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On Robust Stability of Time-Variant Discrete-Time Nonlinear Systems With Bounded Parameter Perturbations

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Abstract—The upper and lower bounds for asymptotic stability (AS) of a time-variant discrete-time nonlinear system with bounded parameter perturbations are provided. The analysis is undertaken for a class of nonlinear relaxation systems with the saturation nonlinearity of sigmoid type. Based upon the theory of convex polytopes and underlying linear relaxation equation, the bounds of the stability region for such a nonlinear system are derived for every time instant.

Index Terms—Convergence, nonlinear systems, polytopes, stability.

I. INTRODUCTION

ASYMPTOTIC stability (AS) and global asymptotic stability (GAS) are important issues in system theory [1]–[3]. For linear systems, the notions of AS and GAS have the same meaning, due to linearity [1], [4], [5], whereas in nonlinear system theory this is not the case because of possible existence of local minima.

The problem of GAS of a time-variant mth-order difference equation

$$y(n) = \mathbf{a}^T(n)\mathbf{y}(n-1)$$
$$= a_1(n)y(n-1) + \cdots + a_m(n)y(n-m) \quad (1)$$

for $\|\mathbf{a}(n)\|_1 < 1$ has a special importance, and was addressed in [6]. Further, the condition of convexity on the set $\mathcal{C}_0$ of the initial values $[y(n-1), \ldots, y(n-m)]^T \in \mathbb{R}^m$ and on the set $\mathcal{A} \in \mathbb{R}^m$ of all allowable values of $\mathbf{a}(n) = [a_1(n), \ldots, a_m(n)]^T$ was imposed and the results from [6] for $a_i \geq 0$, $i = 1, \ldots, n$ were derived as a pure consequence of convexity of the sets $\mathcal{C}_0$ and $\mathcal{A}$ [7]. The state–space approach to the system (1) was further addressed in [8] and [7]. The main result in [6] was that the system (1) is asymptotically stable if $|a_i(n)| \leq a_v^+$, $i = 1, \ldots, m$, $\forall n$ and $\sum_{i=1}^{m} a_v^+ < 1$. This was further elaborated in [7] from the viewpoint of convex sets and the conditions of aperiodic and asymptotic stability were derived.

For nonlinear systems, the convergence of relaxation systems has been shown in the sense of contraction mapping (CM) and fixed point iteration (FPI) techniques [9], [10], especially for neural networks representing nonlinear systems. However, the results for convergence of the relaxation system in the nonlinear case, in the sense of allowing the parameters of the nonlinear system to belong to a certain region, are still in their infancy. Let us therefore consider a system described by

$$y(n) = \Phi(\mathbf{a}^T(n)\mathbf{y}(n-1))$$
$$= \Phi(a_1(n)y(n-1) + \cdots + a_m(n)y(n-m)) \quad (2)$$

where Φ is some saturation nonlinear function which belongs to a class of logistic functions, as shown in Fig. 1. In this paper we show that depending upon the slope b of ϕ for a nonlinear system whose parameters satisfy $|a_i(n)| \leq a_v^+$, $i = 1, \ldots, m$, $\forall n$ and $\sum_{i=1}^{m} a_v^+ < 1$, equation (2) can either converge or diverge and derive, respectively, the upper and lower bounds for the regions of convergence in the parameter space of the vector $\mathbf{a}(n)$.

II. THEORETICAL BACKGROUND

A subset $C \subset \mathbb{R}^d$ is called a convex set if $\alpha_1 x_1 + \alpha_2 x_2$ belongs to C for all $x_1, x_2 \in C$ and all $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = 1$ and $\alpha_1, \alpha_2 \geq 0$. By a convex polytope, or simply a polytope, we mean a set which is the convex hull $\text{conv}\{x_1, \ldots, x_n\}$ of a nonempty finite set $\{x_1, \ldots, x_n\}$, i.e., the set of all combinations

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$
$$\lambda_i \geq 0 \quad i = 1, 2, \ldots, n \quad \sum_{i=1}^{n} \lambda_i = 1. \quad (3)$$

A hyperplane $H(y, \alpha)$ is defined as

$$H(y, \alpha) = \{x \in \mathbb{R}^d \mid \langle x, y \rangle = \alpha, \text{ for } y \in \mathbb{R}^d, \alpha \in \mathbb{R}\}. \quad (4)$$

A halfspace $K(y, \alpha)$ is defined as

$$K(y, \alpha) = \{x \in \mathbb{R}^d \mid \langle x, y \rangle \leq \alpha, \text{ for } y \in \mathbb{R}^d, \alpha \in \mathbb{R}\} \quad (5)$$

where $\langle \cdot, \cdot \rangle$ denotes the vector inner product. In order to derive boundaries of a set whose closure is the relaxation equation (1), our aim is to show that the coefficient vectors $\mathbf{a}(n)$ represent normal vectors of halfspaces which are supporting halfspaces of a set of allowable output values of (1).

Let C be a nonempty closed convex set in $\mathbb{R}^d$. A supporting halfspace of C is a closed halfspace $K \in \mathbb{R}^d$ such that $C \subset K$ and $H \cap C \neq \emptyset$, where H denotes the bounding hyperplane of K. A supporting hyperplane of C is a hyperplane $H \in \mathbb{R}^d$ which bounds a halfspace. Analytically, a hyperplane $H(y, \alpha)$

is a supporting hyperplane of a nonempty closed convex set C if and only if $\alpha = \max_{x \in C} \langle x, y \rangle$ or $\alpha = \min_{x \in C} \langle x, y \rangle$.

A subset $Q \subset \mathbb{R}^d$ is called a polyhedral set if Q is the intersection of a finite number of closed halfspaces, or $Q = \mathbb{R}^d$. This means that every polyhedral set $Q \in \mathbb{R}^d$ is closed and convex and has a representation $Q = \bigcap_{i=1}^n K(x_i, \alpha_i)$, with the boundary bdQ defined as $bdQ = \bigcup_{i=1}^n K(x_i, \alpha_i) \cap Q$. It is clear now that a nonempty bounded polyhedral set in $\mathbb{R}^d$ is a polytope.

The upshot of the previous analysis is that every compact convex set C has an external representation as the intersection of closed halfspaces, namely, the supporting halfspaces, and an internal representation as the convex hull of a (unique) minimal set, namely, the set of extreme points (vertices). The sets which have a finite internal representation are polytopes. The sets which have finite external representations are sets which are intersections of a finite number of closed halfspaces. These sets are called polyhedral sets. We will assume that the solution set of (1) is nonempty and will refer to solution sets as polytopial rather than polyhedral sets.

III. GEOMETRIC STABILITY CONSIDERATION IN THE LINEAR CASE

As the condition for GAS given in [6] was $\sum_i |a_i(n)| < 1 \Leftrightarrow \|\mathbf{a}(n)\|_1 < 1$, the set of equations $y(n) = \mathbf{a}(n)^T \mathbf{y}(n-1)$, $n = 1, 2, \ldots$ represent hyperplanes [11], [12] whose intersection builds convex polytopes [11], [12], [5], [13]. Actually, polytopes represent a convex hull of a family of points in a vector space [11], [12]. In other words, for $M \subset \mathbb{R}^n$ and $S = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_1, \ldots, \lambda_n \geq 0, \sum_{i=1}^n \lambda_i = 1\}$, a polytope may be defined as a mapping

$$\varphi : M \times S \to \mathbb{R}^d$$

$$\varphi((x_1, \ldots, x_n), (\lambda_1, \ldots, \lambda_n)) = \sum_{i=1}^n \lambda_i x_i, \quad x_1, \ldots, x_n \in M \tag{6}$$

which is precisely the convex hull of M. Now, the set $M \times S$ is compact by the compactness of φ and S and φ is continuous. Hence, we can expect that for any sequence in M, there is a subsequence which converges to a point in M. That is compactness which ensures convergence of sequences within a polytope.

An example in $\mathbb{R}^2$, is shown in Fig. 2. A very important point toward the derivation of bounds for (2) is the following claim.

Claim 1 [12]: A subset $C \subset \mathbb{R}^d$ is convex if and only if any convex combination of points from C is again in C.

Therefore, the hyperspaces $K_i = K(y, \mathbf{a}^T(i)\mathbf{y}(i-1))$ form a convex polytope, as shown in Fig. 2.

IV. AS RESULT FOR THE NONLINEAR CASE

Notice that the nonlinear curve in Fig. 1 is concave in the right halfplane, and convex in the left halfplane. That means that any line connecting two points on the curve lies, respectively, below and above the curve, i.e., $|\Phi(x)| < |x|$ for $b < 4$ and $|\Phi(x)| > |x|$ for some subsets of x and $b > 4$. Now, for the relaxation Eq. (2), if the curve in Fig. 1 lies below the line $y = x$, let us consider the situation
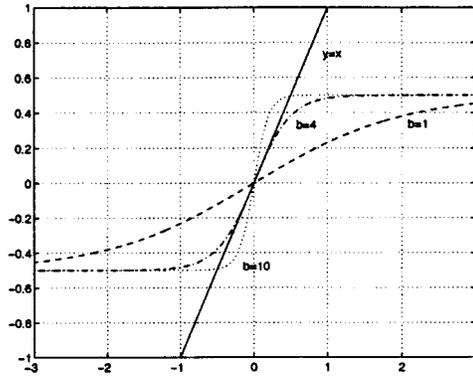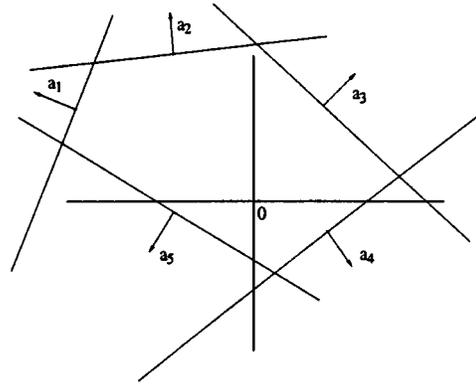

Fig. 1. Various logistic functions $\Phi(b\,x)$.

Fig. 2. Solution polytope of $y(n) = \mathbf{a}^T(n)\mathbf{y}(n-1)$ in $\mathcal{R}^2$, $n = 1, 2, \ldots, 5$.

shown in Fig. 3. Obviously, we have the case of contraction [14]–[16]. Here, for x in the appropriate convex polytope, $d_1 = \|H(x, \mathbf{a}^T(4)\mathbf{y}(3))\|$ and $d_2 = \|H(x, \mathbf{a}^T(5)\mathbf{y}(4))\|$ and $d_1 > d_2$. Hence, the inner circles, whose radii are the norms $\|H_i\| = \|H(x, \mathbf{a}^T(i)\mathbf{y}(i-1))\|$, are the supremum points of the boundary for the nonlinear case (2). Therefore, after every iteration in (1) the convex polytope C_i which represents the boundary is $C_i = \bigcap_{n=1}^i K(x, \mathbf{a}^T(n)\mathbf{y}(n-1))$ and the distance $\|H(x, \mathbf{a}^T(i)\mathbf{y}(i-1)) - \mathbf{0}\| = \|H(x, \mathbf{a}^T(i)\mathbf{y}(i-1))\|$ is always greater than the one defined by $\|\Phi(\mathbf{a}^T(i)\mathbf{y}(i-1))\|$. In other words

$$\|H(x, \mathbf{a}^T(i)\mathbf{y}(i-1)) - \mathbf{0}\| = \|H(x, \mathbf{a}^T(i)\mathbf{y}(i-1))\|$$
$$= \sup_i \|\Phi(\mathbf{a}^T(i)\mathbf{y}(i-1))\|. \tag{7}$$

This allows us to state the following theorem.

Theorem 1 (The Upper Bound): The upper bound $\sup_i \Phi(\mathbf{a}^T(i)\mathbf{y}(n-1))$ of the set of allowable values for the nonlinear system (2) is the ball $\mathcal{B}_i$ whose radius is defined by

$$\sup_i \Phi(\mathbf{a}^T(i)\mathbf{y}(i-1)) = \min_i \|H(x, \mathbf{a}^T(i)\mathbf{y}(i-1))\|. \tag{8}$$

In a similar manner, for the system where the nonlinear curve in Fig. 1 crosses the line $y = x$, we have

$$\|H(x, \mathbf{a}^T(i)\mathbf{y}(i-1)) - \mathbf{0}\| = \|H(x, \mathbf{a}^T(i)\mathbf{y}(i-1))\|$$
$$= \inf_i \|\Phi(\mathbf{a}^T(i)\mathbf{y}(i-1))\| \tag{9}$$

and the corresponding theorem.

MANDIC & CHAMBERS: ON ROBUST STABILITY OF TIME-VARIANT DISCRETE-TIME NONLINEAR SYSTEMS WITH BOUNDED PARAMETER PERTURBATIONS
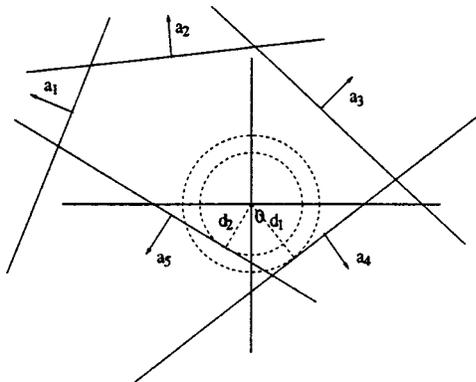
187

Fig. 3. Boundaries of the solution balls of $y(n) = \Phi(\mathbf{a}^T(n)\mathbf{y}(n-1))$ in $\mathcal{R}^2$.

Theorem 2 (The Lower Bound): The lower bound $\inf_i \Phi(\mathbf{a}^T(i)\mathbf{y}(n-1))$ of the set of allowable values for the nonlinear system (2) is the ball $\mathcal{B}_i$ whose radius is defined by

$$\inf_i \Phi(\mathbf{a}^T(i)\mathbf{y}(i-1)) = \max_i \|H(x, \mathbf{a}^T(i)\mathbf{y}(i-1))\|. \quad (10)$$

For the multidimensional case we have balls instead of circles and hyperdiamonds instead of polygons. The following generalization for the multidimensional case readily stems from the previous analysis. Notice that in the first case the system converges, whereas in the second case it diverges. The radii of the balls $\mathcal{B}_i$, $i = 1, 2, \ldots$ given in Theorems 1 and 2 are the normal distance between the origin and the appropriate hyperplanes H_i, $i = 1, 2, \ldots$ at the time instant i.

Notice that in the case of convergence of (2)

$$\mathcal{B}_{i+1} \subset \mathcal{B}_i \quad (11)$$

whereas in the case of divergence of (2)

$$\mathcal{B}_i \subset \mathcal{B}_{i+1} \quad (12)$$

which define the relation of nesting within the set of balls $\mathcal{B}_i$, $i = 1, 2, \ldots, i, \ldots$, i.e., for $r > i > k$ we have $\cdots \subset \mathcal{B}_r \subset \mathcal{B}_{r-1} \subset \cdots \subset \mathcal{B}_{i+1} \subset \mathcal{B}_i \subset \cdots \subset \mathcal{B}_{k+1} \subset \mathcal{B}_k \subset \mathcal{B}_{k-1} \subset \cdots$.

Example 1: For the logistic nonlinearity given by

$$\Phi(x) = \frac{1}{1 + e^{-bx}} \quad (13)$$

contractivity for $\gamma < 1$ is defined by [10]

$$|\Phi(x) - \Phi(y)| \leq \gamma|x - y| = |\Phi'(\xi)(x - y)|$$
$$= |\Phi'(\xi)|\|x - y\|, \quad \xi \in (x, y) \quad (14)$$

The first derivative of Φ is

$$\Phi'(x) = \frac{be^{-bx}}{(1 + e^{-bx})^2} \quad (15)$$

which is strictly positive and whose maximum value is $b/4$ for $x = 0$. Hence, the stability condition of (2) becomes $b \cdot (m a_v^+) \leq 4$, which converges or diverges depending whether Φ is respectively a contraction ($b < 4$) or an expansion ($b > 4$). Hence, the upper and lower bounds for this case stem directly from the underlying linear process, as shown in Theorems 1 and 2. Notice that in this case AS implies GAS.

V. CONCLUSION

The upper and lower bounds for the set of output values for a class of nonlinear relaxation equations with $\|\mathbf{a}(n)\|_1 < 1$ are derived by starting from the underlying linear system and proceeding to the final result. It has been shown that the upper and lower bounds for robust convergence in the case of parameter uncertainties are, respectively, the minimal and maximal distance between the origin and the halfplane which is the bound of a halfspace adjacent to the coefficient vector $\mathbf{a}(i)$ at the time instant i. That being the case, the sets of allowable values defined as above exhibit nesting features which make the solution sets, which are convex balls $\mathcal{B}_i \subset \mathcal{C}_i \subset \mathbb{R}^d$, converge toward $\mathbf{0} \in \mathbb{R}^d$.

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