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SOLUTIONS TO A CLASS OF MULTIDIMENSIONAL SPDES

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Abstract

In this paper, we consider a class of multi-dimensional SPDEs of parabolic type with space-time white noise. We discretize the space-time white noise to independently identically distributed time white noise located on configuration space and seek the solution to the non-linear SPDE with such a family of discretized time noises. Then we prove that the laws of the above solutions are tight when the sum over the configuration space of the above family of time noises tends to space-time white noise. Finally we show that any limiting law satisfies the desired SPDE.

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1 Introduction

The study of the multi-dimensional nonlinear stochastic partial differential equation (SPDE) is a difficult problem. The main difficulty is that the solution is a distribution-valued process (see, for example, [26], [11], [20], [1], [19]). Let us consider the following equation for \( x \in R^d \),

\[
\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta u(t, x) + F \cdot \dot{W}(t, x),
\]

(1.1)

where \( F \) is certain functional depending on the solution and \( \dot{W}(t, x) \) is the space-time white-noise on \( R^+ \times R^d \). It is well-known that if we smooth the noise just a little, then we can get nice solutions (see e.g. [2], [25]). However, for equation (1.1) with space-time white noise in multidimension \((d \geq 2)\), there are only results for the linear case where \( F = u(t, x) \) (see [16], [18], [2], [25], [11]) and the case \( F^2 = u(t, x) \) (in which the bracket process of the martingale part is linear in \( u \)). The solution is a distribution in the former case unless some regularity condition on the noise is assumed. In the latter case, the solution is just a measure-valued random variable, which is the density of measure-valued branching processes ([4], [6], [22]). The multi-dimensional situation \((d \geq 2)\) is very different from the one-dimensional case \((d = 1)\) (see [26], [17], [20], [24] for references). We will consider the non-linear case which can be formally written as \( F^2 = G(u(t, x)dxdt)^2u(t, x) \). This form of \( F \) comes naturally as we know up to now the best form of solution is just measure-valued.

In order to give precise description, let us introduce several notations first. Let \( B([0, t] \times R^d) \) be the Borel field of subsets of \([0, t] \times R^d\). For any Borel subset \( B \in B \), define \( W(B) = \int_B W(ds, dx) \) as a generalized Gaussian zero mean random field defined on a probability space \((\Omega, \mathcal{F}, P)\), whose covariance function is \( EW(B)W(C) = \mu(B \cap C) \), where \( \mu \) denotes the Lebesgue measure on \( R^+ \times R^d \). Denote by \( \mathcal{F}_t \) the completion of \( \sigma(W(B) : B \in B([0, t] \times R^d)) \), and let \( \mathcal{S} \) be the \( \sigma \)-field of \( \mathcal{F}_t \)-progressively measurable subset of \( \Omega \times R^+ \).

The following definition of the Itô integral with respect to the white noise is borrowed from [20]: if \( f \in L^2(\Omega \times R^+ \times R^d, \mathcal{S} \otimes \mathcal{B}(R^d), P(d\omega)dt dx) \), define the process \( \int_0^t \int_{R^d} f(s, x)W(ds, dx) \) as a continuous martingale whose associated increasing process is given by \( \int_0^t \int_{R^d} f^2(s, x)dx ds \).

In this paper, a function \( G : [0, +\infty) \rightarrow R \) is said to be positive (negative respectively) if \( G(v) > 0 \) \((G(v) < 0)\) for all \( v \geq 0 \).
$C_k^0(R^d)$ stands for the space of $k$ times continuously differentiable functions with compact support, if $k = 0$ we use the notation $C_0(R^d)$.

Our main result is

Theorem 1.1 Given a positive (or negative), bounded and Lipschitz continuous function $G$ and a function $\phi \in C_0((0,T] \times R^d)$ for some $T > 0$. Let $u_0$ be a non-negative and Lipschitz continuous $C_0(R^d)$ function. Then there is a measure-valued random variable $V(dx, ds)$ such that for any $q \in C^2_0(R^d)$ there is a semimartingale $Q_t^{(q)}$ satisfying for all $t \leq T$ the relations

$$\int_0^t \int_{R^d} q(x)V(dx, ds) = \int_0^t Q_s^{(q)} ds,$$  \hspace{1cm} (1.2)

and

$$Q_t^{(q)} - \int_{R^d} q(x)u_0(x)dx = \frac{1}{2} \int_0^t \int_{R^d} \Delta q(x)V(dx, ds) + M^{(q)}(t),$$  \hspace{1cm} (1.3)

where $M^{(q)}(t)$ is a martingale with respect to $\mathcal{F}_t$ with its bracket given by

$$\langle M^{(q)}, M^{(q)} \rangle_t = \int_0^t \int_{R^d} \phi(s-r, x-z) V(dx, ds) V(dx, ds).$$  \hspace{1cm} (1.4)

Our proof consists of several steps. We first discretize and localize the space-time white noise and consider the equation corresponding to the discretized noise (or so-called colored noise). For the stochastic reaction diffusion equations with colored noise, we prove there exists a mild solution which is continuous in space and time for almost all $\omega$, moreover the solution can be represented by the Feynman-Kac formula. While we denote by $\{u_{n,k}(t, x)\}$ the discretized solutions, we show that the laws of $\{u_{n,k}(t, x)dxdt\}_{n,k}$ form a tight sequence and any of their limits is the law of some measure-valued random variable. Then we identify the limit as the weak solution of the desired nonlinear SPSE given by the integral equations (1.2)–(1.4) with space-time white noise.

## 2 Discretizing the space white noise

Decompose $R^d$ into the squares:

$$L_{k_1, \ldots, k_d}^{(n)} = \{(x_1, x_2, \ldots, x_d) : \frac{1}{n}k_j \leq x_j \leq \frac{1}{n}(k_j + 1), \text{ for } j = 1, 2, \ldots, d\},$$

$$k_1, k_2, \ldots, k_d = 0, \pm 1, \pm 2, \ldots.$$
Denote $\mathcal{K} = (k_1, k_2, \ldots, k_d)$ and define

$$W_n(t, x, \omega) = n^2 \sum_{\mathcal{K}} a(x, \mathcal{K}) \dot{w}(t, \mathcal{K})$$  \hspace{1cm} (2.1)$$

where $w(t, \mathcal{K})$, $\mathcal{K} \in \mathbb{Z}^d$, are independent Wiener processes of 1-parameter $t$ on a probability space $(\Omega, \mathcal{F}, P)$ and $\dot{w}(t, \mathcal{K})$ denotes the Itô derivative of the Wiener process. Assume $a(x, \mathcal{K})$ is a $C^\infty$ function and satisfies the following conditions: 1) $0 \leq a(x, \mathcal{K}) \leq 1$; 2) for any $\mathcal{K}$ such that $L_{\mathcal{K}}^n \subset \{ x \in \mathbb{R}^d : |x_i| \leq n, i = 1, 2, \ldots, d \}$, $a(x, \mathcal{K})$ has a compact support in $L_{\mathcal{K}}^n$ and $\int_{L_{\mathcal{K}}^n} a(x, \mathcal{K}) dx = \frac{1}{n^2} - \frac{1}{n^2}$; 3) for all other $\mathcal{K}$ i.e. $L_{\mathcal{K}}^n \not\subset \{ x \in \mathbb{R}^d : |x_i| \leq n, i = 1, 2, \ldots, d \}$, then $a(x, \mathcal{K}) = 0$ for all $x \in \mathbb{R}^d$. For simplicity we assume that all nontrivial $a(x, \mathcal{K})$ are identical except for a shift. It is easy to see that the discretized noises are concentrated in the set $\{ x \in \mathbb{R}^d : |x_i| \leq n \}$, and there is no noise outside the compact set. However, this does not pose the same restriction to the limit of $W_n$. It will be shown that the limit is the space-time white noise on $[0, +\infty) \times \mathbb{R}^d$.

To study the limit, first we calculate that for any block $B = \{(t, x_1, x_2, \ldots, x_d) : t_1 \leq t \leq t_2, a_j \leq x_j \leq b_j, \ j = 1, 2, \ldots, d\}$,

$$\int_B W_n(t, x) dx dt = \int_{[a_1, b_1] \times \cdots \times [a_d, b_d]} \int_{t_1}^{t_2} \sum_{\mathcal{K}} n^2 a(x, \mathcal{K}) w(dt, \mathcal{K}) dx$$

$$= \int_{t_1}^{t_2} \sum_{\mathcal{K}} \int_{L_{\mathcal{K}} \cap ([a_1, b_1] \times \cdots \times [a_d, b_d])} n^2 a(x, \mathcal{K}) dx w(dt, \mathcal{K})$$

$$= n^2 \sum_{\mathcal{K}} \int_{L_{\mathcal{K}} \cap ([a_1, b_1] \times \cdots \times [a_d, b_d])} a(x, \mathcal{K}) dx (w(t_2, \mathcal{K}) - w(t_1, \mathcal{K})).$$

Therefore,

$$E(\int_B W_n(t, x) dx dt)^2$$

$$= n^d \sum_{\mathcal{K}} \int_{L_{\mathcal{K}} \cap ([a_1, b_1] \times \cdots \times [a_d, b_d])} a(x, \mathcal{K}) dx \int_{L_{\mathcal{K}} \cap ([a_1, b_1] \times \cdots \times [a_d, b_d])} a(x, \mathcal{K}) dx (w(t_2, \mathcal{K}) - w(t_1, \mathcal{K})).$$

$$= n^d \sum_{\mathcal{K}} \int_{L_{\mathcal{K}} \cap ([a_1, b_1] \times \cdots \times [a_d, b_d])} a(x, \mathcal{K}) dx (t_2 - t_1)$$

$$\rightarrow (b_1 - a_1) \times (b_2 - a_2) \times \cdots \times (b_d - a_d) \times (t_2 - t_1), \text{ as } n \rightarrow \infty.$$
Denote by $\mathcal{P}$ the set of all bounded measures on $[0, T] \times \mathbb{R}^d$ and define for $V \in \mathcal{P}$ the functional

$$G(s, x, V) = G\left(\int_0^s \int_{\mathbb{R}^d} \phi(s - r, x - z))V(\,dz, \,dr)\right),$$

(2.2)

where $G$ is a positive (or negative), bounded and Lipschitz continuous function, $\phi \in C_0((0, T] \times \mathbb{R}^d)$. In the following equation, $\mathcal{G}$ is simply $G(\ldots, u_{n,k}dzdr)$. We consider

$$du_{n,k}(t, x) = \frac{1}{2} \Delta u_{n,k}(t, x)dt + \sum_K a(x, K)u_{n,k}(t, x)$$

$$\times (\frac{1}{k} + \int_{L_K} a^2(z, K)G^2u_{n,k}(t, z)dz)^{\frac{1}{2}} dw(t, K),$$

$$u_{n,k}(0, x) = u_0(x).$$

(2.3)

The coefficient of the above equation is not smooth in $u$ for $u \in \mathbb{R}$. We modify it by making the following smooth extension. For fixed $n$ and $k$, take a bounded and Lipschitz continuous function $\tilde{G}$ of two variables and introduce

$$\tilde{G}_K(t, x, u_{n,k}) = a(x, K)\tilde{G}(\int_{L_K} a^2(z, K)G^2u_{n,k}(t, z)dz)$$

$$\times (\frac{1}{k} + \int_{L_K} a(z, K)u_{n,k}(t, z)dz)^{\frac{1}{2}} dw(t, K),$$

(2.4)

in such a way that for non-negative $u_{n,k}$ (2.3) coincides with

$$du_{n,k}(t, x) = \frac{1}{2} \Delta u_{n,k}(t, x)dt + \sum_K \tilde{G}_K u_{n,k} dw(t, K).$$

(2.5)

The mild solution of the latter equation is defined as the solution of the following integral equation if exists (see [3])

$$u_{n,k}(t, x) = \int_{\mathbb{R}^d} p(y, t, x)u_0(y)dy$$

$$+ \sum_K \int_0^t \int_{\mathbb{R}^d} p(y, t - s, x)u_{n,k}(s, y)\tilde{G}_K(u_{n,k}, s, y)dydw(s, K);$$

(2.6)

here $p(y, t, x)$ is the heat kernel of the Laplacian operator $\frac{1}{2} \Delta$ on $\mathbb{R}^d$:

$$p(y, t, x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left\{-\frac{|y - x|^2}{2t}\right\}.$$
3 Nonlinear functional SPDEs with time white noise

In this section we prove the existence of non-negative solution to (2.6). For simplifying our notations, we may just assume (2.6) is of the form

\[ v(t, x) = \int_{R^d} p(y, t, x)u_0(y)dy + \sum_{K} \int_0^t \int_{R^d} p(y, t - s, x)a(y, K)\tilde{F}(v, s, y)dydw(s, K). \] (3.1)

where

\[ \tilde{F}(v, t, x) = \check{G}(\int_0^t \int_{R^d} \phi(t - r, x - z)\nu(r, z)drdz)v(t, x). \] (3.2)

Then the proof of the existence of a solution to the equation (2.5) is completely the same as that in case of (3.1), for two extra terms involved in (2.5) do not give any difficulties. Assume that \( u_0 \in C(R^d) \) is bounded and Lipschitz continuous.

The existence of the linear stochastic partial differential equations with colored noise was considered by many authors (see e.g. [23], [12]). But their results do not apply as \( \tilde{F} \) in (3.1) violates Lipschitz condition.

We will frequently use the following version of the Burkholder inequality: Assume \( f(s) \) is \( \mathcal{F}_s \)-measurable, then for any \( p \geq 1 \),

\[ E(\int_0^t f(s)dw(s))^{2p} \leq B_p t^{p - 1} \int_0^t E f^{2p}(s)ds. \] (3.3)

Here \( B_p = (2p - 1)^p \).

We also need the following estimation:

**Lemma 3.1** For any \( U \in L^2(R^d) \), and for all positive \( s < t \) and \( \delta < 1 \) the inequality holds

\[ \left| \int_{R^d} [p(x, t, y) - p(x, s, y)]\int_{R^d} p(y, \delta, z)U(z)dz \, dy \right| \leq C(t - s) \left( \int_{R^d} U^2(z)dz \right)^{\frac{1}{2}}. \]

**Proof.** Thanks to the semigroup property of the operator \( \partial/\partial t - \Delta \) it suffices to show that

\[ \left| \frac{\partial}{\partial t} \int_{R^d} p(x, t, y)U(y)dy \right| \leq C \delta^{-\frac{d}{2}} \left( \int_{R^d} U^2(z)dz \right)^{\frac{1}{2}}. \]

for all \( t \geq \delta \). By the Schwartz inequality we get

\[ \int_{R^d} p(y, \delta/2, z)U(z)dz \leq C \delta^{-\frac{d}{2}} \left( \int_{R^d} U^2(z)dz \right)^{\frac{1}{2}}. \] (3.4)
Then for any $\tilde{U}(x) \in L^\infty(R^d)$, one can obtain after simple calculations that

$$\left| \frac{\partial}{\partial t} \int_{R^d} p(x, t, y) \tilde{U}(y) dy \right| \leq C\|\tilde{U}\|_{L^\infty(R^d)}$$

for all $t \geq 1$. By the scaling arguments this implies

$$\left| \frac{\partial}{\partial t} \int_{R^d} p(x, t, y) \tilde{U}(y) dy \right| \leq C\delta^{-1}\|\tilde{U}\|_{L^\infty(R^d)}$$ (3.5)

for all $t \geq \delta/2$. Combining (3.4) and (3.5) and taking into account once again the semigroup property we derive the estimate required. 

Assume $p \geq 2$ is an integer, $F \in L^p(\Omega \times R^d)$. Then by Burkholder inequality and Lemma 3.1, for any $s$ and $t$, $\delta \leq s < t \leq T$, there is a generic constant $C > 0$ such that

$$E\left| \int_0^{s-\delta} \int_{R^d} p(x, t-r, y)a(y, K)F(y)dydW_r \right.$$

$$- \int_0^{s-\delta} \int_{R^d} p(x, s-r, y)a(y, K)F(y)dydW_r \right|^p$$

$$\leq CE\left\{ \int_0^{s-\delta} \left\{ \left| \int_{R^d} [p(x, t-r, y) - p(x, s-r, y)]a(y, K)F(y)dy \right| \right\}^p dr \right\}$$

$$= CE\left( \int_0^{s-\delta} \left\{ \left| \int_{R^d} [p(x, t-r, y) - p(x, s-r, y - \delta)] \right| \right\}^p dr \right)$$

$$\leq CE||s-\delta||\left\{ \frac{(t-s)}{\delta} \int_{R^d} a^2(z, K)F^2(z)dz \right\}^{\frac{p}{2}}$$

$$\leq C\left( \frac{(t-s)^{\frac{p}{2}}}{\delta^{(4+dp)/2}} \right) \int_{R^d} E|F|^p(z)dz.$$ (3.6)

On the other hand, by Burkholder inequality again and Jensen’s inequality for the same $s$ and $t$ we have

$$E\int_{s-\delta}^{t} \int_{R^d} p(x, t-r, y)a(y, K)F(y)dydW_r$$

$$\leq C\left( t - s + \delta \right)\int_{s-\delta}^{t} E\left| \int_{R^d} p(x, t-r, y)a(y, K)F(y)dy \right|^p dr$$

$$+ C\delta^{\frac{p-2}{2}} \int_{s-\delta}^{t} E\left| \int_{R^d} p(x, s-r, y)a(y, K)F(y)dy \right|^p dr$$

$$\leq C\left( t - s + \delta \right)\int_{s-\delta}^{t} E\left| \int_{R^d} p(x, t-r, y)a^p(y, K)|F(y)|dy \right| dr$$

$$+ C\delta^{\frac{p-2}{2}} \int_{s-\delta}^{t} E\left| \int_{R^d} p(x, s-r, y)a^p(y, K)|F(y)|dy \right| dr.$$
\[ C|t - s + \delta|^\frac{p-2}{2} \int_{s-\delta}^t \int_{\mathbb{R}^d} p(x, t - r, y) a^p(y, \mathcal{K}) E|F(y)|^p dy dr \]
\[ + C \delta |p - 2| \int_{s-\delta}^s \int_{\mathbb{R}^d} p(x, s - r, y) a^p(y, \mathcal{K}) E|F(y)|^p dy dr. \]  
(3.7)

**Theorem 3.2** Given a bounded and Lipschitz continuous function \( \tilde{G} \) and a function \( \phi \in C_0([0, T] \times \mathbb{R}^d) \), define \( \tilde{F} \) by (3.2) and assume \( u_0 \) is bounded and Lipschitz continuous. Then there exists a non-negative continuous solution \( v(t, x) \) to equation (3.1).

**Proof.** To simplify our notation, we denote by \( C \) a generic constant of which the value may vary in different places. We prove the theorem in several steps:

(A) Let \( \psi_m(v) \) be a sequence of real-valued smooth functions such that
1) \( \psi_m(u) = 1 \) when \( u < m - 1 \);
2) \( \psi_m(u) = 0 \) when \( u > m \);
3) \( 0 \leq \psi_m(u) \leq 1 \) when \( m - 1 \leq u \leq m \).

Denote \( \tilde{F}_m(v, t, x) = \psi_m(v) \tilde{F}(v, t, x) \). Then \( \tilde{F}_m(v, t, x) \) is Lipschitz in its arguments. We first show that (3.1) has a solution when \( \tilde{F} \) is replaced by \( \tilde{F}_m \). Denote by \( S_0 \) the set of all bounded continuous random fields of the form \( v(t, x, \omega) \), and by \( \bar{S} \) the completion of \( S_0 \) under the norm \( |||v||| = \sup_{0 \leq t \leq T} \sup_x [Ev^p(t, x)]^{\frac{1}{p}} \).

Furthermore, let \( S \) be the set of all the elements in \( \bar{S} \) which have a continuous in \((t, x)\) version.

Define a map: \( \theta : S \to \bar{S} \) by

\[
\theta(v)(t, x) = \int_{\mathbb{R}^d} p(y, t, x) u_0(y) dy + \sum_k \int_0^t \int_{\mathbb{R}^d} p(y, t - s, x) a(y, \mathcal{K}) \tilde{F}_m(v, s, y) dy dw(s, \mathcal{K}).
\]

Then for any \( v_1, v_2 \in S \), integer \( p \geq 2 \) and \( t \leq T \) the inequality holds

\[
E(|\theta(v_1)(t, x) - \theta(v_2)(t, x)|)^p \leq C E \sum_k \int_0^t \int_{\mathbb{R}^d} p(y, t - s, x) a(y, \mathcal{K}) |(\tilde{F}_m(v_1, s, y) - \tilde{F}_m(v_2, s, y))| dy ds
\]
\[
\leq C \sum_k E \int_0^t \int_{\mathbb{R}^d} p(y, t - s, x) |(\tilde{F}_m(v_1, s, y) - \tilde{F}_m(v_2, s, y))|^p dy ds
\]

8
\[
\leq Ct \sup_{0 \leq t \leq T} \sup_x E[[\hat{F}_m(v_1, t, x) - \hat{F}_m(v_2, t, x)]]^p
\leq Ct \sup_{0 \leq t \leq T} \sup_x E(|v_1(t, x) - v_2(t, x)|)^p.
\] (3.8)

In the above deduction the identity \( \int_{\mathbb{R}^d} p(y, t-s, x)dy = 1 \) and the Jensen’s and Burkholder inequalities have been used. Therefore, we obtain the estimate

\[
|||\theta(v_1) - \theta(v_2)||| \leq (CT)^{\frac{1}{p}}|||v_1 - v_2|||.
\] (3.9)

We choose \( T \) sufficiently small so that \( CT < 1 \). Then, \( \{\theta^p u_0\} \) form a Cauchy sequence which converges to some point \( v \) in \( \bar{S} \). The above procedure can be extended to any finite interval \([0, T]\) by considering a finite number of sufficiently small intervals. Next, we are going to show that \( v \) is continuous in \((x, t)\).

(B) Let us establish an equicontinuity estimate. From (3.6) and (3.7) and the triangular inequality,

\[
E[\theta(v)(t, x) - \theta(v)(s, x)]^p \\
\leq C|t - s + \delta|^{\frac{p-2}{2}} \int_{s-\delta}^t \int_{\mathbb{R}^d} p(x, t-r, y)a^p(y, \mathcal{K})E\tilde{F}_m^p(y)dydr \\
+ C\delta^{\frac{p-2}{2}} \int_{s-\delta}^t \int_{\mathbb{R}^d} p(x, s-r, y)a^p(y, \mathcal{K})E\tilde{F}_m^p(y)dydr \\
+ C(t-s)^{\frac{p}{2}} \int_{\mathbb{R}^d} E\tilde{F}_m^p(z)dz.
\] (3.10)

Taking \( \delta^{\frac{p(p-2)}{2}} = (t-s)^{\frac{p}{2}} \) in (3.10), we deduce that there is a constant \( C > 0 \) such that

\[
E[\theta(v)(t, x) - \theta(v)(s, x)]^p \leq C(t-s)^p|||v|||.
\] (3.11)

For any \( x_1, x_2 \in \mathbb{R}^d \),

\[
\theta(v)(t, x_1) - \theta(v)(t, x_2) \\
= \int_{\mathbb{R}^d} p(y, t, x_1)u_0(y)dy - \int_{\mathbb{R}^d} p(y, t, x_2)u_0(y)dy \\
+ \sum_{\mathcal{K}} \int_0^t \int_{\mathbb{R}^d} p(y, t-s, x_1)a(y, \mathcal{K})\hat{F}_m(v, s, y)dydw(s, \mathcal{K}) \\
- \sum_{\mathcal{K}} \int_0^t \int_{\mathbb{R}^d} p(y, t-s, x_2)a(y, \mathcal{K})\hat{F}_m(v, s, y)dydw(s, \mathcal{K}) \\
= \int_{\mathbb{R}^d} p(y, t, x_1)u_0(y)dy - \int_{\mathbb{R}^d} p(y, t, x_2)u_0(y)dy \\
+ \sum_{\mathcal{K}} \int_0^t \int_{\mathbb{R}^d} p(t-s, y)a(y + x_1, \mathcal{K})\hat{F}_m(v, s, y + x_1)dydw(s, \mathcal{K}) \\
- \sum_{\mathcal{K}} \int_0^t \int_{\mathbb{R}^d} p(t-s, y)a(y + x_2, \mathcal{K})\hat{F}_m(v, s, y + x_2)dydw(s, \mathcal{K})
\]
\[
\begin{align*}
= \int_{R^d} p(y, t, x_1)u_0(y)dy - \int_{R^d} p(y, t, x_2)u_0(y)dy \\
+ \sum_{K} \int_{R^d}^{t} \int_{R^d} p(t-s, y)(a(y + x_1, K) - a(y + x_2, K)) \\
\tilde{F}_m(v, s, y + x_1)dydw(s, K) \\
+ \sum_{K} \int_{R^d}^{t} \int_{R^d} p(t-s, y) a(y + x_2, K) \\
\{\tilde{F}_m(v, s, y + x_1) - \tilde{F}_m(v, s, y + x_2)\}dydw(s, K).
\end{align*}
\]

We need moment estimates. To obtain them we use Burkholder inequality (3.3) and Hölder inequality, as follows

\[
\begin{align*}
E|\theta(t, x_1) - \int_{R^d} p(t, x_1)u_0(y)dy| \\
&\quad - |\theta(t, x_2) - \int_{R^d} p(t, x_2)u_0(y)dy|^p \\
\leq \ C t^{\frac{p}{2} - 1} \sum_{K} \int_{0}^{t} E \int_{R^d} p(t-s, y)(a(y + x_1, K) - a(y + x_2, K)) \\
&\quad - a(y + x_2, K))\tilde{F}_m(v, s, y + x_1)dydw(s, K) \\
&\quad + C t^{\frac{p}{2} - 1} \sum_{K} \int_{0}^{t} E \int_{R^d} p(t-s, y) a(y + x_2, K) \\
&\quad \{\tilde{F}_m(v, s, y + x_1) - \tilde{F}_m(v, s, y + x_2)\}dydw(s, K) \\
\leq \ C t^{\frac{p}{2} - 1}|x_1 - x_2|^p \\
&\quad + C t^{\frac{p}{2} - 1} \int_{0}^{t} \sup_{0 \leq r \leq s} \sup_{0 < |x_1 - x_2| < h_0} E|v(r, x_1) - v(r, x_2)|^pds. \\
\end{align*}
\]

By the triangular inequality,

\[
\begin{align*}
E|\theta(t, x_1) - \theta(t, x_2)|^p \\
\leq \ C |x_1 - x_2|^p + C t^{\frac{p}{2}}|x_1 - x_2|^p \\
&\quad + C t^{\frac{p}{2} - 1} \int_{0}^{t} \sup_{0 \leq r \leq s} \sup_{0 < |x_1 - x_2| < h_0} E|v(r, x_1) - v(r, x_2)|^pds \\
\leq \ C |x_1 - x_2|^p + C t^{\frac{p}{2}} \sup_{0 \leq r \leq s} \sup_{0 < |x_1 - x_2| < h_0} E|v(r, x_1) - v(r, x_2)|^pds.
\end{align*}
\]

By the induction principle

\[
E|(\theta^n \hat{u}_0)(t, x_1) - (\theta^n \hat{u}_0)(t, x_2)|^p \leq \sum_{k=0}^{n} C (C t^{\frac{p}{2}})^k |x_1 - x_2|^p,
\]
Thus, taking $\hat{u}_0(t, x) = u_0(x)$. When $Ct_1^{p/2} < 1$, and $0 \leq t \leq t_1$, this gives

$$
\sup_{0 \leq s \leq t < 0 < |x_1 - x_2| < h_0} \sup_{x} E[|\theta^n(\hat{u}_0)(t, x_1) - \theta^n(\hat{u}_0)(t, x_2)|^p] \leq C|x_1 - x_2|^p.
$$

(3.13)

The estimate holds for any $p \geq 2$.

By (3.11), (3.13) and Kolmogorov’s criterion (see e.g. [7], [12]), $\{\theta^n \hat{u}_0\}_n$ form an equicontinuous family when $p$ is sufficiently large. So there is a subsequence which converge in law to a continuous random field. Thus we deduce the limit $v$ obtained in part (A), has a continuous version. We have for $0 \leq t \leq t_1$,

$$
v(t, x) = \int_{R^d} p(y, t, x)u_0(y)dy + \sum_{\kappa} \int_0^t \int_{R^d} p(y, t - s, x)a(y, \kappa)\tilde{F}_m(v, s, y)dydw(s, \kappa).
$$

(3.14)

This formula can be extended to $0 \leq t \leq T$ for any given $T > 0$, by considering finite number of intervals $[t_1, 2t_1], [2t_1, 3t_1], \ldots$

(C) Denote by $v_m$ the solution to (3.14). It is easy to check that (3.14) is equivalent to the Feynman-Kac formula (see e.g. [12], [8])

$$
v_m(t, x) = \hat{\mathcal{E}}u_0(X_{t,0}(x)) \exp\{-\frac{1}{2} \sum_{\kappa} \int_0^t a^2(X_{t,s}(x), \kappa) \tilde{F}_m^2(v_m, s, X_{t,s}(x))ds
$$

$$
+ \sum_{\kappa} \int_0^t a(X_{t,s}(x), \kappa)(\tilde{F}_m(v_m, s, X_{t,s}(x))dw(s, \kappa))\}
$$

(3.15)

where $X_{t,s}$ is the inverse of the Brownian flow in $R^d$ on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{P}})$ and $\hat{\mathcal{E}}$ denotes the expectation over that probability space. The stochastic integral $\int_0^t a\tilde{F}_m(s, X_{t,s}(x))dydw(s, \kappa)$ is defined to be Stratonovich integral on the product probability space $(\Omega \times \hat{\Omega}, \mathcal{F} \times \hat{\mathcal{F}}, \mathcal{P} \times \hat{\mathcal{P}})$. It coincides with the Ito integral as $X_{t,s}(x)$ is independent of $w(s, \kappa)$ for any $\kappa$. Hence $v_m(t, x) \geq 0$. Since $a(X_{t,s}(x), \kappa)(\tilde{F}_m(v_m, s, X_{t,s}(x))$ is bounded, it is easy to see that for any $1 \leq p < \infty$,

$$
\sup_{m,t,x} E|v_m(t, x)|^p < \infty.
$$

(3.16)

Thus, taking $t - t' = |x_1 - x_2|^\alpha$ for a constant $\alpha > 0$, we have

$$
E|\sum_{\kappa} \int_0^t \int_{R^d} (p(y, t - s, x_1) - p(y, t - s, x_2))a(y, \kappa)\tilde{F}_m(v_m, s, y)dydw(s, \kappa)|^p
$$

$$
\leq CE \int_0^{t'} \int_{R^d} (p(y, t - s, x_1) - p(y, t - s, x_2))\tilde{F}_m(v_m, s, y)dy\]
$$

$$
ds
$$
That is to say, there are constants $C$ and $c_1, c_2$ such that
\[
E|v_m(t_2, x_2) - v_m(t_1, x_1)|^p \leq C(|x_1 - x_2|^{c_1p} + |t_2 - t_1|^{c_2p}).
\]
This inequality holds for any $p \geq 2$. Note that $C$ may depend on $p$, but $c_1$ and $c_2$ are independent of $p$. By Kolmogorov’s criterion, $\{v_m\}$ form a weakly tight sequence. Therefore, we can find a subsequence which converges in law to some continuous random field $v$. It is easy to see that $v$ is the desired solution.

We will use the following lemma in the next section.

**Lemma 3.3** If $u_0(x)$ is bounded and Lipschitz continuous and has compact support $\mathcal{U} \subset \mathbb{R}^d$, then the solution $u_{n,k}(t, x)$ to (2.3) satisfies the following inequality
\[
C_1 \exp\{-\frac{(\text{dist}(x, \mathcal{U}))^2}{2t}\} \leq E u_{n,k}(t, x) \leq C_2 \exp\{-\frac{(\text{dist}(x, \mathcal{U}))^2}{2t}\},
\]
with two constants $C_2 \geq C_1 > 0$ independent of $n$ and $k$. Therefore,
\[
E \int_{\mathbb{R}^d} u_{n,k}(t, x) dx < \infty.
\]

**Proof.** Taking mathematical expectation on both sides of (2.6), we have
\[
E u_{n,k}(t, x) = \int_{\mathbb{R}^d} p(y, t, x) u_0(y) dy.
\]
Thus $E u_{n,k}(t, x)$ is the solution of the deterministic heat equation with initial condition $u_0(x)$, and we get the statement of Lemma from the property of the heat kernel.

### 4 Tightness results and the convergence to the solution of SPDEs

Still let $\mathcal{P}$ be the set of all bounded measures on $[0, T] \times \mathbb{R}^d$. Let $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$. Denote by $\mathcal{P}_b$ the set of all measures on $[0, T] \times \overline{\mathbb{R}^d}$ bounded by positive $b$. So $\mathcal{P}_b$
is a compact polish space equipped with the topology of measure convergence, which is the least fine topology to make all the mappings of $P_b$ into $R$: $\mu \rightarrow \int_{[0,T] \times \mathbb{R}^d} f(t,x)\mu(dx,dt)$ continuous for all bounded continuous functions $f(t,x)$ defined on $[0,T] \times \mathbb{R}^d$ (see, for example, [5] III, 60). Denote $\hat{P} = \bigcup_b P_b$. When we equip $\hat{P}$ again with the topology of measure convergence, it is easy to see that for each $0 < b < \infty$,

$$\{\mu \in \hat{P} : \mu([0,T] \times \mathbb{R}^d) < b\}$$

is an open set and its closure is $\mathcal{P}_b$. Thus $\hat{P}$ is a locally compact polish space. Denote by $\overline{\mathcal{P}}$ its one-point compactification. Consider $V_{n,k}(dt,dx) = u_{n,k}(t,x)dtdx$ as a sequence of random variables taking values in $\mathcal{P}$. Here $u_{n,k}(t,x)$ is the solution to (2.3) which exists and is nonnegative.

**Theorem 4.1** Suppose $G$ is a bounded and Lipschitz continuous function, and $u_0 \in C_0([0,T])$ is bounded, Lipschitz continuous and nonnegative. Then there is a subsequence $V_{n_j,k_i}$ which converges to a $\mathcal{P}$-valued random variable $V$ in law, as $i \to \infty$ and then $j \to \infty$.

**Proof.** We prove the theorem in two steps.

1) Denote by $P_{n,k}$ the laws of $V_{n,k}$ on $\mathcal{P}$. Since $\mathcal{P}$ is compact and separable, there is a subsequence $P_{n_j,k_i}$ which converges to some $P_\infty$. This convergence can be split to the following: first, for any fixed $n$, there exists a subsequence $k_i(n) \to \infty$ as $i \to \infty$ such that $P_{n,k_i(n)} \to P_n$ as $i \to \infty$; second, there exists a subsequence $n_j \to \infty$ as $j \to \infty$ such that $P_{n_j} \to P_\infty$ as $j \to \infty$. We are going to show that $P_\infty$ is carried by $\mathcal{P}$. By the Chebyshev inequality,

$$P[V_{n_j,k_i}([0,T] \times \mathbb{R}^d) > c] \leq c^{-1}E[V_{n_j,k_i}([0,T] \times \mathbb{R}^d)]$$

$$= c^{-1}\int_0^T \int_{\mathbb{R}^d} Eu_{n_j,k_i}(t,x)dxdt.$$ 

Since $\{\nu \in \mathcal{P} : \nu([0,T] \times \mathbb{R}^d) > c\}$ is open in $\mathcal{P}$,

$$P_\infty[\nu \in \mathcal{P} : \nu([0,T] \times \mathbb{R}^d) > c] \leq c^{-1}\liminf_{j \to \infty} \liminf_{i \to \infty} \int_0^T \int_{\mathbb{R}^d} Eu_{n_j,k_i}(t,x)dxdt$$

$$= c^{-1}\int_0^T \int_{\mathbb{R}^d} H(t,x)dxdt,$$  

(4.1)

where we have used the fact that weak convergent measures reduce their probabilities in the limit on open sets (see, for example, [9] p.108), Fatou’s lemma and Lemma 3.3. As above $H(t,x)$ is the solution of the deterministic heat equation with the initial condition $H(0,x) = u_0(x)$, so $\int_0^T \int_{\mathbb{R}^d} H(t,x)dxdt < \infty$, and the right-hand side of the above inequality tends to 0 as $c \to \infty$. That is, $P_\infty$ is concentrated by $\hat{P}$. 

13
2) Now let us show that $P_\infty$ is concentrated on $\mathcal{P}$. Indeed, for fixed positive pairs $\kappa$ and $c$,

$$\{\nu \in \mathcal{P} : \nu([0, T] \times \{x \in \mathbb{R}^d : |x| > c\}) > \kappa\}$$

is open in $\hat{\mathcal{P}}$. We have easily

$$P_{n,j,k}[V_{n,j,k}([0, T] \times \{x \in \mathbb{R}^d : |x| > c\}) > \kappa] \leq \kappa^{-1} E\left[\int_0^T \int_{|x|>c} u_{n,j,k}(t, x)dxdt\right].$$

So, similarly to (4.1), one can obtain

$$P_{\infty}[\nu \in \hat{\mathcal{P}} : \nu([0, T] \times \{x \in \mathbb{R}^d : |x| > c\}) > \kappa] \leq \kappa^{-1} \liminf_{j \to \infty} \liminf_{i \to \infty} E\left[\int_0^T \int_{|x|>c} u_{n,j,k}(t, x)dxdt\right]$$

$$\to 0,$$  

(4.2)

uniformly as $c \to \infty$, due to the exponential decay of $H(t, x)$ when $|x| \to \infty$. Thus, we deduce that $P_\infty$ is concentrated on $\mathcal{P}$.

That is to say that there exists $V \in \mathcal{P}$ such that $u_{n,j,k}$ converges to $V \in \mathcal{P}$ in law as $i \to \infty$ and then $j \to \infty$.

Proof of Theorem 1.1. Assume $G$ is positive in the proof. If $G$ is negative, the proof is same. Since $\mathcal{P}$ is a Polish space, by the celebrated Skorohod’s Lemma, we can assume that $V_{n,k}$ and $V$ are all defined on the same probability space and $V_{n,j,k}$ converges to $V$ almost surely. So $u_{n,j,k}dxdt \to V(dx, dt)$ weakly for a.e. $\omega$ i.e. for any bounded and continuous function $\psi(t, x)$

$$\int_0^T \int_{\mathbb{R}^d} \psi(s, x)u_{n,j,k}(s, x)dxds \to \int_0^T \int_{\mathbb{R}^d} \psi(s, x)V(dx, ds).$$  

(4.3)

Here and throughout the proof all the limits are taken first in $i \to \infty$ and then in $j \to \infty$.

Consider equation (2.3) and the subsequence $u_{n,j,k}(t, x)$ of its solution. Let $J$ be a countable dense set in $C^2_0$. Given any $q \in J$, it is easy to see from the integral by parts formula that

$$\int_{\mathbb{R}^d} \int_0^t q(x)\frac{\partial}{\partial s} u_{n,j,k}(s, x)dsdx = N_{t}^{(i,j)} + M_{t}^{(i,j)}$$  

(4.4)

where

$$N_{t}^{(i,j)} = \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \Delta q(x)u_{n,j,k}(s, x)dxds$$
is a process of bounded variation and

\[
M_t^{(i,j)} = \sum_K \int_0^t \int_{\mathbb{R}^d} q(x) a(x, K) u_{n_j,k_i}(s, x) \times \left( \frac{f_{L_K} a^2(z, K) u_{n_j,k_i}(s, z) G^2 dz}{\frac{1}{K^2} + f_{L_K} a(z, K) u_{n_j,k_i}(s, z) dz} \right)^{1/2} dx dw_K(s)
\]

is a martingale. Moreover,

\[
\sup_{i,j} E[|N_T^{(i,j)}|] < \infty \tag{4.5}
\]

and

\[
\begin{align*}
\sup_{(i,j)} E[\langle M^{(i,j)}, M^{(i,j)} \rangle_T] &= \sup_{(i,j)} \left\{ \int_0^T \int_{\mathbb{R}^d} q(x) a(x, K) u_{n_j,k_i}(s, x) \times \left( \frac{f_{L_K} a^2(z, K) u_{n_j,k_i}(s, z) G^2 dz}{\frac{1}{K^2} + f_{L_K} a(z, K) u_{n_j,k_i}(s, z) dz} \right)^{1/2} dx \right\} ds < \infty. \tag{4.6}
\end{align*}
\]

Thus, both \(\{N^{(i,j)}\}\) and \(\{M^{(i,j)}\}\) are tight under pseudo-path topology ([15]). Hence the left-hand side of (4.4)

\[
\int_{\mathbb{R}^d} \int_0^t q(x) \frac{\partial}{\partial s} u_{n_j,k_i}(s, x) ds dx = \int_{\mathbb{R}^d} q(x) u_{n_j,k_i}(s, x) dx - \int_{\mathbb{R}^d} q(x) u_0(x) dx
\]

is also tight and any of its limits is a semi-martingale. For fixed \(n_j\), we take a subsequence, still denoted by \(u_{n_j,k_i}\), such that \(\int_{\mathbb{R}^d} q(x) u_{n_j,k_i}(t, x) dx\) converge to a semimartingale \(Q^{(q,j)}\) for each \(q \in J\) under pseudo-path topology in law. By taking a subsequence still denoted by \(\{Q^{(q,j)}\}\), we get, when \(j \to \infty\), \(Q^{(q,j)} \to Q^{(q)}\) which is a semimartingale for each \(q \in J\). Since \(J\) is dense in \(C_0^2\), \(\int_{\mathbb{R}^d} q(x) u_{n_j,k_i}(t, x) dx\) converges to a semimartingale \(Q^{(q)}\) for each \(q \in C_0^2\) under pseudo-path topology in law. The above fact can be easily seen from the fact that the pseudo-path topology is equivalent to the convergence in measure \(ds\) in this case (see [15] for details), and Skorohod’s lemma mentioned in the beginning of this proof. Although \(q(x) \chi_{[0,t]}(s)\) for fixed \(t\) is not a continuous function in \((x,s) \in R^d \times [0,T]\), but we can still prove the following lemma.

**Lemma 4.2** For any \(q \in C_0(R^d)\), and \(t \leq T\),

\[
\lim_{j} \lim_{i} \int_{0}^{t} \int_{\mathbb{R}^d} q(x) u_{n_j,k_i}(s, x) dx ds = \int_{0}^{t} \int_{\mathbb{R}^d} q(x) V(dx, ds), \quad \text{almost surely.}
\]
We continue the proof of the theorem and leave the proof of the lemma to the end of the section. Lemma 4.2 implies

$$\lim \lim_{i \to \infty} \int_0^t \int_{R^d} \Delta q(x) u_{n_j,k_i}(s,x) \, dx \, ds = \int_0^t \int_{R^d} \Delta q(x) V(dx, ds) \quad (4.7)$$

for any $q \in C^2_0$.

The remaining question is to identify $Q^{(q)}$. We have

$$Q^{(q)}_t - Q^{(q)}_0 = \lim \lim_{i \to \infty} N_t^{(i,j)} + \lim \lim_{i \to \infty} M_t^{(i,j)}$$

$$= \int_0^t \int_{R^d} \Delta q(x) V(dx, ds) + \lim \lim_{i \to \infty} M_t^{(i,j)}. \quad (4.8)$$

Let us consider the martingale part now. Select in each $L_K$ a point $x_K$. By the uniform continuity of $q(\cdot)$,

$$< M_{n_j,k_i}, M_{n_j,k_i} >_t$$

$$= \sum_K \int_0^t \left\{ \int_{L_K} q(x) a(x, z, K) u_{n_j,k_i}(s,x) \left( \int_{\frac{1}{k_i}} + \int_{L_K} a^2(z, K) G^2 u_{n_j,k_i}dz \right)^2 dx \right\} ds$$

$$= \sum_K \int_0^t \left\{ \int_{\frac{1}{k_i}} a^2(z, K) G^2 u_{n_j,k_i}dz \right\} \times \left\{ \left( \int_{L_K} [q(x_K) + \epsilon(K)] a(x, K) u_{n_j,k_i}(s,x)dx \right)^2 ds \right\}$$

$$= \sum_K \left\{ \int_{\frac{1}{k_i}} + \int_{L_K} a^2(z, K) G^2 u_{n_j,k_i}dz \right\} \times \left\{ \left( \int_{L_K} [q(x_K) + \epsilon(K)] a(x, K) u_{n_j,k_i}(s,x)dx \right)^2 ds \right\}$$

$$\leq 1. \quad (4.9)$$

There are only two kinds of subsequences as $i \to \infty$: 1) a subsequence of $\int_{L_K} a(z, K) u_{n_j,k_i}(t,z)dz$ tending to $0$ in which case $\left[ \int_{\frac{1}{k_i}} + \int_{L_K} a^2(z, K) G^2 u_{n_j,k_i}dz \right]$ tending to $0$ along the same subsequence; 2) or a subsequence of $\left\{ \int_{L_K} [q(x_K) + \epsilon(K)] a(x, K) u_{n_j,k_i}(s,x)dx \right\}$ tends to $1$. Considering both cases yields,

$$\lim_{i \to \infty} \sum_K \int_0^t [q(x_K) + \epsilon(K)] a^2(z, K) G^2 u_{n_j,k_i}(t,z)dz \leq 1. \quad (4.10)$$
Therefore, by the uniform continuity of $q$,
\[
\lim_{j \to \infty} \lim_{i \to \infty} \left< M_{n_j,k_i}, M_{n_j,k_i} > (t) \right>
= \lim_{j \to \infty} \lim_{i \to \infty} \sum_{K} \int_{0}^{t} \left[ \int_{L_K} a^2(x, \mathcal{K})q^2(x)G^2 u_{n_j,k_i}dx \right]ds
\]
\[
= \lim_{j \to \infty} \lim_{i \to \infty} \sum_{K} \int_{0}^{t} \left[ \int_{L_K} q^2(x)G^2 u_{n_j,k_i}dx \right]ds
\]
\[
- \lim_{j \to \infty} \lim_{i \to \infty} \sum_{K} \int_{0}^{t} \left[ \int_{L_K} (1 - a^2(x, \mathcal{K}))q^2(x)G^2 u_{n_j,k_i}dx \right]ds.
\] (4.11)

First applying again Lemma 4.2 which is still valid for time continuously dependent $q(s, x)$ ($C_0$ in $x$),
\[
\lim_{j \to \infty} \lim_{i \to \infty} \sum_{K} \int_{0}^{t} \left[ \int_{L_K} q^2(x)G^2 u_{n_j,k_i}dx \right]ds
= \lim_{j \to \infty} \lim_{i \to \infty} \int_{0}^{t} \left[ \int_{R^d} q^2(x)G^2 \left( \int_{0}^{s} \int_{R^d} \phi(s - r, x - z)u_{n_j,k_i}(r, z)dzdr \right) u_{n_j,k_i}(s, x)dx \right]ds
= \int_{0}^{t} \int_{R^d} q^2(x)G^2 \left( \int_{0}^{s} \int_{R^d} \phi(s - r, x - z)V(dz, dr) \right)V(dx, ds).
\] (4.12)

Here we have also used
\[
\int_{0}^{s} \int_{R^d} \phi(s - r, x - z)u_{n_j,k_i}(r, z)dzdr \to \int_{0}^{s} \int_{R^d} \phi(s - r, x - z)V(dz, dr).
\]

This can be seen by extending $\phi(s - r, x - z)$ for fixed $(s, x)$ smoothly to $(r, z) \in [0, T] \times R^d$ by letting $\phi(s - r, x - z) = 0$ for $s \leq r \leq T$ and then applying (4.3). By the same argument, it is easy to check that for fixed $s$, \( \int_{0}^{s} \int_{R^d} \phi(s - r, x - z)V(dz, dr) \) is $C^2$ in $x$. Secondly, by Fatou's Lemma, and the definition of $a(x, \mathcal{K})$,
\[
E \lim_{j \to \infty} \lim_{i \to \infty} \sum_{K} \int_{0}^{t} \left[ \int_{L_K} (1 - a^2(x, \mathcal{K}))q^2(x)G^2 u_{n_j,k_i}dx \right]ds
\]
\[
\leq \lim_{j \to \infty} \lim_{i \to \infty} \sum_{K} \int_{0}^{t} \left[ \int_{L_K} (1 - a^2(x, \mathcal{K}))q^2(x)E(G^2 u_{n_j,k_i})dx \right]ds
\]
\[
\leq 2\|F\|_{\infty}^2 \|q\|_{\infty}^2 \|H\|_{\infty} t \lim_{j \to \infty} \sum_{K} \int_{L_K} (1 - a(x, \mathcal{K}))dx
\]
\[
= 2\|F\|_{\infty}^2 \|q\|_{\infty}^2 \|H\|_{\infty} t \lim_{j \to \infty} \frac{1}{n_j^{\delta}}
\]
\[
= 0.
\]
That is to say
\[
\lim_{j \to \infty} \lim_{i \to \infty} \sum_K \int_0^t \left[ \int_{L_K} (1 - \alpha^2(x, \mathcal{K}))q^2(x)G^2 u_{n_j,k_i} dx \right] ds = 0 \quad a.s.. \tag{4.13}
\]
It follows from (4.11)–(4.13) that
\[
\lim_{j \to \infty} \lim_{i \to \infty} < M_{n_j,k_i} < (t) = \int_0^t \int_{\mathbb{R}^d} q^2(x)G^2 \left( \int_0^s \int_{\mathbb{R}^d} \phi(s-r,x-z)V(dx,dr) \right)V(dx,ds).
\]
This completes the proof of Theorem.

Proof of Lemma 4.2. Without losing generality, we assume \( q \geq 0 \). Let \( g_{1\epsilon}(s) \leq \chi_{[0,t]}(s) \leq g_{2\epsilon}(s) \) be lower and upper regularizers of the step function \( \chi_{[0,t]}(s) \) on \([0,T]\) such that
\[
\int_0^T (g_{2\epsilon}(s) - g_{1\epsilon}(s)) ds \leq \epsilon.
\]
Then,
\[
\int_0^T \int_{\mathbb{R}^d} g_{1\epsilon}(s)q(x)u_{n_j,k_i}(s,x)dxds \leq \int_0^t \int_{\mathbb{R}^d} q(x)u_{n_j,k_i}(s,x)dxds \leq \int_0^T \int_{\mathbb{R}^d} g_{2\epsilon}(s)q(x)u_{n_j,k_i}(s,x)dxds.
\]
If we denote
\[
L_{n_j}^+ = \limsup_{i \to \infty} \int_0^t \int_{\mathbb{R}^d} q(x)u_{n_j,k_i}(s,x)dxds,
\]
\[
L_{n_j}^- = \liminf_{i \to \infty} \int_0^t \int_{\mathbb{R}^d} q(x)u_{n_j,k_i}(s,x)dxds,
\]
and pass to the limit in the above inequality as \( i \to \infty \), then we obtain
\[
\int_0^T \int_{\mathbb{R}^d} g_{1\epsilon}(s)q(x)V_{n_j}(dx,ds) \leq L_{n_j}^- \leq L_{n_j}^+ \leq \int_0^T \int_{\mathbb{R}^d} g_{2\epsilon}(s)q(x)V_{n_j}(dx,ds), \tag{4.14}
\]
where \( V_{n_j} = \lim_{i \to \infty} V_{n_j,k_i} \). On the other hand,
\[
\int_0^T \int_{\mathbb{R}^d} g_{1\epsilon}(s)q(x)V_{n_j}(dx,ds) \leq \int_0^T \int_{\mathbb{R}^d} q(x)V_{n_j}(dx,ds) \leq \int_0^T \int_{\mathbb{R}^d} g_{2\epsilon}(s)q(x)V_{n_j}(dx,ds)
\]
and
\[
E \int_0^T \int_{\mathbb{R}^d} (g_{2\epsilon}(s) - g_{1\epsilon}(s))q(x)V_{n_j}(dx, ds)
\]
\[
= E \lim_{i \to \infty} \int_0^T \int_{\mathbb{R}^d} (g_{2\epsilon}(s) - g_{1\epsilon}(s))q(x)u_{n_j,k_i}(s,x)dx ds
\]
\[
= \int_0^T (g_{2\epsilon}(s) - g_{1\epsilon}(s)) \int_{\mathbb{R}^d} q(x)H(s,x)dx ds
\]
\[
\leq M \epsilon,
\]
here \(H(s,x)\) is the solution of the deterministic heat equation with \(H(0,x) = u_0(x)\) and \(M = \sup_{0 \leq s \leq T} \int_{\mathbb{R}^d} q(x)H(s,x)dx\). Therefore,
\[
E \left| L_{n_j}^+ - \int_0^t \int_{\mathbb{R}^d} q(x)V_{n_j}(dx, ds) \right| + E \left| L_{n_j}^- - \int_0^t \int_{\mathbb{R}^d} q(x)V_{n_j}(dx, ds) \right| \leq 2M \epsilon.
\]
Since \(\epsilon\) is arbitrary chosen, we deduce
\[
\lim_{i \to \infty} \int_0^t \int_{\mathbb{R}^d} q(x)u_{n_j,k_i}(s,x)dx ds = \int_0^t \int_{\mathbb{R}^d} q(x)V_{n_j}(dx, ds).
\]
One can pass to the limit in \(j\) exactly in the same way, and the statement required follows.

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\section*{References}


