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Normal families and value distribution in connection with composite functions

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Abstract

We prove a value distribution result which has several interesting corollaries. Let \( k \in \mathbb{N} \), let \( \alpha \in \mathbb{C} \) and let \( f \) be a transcendental entire function with order less than \( 1/2 \). Then for every nonconstant entire function \( g \), we have that \( (f \circ g)^{(k)} - \alpha \) has infinitely many zeros. This result also holds when \( k = 1 \), for every transcendental entire function \( g \). We also prove the following result for normal families. Let \( k \in \mathbb{N} \), let \( f \) be a transcendental entire function with \( \rho(f) < 1/k \), and let \( a_0, \ldots, a_{k-1}, a \) be analytic functions in a domain \( \Omega \). Then the family of analytic functions \( g \) such that

\[
(f \circ g)^{(k)}(z) + \sum_{j=0}^{k-1} a_j(z)(f \circ g)^{(j)}(z) \neq a(z),
\]

in \( \Omega \), is a normal family. \(^1\)

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1 Introduction

In [1], Hinchliffe proves the following result which provides a criterion for normal families in connection with composite functions.

**Theorem 1.1 (Hinchliffe, [1]):** Let \( f \) be a transcendental meromorphic function in the plane, and let \( \Omega \) be a domain in \( \mathbb{C} \). If \( \mathbb{C}^* \setminus f(\mathbb{C}) = \emptyset, \{\infty\} \) or \( \{\alpha, \beta\} \), where \( \alpha \) and \( \beta \) are two distinct values in \( \mathbb{C}^* = \mathbb{C} \cup \{\infty\} \), then the family \( \mathcal{G} = \{g : g \text{ is analytic in } \Omega, f \circ g \text{ has no fixpoints in } \Omega\} \) is a normal family in \( \Omega \).

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We note that this criterion is that \((f \circ g)(z) \neq z\) in \(\Omega\), or that \((f \circ g)^{(0)}(z) - a(z)\) has no zeros in \(\Omega\), where \(a(z) \equiv z\), for \(g \in G\). Theorem 1.1 then motivates the idea of a criterion for normal families in connection with composite functions involving \((f \circ g)^{(k)}(z) \neq 0\), for \(k \in \mathbb{N}\). This idea is reinforced by the following theorem and corollary by Langley and Zheng, where \(n.e.\) is used as an abbreviation for “nearly everywhere”, that is, to denote the phrase “outside a set of finite measure”.

**Theorem 1.2 (Langley and Zheng, [2])** Let \(k \in \mathbb{N}\). Suppose that \(f\) and \(g\) are transcendental entire functions of finite order. Suppose also that
\[
\mathcal{N}(r, 1/(f \circ g)^{(k)}) = O(T(r, g)) \quad (n.e.) .
\]
Then
\[
T(r, f) \neq o(r^{1/k}) \quad \text{as} \quad r \to \infty.
\]

**Corollary 1.3** Let \(k \in \mathbb{N}\). Suppose that \(f\) is a transcendental entire function such that \(\rho(f) < 1/k\). Suppose that \(g\) is an entire function of finite order such that
\[
(f \circ g)^{(k)}(z) \neq 0
\]
on \(\mathbb{C}\). Then \(g\) is a polynomial.

We note that the example \(f(z) = e^z\) shows that Corollary 1.3 cannot be strengthened to \(\rho(f) \leq 1/k\).

And so, given a transcendental function \(f\) with \(\rho(f) < 1/k\) for some \(k \in \mathbb{N}\), the Bloch Principle (see [3]), Theorem 1.1 and Corollary 1.3 motivate the question whether the family \(G\) of analytic functions \(g\) in a domain \(\Omega\), such that \((f \circ g)^{(k)}(z) \neq 0\) in \(\Omega\), or more generally, \((f \circ g)^{(k)}(z) \neq Q(z)\) for some analytic function \(Q\), is a normal family. This is true, and is a special case of the following result.

**Theorem 1.4** Let \(k \in \mathbb{N}\). Let \(f\) be a transcendental entire function with \(\rho(f) < 1/k\). Let \(a_0, \ldots, a_{k-1}, a\) be analytic functions in a domain \(\Omega\). Then
\[
G = \{g : g \text{ is analytic in } \Omega, (f \circ g)^{(k)}(z) + \sum_{j=0}^{k-1} a_j(z)(f \circ g)^{(j)}(z) \neq a(z) \text{ in } \Omega\}
\]
is a normal family in \(\Omega\).

In the proof of Theorem 1.4, we use the following theorem, which is an interesting value distribution result in its own right.

**Theorem 1.5** Let \(k \in \mathbb{N}\). Let \(f\) be a transcendental entire function with \(\rho(f) < 1/2\). Let \(g\) and \(Q\) be polynomials, with \(g\) nonconstant. Then
\[
(f \circ g)^{(k)} - Q
\]
has infinitely many zeros.

We note that in Theorem 1.5, we must have that \(\rho(f) < 1/2\), since we apply a theorem of cos \(\pi \rho\) type. However, if \(Q \equiv 0\), we can prove Theorem 1.5 for \(\rho(f) < 1\), for the extended case where \(g\) is a nonconstant entire function. We state the result as follows.
**Theorem 1.6** Let \( f \) be a transcendental entire function with \( \rho(f) < 1 \). Let \( g \) be a nonconstant entire function. Then \((f \circ g)'\) has infinitely many zeros.

From Theorem 1.5 and Theorem 1.6, we prove the following corollary which strengthens Corollary 1.3 and which is used in the proof of Theorem 1.4.

**Corollary 1.7** Let \( k \in \mathbb{N} \). Suppose that \( f \) is a transcendental entire function such that \( \rho(f) < 1/k \). Suppose that \( g \) is an entire function of finite order such that

\[
(f \circ g)^{(k)}(z) \neq 0
\]
on \( \mathbb{C} \). Then \( g \) is constant.

Finally, we note that Theorem 1.5 and Theorem 1.6 have the following corollaries.

**Corollary 1.8** Let \( k \) be an integer, \( k \geq 2 \). Let \( f \) be a transcendental entire function with \( \rho(f) < 1/k \). Let \( \alpha \in \mathbb{C} \). Then for every nonconstant entire function \( g \),

\[
(f \circ g)^{(k)} - \alpha
\]

has infinitely many zeros.

Again, although the \( k = 1 \) case is omitted in Corollary 1.8, we can prove the \( k = 1 \) case when \( g \) is a transcendental entire function. We state the result as follows.

**Corollary 1.9** Let \( f \) be a transcendental entire function with \( \rho(f) < 1 \). Let \( \alpha \in \mathbb{C} \). Then for every transcendental entire function \( g \),

\[
(f \circ g)' - \alpha
\]

has infinitely many zeros.

Since the proof of Theorem 1.4 depends on Theorem 1.5, Theorem 1.6 and Corollary 1.7, we prove these results in § 2, § 3 and § 4 respectively. We then prove Theorem 1.4 in § 5. Finally, we prove Corollary 1.8 and Corollary 1.9 in § 6.

## 2 Proof of Theorem 1.5

The following lemma is a version of Taylor’s theorem and is easily proved by induction.

**Lemma 2.1** If \( f \) is an entire function and \( a \in \mathbb{C} \), then for \( k \in \mathbb{N} \) we have

\[
f(z) = f(a) + (z - a)f'(a) + \ldots + \frac{(z - a)^{k-1}}{(k-1)!}f^{(k-1)}(a) + \int_a^z \frac{(z - t)^{k-1}}{(k-1)!}f^{(k)}(t)dt.
\]

We also need the following lemma. We include the proof here for completeness.
Lemma 2.2 Let $k \in \mathbb{N}$. Let $P_1$ and $P_2$ be polynomials of degree $m$ and $n$ respectively, with $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Then we can choose a straight line $\Gamma$ from 0 to $\infty$ such that

$$I = \left| \int_0^z \frac{(z-t)^{k-1}}{(k-1)!} P_1(t) e^{P_2(t)} dt \right| \leq c|z|^{k+m},$$

as $z \to \infty$ along $\Gamma$, for some positive constant $c$.

**Proof** The behaviour of $P_2$ is dominated by the leading term $b_n t^n$. Setting $t = re^{i\theta}$, we have that $|e^{b_n t^n}| = e^{(\alpha \cos(n\theta) + \beta \sin(n\theta)) t^n}$ for some $\alpha, \beta \in \mathbb{R}$, not both 0. Then choose $\theta$ such that $\alpha \cos(n\theta) + \beta \sin(n\theta) = -d < 0$, and let $\Gamma$ be the straight line $z = re^{i\theta}$, for $0 \leq r < \infty$. Then for $t$ on $\Gamma$ between 0 and $z$ we have that $|e^{P_2(t)}| = e^{-dr^n+O(r^{n-1})} \to 0$ as $r \to \infty$, for fixed $\theta$ as above. Thus $|e^{P_2(t)}| \leq c_0$ for some positive constant $c_0$, and since $P_1$ has degree $m$, we have that for $t$ on $\Gamma$ between 0 and $z$, $|P_1(t)| \leq c_1 |t|^m$ for some positive constant $c_1$, as $t \to \infty$. The result follows.

Finally, we need a theorem of $\cos \pi \rho$ type, as follows. We refer the reader to [4] for further reading.

**Theorem 2.3 ([4])** Let $f$ be a nonconstant entire function with $\rho(f) = \rho < 1/2$. For $r > 0$, define $A(r)$ and $B(r)$ as follows

$$A(r) = \inf \{ \log |f(z)| : |z| = r \}$$
$$B(r) = \sup \{ \log |f(z)| : |z| = r \}.$$

If $\rho < \alpha < 1/2$, then

$$\log \text{dens}\{ r : A(r) > (\cos \pi \alpha) B(r) \} \geq 1 - \rho/\alpha$$

where if $E$ is a subset of $(1, +\infty)$ the lower logarithmic density of $E$ is defined by

$$\log \text{dens}(E) = \liminf_{r \to \infty} \left( \int_1^r \frac{\chi(t) dt}{t} \right) / \log r$$

where $\chi(t)$ is the characteristic function of $E$.

We now prove Theorem 1.5.

**Proof of Theorem 1.5** We use a proof by contradiction. Suppose that $(f \circ g)^{(k)} - Q$ has $m$ zeros in $\mathbb{C}$, for some $m \in \mathbb{N} \cup \{0\}$. Then we can write

$$(f \circ g)^{(k)}(z) - Q(z) = P_1(z) e^{P_2(z)}$$

for some polynomials $P_1$ and $P_2$ of degree $m$ and $n$ respectively, with $m$ as above and $n \in \mathbb{N}$. Then, by Lemma 2.1, we have for $a = 0$ that

$$(f \circ g)(z) = (f \circ g)(0) + z(f \circ g)'(0) + \ldots + \frac{z^{k-1}}{(k-1)!} (f \circ g)^{(k-1)}(0) +$$
$$+ \int_0^z \frac{(z-t)^{k-1}}{(k-1)!} (f \circ g)^{(k)}(t) dt$$
$$= Q_{k-1}(z) + \int_0^z \frac{(z-t)^{k-1}}{(k-1)!} (Q(t) + P_1(t) e^{P_2(t)}) dt,$$

where $Q_{k-1}(z)$ is a polynomial of degree $k-1$. If $k = 1$, then $P_1(z) = P_2(z)$, but if $k \geq 2$, then $P_1(z)$ and $P_2(z)$ are both polynomials of degree $m$ and $n$ respectively, with $m$ as above and $n \in \mathbb{N}$. Then, by Lemma 2.1, we have for $a = 0$ that

$$(f \circ g)(z) = (f \circ g)(0) + z(f \circ g)'(0) + \ldots + \frac{z^{k-1}}{(k-1)!} (f \circ g)^{(k-1)}(0) +$$
$$+ \int_0^z \frac{(z-t)^{k-1}}{(k-1)!} (f \circ g)^{(k)}(t) dt$$
$$= Q_{k-1}(z) + \int_0^z \frac{(z-t)^{k-1}}{(k-1)!} (Q(t) + P_1(t) e^{P_2(t)}) dt,$$
where \( Q_{k-1} \) is a polynomial of degree at most \( k - 1 \). Then
\[
|f \circ g(z)| \leq |Q_{k-1}(z)| + \left| \int_0^z (z-t)^{k-1} \frac{Q(t)dt}{(k-1)!} \right| + \left| \int_0^z (z-t)^{k-1} P_1(t) e^{P_2(t)} dt \right|.  \tag{2}
\]
For the remainder of this proof, we use \( c_j \) to denote positive constants.

Since \( Q_{k-1} \) is a polynomial of degree at most \( k - 1 \), and since \( Q \) is a polynomial of degree \( q \) say, \( q \geq 0 \), we have that \( |Q_{k-1}(z)| \leq c_1|z|^{k-1} \) as \( z \to \infty \) and that \( |Q(t)| \leq c_2|t|^q \) as \( t \to \infty \) on any straight line \( \Gamma \) between 0 and \( z \). Then, as \( t \to \infty \), we have that integrating along any straight line \( \Gamma \) between 0 and \( z \) gives
\[
\left| \int_0^z (z-t)^{k-1} \frac{Q(t)dt}{(k-1)!} \right| \leq c_2|z|^{k-1} \int_0^z \frac{O(|t|^q)dt}{(k-1)!} \leq c_3|z|^{k+q}
\]
since \( |z-t| \leq |z| \).

In particular, by Lemma 2.2, we can choose a straight line path \( \Gamma \) from 0 to \( \infty \) such that
\[
\left| \int_0^z \frac{Q(t)dt}{(k-1)!} \right| \leq c_4|z|^{k+m}
\]
as \( z \to \infty \) along \( \Gamma \). Then we have that \( |f \circ g(z)| \leq c_5|z|^{k+q+m} \), which gives
\[
\log |f \circ g(z)| \leq c_6 \log |z| \quad \tag{3}
\]
as \( z \to \infty \) along \( \Gamma \).

Since \( \rho = \rho(f) < 1/2 \), we can apply Theorem 2.3 to \( f \). For \( r > 0 \), define \( A(r) \) and \( B(r) \) as in Theorem 2.3. Then for \( \rho < \alpha < 1/2 \) we have
\[
\log \text{dens}\{r : A(r) > (\cos \pi \alpha)B(r)\} \geq 1 - \rho/\alpha.  \tag{4}
\]
Next, since \( g \) is a polynomial and is nonconstant, we have that \( |g(z)| \geq c_7|z| \) as \( z \to \infty \), then by (3), we have that
\[
\log |f(g(z))| = \log |f \circ g(z)| \leq c_6 \log |z| \leq c_8 \log |g(z)| \quad \tag{5}
\]
as \( z \to \infty \) along \( \Gamma \). Now choose \( R \) large such that \( R \in \{r : A(r) > (\cos \pi \alpha)B(r)\} \). Choose \( w \) such that \( |w| = R \) and \( w = g(z) \) for some \( z \) on \( \Gamma \). Then by (5), we have that
\[
(\cos \pi \alpha)B(R) < A(R) \leq \log |f(w)| \leq c_8 \log R.
\]
This is a contradiction since \( f \) is a transcendental function, which implies that \( B(R)/\log R \to +\infty \) as \( R \to \infty \).

## 3 Proof of Theorem 1.6

We need the following lemma.

**Lemma 3.1** If \( f \) is a transcendental entire function with \( \rho(f) < 1 \), then \( f' \) has infinitely many zeros.

**Proof of Theorem 1.6** Since \( g \) is a nonconstant entire function, we have by Picard’s theorem that \( g \) omits at most one value in \( \mathbb{C} \). Since \( f \) is a transcendental entire function with \( \rho(f) < 1 \), we have by Lemma 3.1 that \( f' \) has infinitely many zeros. Then since \( g \) omits at most one of these zeros, we have that \( f'(g(z)) \) has infinitely many zeros. Therefore, since \( (f \circ g)'(z) = f'(g(z))g'(z) \), we have that \( (f \circ g)' \) has infinitely many zeros.
4 Proof of Corollary 1.7

Proof of Corollary 1.7 By Corollary 1.3, we have that \( g \) is a polynomial. However, by Theorem 1.6 for \( k = 1 \) and by Theorem 1.5 for \( k \geq 2 \), if \( g \) is a nonconstant polynomial then \( (f \circ g)^{(k)} \) has infinitely many zeros. Therefore \( g \) is constant.

5 Proof of Theorem 1.4

First we need the following result, which is a version of Hurwitz’ theorem (see [3]).

Lemma 5.1 Let \( (f_n) \) be a sequence of analytic functions on a domain \( \Omega \), which converge spherically uniformly on compact subsets to a function \( f \). Let \( (s_n) \) be a sequence of analytic functions tending to 0 on some disc \( B(\alpha, \delta) = \{ z : |z - \alpha| < \delta \} \subseteq \Omega \), for some \( \delta > 0 \). If \( f \not\equiv 0 \) and \( f(\alpha) = 0 \), then for large \( n \), we have \( f_n(z) = s_n(z) \) for some \( z \) near \( \alpha \).

Next, we note that Lemma 5.1 has the following corollary, which we will use in the proof of Theorem 1.4. We provide a proof for completeness.

Corollary 5.2 Let \( k \in \mathbb{N} \). Let \( \Omega \) be the open unit disc \( B(0,1) \). Let \( a \) be an analytic function on \( \Omega \). Let \( (f_n) \) be a sequence of analytic functions on \( \Omega \), such that \( f_n(z) \not\equiv a(z) \) on \( \Omega \). Let \( (z_n) \) be a sequence of points tending to \( z_0 \in \Omega \), and let \( (\rho_n) \) be a positive sequence tending to 0. Suppose \( g \) is an entire function such that

\[
\lim_{n \to \infty} \rho_n^k f_n(z_n + \rho_n z) = g(z)
\]

locally uniformly on \( \mathbb{C} \). Then either \( g \equiv 0 \) on \( \mathbb{C} \), or \( g(z) \not\equiv 0 \) on \( \mathbb{C} \).

Proof Suppose there exists \( \alpha \in \mathbb{C} \) such that \( g(\alpha) = 0 \). If \( g \equiv 0 \), then we are done. Otherwise, we note that \( z_n + \rho_n z \in \Omega \) for \( n \) large, and that \( (\rho_n^k f_n) \) is a sequence of analytic functions which converge to \( g \) locally uniformly on \( \mathbb{C} \). We note also that since \( a \) is analytic, and therefore bounded near \( z_0 \), and since \( (\rho_n^k a(z_n + \rho_n z)) \) is a sequence tending to 0, then \( (\rho_n^k a(z_n + \rho_n z)) \) is a sequence of functions tending to 0, for \( n \) large, on \( B(\alpha, \delta) \), for some \( \delta > 0 \). Then by Lemma 5.1, we obtain

\[
\rho_n^k f_n(z_n + \rho_n z) = \rho_n^k a(z_n + \rho_n z)
\]

for \( n \) large, for some \( z \) near \( \alpha \). Since \( (\rho_n^k) \) is a positive sequence, we therefore have that \( f_n(z_n + \rho_n z) = a(z_n + \rho_n z) \), which is a contradiction since \( z_n + \rho_n z \in \Omega \) for \( n \) large.

We need the following lemma, which is called the Zalcman lemma (see [5]).

Theorem 5.3 A family \( \mathcal{G} \) of analytic functions in the open unit disc \( B(0,1) \) is not normal at the origin, if and only if there exist a sequence of functions \( g_n \in \mathcal{G} \), a sequence of points \( z_n \to 0 \), a positive sequence \( \rho_n \to 0 \) and a non-constant entire function \( g \) on \( \mathbb{C} \) such that

\[
\lim_{n \to \infty} g_n(z_n + \rho_n z) = g(z)
\]

locally uniformly on \( \mathbb{C} \), with respect to the spherical metric, such that the spherical derivative of \( g \) is bounded, \( g^2(z) \leq g^2(0) = 1 \).

Finally, we need the following lemma which is an immediate consequence of the definition of the order of a meromorphic function, using the Ahlfors-Shimizu form of the Nevanlinna characteristic.
Lemma 5.4 Let $f$ be a meromorphic function with bounded spherical derivative. Then the order of $f$ is at most 2.

We now prove Theorem 1.4.

Proof of Theorem 1.4 Since normality is a local property, we can assume, without loss of generality, that $\Omega$ is a disc and $a_0, \ldots, a_{k-1}, a$ are bounded on $\Omega$. Using a linear change of variables $h(z) = g(\alpha + \beta z)$, and the fact that $(f \circ h)^{(j)}(z) = \beta^j(f \circ g)^{(j)}(\alpha + \beta z)$, for a suitable choice of $\alpha, \beta \in \mathbb{C}$, we may assume that $\Omega$ is $B(0, 1)$. Suppose that $G$ is not normal on $\Omega$. Then $G$ must be not normal at least one point in $\Omega$, and without loss of generality we can suppose that $G$ is not normal at 0.

Since $G$ is a family of analytic functions, we can apply Lemma 5.3. Then there exist points $(z_n)$ tending to 0, a sequence $(g_n)$ in $G$, a positive sequence $(\rho_n)$ tending to 0 and a nonconstant entire function $g$ such that
\begin{equation}
\rho_n(z_n + \rho_n z) \to g(z)
\end{equation}
locally uniformly on $\mathbb{C}$, with respect to the spherical metric, with $g^2(z) \leq 1$. Then since $g$ has bounded spherical derivative, we have by Lemma 5.4 that $g$ is a function of finite order.

Next, since $f$ is an entire function, we have that
\begin{equation}
(f \circ h_n)(z) \to (f \circ g)(z),
\end{equation}
locally uniformly on $\mathbb{C}$. Then by the Weierstrass theorem (see [3]), for $j \in \mathbb{N},$
\begin{equation}
(f \circ h_n)^{(j)}(z) = \rho_n^j(f \circ g_n)^{(j)}(z_n + \rho_n z) \to (f \circ g)^{(j)}(z),
\end{equation}
locally uniformly on $\mathbb{C}$. However, since each $g_n \in G$, we have that for $z_n + \rho_n z \in \Omega,$
\begin{align}
F_n(z) &= (f \circ g_n)^{(k)}(z_n + \rho_n z) + \sum_{j=0}^{k-1} a_j(z_n + \rho_n z)(f \circ g_n)^{(j)}(z_n + \rho_n z) \neq a(z_n + \rho_n z).
\end{align}

Then we have that
\begin{equation}
\rho_n^k F_n(z) = \rho_n^k(f \circ g_n)^{(k)}(z_n + \rho_n z) + \sum_{j=0}^{k-1} \rho_n^{k-j} a_j(z_n + \rho_n z) \rho_n^j(f \circ g_n)^{(j)}(z_n + \rho_n z)
\end{equation}
\begin{equation}
\neq \rho_n^k a(z_n + \rho_n z).
\end{equation}

Next, since $\rho_n^{k-j} \to 0$ for $j = 0, \ldots, k-1$, and since the $a_j$ are assumed bounded on $\Omega$, we have by (7) that
\begin{equation}
\lim_{n \to \infty} \rho_n^k F_n(z) = (f \circ g)^{(k)}(z)
\end{equation}
locally uniformly on $\mathbb{C}$. However, we can write $F_n(z) = G_n(z_n + \rho_n z)$ where $G_n(z) \neq a(z)$ on $\Omega$, and so we have by Corollary 5.2, that either $(f \circ g)^{(k)}(z) \equiv 0$ on $\mathbb{C}$, or $(f \circ g)^{(k)}(z) \neq 0$ on $\mathbb{C}$.

Case 1: $(f \circ g)^{(k)}(z) \equiv 0$ on $\mathbb{C}$.
Then integrating this equation $k-1$ times, we have that $(f \circ g)'(z) = P_{k-2}(z)$, where $P_{k-2}$ is a polynomial of degree at most $k-2$. Since $P_{k-2}$ has at most $k-2$ zeros, counting multiplicities, $(f \circ g)'(z) = f'(g(z)).g'(z)$ has also. However, $f$ is a transcendental entire function and $\rho(f) < 1$, and so by Lemma 3.1, we have that $f'$ has infinitely many zeros on $\mathbb{C}$. Then since $g$ is a nonconstant entire function, we must have that $g$ omits infinitely many zeros of $f'$ on $\mathbb{C}$, which is a contradiction by Picard's Theorem.
Case 2: $(f \circ g)^{(k)}(z) \neq 0$ on $\mathbb{C}$.

Suppose first that $k = 1$. Then by Theorem 1.6, we have that $(f \circ g)'$ has infinitely many zeros on $\mathbb{C}$ and so we have a contradiction.

Suppose second that $k \geq 2$. Then since $f$ is a transcendental entire function with $\rho(f) < 1/k$ and $g$ is a nonconstant entire function of finite order, we have by Corollary 1.7 that $(f \circ g)^{(k)}$ has at least one zero in $\mathbb{C}$. Therefore we have a contradiction.

Therefore $\mathcal{G}$ is a normal family.

6 Proof of Corollary 1.8 and Corollary 1.9

Proof of Corollary 1.8 Suppose that $(f \circ g)^{(k)} - \alpha$ has finitely many zeros.

Suppose first that $g$ is a function of finite order. If $g$ is a polynomial, then we have a contradiction by Theorem 1.5. If $g$ is a transcendental function, then since $N(r, 1/((f \circ g)^{(k)} - \alpha)) = O(\log r) = o(T(r, g))$ and since $T(r, f) = o(r^{1/k})$, then we have a contradiction by Theorem 1.2.

Suppose second that $g$ is a function of infinite order. Then, by Lemma 5.4, we have that $g$ has unbounded spherical derivative, that is, we can choose a sequence of points $(\alpha_n)$ tending to $\infty$ such that $g^\sharp(\alpha_n) \to \infty$ as $n \to \infty$. Then the family of functions $\{g_n(z) = g(\alpha_n + z) : n \in \mathbb{N}\}$ is not a normal family on the open unit disc $B(0,1)$. Then by Theorem 1.4 we have a contradiction.

Proof of Corollary 1.9 Suppose that $(f \circ g)' - \alpha$ has finitely many zeros. If $g$ is a function of finite order, then by the argument in Corollary 1.8, we have that $g$ is a polynomial. This is a contradiction since $g$ is a transcendental function. If $g$ is a function of infinite order, then by the argument in Corollary 1.8, we have a contradiction by Theorem 1.4.

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