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The existence of Rayleigh-Bloch surface waves

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The existence of pure Rayleigh-Bloch surface waves for a wide class of geometries is proved.

1. Introduction

We are concerned here with the existence of waves propagating along a diffraction grating in two dimensions, in the absence of any incident wave field. Specifically, we consider a grating in the form of a periodic continuous curve Γ of the form \( x = g(y) \), where \( g(y) \) has period \( d \) and \( |g(y)| \) is bounded. (We allow discontinuities in \( g(y) \), in which case Γ is assumed to be parallel to the \( x \)-axis at a point of discontinuity.) We then ask whether non-trivial solutions to the Helmholtz equation \( \nabla^2 \phi + k^2 \phi = 0 \) exist in \( x > g(y) \), which satisfy homogeneous boundary conditions on Γ, and which decay exponentially as \( x \to \infty \). Such solutions are known as pure Rayleigh-Bloch surface waves. The values of \( k^2 \) at which they occur are eigenvalues of the operator \( -\nabla^2 \) subject to the appropriate boundary conditions.

This problem derives from the study of acoustic and electromagnetic waves, and also arises in the analysis of water waves propagating along periodic coastlines in an ocean of constant depth (in the which case the two-dimensional Helmholtz equation arises after factoring out the depth dependence).

No general existence result appears to exist if the boundary condition on Γ is of Neumann type, though in Bonnet-Bendhia & Starling (1994) (hereafter denoted by BBS) it was proved that there are geometries for which pure Rayleigh-Bloch surface waves do exist. Furthermore, such modes have been observed experimentally (Barlow & Karbowiak 1954) and computed numerically for many geometries (see Porter & Evans 1999, and the references cited therein). It should be noted that BBS treat the more general problem in which the boundary Γ is contained within a periodic dielectric medium. The existence of surface waves for diffraction gratings consisting of a periodic dielectric strip (of constant width) on a flat boundary (either Neumann or Dirichlet) is discussed in Grikurov et al. (2000).

The solutions which have been computed have a dominant wavenumber \( \beta \) and this acts as a cut-off for the problem, in that for \( k^2 < \beta^2 \) waves cannot propagate away from the grating. This allows us to use a standard variational approach to search for eigenvalues \( k^2 \) in the interval \( (0, \beta^2) \). The analytic framework for this problem has been set up in BBS, and in this note we simply construct suitable test functions which can be used to prove the existence of such eigenvalues for a wide class of diffraction gratings, using the techniques described in Evans, Levitin & Vassiliev (1994).
2. Proof of existence

Let $\Omega$ be the domain \{\{(x, y) : x > g(y), 0 < y < d\}\}. Let $H^1_b(\Omega)$ be the space of functions $\{\phi \in L^2(\Omega), |\nabla \phi| \in L^2(\Omega)\}$ which also satisfy the boundary conditions

$$\phi|_{y=d} = e^{i\beta d} \phi|_{y=0}, \quad \frac{\partial \phi}{\partial y}|_{y=d} = e^{i\beta d} \frac{\partial \phi}{\partial y}|_{y=0}. \quad (2.1)$$

It is evident that we only need to consider $\beta d$ in an interval of length $2\pi$ and it turns out that it is convenient to restrict $\beta d$ to lie in the interval $(-\pi, \pi]$. The space $H^1_{\beta,0}(\Omega)$ is the subspace of $H^1_b(\Omega)$ which consists of those functions which also satisfy $\phi = 0$ on $\Gamma$.

We denote by $A_N$ (resp. $A_D$) the positive self-adjoint operator $-\nabla^2$ on $\Omega$ restricted to functions in $H^1_b(\Omega)$ (resp. $H^1_{\beta,0}(\Omega)$). Our aim is to establish whether $A_N$ has any eigenvalues. The spectrum of an operator $A$, $\sigma(A)$, (i.e. the set containing all the values of $k^2$ for which $A - k^2 I$ does not have a bounded inverse) is made up of the discrete spectrum, containing any isolated eigenvalues of finite multiplicity, and the essential spectrum, $\sigma_{\text{ess}}(A)$. It is proved in BBS that

$$\sigma_{\text{ess}}(A_N) = \sigma(A_D) = [\beta^2, \infty). \quad (2.2)$$

It immediately follows that there are no eigenvalues $k^2 < \beta^2$ for the Dirichlet problem. From the variational formulation of the problem set up in BBS we also know that the lowest point of the spectrum of $A_N$ is $\inf Q[\psi]$, where the Rayleigh quotient $Q[\psi]$ is given by

$$Q[\psi] = \frac{\int_{\Omega} |\nabla \psi|^2 \, d\Omega}{\int_{\Omega} |\psi|^2 \, d\Omega} \quad (2.3)$$

and the infimum is taken over all functions in $H^1_b(\Omega) \setminus \{0\}$. Thus if we can find a function $\psi \in H^1_b(\Omega) \setminus \{0\}$ for which $Q[\psi] < \beta^2$, then an eigenvalue $k^2 < \beta^2$ must exist.

The boundary curve $\Gamma$ is of the form $x = g(y)$ where $g(y)$ has period $d$ and it is assumed that $\Gamma$ has a well-defined normal at all but a finite number of points. The results in (2.2) rely on the fact that if the domain $\Omega$ were truncated at $x = X$ then the spectrum of the Laplacian would be entirely discrete. To ensure that this is the case we assume that $\Omega$ has no outward pointing cusps (i.e. the boundary $\Gamma$ has no inward pointing cusps). We will assume that the line $x = 0$ is positioned so that $\max g(y) = -\min g(y) = a$, say (the maximum value may be attained more than once in any one period of $g$). Let $y_1$ be the smallest value of $y$ in $[0, d]$ for which $g(y) = a$ and let $y_2$ be the largest such value. Clearly $0 \leq y_1 \leq y_2 \leq d$. For a given periodic boundary, excluding the case $g(y) \equiv 0$, we can always choose axes so that

$$0 < y_1 = d - y_2 < d. \quad (2.4)$$

We then divide the domain $\Omega$ into four regions. Region I is $\{0 < y < y_1, g(y) < x < a\}$, region II is $\{y_2 < y < d, g(y) < x < a\}$, region III is $\{y_1 < y < y_2, g(y) < x < a\}$, and region IV is $\{0 < y < d, a < x\}$. Region III may be disconnected and may of course be empty. These remarks are illustrated in Figure 1.
The existence of Rayleigh-Bloch surface waves

Figure 1. First type of geometry under consideration.

We introduce a positive parameter $\epsilon$ and define our test function by

$$
\psi = \begin{cases} 
    e^{i\beta y} + \frac{\epsilon}{2} \cos\left(\pi x/a\right) & \text{in I} \\
    e^{i\beta y} + \epsilon^{1/2} e^{i\beta d} \cos\left(\pi x/a\right) & \text{in II} \\
    e^{i\beta y} & \text{in III} \\
    e^{i\beta y} e^{-\epsilon(x-a)} & \text{in IV}.
\end{cases}
$$

(2.5)

This function satisfies (2.1) and clearly belongs to $H^1_{\beta}(\Omega)\setminus\{0\}$. A straightforward calculation reveals that, for $\beta \neq 0$,

$$
\beta^{-2} Q[\psi] = \frac{1}{2} \frac{d\epsilon^{-1} + \int_{y_1}^{y_2} [a - g(y)] \, dy + O(\epsilon)}{\frac{1}{2} d\epsilon^{-1} + \int_{y_1}^{y_2} [a - g(y)] \, dy + 4\epsilon^{1/2} \frac{\pi}{d} J + O(\epsilon)} = 1 - \frac{8\epsilon^3}{J} + O(\epsilon^2),
$$

(2.6)

where

$$
J = \frac{a}{\pi d} \int_{0}^{y_1} \left(2 - \sin\frac{\pi g(y)}{2a} - \sin\frac{\pi g(d-y)}{2a}\right) \cos \beta y \, dy.
$$

(2.7)

The integrand in this expression is strictly positive on $(0, y_1)$ since $y_1 \leq \frac{1}{2}d$, $|\beta d| \leq \pi$, and both $|g(y)|$ and $|g(d-y)|$ are strictly less than $a$ for $y \in (0, y_1)$. Thus $J > 0$ and $\epsilon$ can be chosen sufficiently small so that $Q[\psi] < \beta^2$.

This completes the existence proof for the geometries described above, which includes some, but not all, of the geometries for which pure Rayleigh-Bloch modes have been constructed numerically. For example, it includes an infinite array of circles (for which Rayleigh-Bloch modes were constructed in McIver, Linton & McIver 1998) since the line of symmetry of the array can be treated as a Neumann boundary. However, it does not include the infinite array of staggered plates considered by Koch (1983), nor the angled ellipses treated in Porter & Evans (1999). Existence can be proved for these and similar cases as follows.

Consider a periodic array of closed contours, each one defined by two curves $y = f_1(x)$ and $y = f_2(x)$ ($0 < f_1(x) \leq f_2(x) < d$) as illustrated in Figure 2. As in the previous case, we assume that the boundary has a well-defined normal at all but a finite number of points but we allow discontinuities in $f_1$ and $f_2$, in which case the boundary is assumed to be parallel to the $y$ axis at a point of discontinuity.

The fluid domain is again divided into four regions as shown in Figure 2 and we use
the test function

\[ \psi = \begin{cases} 
  e^{i\beta y} + \epsilon^{1/2} \cos(\pi x/2a) & \text{in I} \\
  e^{i\beta y} + \epsilon^{1/2} e^{i\beta d} \cos(\pi x/2a) & \text{in II} \\
  e^{i\beta y} e^{\epsilon(x-a)} & \text{in III} \\
  e^{i\beta y} e^{-\epsilon(x-a)} & \text{in IV} 
\end{cases} \tag{2.8} \]

and find that, for \( \beta \neq 0 \),

\[ \beta^{-2} Q[\psi] = 1 - 2\epsilon^{3/2} J + O(\epsilon^2), \tag{2.9} \]

where

\[ J = \frac{1}{\beta d} \int_{-a}^{a} \cos \frac{\pi x}{2a} \left( \sin[\beta f_1(x)] + \sin[\beta(d - f_2(x))] \right) \, dx. \tag{2.10} \]

Since \( |\beta d| \leq \pi \), the integrand is strictly positive for \( x \in (-a, a) \) and so \( J > 0 \). Hence \( \epsilon \) can be chosen sufficiently small so that \( Q[\psi] < \beta^2 \) in this case also.

The types of staggered-plate array treated by Koch (1983) are not covered by geometries of the above type because the plates overlap so that each period contains sections from more than one plate in the array. However, such situations are easily included by defining \( a \) to be the maximum value which allows just one plate in the region \( |x| < a \), \( 0 < y < d \), as shown in Figure 3. We can then proceed exactly as in the previous example, the presence of parts of the geometry in regions I and IV making no difference.

We have proved that pure Rayleigh-Bloch surface waves exist for an extremely wide class of periodic geometries, but have not attempted to characterize the most general
The existence of Rayleigh-Bloch surface waves

form of geometry for which modes exist. Some generalizations are obvious. For example, in the any of the examples above we can clearly add further closed contours within regions I and II.

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REFERENCES


