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A New Method to Construct Integrable Equations

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Oct. 2005

Abstract

We use the prolongation method to construct pairs of compatible equations in two independent variables. A common potential is then used to construct an equation in three independent variables. The AKNS–system still contains a spectral parameter.
1) Introduction

It is well-known that the sine-Gordon equation is compatible with the modified Korteweg-de Vries equation. By this we mean that, if a function \( \varphi \) satisfies

\[
\varphi_{xx} = \sin \varphi
\]
\[
\varphi_t = \varphi_{xxx} + \frac{1}{2} \varphi_x^3
\]

then the \( t \)-derivative of the first equation is equal to the \( xz \)-derivative of the second equation. These equations are not genuinely three-dimensional since the function satisfies one equation in one pair of variables and the other equation in the other pair. Both equations admit a potential, viz.

\[
p_x = -\cos \varphi \quad , \quad p_z = \frac{1}{2} \varphi_z^2
\]
\[
q_x = -\cos \varphi \quad , \quad q_t = \sin \varphi \quad \varphi_{xx} - \frac{1}{2} \cos \varphi \quad \varphi_x^2
\]

Since \( p_x = q_x \) we may, without loss of generality, set \( p = q \). The introduction of the potential does not, however, lead to a new equation in three variables since the \( z \)-derivatives decouple from the others. We shall show below, that there are compatible integrable equations for which this is not the case. There, the introduction of a potential leads to an integrable equation in more than two dimensions.

2) Equations with \( A_1^{(1)} \) as Prolongation Algebra

The use of the Estabrook-Wahlquist prolongation method has been discussed in many places and we point the reader to some of the main texts [1]. Here, we consider equations with the loop algebra \( A_1^{(1)} \) as prolongation algebra. Recall that this Lie-algebra is described by the generators \( X_i, Y_i, Z_i, i \in \mathbb{Z} \), and the commutation relations

\[
\begin{align*}
[X_i, Y_k] &= Z_{i+k} \quad , \quad [X_i, Z_k] = -Y_{i+k} \quad , \quad [Y_i, Z_k] = X_{i+k} \\
[X_i, X_k] &= [Y_i, Y_k] = [Z_i, Z_k] = 0
\end{align*}
\]

from which the structure constants \( C_{ik}^{\ell} \) can be read off. One now considers the Cartan-Maurer forms for this algebra. They can be written in terms of 1-forms \( \xi, \eta \) and \( \vartheta \) concisely as

\[
d\xi = \eta \vartheta \quad , \quad d\eta = \xi \eta \quad , \quad d\vartheta = -\xi \vartheta
\]

where we have omitted the "\( \lambda \)" which normally appears between differential forms. Also, the forms are to be considered as formal (to avoid issues of convergence) expansions in a spectral parameter \( \lambda \) such as

\[
\xi = \sum_{i=-\infty}^{\infty} \xi_i \lambda^i
\]
The Cartan–Maurer forms hence are spelt out in detail in terms of the coefficients of $\lambda^i$ as, $i = -\infty \ldots \infty$,

$$d\xi_i = \sum_{n=-\infty}^{\infty} \eta_n \delta_{i-n}, \quad d\eta_i = \sum_{n=-\infty}^{\infty} \xi_n \eta_{i-n}, \quad d\delta_i = -\sum_{n=-\infty}^{\infty} \xi_n \delta_{i-n} \quad (5)$$

To generate integrable equations, i.e., those which admit an infinite dimensional prolongation algebra which is the definition we shall adopt here, one sets almost all – al but a finite number – of $\xi_i$, $\eta_j$ and $\delta_k$ to zero. In terms of a bounded subset $S \subset \mathbb{Z}^3$ and a multi-index $s = \{i, j, k\}$ referring to the indices of $\xi_i$, $\eta_j$ and $\delta_k$ this splits the system (5) naturally into the $\rho$ and $\sigma$ system

$$\rho: \quad d\xi_i = \sum \eta_n \delta_{i-n}, \quad d\eta_i = \sum \xi_n \eta_{i-n}, \quad d\delta_i = -\sum \xi_n \delta_{i-n}, \quad i, n, i-n \in S \quad (6)$$

$$\sigma: \quad 0 = \sum \eta_n \delta_{i-n}, \quad 0 = \sum \xi_n \eta_{i-n}, \quad 0 = -\sum \xi_n \delta_{i-n}, \quad n, i-n \in S, \ i \notin S$$

It should be noted that by construction

$$d\rho = 0 \text{ mod} (\rho, \sigma), \quad d\sigma = 0 \text{ mod} (\rho, \sigma) \quad (7)$$

i.e., there is an integral manifold to the system. The $\sigma$-system is solved algebraically, the number of independent 1-forms is Cartan's genus $g$. The coefficients when expressing the various 1-forms in terms of the linearly independent ones become, at a later stage, functions on the integral manifold. The differential $\rho$-system yields not only exact 1-forms to be used as coordinate differentials but also equations between the above-mentioned coefficients which combine to nonlinear equations.

It should be stressed that not every choice of $S$ yields an equation in, at least, 2 dimensions or, for that matter, an “interesting” equation. So far, we have no a priori method for deciding this [2]; nevertheless, “trial and error” methods still yield interesting results. For instance, the example (1) cited in the introduction – to be precise, the Sinh–Gordon equation and the other sign in the mKdV equation, resulting from (1) by $\phi \to i\phi$ – is provided by $\{\eta_{-1}, \delta_{-1}, \xi_0, \eta_1, \delta_1, \xi_2, \eta_3, \delta_3\} \neq 0$, all others $= 0$.

In terms of an AKNS-system, the equations (6) follow from the integrability of the system

$$d\Phi = F \Phi, \quad F = \begin{pmatrix} \xi & \eta \\ \delta/2 & -\xi \end{pmatrix} \quad (8)$$

Here, $F$ is a matrix-valued 1-form expanded in powers of $\lambda$. Let $x_i$, $i = 1 \ldots g$, be the coordinates on the solution manifold. With

$$M_i = \partial_{x_i} F \quad (9)$$

(N.b., the $M_i$ are still power series in $\lambda$), we get $g$ equations

$$\partial_{x_i} \Phi = M_i \Phi \quad (10)$$
with \(g(g-1)/2\) integrability conditions which are to be sorted with respect to \(\xi\). The classical AKNS-system has \(g = 2\) and thus only one integrability condition. There is, however, no reason why these notions should not be generalized to systems with genus \(\geq 2\).

In what follows, we shall examine two sets of choices for \(S\) which both, eventually, will yield non-trivial equations in 3 dimensions which are, by construction, integrable.

3) A Generalization of the (m)KdV Equation

This example is provided by keeping \(\{(\xi, \eta, \theta)_0, (\xi, \eta, \theta)_1, (\xi, \eta, \theta)_2, \xi_3\}\) and setting all other 1-forms to 0. The \(\phi - \sigma\)-systems become

\[
\begin{align*}
\phi: & \quad d\xi_0 = \eta_0 \theta_0 \\
& \quad d\eta_0 = \xi_0 \eta_0 \\
& \quad d\theta_0 = -\xi_0 \eta_0 \\
& \quad d\xi_1 = \eta_0 \theta_1 + \eta_1 \theta_0 \\
& \quad d\eta_1 = \xi_0 \eta_1 + \xi_1 \eta_0 \\
& \quad d\theta_1 = -\xi_0 \eta_1 - \xi_1 \eta_0 \\
& \quad d\xi_2 = \eta_0 \theta_2 + \eta_2 \theta_1 + \eta_2 \theta_0 \\
& \quad d\eta_2 = \xi_0 \eta_2 + \xi_1 \eta_1 + \xi_2 \eta_0 \\
& \quad d\theta_2 = -\xi_0 \eta_2 - \xi_1 \eta_1 - \xi_2 \eta_0 \\
& \quad d\xi_3 = \eta_1 \theta_3 + \eta_2 \theta_1
\end{align*}
\]

(12)

\[
\begin{align*}
\sigma: & \quad \eta_0 \xi_3 = \xi_1 \eta_2 + \xi_2 \eta_1 \\
& \quad \eta_0 \xi_3 = \xi_1 \theta_2 + \xi_2 \theta_1 \\
& \quad \eta_2 \theta_3 = 0 \\
& \quad \eta_1 \xi_3 = \xi_2 \eta_2 \\
& \quad \eta_1 \xi_3 = \xi_2 \theta_2 \\
& \quad \eta_2 \xi_3 = 0 \\
& \quad \theta_2 \xi_3 = 0
\end{align*}
\]

(13)

(N.b., the ordering of the 1-forms \(\xi_i\) etc. is as given by the computer algebra system we are using.) The last two equations of the \(\sigma\)-system tell us that \(\eta_2\) and \(\theta_2\) must be proportional to \(\xi_3\). Hence, introducing 0-forms \(a\) and \(b\), \(\eta_2 = a \xi_3\) and \(\theta_2 = b \xi_3\). Proceeding in this manner in finding the algebraic solution of the \(\sigma\)-system we get the solution - \(a, b, e, f, g, h\) being 0-forms,

\[
\begin{align*}
\eta_2 &= a \xi_3 \\
\theta_2 &= b \xi_3 \\
\eta_1 &= a \xi_2 + c \xi_3 \\
\theta_1 &= b \xi_2 + f \xi_3 \\
\eta_0 &= a \xi_1 + e \xi_2 + g \xi_3 \\
\theta_0 &= b \xi_1 + f \xi_2 + h \xi_3
\end{align*}
\]

(14)
and hence the system admits four linearly independent 1-forms, $\xi_0 \ldots \xi_3$; the solution manifold is 4-dimensional.

Putting this result into the $\varphi$-system, we first notice that

$$d\xi_3 = d\xi_2 = 0.$$  

Thus we set

$$\xi_3 = dt, \quad \xi_2 = dy$$  \hspace{1cm} (15a)

with $t$ and $y$ being coordinates on the solution manifold. Some further calculations reveal new exact 1-forms $dx$ and $dz$ such that

$$\xi_1 = -a \ b \ dt + dx, \quad \xi_0 = -a \ b \ dy - (a \ f + b \ e) \ dt + dz$$  \hspace{1cm} (15b)

We use $x, y, z$ and $t$ as parameters on the integral manifold. This manifold is now described by the $\varphi$-system which contains forms such as "$da \ dx$" or "$dx \ dt$". Using $x, y, z$ and $t$ as coordinates on the solution manifold, the $0$-forms $a, b, e, f, g, h$ become functions of the coordinates; to distinguish functions from forms we shall use the corresponding capital letters. The following relations are obtained by sorting the $\varphi$-system with respect to the 2-forms "$dx \ dt$" etc..

$$E = A_x$$  
$$F = -B_x$$  
$$G = A_y + A^2 \ B$$  
$$H = -B_y + A \ B^2$$  

and the main equations are

$$A_y = A_{xx} - A^2 \ B$$  \hspace{1cm} (17a)  
$$B_y = -B_{xx} + A \ B^2$$  

$$A_t = A_{xxx} - 3 \ A \ B \ A_x$$  
$$B_t = B_{xxx} - 3 \ A \ B \ B_x$$  \hspace{1cm} (17b)

The $z$-dependence corresponds to a simple scaling $A_z = A$ and $B_z = -B$. The equations (17a) are a real version of the nonlinear Schrödinger equation - the original one to be obtained by the substitutions $y \rightarrow iy, A \rightarrow \psi, B \rightarrow \psi^*$. On the other hand, equations (17b) generalizes the Korteweg-de Vries equation (for $B = 1$) or the modified Korteweg-de Vries equation (for $B = \pm A$). Both equations (17a, 17b) have been known for a long time. What is new, is that both equations are compatible and that both admit a potential, viz.
\[ P_x = A \ B \]
\[ P_y = B \ A_x - A \ B_x \]

\[ Q_x = A \ B \]
\[ Q_t = A \ B_{xx} + B \ A_{xx} - A_x \ B_x - \frac{3}{2} A^2 B^2 \]

Disregarding functions of integration we have \( Q = P \). Eliminating \( A, B \) and their derivatives we get one equation for \( P \)

\[ P_t = P_{xxx} - \frac{3}{2} P_x^2 + \frac{3}{4} P_x \left( P_y^2 - P_x^2 \right) \]  \hspace{1cm} (19)

This is a new generalization of the KdV equation in three dimensions. By construction, this equation is integrable in the sense that its integrability arises from the integrability of a system \((17a, 17b)\) – one has to eliminate \( A \) and \( B \) in favour of \( P_x \) and \( P_y \).

Note that, here, the system corresponding to \((8)\), respectively \((9)\), still contains a spectral parameter \( \lambda \). The standard procedure to generalize a given AKNS-system is to replace \( \lambda \) by \( \partial_z \). This can, of course, also be applied here. In this manner one would obtain a 4-dimensional system of equations which is guaranteed to be integrable. The practical calculations have, so far, proven to be prohibitively long.

## 4 Another new System

In this section we shall consider the system arising from \( \{ (\xi, \eta, \vartheta)_1, (\xi, \eta, \vartheta)_2, (\xi, \eta, \vartheta)_3 \} \)

\[ = 0, \text{ all others} = 0 \]. The systems are

\[ p: \]
\[ \begin{align*}
  d\xi_1 &= 0 \\
  d\eta_1 &= 0 \\
  d\vartheta_1 &= 0 \\
  d\xi_2 &= \eta_1 \ \vartheta_1 \\
  d\eta_2 &= \xi_1 \ \eta_2 \\
  d\vartheta_2 &= -\xi_1 \ \vartheta_1 \\
  d\xi_3 &= \eta_1 \ \vartheta_2 + \eta_2 \ \vartheta_1 \\
  d\eta_3 &= \xi_1 \ \eta_2 + \xi_2 \ \eta_1 \\
  d\vartheta_3 &= -\xi_1 \ \vartheta_2 - \xi_2 \ \vartheta_1
\end{align*} \] \hspace{1cm} (20)

\[ \sigma:\]
\[ \begin{align*}
  \eta_1 \ \vartheta_3 &= \eta_1 \ \eta_3 - \eta_2 \ \vartheta_2 \\
  \xi_1 \ \eta_3 &= \eta_1 \ \xi_3 - \xi_2 \ \eta_2 \\
  \xi_1 \ \vartheta_3 &= \eta_1 \ \xi_3 - \xi_2 \ \vartheta_3 \\
  \eta_2 \ \vartheta_3 &= \eta_2 \ \eta_3 \\
  \xi_2 \ \eta_3 &= \eta_2 \ \xi_3 \\
  \xi_2 \ \vartheta_3 &= \eta_2 \ \xi_2 \\
  \vartheta_3 \ \eta_3 &= 0 \\
  \xi_3 \ \eta_3 &= 0 \\
  \xi_3 \ \vartheta_3 &= 0
\end{align*} \] \hspace{1cm} (21)
Again, the last three equations of the $\sigma$-system tell us that $(\xi, \eta, \phi)_3$ must be mutually proportional. The algebraic solution of the $\sigma$-system is given by:

\[
\begin{align*}
\eta_3 &= a \xi_3 \\
\phi_3 &= b \xi_3 \\
\eta_2 &= a \xi_2 + e \xi_3 \\
\phi_2 &= b \xi_2 + f \xi_3 \\
\eta_1 &= a \xi_1 + e \xi_2 + g \xi_3 \\
\phi_1 &= b \xi_1 + f \xi_2 + h \xi_3
\end{align*}
\]

(22)

It should be noted here, that the choice of keeping the $\xi_i$'s as basic 1-forms is merely a matter of taste, any other combination could have been used with equal justification.

There is also a more serious matter to be considered. The system obviously has genus 3 and there are three equally obvious exact 1-forms, $(\xi, \eta, \phi)_1$, in the $\rho$-system. In this particular instance, however, we choose, guided by experience gained from the mKdV-equation and the Harry Dym-equation, to use the $\xi_i$'s, and exact 1-forms constructed from them, as basic 1-forms resp. coordinate differentials. Inserting the solution of the $\sigma$-system into the $\rho$-system we find that

\[
\xi_3 = \frac{\text{dt}}{\sqrt{1 + 2 a b}}
\]

(23)

This suggests the substitution

\[
a \rightarrow e^b \tan a \quad , \quad b \rightarrow \tan a e^{-b} / 2
\]

(24)

which leaves us with (the attribution of a $\sqrt{2}$ to $e^{sb}$'s is, again, a matter of taste.)

\[
\begin{align*}
\xi_3 &= \cos a \ dt \\
\eta_3 &= \sin a \exp(b) \ dt \\
\phi_3 &= \sin a \exp(-b) \ dt / 2
\end{align*}
\]

The relevant equations from the $\rho$-system yield yet another exact 1-form, viz.

\[
\xi_2 = \cos a \left( dx - \left( e \exp(-b) / 2 + f \exp(b) \right) \sin a \cos a \ dt \right)
\]

(25a)

As the third exact 1-form we take

\[
\xi_1 = dz
\]

(25b)

Again we replace 1-forms by functions and obtain
\[ E = e^B ( A_z + \tan A \ B_z ) \]
\[ F = \frac{1}{2} e^B ( - A_z + \tan A \ B_z ) \]
\[ G = \frac{e^B}{\cos A} ( A_x + \tan A \ B_x ) \]
\[ H = \frac{1}{2} e^B \cos A ( - A_x + \tan A \ B_x ) \]

and the main equations become

\[ A_x = \cos^2 A \left( \sin A \ B_{zz} + \cos A \ A_z \ B_z \right) \tag{27a} \]
\[ B_x = \cos A \left( \cos A \sin A \ A_{zz} + B_z^2 - A_z^2 \right) \]
\[ A_t = \cot A \sin A \ B_{zz} + \left( \frac{3}{2} \cos^2 A - 2 \right) A_x \ B_z + A_z \ B_x \tag{27b} \]
\[ B_t = \cot A \ A_{zz} - A_x \ A_z + \frac{3}{2} \cos^2 A \ B_x \ B_z \]

These equations are not truly coupled since the \( x \)-derivatives in the second set of equations can be eliminated by the first set.

There is, of course, the problem of which 1-form to choose as coordinates. In terms of parametrization of the solution manifold, one is as good as the other. Indeed, one could, with equal justification, have used \( \xi_1, \tau_1 \) and \( \vartheta_1 \) as coordinate differentials. In this case, \( dx \) and \( dt \) would have become potentials of another set of equations which, however, would have described the same integral manifold. Transformations exchanging a coordinate and a potential are known to relate apparently different equations, for instance the mKdV equation and the Harry Dym–equation.

Experience has it that it is, generally, preferable to keep the \( \xi_1 \) as independent 1-forms even though no mathematical argument in this direction has been found.

Also, at first sight it is not obvious which of the two potentials related to \( \tau_1 \) and \( \vartheta_1 \) to use when trying to construct a 3-dimensional equation from the two sets of 2-dimensional equations (27a, 27b). In any case, it should be one potential related to both equations, up to functions of integration; it is clear that, since \( \tau_1 \) and \( \vartheta_1 \) are linearly independent, they do not correspond to a common potential. There is, however, another potential hidden in the system (27a, 27b) – in fact, there are two, but, since their \( z \)-derivatives agree, we use the argument from the previous section, disregard appearing functions of integration and treat them as the same. We then find:

\[ C_x = \sin^2 A \ B_z \]
\[ C_z = \cos^{-1} A \]
\[ C_t = \sin A \cos A \ A_{zz} + \left( \frac{1}{2} \cos^2 A - 1 \right) A_z^2 - \frac{3}{2} \cos^2 A \sin^2 A \ B_z^2 \]

\( C_x \) and \( C_z \) are independent functions and, eliminating \( A \) and \( B \) from (28), we get

\[ C_t = \frac{1}{C_z^3} C_{zzz} + \frac{3}{2} \left( \frac{2}{C_z^2} - 1 \right) C_{zz} - \frac{1}{2} \left( 1 - C_z^2 \right) C_x^2 \tag{29} \]
While this is certainly a complicated equation, it is integrable by construction.

5) Concluding Remarks

We should remark that one can recover equation (27a) from the system \( \{(\xi, \eta, \varphi)\}_1, (\xi, \eta, \varphi)_2 \neq 0 \). On the other hand, the system \( \{(\xi, \eta, \varphi)_1, (\xi, \eta, \varphi)_2, (\eta, \varphi)_3 \} = 0 \) yields a complicated 2-dimensional system of three coupled equations for three variables. Similarly, the system \( \{(\eta, \varphi)_1, (\xi, \eta, \varphi)_2, (\xi, \eta, \varphi)_3 \} \) gives a rather lengthy system of four equations in four dependent variables. Compared to these additional systems of equations, the system (17a, 17b) can be described as "simple".

These observations go to show that, while the present paper demonstrates that there are hitherto unsuspected ways to derive integrable equations in more than 2 dimensions, "simple" integrable systems such as the classical equations – sine-Gordon, KdV, nonlinear Schrödinger, to name but some of them – are few and far between.

Furthermore, attention should be drawn to the fact that there is no method is known to discern whether a given equation in 3 dimensions is actually integrable. The present study may shed some light on this problem. Moreover, keeping in mind that AKNS-systems pertaining to equations (17a, 17b) and (27a, 27b) still contain a spectral parameter \( \lambda \), those systems lend themselves to be generalized to 4 dimensions by the standard replacement \( \lambda \to \varphi \) and this may open a route to integrable 4-dimensional equations.

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