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A Design Optimization Procedure for Epicyclic Gears*

John D. ANDREWS**

This paper describes a design optimization procedure to increase the working life of epicyclic gear units by minimizing the tensile fillet stresses experienced in the gear system components. An objective function is formulated together with 69 constraints (inequality and equality) which ensure that a functional system design is achieved. For the example presented, the optimum design was produced in only three iterations and reduced the maximum stress in the most highly stressed component by 54%.

Key Words: Epicyclic Gears, Tensile Fillet Stresses, Design Optimization Procedures

1. Introduction

As the requirement for greater transmitted power has increased, so has the use of epicyclic gear systems. The advantages of transmitting a load by more than a single set of tooth contacts mean that epicyclic gears are frequently used in engineering applications such as marine propulsion systems and turbo generators. When designing gear systems, as with other engineering components, a design process which consists of preliminary design, analysis, appraisal and redesign is carried out. The preliminary design will be produced making use of engineering judgement, operational experience and the relevant codes of practice. This design must then be assessed to determine its ability to meet the design intention. Since it is generally recognized that one of the primary causes of gear tooth failure is the presence of large tensile fillet stresses, this factor will determine the adequacy of the design, although other factors can be considered simultaneously. If tensile fillet stresses for the design are too large then the gear system will be modified and re-assessed. The redesign, analysis and appraisal stages will be repeated until an acceptable design is attained.

There are many alternative methods which can be used in gear design to assess the gear strength by entering the design parameters into formulae derived from cantilever beam theory with the addition of semi-empirical stress concentration factors. Several researchers[11-14] have stated the deficiencies of such approaches. The inaccuracy of these formulae means they are unable to adequately predict the effects of design parameter changes on the fillet stresses. This difficulty can be eliminated by making use of the finite-element method to analyze any proposed gear design[15].

Another point which requires consideration in the traditional design process outlined above is the question of when a design is regarded as adequate and is accepted. Designs are commonly adopted when the critical stress achieves a value below some preset target. Unfortunately these may not result in good

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designs since gear systems with considerably lower stress levels and hence longer life expectancies may be achievable. From the root stress point of view, what is required is an optimal design where maximum tensile fillet stresses are the minimum achievable such that the design performs as required, satisfying the functional and financial constraints. When epicyclic gear systems are to be designed the stresses resulting in sun, planet and annulus gears must all be considered. An objective function is formulated to represent the total stress in the system by combining the maximum fillet stresses in each component.

Structural optimization techniques have been applied to a variety of engineering systems such as connecting rods[8], dams[71], pressure vessels[84] and spur gears[83]. These works have shown that the design derivatives required to formulate the objective function can be obtained efficiently using the finite-element method. This paper extends the methods to consider systems where the stress levels in more than one component must be considered.

2. Epicyclic Gear Design Specification

Epicyclic gears enable a load to be transmitted by several sets of tooth contacts and can hence be used in higher load applications than conventional gearing. To ensure that any one gear does not experience overload it is imperative that the load be shared equally among the planets.

For the epicyclic system considered in this example the planets are mounted on intermediate shafts located on a common carrier (Fig. 1). The drive is then applied to the sun and output taken from the carrier with the annulus fixed. Providing that a reduction ratio between 3.5 and 5.0 can be achieved a variable input can be supplied to yield the desired output.

For equal load sharing among the planets they are equally spaced on the carrier. A torque of 120 Nmm/mm is applied to the input member. All gear components are made of steel.

The design parameters which can be varied during the optimization process are those which define the form of the individual gear tooth shapes and the distance between gear centers. Since the sun and planet gears are external forms and can be generated from a common basic rack, their design parameters are identified by subscripts s, p and t for the sun, planet and rack, respectively. The annulus is an internal gear which is produced by a pinion cutter which is part of the meshing system conjugate to the common basic rack. Variables with subscripts a and c refer to the annulus and cutter, respectively.

For an epicyclic system whose components are involute spur gears, in which the number of planets is fixed (in this case 3), there are sixteen design variables:

- basic rack:
  - pressure angle $\alpha_t$
  - module $m_t$
  - tip radius factor $r_{th}$
  - median height factor $h_{0t}$

- pinion cutter:
  - number of teeth $Z_c$
  - tip radius $r_{ac}$
  - addendum factor $a_{0c}$

- gear teeth:
  - Sun $Z_s$
  - Planet $Z_p$
  - Annulus $Z_a$
  - addendum factor $a_{0s}$, $a_{0p}$, $a_{0a}$
  - modification factor $s_{0s}$, $s_{0p}$, $s_{0a}$

A full description of these parameters can be found in Refs. (11) and (12).

The design algorithm is required to produce an epicyclic gear unit by selecting the optimal settings for the design variables defined above. This resulting design will be optimal with respect to the maximum fillet stresses occurring in each gear in the system. Maximum tensile fillet stress in the most highly stressed member will be minimized whilst also reducing the tensile fillet stresses in the remaining two gears to the smallest possible value. By this means the life expectancy of the gear system will be maximized.

2.1 Initial design specification and design limits

To limit the possible values which the design variables can assume, upper and lower limits are specified. These bounds together with the initial value specified for each parameter are listed below.

12 ≤ $\alpha_t$ ≤ 30
2 ≤ $m_t$ ≤ 8
0.0 ≤ $r_{th}$ ≤ 0.7
25 ≤ $Z_s$ ≤ 55
25 ≤ $Z_p$ ≤ 55
25 ≤ $Z_a$ ≤ 55
-0.5 ≤ $s_{0s}$ ≤ 0.5
-0.5 ≤ $s_{0p}$ ≤ 0.5
-0.5 ≤ $s_{0a}$ ≤ 0.5
12 ≤ $a_{0s}$ ≤ 20
20 ≤ $a_{0p}$ ≤ 25
40 ≤ $a_{0a}$ ≤ 50
For the design optimization carried out here, the form of the pinion cutter, the addendum factors of each of the epicyclic gear elements and the median height of the basic rack are all fixed and retain their initial values throughout the iterative procedure. All other variables are free to take any value which lies between the upper and lower limits prescribed for the design.

3. The Objective Function

The objective in optimizing the epicyclic system design was to increase the life expectancy of the gear unit by minimizing the fillet stresses in the gear system.

The general structural design problem is:

\[ \text{Minimize } Q(x) \]  

subject to constraint functions dependent upon the design vector \( x \). \( Q \) is a function of the maximum tensile fillet stress in each component of the epicyclic system. The formulation of the objective function must account for the maximum stress resulting in each component in such a way that its minimization will produce a design which gives the lowest stress level in every member. For an epicyclic gear unit, the maximum stresses in sun, planet and annulus teeth must all be included in the objective function used. The mathematical specification of the function representing the maximum fillet stress in each gear tooth must also be decided.

The geometry of each gear tooth is defined by the relevant set of design parameters. The variables contained in these sets fall into two categories for each tooth: those which determine the geometry of that particular tooth form and those which are common to more than one of the teeth. By placing the variables into these separate design vectors the root fillet stress equations can be expressed as

\[ \begin{align*}
\alpha & = f(x_1, x_2) \\
\beta & = f(x_3, x_4) \\
\gamma & = f(x_5, x_6) \\
\delta & = f(x_7, x_8)
\end{align*} \]  

where \( x_1 = (Z_o, a_o, s_o) \), \( x_2 = (Z_o, a_o, s_o) \) and \( x_6 = (Z_o, r_o, a_o, s_o) \) are the independent vectors of design parameters relating to sun, planet and annulus, respectively, and \( x_5 = (a_i, m_i) \) and \( x_7 = (r_o, h_o) \) are design parameters 'shared' by more than one gear.

In minimizing the maximum tensile fillet stress in the most highly stressed gear, it is possible that the value then assigned to a common variable will increase the fillet stresses in one of the other gears producing an overall detrimental effect. A sequential optimization process which can cope with this situation is presented in detail in Ref. (13). An objective function defined as the sum of the three fillet stress components is used together with additional constraints to represent the total stress in the gear system. The extra constraints ensure that changes made to common design variables produce a beneficial effect to the system as a whole. Therefore

\[ Q = \alpha + \beta + \gamma. \]  

In the interest of efficiency the most accurate approximation which expresses the maximum tensile stress in each tooth in terms of the design parameters is required. Linear and quadratic forms of the objective function were investigated in Ref. (6) and it was found that the extra complexity of the nonlinear function produced a much faster rate of convergence. As such, the maximum tensile fillet stress in the sun, planet and annulus gears was represented by a quadratic function of the design variables. This polynomial is chosen to be as simple as possible by ignoring the cross-multiplication terms.

\[ \sigma_{\text{max}}(x) = \frac{n}{2} a_i x_i^2 + \frac{n}{2} a_i x_i x_j + a_i \]  

The optimization process assumes this representation of the maximum tensile fillet stresses to be accurate only within a small neighborhood of the current design point. Design vectors are optimized within these bounds and the resulting new design is re-analyzed. This process is repeated until convergence is obtained. Evaluation of each polynomial coefficient \( a_i \) is achieved by differentiation of the function at design points \( j \) and \( j+1 \) obtained at successive iterations (a linear scheme is used to start the procedure) giving

\[ \frac{\partial \sigma_{\text{max}}(x)}{\partial x_i} = 2a_i x_i + a_i + \sigma_{i+1} \]  

\[ \frac{\partial \sigma_{\text{max}}(x)}{\partial x_i} = 2a_i x_i + a_i + \sigma_{i+1} \]  

\[ i = 1, \ldots, n. \]

Therefore

\[ a_i = \left( \frac{\partial \sigma_{\text{max}}(x)}{\partial x_i} \right)_{x_i} - \frac{\partial \sigma_{\text{max}}(x)}{\partial x_i} \]  

\[ a_i = \left( \frac{\partial \sigma_{\text{max}}(x)}{\partial x_i} \right)_{x_i} - \frac{\partial \sigma_{\text{max}}(x)}{\partial x_i} \]  

and

\[ a_{i+1} = \left( \frac{\partial \sigma_{\text{max}}(x)}{\partial x_i} \right)_{x_i} - 2a_i x_i. \]

The analysis of the gear design at each iteration is carried out using the finite-element method. This gives the values of \( \sigma_{\text{max}} \) for the sun, planet and annulus at the current design point \( x^j \). By changing each design parameter \( x_i \) by \( \Delta x_i \) in turn and re-evaluating this design with finite elements will yield \( \sigma_{\text{max}}(x_i + \Delta x_i, \ldots, x_i + \Delta x_i, \ldots, x_i) \). A finite difference formula is then used to determine the first derivative coefficient, i.e.,

\[ \frac{\partial \sigma_{\text{max}}(x)}{\partial x_i} = \sigma_{\text{max}}(x_1, \ldots, x_i - \Delta x_i, x_i + 2\Delta x_i, \ldots, x_n) - \sigma_{\text{max}}(x_1, \ldots, x_i - \Delta x_i, x_i, \ldots, x_n) \]  

\[ / \Delta x_i \]
Since several finite-element analyses are carried out in this procedure an automatic mesh generation program has been written which uses the design vector and element density specifications to define the finite-element mesh over each gear. Without this computer program the design scheme would not be practical.

From the first derivatives, coefficients $a_{k+n}$ are evaluated using Eq. (6). Second derivative terms $a_i$ are obtained from Eq. (5) to specify the complete quadratic form of the objective function. Thus the objective functions can be formulated directly from the finite-element results.

4. Formulation of Constraints

The parameters which define the elements of the epicyclic system must be constrained to ensure that the requirements of the gear system design are achieved. Two types of constraint are applied: behavioral (implicit) constraints and side (explicit) constraints.

The behavioral or implicit constraint forms cannot generally be expressed as functions of the design vector, for example, limits imposed on the stresses which can be determined only by a full structural analysis.

Side constraints are expressed explicitly as algebraic functions of the design variables and can be evaluated for any design vector $x$. For the optimization of the epicyclic gear two forms of side constraints were imposed: the first set holds the design between fixed limits and ensures functionality, the second set of constraints is used to control the mathematical optimization technique.

Within the first set of side constraints are the barrier constraints normal to the $x_i$ axis used to define the limits of the design space, i.e.,

$$x_{	ext{min}} \leq x_i \leq x_{\text{max}} \quad i = 1, \ldots, n. \tag{7}$$

The numerical optimization scheme will form a sequence of mathematical models by minimizing the objective function near the current ($j$th) design point $x^j$. As these approximations to the objective function are valid only in the locality of this design point, restrictions must be placed on the range of each design variable within which the objective function is considered sufficiently accurate. This move limit method is achieved by placing restrictions on the design vector $x$ such that the step length taken between the current design point $x$ and the updated design point $x^j+\Delta x_i$ is held within some prescribed limit, i.e.,

$$x_i - |\Delta x_i| \leq x^j + \Delta x_i \leq x_i + |\Delta x_i| \quad i = 1, \ldots, n. \tag{8}$$

where $\Delta x_i$ and $\Delta x_i$ are the upper and lower step lengths for the variable $x_i$ at the design point $j$, respectively.

4.1 Application-dependent constraints

In addition to the sets of constraints which provide the absolute limits within which each design parameter can vary and those which govern the convergence of the mathematical optimization process, it is necessary to include conditions to ensure that a practical gear design results. Problem-dependent constraints and compatibility constraints are utilized for this purpose. Problem-dependent constraints are concerned with the precise nature of the design specification. For the epicyclic unit a reduction ratio which lies between prescribed limits is required. It is also necessary that the three planet gears are equally spaced around the carrier. These are features specific to this design problem and are not absolute requirements of all gear systems. Conditions which are necessary or desirable in all planetary gear systems are included in the compatibility constraints. These ensure that the design is practicable and that the tooth forms can be manufactured and will mesh together to form a working system. Placed in this category of constraints are conditions which guarantee the following:

(i) The basic rack exists.
(ii) Constant rotary motion is achieved.
(iii) Tooth profiles are generated without undercut.
(iv) Tooth tip thickness is acceptable.
(v) The specified clearance exists between engaging tooth forms.
(vi) Tip interference does not occur between the internal tooth and mating pinion forms.
(vii) Interference does not occur between annulus/planet or planet/sun teeth in contact.
(viii) Positive backlash exists for each mating tooth pair.
(ix) Sun/planet and planet/annulus center distances are equal.

These constraints are developed in detail in the following two sections; the first describes the problem-dependent constraints, the second defines the compatibility constraints.

4.2 Problem-dependent constraints

4.2.1 Acceptability of the reduction ratio

For a fixed annulus planetary system with the drive applied to the sun gear and output taken from the carrier, the ratio of the speed of the sun to that of the carrier is

$$r_{ac} = \frac{Z_s + Z_a}{Z_s} \tag{9}$$

As this must fall between the prescribed limits of $r_{\text{min}}$ and $r_{\text{max}}$, the following inequality constraints must be satisfied:
4.2.2 Equal spacing of the planets

For equal load sharing between the three planets they must be equally spaced on the carrier. If the first of these planets is located in position and the carrier rotated through one third of a revolution with the sun wheel fixed, then this position is that at which the next planet is to be inserted. It will only be possible to assemble the second planet if the teeth on the annulus have advanced an integer number of pitches. The annulus will have rotated through
\[
\frac{1}{3} \left( 1 + \frac{Z_e}{Z_a} \right) \text{ revolution,}
\]
and since one annulus revolution corresponds to \( Z_e \) pitches,
\[
\frac{Z_a + Z_e}{3} \text{ must be an integer.}
\]

4.3 Compatibility constraints

4.3.1 The basic rack exists

Four of the parameters used to define the design vector, \( \alpha \), \( m_r \), \( r_{\phi} \), and \( h_{\phi} \) relate to the straight-sided rack which is used to generate the gear. As shown in Fig. 2 the conditions \( m_r > 0 \), \( h_{\phi} > 0 \), \( r_{\phi} > 0 \) and \( \alpha > 0 \) are prerequisites of the rack. It is also necessary that some distance exists between teeth at the root circle (c). Expressing c in terms of \( h_{\phi} \) and \( \alpha \) gives the first compatibility constraint:
\[
\frac{\alpha}{4} - h_{\phi} \tan \alpha \geq 0
\]

It is also evident that the rack tip radius cannot be greater than a value \( r_{\phi} \) which prescribes a completely circular rack tooth tip. Therefore it is necessary to add \( r_{\phi} \leq r_{\phi} \) to the constraints. This is achieved by expressing \( r_{\phi} \) in terms of \( r_{\phi} \), \( h_{\phi} \), and \( \alpha \):
\[
\frac{\alpha \cos \alpha}{4(1 - \sin \alpha)} - \frac{h_{\phi} \sin \alpha}{1 - \sin \alpha} - r_{\phi} \geq 0
\]

4.3.2 Constant rotary motion is achieved

For engaging gear teeth to produce constant rotary motion, the contact ratio, that is the ratio of the contact path length to the base pitch, must achieve a value greater than or, in the limiting case, equal to unity. In the epicyclic system, gear tooth contact takes place between two sets of gear pairs, the sun/planet and the planet/annulus. Uniform rotary motion for the system requires that acceptable contact ratios are achieved for both sets of contact conditions.

The situation for external gears in contact operating at extended centers, applicable to the sun/planet interface, is shown in Fig. 3 with a contact ratio given by
\[
C_{sp} = \frac{A_s N_s + A_p N_p - N_a N_e}{2 m_r \cos \alpha} = \frac{1_s + 1_p - 1_f}{2 m_r \cos \alpha},
\]

where
\[
1_s = \frac{A_s N_s}{A_s N_s + A_p N_p - N_a N_e} = \left( \frac{Z_e}{2} \cos \alpha \right)^\frac{1}{2},
\]
and
\[
1_p = \frac{A_p N_p}{A_s N_s + A_p N_p - N_a N_e} = \left( \frac{Z_e}{2} \cos \alpha \right)^\frac{1}{2},
\]

1_s and 1_p being the length along the path of contact from the base circle to the addendum circles for the sun and planet gears, respectively, and \( l_f \) the length of the common tangent to the base circles.

The contact ratio for an internal gear and pinion as in the planet/annulus situation is given by
\[
C_{sp} = \frac{A_a Z_a}{P_s} = \frac{1_s - 1_p + l_f}{2 m_r \cos \alpha},
\]
as shown in Fig. 4.

![Fig. 2 Gear tooth parameter definitions](image)

![Fig. 3 External gear pair contact conditions](image)
1. If \( A \), \( A_2 \), \( N \), \( l \), and \( N_0 \) are defined above by Eq. (16), \( l = a_n \) is given by

\[
l = a_n N_0 = m_t \left[ \left( \frac{Z - A_1 + \delta A_1 - a_0}{2} \right)^2 \cos \alpha_i \right]^{1/2}
\]

and

\[
l = A_2 N_0 = m_t \left[ \left( \frac{Z - A_2 + \delta A_2 - a_0}{2} \right) \sin \alpha_i \right]
\]

(20)

where \( \alpha_i \) is given by

\[
\alpha_i = \cos^{-1} \left[ \left( \frac{Z - A_2 + \delta A_2 - a_0}{Z - A_2 + \delta A_2 - 2 s_{s_0} - a_0} \right) \cos \alpha_i \right]
\]

(21)

thus providing the constraints

\[
C_{ns} \geq 1.0
\]

(22)

The constraints must also ensure that the annulus/cutter contact ratio, \( C_{ns} \), achieves an acceptable value. \( C_{ns} \) can be calculated from Eq. (18) by replacing the planet gear variables with the equivalent variables which define the cutter geometry. It must be noted that \( s_{s_0} - a_0 = 0 \) and that \( \alpha_i \) is evaluated from

\[
C_{ns} = \frac{Z}{2} \sin \alpha_i - \frac{\left[ h_{01} - s_{s_0} \right]}{\sin \alpha_i} \geq 0
\]

(25)

A check is also carried out on the contact ratios between the rack and the sun, planet, and pinion cutter. These also must have values greater than unity. The contact ratio between any rack and pinion is given by

\[
r_{r} = \frac{Z}{2} \sin \alpha_i - \left[ \frac{Z}{2} + s_{s_0} + A_0 \right] \cos \alpha_i
\]

(24)

By inserting the correct gear tooth parameters, this formula provides the ratios between the rack and the sun, planet, and pinion cutter.

4.3.3 Tooth profiles are generated without undercut To ensure that a full working depth is achieved for all gear teeth, it is necessary to check that undercutting has not occurred during the generation process. As an involute curve is not defined below the base circle, conjugate tooth action cannot take place below it. It the straight-sided rack acts against the involute curve of the gear and the corners extend too far below the base circle, interference results and the tooth profile is undercut. To avoid this, the corner of the rack or equivalent hob form are only allowed to extend a limited distance below the base circle.

When teeth are modified by displacing the rolling line from the median of the generating rack, the active profiles of the teeth are formed by a different portion of the involute curve. This property can be used to ensure that undercutting is eliminated. Figure 5 shows the general form of the generating rack. Involute profiles will be generated down to a point where the horizontal line through \( c_1 \) passes through the interference point \( I \) providing an involute curve down to the base circle. If the line through \( c_1 \) passes the interference point, undercutting occurs, destroying part of the previously generated involute. For the sun and planet gears undercut is avoided provided that the following two constraints are satisfied:

\[
\frac{Z}{2} \sin \alpha_i - \left[ h_{s} - s_{s_0} - (1 - \sin \alpha_i) r_{r} \right] \geq 0
\]

(25)

and

\[
\frac{Z}{2} \sin \alpha_i - \left[ h_{s} - s_{s_0} - (1 - \sin \alpha_i) r_{r} \right] \geq 0.
\]

For internal gears, since the involute profile cannot exist below its base circle,

\[
r_{r} = r_{r'} \leq 0,
\]

giving

\[
\left( \frac{Z - A_2 + s_{s_0} - a_0}{2} \right) - \left( \frac{Z}{2} \cos \alpha_i \right) \geq 0.
\]

(27)

4.3.4 The tooth tip thickness is acceptable

The tip width on all teeth must be greater than zero.

Fig. 4 Internal gear pair contact conditions

Fig. 5 Undercutting conditions for a rack-generated gear
In practice it must be greater than a minimum prescribed value \( w_t = w_{0\, m} \). Inadequate crest widths may leave the tooth liable to fail. This is especially true if the gears are case-hardened; the tips then become over-carburized and brittle. For external gears the tip width, \( w_t \), is

\[
\begin{align*}
  w_t &= 2m_t \cos^{-1} \left( \frac{r_u \tan \alpha_t}{Z} \right) + \text{INV} \alpha_t \\
  &= -\text{INV} \left( \cos^{-1} \left( \frac{r_u}{r_{tip}} \right) \right),
\end{align*}
\]

and for internal gears

\[
\begin{align*}
  w_t &= 2m_t \cos^{-1} \left( \frac{r_u \tan \alpha_t}{Z} \right) - \text{INV} \alpha_t \\
  &= +\text{INV} \left( \cos^{-1} \left( \frac{r_u}{r_{tip}} \right) \right).
\end{align*}
\]

These are strictly the thicknesses measures around the tip circle but this is indistinguishable from the chord in practical terms. Thus

\[
\begin{align*}
  w_t &\geq w_{\text{min}}. \quad \text{(30)}
\end{align*}
\]

For the example described in this paper, the minimum tip thickness \( w_{\text{min}} \) is taken to be 0.25 \( m_t \).

### 4.3.5 The specified clearance exists between engaging tooth forms

For gears to rotate together correctly there must be a positive clearance between the tooth tips and the root circles of the gear and pinion. The clearance obtained is dependent upon the center distance of the gear pair. In this example the center distance is fixed by increasing the standard center distance by the sum of the profile modifications. This produces a tooth pair which engages at the full working depth. The clearances as illustrated in Fig. 6 are

\[
\begin{align*}
  c_{ps} &= h_{ps} - a_{ps} \geq c_{\text{min}} \quad \text{(31)} \\
  c_{sp} &= h_{sp} - a_{sp} \geq c_{\text{min}}, \quad \text{(32)} \\
  c_{ps} &= h_{ps} - a_{ps} \geq c_{\text{min}} \quad \text{(33)} \\
  c_{sp} &= h_{sp} - a_{sp} \geq c_{\text{min}}. \quad \text{(34)}
\end{align*}
\]

\( c_{\text{min}} \) is the factor specified for the minimum acceptable clearance and has a value of 0.2 for the epicyclic system under consideration.

### 4.3.6 Tip interference does not occur between the internal tooth and mating pinions

When the difference in the number of teeth for the planet and annulus is small, tip interference may occur. Figure 7 shows the condition of meshing for the annulus and planet. The pinion and internal gear are shown with their center line coinciding with the line of centers \( OQ \). On rotation of the planet, its tip \( a_t \) will travel and cross the tip circle of the annulus at \( a_s \). During this time point \( b_1 \) on the internal gear will travel to position \( b_2 \). If tip interference is not present \( b_2 \) must be in front of \( a_2 \).

The rotation of the planet pinion \( \delta_p \) while the tip moves from \( a_t \) to \( a_s \) is

\[
\delta_p = a + \gamma_p.
\]

The corresponding rotation of the annulus is

\[
\delta_a = (a - \gamma_a) \frac{Z_p}{Z_a}, \quad \text{(35)}
\]

where

\[
\begin{align*}
  a &= \cos^{-1} \left[ \left( \frac{Z_a}{2} + s_{oa} - a_{oa} \right)^2 - \left( \frac{Z_p}{2} + s_{op} + a_{op} \right)^2 \\
  &- \left( \frac{Z_a}{2} - s_{oa} \right)/ \left( \frac{Z_a}{2} + s_{op} + 2a_{op} \right) \right] \\
  \gamma_p &= \frac{Z_a}{2} + 2s_{op} \tan \alpha_t + \text{INV} \alpha_t - \text{INV} \alpha_p \quad \text{(36)}
\end{align*}
\]

\[
\begin{align*}
  \gamma_a &= \cos^{-1} \left[ \frac{Z_p \cos \alpha_t + 2s_{op} + 2a_{op}}{Z_a + 2s_{oa} + 2a_{oa}} \right],
\end{align*}
\]

The angular advance of point \( \beta_2 \) is \( \delta_a + \gamma_a \) so that tip clearance \( c_{lp} \) can be evaluated by

\[
\begin{align*}
  c_{lp} &= m_t \left( \frac{Z_a}{2} + s_{oa} - a_{oa} \right) \left( \delta_a + \gamma_a - \beta \right), \quad \text{(37)}
\end{align*}
\]

where

\[
\begin{align*}
  \gamma_a &= \frac{Z_a}{2} + 2s_{oa} \tan \alpha_t + \text{INV} \alpha_t - \text{INV} \alpha_a \\
  \alpha_a &= \cos^{-1} \left[ \frac{Z_p \cos \alpha_t}{Z_a + 2s_{oa} - 2a_{oa}} \right]
\end{align*}
\]

and

![Fig. 6 Gear tooth clearances](image)

![Fig. 7 Determination of tip clearance](image)
\begin{equation}
\beta = \sin^{-1}\left[\frac{(Z_a + Z_p + 2s_p) \sin \alpha}{Z_a + 2s_a - 2a_o}\right].
\end{equation}

Hence, the additional constraint
\begin{equation}
c_b \geq 0.0.
\end{equation}

Tip interference can also occur between the annulus and pinion cutter during the annulus machining process. The effect would be to remove part of the previously generated profile. To avoid this
\begin{equation}
c_b \geq 0.0.
\end{equation}

By replacing all variables which relate to the pinion, gear, and \( \beta \) defined as for Eq. (37) above with the exception that
\begin{equation}
\alpha = \cos^{-1}\left[\left(\frac{Z_a}{2} + s_a - a_o \cos \alpha_1\right)^2 - \left(\frac{Z_a}{2} + a_o \cos \alpha_1\right)^2\right]
\end{equation}
\begin{equation}
\times \left(\frac{Z_a - Z_0}{2} \cos \alpha_1\right) / \left(\frac{Z_a}{2} + a_o \cos \alpha_1\right),
\end{equation}

where \( \alpha \) is given in Eq. (23).

4.3.7 Interference does not occur between annulus/planet or planet/sun teeth in contact

Interference between the engaging members of the epicyclic system along the line of action also requires elimination. For each tooth pair in contact, interference will not occur provided that two conditions are satisfied. The annulus/planet contact is illustrated in Fig. 4. For the internal gear pair the conditions are as follows.

\textbf{CONDITION 1}
\begin{equation}
A_0N_0 \geq C_0N_0 + N_pN_x
\end{equation}
\begin{equation}
C_0N_x = m_t \left(\frac{Z_a}{2} \sin \alpha_1\right) - \left(\frac{h_u - s_p - (1 - \sin \alpha_1) \gamma_v}{\sin \alpha_1}\right)
\end{equation}

Expressions for \( A_0N_0 \) and \( N_pN_x \) are given by Eqs. (19) and (20), respectively.

\textbf{CONDITION 2}
\begin{equation}
A_pN_0 \geq A_0N_0 + N_pN_x
\end{equation}
\begin{equation}
A_0N_x = m_t \left(\left[\frac{Z_a}{2} - Z_p + s_p - s_a \cos \alpha_1\right] \sin \alpha_1\right) + \left[\left(\frac{Z_a}{2} + a_o - r_a \cos \alpha_1\right)^2 - \left(\frac{Z_a}{2} + a_o \cos \alpha_1\right)^2\right] \sin \beta.
\end{equation}

\( \alpha \) is obtained from Eq. (23).

When two external gears mesh together, as shown in Fig. 3, the conditions become

\textbf{CONDITION 1}
\begin{equation}
A_0N_0 \geq C_0N_0 + N_pN_x
\end{equation}
\begin{equation}
C_0N_0 \text{ and } C_0N_x \text{ are given by Eqs. (15) and (40), respectively.}
\end{equation}

\( \alpha \) is the working pressure angle given by
\begin{equation}
\alpha \text{ is the working pressure angle given by}
\end{equation}
\begin{equation}
\alpha = \cos^{-1}\left[\left(\frac{Z_a + Z_p}{Z_a + Z_p + 2(s_a + s_p)}\right) \cos \alpha_1\right].
\end{equation}

\textbf{CONDITION 2}
\begin{equation}
A_0N_0 - C_0N_0 \geq A_0N_0
\end{equation}

\( A_0N_0 \) is given by Eq. (16) whilst \( C_0N_0 \) takes exactly the same form as Eq. (40) with the planet gear parameters.

4.3.8 Positive backlash exists between each mating pair

Backlash is the shortest distance between the trailing, nonworking surfaces of meshing teeth. Backlash (measured perpendicular to the tooth flanks) is provided by making appropriate adjustments to the tooth thickness of either or both teeth of the mating pair. When two nonmodified gears mesh at standard center distance, zero nominal backlash exists. This would normally be subject to tolerances but as the surface finish and gear alignment are subject to the manufacturing process chosen and cannot be included in the finite-element model a nonzero nominal backlash, \( b \), is assumed to be acceptable. Modified external gears operating at extended centers have a nonzero nominal backlash \( b \) given by
\begin{equation}
b = m_t \left[\left(\frac{Z_a}{2} + Z_p\right) \cos \alpha_1(\text{INV} \alpha - \text{INV} \alpha_0)\right] - 2(s_a + s_p) \sin \alpha_1,
\end{equation}

where \( \alpha_0 \) is the working pressure angle given by Eq. (43).

For the annulus/planet meshing conditions the backlash is
\begin{equation}
b = m_t \left[\left(\frac{Z_a}{2} - Z_p\right) \cos \alpha_1(\text{INV} \alpha - \text{INV} \alpha_0)\right] - 2(s_a - s_p) \sin \alpha_1.
\end{equation}

In this case the working pressure angle \( \alpha_0 \) is represented by Eq. (21).

4.3.9 Sun/planet and planet/annulus center distances are equal

For gears acting at extended centers or standard centers depending on the modification factors, the center distance between the sun and planet is
\begin{equation}
d = \frac{m_t}{2} \left(\frac{Z_a}{2} + Z_p + 2(s_a + s_p)\right).
\end{equation}

For the planet and annulus the center distance is
\begin{equation}
d = \frac{m_t}{2} \left(\frac{Z_a}{2} - Z_p - 2(s_a - s_p)\right).
\end{equation}

In the epicyclic system these distances must be equal giving the equality constraint
\begin{equation}
Z_a + 2Z_p = 2(s_a + 2(s_a + s_p)) = 0.0.
\end{equation}

5. Sequential Optimization Algorithm

1. Select some feasible initial design vector \( x^0 \). Generate sun, planet and gear tooth geometries and superimpose finite-element meshes upon each using the automatic mesh generation programs for external
and internal gears. Add the relevant data for material properties and loading to each finite-element mesh and evaluate $\sigma_x$, $\sigma_y$, and $\sigma_z$, the maximum tensile fillet stresses in the sun, planet, and annulus, respectively.

2. Choose the forms of the stress functions $\sigma(x')$, $\sigma_y(x')$ and $\sigma_z(x')$ which most accurately represent the behavior of the maximum tensile fillet stresses in terms of the design variables.

3. Combine the gear tooth stress functions to form the overall objective function

$$Q(x') = \sigma_x(x') + \sigma_y(x') + \sigma_z(x').$$

4. For generality let the stress functions be interchangeable so that $\sigma_x(x'), \sigma_y(x')$ and $\sigma_z(x')$ become $\sigma_i(x')$, $i = 1, 2, 3$ placed in order such that $\sigma_1(x') \geq \sigma_2(x') \geq \sigma_3(x').$

Calculate the design derivatives for each variable of each gear tooth and formulate the stress functions, then define the objective function $Q(x').$

5. Incorporate the compatibility constraints, barrier constraints, move limit constraints and problem-dependent constraints to completely specify the optimization problem.

6. Minimize the total stress function $Q(x')$ such that imposed inequality and equality constraints $g_i(x') \geq 0$, $i = 1, ..., m$ and $h_k(x') = 0$, $k = 1, ..., n$ are satisfied along with the integer function $I(x')$, to obtain new design vector $x^{(n)}$. 

7. Generate finite-element meshes for the tooth forms represented by the new design vector and use the finite-element method to evaluate the maximum tensile fillet stresses $\sigma_x^{(n)}$, $\sigma_y^{(n)}$ and $\sigma_z^{(n)}$.

8. Check for convergence. If $Q^{(n)} - Q^l \geq 0$ and the location of $\sigma_x^{(n)}$, $\sigma_y^{(n)}$ and $\sigma_z^{(n)}$ have not changed, return the move limit constraints and return to step 6. If the location of the maximum fillet stress in any of the component gear forms has moved to an adjacent sampling point then continue to step 10.

9. Check that the implicit constraints, $\sigma_i - \sigma_i^{(n)} \geq 0$ for $i = 1, 2, 3$, are satisfied. If not, return the move limit constraints and return to step 6.

10. If the sequence of design vectors have reached a point where the corresponding sequence of values of $Q^l$ has converged to a minimum or is below the critical value specified for the application then stop. Otherwise re-enter the algorithm at step 3 for the next iteration of the design process with a new updated design vector $x^{(n)}$.

6. Solving the Optimization Problem

Minimizing an objective function derived from the fillet stress functions of three gear teeth requires additional to ensure that a “trade-off” situation where a parameter modification decreases the fillet stress in one gear but increases it in another will be acceptable only when the result is beneficial to the whole system.

The final optimization problem is to minimize the objective function subject to 67 inequality constraints and 2 equality constraints, variables being both real and integer values. The inclusion of the integer variables together with the highly nonlinear constraint forms excluded the more sophisticated optimization algorithms developed for modern digital computers.

The method used to solve the problem was a simple grid-sampling method. Each parameter of the design vector was examined to find the amount by which it would be required to change before a change in the manufactured design would become practicable. A lattice of possible design points was then constructed in the current design space. The values of the objective function for each point on the feasible design space lattice were then directly compared. However, for this problem there are nine design variables, and one of the disadvantages of such an approach is the CPU time required, even on powerful modern computers, to carry out the search process. If the design function and all of the 69 constraints are evaluated at each of the lattice design points then this takes a considerable amount of computer power. Even a moderate design lattice with only seven acceptable values for each of the nine design parameters results in the objective function and the constraints being evaluated 40 353 607 times—a task which cannot be completed in a realistic time scale. To reduce this to a reasonable problem, the two constraint functions which were most frequently violated, the equality constraint (46) and the integer function constraint (11), were first evaluated and checked to ensure that the present design vector is feasible. The objective function and the remaining constraints were only then evaluated providing these initial checks were satisfied.

7. Epicyclic System Optimization

7.1 Analysis of the initial design

Finite-element analyses were performed on all gears in the system to obtain their maximum tensile fillet stress. Examining those predicted at the fillet surface nodal points results in maximum values of 260 N/mm², 238 N/mm² and 221 N/mm² in the planet, sun and annulus, respectively. Since the Gauss points located near the tooth surface have been shown to reflect the trends in the surface stresses and are more accurate, they have been used in the optimization. For the initial design the Gauss point stresses were 215, 215 and 186 N/mm² in the sun, planet and annulus gears, respectively. These stresses are the values in Table 1 which describes the results obtained during the optimization process for each of the design vectors produced. The initial design is iteration 0.
Table 1  Variation of the design parameters during the optimization

<table>
<thead>
<tr>
<th>Parameter values</th>
<th>Stresses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iteration</td>
<td>$x_1$</td>
</tr>
<tr>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>1</td>
<td>18</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
</tr>
</tbody>
</table>

7.2 Design optimization
To utilize the quadratic function to represent the gear tooth fillet stresses requires design derivatives generated at two design points. In order to obtain this information, the first step in the design optimization was carried out using linear representation. Subsequent iterations then employed the quadratic form of the objective function. Using this scheme the optimal design was produced in 3 iterations as shown in Table 1 with the maximum Gauss point tensile fillet stress in every component being reduced from 215 N/mm$^2$ to only 103 N/mm$^2$. The most highly stressed component in the system alternated between the two external gear forms with the planet being the most highly stressed in the final design. Maximum tensile fillet stresses for individual gear teeth and the overall objective function are shown at each iteration in Fig. 8.

The value assigned to the design variables at each step in the design process, indicated in Table 1, successfully reduced the maximum stress in the epicyclic system. This achievement could be attributed to one or both of the following factors. Firstly, the design changes made by the optimization procedure may have produced gear teeth which are stronger and more able to support high loading without the fillet stress intensities becoming unacceptable high.

Secondly, the factor which will affect the fillet stress intensities is the value of the load applied to transmit the necessary torque. The load applied to each tooth is dependent on both the moment arm and the torque and the working pressure angle between the teeth in contact. Both moment arm and the torque are determined by the design variables, with the moment arm being a function of $Z_s$ and $m_i$, and the working pressure angle a function of $Z_s$, $Z_p$, $s_{sp}$, $s_{sp}$ and $\alpha_i$ for the sun/planet and $Z_p$, $Z_s$, $s_{sp}$, $s_{0s}$ and $\alpha_i$ for the planet/annulus. Fillet stresses will be reduced if either the load moment arm increases or the working pressure angle decreases.

Load applied to the final design solution to achieve the required torque was a 33.5 percent reduc-

![Fig. 8 Variation of stress for each iteration in the optimization process](image)

When compared to that of the initial design, producing a corresponding reduction of gear stress which is expected to be of the same order of magnitude.

At each iteration the tensile fillet stresses in the sun, planet and annulus gears were factored to indicate the value they would have attained if the load were equal in magnitude to that of the initial design. This showed that the stress in each gear would still steadily decrease during the optimization. Therefore the design variables have been manipulated in a way which increases the strength of the tooth design in addition to reducing its load. Even with the load scaled in this way, stresses in the sun, planet and annulus were reduced by 29, 28 and 33 percent, respectively, when compared to their initial values.

The percentage improvement due to reduced moments and increased gear root strength indicates that both factors are equally significant to the overall reduction in stress in the final gear design.

The optimization process for the epicyclic gear system did not produce a sequence of objective function values which converged to a minimum point on
the plot shown in Fig 8. It appears that further improvements in the design could be produced. However, any change in the design vector which reduced the overall fillet stress function violated the design constraints and the optimization process was halted.

8. Conclusions

(1) The solution of the optimization problem defined by an objective function which was the sum of functions representing the maximum fillet stresses in sun, planet and annulus gears proved to be successful in minimizing the maximum bending stress in an epicyclic gear system. Maximum tensile fillet stress in the system occurred in the planet gear teeth. This was reduced by 54 percent from 260 to 120 N/mm² for surface stresses in only 3 iterations. Sun and annulus gear teeth also experience decreases in their maximum tensile fillet stresses of 50 and 58 percent respectively.

(2) Utilization of this type of design process to allow the optimum set of design parameters to be selected could significantly improve the life expectancy of gear units. If other factors were more critical for an application, such as surface contact stresses, then a similar approach could be adopted with a different objective function.

(3) It is generally well known which parameters to increase or decrease in order to reduce the fillet stress intensities. This algorithm provides the knowledge of how much each parameter should be changed without violating the conditions required for gear manufacture and operation. Where a change in a parameter affects the stresses in a manner which is beneficial for one tooth form but detrimental for another a trade-off situation results. Within the automatic algorithm proposed here this trade-off situation is automatically resolved to produce the most beneficial effect for the entire gear system.

References

(1) Shotter, B. A., A New Approach to Gear Tooth