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Three-dimensional travelling gravity-capillary water waves

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1 Introduction

1.1 The hydrodynamic problem

The classical water-wave problem concerns the irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. The fluid motion is described by the Euler equations in a domain bounded below by a rigid horizontal bottom \( y = -h \) and above by a free surface which is described as a graph \( y = \eta(x, z, t) \), where the function \( \eta \) depends upon the two horizontal spatial directions \( x, z \) and time \( t \). In terms of an Eulerian velocity potential \( \phi(x, y, z, t) \) the mathematical problem is to solve the equations

\[
\begin{align*}
\phi_{xx} + \phi_{yy} + \phi_{zz} &= 0, & -h < y < \eta, \\
\phi_y &= 0, & \text{on } y = -h, \\
\phi_y &= \eta_t + \eta_x \phi_x + \eta_z \phi_z, & \text{on } y = \eta
\end{align*}
\]

and

\[
\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta
- T \left[ \frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_x
- T \left[ \frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}} \right]_z = 0 \quad \text{on } y = \eta,
\]

where \( g \) and \( T \) are respectively the acceleration due to gravity and the coefficient of surface tension. The above formulation describes three-dimensional gravity-capillary waves on water of finite depth, but several variations upon this theme are possible. Solutions which do not depend upon the spatial coordinate \( z \) are called two-dimensional water waves, solutions with \( T = 0 \) are called gravity waves and the limiting case \( h \to \infty \) is the infinite-depth problem. Travelling waves are water waves of the special form \( \eta(x, z, t) = \eta(x - c_1 t, z - c_2 t) \),

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$\phi(x, y, t) = \phi(x - c_1 t, y, z - c_2 t)$; in other words they are uniformly translating in the horizontal direction with velocity $c = (c_1, c_2)$. In this paper we survey some recent mathematical results concerning three-dimensional travelling water waves, and in keeping with convention we continue to write $x$ and $z$ as abbreviations for $x - c_1 t, z - c_2 t$.

The travelling water-wave problem is one of the classical problems in applied mathematics, and there is a huge and growing literature in this area. However virtually all the published results on three-dimensional travelling water waves concentrate on numerical studies or approximations by simpler model equations, and there has been far less rigorous mathematical study of the complete hydrodynamic problem formulated above (a comprehensive survey of the existing literature on model equations is given in the review papers by Aklyas [1] and Dias & Kharif [8]). Functional-analytic techniques for partial differential equations are required to make progress with the exact equations, and in this paper we survey the currently available mathematical results.

1.2 A gallery of three-dimensional travelling water waves

The three-dimensional travelling waves for which rigorous mathematical results are available fall into two broad categories.

(i) **Doubly periodic travelling waves** are periodic in each of two distinct horizontal directions (see Figure 1). There are existence theories for doubly periodic travelling waves with arbitrary fundamental domain.

![Fig. 1](image) A doubly periodic travelling wave exhibiting a typical ‘hexagonal’ pattern; the arrow shows the direction of wave propagation.

(ii) Travelling water waves which have a pulse-like profile in a distinguished horizontal direction are called **solitary waves**. There are three types of **classical solitary waves** which have their pulse-like profile in the direction of propagation (Figure 2). A **line solitary wave** is spatially homogeneous in the direction transverse to its direction of propagation, while a **periodically modulated solitary wave** is periodic in the transverse direction. A **fully localised solitary wave** on the other hand decays to zero in all spatial directions. There also exist solitary waves whose pulse-like profile lies in a direction different to that of their direction of propagation; Figure 3 shows two examples of **oblique periodically modulated solitary waves** of this kind. The figure also illustrates another possible
feature of a solitary wave, namely that its pulse-like profile may be made up of multiple individual pulses; waves of this type are called *multi-pulse solitary waves*.

**Fig. 2** Clockwise from top left: line, periodically modulated and fully localised solitary waves.

**Fig. 3** These solitary waves have a one-pulse (left) and two-pulse (right) profile in one horizontal direction and are periodic in another.

In this paper we outline the existence theories for the water waves sketched in Figures 1–3, all of which are variational in nature and all of which require that $T > 0$ (so that gravity waves are excluded).
1.3 A variational principle

The key to all currently available existence theories for three-dimensional travelling water waves is the observation that the hydrodynamic problem follows from the variational principle

\[ \delta \iint \left( \int_{-h}^{h} \left( -c_1 \phi_x - c_2 \phi_z + \frac{1}{2} (\phi_x^2 + \phi_y^2 + \phi_z^2) \right) \, dy + \frac{1}{2} g \eta^2 + T \left( \sqrt{1 + \eta_x^2 + \eta_z^2} - 1 \right) \right) \, dx \, dz = 0, \]

in which the variation is taken in \((\eta, \phi)\) (Luke [27]). This variational principle can be exploited in two ways.

(i) It may be possible to use the direct methods of the calculus of variations to find critical points of the variational functional and hence solutions of the problem. This approach is used in the existence theories for doubly periodic waves by Craig & Nicholls [5] (see Section 4.1) and fully localised solitary waves by Groves & Sun [16] (see Section 4.2).

(ii) The phrase spatial dynamics refers to an approach where a system of partial differential equations governing a physical problem is formulated as a (typically ill-posed) evolutionary equation

\[ u_\xi = L(u) + N(u), \]

in which an unbounded spatial coordinate plays the role of the time-like variable \(\xi\). The method, which was introduced by Kirchgässner [23], has become the basis for a wide range of problems in the applied sciences and has proved particularly fruitful in the study of local bifurcation phenomena for two-dimensional travelling water waves (see the review article by Dias & Iooss [7]).

Groves & Haragus [12] have recently applied spatial-dynamics methods to three-dimensional travelling water waves. In their approach the functional in the above variational principle is regarded as an action functional in which an arbitrary horizontal spatial direction \(\xi\) (which is \textit{a priori} unbounded) plays the role of the time-like variable; classical Legendre-transform theory is then used to re-formulate the associated Euler-Lagrange equations as a Hamiltonian system, solutions of which can be found by a variety of techniques from the theory of evolutionary equations. This method is explained in detail in Sections 2 and 3 below; it leads to existence theories for doubly periodic and (multi-pulse) periodically modulated solitary waves.

2 Spatial dynamics

In this section we describe the theory of spatial dynamics for three-dimensional travelling water waves which was introduced by Groves & Mielke [14] and developed by Groves & Haragus [12]. Here we suppose without loss of generality that \(c = (c, 0)\), so that \(x\) is the direction of wave propagation and \(z\) is the transverse spatial direction.

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2.1 Formulation of the evolutionary equation

Let us begin by introducing an oblique horizontal coordinate system \((\xi, Z)\), where \(\xi\) and \(Z\) are the horizontal directions making angles of respectively \(\theta_1\) and \(\theta_2\) with the positive \(x\) axis (see Figure 4); we consider waves which are periodic with fixed frequency \(h\nu\) in the \(Z\) direction. In the new three-dimensional coordinate system \((\xi, y, Z)\), the coordinates \(y\) and \(Z\) are bounded (since the water is finitely deep and the waves are periodic in \(Z\)), while no restrictions are placed upon \(\xi\), which may therefore be regarded as unbounded and hence 'time-like'.

A formulation of the hydrodynamic problem as a Hamiltonian evolutionary system in which \(\xi\) is the ‘time-like’ variable is found by exploiting the variational principle (5). We proceed by seeking solutions of the form

\[
\eta(x, z) = \eta(\xi, Z), \quad \phi(z, y, z) = \phi(\xi, y, Z),
\]

where

\[
\xi = \sin \theta_2 x - \cos \theta_2 z, \quad Z = \sin \theta_1 x - \cos \theta_1 z
\]

and \(\eta, \phi\) are \(2h\pi/\nu\)-periodic in \(Z\), and using the change of variable

\[
y = Y \left( h + \eta(\xi, Z) \right) - h, \quad \phi(\xi, y, Z) = \Phi(\xi, Y, Z),
\]

which transforms the variable fluid domain \((-h < y < \eta(x, z))\) into the fixed domain \(\{0 < Y < h\}\); at this stage it is also convenient to introduce dimensionless variables

\[
(\xi', Y', Z') = \frac{1}{h} (\xi, Y, Z), \quad \eta'(\xi', Z') = \frac{1}{h} \eta(\xi, Z), \quad \Phi'(\xi', Y', Z') = \frac{1}{ch} \Phi(\xi, Y, Z).
\]

In this fashion one obtains the new variational principle

\[
\delta \mathcal{L} = 0, \quad \mathcal{L} = \int_{-\infty}^{\infty} \left( \int_0^{2\pi/\nu} L(\eta, \Phi, \eta_\xi, \Phi_\xi) \, dZ \right) \, d\xi,
\]

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in which

\[
L(\eta, \Phi, \eta_\xi, \Phi_\xi) = 
\int_0^{2\pi/\nu} \int_0^1 \left\{ -\sin \theta_2 \left( \Phi_\xi - \frac{Y \eta_\xi}{1 + \eta} \Phi_Y \right) - \sin \theta_1 \left( \Phi_Z - \frac{Y \eta_Z}{1 + \eta} \Phi_Y \right) + \frac{1}{2} \left( \Phi_\xi - \frac{Y \eta_\xi}{1 + \eta} \Phi_Y \right)^2 + \frac{\Phi_Y^2}{2(1 + \eta)^2} + \frac{1}{2} \left( \Phi_Z - \frac{Y \eta_Z}{1 + \eta} \Phi_Y \right)^2 \right. \\
+ \cos(\theta_1 - \theta_2) \left( \Phi_\xi - \frac{Y \eta_\xi}{1 + \eta} \Phi_Y \left( \Phi_Z - \frac{Y \eta_Z}{1 + \eta} \Phi_Y \right) \right) \left( 1 + \eta \right) dY dZ \\
+ \int_0^{2\pi/\nu} \left\{ \frac{1}{2} \alpha \eta^2 + \beta \left( 1 + \eta_\xi^2 + \eta_Z^2 + 2 \cos(\theta_1 - \theta_2) \eta_\xi \eta_Z \right)^{1/2} \right\} dZ,
\]

\( \alpha = gh/c^2, \beta = T/\hbar c^2 \) and the primes have been dropped for notational simplicity.

Observe that the variational principle (7) takes the form of Hamilton’s principle for an action functional in which \( \xi \) is the time-like variable, \( (\eta, \Phi) \) are the coordinates and \( (\eta_\xi, \Phi_\xi) \) the corresponding velocities. The next step is to carry out the Legendre transform according to the classical theory. We introduce new variables \( \omega \) and \( \zeta \) by the formulae

\[
\omega = \frac{\delta L}{\delta \eta_\xi}, \quad \zeta = \frac{\delta L}{\delta \Phi_\xi},
\]

in which the variational derivatives are taken in respectively \( L^2(0, 2\pi/\nu) \) and \( L^2(\Sigma) \), where \( \Sigma = (0, 1) \times (0, 2\pi/\nu) \), and define the Hamiltonian function by

\[
H(\eta, \omega, \Phi, \zeta) = \int_0^{2\pi/\nu} \int_0^1 \eta_\xi \Phi_\xi dY dZ + \int_0^{2\pi/\nu} \omega_{\eta_\xi} d\zeta - L(\eta, \Phi, \eta_\xi, \Phi_\xi);
\]

Hamilton’s equations are

\[
\eta_\xi = \frac{\delta H}{\delta \omega}, \quad \omega_\xi = -\frac{\delta H}{\delta \eta}, \quad \Phi_\xi = \frac{\delta H}{\delta \zeta}, \quad \zeta_\xi = -\frac{\delta H}{\delta \Phi}.
\]

Writing down these equations, one finds that \( (\eta, \omega, \Phi, \zeta) = (0, 0, 0, -\sin \theta_2) \) is always a solution (the state of rest in the hydrodynamic problem); we therefore introduce the new variable \( \Psi = \zeta + \sin \theta_2 \) and obtain the Hamiltonian

\[
H(\eta, \omega, \Phi, \Psi) = 
\int_0^{2\pi/\nu} \int_0^1 \left\{ \sin \theta_2 \Psi + \sin \theta_1 (1 + \eta) \left( \Phi_Z - \frac{Y \eta_Z}{1 + \eta} \Phi_Y \right) + \frac{(\Psi - \sin \theta_2)^2}{2(1 + \eta)} - \frac{\Phi_Y^2}{2(1 + \eta)} \right. \\
\left. - \frac{\Phi_Z^2}{2(1 + \eta)} \right\} d\Phi dY dZ \\
+ \int_0^{2\pi/\nu} \left\{ -\frac{1}{2} \alpha \eta^2 - \cos(\theta_1 - \theta_2) \eta_\xi \sin \theta_2 \right\} \left( \Phi_Z - \frac{Y \eta_Z}{1 + \eta} \Phi_Y \right) \left( 1 + \eta_\xi \right) \left( 1 + \eta_\xi \right) \right\} dZ.
\]

(8)
in which

$$W = \omega + \frac{1}{1 + \eta} \int_{0}^{1} Y \Phi_Y (\Psi - \sin \theta_2) \, dY. \tag{9}$$

Hamilton’s equations are given explicitly by

\[
\begin{align*}
\eta_\xi &= W \left( \frac{1 + \sin^2(\theta_1 - \theta_2)\eta_Z^2}{\beta^2 - W^2} \right)^{1/2} - \cos(\theta_1 - \theta_2) \eta_Z, \\
\omega_\xi &= \frac{W}{(1 + \eta)^2} \left( \frac{1 + \sin^2(\theta_1 - \theta_2)\eta_Z^2}{\beta^2 - W^2} \right)^{1/2} \int_{0}^{1} Y \Phi_Y (\Psi - \sin \theta_2) \, dY \\
&+ \alpha \eta - \sin^2(\theta_1 - \theta_2) \left[ \eta_Z \left( \frac{\beta^2 - W^2}{1 + \sin^2(\theta_1 - \theta_2)\eta_Z^2} \right)^{1/2} \right]_Z \\
&- \frac{1}{2} \sin^2 \theta_2 + (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1) \Phi_Z |_{Y=1} - \cos(\theta_1 - \theta_2) \omega_Z \\
&+ \int_{0}^{1} \left\{ \frac{(\Psi - \sin \theta_2)^2}{(1 + \eta)^2} - \frac{\Phi_Y}{2(1 + \eta)} \right. \\
&\left. + \frac{1}{2} \sin^2(\theta_1 - \theta_2) \left( \Phi_Z - \frac{Y \eta_Z \Phi_Y}{1 + \eta} \right) \right\} \Phi_Y (\Psi - \sin \theta_2) \, dY, \tag{10}
\end{align*}
\]

\[
\begin{align*}
\Phi_\xi &= \frac{\Psi - \sin \theta_2 + Y \Phi_Y W}{1 + \eta} \left( \frac{1 + \sin^2(\theta_1 - \theta_2)\eta_Z^2}{\beta^2 - W^2} \right)^{1/2} \\
&+ \sin \theta_2 - \cos(\theta_1 - \theta_2) \Phi_Z, \\
\Psi_\xi &= -\frac{\Phi_Y}{1 + \eta} - \cos(\theta_1 - \theta_2) \Psi_Z \\
&+ ((Y \Psi)_Y - \sin \theta_2) \frac{W}{1 + \eta} \left( \frac{1 + \sin^2(\theta_1 - \theta_2)\eta_Z^2}{\beta^2 - W^2} \right)^{1/2} \\
&+ \sin^2(\theta_1 - \theta_2) \left( Y \eta_Z \left( \Phi_Z - \frac{Y \eta_Z \Phi_Y}{1 + \eta} \right) \right)_Y \\
&- \sin^2(\theta_1 - \theta_2) \left( (1 + \eta) \left( \Phi_Z - \frac{Y \eta_Z \Phi_Y}{1 + \eta} \right) \right)_Z \tag{11}
\end{align*}
\]

together with the boundary conditions

\[
\Phi_Y = 0 \quad \text{on} \quad Y = 0, \quad \tag{14}
\]

\[
\frac{-\Phi_Y}{1 + \eta} + (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1) \eta_Z + \sin^2(\theta_1 - \theta_2) \left( \Phi_Z - \frac{\eta_Z \Phi_Y}{1 + \eta} \right) \eta_Z \Phi_Y \\
+ (\Psi - \sin \theta_2) \frac{W}{1 + \eta} \left( \frac{1 + \sin^2(\theta_1 - \theta_2)\eta_Z^2}{\beta^2 - W^2} \right)^{1/2} = 0 \quad \text{on} \quad Y = 1, \tag{15}
\]

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the second of which is nonlinear. (An integration by parts with respect to the vertical coordinate \(Y\) is required to compute Hamilton’s equations, and boundary conditions at \(Y = 0\) and \(Y = 1\) emerge during this procedure.)

Equations (10)–(13) constitute a quasilinear evolutionary system

\[ u_\xi = Lu + N(u) \tag{16} \]

in the infinite-dimensional phase space

\[ \mathcal{X}_s = H^s_{\text{per}}(0, 2\pi/\nu) \times H^s_{\text{per}}(0, 2\pi/\nu) \times H^s_{\text{per}}(\Sigma) \times H^s_{\text{per}}(\Sigma), \quad s \in (0, 1/2), \]

where

\[ H^s_{\text{per}}(0, 2\pi/\nu) = \{ u \in H^s_{\text{loc}}(\mathbb{R}) : u(Z + 2\pi/\nu) = u(Z), \ Z \in \mathbb{R} \}, \]

\[ H^s_{\text{per}}(\Sigma) = \{ u \in H^s_{\text{loc}}((0, 1) \times \mathbb{R}) : u(Y, Z + 2\pi/\nu) = u(Y, Z), \ Y \in (0, 1), \ Z \in \mathbb{R} \}; \]

the domain \(D_s\) of the vector field on the right-hand side of (16) is the subset of \(X_{s+1}\) defined by the boundary conditions (14), (15). Furthermore, equation (16) represents Hamilton’s equations for the Hamiltonian system \((M, \Omega, H)\), where \(M = \mathcal{X}_s, \ \Omega\) is the canonical 2-form with respect to the \(L^2(0, 2\pi/\nu) \times L^2(0, 2\pi/\nu) \times L^2(\Sigma) \times L^2(\Sigma)\) inner product and \(H\) is given by the formula (8). A detailed explanation of these facts is presented by Groves & Haragus [12, §2.2].

Observe that equation (16) (or equivalently the system (10)–(13)) is reversible, that is the Hamiltonian vector field anticommutes with the reverser \(R : \mathcal{X}_s \to \mathcal{X}_s\) defined by

\[ R(\eta(Z), \omega(Y, Z), \Phi(Y, Z), \Psi(Y, Z)) = (\eta(-Z), -\omega(-Z), -\Phi(Y, -Z), \Psi(Y, -Z)) \]

and \(D_s\) is invariant under the action of \(R\). There are additional discrete symmetries in the special cases \((\theta_1, \theta_2) = (0, \pm \pi/2)\) and \((\theta_1, \theta_2) = (\pm \pi/2, 0)\). In the former case equation (16) is invariant under the reflection \(S_1 : \mathcal{X}_s \to \mathcal{X}_s\) given by

\[ S_1(\eta(Z), \omega(Y, Z), \Phi(Y, Z), \Psi(Y, Z)) = (\eta(-Z), \omega(-Z), \Phi(Y, -Z), \Psi(Y, -Z)), \]

while in the latter case it is invariant under the reflection \(S_2 : \mathcal{X}_s \to \mathcal{X}_s\) given by

\[ S_2(\eta(Z), \omega(Y, Z), \Phi(Y, Z), \Psi(Y, Z)) = (\eta(-Z), \omega(-Z), -\Phi(Y, -Z), -\Psi(Y, -Z)). \]

Before proceeding to the task of finding solutions to the evolutionary equation (16), it is helpful to consider the issue of interpreting its solutions as water waves. Figure 5 shows several examples of orbits in its (infinite-dimensional) phase space; the key to understanding the qualitative nature of the corresponding water waves is the observation that each point in the phase space corresponds to a wave which is periodic in the spatial direction \(Z\), while the behaviour of a solution in ‘time’ determines the profile of the wave in the spatial direction \(\xi\). According to this recipe, periodic solutions of the reduced system correspond to water waves which are periodic in the \(\xi\) direction. Since these waves are also periodic in the \(Z\) direction we obtain doubly periodic water waves (Figure 5(a)). Similarly, homoclinic solutions of the reduced system correspond to water waves whose profile in the \(\xi\) direction decays to zero (that is, the undisturbed state of the water) as \(\xi \to \pm \infty\); such waves are periodic in the \(Z\) direction and have a solitary-wave profile in the \(\xi\) direction (Figure 5(b)). Solutions which are homoclinic to periodic solutions (Figure 5(c)) likewise correspond to water waves which are periodic in the \(Z\) direction and whose pulse-like profile in the \(\xi\) direction decays to a periodic oscillation as \(\xi \to \pm \infty\); these waves are termed generalised solitary waves.

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(a) A periodic solution of the evolutionary equation (above left) generates a wave which is periodic in the $\xi$ direction (below left); the free surface is also periodic in the $Z$ direction (right).

(b) A homoclinic solution of the evolutionary equation (above left) generates a wave with a solitary-wave profile in the $\xi$ direction (below left); the free surface is also periodic in the $Z$ direction (right).

(c) A solution to the evolutionary equation which is homoclinic to a periodic orbit (above left) generates a wave with a generalised solitary-wave profile in the $\xi$ direction (below left); the free surface is also periodic in the $Z$ direction (right).

**Fig. 5** Interpreting solutions of the evolutionary equation.
2.2 Water waves as a paradigm for finite-dimensional Hamiltonian systems

The hydrodynamic problem depends upon the angles $\theta_1$ and $\theta_2$, the frequency $\nu$ in the $Z$ direction and the dimensionless physical parameters $\alpha = gh/c^2$ and $\beta = T/hc^2$, where $g$, $h$, $c$ and $T$ denote respectively the acceleration due to gravity, the depth of the fluid in its undisturbed state, the speed of the travelling wave and the coefficient of surface tension. We now seek bifurcation phenomena by fixing a value $(\alpha_0, \beta_0, \theta_{10}, \theta_{20}, \nu_0)$ of $(\alpha, \beta, \theta_1, \theta_2, \nu)$ and perturbing around it with a bifurcation parameter $\lambda \in \mathbb{R}^5$. The first step is to compute the spectrum of the linear operator $L$ in the evolutionary equation (16) at $\lambda = 0$; we henceforth denote this operator by $L_0$.

Lemma 2.1 The spectrum $\sigma(L_0)$ of $L_0$ consists entirely of isolated eigenvalues. A complex number $\sigma$ is an eigenvalue of $L_0$ with corresponding eigenvectors in the $n$th Fourier mode if and only if

$$ (\sigma \sin \theta_{20} + i \nu_0 \sin \theta_{10})^2 \cos \gamma = (\alpha_0 - \beta_0 \gamma^2) \gamma \sin \gamma, $$

where $\gamma^2 = \sigma^2 - n^2 \nu_0^2 + 2 \cos(\theta_{10} - \theta_{20})i \nu_0 \sigma$. In particular, one finds that

(i) $\sigma(L_0) \cap \mathbb{R}$ is a finite set;

(ii) the set $\sigma(L_0)$ is symmetric with respect to the real and imaginary axes, that is $\sigma(L_0) = -\sigma(L_0)$;

(iii) the set $\sigma(L_0) \setminus \mathbb{R}$ is always bounded away from the imaginary axis in two wedge-shaped regions which qualitatively resemble those shown in Figure 6.

![Fig. 6](image)

Fig. 6 The imaginary spectrum of the linear operator $L_0$ consists of a finite number of eigenvalues, while the rest of the spectrum is contained in wedge-like regions (shaded) bounded away from the imaginary axis.

A imaginary eigenvalue $i\kappa$ of $L_0$ with corresponding eigenvector in the $n$th Fourier mode corresponds to a linear travelling water wave of the form $\eta(x, z) = \eta_{\kappa,n}e^{ikx+i\ell z}$ with

$$ k = \sin \theta_2 \kappa + \sin \theta_1 \nu \kappa, \quad \ell = -\cos \theta_2 \kappa - \cos \theta_1 \nu \kappa, $$

and it is well known that a solution of this kind exists if and only if $k$ and $\ell$ satisfy the classical dispersion relation

$$ D(k, \ell) = -k^2 + (\alpha + \beta(k^2 + \ell^2)) \sqrt{k^2 + \ell^2} \tanh \sqrt{k^2 + \ell^2} = 0. $$
The above observation has an elegant geometric interpretation: imaginary eigenvalues $i\kappa$ correspond to intersections in the $(k, \ell)$ plane of the real branches $C_{tr}$ of the dispersion relation with the parallel lines $K_n$, $n = 0, \pm 1, \pm 2, \ldots$ given in parametric form by

$$K_n = \{(k, \ell) \in \mathbb{R}^2 : k = \sin \theta_2 \kappa + \sin \theta_1 n \nu, \ell = -\cos \theta_2 \kappa - \cos \theta_1 n \nu, \kappa \in \mathbb{R}\}$$

(see Figure 7(b)). A point of intersection of $K_n$ and $C_{tr}$ corresponds to an imaginary mode $n$ eigenvalue $i\kappa$; its imaginary part is the value of the parameter $\kappa$ at the point of intersection, that is the value of $K_0$ in the $(K_0, L)$ coordinate system at the intersection, where

$$L = \{(k, \ell) \in \mathbb{R}^2 : k = \sin \theta_1 \mu, \ell = -\cos \theta_1 \mu, \mu \in \mathbb{R}\}.$$

The geometric multiplicity of the eigenvalue $i\kappa$ is given by the number of distinct lines in the family $\{K_n\}$ which intersect $C_{tr}$ at this parameter value, and a tangent intersection between $K_n$ and $C_{tr}$ indicates that each eigenvector in mode $n$ has an associated Jordan chain of length 2.

The $(\beta, \alpha)$ parameter plane is divided into regions I, II and III in which $C_{tr}$ has respectively zero, one and two nontrivial bounded branches (see Figure 7(a)). These regions are delineated by the line \{\alpha = 1\} and the curve

$$\Gamma = \left\{ (\beta, \alpha) = \left(-\frac{1}{2 \sinh^2 \kappa} + \frac{1}{2 \kappa \tanh \kappa}, \frac{\kappa^2}{2 \sinh^2 \kappa} + \frac{\kappa}{2 \tanh \kappa}\right) : \kappa \in [0, \infty) \right\}.$$

Regions II and III are in fact each divided into two subregions, at the mutual boundary of which the qualitative shape of the branches changes, namely from convex to nonconvex. The dimension of the central subspace is given by the number of intersections of a discrete family of parallel lines with a bounded set of curves and is therefore always finite (see Figure 7(b)). Notice, however, that in the case of gravity waves ($\beta = 0$) the set $C_{tr}$ is always unbounded, so that the linear operator always has infinitely many imaginary eigenvalues.

The key step in our local bifurcation theory for equation (16) is the following reduction principle, which asserts that the hydrodynamic problem is locally equivalent to a Hamiltonian system with finitely many degrees of freedom whose dimension is equal to the number of imaginary eigenvalues of $L_0$. Its proof relies upon a reduction theorem for quasilinear systems due to Mielke [28], for which the spectral information in Lemma 2.1 is a central hypothesis, and a change of variable which converts equation (16) into an equivalent evolutionary equation with linear boundary conditions; full details are given by Groves & Haragus [12, §2.3]. (The fact that $\beta > 0$ is essential here since it ensures that $X^{\kappa}_c$ is finite-dimensional.)

**Theorem 2.2** Let $X^{\kappa}_c$ denote the centre subspace of $X_c$ determined by the operator $L_0$, so that $X^{\kappa}_c$ is finite-dimensional. There exist neighbourhoods $\Lambda$ of 0 in $\mathbb{R}^5$ and $U$, $U_1$ of 0 in $X_{\kappa+1}, D(L_0) \cap X^{\kappa}_c$ together with a reduction function $h \in U_1 \times \Lambda \rightarrow X_{\kappa+1}$ which satisfies $h(0, 0) = 0, d_1 h(0, 0) = 0$ and has the following properties. For each $\lambda \in \Lambda$ the graph

$$M^{\kappa}_\Lambda = \{(x_1 + h(x_1, \lambda), \lambda) \in U \cap D_s : x_1 \in U_1\}$$

is a locally invariant two-dimensional manifold of (16), every small, bounded solution of (16) lies on $M^{\kappa}_\Lambda$, and $M^{\kappa}_\Lambda$ is a symplectic submanifold of $M$. Moreover, the flow determined by the Hamiltonian system $(M^{\kappa}_\Lambda, \tilde{\Omega}, \tilde{H})$, where the tilde denotes restriction to $M^{\kappa}_\Lambda$, coincides with the flow on $M^{\kappa}_\Lambda$ determined by $(M, \bar{\Omega}, H)$; in particular the reversibility and all discrete symmetries of $(M, \bar{\Omega}, H)$ are inherited by $(M^{\kappa}_\Lambda, \tilde{\Omega}, \tilde{H})$. 

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The reduction method is particularly well suited to bifurcation scenarios in which the number of imaginary eigenvalues changes upon varying a parameter through a critical value. By varying the angles $\theta_1, \theta_2$, together with the physical parameters $\beta, \alpha$ and the frequency $\nu$ of the waves in the $Z$ direction, one can systematically compile a complete catalogue of bifurcation scenarios which are associated with a change in the number of imaginary eigenvalues or, equivalently, a change in the number of points of intersection between the family of parallel lines $\{K_n\}_{n \in \mathbb{Z}}$ and the dispersion curve $C_{dr}$. The catalogue is extensive, containing virtually all bifurcations and resonances known in Hamiltonian systems theory (see Groves & Haragus [12, §3]). In this sense one can regard the present version of the water-wave problem as a paradigm for finite-dimensional Hamiltonian systems, and this observation has a significant consequence. There is a cornucopia of nonlinear bifurcation theories for finite-dimensional Hamiltonian systems, each associated with a particular bifurcation scenario. A wealth of existence theories for travelling water waves can therefore be found by selecting a nonlinear bifurcation theory and applying it to the hydrodynamic problem via the reduced equations associated with the relevant bifurcation scenario.

### 2.3 Doubly periodic waves

A $2\pi/\kappa$ periodic solution of the reduced Hamiltonian system corresponds to a doubly periodic travelling water wave which is $2\pi/\kappa$ periodic in $\xi$ and $2\pi/\nu$ periodic in $Z$ (see Figure 5(a)). Such solutions may be found by applying an appropriate version of the Lyapunov centre theorem, the standard form of which is applicable to a finite-dimensional Hamiltonian system with a pair of non-resonant imaginary eigenvalues and asserts the existence of a family of small-amplitude periodic solutions with frequencies near the magnitude of these eigenvalues.

The following theorem is a representative application of the standard Lyapunov centre theorem to periodic solutions controlled by mode $\pm 1$ imaginary eigenvalues. It has the character of an ‘inverse’ result in which the fundamental domain of a doubly periodic surface wave is specified and values of the physical parameters $\alpha$ and $\beta$ are found which guarantee the
existence of the desired wave. The parameters are selected so that $\pm \i \kappa$ are mode $\pm 1$ eigenvalues which are not in nonsemisimple resonance with any other mode $\pm 1$ eigenvalues or in semisimple resonance with eigenvalues in any of the other Fourier modes.

**Theorem 2.3** Choose angles $\theta_{10}, \theta_{20}$ and frequencies $\kappa$ and $\nu_0$ in respectively $\xi$ and $Z$ so that $\kappa + \nu_0 \cos \theta_{10} \neq 0$ if $\theta_20 = 0$, $\kappa + \nu_0 \sin \theta_{10} \neq 0$ if $\theta_{20} = \pi/2$ and $\kappa - \nu_0 \sin \theta_{10} \neq 0$ if $\theta_{20} = -\pi/2$. Suppose that $(\beta_0, \alpha_0)$ lies on the line

$$a_0 + \gamma_0^2 \beta_0 = \frac{(\sin \theta_{20} \kappa + \sin \theta_{10} \nu_0)^2}{\gamma \tanh \gamma},$$

where $\gamma^2 = \kappa^2 + \nu_0^2 + 2 \cos(\theta_{10} - \theta_{20}) \nu_0 \kappa$, and does not lie on any of the lines

$$a_0 + \gamma_{m,n}^2 \beta_0 = \frac{(m \sin \theta_{20} \kappa + n \sin \theta_{10} \nu_0)^2}{\gamma_{m,n} \tanh \gamma_{m,n}}, \quad (m, n) \in \mathbb{N}_0 \times \mathbb{N}_0 \setminus \{(1, 1)\},$$

where $\gamma_{m,n}^2 = m^2 \kappa^2 + n^2 \nu_0^2 + 2 \cos(\theta_{10} - \theta_{20}) \nu_0 \kappa \nu_0 \kappa$, or on the line

$$(\tanh \gamma + \gamma \sech^2 \gamma) a_0 + (3 \gamma^2 \tanh \gamma + \gamma^3 \sech^2 \gamma) \beta_0 = \frac{2 \gamma \sin \theta_{20} (\sin \theta_{20} \kappa + \sin \theta_{10} \nu_0)}{\kappa + \nu_0 \cos(\theta_{10} - \theta_{20})}$$

if $\kappa + \nu_0 \cos(\theta_{10} - \theta_{20}) \neq 0$.

The reduced Hamiltonian system at $\mu = 0$ possesses a periodic orbit on the energy surface $\{ \widetilde{H}_0^\mu = \epsilon \}$ for each sufficiently small value of $\epsilon > 0$. Each of these periodic orbits corresponds to a travelling water wave which is periodic in $\xi$ and $Z$ with frequencies respectively near $\kappa$ and equal to $\nu_0$.

There are generalisations of the Lyapunov centre theorem which deal with the ‘resonant’ case where an integer multiple of the basic pair of imaginary eigenvalues $\pm i \kappa$ is itself a pair of imaginary eigenvalues. A pair $\pm i \omega$ of imaginary eigenvalues of a Hamiltonian system has an associated index (the *Krein signature*), which may be positive or negative and controls a sign in the quadratic part of the Hamiltonian. The precise combination of the Krein signatures plays an important role in Lyapunov centre theorems for resonant cases. There is for example the result due to Weinstein [34] and further developed by Moser [29], which states that the nonresonance condition on the eigenvalues can be replaced by the requirement that the quadratic part of the Hamiltonian is positive-definite (in other words, that all Krein signatures are equal). Cases in which non-trivial combinations of Krein signatures arise have to be treated on an individual basis by methods such as normal-form and singularity theory (e.g. see Duistermaat [9]). In fact we have sufficient freedom in our choice of parameters for the travelling water-wave problem to generate practically any resonant case with any combination of the Krein signatures, and in this sense it is also a paradigm for the Lyapunov centre theorem.

Existence theories of the kind given in Theorem 2.3 have also been presented by Craig & Nicholls [5] using a variational Lyapunov-Schmidt reduction of the equations for travelling water waves; their method is described in more detail in Section 4.1 below. Both approaches give existence theories for doubly periodic water waves with arbitrary fundamental domain, in particular for the ‘short-crested waves’ described by Hammack, Scheffner & Segur [17] which are spatially periodic in two orthogonal directions, one of which coincides with the direction of propagation (take $(\theta_1, \theta_2) = (0, \pm \pi/2)$ and for the doubly periodic waves described by Reeder & Shinbrot [32] whose fundamental domain is a ‘symmetric diamond’ (choose $\theta_1 = \pm \theta_2, \kappa = \nu$).
2.4 Solitary waves

![Diagram of solitary waves]

**Fig. 8** A Hamiltonian-Hopf bifurcation arises as the frequency $\nu$ of the waves in the $Z$ direction is varied through a critical value $\nu_c$.

The Hamiltonian-Hopf bifurcation occurs in a Hamiltonian system with two degrees of freedom when four complex eigenvalues become imaginary by colliding in pairs on the imaginary axis as a parameter is varied; the bifurcation is of interest here since it is associated with a rich homoclinic bifurcation theory which forms the basis of an existence theory for periodically modulated solitary waves (cf. Figure 5(b)). A Hamiltonian-Hopf bifurcation involving only mode 1 and mode 1 eigenvalues can be generated in our hydrodynamic problem by varying the frequency $\nu$ of the waves in the $Z$ direction through a critical value $\nu_c$ (see Figure 8). Introducing a bifurcation parameter by writing $\nu = \nu_c + \mu$, one obtains a reduced Hamiltonian system with two degrees of freedom (since the centre subspace of $L$ at $\mu = 0$ is four-dimensional), and the reduction procedure captures all small bounded solutions of the water-wave problem for all sufficiently small values of $\mu$. Observe that there are two degenerate versions of the Hamiltonian-Hopf bifurcation (both of which are included in the theory below), in which the eigenvalues are zero at criticality; in one of these cases the eigenvalues are complex for $\mu > 0$ while in the other ($\theta_1, \theta_2 = (\pm \pi/2, 0)$) they are real.

The two-degree-of-freedom reduced Hamiltonian system is conveniently studied using complex coordinates $(A, B)$ and a normal-form transformation. The Birkhoff normal-form theory states that for each $n_0 \geq 2$ there is a near-identity, analytic, symplectic change of coordinates with the property that

$$
\begin{align*}
\tilde{H}^\mu(A, B) &= iq(AB - \bar{A}\bar{B}) + |B|^2 \\
&\quad + H_{NF}(|A|^2, i(AB - \bar{A}\bar{B}), \mu) + O(|(A, B)|^2 |(\mu, A, B)|^{n_0})
\end{align*}$$

(18)

in the new coordinates; the function $H_{NF}$ is a real polynomial of order $n_0 + 1$ which satisfies

$$
H_{NF}(|A|^2, i(AB - \bar{A}\bar{B}), \mu) = O(|(A, B)|^2 |(\mu, A, B)|).
$$

In these coordinates Hamilton’s equations for the reduced system are given by

$$
A_\xi = \frac{\partial H}{\partial B}, \quad B_\xi = -\frac{\partial H}{\partial A}
$$

(19)

(together with their complex conjugates) and the action of the reverser $R$ is given by $(A, B) \mapsto (\bar{A}, -\bar{B})$. 

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A homoclinic bifurcation theory for the above Hamiltonian system was introduced by Iooss & Kirchgässner [19], who noticed that the ‘truncated normal form’ obtained by neglecting the $O((|A, B|)^2 |(\mu, A, B)|^{10})$ remainder term in equation (18) is completely integrable ($H$ and $AB - \bar{A}B$ are conserved quantities) and invariant under the rotation $(A, B) \mapsto (Ae^{i\theta}, Be^{i\theta})$, $\theta \in \mathbb{R}$. For $\mu > 0$ it admits a circle of homoclinic solutions related by rotation, and an application of the implicit-function theorem shows that two of these solutions persist when the remainder terms are reinstated; the two persisting solutions are characterised by the fact that they are symmetric, that is their orbits are invariant under the action of the reverser $R$ (see Iooss & Pérouème [20]). Buffoni & Groves [4] showed that there are in fact infinitely many homoclinic solutions which resemble multiple copies of Iooss & Kirchgässner’s solutions. These multipulse homoclinic solutions are obtained as critical points of an action functional associated with the above Hamiltonian system. Their proof is obtained by an argument in which several copies of a ‘primary’ homoclinic orbit are ‘glued’ together to produce a multipulse homoclinic orbit; the existence of the latter is confirmed by topological methods which use the variational structure of the problem (and in particular mountain-pass arguments) in a crucial way.

The following theorem states the homoclinic bifurcation results more precisely; the conditions on the signs of the coefficients in the normal form have been verified by Groves & Haragus [12].

**Theorem 2.4** Suppose that the coefficients $c_1$ and $c_3$ in the expansion

$$H_{NF} = \mu c_1 |A|^2 + \mu ic_2 (AB - \bar{A}B) + c_3 |A|^4$$

$$+ i c_4 |A|^2 (AB - \bar{A}B) - c_5 (AB - \bar{A}B)^2 + \mu^2 c_6 |A|^2 + \mu^2 i c_7 (AB - \bar{A}B) + \ldots$$

are respectively negative and positive.

(i) For each sufficiently small, positive value of $\mu$ the two-degree-of-freedom Hamiltonian system (19) has two distinct symmetric homoclinic solutions.

(ii) Suppose that $(\theta_1, \theta_2) \neq (\pm \pi/2, 0)$. For each sufficiently small, positive value of $\mu$ the two-degree-of-freedom Hamiltonian system (19) has an infinite number of geometrically distinct homoclinic solutions which generically resemble multiple copies of one of the homoclinics in part (i).

The homoclinic solutions identified above correspond to travelling water waves which are $2\pi/(\nu_c + \mu)$ periodic in $\mathbb{Z}$ and have an envelope solitary-wave profile in the $\xi$ direction (a periodic wave train modulated by an exponentially decaying envelope) whose amplitude is of order $\sqrt{c_1}$; they are sketched in Figures 9 and 10.

Finally note that the Hamiltonian-Hopf bifurcation at $\nu = \nu_c$ in region III of the $(\beta, \alpha)$ parameter plane may be followed by a second Hamiltonian-Hopf bifurcation as $\nu$ is decreased (see Figure 7(a)). The above theory also applies in this case, but the signs of the bifurcation parameter $\mu$ and the coefficient $c_1$ are reversed.

Generalised solitary waves correspond to solutions of the reduced Hamiltonian system which are homoclinic to periodic orbits (see Figure 5(c)), and homoclinic bifurcation of this type is associated with an $(i\omega_0)^2 i\omega_1$ resonance in Hamiltonian systems with three degrees of freedom (four complex eigenvalues become imaginary by colliding in pairs on the imaginary axis in the presence of an additional pair of imaginary eigenvalues). This bifurcation can also
Fig. 9  The homoclinic bifurcation theory of Iooss & Kirchgässner yields waves with an envelope solitary-wave profile in the $\xi$ direction (left); the corresponding free surface of the water is in addition periodic in the $Z$ direction (right).

Fig. 10  The homoclinic bifurcation theory of Buffoni & Groves yields waves with a multi-pulse envelope solitary-wave profile in $\xi$ direction; a two-pulse profile is shown here (left) together with the corresponding free surface of the water (right).

be generated by varying $\nu$ through $\nu_c$ (choose $\theta_2$ so that $K_0$ intersects $C_{dr}$ in two points in Figure 8). Details of nonlinear bifurcation theories for this bifurcation scenario based upon normal-form and persistence arguments are given by Groves & Haragus [12, pp. 439–441].

3  Connections with two-dimensional travelling waves

In this section we continue working in the spatial dynamics framework introduced in Section 2. Here our interest lies in the ‘rectangular’ choice of coordinate system $(\xi, Z) = (z, x)$, so that $(\theta_1, \theta_2) = (\pi/2, 0)$ and the waves are $2\pi/\nu$ periodic in their direction of propagation with the transverse direction playing the role of ‘time’. This choice has recently been used in several investigations of three-dimensional solitary waves, where it reveals unexpected connections with the two-dimensional ($z$ independent) water-wave problem.
3.1 Three-dimensional solitary waves and two-dimensional Stokes waves

A straightforward calculation using equation (17) with \((\theta_1, \theta_2) = (\pi/2, 0)\) shows that all eigenvalues of \(L\) are real or imaginary and that for each \(k\) there are either zero or two non-zero imaginary mode \(k\) eigenvalues. Observe that all eigenvalues are geometrically double since the combination of periodicity and translation invariance of our evolutionary equation in the \(x\) direction generates an \(O(2)\) symmetry. Figure 11(a) illustrates how the number of imaginary eigenvalues changes at each of a countably infinite number of straight lines \(C_1, C_2, \ldots\) in \((\beta, \alpha)\) parameter space; at each point of the line \(C_k\) two real mode \(k\) eigenvalues become imaginary by passing through zero in a nonsemisimple resonance. We now make all eigenvalues geometrically simple by exploiting the \(Z_2\) symmetry with reflector \(S_2(\eta(x), \omega(x), \Phi(x, Y), \Psi(x, Y)) = (\eta(-x), \omega(-x), -\Phi(-x, Y), -\Psi(-x, Y))\); we restrict attention to those solutions which are invariant with respect to this symmetry, so that their \(\eta, \omega\) and \(\Phi, \Psi\) components are described by respectively cosine and sine series in \(x\).

The line \(C_k\) consists of those values of \(\alpha\) and \(\beta\) which satisfy the equation

\[
(\alpha + \beta \nu^2 k^2) \sinh \nu k = \nu k \cosh \nu k;
\]

it connects the point \((\frac{1}{k} \coth \nu k, 0)\) on the \(\beta\) axis with the point \((0, \nu k \coth \nu k)\) on the \(\alpha\) axis. Starting from the \(\beta\) axis and moving left, we find that the line \(C_k\) intersects each of the lines \(C_1, \ldots, C_{k-1}, C_{k+1}, C_{k+2}, \ldots\) in turn before arriving at the \(\alpha\) axis, and we denote the point of intersection of \(C_i\) and \(C_j\), \(i > j\) by \(P_{i,j}\) (see Figure 11). A Hamiltonian \(0^2\) resonance therefore occurs at those points of \(C_1\) to the right of \(P_{1,2}\) and those points of \(C_j\) between \(P_{j-1,j}\) and \(P_{j,j+1}\) for \(j \geq 2\); the imaginary spectrum consists of a geometrically simple zero eigenvalue which has a Jordan chain of length 2. The points \(P_{j,j+1}, j = 1, 2, \ldots\) on the other hand are codimension-two points at which a Hamiltonian \(0^20^2\) resonance takes place; here the imaginary spectrum consists of a geometrically simple mode \(k\) zero eigenvalue and a geometrically simple mode \((k + 1)\) zero eigenvalue, each with a Jordan chain of length 2, so that the zero eigenspace and generalised eigenspace are respectively two- and four-dimensional. Notice that mode \(i\) and mode \(j\) eigenvalues can cross on the imaginary or real axes in a semisimple resonance; in particular mode \(k\) and mode \(k + 1\) eigenvalues cross on the imaginary axis in a Hamiltonian semisimple \(1 : 1\) resonance and on the real axis in a Hamiltonian real semisimple \(1 : 1\) resonance at points of curves \(I_{k,k+1}\) and \(R_{k,k+1}\) which emanate from \(P_{k,k+1}\) (see Figure 11(b)). The curves \(I_{k,k+1}\) and \(R_{k,k+1}\) are in fact globally multi-branched and can be parameterised by the square of the magnitude of the eigenvalues undergoing the semisimple \(1 : 1\) resonance at points of these curves.)

The lines \(C_k\) also play a central role in bifurcation theory for the two-dimensional travelling water waves. We have characterised points on the line \(C_k\) in \((\beta, \alpha)\) parameter space as values of the physical parameters at which a mode \(k\) zero eigenvalue exists. A mode \(k\) zero eigenvalue arises when the linearised travelling water-wave problem admits a solution whose \(x\) and \(z\) dependence is described by \(e^{ikx}e^{i\nu z}\), and a solution of this form is clearly a periodic solution of the two-dimensional \((z\) independent) linear travelling water-wave problem with frequency \(k\nu\). A necessary condition for the bifurcation of nonlinear Stokes waves (nonlinear two-dimensional periodic waves) with this frequency from the uniform flow is therefore satisfied at points on this line. This observation was taken up by Jones [22], who used a formulation of the two-dimensional travelling water-wave problem as an integral equation to analyse primary and secondary bifurcation phenomena for Stokes waves. (This reference is
Fig. 11 (a) The line $C_k$ consists of points in $(\beta, \alpha)$ parameter space at which two real mode $k$ eigenvalues become imaginary by passing through zero in a non-semisimple resonance, and real mode $k$ and $k + 1$ eigenvalues pass through each other in a semisimple resonance at the points on the dashed lines. Mode 1, 2 and 3 eigenvalues are represented respectively by black, grey and white dots. The shaded regions indicate the parameter regimes where multipulse homoclinic bifurcation is detected. (b) Behaviour of mode $k$ (black) and mode $k + 1$ (grey) eigenvalues in a neighbourhood of the codimension-two point $P_{k,k+1}$.

actually a modification of previous work by Jones & Toland [21] for water of infinite depth.) Jones was particularly interested in the points $P_{i,j}$ at which $C_i$ intersects $C_j$. Here we have characterised these points as points at which mode $i$ and mode $j$ zero eigenvalues of our evolutionary system simultaneously exist, but they can also be understood as points associated with the simultaneous existence of the $i$th and $j$th harmonics of a fundamental periodic two-dimensional travelling wave; the corresponding nonlinear waves are called Wilton ripples.

In fact our reduced system displays single-pulse homoclinic bifurcation phenomena near the Hamiltonian $0^2$ resonance points and multipulse homoclinic bifurcation phenomena near the points $P_{k,k+1}$, $k \geq 1$ (see below). There is therefore an intimate connection between the bifurcations of ‘simple’ Stokes waves and single-pulse three-dimensional solitary waves and between ‘complex’ Wilton ripples and multipulse three-dimensional solitary waves; the three-dimensional solitary waves in question are periodic in the direction of propagation and have a pulse-like profile in the transverse direction.

Introducing a bifurcation parameter $\mu$ by writing $(\beta, \alpha) = (\beta_0, \alpha_0 + \mu)$, where $(\beta_0, \alpha_0)$ is one of the points associated with a Hamiltonian $0^2$ resonance (see above), we obtain a two-dimensional reduced Hamiltonian system. Several types of small, bounded solutions are found on its phase portrait for $\mu > 0$, namely nonzero equilibrium, periodic and homoclinic solutions (Groves [11]). Recall that each point in phase space corresponds to a periodic function of $x$; the dynamics in $z$ describes the wave profile as a function of $z$. Equilibria are constant, that is time-independent solutions of the reduced system and therefore correspond
to water waves which are constant in the $z$ direction and periodic in the $x$ direction, that is Stokes waves. Similarly, periodic solutions of the reduced system correspond to doubly periodic water waves which have a ‘rectangular’ fundamental domain, while homoclinic solutions of the reduced system correspond to solitary waves which are periodic in the direction of propagation and have a pulse-like profile in the transverse direction. The phase portrait and the three types of water waves arising from it are sketched in Figure 12. Observe that the phase portrait is symmetric about the $q$ axis due to the reversibility; the action of the reverser $R$ is given by $(q, p) \mapsto (q, -p)$. It is also symmetric about the $p$ axis due to the $\mathbb{Z}_2$ symmetry $(q, p) \mapsto (-q, -p)$, which corresponds to the physical transformation $x \mapsto x + \pi/k\nu$, that is translation through half a period; the orbits on the left- and right-hand sides of the phase portrait therefore correspond to the same solutions of the hydrodynamic problem.

![Phase portrait and water waves](image)

**Fig. 12** The behaviour in 'time' of solutions to the spatial dynamics formulation determines the wave profile in the $z$ direction; all waves are periodic in the $x$ direction. Nonzero equilibrium, periodic and homoclinic solutions generate waves whose profile in the $z$ direction is respectively constant, periodic and pulse-like (from top).

Writing $(\beta, \alpha) = (\beta_0 + \epsilon_1, \alpha_0 + \epsilon_2)$, where $(\beta_0, \alpha_0) = P_{k,k+1}$, we obtain a four-dimensional reduced Hamiltonian system which captures the small-amplitude dynamics of the present water-wave problem in a full neighbourhood of $P_{k,k+1}$. Of particular interest is
\[ \epsilon_1 = \hat{\beta}_{k,k+1} \mu^2, \quad \epsilon_2 = (\hat{\alpha}_{k,k+1} + \delta) \mu^2, \]

where \( \hat{\beta}_{k,k+1}, \hat{\alpha}_{k,k+1} \) are the coefficients of \( \kappa^2 \) in the Taylor expansion of the parameterisation of \( R_{k,k+1} \) in terms of the magnitude \( \kappa \) of the eigenvalues undergoing the semisimple \( 1 : 1 \) resonance near \( P_{k,k+1} \). The former parameter \( \delta \) plays the role of a bifurcation parameter (varying \( \delta \) through zero from above we cross the critical curve \( R_{k,k+1} \) in parameter space from above), while the latter parameter \( \mu \) indicates the distance from the point \( P_{k,k+1} \).

Let us first discuss the reduced equations for \( k \geq 2 \), using coordinates \( (q_1, p_1) \) and \( (q_2, p_2) \) which are associated at the linear level with respectively the mode \( k \) and mode \( k + 1 \) eigenvalues. The equations are reversible and have a \( \mathbb{Z}_2 \) symmetry, the action of whose reflector \( T \) corresponds to \( x \mapsto x + \pi / k \nu \), that is translation through half a period. It follows from the combinatorics of the \( k \) : \( k + 1 \) mode interaction that both \( \{(q_1, p_1) = (0,0)\} \) and \( \{(q_2, p_2) = (0,0)\} \) are invariant subspaces under the flow generated by the Taylor expansion of the Hamiltonian vector field to every order; these subspaces are equipped with the \( \mathbb{Z}_2 \) symmetries with reflectors \( U_2 : (q_2, p_2) \mapsto (-q_2, -p_2) \) and \( U_1 : (q_1, p_1) \mapsto (-q_1, -p_1) \) respectively. The phase space of this ‘truncated Hamiltonian system’ in the \((q_1, p_1)\) and \((q_2, p_2)\) coordinate planes is qualitatively the same as that shown in Figure 12, and transversality arguments show that all solutions persist for the complete Hamiltonian system. The corresponding water waves are at leading order associated with respectively the \( k \)th and \((k + 1)\)th harmonics in the \( x \) direction, and the actions of the reflectors \( U_1, U_2 \) correspond to respectively \( x \mapsto x + \pi / k \nu \) and \( x \mapsto x + \pi / (k + 1) \nu \), that is translation through half a period.

In the complete four-dimensional phase space the reduced equations admit another single pulse transverse homoclinic orbit \( u_{k,k+1} \), which at leading order is associated with the \( k \)th and \((k + 1)\)th harmonics in the \( x \) direction (see Figure 13); a further homoclinic orbit is obtained using the reflector \( T \). (These orbits are obtained as perturbations of explicit homoclinic solutions to suitably scaled equations at \( (\delta, \mu) = (0,0) \).) Groves & Sandstede [15] have shown that for \( \delta < 0 \) there are also infinitely many homoclinic solutions which resemble multiple copies of \( u_{k,k+1} \) and \( Tu_{k,k+1} \) glued together in a strictly alternating sequence. These multipulse homoclinic solutions are obtained by the homoclinic Lyapunov-Schmidt theory which reduces the existence question to a bifurcation equation for \( N - 1 \) ‘times of flight’ of orbits close to the primary homoclinic solutions \( u_{k,k+1} \) and \( Tu_{k,k+1} \); the solvability of the bifurcation equation (which has a transparent structure due to the Hamiltonian structure and reversibility) is addressed using asymptotic information from the ‘tails’ of the homoclinic orbits. The corresponding travelling water waves are \( 2\pi / \nu \) periodic in \( x \) and have a large-scale structure consisting of a multipulse profile in \( z \) whose successive pulses are out of phase by \( \pi / \nu \) (see Figure 14).

The solution set of the reduced equations for \( k = 1 \) is slightly different due to the different combinatorics of the \( 1 \) : \( 2 \) mode interaction. The equations are again reversible and have a \( \mathbb{Z}_2 \) symmetry, the action of whose reflector \( T : (q_1, p_1) \mapsto (-q_1, -p_1) \) corresponds to \( x \mapsto x + \pi / k \nu \), that is translation through half a period. The flow in the invariant subspace \( \text{Fix} \ T = \{(q_1, p_1) = (0,0)\} \) has a phase portrait qualitatively the same as that shown in Figure 12; the corresponding water waves are at leading order associated with the second harmonic in the \( x \) direction and the subspace is equipped with the \( \mathbb{Z}_2 \) symmetry with reflector \( U_2 : (q_2, p_2) \mapsto (-q_2, -p_2) \) (yielding a translation \( x \mapsto x + \pi / 2 \nu \) through half a period).
This periodic travelling wave is associated with two higher harmonics in the $x$ direction; it has a pulse-like profile in the $z$ direction. The arrow shows the direction of propagation.

Sketches of single- and multipulse travelling water waves near $P_{k,k+1}$; for clarity only the first harmonic is shown. Successive pulses in the $z$ direction are out of phase by one half of the period in the $x$ direction.

In the complete four-dimensional phase space the reduced equations admit another unipulse transverse homoclinic orbit $u_{1,2}$ which at leading order is associated with the first and second harmonics in the $x$ direction; a further homoclinic orbit is obtained using the reflector $T$. In this case the homoclinic Lyapunov-Schmidt theory is applicable for $\delta > 0$ and yields the existence of infinitely many homoclinic solutions which resemble multiple copies of $u_{k,k+1}$ and $Tu_{k,k+1}$ glued together in a strictly alternating sequence; the corresponding travelling water waves are $2\pi/\nu$ periodic in $x$ and have a large-scale structure consisting of a multipulse profile in $z$ whose successive pulses are out of phase by $\pi/\nu$. 
3.2 A dimension-breaking phenomenon

The term dimension-breaking phenomenon describes the spontaneous emergence of a spatially inhomogeneous solution of a partial differential equation from a solution which is homogeneous in one or more spatial dimensions. Groves, Haragus & Sun [13] recently examined a secondary bifurcation in which a family of classical periodically modulated solitary waves (which have a pulse-like profile in the direction of propagation and are periodic in the transverse direction) emerges from a classical line solitary wave (which does not depend upon the transverse direction); the two types of solitary wave involved in this dimension-breaking scenario are illustrated in Figure 2(above).

Let us begin by stating an existence result for the line solitary waves which undergo this dimension-breaking phenomenon. The following theorem was established by Amick & Kirchgässner [2] (see also Kirchgässner [24] and Sachs [33]); the stated expressions for η* and Φ* are obtained by applying the change of variable (6) to the asymptotic expressions for the solution of the hydrodynamic problem obtained by Amick & Kirchgässner.

**Theorem 3.1** Suppose that β > 1/3 and α = 1 + ε. The hydrodynamic problem admits a line solitary-wave solution (η*(x), Φ*(x, Y)), where

\[ η^*(x) = -ε sech^2 \left( \frac{e^{1/2}x}{2(β - 1/3)^{1/2}} \right) + R_1(ε^{1/2}x), \]

\[ Φ^*(x, Y) = -2(β - 1/3)^{1/2}ε^{1/2} \tanh \left( \frac{e^{1/2}x}{2(β - 1/3)^{1/2}} \right) + \frac{1}{2} ε \left( Y^2 - \frac{1}{3} \right) \left( sech^2 \left( \frac{e^{1/2}x}{2(β - 1/3)^{1/2}} \right) \right)_x + R_2(ε^{1/2}x, Y), \]

in which R_1, R_2 are O(ε^2) functions of their arguments in respectively C^b_{1,ε}(R) and C^b_{0,ε}(R × (0, 1)) for any k ≥ 0; they decay exponentially to zero with exponential rate ε^* as x → ±∞, where ε^* is a real positive number strictly less than (4(β - 1/3))^{-1/2}, and are respectively even and odd in their first argument (so that η*(-x) = η*(x), Φ*(-x, Y) = -Φ*(x, Y)).

In order to detect a dimension-breaking phenomenon in which these line solitary waves become periodic in the z direction we study our Hamiltonian evolutionary system

\[ u_z = Lu + N(u) \]  

(22)
in a different phase space, namely a space of symmetric functions which decay to zero as x → ±∞, so that all its solutions are symmetric solitary waves. In particular, equilibrium and periodic solutions of (22) correspond to respectively line solitary waves (which do not depend upon the transverse spatial coordinate z) and periodically modulated solitary waves (which are periodic in the transverse spatial coordinate z). We therefore replace \( X_s \) with

\[ \tilde{X}_s = H^*_c(\Sigma) \times H^*_a(\Sigma) \times S^*(\Sigma) \times H^*_a(\Sigma), \quad \Sigma = \mathbb{R} \times (0, 1) \]

where

\[ H^*_c(\Sigma) = \{ u \in H^*(\Sigma), u(x, y) = u(-x, y) \text{ for all } (x, y) \in \Sigma \}, \]

\[ H^*_a(\Sigma) = \{ u \in H^*(\Sigma), u(x, y) = -u(-x, y) \text{ for all } (x, y) \in \Sigma \} \]
(\(H^s_\varepsilon(\mathbb{R})\), \(H^s_0(\mathbb{R})\) are defined in the same fashion), and

\[
\mathcal{S}^s(\Sigma) = \{ u \in L^2_{\text{loc}}(\Sigma) : u_x, u_y \in H^s(\Sigma), u(x, y) = -u(-x, y) \text{ for all } (x, y) \in \Sigma \}, \tag{23}
\]

which is a Banach space with respect to the norm

\[\| u \|_{\mathcal{S}^s} := \left( \| u_x \|_s^2 + \| u_y \|_s^2 \right)^{1/2};\]

here the fact that \(\Phi\) is odd with respect to \(x\) is exploited to create \(\mathcal{S}^s\), and we impose evenness or oddness constraints on the remaining components of \((\eta, \omega, \Phi, \xi)\) in order to ensure their compatibility. (We cannot use \(H^{s+1}_\varepsilon(\Sigma)\) for the \(\Phi\) component of the phase space since only the derivatives of \(\Phi\) necessarily decay to zero as \(x \to \pm \infty\); indeed, an examination of the asymptotic formula for \(\Phi^*\) shows that it tends to different limits as \(x \to \pm \infty\).) The domain \(\mathcal{D}_s\) of the vector field on the right-hand side of (22) is now understood as the subset of \(\tilde{X}_{s+1}\) defined by the boundary conditions (14), (15) (with \((\theta_1, \theta_2) = (\pi/2, 0)\) and \(Z = x\)).

The line solitary wave described in Theorem 3.1 defines an equilibrium solution \(u^* = (\eta^*, 0, \Phi^*, 0)\) to (22), and we may use a translation

\[u(z) = u^* + w(z)\]

to obtain the new Hamiltonian system

\[w_z = L^*w + N^*(w). \tag{24}\]

The following result by Groves, Haragus and Sun [13] concerns the spectrum of the linear operator \(L^*\).

**Theorem 3.2** The spectrum of \(L^*\) consists of two simple imaginary eigenvalues \(\pm \im k,\) where \(k\) is \(O(\varepsilon)\), together with essential spectrum along the whole of the real axis (Figure 15).

![Fig. 15](image)

The spectrum of the linear operator \(L^*\) for small, non-negative values of \(\varepsilon\): two simple imaginary eigenvalues of \(O(\varepsilon)\) together with essential spectrum along the real axis.

At this point it is helpful to recall the classical Lyapunov centre theorem that a finite-dimensional Hamiltonian system with nonresonant imaginary eigenvalues \(\pm \im k\) has a family of small-amplitude periodic solutions with frequency near \(k\). Iooss [18] has recently established a result of this kind for reversible systems in infinite-dimensional settings for which the nonresonance condition is violated at the origin due to the presence of essential spectrum.
Theorem 3.3 Consider a quasilinear, reversible evolutionary equation
\[ u_t = Lu + N(u) \]
in the phase space \( X \), where the domain \( D \) of \( L \) is a dense linear subspace of \( X \) and \( N \) is a smooth \( X \)-valued function of a neighbourhood of the origin in \( D \). Suppose that the linear operator \( L \) has a pair \( \pm ik \) of simple imaginary eigenvalues, that 0 is contained in its essential spectrum, and that

(i) all nonzero integer multiples of \( \pm ik \) lie in the resolvent set of \( L \);
(ii) \( L \) satisfies the estimate \( \| (L - i\lambda I)^{-1} \| = O(\lambda^{-1}) \) as \( \lambda \to \pm \infty \);
(iii) the range of the nonlinearity \( N \) lies in the range of \( L \), so that the equation \( Lv = -N(u) \) is solvable for each function \( u \) in the domain of \( N \).

Under these conditions the above evolutionary equation admits a family of small-amplitude periodic solutions whose frequency is near \( k \).

Theorem 3.4 Suppose that \( \beta > 1/3 \) and \( \alpha = 1 + \epsilon \). There exists a positive constant \( \omega_0 \) in the interval \( (0, 1/(4(\beta - 1/3)^{1/2})) \) and a small neighbourhood \( N_\epsilon \) of the origin in \( \mathbb{R} \) such that a family of periodically modulated solitary waves \( \{ (\eta_a(x, z), \Phi_a(x, Y, z)) \}_{a \in N_\epsilon} \) emerges from the line solitary wave \( (\eta^*(x), \Phi^*(x, Y)) \) in a dimension-breaking bifurcation. Here
\[ \eta_a(x, z) = \eta^*(x) + \epsilon \eta^*_a(\epsilon^{1/2}x, z), \]
in which \( \eta^*_a(\cdot, \cdot) \) has amplitude of \( O(|a|) \) and is even in both arguments and periodic in its second argument with frequency \( \epsilon k_\epsilon + O(|a|^2) \); the positive number \( k_\epsilon \) satisfies \( |k_\epsilon - \omega_0| = O(\epsilon^{1/4}) \).

4 Variational methods

In this section we turn to existence theories which are obtained by applying the direct methods of the calculus of variations to the functional in the variational principle (5). There are currently two existence theories of this kind, namely the result for doubly periodic waves due to Craig & Nicholls [5] and the treatment of fully localised solitary waves due to Groves & Sun [16]. The key step in both theories is the use of a reduction principle to convert this intractable variational principle, whose corresponding Euler-Lagrange equations are quasilinear, to a locally equivalent variational principle whose mathematical properties are much more favourable, namely a finite-dimensional variational problem (Section 4.1) and a semilinear variational problem (Section 4.2).
4.1 Doubly periodic waves

Let us now discuss the small-amplitude existence theory for doubly periodic travelling gravity-capillary water waves given by Craig & Nicholls [5]. Their proof is independent of that given in Section 2.3 and also yields doubly periodic waves with arbitrary fundamental domain $\Gamma$.

Craig & Nicholls begin with the observation by Zakharov [35] that the free-surface elevation $\eta(x, z, t)$ and Dirichlet data at the free surface $\xi(x, z) = \phi(x, \eta(x, z), z)$ completely determine the wave motion, and indeed the variational principle (5) can be written in terms of these variables as

$$\delta \int_{\Gamma} \left\{ \frac{1}{2} \xi G(\eta) \xi + \frac{1}{2} \eta^2 + T(\sqrt{1 + \eta_x^2 + \eta_z^2} - 1) - c_1 \eta \xi_x - c_2 \eta \xi_z \right\} \, dx \, dz = 0,$$

where $G(\eta)$ is the Dirichlet-Neumann operator defined by $G(\eta) \xi = \nabla \phi(-\eta_x, -\eta_z, 1)|_{y=\eta}$ and the potential function $\phi$ is the harmonic extension of $\xi$ into $D_\eta$ with Neumann data at $y = -h$. This variational principle can be interpreted in a different manner, namely that travelling waves are critical points of the energy

$$E(\eta, \xi) = \int_{\Gamma} \left\{ \frac{1}{2} \xi G(\eta) \xi + \frac{1}{2} \eta^2 + T(\sqrt{1 + \eta_x^2 + \eta_z^2} - 1) \right\} \, dx \, dz,$$

subject to fixed values of the two impulse functionals

$$I_1(\eta, \xi) = \int_{\Gamma} \eta \xi_x \, dx \, dz, \quad I_2(\eta, \xi) = \int_{\Gamma} \eta \xi_z \, dx \, dz;$$

the wave speeds $c_1$ and $c_2$ are the Lagrangian multipliers in the variational principle

$$\delta (E - c_1 I_1 - c_2 I_2) = 0. \quad (25)$$

Observe that $E$, $I_1$ and $I_2$ are invariant under the torus action $T_{\alpha_1, \alpha_2} \eta(x, z), \xi(x, z) = (\eta(x + \alpha_1, z + \alpha_2), \xi(x + \alpha_1, z + \alpha_2))$, so that any critical point belongs to a torus of critical points. Craig & Nicholls consider (25) as an operator equation $F(\eta, \xi, c) = 0$, where $F : \mathcal{X} \times \mathbb{R}^2 \rightarrow \mathcal{Y}$ is defined by

$$F(\eta, \xi, c) = (\delta_{\eta}(H - c_1 I_1 - c_2 I_2), \delta_{\xi}(H - c_1 I_1 - c_2 I_2))$$

and $\mathcal{X} = H_{\text{per}}^{s+2}(\Gamma) \times H_{0, \text{per}}^{s+1}(\Gamma)$, $\mathcal{Y} = H_{\text{per}}^s(\Gamma) \times H_{0, \text{per}}^s(\Gamma)$, $s > 1$; a precise description of the mapping properties of $G(\eta)$ is used to establish that $F$ indeed maps $\mathcal{X}$ into $\mathcal{Y}$.

The linearised operator $\mathcal{L}^c \equiv \delta_1 F[0, c]$ is readily analysed using Fourier series, that is writing

$$\eta(x, z) = \sum_{(\mu, k) \in \Gamma'} \hat{\eta}_{\mu, k} e^{i\mu x} e^{ikz}, \quad \xi(x, z) = \sum_{(\mu, k) \in \Gamma' \setminus \{0\}} \hat{\xi}_{\mu, k} e^{i\mu x} e^{ikz},$$

where $\Gamma'$ is the dual to the lattice defining the doubly periodic domain and $\hat{\eta}_{\mu, k}, \hat{\xi}_{\mu, k}$ satisfy the reality condition that $\overline{\hat{\eta}_{\mu, k}} = \hat{\eta}_{-\mu, -k}, \overline{\hat{\xi}_{\mu, k}} = \hat{\xi}_{-\mu, -k}$. One finds that

$$\mathcal{L}^c \left( \begin{array}{c} \eta \\ \xi \end{array} \right) = \sum_{(\mu, k) \in \Gamma'} \mathcal{L}^c_{\mu, k} \left( \begin{array}{c} \hat{\eta}_{\mu, k} \\ \hat{\xi}_{\mu, k} \end{array} \right) e^{i\mu x} e^{ikz}, \quad (26)$$
where \( \dot{\xi}_{0,0} = 0 \) and
\[
\mathcal{L}_{c,k}^0 = \begin{pmatrix}
g + T(\mu^2 + k^2) & -i(c_1 \mu + c_2 k) \\
i(\mu^2 + k^2) / 2 \tanh(\mu^2 + k^2)^{1/2}h & (\mu^2 + k^2)^{1/2} \tanh(\mu^2 + k^2)^{1/2}h
\end{pmatrix},
\]
The operator \( \mathcal{L}_{0,0}^0 : \mathbb{R} \to \mathbb{R} \) is clearly invertible, while the self-adjoint operator \( \mathcal{L}_{c,k}^0 : \mathbb{C}^2 \to \mathbb{C}^2 \) is singular (with a one-dimensional kernel) if and only if the wavenumber \((\mu, k)\) and wave velocity \((c_1, c_2)\) satisfy the classical dispersion relation
\[
\Delta_{c,\mu,k} = (c_1 \mu + c_2 k)^2 - (g + T(\mu^2 + k^2))(\mu^2 + k^2)^{1/2} \tanh(\mu^2 + k^2)^{1/2}h = 0;
\]
in this case we have the orthogonal decomposition \( \mathbb{C}^2 = \ker \mathcal{L}_{c,k}^0 \oplus \operatorname{Im} \mathcal{L}_{c,k}^0 \). Observe that \( \Delta_{c,\mu,k} = \Delta_{c,-\mu,-k} \), so that the nullity of \( \mathcal{L}^c \) is twice the number \( N \) of solutions \((\mu_1, k_1), \ldots, (\mu_N, k_N)\) of \( \Delta_{c,\mu,k} = 0 \) which are normalised such that \( c_1 \mu_1 + c_2 k_1 > 0 \); a straightforward calculation shows that for any \( \Gamma^* \) there exists \((c_1, c_2)\) such that \( 2 \leq N < \infty \). (The fact that \( T > 0 \) is crucial in this respect: in the case \( T = 0 \) (gravity waves) the kernel of the linear operator is infinite-dimensional, and existence theories for doubly periodic gravity waves are therefore likely to encounter small-divisor problems. This observation has already been made in a short note by Plotnikov [31], and although Plotnikov gave a sketch of an existence proof for doubly periodic travelling gravity waves using superconvergence methods this problem remains essentially open.) This construction yields the orthogonal decomposition
\[
L^2(\Gamma) = \ker \mathcal{L}^c \oplus (\ker \mathcal{L}^c)^{\perp}
\]
of \( L^2(\Gamma) \), which in turn induces the decompositions
\[
\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2, \quad \mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2,
\]
where
\[
\mathcal{X}_1 = \mathcal{Y}_1 = \ker \mathcal{L}^c, \quad \mathcal{X}_2 = \mathcal{X} \cap (\ker \mathcal{L}^c)^{\perp}, \quad \mathcal{Y}_2 = \mathcal{Y} \cap (\ker \mathcal{L}^c)^{\perp} = \operatorname{Im} \mathcal{L}^c.
\]
The following result is established by a series of direct estimates using the explicit representation (26) of \( \mathcal{L}^c \) in terms of Fourier series.

**Proposition 4.1** The operator \( \mathcal{L}^c : \mathcal{X}_2 \to \mathcal{Y}_2 \) admits a continuous inverse.

We proceed using the Lyapunov-Schmidt reduction. Choose \( e^* \) so that \( \dim \ker \mathcal{L}^c e^* \geq 4 \) (see above) and let \( P : L^2(\Gamma) \to L^2(\Gamma) \) be the orthogonal projection of \( L^2(\Gamma) \) onto \( \ker \mathcal{L}^c \). Applying \( P \) and \( I - P \) to the bifurcation equation
\[
F(u, c) = \mathcal{L}^c u + \mathcal{N}(u, c) = 0,
\]
we obtain the equivalent pair of equations
\[
P \mathcal{N}(u_1 + u_2, c) = 0, \quad \mathcal{L}^c u_2 + (I - P) \mathcal{N}(u_1 + u_2, c) = 0,
\]
which may alternatively be written as
\[
P(\delta E(u_1 + u_2, c) - c \delta I(u_1 + u_2, c)) = 0,
\]
where \( u_1 = Pu \), \( u_2 = (I - P)u \) and \( c.I = c_1I_1 + c_2I_2 \). The second equation can be locally solved by means of the implicit-function theorem for \( u_2 \) as a smooth function of \((u_1, c)\) in a neighbourhood of \((0, c^*)\) in \( A_1 \times \mathbb{R}^2 \); substituting \( u_2 = u_2(u_1, c) \) into the first equation, we obtain the \textit{reduced equation for } \( u_1 \), namely

\[
PN(u_1 + u_2(u_1, c), c) = 0.
\] (27)

**Proposition 4.2** There exists a smooth function \( c = c(u_1) \) which satisfies \( c(0) = c^* \) and has the property that the solutions of equation (27) are precisely the critical points of \( \tilde{E}(u_1) = E(u_1 + u_2(u_1, c(u_1))) \) subject to the constraints that

\[
\tilde{I}_1(u_1) = I_1(u_1 + u_2(u_1, c(u_1))) = a_1, \quad \tilde{I}_2(u_1) = I_2(u_1 + u_2(u_1, c(u_1))) = a_2.
\]

Here \( a_1, a_2 \) are sufficiently small constants such that \((a_1, a_2)\) is not a multiple of any \((\mu_j, k_j)\), \( j = 1, \ldots N \) and \( \{\delta \tilde{I}(u_1), \delta \tilde{I}(u_1)\} \) is linearly independent at each point of \( S_a = \{u_1 : I_1 = a_1, I_2 = a_2\} \).

**Proof.** The strategy is to produce a function \( c = c(u_1) \) such that

\[
\int_{\Gamma} \left( P(\delta E(u_1 + u_2(u_1)) - c(u_1).\delta I(u_1 + u_2(u_1))), w_1 \right) = 0
\] (28)

for all \( w_1 \in \mathcal{Y}_1 \) if and only if \( u_1 \) is a critical point of the functional

\[ J(u_1) = \tilde{E}(u_1) - c(u_1).\left( I(u_1) - a \right) \]

on the set \( S_a \). Here \((\cdot, \cdot)\) is the usual \( \mathbb{R}^2 \) inner product and \( u_2(u_1) \) is an abbreviation for \( u_2(u_1, c(u_1)) \).

Craig & Nicholls [5, §4.3] use the implicit-function theorem to construct a smooth function \( c = c(u) \) with \( c(0) = c^* \) such that

\[
\int_{\Gamma} \left( P(\delta E(u_1 + u_2(u_1)) - c(u_1).\delta I(u_1 + u_2(u_1))), \delta \tilde{I}_j(u_1) \right) = 0, \quad j = 1, 2
\]

whenever \( u_1 \in S_a \) (the condition that \((a_1, a_2)\) is not a multiple of \((\mu_1, k_1), \ldots, (\mu_N, k_N)\) is used in their construction); it follows that (28) is satisfied whenever \( u_1 \) is a normal vector to \( S_a \) at the point \( u_1 \). Suppose on the other hand that \( w_1 \) is a tangent vector to \( S_a \) at the point \( u_1 \).
We find that
\[ \frac{dJ[u_1]}{w_1} = \int_\Gamma \langle \delta E(u_1 + u_2(u_1)) - c(u_1)\delta I(u_1 + u_2(u_1)), w_1 \rangle 
+ \int_\Gamma \langle \delta E(u_1 + u_2(u_1)) - c(u_1)\delta I(u_1 + u_2(u_1)), du_2[w_1](w_1) \rangle 
+ \int_\Gamma \langle du_1[w_1].(\bar{I} - a) \rangle 
= \int_\Gamma \langle P(\delta E(u_1 + u_2(u_1)) - c(u_1)\delta I(u_1 + u_2(u_1))), w_1 \rangle 
+ \int_\Gamma \langle (I - P)(\delta E(u_1 + u_2(u_1)) - c(u_1)\delta I(u_1 + u_2(u_1))), du_2[w_1](w_1) \rangle 
= \int_\Gamma \langle P(\delta E(u_1 + u_2(u_1)) - c(u_1)\delta I(u_1 + u_2(u_1))), w_1 \rangle, \]
in which the facts that \( w_1 = Pw_1, u_2 = (I - P)u_2 \) are used in the second step. \qed

We thus arrive at the reduced variational principle of finding the critical points of a smooth functional \( \bar{E} \) on a two-codimensional compact submanifold \( S_\alpha \) of \( 2N \) dimensional real space; the problem is equivariant with respect to a torus action \( T_\alpha \). One finds that \( S_\alpha \) is given geometrically as the intersection of two ellipsoids, and we can this feature to conclude the existence of periodic orbits. When \( N = 2 \) the set \( S_\alpha \) is homeomorphic to a two-dimensional torus and the orbit under \( T_\alpha \) of any point of \( S_\alpha \) is the whole of \( S_\alpha \), so that \( \bar{E}|_{S_\alpha} \) is constant and has one \( T_\alpha \) equivariant critical point. The ‘symmetric diamond’ solution of Reeder and Shinbrot [32] is a special case with \((\mu_1, k_1) = (\kappa, \ell), (\mu_2, k_2) = (\kappa, -\ell), \kappa \neq \ell \) and \( a_1 = a_2 \). When \( N > 2 \) the set \( S_\alpha \) is homeomorphic to the product of two spheres, and it follows that there are at least \( \text{ind}_{T_\alpha} S_\alpha + 1 \) distinct \( T_\alpha \) equivariant critical points of \( \bar{E} \) on \( S_\alpha \), where \( \text{ind}_{T_\alpha} S_\alpha \) is a \( T_\alpha \) equivariant cohomological index of \( S_\alpha \). Craig & Nicholls show that it is always possible to choose the index so that \( \text{ind}_{T_\alpha} S_\alpha = N - 2 \).

Altogether, this analysis constitutes the following existence result, and in fact a similar procedure yields the corresponding theorem for water of infinite depth.

**Theorem 4.3** For any given fundamental domain \( \Gamma \) and values of \( g, T \) and \( h \) there exists a velocity \( c = (c_1, c_2) \) and a nontrivial periodic travelling wave with velocity \( c \) and fundamental domain \( \Gamma \).

This method is a generalisation of the proof of the Lyapunov centre theorem by Moser [29], in which one seeks periodic solutions of a finite-dimensional Hamiltonian system near an elliptic equilibrium. Such solutions are characterised as critical points of an action functional subject to fixed values of the averaged Hamiltonian; the unknown period is the Lagrange multiplier and all critical points are members of a sphere of critical points due to autonomy. The Lyapunov-Schmidt method can be employed to reduce the bifurcation equation (the Euler-Lagrange equation for the variational problem) to a finite-dimensional problem which is equivariant with respect to a circle action; a suitable functionalisation of the unknown period yields a corresponding reduced variational principle which is treated topologically using Liusternik-Schnirelman category theory.
4.2 Fully localised solitary waves

Finally, we summarise the existence theory for the fully localised solitary wave (Figure 2(below)) given by Groves & Sun [16]. Without loss of generality we take $c = (c, 0)$, so that $x$ is the direction of wave propagation and $z$ is the transverse spatial direction. The hydrodynamic problem depends upon two dimensionless parameters $\alpha = gh/c^2$ and $\beta = T/hc^2$ (see below); we suppose that $\beta > 1/3$ and introduce a bifurcation parameter $\varepsilon$ by writing $\alpha = 1 + \varepsilon$.

The change of variable

$$y = \frac{Y}{h}(h + \eta(x, z)) - h, \quad \phi(x, y, z) = \Phi(x, Y, z)$$

transforms the variable fluid domain $\{-h < y < \eta(x, z)\}$ into the fixed domain $\{0 < Y < h\}$, and it is convenient to introduce the scaled, dimensionless variables

$$(x', Y', z') = \frac{1}{h}(\varepsilon^{1/2}x, Y, \varepsilon z),$$

$$\eta'(x', z') = \frac{1}{\varepsilon h}\eta(x, z), \quad \Phi'(x', Y', z') = \frac{1}{\varepsilon^{1/2} h} \Phi(x, Y, z).$$

Writing the variational principle (5) in terms of the new coordinates, one finds that

$$\delta V(\rho, \Phi) = 0,$$

where

$$V(\rho, \Phi) = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \varepsilon \alpha \eta^2 + \beta \varepsilon^{-1} [\sqrt{1 + \varepsilon^3 \eta_x^2 + \varepsilon^4 \eta_z^2} - 1] + \varepsilon \int_0^1 (\eta_x Y \Phi_Y - \eta \Phi_x) \, dY ight\} \, dx \, dz,$$

$$\alpha = gh/c^2 = 1 + \varepsilon, \quad \beta = T/hc^2$$

and the primes have been dropped for notational simplicity.

The Euler-Lagrange equation for $\eta$ is

$$(1 + \varepsilon)\eta + \beta \varepsilon \eta_{xx} + \beta \varepsilon^2 \eta_{zz} = N_1(\eta, \Phi),$$

(29)

while that for $\Phi$ takes the form of the boundary-value problem

$$\varepsilon \Phi_{xx} - \varepsilon^2 \Phi_{zz} - \Phi_{YY} = N_2(\eta, \Phi), \quad 0 < Y < 1,$$

(30)

$$\Phi_Y = 0, \quad Y = 0,$$

(31)

$$\Phi_Y + \varepsilon \eta_x = N_3(\eta, \Phi), \quad Y = 1,$$

(32)

where the symbol $N_i(\eta, \Phi)$ denotes a nonlinear function of $\eta$ and $\Phi$. We seek a solution $(\eta, \Phi)$ of these equations which decays to zero as $|(x, z)| \to \infty$. Our strategy in solving (29)–(32) is to reduce the problem to a pseudodifferential equation for a single variable $\Phi_1 = \Phi_1(x, z)$ which is itself an Euler-Lagrange equation; a solution is found by applying the direct methods of the calculus of variations to the relevant variational functional. Here we proceed formally and defer to Groves & Sun [16] for the full technical details of the reduction procedure.
The first step is to apply the implicit-function theorem to solve equation (29) for \( \eta \) as a function of \( \Phi \) and substitute \( \eta = \eta(\Phi) \) into equations (30)–(32). Taking Fourier transforms of the ‘reduced’ version of (30)–(32) with \( \eta = \eta(\Phi) \), we obtain the equations

\[
\begin{align*}
-\hat{\Phi}_{YY} + \eta^2 \hat{\Phi} &= \hat{H}(\Phi), & 0 < Y < 1, \\
\hat{\Phi}_Y &= 0, & Y = 0, \\
\hat{\Phi}_Y - \frac{\epsilon \mu^2 \hat{\Phi}}{1 + \epsilon + \beta q^2} &= \hat{h}(\Phi), & Y = 1,
\end{align*}
\]

where \( (\mu, k) \) is the independent variable associated with the Fourier transform in \((x, z)\) and \( q^2 = \epsilon \mu^2 + \epsilon^2 k^2 \); the nonlinear functions \( H, h \) are defined by

\[
H(\Phi) = e^{-\frac{1}{2}} N_2(\eta(\Phi), \Phi), \quad \hat{h}(\Phi) = e^{-\frac{1}{2}} \hat{N}_3(\eta(\Phi), \Phi) - \frac{i \epsilon \mu}{1 + \epsilon + \beta q^2} \hat{N}_1(\eta(\Phi), \Phi).
\]

This boundary-value problem can be re-formulated as the integral equation

\[
\hat{\Phi} = -\int_0^1 G(y, \zeta) \hat{H}(\Phi) \, d\zeta - G(y, 1) \hat{h}(\Phi),
\]

in which \( G(y, \zeta) \) is the Green’s function associated with the linear operator defined by its left-hand side, and we proceed by decomposing \( G \) into a leading-order singular part and a remainder term according to the formula

\[
G(y, \zeta) = -\frac{1 + \epsilon}{\epsilon^2 Q} + e^{-2} G_1(y, \zeta),
\]

where

\[
Q = k^2 (1 + \epsilon) + \mu^2 + (\beta - 1/3) \frac{q^4}{\epsilon^2} + c_0 \frac{q^6}{\epsilon^2}
\]

and \( c_0 = \beta/2 - 2(1 + \epsilon)/15 \). Consider the equations

\[
\begin{align*}
\hat{\Phi}_1 &= 1 + \epsilon \left( \int_0^1 \hat{H}(\Phi_1 + \Phi_2) \, d\zeta + \hat{h}(\Phi_1 + \Phi_2) \right), \\
\hat{\Phi}_2 &= -\int_0^1 e^{-2} G_1(y, \zeta) \hat{H}(\Phi_1 + \Phi_2) \, d\zeta - e^{-2} G_1(y, 1) \hat{h}(\Phi_1 + \Phi_2),
\end{align*}
\]

where \( \Phi_1 = \Phi_1(x, z) \) and \( \Phi_2 = \Phi_2(x, y, z) \). Clearly any solution \((\Phi_1, \Phi_2)\) of this pair of equations yields a solution \( \Phi = \Phi_1 + \Phi_2 \) of (33), and conversely any solution \( \Phi \) of (33) can be decomposed into a sum \( \Phi = \Phi_1 + \Phi_2 \), where \( (\Phi_1, \Phi_2) \) solve (34), (35) (the functions \( \Phi_1 \) and \( \Phi_2 \) are calculated from the formulae obtained by replacing \( \Phi_1 + \Phi_2 \) by \( \Phi \) on the right-hand sides of (34), (35)). Equation (33) is therefore equivalent to (34), (35).

Finally, we solve equation (35) for \( \Phi_2 \) as a function of \( \Phi_1 \) using the implicit-function theorem. Inserting \( \Phi_2 = \Phi_2(\Phi_1) \) into (34), we obtain the reduced equation for \( \Phi_1 \), which may be re-formulated as the differential equation

\[
\frac{\epsilon^2}{1 + \epsilon} \left[ -c_0 \epsilon (\partial_2^2 + \epsilon \partial_2^3) + (\beta - \frac{1}{3})(\partial_2^2 + \epsilon \partial_2^3)^2 - (1 + \epsilon) \partial_2^2 - \partial_2^3 \right] \Phi_1
\]

\[
= \int_0^1 H(\Phi_1 + \Phi_2(\Phi_1)) \, dY + h(\Phi_1 + \Phi_2(\Phi_1)).
\]
We now identify the variational structure of (36). Recalling that (29) and (30)–(32) correspond to the Euler-Lagrange equations for $V$ with respect to $\eta$ and $\Phi$, that is
\[ d_1 V[\eta, \Phi] = 0, \quad d_2 V[\eta, \Phi] = 0, \]
we find that the ‘reduced’ version of (30)–(32) with $\eta = \eta(\Phi)$ is the Euler-Lagrange equation for the functional $W = V(\eta(\Phi), \Phi)$, since
\[ dW[\Phi] = d_1 V[\eta(\Phi), \Phi](d\eta[\Phi]) + d_2 V[\eta(\Phi), \Phi], \]
in which the second line follows by the defining property of $\eta(\Phi)$ as a solution of the Euler-Lagrange equation for $V$ with respect to $\eta$. This calculation shows that the elimination of $\eta$ qualifies as ‘natural’ with respect to the variational structure.

Equation (34) is equivalent to the differential equation
\[
\frac{\varepsilon^2}{1 + \varepsilon} \left[ -\varepsilon_0 \varepsilon (\partial_x^2 + \varepsilon \partial_z^2)^3 + (\beta - \frac{1}{3})(\partial_x^2 + \varepsilon \partial_z^2)^2 - (1 + \varepsilon) \partial_x^2 - \partial_x \right] \Phi_1
\]
\[ = \int_0^1 H(\Phi_1 + \Phi_2) \, dY + h(\Phi_1 + \Phi_2), \quad 0 < Y < 1, \quad (37) \]
while (35) is equivalent to the boundary-value problem
\[ -\Phi_{2Y} + q^2 \Phi_2 + \frac{1 + \varepsilon}{\varepsilon^2 Q S} \left( \int_0^1 q^2 \Phi_2 \, dY - \frac{\varepsilon^2 \Phi_2|_{Y=1}}{1 + \varepsilon + \beta q^2} \right) \]
\[ = \hat{H}(\Phi_1 + \Phi_2), \quad 0 < Y < 1, \quad (38) \]
\[ \Phi_{2Y} = 0, \quad Y = 0, \quad (39) \]
\[ \Phi_{2Y} - \frac{\varepsilon^2 \Phi_2}{1 + \varepsilon + \beta q^2} - \frac{(1 + \varepsilon) e \mu^2}{\varepsilon^2 Q (1 + \varepsilon + \beta q^2) S} \left( \int_0^1 q^2 \Phi_2 \, dY - \frac{\varepsilon^2 \Phi_2|_{Y=1}}{1 + \varepsilon + \beta q^2} \right) \]
\[ = \hat{h}(\Phi_1 + \Phi_2), \quad Y = 1, \quad (40) \]
in which
\[ S = 1 - \frac{q^2 (1 + \varepsilon)}{\varepsilon^2 Q} + \frac{(1 + \varepsilon) e \mu^2}{\varepsilon^2 Q (1 + \varepsilon + \beta q^2)}. \]
The left-hand side of (37) defines a formally self-adjoint operator acting upon $\Phi_1(x, z)$ which is associated with the quadratic form
\[ Q_1(\Phi_1) = \frac{\varepsilon^2}{2(1 + \varepsilon)} \int_{\mathbb{R}^2} \left\{ \varepsilon_0 (e \Phi_{1xxx}^2 + 3e^2 \Phi_{1xxx}^2 + 3e^3 \Phi_{1xxx} + e^4 \Phi_{1xxx}) + (\beta - 1/3)(\Phi_{1xx}^2 + 2e \Phi_{1xx}^2 + e^2 \Phi_{1xx}^2) + (1 + \varepsilon) \Phi_{1x}^2 \right\} \, dx \, dz, \]
and similarly the left-hand side of the boundary-value problem (38)–(40) defines a formally self-adjoint operator acting upon $\Phi_2(x, Y, z)$ which is associated with the quadratic form

$$Q_2(\Phi_2) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ \int_0^1 (|\hat{\Phi}_2^Y| + q^2|\hat{\Phi}_2|) \, dY - \frac{\epsilon \mu^2 |\hat{\Phi}_2|^2}{1 + \epsilon + \beta q^2} \right\} d\mu \, dk.$$

These observations, together with the fact that

$$dW_{NL}[\Phi](\Psi) = \int_{\mathbb{R}^2} \left\{ \int_0^1 H(\Phi) \Psi \, dY + h(\Phi) \Psi |_{Y=1} \right\} \, dx \, dz,$$

where $W_{NL}(\Phi)$ denotes the non-quadratic part of $W(\Phi)$, show that (37) and (38)–(40) (or equivalently (34) and (35)) are the Euler-Lagrange equations for respectively $\Phi_1$ and $\Phi_2$ of the functional

$$\tilde{W}(\Phi_1, \Phi_2) = Q_1(\Phi_1) + Q_2(\Phi_2) + W_{NL}(\Phi_1 + \Phi_2).$$

**Proposition 4.4** The reduced equation (36) for $\Phi_1$ is the Euler-Lagrange equation of the functional

$$I(\Phi_1) = Q_1(\Phi_1) + Q_2(\Phi_2(\Phi_1)) + W_{NL}(\Phi_1 + \Phi_2(\Phi_1)).$$

**Proof.** The calculation

$$dI[\Phi_1 + \Phi_2(\Phi_1)](\Psi_1)$$

$$= (dQ_1[\Phi_1] + dW_{NL}[\Phi_1 + \Phi_2(\Phi_1)])(\Psi_1)$$

$$+ (dQ_2[\Phi_2] + dW_{NL}[\Phi_1 + \Phi_2])(d\Phi_2[\Phi_1](\Psi_1))$$

$$= (dQ_1[\Phi_1] + dW_{NL}[\Phi_1 + \Phi_2(\Phi_1)])(\Psi_1),$$

in which the second equality follows by defining property of $\Phi_2(\Phi_1)$ as a solution of (38)–(40), shows that (36) is the Euler-Lagrange equation for the reduced functional $I(\Phi_1)$. \(\square\)

According to Proposition 4.4 the set of solutions to (36) which decay to zero as $|(x, z)| \to \infty$ and the set of critical points of the functional $I$ coincide. We now show that the latter set is nonempty by applying the direct methods of the calculus of variations; for convenience we actually study the equivalent functional $K = \epsilon^{-2}I$.

The calculus of variations offers a variety of results for studying functionals of the type

$$\mathcal{J}(u) = \int_S J(u) \, dx^n$$

which are defined on spatially extended domains $S$ (that is subsets of $\mathbb{R}^n$ which are unbounded in one or more spatial directions). A problem of this kind is typically treated in two stages. Firstly one establishes the existence of a *Palais-Smale* sequence $\{u_m\}$ with the property that $\mathcal{J}(u_m) \to a$, $\mathcal{J}'(u_m) \to 0$ as $m \to \infty$ for some nonzero constant $a$, so that $\{u_m\}$ is a sequence of successively better approximations to a putative critical point $u \neq 0$ with $\mathcal{J}(u) =$
$a$, $J'(u) = 0$. The second step is to study the convergence properties of $\{u_m\}$ (note that weaker results than the strong convergence of $\{u_m\}$ are sufficient to guarantee the existence of a nonzero critical point). The \textit{concentration-compactness principle} of Lions \cite{Lions1, Lions2} is frequently helpful in this respect; it has been applied with great success to the following class of problems collectively known as ‘the coercive, semilinear, locally compact case’. Suppose that $J$ is a smooth functional on $X(S)$, where $X(U)$ is a Sobolev space of functions defined upon the spatial domain $U \subseteq \mathbb{R}^n$. Let us write

$$J(u) = J_2(u) + J_{NL}(u),$$

where $J_2 : X(S) \to \mathbb{R}$ is the quadratic part of $J$, and suppose that $J_{NL}$ extends to a smooth functional $J_{NL} : Y(S) \to \mathbb{R}$, where

(i) (‘coerciveness’) $J_2$ is equivalent to the $X(S)$-norm;

(ii) (‘semilinearity’) $Y(S)$ is continuously embedded in $X(S)$;

(iii) (‘local compactness’) $Y(U)$ is compactly embedded in $X(U)$ for every bounded subset $U$ of $\mathbb{R}^n$.

Standard recipes for the construction of a critical point from a Palais-Smale sequence for a functional satisfying these criteria are available (see in particular de Bouard & Saut \cite{BouardSaut}, who treat model equations for three-dimensional water waves).

Let us now identify function spaces in which the functional $K$ falls into the ‘the coercive, semilinear, locally compact case’ described above. Let $X(\mathbb{R}^2)$ be the Hilbert space $\{u : \|u\| < \infty\}$, where

$$\langle u, v \rangle = \int_{\mathbb{R}^2} \left\{ c_0 (\varepsilon u_{xxxx} v_{xxxx} + 3 \varepsilon^2 u_{xxxx} v_{xxx} + 3 \varepsilon^3 u_{xxxx} v_{xxx} + \varepsilon^4 u_{xxxx} v_{xxx}) + (\beta - \frac{1}{4}) (u_{xxxx} v_{xx} + 2 \varepsilon u_{xxxx} v_{xx} + \varepsilon^2 u_{xxxx} v_{xx}) + u_{xx} v_{x} + (1 + \varepsilon) u_{xx} v_{x} \right\} \, dx \, dz$$

and $U^{l,p}_\varepsilon(\mathbb{R}^2)$, $\delta \geq 0$, $p \geq 2$ be the Banach space $\{u : \|u\|_{U^{l,p}_\varepsilon} < \infty\}$, where

$$\|u\|_{U^{l,p}_\varepsilon} = \|\mathcal{F}^{-1}[(1 + \mu^2 + \varepsilon k^2)\frac{1}{2}\mathcal{F}u_x]||_p + \varepsilon^{\frac{1}{2}} \|\mathcal{F}^{-1}[(1 + \mu^2 + \varepsilon k^2)\frac{1}{2}\mathcal{F}u_x]||_p,$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ denote respectively the Fourier and inverse Fourier transform and $\| \cdot \|_p$ is the $L^p(\mathbb{R}^2)$ norm. The local space $X(U)$, where $U$ is a compact subset of $\mathbb{R}^2$, is defined by replacing $\mathbb{R}^2$ by $U$ in the definition of the inner product, and since $\|u\|_{U^{l,p}_\varepsilon}$ is given in terms of $L^p(\mathbb{R}^2)$ norms of derivatives when $\delta$ is an integer we can define $U^{l,p}_\varepsilon(U)$ by interpolation. The following theorem, which is the central result of Groves & Sun \cite{GrovesSun}, shows that $K$ meets conditions (i)–(iii) above with $X(S) = X(S)$ and $Y(S) = U^{0,2}_\varepsilon(S) \cap U^{0,4}_\varepsilon(S) \cap U^{0,8}_\varepsilon(S)$. Notice that the functionals are defined only upon a neighbourhood $B_M(0)$ of the origin in the various function spaces due to the local nature of the reduction procedure; it is however possible to take $M$ arbitrarily large (at the expense of taking a smaller value of $\varepsilon$).
Theorem 4.5

(i) Choose \( M > 0 \). For each sufficiently small value of \( \delta \), each sufficiently large value of \( p \) and each sufficiently small value of \( \varepsilon \), the functional \( K \) is a smooth mapping \( \tilde{B}_M(0) \subset X(\mathbb{R}^2) \to \mathbb{R} \). Furthermore, writing

\[
K(\Phi_1) = K_2(\Phi_1) + K_3(\Phi_1) + K_4(\Phi_1),
\]

one finds that \( K_3 \) and \( K_4 \) extend to smooth functionals \( \tilde{B}_M(0) \subset U^0_\varepsilon(\mathbb{R}^2) \cap U^0_\varepsilon(\mathbb{R}^2) \cap U^{\delta,p}_\varepsilon(\mathbb{R}^2) \to \mathbb{R} \) and

\[
|K_4(\Phi_1)| \leq c \varepsilon^{\frac{1}{2}} P_4(\|\Phi_1\|), \quad (41)
\]

in which the symbols \( \Delta \) and \( P_4 \) denote respectively a quantity which is \( O(\delta + 1/p) \) and a polynomial which has unit positive coefficients and no monomials of degree less than four.

(ii) The space \( X(S) \) is continuously embedded in \( U^{\delta,p}_\varepsilon(S) \) for \( \delta \in [0,1] \), \( p \geq 2 \). The embedding is compact whenever \( S \) is bounded.

The final ingredient is the existence of a Palais-Smale sequence for \( K \), and here the key is the mountain-pass lemma as stated by Brezis & Nirenberg [3, p. 943].

Lemma 4.6 Consider a Banach space \( X \) and a functional \( J \in C^1(X,\mathbb{R}) \) with the properties that \( J(0) = 0 \), that \( 0 \) is a strict local minimum of \( J \) and that there is an element \( x \in X \) with \( J(x) < 0 \). There exists a Palais-Smale sequence \( \{x_m\} \subset X \) such that \( J(x_m) \to a \), \( J'(x_m) \to 0 \) as \( m \to \infty \), where

\[
a = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} J(\gamma(s)), \quad \Gamma = \{ \gamma \in C([0,1],X) : \gamma(0) = 0, J(\gamma(1)) < 0 \}. \]

It is not possible to apply Lemma 4.6 directly to \( K : \tilde{B}_M(0) \subset X(\mathbb{R}^2) \to \mathbb{R} \) since it is not defined upon the whole of \( X(\mathbb{R}^2) \). Notice however that it does meet the geometric requirements of the lemma: the estimates (41) and

\[
|K_2(\Phi_1)| = \frac{1}{2(1 + \varepsilon)} \|\Phi_1\|^2, \quad |K_3(\Phi_1)| \leq c \|\Phi_1\|^{3}_{L^{0,3}} \leq c \|\Phi_1\|^3
\]

show that \( 0 \) is a strict local minimum of \( K \), and choosing \( \Phi_1^* \) such that \( K_3(\Phi_1^*) \neq 0 \), we find that there exists a real number \( \lambda^* \) which has the property that \( J(\lambda^* \Phi_1^*) < 0 \). One proceeds by extending \( K \) to a smooth functional \( \tilde{K} : X(\mathbb{R}^2) \to \mathbb{R} \) in such a way that \( \tilde{K} \) and \( \tilde{K} \) coincide on a sufficiently large neighbourhood \( \mathcal{N} \) of the origin. The new functional therefore inherits the geometric structure of \( K \) and can be treated using Lemma 4.6; the resulting Palais-Smale
sequence for $\tilde{K}$ can be selected so that it lies in $N$ and is therefore a Palais-Smale sequence for $K$.

The existence proof is completed using one of the standard recipes for constructing a critical point from a Palais-Smale sequence for a functional in ‘the coercive, semilinear, locally compact case’. Groves & Sun [16] use a strategy similar to that employed by Groves [10] for a model equation for water waves; the present problem does however have some additional technical difficulties, in particular the fact that $K$ is constructed in terms of nonlocal operators.

References