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On the real zeroes of the Hurwitz zeta-function and Bernoulli polynomials

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Abstract. The behaviour of real zeroes of the Hurwitz zeta function

\[ \zeta(s,a) = \sum_{r=0}^{\infty} (a + r)^{-s} \quad a > 0 \]

is investigated. It is shown that \( \zeta(s,a) \) has no real zeroes \((s = \sigma, a)\) in the region \(a > \frac{\sigma^2}{2\pi} + \frac{1}{4\pi} \log(-\sigma) + 1\) for large negative \(\sigma\). In the region \(0 < a < \frac{\sigma^2}{2\pi}\) the zeroes are asymptotically located at the lines \(\sigma + 4a + 2m = 0\) with integer \(m\). If \(N(p)\) is the number of real zeroes of \(\zeta(-p,a)\) with given \(p\) then

\[ \lim_{p \to \infty} \frac{N(p)}{p} = \frac{1}{\pi e} \]

As a corollary we have a simple proof of Inkeri’s result that the number of real roots of the classical Bernoulli polynomials \(B_n(x)\) for large \(n\) is asymptotically equal to \(\frac{2n}{\pi^2}\).

1 Introduction.

The classical Hurwitz zeta-function is defined for any positive real \(a\) as an analytic continuation of the series

\[ \zeta(s,a) = \sum_{r=0}^{\infty} (a + r)^{-s}. \]

When \(a = 1\) it reduces to the Riemann zeta-function.
We should note that sometimes the definition of the Hurwitz zeta-function is restricted to $0 < a \leq 1$ (see e.g. [1, 2]). From our point of view this is not natural and we follow the definition of the Hurwitz zeta-function from [3] where all positive $a$ are allowed (cf. also the original Hurwitz paper [4]).

It is known (see e.g. [2], volume 1, page 27) that in the special cases when $s$ is a negative integer this function (as a function of the parameter $a$) reduces, up to a factor, to a Bernoulli polynomial: explicitly when $s = -m, m = 0, 1, 2, 3, ...$

$$
\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1}.
$$

The Bernoulli polynomials $B_k(a)$ can be defined through the generating function:

$$
\frac{ze^{za}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(a)}{k!} z^k
$$

giving, for example:

$$
B_0(a) = 1 \quad B_1(a) = a - \frac{1}{2} \quad B_2(a) = a^2 - a + \frac{1}{6} \quad B_3(a) = a^3 - \frac{3a^2}{2} + \frac{a}{2} \ldots
$$

Bernoulli polynomials possess many interesting properties and arise in many areas of mathematics (see [1, 2]).

Inkeri [5] proved a remarkable fact that the number $N(n)$ of real roots of the Bernoulli polynomials $B_n$ for large $n$ asymptotically equals $\frac{2n\pi}{\sqrt{3}}$. More precise estimates for $N(n)$ have been found by Delange [6, 7].

In this paper we investigate the behaviour of the real zeroes of the Hurwitz zeta function $\zeta(s, a)$ in the upper half-plane $a > 0$. As a corollary we have a simple proof of Inkeri’s result. Our approach is different from [5, 6, 7] and we believe is more elementary. It is based on the remarkable Hurwitz representation of the $\zeta(s, a)$ on the interval $0 < a \leq 1$ and $\text{Re}(s) = \sigma < 0$:

$$
\zeta(s, a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{r=1}^{\infty} \frac{\sin(2\pi ra + \frac{1}{2}\pi s)}{r^{1-s}},
$$

where $\Gamma$ is the Euler gamma-function (see e.g [2], volume 1, page 26).

Our main observation is that for a large negative $\sigma$ this formula gives a good approximation for $\zeta(s, a)$ on a much wider interval: $0 < a < -\frac{\sigma}{2\pi e}$. As a result we prove that

$$
\frac{\zeta(\sigma, a)}{Q(\sigma)} = \sin(2\pi a + \frac{1}{2}\pi\sigma) + o(1) \quad \text{as \ } \sigma \to -\infty \quad \text{provided \ } 0 < a < -\frac{\sigma}{2\pi e}
$$

where $Q(s) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}$. We show also that in the region $a > -\frac{\sigma}{2\pi} + \frac{1}{4\pi\sigma} \log(-\sigma) + 1$ the Hurwitz zeta-function has no real zeroes for large negative $\sigma$. 

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Our results are illustrated in Figure 1 which shows the behaviour of the real zeroes of $\zeta(\sigma,a)$. Since $\frac{1}{2\pi e}$ is a small number we have chosen different scales on the axes to make the picture more illuminating. Notice that $\zeta(s,1)$ coincides with the Riemann zeta-function $\zeta(s)$ which only has real zeroes (for $s < 0$) at negative even numbers $s = -2, -4, -6, \ldots$. Also, using the well-known identity $\zeta(s,1/2) = (2^s - 1)\zeta(s)$, we see that the only real zeroes of $\zeta(s,a)$ on $a = 1$ are $s = -2, -4, -6, \ldots$ and on $a = \frac{1}{2}$ they are $s = 0, -2, -4, \ldots$. As we have mentioned above when $s$ is a non-positive integer $\zeta(s,a)$ reduces to a Bernoulli polynomial. We have used this fact to compute numerically the corresponding values of $\zeta(s,a)$ and to draw the picture in the regions II and IV. When $s = 1$ the Hurwitz zeta-function has a simple pole with the residue 1 and for $\sigma = \text{Re}(s) > 1$ it is given by the convergent series with positive elements and therefore has no zeroes.

2 Asymptotic behaviour of the Hurwitz zeta-function $\zeta(\sigma,a)$ for large negative $\sigma$.

The Hurwitz zeta-function (or generalised Riemann zeta-function) is defined as a series

\[
\zeta(s,a) = \sum_{r=0}^{\infty} (a + r)^{-s} \quad a > 0
\]
in the complex domain \( \text{Re}(s) > 1 \) and can be analytically continued to a meromorphic function in the whole complex plane with the only pole at \( s = 1 \) (see [1],[2],[3]). When \( a = 1 \) it reduces to the Riemann zeta-function

\[
\zeta(s) = \sum_{k=1}^{\infty} k^{-s}.
\]

The Hurwitz zeta-function can be extended to the whole of the complex \( s \)-plane through the formula

\[
\zeta(s, a) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\gamma} (z^{s-1} e^{-az})^{-1} dz \quad a > 0
\]

in which the integral is taken over a curve starting at ‘infinity’ on the real axis, encircles the origin in a positive direction and returns to the starting point (see [2]). By using an alternative integral formulation for \( \zeta(s, a) \) it can be shown that \( \zeta(s, a) \) is analytic everywhere except for the simple pole at \( s = 1 \).

The Hurwitz zeta-function obviously satisfies the functional relation:

\[
\zeta(s, a) = \zeta(s, n+a) + \sum_{r=0}^{n-1} (r+a)^{-s} \quad n = 1, 2, \ldots
\] (1)

Since each term in this relation is analytic we can assume this relation is true for the whole of the complex \( s \)-plane, except for \( s = 1 \).

In this paper we restrict ourselves to the case when \( s \) is real: \( s = \sigma \in \mathbb{R} \). When \( \sigma \) is negative Hurwitz has found the following Fourier representation for \( \zeta(\sigma, a) \) on the interval \( 0 < a \leq 1 \):

\[
\zeta(\sigma, a) = \frac{2\Gamma(1-\sigma)}{(2\pi)^{1-\sigma}} \sum_{r=1}^{\infty} \frac{\sin(2\pi r a + \frac{1}{2} \pi \sigma)}{r^{1-\sigma}}
\] (2)

From this formula we see that

\[
\frac{\zeta(\sigma, a)}{Q(\sigma)} = \sin(2\pi a + \frac{1}{2} \pi \sigma) + o(1) \quad \text{when} \quad \sigma \to -\infty,
\]

where \( Q(\sigma) = \frac{2\Gamma(1-\sigma)}{(2\pi)^{1-\sigma}} \). Our first theorem proves that this is actually true on a much larger interval.

As part of the theorem proofs we will use the following inequality for the function \( S(p, n) = 1^p + 2^p + \ldots + n^p \):

\[
S(p, n) < n^p \left( \frac{1 - e^{-p}}{1 - e^{-p/n}} \right)
\]

Indeed,

\[
S(p, n) = n^p \left( 1 + (1 - \frac{1}{n})^p + (1 - \frac{2}{n})^p + \ldots + (1 - \frac{n-1}{n})^p \right)
\]

But since \( 1 - x < e^{-x} \) for \( x < 1 \) we have

\[
1 - \frac{1}{n} < e^{-1/n} \quad 1 - \frac{2}{n} < e^{-2/n} \quad \ldots \quad 1 - \frac{n-1}{n} < e^{-(n-1)/n}
\]
and therefore

\[ S(p, n) < n^p \left(1 + e^{-p/n} + e^{-2p/n} + \ldots + e^{-(n-1)p/n}\right) = n^p \left(1 - e^{-p/1 - e^{-p/n}}\right). \]

We shall also need an estimate for the sum of the series: \( \sum_{r=2}^{\infty} \frac{1}{r^{1+p}}. \) Now

\[ \frac{1}{2^{1+p}} + \frac{1}{3^{1+p}} + \frac{1}{4^{1+p}} + \ldots = \frac{1}{2 p} + \frac{1}{3 \cdot 2^p} + \frac{1}{4 \cdot 4^p} + \ldots \]

But \( \left(\frac{3}{n}\right)^p < \frac{1}{n} \) if \( p > 4 \) and \( n \geq 3 \) so we easily deduce

\[ \sum_{r=2}^{\infty} \frac{1}{r^{1+p}} < \left(\zeta(2) - \frac{3}{4}\right)^2^{-p} \quad p > 4 \]

Finally, we will also make particular use of Stirling’s inequality for the gamma function:

\[ (2\pi p)^{1/2} p^p e^{-p} < \Gamma(1+p) < (2\pi p)^{1/2} p^p e^{-p} e^{1/2} \quad p \gg 1 \]

**Theorem 1.** Let \( \sigma = -p, p \geq 0 \) and \( 0 < a < \alpha p \) for some positive \( \alpha \) then the Hurwitz zeta-function satisfies the inequality

\[ \left| \frac{\zeta(-p,a)}{Q(-p)} - \sin(2\pi a - \frac{1}{2} \pi p) \right| < C_1 p^{-1/2}(2\pi e\alpha)^p + C_2 2^{-p}, \quad (3) \]

where \( C_1, C_2 \) are constants, which do not depend on \( p \). In particular, on the interval \( 0 < a < \frac{1}{2\pi e p} \) we have the asymptotic behaviour

\[ \frac{\zeta(-p,a)}{Q(-p)} = \sin(2\pi a - \frac{1}{2} \pi p) + o(1) \quad \text{when} \quad p \to \infty. \]

**Proof**

Let us represent \( a \) as \( n + b, \quad 0 < b \leq 1 \) and with \( n \) integer. It follows from the functional relation (1) that

\[ \left| \frac{\zeta(\sigma,a)}{Q(\sigma)} - \frac{\zeta(\sigma,b)}{Q(\sigma)} \right| = \left| \frac{\zeta(\sigma,n + b)}{Q(\sigma)} - \frac{\zeta(\sigma,b)}{Q(\sigma)} \right| \leq \frac{1}{|Q(\sigma)|} \sum_{r=0}^{n-1} (r + b)^{-\sigma} \]

Since \( 0 < b \leq 1 \) we obviously have

\[ \sum_{r=0}^{n-1} (r + b)^{-\sigma} \leq S(p, n) \quad 0 < p = -\sigma \]

and, as we obtained above,

\[ S(p, n) < n^p \left(1 - e^{-p/1 - e^{-p/n}}\right). \]

Also, from the Stirling formula for the \( \Gamma \)-function we have the following asymptotically exact inequality \( \Gamma(1+p) > (2\pi p)^{1/2} p^p e^{-p} \) and therefore

\[ \frac{1}{Q(-p)} = \frac{(2\pi)^{1+p}}{2\Gamma(1+p)} < \left(\frac{2\pi e}{p}\right)^p \frac{\pi}{\sqrt{2\pi p}} \]
Thus for a large $p$

$$\left| \frac{\zeta(-p, a)}{Q(-p)} - \frac{\zeta(-p, b)}{Q(-p)} \right| < \left\{ \frac{2\pi e n}{p} \right\}^p \frac{\pi}{\sqrt{2\pi p}} \left( \frac{1 - e^{-p}}{1 - e^{-p/n}} \right)$$

$$< \left\{ \frac{2\pi e n}{p} \right\}^p \frac{\pi}{\sqrt{2\pi p}} \left( \frac{1}{1 - e^{-p}} \right)$$

if \( \frac{n}{p} < \alpha \) \hspace{1cm} (4)

which is true since \( \frac{n}{p} < \frac{a}{p} < \alpha \) by assumption.

However, from Hurwitz’ formula (2) it follows that

$$\frac{\zeta(-p, b)}{Q(-p)} - \frac{\zeta(-p, a)}{Q(-p)} = \sum_{r=1}^{\infty} \frac{\sin(2\pi r b - \frac{1}{2} \pi p)}{r^{1+p}}$$

Therefore

$$\left| \frac{\zeta(-p, b)}{Q(-p)} - \sin(2\pi b - \frac{1}{2} \pi p) \right| = \left| \frac{\zeta(-p, b)}{Q(-p)} - \sin(2\pi a - \frac{1}{2} \pi p) \right| < \sum_{r=2}^{\infty} r^{-p-1} < (\zeta(2) \frac{3}{4}) 2^{-p}$$

if $p > 4$. The estimates (4) and (5) imply the theorem.

By a slight modification of the previous arguments we can prove the following result.

**Theorem 2** In the region

$$a > \frac{p}{2\pi e} + \frac{1}{4\pi e} \log p + 1$$

\(\zeta(-p, a)\) is negative if $p$ is sufficiently large.

**Proof** From the same functional relation (1)

$$\frac{\zeta(-p, a)}{Q(-p)} < \frac{\zeta(-p, b)}{Q(-p)} - (a - 1)^p$$

Now assuming that $p$ is sufficiently large we can use Stirling’s inequality \(\Gamma(1 + p) < \sqrt{2\pi p} \left( \frac{p}{\pi} \right)^p e^{\frac{1}{12p}}\), so

$$Q(-p) = \frac{2\Gamma(1 + p)}{(2\pi)^{1+p}} < \frac{\sqrt{2\pi p}}{\pi} \left( \frac{p}{2\pi e} \right)^p e^{\frac{1}{12p}}$$

However, we know from the Hurwitz formula, that when $p \to \infty$

$$\frac{\zeta(-p, b)}{Q(-p)} = \sin(2\pi b - \frac{1}{2} \pi p) + o(1)$$

Therefore if \( \frac{(a-1)^p}{Q(-p)} > 1 \) and $p$ large enough then \( \zeta(-p, a) < 0 \). But as we have shown

$$\frac{(a-1)^p}{Q(-p)} > \left( \frac{2\pi e (a - 1)}{p} \right)^p \sqrt{\frac{\pi}{2p}} e^{-\frac{1}{12p}}$$

which is greater than 1 if

$$a - 1 > \frac{p}{2\pi e} \left( \frac{2\pi}{p} \right)^{\frac{1}{p^2}} e^{\frac{1}{12p}} = \frac{p}{2\pi e} \left( \frac{\log(2\pi) - \log n}{2p} \right) \left( \frac{1}{12p^2} \right)$$

Now, using the inequality \( e^x < 1 + x + x^2 \) for sufficiently small $x$ we find that to guarantee that \( \frac{(a-1)^p}{Q(-p)} > 1 \) it is enough to demand that

$$a > \frac{p}{2\pi e} + \frac{1}{4\pi e} \log p + 1.$$ 

This implies the theorem.
The Real Zeroes of the Hurwitz Zeta-function and Bernoulli Polynomials.

Let us now fix $\sigma = -p$ and consider $\zeta(-p,a)$ as a function of $a$. It follows from theorem 2 that the zeroes of this function for large $p$ are located in the interval $0 < a < \frac{p - 1}{2\pi} + \frac{1}{4\pie} \log p + 1$. For given $p$ let $N(p)$ be the number of real zeroes of $\zeta(-p,a)$, and $A(p)$ be the largest of these zeroes.

**Theorem 3** For $p$ sufficiently large

\[
\frac{p - 1}{2\pi} - \frac{1}{2} < A(p) < \frac{p}{2\pi}e + \frac{1}{4\pie} \log p + 1 \tag{6}
\]

\[
\frac{p - 1}{\pi} - \frac{1}{2} < N(p) < \frac{p - 1}{\pi}e + \frac{1}{2} \log p + 2 \pi e + 2 \tag{7}
\]

The zeroes of $\zeta(-p,a)$ on the interval $0 < a < \frac{p - 1}{2\pi}$ are simple and close to the half-integer lattice: $a = \frac{p}{4} + \frac{l}{2}, \ l \in \mathbb{Z}$.

**Proof** Let us introduce the function $Z_p(a) = \frac{\zeta(-p,a)}{\zeta(-p)}$. From theorem 1 it follows that

\[
Z_p(a) = \sin(2\pi a - \frac{1}{2}\pi p) + o(1)
\]

on the interval $I_p : 0 < a < \frac{p}{2\pi}$ when $p \to \infty$. Actually this is true also for the $k^{\text{th}}$ derivative of $Z_p(a)$ but on a smaller interval $I_{p-k}$. Indeed, from the definition of the Hurwitz zeta-function we see that

\[
\frac{\partial}{\partial a} \zeta(s,a) = (-s)\zeta(s+1,a).
\]

From the property of the $\Gamma$-function $\Gamma(p+1) = p\Gamma(p)$ it follows that

\[
Q(-p) = \frac{2\Gamma(1+p)}{(2\pi)^{1+p}} = \frac{p}{2\pi} Q(-p+1).
\]

Thus the derivative of $Z_p(a)$ is equal to

\[
Z_p'(a) = 2\pi Z_{p-1}(a) = 2\pi \sin(2\pi a - \frac{1}{2}\pi(p-1)) + o(1) = 2\pi \cos(2\pi a - \frac{1}{2}\pi p) + o(1)
\]

on the interval $I_{p-1}$. Similarly we have for the $k^{\text{th}}$ derivative of $Z_p(a)$

\[
Z_p^{(k)}(a) = \sin^{(k)}(2\pi a - \frac{1}{2}\pi p) + o(1)
\]

on the interval $I_{p-k}$.

In particular, on the interval $I_{p-1}$ the function $Z_p(a)$ (and its derivative) tend to $\sin(2\pi a - \frac{1}{2}\pi p)$ (and its derivative) when $p \to \infty$ which ensures that for large $p$ all the roots of $\zeta(-p,a)$ on this interval are simple and located near the points $a = \frac{p}{4} + \frac{l}{2}, \ l \in \mathbb{Z}$. This implies the last statement of the theorem and the lower estimates of (6) and (7).

The upper estimates for $A(p)$ follows directly from theorem 2. To prove the upper estimates for $N(p)$ we need the following simple lemma.

**Lemma** If a function $f(x)$ (with a continuous $n^{\text{th}}$ derivative) on some interval $(a,b)$ has the property that the sign of the $n^{\text{th}}$ derivative is constant throughout the interval then $f$ has no more than $n$ roots on this interval.
Now we apply this lemma to the function \( Z_p(a) \) on the interval \( J_p : (\frac{-p+1}{2\pi e}, \frac{-p+1}{2\pi e} + \frac{1}{4\pi e} \log p + 1) \) to estimate the number of roots there. The idea of this calculation is clear from Figure 2 (in which \( \kappa \equiv \frac{1}{2\pi e} \)).

\[
\begin{align*}
a &= \kappa (p + \frac{1}{2} \log p) + 1 \\
p &= -\sigma
\end{align*}
\]

Figure 2

Using the fact that \( Z_p^{(n)}(a) = (2\pi)^n Z_{p-n}(a) \) we differentiate \( Z_p(a) \) many times until we have a negative function and then apply the lemma. As one can see from Figure 2 if \( n > |y| \) then \( Z_p^{(n)}(a) \) will be negative in the interval \( J_p \) and, as such, cannot have more than \( n \) simple roots in this interval. Now \( y \) is the solution to the equation

\[
\kappa \left( (p - y) + \frac{1}{2} \log(p - y) \right) + 1 = \kappa (p - 1)
\]

or

\[
-y + \frac{1}{2} \log(p - y) + (\kappa^{-1} + 1) = 0
\]

We claim that the solution to this equation for large \( p \) satisfies the inequality

\[
y < \frac{1}{2} \log p + 2\pi e + 1
\]

Indeed the function \( F(y) = -y + \frac{1}{2} \log(p - y) + (2\pi e + 1) \) is monotonically decreasing and

\[
F\left( \frac{1}{2} \log p + 2\pi e + 1 \right) = \frac{1}{2} \log p - 2\pi e - 1 + \frac{1}{2} \log(p - \frac{1}{2} \log p - 2\pi e - 1) + (2\pi e + 1)
\]

\[
= \frac{1}{2} \log(1 - \frac{1}{2p} \log p - \frac{2\pi e + 1}{p}) < 0
\]

for large \( p \). Thus according to the lemma \( Z_p(a) \) has no more than \( \frac{1}{2} \log p + 2\pi e + 1 \) roots on the interval \((\frac{-p+1}{2\pi e}, \frac{-p+1}{2\pi e} + \frac{1}{4\pi e} \log p + 1)\). Since on the interval \([0, \frac{p+1}{2\pi e}]\) we have no more than \( \frac{p+1}{2\pi e} + 1 \) zeroes this proves theorem 3.

As we have already mentioned, when \( p = -\sigma = m, m \in \mathbb{Z}^+ \) the Hurwitz-zeta function reduces to certain polynomials related in a simple way to the Bernoulli polynomials:

\[
\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1}
\]
Theorem 3 applied to these special values of \( p \) gives some estimates on the real positive roots of the Bernoulli polynomials but because of their well-known symmetry properties:

\[
B_m(1 - a) = (-1)^m B_m(a)
\]

we can immediately extend this result for all real roots of \( B_m(a) \). In particular if \( N(m) \) is the number of all real roots of \( B_m(a) \) and \( A(m) \) is the largest of these roots then from Theorem 3 it follows that for a large \( m \)

\[
\frac{m}{2\pi e} - \left( \frac{1}{\pi e} + \frac{1}{2} \right) < A(m) < \frac{m}{2\pi e} + \frac{1}{4\pi e} \log m + \left( 1 - \frac{1}{2\pi e} \right)
\]

\[
\frac{2m}{\pi e} - \left( \frac{2}{\pi e} + 2 \right) < N(m) < \frac{2m}{\pi e} + \log m + \left( 4\pi e - \frac{4}{\pi e} \right)
\]

**Corollary (K. Inkeri).**

\[
\lim_{m \to \infty} \frac{N(m)}{m} = \frac{2}{\pi e}, \quad \lim_{m \to \infty} \frac{A(m)}{m} = \frac{1}{2\pi e}.
\]

**Remark.** H. Delange in [6, 7] has found sharper estimates for \( A(m) \) and \( N(m) \). In particular he showed that the additional logarithmic terms exist in both upper and lower bounds. This should be true also for the real zeroes of the Hurwitz zeta-function but it does not follow from our elementary arguments.

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**References**


