \Gamma_q K_q)^{m_q} - \frac{1}{a} P_q < 0 \tag{19}$$

Hence, the minimum dwelling times for each sampling rate are given by $m_q h_q$.

Remark 1: It shall be noticed that the result in Theorem 1 are mainly for theoretic interests, i.e how many possible unstable switching sequence. Eq. (15) and (16) together with Eq. (17) form the required conditions for searching m, n, P_1, P_2 and a. For a fixed pair (m, n), after re-scaling P_2 by P_2/a, these equations can be converted into LMIs and the feasibility can be tested using existing software package (Boyd *et al.*, 1994). The right pair of positive number m, n can be found by increasing m and n iteratively and test its feasibility until it is feasible. There are more complicated methods for searching piecewise Lyapunov quadratic functions for hybrid systems like the variable sampled-data system discussed in this paper; for example see (Johansson and Rantzer, 1998). In real implementation, to avoid unstable behavior caused by variable sampling rate, as discussed above, only Eq. (16) with Eq.(17) are required for finding the minimum dwelling times m. Less computation is required in this case.

4 Controller design

When restrictions on sampling rate variations are not desirable, a controller that is stable against the variation in sampling rate has to be found. Furthermore, as in the example in Section 2, not only stability but also performance are interested in engineering. This section will develop a method to design control law for sampled-data systems with variable sampling rate, which not only stabilizes the system at all possible switching strategies but also achieves optimal performance in certain sense.

To achieve this, instead of minimizing a continuous objective function over the infinite horizon as in (7), the performance is minimised only over one sampling interval. To compensate for the remaining cost, a terminal penalty is added to the performance index. Minimizing the cost over only one sampling period is more sensible since the sampling rate may change after one sampling period anyway, i.e., after a sampling interval, a different subsystem might be chosen. Since the terminal penalty has to be at least as big as the remaining worst case cost (as will be shown later, this is due to stability requirement), we have

$$x(k)^T P x(k) \geq \min_u \int_{kh}^{kh+h_q} (x(t)^T Q_c x(t) + u(t)^T R u(t)) dt + x(k+1)^T P x(k+1) \tag{20}$$

$$\forall \quad q = \{1, 2, \ldots, N\}$$

where $x(k)$ denotes the state at the kth sampling time and kh denotes the time period from initial time to kth sampling, depending on the past sampling rate history. The solution gives an optimal, piecewise constant state feedback controller for the hybrid system, which is stable regardless of the scheduling.

The first step in solving (20) is to discretize the objective function. This is done similarly as in (Åström and Wittenmark, 1997). The discretized objective function over one sampling interval with terminal penalty is

$$x(k)^T P x(k) \geq \min_u \left(x(k)^T Q_{1,q} x(k) + 2x(k)^T Q_{12,q} u(k) + u(k)^T Q_{2,q} u(k) \right) + \\ + x(k+1)^T P x(k+1) \quad (21)$$

$$\forall \quad q \in \{1, 2, \ldots, N\}$$

where $Q_{1,q}, Q_{12,q}$ and $Q_{2,q}$ are defined in (9-11).

One of the main results in this paper is stated in Theorem 2.

Theorem 2: Consider a continuous time system (1) controlled with variable sampling period, $h_q, q \in \{1, 2, \ldots, N\}$, and the performance index is given by (7) where non-zero state is detectable. Suppose that there exists $P = P^T > 0$, K_q, $q \in \{1, 2, \ldots, N\}$ such that

$$(\Phi_q + \Gamma_q K_q)^T P (\Phi_q + \Gamma_q K_q) - P + Q_{1,q} + Q_{12,q} K_q + K_q^T Q_{12,q}^T + K_q^T Q_{2,q} K_q \leq 0 \quad (22)$$

$$\forall \quad q \in \{1, 2, \ldots, N\}$$

where $Q_{1,q}, Q_{12,q}$, $Q_{2,q}$ are defined in (9-11). When at the sampling period h_q, the control law

$$u(k) = K_q x(k) \quad (23)$$

is applied, the closed-loop system with variable sampling rate is always stable for all switching strategies among its sampling rates. Furthermore, the performance of the sampled-data system with variable sampling rate is bounded by $x_0^T P x_0$ where x_0 denotes the initial state.

Proof: At the time instant k, suppose that the sampling period h_q is adopted and the corresponding control (23) is applied where K_q satisfies condition (22). The corresponding discrete time system at the kth sampling period is given by

$$\begin{aligned} x(k+1) &= \Phi_q x(k) + \Gamma_q u(k) \\ &= (\Phi_q + \Gamma_q K_q) x(k) \end{aligned} \quad (24)$$

Choose $V(x(k)) = x(k)^T P x(k)$ as a Lyapunov candidate for the sampled-data system with variable sampling rate since $P = P^T > 0$. The difference of the Lyapunov function along the trajectory of the dynamic system is given by

$$\begin{aligned}\Delta V(x(k)) &\equiv V(x(k+1)) - V(x(k)) \\ &= x(k)^T \left((\Phi_q + \Gamma_q K_q)^T P (\Phi_q + \Gamma_q K_q) - P \right) x(k) \\ &\leq -x(k)^T \begin{bmatrix} I & K_q^T \end{bmatrix} Q_q \begin{bmatrix} I \\ K_q \end{bmatrix} x(k) \end{aligned} \tag{25}$$

with

$$Q_q = \begin{bmatrix} Q_{1,q} & Q_{12,q} \\ Q_{12,q}^T & Q_{2,q} \end{bmatrix}, \quad \forall \quad q \in Q = \{1, 2, \ldots, N\} \tag{26}$$

The last inequality in the above follows from condition (22). After substituting (9-11) into (26), Eq. (25) becomes

$$\begin{aligned}\Delta V(x(k)) \leq\ & -x(k)^T \int_0^{h_q} (\Phi_s + \Gamma_s K_q)^T Q_c (\Phi_s + \Gamma_s K_q) ds x(k) - \\ & x(k)^T \int_0^{h_q} (\Gamma_s K_q)^T R (\Gamma_s K_q) ds x(k) \end{aligned} \tag{27}$$

Eq. (27) implies that $\Delta V(x(k)) \leq 0$. Furthermore, since the non-zero state is detectable in the performance index, the first item in the left side of Eq.(27) is equal to zero only when $x \equiv 0$. This implies that $\Delta V(x) < 0$ for all non-zero state. Hence, the sampled-data system with variable sampling period h_q, $q = [1, \ldots, N]$, is stable for all possible switching strategies when the control law (23) is applied under the corresponding sampling rate.

We are now in the stage of showing that the performance defined in (7) is bounded by $x_0^T P x_0$.

It follows from (22) that

$$\begin{aligned}x(k)^T P x(k) &\geq x(k)^T (\Phi_q + \Gamma_q K_q)^T P (\Phi_q + \Gamma_q K_q) x(k) \\ &\quad + x(k)^T (Q_{1,q} + Q_{12,q} K_q + K_q^T Q_{12,q}^T + K_q^T Q_{2,q} K_q) x(k) \\ &= x(k+1)^T P x(k+1) + \\ &\quad + x(k)^T Q_{1,q} x(k) + 2x(k)^T Q_{12,q} u(k) + u(k)^T Q_{2,q} u(k) \end{aligned} \tag{28}$$

By repeating the above process from $k = 0$ to ∞, one has

$$\begin{aligned}x_0^T P x_0 &\geq x(\infty)^T P x(\infty) + \sum_{k=0}^{\infty} x(k)^T Q_{1,q} x(k) + 2x(k)^T Q_{12,q} u(k) + u(k)^T Q_{2,q} u(k) \\ &= x(\infty)^T P x(\infty) + \int_0^{\infty} x(t)^T Q_c x(t) + u(t)^T R u(t) dt \end{aligned} \tag{29}$$

It should be noticed that $Q_{1,q}, Q_{12,q},\ Q_{2,q}$ in the above equation are not constant matrices, which varies with the sampling rate employed for each sampling instant. Since the closed-loop system with variable sampling rate is stable, $x(\infty)$ approaches zero. Hence under all possible switching among the different sampling rates, following Eq. (29), one has

$$J = \int_0^\infty x(t)^T Q_c x(t) + u(t)^T R u(t) dt \leq x_0^T P x_0 \tag{30}$$

which implies that the performance of the sampled-data system with variable sampling rate is bounded by $x_0^T P x_0$.

$\square$

Remark 2: Theorem 2 gives the upper bound for the performance of a sampled-data system switching between different sampling rate and establishes its stability. This result is obtained based on Eq. (20), i.e trying to optimise the performance in one step ahead with certain terminal performance. At the first glance, it seems it is a bd idea to do one step ahead optimisation. However as well known in model predictive control literature (Bitmead *et al.*, 1990), for a linear system, when the terminal term is properly chosen and there are no constraints on control and state, the same optimal performance as in LQR can be achieved for this scheme. Actually, when there is no switch, i.e. $q = 1$, the solution presented in Theorem 2, i.e. Eq. (22), reduces to the fake Riccati algebraic equation associated with LQR with the performance cost $x_0^T P x_0$ (Boyd *et al.*, 1994; Bitmead *et al.*, 1990). By minimising the cost function (e.g. the trace of the matrix P) as in the next section, the optimal LQR is resulted. In other words, the proposed controller reduces to the optimal LQR controller when the sampling rate does not change.

5 Controller synthesis using LMI's

Theorem 2 points out that if a piecewise state feedback controller satisfying (22) is found, it can be guaranteed that the controlled closed loop system is stable for all variations among h_q, $q \in \{1, 2, \ldots, N\}$ and then the performance is bounded by P. This section will develop a procedure to find the feedback gain, K_q, $q \in \{1, 2, \ldots, N\}$, and the corresponding performance bound.

Condition (22) can be re-written as

$$\begin{bmatrix} \Phi_q + \Gamma_q K_q \\ I \\ K_q \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 \\ 0 & Q_{1,q} & Q_{12,q} \\ 0 & Q_{12,q}^T & Q_{2,q} \end{bmatrix} \begin{bmatrix} \Phi_q + \Gamma_q K_q \\ I \\ K_q \end{bmatrix} - P \quad \leq \quad 0 \qquad (31)$$

$$\forall \quad q \in \{1, 2, \dots, N\}$$

Applying Schur's complement to the above expression, one obtains

$$\begin{bmatrix} P & (\Phi_q + \Gamma_q K_q)^T & \begin{bmatrix} I & K_q^T \end{bmatrix} \\ (\Phi_q + \Gamma_q K_q) & P^{-1} & 0 \\ \begin{bmatrix} I \\ K_q \end{bmatrix} & 0 & Q_q^{-1} \end{bmatrix} \geq 0$$

$$\forall \quad q \in \{1, 2, \dots, N\}$$

where

$$Q_q = \begin{bmatrix} Q_{1,q} & Q_{12,q} \\ Q_{12,q}^T & Q_{2,q} \end{bmatrix}$$

Multiplying the above inequality from left and right with

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and setting $W_0 = P^{-1}$, $W_q = K_q P^{-1}$, we obtain the controller synthesis Linear Matrix Inequalities (LMI's)

$$\begin{bmatrix} W_0 & (\Phi_q W_0 + \Gamma_q W_q)^T & \begin{bmatrix} W_0 & W_q^T \end{bmatrix} \\ \Phi_q W_0 + \Gamma_q W_q & W_0 & 0 \\ \begin{bmatrix} W_0 \\ W_q \end{bmatrix} & 0 & Q_q^{-1} \end{bmatrix} \geq 0 \qquad (32)$$

$$\forall \quad q \in \{1, 2, \dots, N\}$$

in $W_0 = W_0^T > 0$ and W_q. The solution of the LMI's (32) gives the state feedback gains $K_q = W_q W_0^{-1}$ $\forall \quad q \in \{1, 2, \dots, N\}$. Applying the state feedbacks gives a stable closed loop system when the sampling period varies among h_q, $\forall \quad q \in \{1, 2, \dots, N\}$.

However, in addition to stabilizing the system, we also intend to minimize the cost for driving the states to the origin in terms of the given objective function (7). Since according to Theorem 2, the performance under the variable sampling rate is bounded by $x_0^T P x_0$. Therefore, we would like to minimize the trace of $P = W_0^{-1}$. Unfortunately, this is a non-convex optimization problem. Instead of minimizing Trace(W_0^{-1}),

$$\log\det W_0^{-1} \tag{33}$$

is minimized subject to (32) (Boyd *et al.*, 1994). It can be shown that this is a convex optimization problem (Boyd *et al.*, 1994).

Remark 3: The performance cost is proved to be bounded by $x_0^T P x_0$ under all possible switching sequences and the procedure based on the LMIs is presented to find a set of gains to minimise the bound of the cost function. This common matrix P is required to satisfy a set of LMIs and sometime it might be conservative. This is a currently widely used method for many areas such as robust control (see **?**). Since the system considered in this paper can randomly switch from one sampling rate to another sampling rate due to available computing resources (similar to time-varying uncertainties), the result is not very conservative (also see discussed in Remark 2).

6 Illustrative example revisited

This section revisits the illustrative example in Section 2 using the control synthesis procedure developed in Section 4 and 5.

The same sampling periods are considered, i.e., $h_1 = 0.002s$, $h_2 = 0.0312s$. Under these sampling rates, the discrete-time system are given by (3) with (5) and (6). With the same weighting matrices as in the introductory example, after calculating $Q_{1,q}$, $Q_{12,q}$ and $Q_{2,q}$ by (9)-(11), the matrix

$$Q_q = \begin{bmatrix} Q_{1,q} & Q_{12,q} \\ Q_{12,q}^T & Q_{2,q} \end{bmatrix}$$
$$\forall \quad q \in \{1,2\}$$

are given by

$$Q_1 = \begin{bmatrix} 5329.5 & -394.6 & -0.529 \\ -394.6 & 39.5 & 0.0395 \\ -0.529 & 0.0395 & 0.1001 \end{bmatrix} \tag{34}$$

and

$$Q_2 = \begin{bmatrix} 3137000 & -4.6437 & -313.70 \\ -4.6437 & 309.39 & 0.000864 \\ -313.70 & 0.000864 & 1.5914 \end{bmatrix} \tag{35}$$

Solve the minimisation problem (33) subject to (32) obtains $W_0 = W_0^T > 0$ and W_1, W_2. Then the state feedback gains are calculated by $K_q = W_q W_0^{-1}$, $\forall q \in \{1, 2\}$, given by

$$K_1 = \begin{bmatrix} 14847.1 & -12.419 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 969.386 & 9.3209 \end{bmatrix}$$

Applying these state feedback gains guarantees stability and robustness against all possible variations of the sampling periods between h_1 and h_2. Furthermore, the cost under all possible variations of the sampling rate is bounded by

$$P = W_0^{-1} = \begin{bmatrix} 13959032 & 134798.2 \\ 134798.2 & 14043.47 \end{bmatrix} \tag{36}$$

The time response of the closed-loop system with the same sampling sequence as in Fig.1 under the proposed control scheme is shown in Fig. 2, which clearly indicates that the closed-loop system is stable with satisfactory performance.

7 Conclusion

This paper concerns optimal control of sampled-data systems with variable sampling rate. This kind of problem arises from several situations including real-time digital control where computing resources are used for different tasks. It was shown that sampling a continuous-time system at different sampling rates results in different discrete-time systems and changing the sampling rate might destroy the stability of the system. This was highlighted by an example where controllers was designed by minimizing the same continuous-time loss function for an open-loop stable system at two sampling rates. This leads to two stable closed-loop systems, however, it was shown that the closed-loop system might be unstable when the sampling changes between these two rates.

Two approaches are adopted to overcome this problem. It was shown that both of them can guarantee stability of the sampled-data system with variable

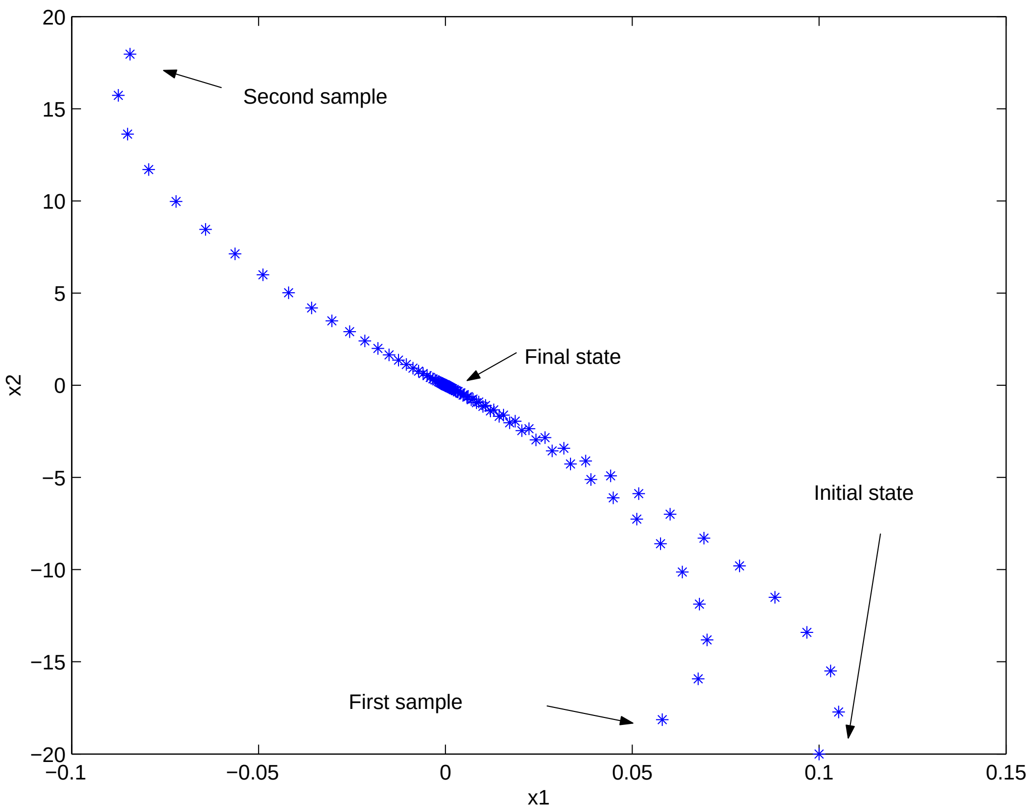

Figure 2: Time response under the developed control scheme with variable sampling rate

sampling rate, in particular, the second approach also minimises the bound of the cost of the system under all possible switching sequences. The first approach shows that restrictions on switching (scheduling) strategies can be imposed so as to guarantee stability. For cases where such restrictions cannot be imposed, a different controller design was proposed. It was suggested that the objective function had to be minimized only over one sampling period instead of minimizing over the infinite horizon. It was shown that when a proper chosen terminal penalty was added, which should be greater than or equal to the remaining cost for the worst case variations in sampling rate, the system is always stable under all possible variations of sampling rates. The results developed in this paper are quite useful for embedded systems and real-time digital control of continuous-time systems.

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References

Åström, K.J. and B. Wittenmark (1997). *Computer-Controlled Systems.* Third Edition. Prentice Hall. New Jersey, NJ.

Bitmead, R. R., M. Gevers and V. Wertz (1990). *Adaptive Optimal Control: The Thinking Man's GPC.* Prentice-Hall. New York.

Boyd, S., L. El Ghaoui, E. Ferson and V. Balakrishnan (1994). *Linear Matrix Inequalities in Systems and Control Theory.* AIAM. Philadelphia, PA.

Branicky, M.S. (1998). Multiple lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. on Automatic Control* **43**(4), 475–482.

Cervin, A. (2000). Towards the integration of control and real-time scheduling design. Technical Report TFRT-3226. Box 118, S-221 00 Lund, Sweden. Licentiate thesis.

Chen, W.-H. and D.J. Ballance (2002). On a switching control scheme for nonlinear systems with ill-defined relative degree. *Systems and Control Letters* **47**(2), 159–166.

Eker, J. (1999). Flexible Embedded Control Systems. PhD thesis. Dept. of Automatic Control, Lund Institute of Technology. Box 118, S-221 00 Lund, Sweden.

Johansson, K.H., J. Lygeros, S. Sastry and M. Egerstedt (1999*a*). Simulation of zeno hybrid automata. In: *Proceedings of Conference on Decision and Control*. Phoenix.

Johansson, K.H., M. Egerstedt, J. Lygeros and S. Sastry (1999*b*). On the regularization of zeno hybrid automata. *System & Control Letters* **38**, 141–150.

Johansson, M. and A. Rantzer (1998). Computation of piecewise quadratic lyapunov functions for hybrid systems. *IEEE Transactions on Automatic Control* **43**(4), 555–559.

Owens, D.H. (1996). Adaptive stabilization using a variable sampling rate. *Internatinal Journal of Control* **63**(1), 107–117.

Schinkel, M., W. H. Chen and A. Rantzer (2002). Optimal control for systems with varying sampling rate. In: *Proceedings of American Control Conference*. Anchorage.

Wu, F. (2003). Robust quadratic performance for time-delayed uncertain linear systems. *IInternational Journal of Robust and Nonlinear Control* **13**(2), 153–172.

Ye, H., A. N. Michel and L. Hou (1998). Stability theory for hybrid dynamical systems. *IEEE Trans. on Automatic Control* **43**(5), 461–474.

Yen, J.Y., Y.L. Chen and M. Tomizuka (2002). Variable sampling rate controller design for brushless dc motor. In: *Proceedings of 41st IEEE Conference on Decision and Control*. Las Vegas, USA.

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