Spectral theory of the Laplace operator on manifolds with generalized cusps

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Spectral theory of the Laplace operator on manifolds with generalized cusps

by
Nikolaos Roidos

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Abstract

In this thesis we study the Laplace operator $\Delta$ acting on p-forms, defined on an $n$ dimensional manifold with generalized cusps. Such a manifold consists of a compact piece and a noncompact one. The noncompact piece is isometric to the generalized cusp. A generalized cusp $[1, \infty) \times N$ is an $n$ dimensional noncompact manifold equipped with the warped product metric $dx^2 + x^{-2a} h$, where $N$ is a compact oriented manifold, $h$ is a metric on $N$ and $a > 0$ is a fixed constant. First we regard the cusp separately, where by using separation of variables we determine the spectral properties of the Laplacian and we determine explicitly the structure of the continuous part of the spectral theorem. Using this result, we meromorphically continue the resolvent of the Laplace operator to a certain Riemann surface, which we determine. By standard gluing techniques, the resolvent of the Laplace operator $\Delta$ on the manifold with cusp is meromorphically continued to the same Riemann surface. This enables us to construct the generalized eigenforms for the original manifold without boundary. That describes the continuous spectral decomposition of $\Delta$ and determines some of its important properties, like analyticity and the existence of a functional equation. We also define the stationary scattering matrix and find its analytic properties and its functional equation.
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Chapter 1

Introduction

In this thesis, we study the spectral theory of the Laplace operator on manifolds with noncompact ends. In particular, we consider the Laplace operator, $\Delta$, acting on $p$-forms, defined on a connected $n$-dimensional Riemannian manifold, $M$, which consists of a compact part $M_0$, having possible disconnected boundary $N$, and a noncompact part $M_1$, having the same boundary, which are glued together. The compact part $M_0$ can be any compact connected Riemannian manifold with boundary. The boundary $N$ will be a closed and oriented Riemannian manifold. It can be disconnected, so there are several ends. The noncompact part $M_1$ will be a generalized cusp. A generalized cusp is a noncompact manifold $[1, \infty) \times N$, endowed with the warped product metric $dx^2 + x^{-2a}h$, where $a \in (0, \infty)$ and $h$ is a fixed metric on $N$. Such a cusp is considered to be generalized because the constant $a$ appears in the metric. When $a$ goes to zero, the cusp becomes a cylinder, and when $a$ goes to $\infty$, it could be thought of as approaching the $n$-dimensional hyperbolic cusp. Thus, the full manifold will be of the form $M = M_0 \cup M_1$, and the metric $g$ on $M$ will become $dx^2 + x^{-2a}h$ when it is restricted to $M_1$.

Similar cases treated in the literature are those of the cylindrical end and of the hyperbolic end, which is usually called a hyperbolic cusp. An end $M_1$ is a cylinder when the metric there has the form $dx^2 + h$, and hyperbolic cusp when it has the form $(dx^2 + h)/x^2$. The cases of hyperbolic cusps and cylinders have been well studied (see e.g. work of W. Müller [32], R. Melrose, [27] for background). In both cases, the eigenvalue equation of the Laplace operator can be solved explicitly in the continuous subspace. Then, the generalized eigenforms of the Laplacian can be constructed and their asymptotic behavior can be estimated. Also, scattering theory can be applied to that cases. Thus, in a manifold with one cylindrical end, the spectrum of the Laplacian is a union of a pure point spectrum, consisting of eigenvalues of finite multiplicity, and an
absolutely continuous spectrum, consisting of an infinite union of branches with thresholds at every eigenvalue of the Laplacian at the boundary, where at each branch the multiplicity is equal to the rank of the eigenspace of the correspondent threshold. In a manifold with hyperbolic end, the spectrum of the Laplacian has the same nature, but the absolutely continuous part now is simple and starts from a bound depending on the dimension of the manifold and the degree of the differential forms.

The generalized cusp metric we investigate here has already been treated by S. Golenia and S. Moroianu [15], and F. Antoci [2]. They consider a larger class of metrics than that one of the generalized cusp, and by regarding the spectral properties of the $p$-form Laplacian without solving the eigenvalue equation in the continuous subspace, they clarify the nature of the essential spectrum, when it arises. In [15], they provide also eigenvalue asymptotics (in terms of counting functions), whenever the spectrum is purely discrete. No information about the asymptotic expansion of the generalized eigenforms of the Laplacian is provided.

We deal first with the generalized cusp separately. We fix Dirichlet boundary conditions at $x = 1$ on the $p$-form Laplacian and by using separation of variables, we determine its spectral decomposition. Then, since we have an explicit formula for the continuous part of the spectral theorem of the Laplacian, we meromorphically continue its resolvent, from the resolvent set to some Riemann surface, which we determine. Next, we pass to the full manifold, and by using some standard techniques, we construct a meromorphic continuation of the Laplace resolvent there, to the same Riemann surface, and we obtain some spectral information. Having this resolvent, we construct the generalized eigenforms of the Laplacian and we define the scattering matrix. Our results can be summarized in the following theorem.

**Theorem 1.1.** Let $M = M_0 \cup_N ([1, \infty) \times N)$ be an $n$-dimensional manifold with generalized cusp, with metric on the cusp given by $dx^2 + x^{-2a}h$, $a > 0$. Let $\mathcal{H}^p(N)$ be the space of square integrable harmonic $p$-forms on $N$, and $S$ be the Riemann surface of the function $\log z$. Let $\Delta$ be the Laplace operator acting on smooth $p$-forms on $M$. For any $\theta \oplus \tilde{\theta} \in \mathcal{H}^p(N) \oplus \mathcal{H}^{p-1}(N)$ and any $\lambda \in S$, there exists a $p$-form $E_\lambda(y, \theta \oplus \tilde{\theta})$ on $M$, called $\lambda^2$ generalized eigenform of $\Delta$, with the following properties

1) $E_\lambda(y, \theta \oplus \tilde{\theta})$ is smooth in $y \in M$ and meromorphic in $\lambda \in S$.

2) $(\Delta - \lambda^2 I)E_\lambda(y, \theta \oplus \tilde{\theta}) = 0$, for any $y \in M$ and $\lambda \in S$. 
3) For $x > 1$ and $\lambda \in S$, we have an expansion of the form
$$E_\lambda(y, \theta \oplus \tilde{\theta}) = x^{bp} H^{(1)}_b(\lambda x) \theta + dx \wedge x^{bp-1} H^{(1)}_{bp-1}\lambda_x \tilde{\theta}$$
$$+ x^{bp} H^{(2)}_b(\lambda x) C_{p,\lambda}(\theta) + dx \wedge x^{bp-1} H^{(2)}_{bp-1}\lambda_x C_{p-1,\lambda}(\tilde{\theta}) + \Psi_\lambda(y, \theta \oplus \tilde{\theta}),$$
where $H$ are the Hankel functions,
$$b_p = \frac{a(n - 2p - 1) + 1}{2}$$
and
$$C_{p,\lambda} = \begin{pmatrix} C_{p,\lambda} & 0 \\ 0 & C_{p-1,\lambda} \end{pmatrix} \in \text{End}\left(\mathcal{H}^p(N) \oplus \mathcal{H}^{p-1}(N)\right)$$
is linear, meromorphic in $\lambda \in S$, and is called the (stationary) scattering matrix associated to $E_\lambda(y, \theta \oplus \tilde{\theta})$. For the tail term we have that
$$\Psi_\lambda(y, \theta \oplus \tilde{\theta}) = O\left(x^{bp-1/2} e^{-\mu a + 1/2} x^{a+1}\right), \forall \lambda \in S,$$
where $\mu > 0$ is the square root of the smallest nonzero eigenvalue of the $p$-form Laplacian of the boundary $N$. Also, $E_\lambda(y, \theta \oplus \tilde{\theta})$, $C_{p,\lambda}$ and $\Psi_\lambda(y, \theta \oplus \tilde{\theta})$ are uniquely determined by the above properties.

Next, we use the above theorem, especially the uniqueness from the theorem and the explicit asymptotic behavior of the generalized eigenform at infinity, to prove the properties of the scattering matrix, including its functional equation. The results are stated as follows

**Theorem 1.2.** Let $S$ and $C_{p,\lambda}$ be the Riemann surface and the scattering matrix defined in the previous theorem. For any $\lambda \in S$, we have that
$$C_{p,\lambda}^* \circ C_{p,\lambda} = I \ (\text{unitarity}) \text{ and } C_{p,\lambda} \circ C_{p,\lambda} = I.$$  
If we denote $\gamma_p = a(n - 2p - 1)$, then the scattering matrix satisfies the following functional equation
$$(C_{p,-\lambda} - I) \circ C_{p,\lambda} = \begin{pmatrix} e^{-i\pi\gamma_p} & 0 \\ 0 & e^{-i\pi\gamma_{p-1}} \end{pmatrix} \left(I - C_{p,\lambda}\right).$$
Also, if $*_N$ is the Hodge star operator on the boundary $N$, the following commutation relation holds
$$*_N C_{p,\lambda} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & e^{i\pi \gamma_{n-p-1}} \\ e^{i\pi \gamma_{n-p}} & 0 \end{pmatrix} C_{n-p,\lambda}*_N = 0.$$
This document is organized as follows. In the second chapter, we give some general background on the spectral theory, based on the classical analysis of unbounded self-adjoint operators and on the theory of pseudodifferential operators. We give also a quick introduction to the mathematical scattering theory of self-adjoint operators. In the third chapter we do spectral theory of the Laplacian on the generalized cusp. We collect first some of the geometric properties of the generalized cusp $M_1$, and by using separation of variables, we determine all the spectral information of the Laplace operator there, such as the appropriate space decomposition and the continuous part of the spectral theorem. We also meromorphically continue the resolvent of the Laplacian to the indicated Riemann surface. Finally, at the fourth chapter we pass the information we have found to the full manifold $M$. After using some gluing techniques, we construct the generalized eigenforms of the Laplacian, we define the scattering matrix and find its properties. In the appendix, we have collected some properties of the Bessel functions, since they are used frequently in the document.
Chapter 2

Background on spectral theory

In this chapter we will give some general background on the spectral theory of unbounded self-adjoint operators. We will give definitions and state some standard results which we are going to use later. We start with some material from the classical analysis of operators, we give a quick introduction to scattering theory, and finally we explain some elementary ideas and results from the theory of pseudodifferential operators.

2.1 Spectral theory of unbounded self-adjoint operators

Here we introduce basic definitions and theorems about the spectrum of an unbounded self-adjoint operator. In particular this section develops the most important ideas on spectral theory, by moving towards the notion of the spectral theorem. Also, some other important theorems are recalled, which we will make use of in the remainder of this document.

Definition 2.1. Take a linear operator $T$ from a (usually dense) subset $\text{Dom}(T)$ of a Hilbert space $H_1$ to a Hilbert space $H_2$. If there exists some constant $c$ such that $\|T\psi\|_2 \leq c\|\psi\|_1$ for all $\psi$ in $\text{Dom}(T)$, where $\| \cdot \|_1$ and $\| \cdot \|_2$ are the norms induced by the inner products in $H_1$ and $H_2$ respectively, then $T$ is called bounded, and $\inf\{c : \|T\psi\|_2 \leq c\|\psi\|_1, \forall \psi \in H_1\}$ is called the norm of $T$, denoted by $\|T\|$. In this case, $T$ extends to a bounded operator on all of $H_1$. Otherwise, $T$ is called unbounded and the set $\text{Dom}(T)$ is called the domain of $T$. The set of bounded operators from the space $H_1$ to the space $H_2$ is denoted by $\mathcal{L}(H_1, H_2)$, and by $\mathcal{L}(H_1)$ if $H_1 = H_2$. 
Definition 2.2. Let $T$ be an operator on a Hilbert space $H$. The set

$$\Gamma(T) = \{(\psi, T\psi) : \psi \in \text{Dom}(T)\}$$

is called graph of $T$. The operator $T$ is called closed if its graph is a closed subset of $H \times H$ in the product topology. Also, two operators are equal if their graphs are equal.

Definition 2.3. Let $T_1$ and $T_2$ be two operators on a Hilbert space $H$, such that $\Gamma(T_1) \subset \Gamma(T_2)$. Then $T_2$ is called an extension of $T_1$. An operator is called closable if it has a closed extension. In this case the minimal such extension, in terms of set inclusions, is called the closure of the operator.

The above definitions can apply to an operator acting between two different Hilbert spaces. Now, we restrict to an operator acting from a Hilbert space to itself.

Definition 2.4. Let $T$ be a closed operator on a Hilbert space $H$. The resolvent set $\rho(T)$ of $T$ is the set of points $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ is a bijection from $\text{Dom}(T)$ onto $H$ with bounded inverse. The operator valued function

$$R_\lambda(T) = (T - \lambda I)^{-1} \text{ with } \lambda \in \rho(T),$$

is called the resolvent of $T$. The spectrum $\sigma(T)$ of $T$ is the set $\mathbb{C} \setminus \rho(T)$.

Definition 2.5. Let $T$ be an operator on a Hilbert space $H$ with inner product $(\cdot, \cdot)$. Define the operator $T^*$ on $H$ with domain consisting of elements $\psi \in H$ for which there exists some $\tau \in H$ such that for any $\phi \in \text{Dom}(T)$ we have $(T\phi, \psi) = (\phi, \tau)$. Then define $T^*\psi = \tau$, for the above $\psi$. $T^*$ is called the adjoint of $T$. If $T^* = T$, then $T$ is called self-adjoint.

Recall that a family $\mathcal{A}$ of subsets of a set $A$ is called $\sigma$-ring if its is closed under (probably infinite) unions and set complements. The Borel sets of $\mathbb{R}$ is the smallest family $\mathcal{B}$ of subsets of $\mathbb{R}$ which contains the open intervals, and it is closed under complements and countable unions. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function if the pre-image of any open interval is a Borel set. The family $\mathcal{B}$ is an $\sigma$-ring over $\mathbb{R}$, and a measure $\mu$ on $\mathbb{R}$ with $\sigma$-ring $\mathcal{B}$ is called Borel measure on $\mathbb{R}$ if additionally the following properties are satisfied

$$\mu(A) = \sup\{\mu(C) : C \subset A \text{ and } C \text{ is compact}\} = \inf\{\mu(B) : A \subset B \text{ and } B \text{ is open}\}$$
and
\[ \mu(C) < \infty \] if \( C \) is compact.

A sequence of operators \( T_n \in \mathcal{L}(H_1, H_2) \) converges strongly to an operator \( T \in \mathcal{L}(H_1, H_2) \) if \( \| T\phi - T_n\phi \|_{H_2} \to 0 \) when \( n \to \infty \), for all \( \phi \in H_1 \). If we denote by \( s - \lim \) the strong convergence then we have the following.

**Definition 2.6.** The family of operators \( \{ E_A : H \to H \} \), where \( A \subset \mathbb{R} \) is Borel-measurable, is called a projection-valued measure on a Hilbert space \( H \) if the following properties hold:

1) Each \( E_A \) is self-adjoint and \( E_{A_1} E_{A_2} = E_{A_1 \cap A_2} \).
2) \( E_\emptyset = 0 \) and \( E_{(-\infty,+\infty)} = I \).
3) If \( A = \bigcup_n A_n \), with \( A_n \cap A_m = \emptyset \) for \( n \neq m \), then

\[ E_A = s - \lim_{N \to \infty} \sum_{n=1}^{N} E_{A_n}. \]

If \((\cdot,\cdot)\) is the inner product in \( H \), then for any \( \phi, \psi \in H \), the function \( A \to (\phi, E_A \psi) \) is a Borel-measure, which we denote by \( d(\phi, E_A \psi) \). If \( A = (-\infty, \lambda) \), then we just denote \( E_A \) by \( E_\lambda \). We can state now the spectral theorem, which is in a sense of generalization of diagonalization of a symmetric matrix.

**Theorem 2.7.** (Spectral theorem) There is a one to one correspondence between self-adjoint operators \( T \) and projection-valued measures \( \{ E_\lambda \} \) on a Hilbert space \( H \), given by

\[ (\phi, T\psi) = \int_{\mathbb{R}} \lambda d(\phi, E_\lambda \psi), \]

which we also denote by

\[ T = \int_{\mathbb{R}} \lambda dE_\lambda. \]

Also, if \( f(\cdot) \) is a real valued Borel-function on \( \mathbb{R} \), then the operator defined by
\[ f(T) := \int_{\mathbb{R}} f(\lambda) dE_\lambda, \]
with domain \( D_f = \{ \phi \in H : \int_{\mathbb{R}} |f(\lambda)|^2 d(\phi, E_\lambda \phi) < \infty \} \), is self-adjoint on \( H \).

**Proof.** Theorem VIII.6 in [39].

**Definition 2.8.** A Borel measure \( \mu \) on \( \mathbb{R} \) is called:

1) Continuous if \( \mu(x) = 0 \) for all \( x \in \mathbb{R} \).
2) Pure point measure if \( \mu(A) = \sum_{x \in A} \mu(x) \), for any Borel-set \( A \).
3) Absolutely continuous with respect to Lebesgue measure if there exists an \( L^1_{loc} \) function \( f \) on \( \mathbb{R} \) such that for any Borel function \( g \in L^1(\mathbb{R}, d\mu) \), we have
\[
\int_{\mathbb{R}} g d\mu = \int_{\mathbb{R}} g f dx.
\]
4) Singular relative to Lebesque measure if \(\mu(A) = 0\) for some set \(A\) such that \(\mathbb{R} \setminus A\) has Lebesque measure zero.

Since for any \(\psi \in H\), \(\mu_{\psi} = (\psi, E_\lambda \psi)\) is a Borel measure on \(\mathbb{R}\), according to the previous definition, we have

**Definition 2.9.** Let \(T\) be a self-adjoint operator on a Hilbert space \(H\). We define the following subspaces of \(H\):
1) \(H_{\text{pp}} = \{\psi \in H : \mu_{\psi} \text{ is pure point measure}\}\).
2) \(H_{\text{ac}} = \{\psi \in H : \mu_{\psi} \text{ is absolutely continuous with respect to Lebesque measure}\}\).
3) \(H_{\text{sing}} = \{\psi \in H : \mu_{\psi} \text{ is singular relative to Lebesque measure}\}\).

In a similar way as the Borel measure \(\mu\) can be uniquely decomposed to a sum \(\mu = \mu_{\text{pp}} + \mu_{\text{ac}} + \mu_{\text{sing}}\), where \(\mu_{\text{pp}}, \mu_{\text{ac}}\) and \(\mu_{\text{sing}}\) is pure point, absolutely continuous and singular measure respectively (Lebesque decomposition theorem), we get the following (Theorem VII.4 in [39])

**Theorem 2.10.** Let \(T\) be a self-adjoint operator on a Hilbert space \(H\). Then \(H\) can be uniquely decomposed by \(H = H_{\text{pp}} \oplus H_{\text{ac}} \oplus H_{\text{sing}}\), where \(T|_{H_{\text{pp}}}\) has a complete set of eigenvectors, \(T|_{H_{\text{ac}}}\) has absolutely continuous spectral measure and \(T|_{H_{\text{sing}}}\) has singular spectral measure. The \(T\) depended spaces \(H_{\text{pp}}, H_{\text{ac}}\) and \(H_{\text{sing}}\) are called pure point, absolutely continuous and singular continuous subspaces of \(H\) respectively.

By using the above theorem, we can give the following classification on the spectrum of a self-adjoint operator.

**Definition 2.11.** We define the pure point, continuous, absolutely continuous and continuous singular spectrum of a self-adjoint operator \(T\) on a Hilbert space \(H\) by the following sets respectively:
\[
\sigma_{\text{pp}}(T) = \sigma(T|_{H_{\text{pp}}}),
\]
\[
\sigma_{\text{cont}}(T) = \sigma(T|_{H_{\text{ac}} \oplus H_{\text{sing}}}),
\]
\[
\sigma_{\text{ac}}(T) = \sigma(T|_{H_{\text{ac}}})
\]
and
\[
\sigma_{\text{sing}}(T) = \sigma(T|_{H_{\text{sing}}}).
\]
Obviously, \(\sigma(T) = \sigma_{\text{pp}}(T) \cup \sigma_{\text{ac}}(T) \cup \sigma_{\text{sing}}(T)\).

One other classification of the spectrum of a self-adjoint operator follows by the spectral theorem. If \(A\) is a Borel set in \(\mathbb{R}\) and \(\chi_A\) is a characteristic function on \(A\), i.e. \(\chi_A(x)\) is one if \(x \in A\) and zero otherwise, then the spectral projection of a self adjoint operator \(T\) on the set \(A\) is defined by \(P_A(T) = \chi_A(T)\), in the sense of the spectral theorem. So, we can state the following
2.1 Spectral theory of unbounded self-adjoint operators

**Definition 2.12.** The essential and the discrete spectrum of a self-adjoint operator $T$ on a Hilbert space $H$ are defined respectively by the sets:

$$
\sigma_{\text{ess}}(T) = \{ \lambda \in \mathbb{R} : \text{Ran} P_{(\lambda - \varepsilon,\lambda + \varepsilon)}(T) \text{ is infinite dimensional for all } \varepsilon > 0 \}
$$

and

$$
\sigma_{\text{disc}}(T) = \{ \lambda \in \mathbb{R} : \text{Ran} P_{(\lambda - \varepsilon,\lambda + \varepsilon)}(T) \text{ is finite dimensional for some } \varepsilon > 0 \}.
$$

Recall that a bounded operator $T$ on a Hilbert space $H$ is called positive if $(\psi, T\psi) \geq 0$ for all $\psi \in H$. In this case we denote by $T \geq 0$. We also denote by $T_1 \geq T_2$ if $T_1 - T_2 \geq 0$. If $T \in \mathcal{L}(H)$ and $T \geq 0$, then there exists a unique $\sqrt{T} \in \mathcal{L}(H)$ such that $\sqrt{T} \geq 0$, $\sqrt{T}^2 = T$ and $\sqrt{T}$ commutes with every bounded operator which commutes with $T$ (square root lemma). Thus, since for any $T \in \mathcal{L}(H)$ we have that $T^* T \geq 0$, we denote by $|T| = \sqrt{T^* T}$. Also, if $H$ is separable (i.e. contains a countable dense subset) Hilbert space and $\{\phi_i\}$ is an orthonormal basis in $H$, for any positive operator $T \in \mathcal{L}(H)$ its trace is defined by the following number $\text{tr} T = \sum_i (\phi_i, T\phi_i)$, which is independent of the choice of the orthonormal basis. We define the following important class of operators.

**Definition 2.13.** Let $T$ be a bounded operator in a separable Hilbert space $H$. $T$ is called trace class if $\text{tr} |T|$ is finite. The family of all these operators is denoted by $\mathcal{L}^1$.

The family $\mathcal{L}^1$ defined above is a $*$-ideal of the space $H$. This means that $\mathcal{L}^1$ is a vector space and that $T^* T$, $T^* T$ and $T^*$ belong to $\mathcal{L}^1$, for any $T \in \mathcal{L}^1$ and $T \in \mathcal{L}(H)$ (Theorem VI.19 in [39]). A subset $A$ of a Banach space $B$ is called precompact if its closure is compact. We define next the class of compact operators which contains the class $\mathcal{L}^1$.

**Definition 2.14.** Let $B_1$ and $B_2$ be Banach spaces. An operator $T \in \mathcal{L}(B_1, B_2)$ is called compact if it maps bounded sets in $B_1$ into precompact sets in $B_2$.

If we have two operators $T \in \mathcal{L}(B_1, B_2)$ and $\tilde{T} \in \mathcal{L}(B_2, B_3)$, then $\tilde{T} T \in \mathcal{L}(B_1, B_3)$ is compact if one of $T$, $\tilde{T}$ is compact (Theorem VI.12 in [39]). We give next the notion of relatively compact perturbation of a self-adjoint operator, whose spectral properties are connected to the spectral properties of the perturbed operator.

**Definition 2.15.** Let $T$ be a self-adjoint operator. An operator $\tilde{T}$ is called relatively compact with respect to $T$ if $\text{Dom}(T) \subset \text{Dom}(\tilde{T})$ and $\tilde{T}(T - iI)^{-1}$ is compact.
The importance of a relative compact perturbation of an operator appears in the next theorem.

**Theorem 2.16.** If $\tilde{T}$ is relatively compact with respect to $T$, then

$$\sigma_{ess}(T + \tilde{T}) = \sigma_{ess}(T).$$

**Proof.** This is a consequence of Weyl’s essential spectrum theorem (Theorem XIII.14 in [39]).

We state next some more results which we are going to use later for technical purposes.

**Proposition 2.17.** If $T$ is closed, then $T$ is compact if and only if $T^*T$ is compact.

**Proof.** The one direction comes immediately from Theorem VI.12 in [39]. To prove the other direction, note that from the polar decomposition of a closed operator (Theorem VIII.32 in [39]), there exists a partial isometry $U$ (i.e. $\|U(x)\| = \|x\|$ when $x \in (\text{Ker } U)^\perp$) mapping from $(\text{Ker } T)^\perp$ into $\text{Ran } T$, and a positive self-adjoint operator $|T|$ with the same domain with $T$, which is the square root (in the sense of the spectral theorem) of the unique positive self-adjoint extension of $T^*T$, such that $T = U|T|$. Thus, since $U$ is bounded, it is enough to prove that $|T|$ is compact. Indeed, any positive self-adjoint operator is compact if and only if its square root (in the sense of spectral theorem) is compact. The last follows by the spectral theorem and the fact that a self-adjoint operator $Q$ on a Hilbert space $H$ is compact if and only if there exists a complete orthonormal basis $\{\phi_n\}$ on $H$ such that $Q\phi_n = \lambda_n\phi_n$ and $\lambda_n \to 0$ as $n \to \infty$. To see this, note that the one direction is the Theorem VI.16 in [39]. For the other direction, if $(\cdot, \cdot)$ is the inner product in $H$, take the sequence of finite rank, and hence compact, operators $Q_k = \sum_{n=1}^{k} \lambda_n(\phi_n, \cdot)\phi_n$ which converges to $Q$ in the norm topology, to get the result.

Note that if $T$ is closed, by its polar decomposition we get that $T^*$ is compact if $T$ is compact. Recall that a self-adjoint operator $T$ on a Hilbert space $H$ is said to be bounded from below if there exists some constant $c$ such that $T - cI$ has no negative spectrum. In this case we denote by $T \geq cI$. We can state now the following fundamental result in spectral theory.
**Theorem 2.18.** *(min-max principle)* Let $T$ be a self-adjoint operator bounded from below. Let
\[ \rho_n(T) = \sup_{\phi_1, \ldots, \phi_{n-1}} \inf_{\psi \in \text{Dom}(T), \|\psi\| = 1} (\psi, T\psi). \]
Then, either

a) there are $n$ eigenvalues (counting multiplicity) below the bottom of the essential spectrum of $T$, and $\rho_n(T)$ is the $n$th eigenvalue (counting multiplicity),

or

b) $\rho_n(T)$ is the bottom of the essential spectrum, $\rho_m(T) = \rho_n(T)$ for $m > n$, and there at most $n - 1$ eigenvalues (counting multiplicity) below $\rho_n(T)$.

**Proof.** Theorem XIII.1 in [39].

We state finally the next theorem.

**Theorem 2.19.** *(meromorphic Fredholm theorem)* Let $\Omega$ be a connected open subset of $\mathbb{C}$ and $Q \subset \Omega$ be a discrete set. Let $T(z)$ be an operator valued function which is analytic in $\Omega \setminus Q$ and around any $z_0 \in Q$ has an expansion
\[ T(z) = \sum_{n=-k}^{\infty} T_n(z - z_0)^n, \text{ for some } k \geq 0. \]
Assume also that $T(z)$ is compact if $z \in \Omega \setminus Q$ and that $T_n$ are of finite rank when $n < 0$. Then, either

$I - T(z)$ is not invertible in $\Omega \setminus Q$

or

$I - T(z)$ is invertible for $z \notin Q \cup Q'$, for some discrete set $Q' \subset \Omega$, and extents to an analytic function in $\Omega \setminus Q'$ such that the coefficients of the negative powers in the Laurent expansion of $T(z)$ around any point of $Q'$, are of finite rank (i.e. $(I - T(z))^{-1}$ is meromorphic in $\Omega$ with finite rank residues).

**Proof.** Theorem XIII.13 in [39].
2.2 Abstract of scattering theory

The mathematical scattering theory of self-adjoint operators provides some conclusions about an operator $A$, given some information regarding an operator $B$, where both operators are self-adjoint and act on a Hilbert space $H$. In this section, we give an elementary background motivation for scattering theory, by defining the generalized wave operators, the scattering operator and the (dynamical) scattering matrix associated to the self-adjoint operators $A$ and $B$. Our aim is to explain how the (stationary) scattering matrix we define in chapter four is related to the standard (dynamical) scattering matrix.

**Definition 2.20.** Let $A, B$ be two self-adjoint operators on a Hilbert space $H$. We define the generalized wave operators $W^\pm(A, B)$ by the following limit

$$W^\pm(A, B) = \lim_{t \to \mp \infty} e^{iat} e^{-ibt} P_{ac}(B),$$

when it exists, where $P_{ac}(B)$ is the projection to the absolutely continuous subspace of $B$. If $W^\pm(A, B)$ exist, we denote the spaces $\text{Ran} W^+$ and $\text{Ran} W^-$, by $H_+$ and $H_-$ respectively.

If the wave operators exist, then they are partial isometries from the space $P_{ac}(B)H$ to the spaces $H_\pm$. The spaces $H_\pm$ are invariant under $A$ in the sense that,

$$W^\pm(\text{Dom}(B)) \subset \text{Dom}(A) \text{ and } AW^\pm(A, B) = W^\pm(A, B)B.$$

Also, $H_\pm \subset \text{Ran} P_{ac}(A)$. If $W^\pm(A, B)$ and $W^\pm(B, C)$ exist, then $W^\pm(A, C)$ exists and

$$W^\pm(A, C) = W^\pm(A, B)W^\pm(B, C).$$

Proofs for the above properties can be found in [39] vol. III.

**Definition 2.21.** Take two self-adjoint operators $A, B$ on a Hilbert space $H$. Let $P_{ac}(A)$ and $P_{pp}(B)$ be the projections onto the absolutely continuous and the pure point subspace of $H$, with respect to the spectral decomposition of $A$ and $B$ respectively. Assuming that $W^\pm(A, B)$ exist, we say that they are asymptotically complete if $H_+ = H_- = (P_{pp}(B)H)^\perp$, and complete if $H_+ = H_- = P_{ac}(A)H$.

Note that asymptotic completeness implies completeness and $\sigma_{\text{sing}}(A) = \emptyset$.

Next, we state a theorem which provides the existence and the completeness of the wave operators.

**Theorem 2.22.** Let $A, B$ be two self-adjoint operators on a Hilbert space $H$, such that $A, B \geq (1 - c)I$, for some constant $c$. If $(A + cI)^{-k} - (B + cI)^{-k}$ is trace class for some $k \in \mathbb{R}$, then $W^\pm(A, B)$ exist and are complete.
2.2 Abstract of scattering theory

Proof. This is a consequence of Theorem XI.9 in [39]. □

We can define now the dynamical scattering matrix according to the self-adjoint operators $A$, $B$. We give first the following

**Definition 2.23.** If for the self-adjoint operators $A$, $B$ the operators $W^\pm(A, B)$ exist, then we define the scattering operator $S(A, B)$ by

$$S(A, B) = (W^-(A, B))^*W^+(A, B).$$

From its construction, $S(A, B)$ commutes with $B$ and is unitary if and only if $H_+ = H_-$. Thus, if $W^\pm(A, B)$ are complete, then $S(A, B)$ is a unitary operator in $P_{ac}(B)H$, i.e. in the absolutely continuous subspace of $H$, with respect to $B$. By the spectral calculus for the operator $B$, when it is restricted to $P_{ac}(B)H$, there exists a unique Borel measure $\mu$ on $\mathbb{R}$ such that $P_{ac}(B)H$ is decomposed by the direct integral

$$P_{ac}(B)H = \int_{\sigma_{ac}(B)} h_\lambda d\mu(\lambda),$$

for some family of Hilbert spaces $h_\lambda$, where $B$ acts in every $h_\lambda$ by multiplication with $\lambda$. Then, since $S(A, B)$ and $B$ commute

$$S(A, B) = \int_{\sigma_{ac}(B)} S_\lambda(A, B)d\mu(\lambda),$$

for some uniquely defined operator valued function $S_\lambda(A, B)$, which is called the (dynamical) scattering matrix. Even though $S_\lambda(A, B)$ is called matrix, it is an operator on $h_\lambda$. In the case we consider, we will show that $\dim h_\lambda < \infty$. 
2.3 Pseudodifferential operators

In this section, we run through the foundations of theory of pseudodifferential operators. We state some elementary notions and basic results, without giving details on the construction of the pseudodifferential calculus. Finally, we pass to the elliptic operators on compact manifolds, and prove some important facts related to the Laplacian, which we are going to use later. Proofs of the propositions and the theorems we state, can be found in [41] and [43].

Definition 2.24. Let $X \subseteq \mathbb{R}^n$ be an open set. A function $a(x, \xi) \in C^\infty(X, \mathbb{R}^n)$ is a symbol of order $m \in \mathbb{R}$ if for any multi indices $\alpha, \beta$ and any compact set $K \subset X$ there exists some constant $C_{\alpha, \beta, K}$ such that for all $x \in K$

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta, K} \langle \xi \rangle^{m-|\beta|},$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. We denote this class of functions by $S^m(X, \mathbb{R}^n)$. We also put $S^{-\infty}(X, \mathbb{R}^n) = \cap_m S^m(X, \mathbb{R}^n)$.

Now if $a(x, \xi) \in C^\infty(X, \mathbb{R}^n)$ and $w(x) \in C_0^\infty(X)$, by using the relation

$$e^{-i(x, \xi)} = \langle x \rangle^{-k} (D_\xi)^k e^{-i(x, \xi)}, \text{ where } D_\xi = (\frac{1}{i} \partial_{\xi_1}, \ldots, \frac{1}{i} \partial_{\xi_N}),$$

considered for an even number $k > m+n$, and doing $k$ times integration by parts, we can make the integral below

$$\int_{\mathbb{R}^n} \int_X e^{-i(x, \xi)} w(x) a(x, \xi) dx d\xi,$$

which is called oscillatory integral, to converge absolutely. Hence, according to the Fourier integral operators approach, by using the above class of functions, we give the following definition.

Definition 2.25. An operator $A$ with Schwartz kernel

$$k(x, y) = \int_{\mathbb{R}^n} e^{i(x-y, \xi)} a(x, y, \xi) d\xi,$$

where $a(x, y, \xi) \in S^m(X \times X, \mathbb{R}^n)$, for some open set $X \subseteq \mathbb{R}^n$, is called pseudodifferential operator of order $m$, where the above integral as well as the action of $A$ is to be understood in the sense of the oscillatory integrals. The function $a(x, y, \xi)$ is called the symbol (or total symbol) of $A$, and the class of such operators is denoted by $\Psi^m(X)$. We also put $\Psi^{-\infty}(X) = \cap_m \Psi^m(X)$. 

If \( X \subseteq \mathbb{R}^n \) is open, it can be shown by the properties of a symbol, that the distribution \( k(x,y) \) is smooth away from the diagonal of \( X \times X \), and that \( A \) is a continuous linear map from \( C_c^\infty(X) \) to \( C^\infty(X) \). Equivalently, it is a continuous linear map from \( \mathcal{E}'(X) \to \mathcal{D}'(X) \), where \( \mathcal{E}'(X) \) and \( \mathcal{D}'(X) \) are the dual spaces of \( C^\infty(X) \) and \( C_c^\infty(X) \) respectively. \( \mathcal{E}'(X) \) is identified with the space of compactly supported distributions in \( \mathcal{D}'(X) \). Also, \( \text{supp} \, Au \subset \text{supp} \, u \) for any \( u \in \mathcal{E}'(X) \) (pseudolocality). The symbol of an operator is not unique, but any symbol \( a(x,y,\xi) \in S^m(X \times X, \mathbb{R}^n) \) can be written uniquely in the forms \( b(x,\xi) + c(x,y,\xi) \) or \( \tilde{b}(y,\xi) + \tilde{c}(x,y,\xi) \), with \( b(x,\xi), \tilde{b}(y,\xi) \in S^m(X, \mathbb{R}^n) \) and \( c(x,y,\xi), \tilde{c}(x,y,\xi) \in S^{-\infty}(X \times X, \mathbb{R}^n) \). Finally, recall that a map between two topological spaces is called proper if the preimage of every compact set is compact.

**Definition 2.26.** A symbol \( a(x,y,\xi) \) is properly supported if both projections

\[
P_x, P_y : \overline{\text{supp} \, a(x,y,\xi)} \to \mathbb{R}^n
\]

are proper maps. Similarly, an operator \( A \in \Psi^m(X) \) is called properly supported if both projections \( P_x, P_y : \text{supp} \, K_A \to X \) are proper maps.

Any pseudodifferential operator \( A \) can be written as \( A = B + C \), where \( B \) is properly supported and \( C \in \Psi^{-\infty}(X) \). Properly supported operators have the additional property that can be extended to maps \( A : \mathcal{E}'(X) \to \mathcal{E}'(X) \) and \( A : \mathcal{D}'(X) \to \mathcal{D}'(X) \). Also, their total right symbol can be defined equivalently by the following.

**Definition 2.27.** The symbol (or complete symbol) \( \sigma_A(x,\xi) \) of a properly supported operator \( A \in \Psi^m(X) \) is defined by the relation

\[
\sigma_A(x,\xi) = e^{-ix\xi} A e^{ix\xi}.
\]

If \( A \in \Psi^m(X) \) is properly supported, then its formal adjoint, defined as usually on \( C_c^\infty(X) \) by

\[
(A\psi, \phi) = (\psi, A^*\phi), \quad \psi, \phi \in C_c^\infty(X),
\]

is also a properly supported operator in \( \Psi^m(X) \). Some additional restriction on the class of symbols of operators will give rise to some important properties. We give first the notion of classicality of a pseudodifferential operator. Recall that a function \( f(x) \) on \( \mathbb{R}^n \setminus \{0\} \) is called positively homogeneous of order \( k \in \mathbb{R} \) if \( f(tx) = t^k f(x) \) for all \( x \in X \) and \( t > 0 \).
Definition 2.28. A symbol \( a(x, \xi) \) of order \( m \) is classical if there exists a sequence \( \{a_{m-i}(x, \xi)\}_{i \in \mathbb{N}} \) of positively homogeneous functions \( a_{m-i}(x, \xi) \) of order \( m - i \) in \( \xi \) such that
\[
a(x, \xi) - \sum_{i=0}^{k-1} \psi(\xi)a_{m-i}(x, \xi) \in S^{m-k}(X, \mathbb{R}^n), \forall k,
\]
where \( \psi(\xi) \in C^\infty(\mathbb{R}^n) \) is 0 for \( |\xi| < 1 \) and 1 for \( |\xi| > 2 \). A pseudodifferential operator \( A \) is classical if its symbol is classical. The class of such operators is denoted by \( \Psi_m^{cl}(X) \). In this case, the term \( \psi(\xi)a_m(x, \xi) \) in the above expansion of the symbol is called principal symbol, and it is denoted by \( \sigma_{A,m}(x, \xi) \).

Next, we continue to the notion of the ellipticity of a pseudodifferential operator, which is a property satisfied by the Laplacian as well.

Definition 2.29. A classical pseudodifferential operator is elliptic if its principal symbol is never zero in \( X \times (\mathbb{R}^n \setminus 0) \).

If \( A \) is elliptic and \( u \in E'(X) \), then \( \text{sing supp } Au = \text{sing supp } u \) (elliptic regularity). An important property of elliptic operators is that they are invertible up to a \( \Psi^{-\infty}(X, \mathbb{R}^n) \) (i.e. up to a smoothing) term, as the following theorem states.

Theorem 2.30. If \( A \in \Psi_m^{cl}(X) \) is elliptic and properly supported, then there exists an operator \( B \in \Psi_{-m}^{cl}(X) \), such that \( AB - I, BA - I \in \Psi^{-\infty}(X) \).

Proof. Theorem 5.1 in [41].

Suppose that we have a diffeomorphism \( \kappa : X \to \tilde{X} \), where \( X, \tilde{X} \subseteq \mathbb{R}^n \) are open. Then, if \( A \in \Psi^n(X) \), we can define a operator \( \tilde{A} : C^\infty_0(\tilde{X}) \to C^\infty(\tilde{X}) \) by the following commutative diagram
\[
\begin{array}{ccc}
C^\infty_0(X) & \xrightarrow{A} & C^\infty(\tilde{X}) \\
\kappa^* \uparrow & & \uparrow \kappa^* \\
C^\infty_0(\tilde{X}) & \xrightarrow{\tilde{A}} & C^\infty(\tilde{X})
\end{array}
\]
where \( \kappa^* : C^\infty(\tilde{X}) \to C^\infty(X) \) maps a function \( u \) to \( u \circ \kappa \). It can be proved (Theorem 4.1 in [41]) that \( \tilde{A} \) is a pseudodifferential operator of the same order as \( A \), \( \tilde{A} \) is classical if \( A \) is classical and it is elliptic if \( A \) is elliptic. By using this, we can extend the notion of a pseudodifferential operator in a general \( n \)-dimensional manifold \( M \), equipped with an atlas \( \{U_i, \phi_i\} \), acting on complex smooth vector bundles. Hence, the class \( \Psi^n(M, E, \tilde{E}) \) of pseudodifferential operators of order \( m \) acting between sections of the smooth complex vector bundles \( E \) and \( \tilde{E} \) on \( M \) respectively, the subclass \( \Psi_m^{cl}(M, E, \tilde{E}) \) of the classical ones and the notion of ellipticity are well defined by the following.
Definition 2.31. A pseudodifferential operator $A$ of order $m$ acting from the space $C^\infty_0(M, E)$ of the compactly supported sections of one smooth complex vector bundle $E$ to the space of sections $C^\infty(\tilde{M}, \tilde{E})$ of another smooth complex vector bundle $\tilde{E}$, over a manifold $M$ with an atlas $\{U_i, \phi_i\}$, is a linear continuous map, such that in any local trivialization of $E$, $\tilde{E}$ on some open set $V \subset U_i$, for some $i$, $A$ maps from $C^\infty_0(M, C^{n_1})$ to $C^\infty(M, C^{n_2})$, for some $n_1$, $n_2$, and for any $f, g \in C^\infty_0(V)$, $fAg$ takes the form $(fAg\phi)_i = \sum_j fA_{ij}g\phi_j$, with $fA_{ij}$ pseudodifferential operators of order $m$. The class is denoted by $\Psi^m(M, E, \tilde{E})$, or by $\Psi^m(M, E)$ in case of $E = \tilde{E}$.

If $T^*M$ is the cotangent bundle and $\pi : T^*M \to M$ is the projection of the bundle, then the principal symbol of $A$ belongs to $C^\infty(T^*M \setminus 0, \text{Hom}(\pi^*E, \pi^*\tilde{E}))$, where $\pi^*E$ is the pulled back bundle. An operator $A \in \Psi^m(M, E, \tilde{E})$ is defined to be elliptic if its principal symbol $\sigma_A(x, \xi)$ is invertible for all $x \in M$ and $\xi \neq 0$. The conclusion of Theorem 2.30 holds in the same way for elliptic operators acting on sections of vector bundles.

Theorem 2.32. If $A \in \Psi^m_d(M, E, \tilde{E})$ is elliptic and properly supported, then there exists an operator $B \in \Psi^{-m}_d(M, \tilde{E}, E)$, such that $AB - I \in \Psi^{-\infty}_d(M, E)$ and $BA - I \Psi^{-\infty}_d(M, \tilde{E})$.

It can be shown that for any manifold $M$ there exists a properly supported, classical elliptic pseudodifferential operator $\Lambda_s$, of order $s$, with positive principal symbol (for $\xi \neq 0$), and that the next definition we give does not depend on the choice of $\Lambda_s$.

Definition 2.33. For any Riemannian manifold $M$, any compact set $K \subset M$, and any $s \in \mathbb{R}$ we define the Sobolev spaces

$$H^s_{\text{loc}}(M) = \{u \in \mathcal{D}'(M) : \Lambda_s u \in L^2_{\text{loc}}(M)\},$$

and

$$H^s_{\text{com}}(M) = H^s_{\text{loc}}(M) \cap \mathcal{E}'(M)$$

Also,

$$H^s(K) = \{u \in H^s_{\text{com}}(M) : \text{supp } u \subseteq K\}.$$ 

Also,

$$H^s(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \hat{u}(\xi)\xi^s \in L^2(\mathbb{R}^n)\},$$

where $S'(\mathbb{R}^n)$ is the dual of the Schwartz space and $\hat{u}$ is the Fourier transform of $u$. 
If $M$ is $\mathbb{R}^n$, then it turns out that $H^s(K) = \mathcal{E}'(K) \cap H^s(\mathbb{R}^n)$. If $E$ is a complex vector bundle over a manifold $M$, then by using a local trivialization, we can similarly define the Sobolev spaces $H^s_{\text{loc}}(M, E)$, $H^s_{\text{com}}(M, E)$ and $H^s(K, E)$ over this vector bundle. The importance of the above spaces can be seen in the next theorem.

**Theorem 2.34.** Let $X \subseteq \mathbb{R}^n$ be an open set, $K \subset X$ be compact, $A \in \Psi^m(X)$ and $s \in \mathbb{R}$. Then, $A$ is a continuous linear operator from $H^s_{\text{com}}(M)$ to $H^s_{\text{loc}}(M)$. If $A$ is properly supported, then it extents to a continuous linear operator from $H^s_{\text{com}}(M)$ to $H^s_{\text{com}}(M)$, from $H^s_{\text{loc}}(M)$ to $H^s_{\text{loc}}(M)$ and also from $H^s(K)$ to $H^s(K)$, where $K$ is compact in $\mathbb{R}^n$ depending on $K$.

**Proof.** Proposition 7.5 and Theorem 7.3 in [41].

We also have the following Sobolev embedding theorem.

**Theorem 2.35.** (Rellich) If $l > s$, then the inclusion from $H^l(M)$ to $H^s(M)$ is compact, for any compact manifold $M$.

**Proof.** Theorem 7.4 in [41].

Consider a closed (compact without boundary) manifold $M$, two smooth vector bundles $E$, $\hat{E}$ on $M$, and an elliptic pseudodifferential operator $A$ of order $m$, which belongs to the space $\mathcal{L}(H^s(M, E), H^{s-m}(M, \hat{E})))$, for any $s \in \mathbb{R}$. We can regard $A$ as an unbounded operator in $L^2(M, E)$ with domain $H^m(M, E)$, which turns out to be a closed operator. Then, if we use Theorems 2.30, 2.34 and 2.35, we can prove the following fact for the resolvent $R_\lambda(A)$.

**Theorem 2.36.** Let $M$ be a closed manifold, $E$, $\hat{E}$ be two smooth vector bundles on $M$, and $A \in \Psi^m(M, E, \hat{E})$ be elliptic with $m > 0$ and $\text{Dom}(A) = H^m(M, E)$ in $L^2(M, E)$. Then, for $\lambda \notin \sigma(A)$, the resolvent $R_\lambda(A)$ can be continuously extended from $\mathcal{D}'(M, \hat{E})$, to an elliptic operator in $\Psi^{-m}(M, \hat{E}, E)$. Also, $R_\lambda(A)$ is compact in $L^2(M, \hat{E})$.

**Proof.** Theorem 8.2 in [41].

If the operator $A$ is also self adjoint, then by using the previous theorem we can prove the following about the spectral decomposition of $A$.

**Theorem 2.37.** Let $M$ be a closed manifold, $E$ be a smooth vector bundle on $M$, $A \in \Psi^m(M, E)$ be an elliptic self-adjoint operator and $m > 0$. Then there exists a sequence of real numbers $\{\lambda_i\}_{i \in \mathbb{N}}$, with $|\lambda_i| \to \infty$ when $i \to \infty$, and a sequence of elements $\{\psi_i\}_{i \in \mathbb{N}}$, with $\psi_i \in C^\infty(M, E)$, such that $\psi_i$ form a complete orthonormal system in $L^2(M, E)$, $A\psi_i = \lambda_i\psi_i$ and $\sigma(A) = \bigcup_i \lambda_i$. 
Proof. Theorem 8.3 in [41].

The Laplace operator is defined on the de Rham complex, which is an elliptic complex, which we define as follows.

Definition 2.38. A sequence of vector bundles $E_0, ..., E_k$ over a compact manifold $M$ and a sequence of operators $A_i \in \Psi^m(M, E_i, E_{i+1})$ form an elliptic complex (of order $m$) if $A_{i+1}A_i = 0$ and the image of $a_i$ is equal to the kernel of $a_{i+1}$, where $a_j$ is the principal symbol of $A_j$, for any $i$ (i.e. $\{a_i\}$ is exact).

If we equip every vector bundle $E_i$ with some inner product, then we can define the adjoint operator $A_i^* \in \Psi^m(M, E_{i+1}, E_i)$ of each $A_i$. Also, we define the Laplacians on each bundle $E_i$ by

$$\Delta_i = A_{i-1}^* A_i + A_i^* A_i \in \Psi^{2m}(M, E_i).$$

The principal symbol of $\Delta_i$ is by definition $\delta_i = a_{i-1} a_i^* + a_i^* a_i$. If $(\cdot, \cdot)_i$ is the inner product in $E_i$ and $\delta_i \psi = 0$, for some $\psi \in E_i$, then

$$(\delta_i \psi, \psi)_i = 0 \Rightarrow (a_i \psi, a_i \psi)_{i+1} + (a_{i-1}^* \psi, a_{i-1}^* \psi)_{i-1} = 0$$

which gives that $a_i \psi = a_{i-1}^* \psi = 0$. By assumption, $a_i \psi = 0$ implies $\psi = a_{i-1} \phi$, for some $\phi \in E_{i-1}$. But then

$$0 = (a_{i-1}^* \psi, \phi)_{i-1} = (\psi, a_{i-1} \phi)_i = (\psi, \psi)_i \Rightarrow \psi = 0,$$

which gives the ellipticity for $\Delta_i$. Thus we have proved the following.

Proposition 2.39. On an elliptic complex, given by Definition 2.38, the Laplacians defined by $\Delta_i = A_{i-1} A_{i-1}^* + A_i^* A_i$ are elliptic operators in $\Psi^{2m}(M, E_i)$.

The following result, provides us a useful decomposition.

Theorem 2.40. (Generalized Hodge decomposition theorem) For an elliptic complex, given by Definition 2.38, and for the Laplacians on it, defined by Proposition 2.39, the following decomposition holds

$$C^\infty(M, E_i) = \text{Ker } \Delta_i \oplus A_{i-1} C^\infty(M, E_{i-1}) \oplus A_i^* C^\infty(M, E_{i+1}).$$

Also,

$$\text{Ker } \Delta_i = \text{Ker } A_i \cap \text{Ker } A_{i-1}^*.$$
Proof. Since $\text{Ker} \Delta_i$ is a closed subspace of $C^\infty(M, E_i)$, by Theorem II.3 in [39], we have that

$$C^\infty(M, E_i) = \text{Ker} \Delta_i \oplus (\text{Ker} \Delta_i)^\perp.$$\[101x680]\]

Also, by construction, each $\Delta_i$ is self adjoint, so $\text{Ker} \Delta_i \perp \text{Im} \Delta_i$. Thus,

$$C^\infty(M, E_i) = \text{Ker} \Delta_i \oplus \Delta_i(C^\infty(M, E_i)).$$\[101x505]\]

There is,

$$\Delta_i(C^\infty(M, E_i)) = A_{i-1} A_{i-1}^*(C^\infty(M, E_i)) \oplus A_i^* A_i(C^\infty(M, E_i)) \subset$$\[101x340]\]

$$A_{i-1}(C^\infty(M, E_{i-1})) \oplus A_i^*(C^\infty(M, E_{i+1})).$$\[101x478]\]

By assumption

$$A_{i-1}(C^\infty(M, E_{i-1})) \perp A_i^*(C^\infty(M, E_{i+1})),\[101x478]\]

which together with the orthogonality $\text{Ker} \Delta_i \perp \text{Im} \Delta_i$ guarantees the uniqueness of the decomposition of any element in these three spaces, and the result follows.

The last statement follows by the equality

$$(\Delta_i \psi, \psi)_i = (A_i \psi, A_i \psi)_{i+1} + (A_{i-1}^* \psi, A_{i-1}^* \psi)_{i-1}.$$\[101x505]\]

Let us now consider the de Rham complex on a compact manifold $M$ of dimension $n$, i.e. the bundles $\wedge^p T^* M$, $0 \leq p \leq n$, of $p$-forms on $M$ together with the exterior derivatives $d_p$ acting there. We will show that it is an elliptic complex and we will apply the previous results to the Laplacian induced there, which will be the Laplacian acting on $p$-forms. The operator $d_p$ maps from smooth sections $\Omega^p(M) = C^\infty(M, \wedge^p T^* M)$ to $\Omega^{p+1}(M)$ by the following way

$$d_0 \omega = \sum_{i=1}^n \frac{\partial \omega}{\partial x^i} dx^i, \text{ and } d_p \omega = \sum_I d_0 \omega_I \wedge dx^I \text{ for } p > 0,$$\[101x680]\]

where $\omega = \sum_I \omega_I dx^I$, with $I$ multi index, and $\{x^1, ..., x^n\}$ are the coordinates in some local trivialization of $M$. Since the wedge product $\wedge$ is antisymmetric on one forms, it follows that that $d_{p+1} d_p = 0$. By taking in local coordinates the Fourier transform of a $p$-form $\omega$ and transforming it back we get

$$\omega = \omega_I dx^I = \left(2\pi\right)^{-n} \int \int e^{i(x-y)^I \xi} \omega_I(y) dy d\xi \, dx^I,$$\[101x760]\]
thus
\[
d_p \omega = \left( (2\pi)^{-n} \int \int i\xi e^{i(x-y)\xi} \omega(y) dy d\xi \right) dx^1 \wedge dx^n
\]
\[
= (2\pi)^{-n} \int \int e^{i(x-y)\xi} (i\xi dx^i) \wedge (\omega(y) dx^j) dy d\xi,
\]
where we have omitted summation by using Einstein’s notation. Hence, the symbol of \( d_p \) is given by
\[
i\xi \wedge : \omega \rightarrow i\xi \wedge \omega = i\xi dx^i \wedge \omega.
\]
By the antisymmetry of \( \wedge \) on one forms, we get that the image of a symbol belongs to the kernel of the next one in the complex. Also, if \( \omega \) is in the kernel of the symbol of \( d_p \), then it has the form \( \omega = f dx^1 \wedge ... \wedge dx^n \), which is the image of \((-1)^i \frac{d}{dx_i} dx^1 \wedge ... \wedge \hat{dx^i} ... \wedge dx^n \), where \( \hat{\cdot} \) means that we omitted the corresponding term. Hence the de Rham complex is an elliptic complex and the Laplacians defined there by \( \Delta_p = d_p d_p^* + d_p^* d_p \) are elliptic operators.

If we consider some Riemannian metric \( g \) on \( M \), then we can define the Hodge star operator
\[
* : \wedge^p T^* M \rightarrow \wedge^{n-p} T^* M,
\]
and the Riemannian inner product between \( \omega_1 \) and \( \omega_2 \) in \( \Omega^p(M) \), which is given by
\[
(\omega_1, \omega_2) = \int_M \bar{\omega}_1 \wedge * \omega_2.
\]
According to this inner product, by using Stokes’ theorem, we can calculate the adjoint of \( d_p \), which takes the form \( \delta_{p+1} = (-1)^{np+1} * d^* \). Let the space \( L^2(M, \wedge^p T^* M) \) be the completion of \( \Omega^p(M) \) with respect to the above inner product. From Theorems 2.37 and 2.40 we get

**Corollary 2.41.** Let the Laplacian \( \Delta_p = d_{p-1} \delta_p + \delta_{p+1} d_p \) acting on smooth sections \( \Omega^p(M) \) of the vector bundle of \( p \)-forms, on a closed Riemannian manifold \( M \). Then there exists a sequence of real positive numbers \( \{\lambda_i\}_{i \in \mathbb{N}}, \) with \( \lambda_i \rightarrow \infty \) when \( i \rightarrow \infty \), and a sequence of elements \( \{\psi_i\}_{i \in \mathbb{N}}, \) with \( \psi_i \in \Omega^p(M) \), such that \( \psi_i \) form a complete orthonormal system in \( L^2(M, \wedge^p T^* M) \), \( \Delta_p \psi_i = \lambda_i \psi_i \) and \( \sigma(\Delta_p) = \cup_i \lambda_i \). Also we have the following decomposition
\[
\Omega^p(M) = \text{Ker} \Delta_p \oplus d_{p-1} \Omega^{p-1}(M) \oplus \delta_{p+1} \Omega^{p+1}(M),
\]
and
\[
\text{Ker} \Delta_p = \text{Ker} d_p \cap \text{Ker} \delta_p.
\]
Chapter 3

Spectral theory of the Laplacian on a generalized cusp

3.1 Geometry of the generalized cusp

In this section, we determine some of the geometric properties of the generalized cusp. We state some fundamental results which we are going to use later in the document for technical purposes. We also calculate the expression of the Laplace operator acting on $p$-forms.

Consider a noncompact $n$-dimensional Riemannian manifold $M_1 = [1, \infty) \times N$ with metric

$$g_1 = dx^2 + x^{-2a}h,$$

where $x \in [1, \infty)$, $a \in (0, \infty)$ is a parameter, and $h$ is the pull back under the canonical projection $\pi : [1, \infty) \times N \to N$ of a metric tensor which corresponds to some closed and oriented $n - 1$-dimensional Riemannian manifold $N$. We call the manifold $M_1$ generalized cusp. When $N = S^1$, the curvature of $M_1$ is equal to $-a(a + 1)/x^2$. Let $\wedge^p T^*(M_1)$ be the bundle of $p$-forms on $M_1$. Any element in $\omega \in \wedge^p T^*(M_1)$ can be uniquely decomposed as $\omega = \alpha + dx \wedge \beta$, where $\alpha \in \wedge^p T^*(M_1)$ and $\beta \in \wedge^{p-1} T^*(M_1)$ are sections of the pulled back bundles $\pi^* \wedge^p T^* N$ and $\pi^* \wedge^{p-1} T^* N$ respectively under the canonical projection. Hence, $\wedge^p T^*(M_1)$ is canonically isomorphic to $\wedge^p T^*(N) \oplus \wedge^{p-1} T^*(N)$. So, we can regard $\alpha$ and $\beta$ as sections of the bundles $\wedge^p T^* N$ and $\wedge^{p-1} T^* N$ respectively, which depend on the parameter $x$.

Let $\Omega^p(M_1)$ be the space of smooth $p$-forms on $M_1$ (i.e. smooth sections of the $p$-form bundle) and $\Omega^p_0(M_1)$ its subspace consisting of compactly supported
elements. The $L^2$ inner product is defined by

$$ (\omega_1, \omega_2) = \int_{M_1} \bar{\omega}_1 \wedge \ast \omega_2 $$

for any $\omega_1, \omega_2 \in \Omega^p(M_1)$, where $\ast : \Omega^p(M_1) \to \Omega^{n-p}(M_1)$ is the Hodge star operator, mapping between bundles of smooth $p$-forms. If $\ast_N$ is the Hodge star operator defined in $N$, with respect to the metric $h$, then we have the following.

**Lemma 3.1.** $\ast \omega = x^{-a(n-2p+1)} \ast_N \beta + (-1)^p x^{-a(n-2p-1)} dx \wedge \ast_N \alpha$.

**Proof.** First note that if $\{e^1, ..., e^{n-1}\}$ is a local positively oriented orthonormal basis in $T^*N$, then $\{dx, x^{-a} e^1, ..., x^{-a} e^{n-1}\}$ is a local positively oriented orthonormal basis in $\Omega^1(M_1)$. It suffices to consider $\alpha = \alpha_{i_1...i_p} e^{i_1} \wedge ... \wedge e^{i_p}$ and $\beta = \beta_{j_1...j_{p-1}} e^{j_1} \wedge ... \wedge e^{j_{p-1}}$ expressed in that basis, as general forms will be linear combinations of such terms. Then

$$ \omega = \alpha + dx \wedge \beta = x^{ap} \alpha_{i_1...i_p} x^{-a} e^{i_1} \wedge ... \wedge x^{-a} e^{i_p} + x^{a(p-1)} dx \wedge \beta_{j_1...j_{p-1}} x^{-a} e^{j_1} \wedge ... \wedge x^{-a} e^{j_{p-1}}. $$

Hence

$$ \ast \omega = (-1)^p x^{ap} \alpha_{i_1...i_p} dx \wedge x^{-a} e^{i_p+1} \wedge ... \wedge x^{-a} e^{i_{n-1}} $$

$$ + x^{a(p-1)} \beta_{j_1...j_{p-1}} x^{-a} e^{j_p} \wedge ... \wedge x^{-a} e^{j_{n-1}} $$

$$ = (-1)^p x^{-a(n-2p+1)} \alpha_{i_1...i_p} dx \wedge e^{i_p+1} \wedge ... \wedge e^{i_{n-1}} $$

$$ + x^{-a(n-2p+1)} \beta_{j_1...j_{p-1}} e^{j_p} \wedge ... \wedge e^{j_{n-1}}. $$

We have that

$$ \ast_N \alpha = \alpha_{i_1...i_p} e^{i_p+1} \wedge ... \wedge e^{i_{n-1}}, $$

$$ \ast_N \beta = \beta_{j_1...j_{p-1}} e^{j_p} \wedge ... \wedge e^{j_{n-1}}, $$

and the lemma is proved.

Let us denote by $L^2(M_1, \wedge^p T^* M_1)$ the space of square integrable $p$-forms on $M_1$ with respect to the Riemannian inner product (i.e. the completion of the space $\Omega^p_0(M_1)$ with respect to this inner product), and similarly for the manifold $N$. The following consequence of the previous lemma holds.
Lemma 3.2. If $\omega_1 = \alpha_1 + dx \wedge \beta_1$ and $\omega_2 = \alpha_2 + dx \wedge \beta_2$ are two $p$-forms in $L^2(M_1, \wedge^p T^* M_1)$, then for their inner product we have that

$$(\omega_1, \omega_2) = (\alpha_1, \alpha_2) + (\beta_1, \beta_2).$$

Also, let

$$\gamma_p = a(n - 2p - 1),$$

$\alpha, \tilde{\alpha} \in L^2([1, \infty), x^{-\gamma_p} dx)$ and $\omega, \tilde{\omega} \in L^2(N, \wedge^p T^* N)$. Then,

$$(\alpha \omega, \tilde{\alpha} \tilde{\omega})_{M_1} = (\alpha, \tilde{\alpha})_{L^2([1, \infty), x^{-\gamma_p} dx)} (\omega, \tilde{\omega})_N.$$

Proof. The first part of the lemma is trivial. By Lemma 3.1, the second part follows by the equality

$$(\alpha \omega, \tilde{\alpha} \tilde{\omega}) = \int_{M_1} \alpha \omega \wedge \star (\tilde{\alpha} \tilde{\omega}) = \int_{M_1} \tilde{\alpha} \tilde{\omega} \wedge ((-1)^p x^{-\gamma_p} dx \wedge \star_N \tilde{\alpha} \tilde{\omega}) = \int_{M_1} \tilde{\alpha} \tilde{\omega} x^{-\gamma_p} dx \wedge \tilde{\omega} \wedge \star_N \tilde{\omega} = (\int_1^\infty \tilde{\alpha} \tilde{\omega} x^{-\gamma_p} dx)(\int_N \tilde{\omega} \wedge \star_N \tilde{\omega}).$$

If $d : \Omega^p(M_1) \to \Omega^{p+1}(M_1)$ is the exterior derivative on $M_1$, then the formal adjoint $\delta : \Omega^{p-1}(M_1) \to \Omega^p(M_1)$ of $d$ on $\Omega^p_0(M_1 \setminus \partial M_1)$ with respect to the Riemannian inner product, is given by $\delta = (-1)^{n(p+1)+1} \star d \star$ (cf. [40]). The Laplacian on $M_1$ acting on the space $\Omega_0(M_1 \setminus \partial M_1)$ of smooth compactly supported $p$-forms, with support away from the boundary, is as usually given by

$$\Delta = d\delta + \delta d.$$

$\Delta$ is a positive formally self-adjoint operator. Now, if we denote the exterior derivative on $N$ by $d_N$, we have

$$d\omega = d_N \alpha + dx \wedge (\partial_x \alpha - d_N \beta). \quad (3.2)$$

Let $\delta_N$ be the adjoint of $d_N$ in $N$ with respect to the metric $h$. If we use the property $\star^2 = (-1)^{p(n-p)}$ of the Hodge star operator, the commutativity of $\partial_x$ with $d_N$ and $\star_N$, the Lemma 3.1, and the Equation (3.2), we find that

$$d \star \omega = x^{-\gamma_p} d_N \star_N \beta + dx \wedge \left(-\gamma_{p-1} x^{-\gamma_{p-1}-1} \star_N \beta \right. + x^{-\gamma_p} \partial_x \star_N \beta + \left(-1)^{p+1} x^{-\gamma_p} d_N \star_N \alpha \right),$$
\[ *d *\omega = -\frac{\gamma \beta}{x} (p-1)(p-n) \beta + *N \partial_x \beta \]
\[ + (-1)^{p+1} x^2 \alpha d_N *N \beta \alpha + (-1)^{n-p+1} x^2 \alpha d_N *N \beta, \]

and
\[ \delta \omega = \frac{\gamma \beta}{x} - \partial_x \beta + x^2 \alpha d_N = x^2 \alpha \delta N - x^2 \alpha \delta N \beta. \] (3.3)

Now, from (3.2) and (3.3) we find the expression of the Laplacian on \( \Omega_0(M_1 \setminus \partial M_1) \) to be
\[ \Delta \omega = x^{2a} \Delta_N \alpha - \partial_x^2 \alpha + \frac{\gamma \beta}{x} \partial_x \alpha + \frac{2a}{x} d_N \beta + \]
\[ dx \wedge \left( x^{2a} \Delta_N \beta - \partial_x^2 \beta + \frac{\gamma \beta}{x} \partial_x \beta - \frac{\gamma \beta}{x^2} \beta + \frac{2a}{x} x^2 \alpha \delta N \alpha \right), \] (3.4)

where \( \Delta_N \) is the Laplacian on forms in \( N \), generated by the metric \( h \).
3.2 The Friedrichs extension

In a geodesically complete orientable manifold $\mathcal{M}$, the Laplacian $d\delta + \delta d$ and the first order operator $d + \delta$, defined on $\Omega^p_0(\mathcal{M})$, have unique self-adjoint extensions to $L^2$ (see [5]). We first consider $M_1$, which has a boundary at $x = 1$, and it is at this boundary what extensions we will consider. We choose the closed self-adjoint extension of $d\delta + \delta d$ to be the Friedrichs extension (cf. Theorem X.23, [39] vol.II), which we will show that corresponds to the Laplacian with Dirichlet boundary conditions at $x = 1$.

Recall the definition of the Friedrichs extension. If $\tilde{q} = (\phi, d\delta + \delta d\psi)$ is the quadratic form associated to the Laplacian, we define the inner product $(\phi, \psi)_+ = (\phi, d\delta + \delta d\psi) + (\phi, \psi)$ in the form domain of $\tilde{q}$. The Sobolev space $H^1_0(M_1) \subset L^2(M_1, \wedge^p T^* M_1)$ is defined as the completion of $\Omega^p_0(M_1 \setminus \partial M_1)$ with respect to this inner product. The form $\tilde{q}$ extends to a closed form $q$ in $H^1_0$.

Define a self-adjoint operator $A$, as follows. First, $\text{Dom}(A) = \{ \psi \in H^1_0 : q(\cdot, \psi) \text{ is a bounded linear functional in } L^2(M_1, \wedge^p T^* M_1) \}$. Then, since by the Riesz lemma (Theorem II.4 in [39] vol.I) for any bounded linear functional $f(\cdot)$ in $L^2(M_1, \wedge^p T^* M_1)$ there exists a unique element $\tilde{\psi}$ in $L^2(M_1, \wedge^p T^* M_1)$ such that $f(\cdot) = (\cdot, \tilde{\psi})$, by taking $f(\cdot)$ to be $q(\cdot, \psi)$, we can define $A\psi = \tilde{\psi}$. Then, the operator $A - I$ is the Friedrichs extension of $d\delta + \delta d$, which we keep denoting by $\Delta$, and it has domain

$$\text{Dom}(\Delta) = \{ \psi \in H^1_0 : \exists \tilde{\psi} \in L^2(M_1, \wedge^p T^* M_1) \text{ such that } q(\phi, \psi) = (\phi, \tilde{\psi}), \forall \phi \in L^2(M_1, \wedge^p T^* M_1) \}. \quad (3.5)$$

We have extended $d\delta + \delta d$ from the space of smooth compactly supported $p$-forms, with support away from the boundary. The last assumption will give Dirichlet boundary conditions to the Friedrichs extension $\Delta$, as we can see from the following.

**Lemma 3.3.** If $\omega \in \text{Dom}(\Delta)$, then $\omega|_{\partial M_1} = 0$. Thus, $\Delta$ satisfies Dirichlet boundary conditions.

**Proof.** From (3.5), we have that $\text{Dom}(\Delta) \subset H^1_0$. The space $H^1_0$ is the completion of $\Omega^p_0(M_1 \setminus \partial M_1)$ with respect to the inner product $(\cdot, \cdot)_+$. Hence, $\psi \in H^1_0$ implies $\psi|_{\partial M_1} = 0$, since the trace operator $\tau \psi = \psi|_{\partial M_1}$ is a continuous operator (cf. [43]), which completes the proof. \qed
3.3 The decomposition of the Laplacian

In this section, we decompose the $L^2$ space of $p$-forms on $M_1$ by using the Hodge decomposition of the Laplacian $\Delta_N$, defined on the boundary $N$. The decomposition is given in a way, such that the corresponded subspaces are left invariant under any spectral projection of the Laplacian. Also, we break down the Laplacian on $M_1$ by using an eigenbasis of $\Delta_N$, and express the eigenvalue equation of $\Delta$ restricted to each of the invariant subspaces by an equivalent system of ordinary differential equations.

Since any element $\omega \in \wedge^p T^*(M_1)$ can be written as $\omega = \alpha + dx \wedge \beta$, where $\alpha$ and $\beta$ are elements of the pulled back bundles $\pi^* \wedge^p T^*N$ and $\pi^* \wedge^{p-1} T^*N$ respectively under the canonical projection $\pi : [1, \infty) \times N \to N$, according to the Corollary 2.41 and Lemma 3.2, we have that

$$\Omega^p_\partial(M_1 \setminus \partial M_1) = \left( \text{Ker } \Delta_N|_{C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^p T^*N)} \oplus d_N C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^{p-1} T^*N) \right)$$

$$\oplus dx \wedge \left( \text{Ker } \Delta_N|_{C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^{p-1} T^*N)} \oplus d_N C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^p T^*N) \right)$$

(3.6)

Thus, any $p$-form $\omega$ in $\Omega^p_\partial(M_1 \setminus \partial M_1)$ can be uniquely written in the form

$$\omega = \omega_H + \omega_d + \omega_\delta + dx \wedge (\tilde{\omega}_H + \tilde{\omega}_d + \tilde{\omega}_\delta),$$

where

$$\omega_H, \tilde{\omega}_H \in \text{Ker } \Delta_N, \omega_d, \tilde{\omega}_d \in \text{Im } d_N, \text{ and } \omega_\delta, \tilde{\omega}_\delta \in \text{Im } \delta_N.$$ 

Hence, we may write

$$\omega = (\omega_H + dx \wedge \tilde{\omega}_H) + (\omega_d + dx \wedge \tilde{\omega}_d) + (\omega_\delta + dx \wedge \tilde{\omega}_\delta).$$

According to the above, we give the following

**Definition 3.4.** Define the canonical projection $\pi : M_1 \to N$. Define the spaces

$$\Omega^p_{0,H}(M_1 \setminus \partial M_1) := \{ \omega_H + dx \wedge \tilde{\omega}_H : \omega_H \in \text{Ker } \Delta_N|_{C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^p T^*N)} \},$$

$$\Omega^p_{0,d,\delta}(M_1 \setminus \partial M_1) := \{ \omega_d + dx \wedge \tilde{\omega}_d : \omega_d \in d_N C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^{p-1} T^*N) \},$$

$$\Omega^p_{\partial}(M_1 \setminus \partial M_1) := \{ \omega_H + dx \wedge \tilde{\omega}_H : \omega_H \in \text{Ker } \Delta_N|_{C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^{p-1} T^*N)} \},$$

$$\Omega^p_{\partial}(M_1 \setminus \partial M_1) := \{ \omega_d + dx \wedge \tilde{\omega}_d : \omega_d \in d_N C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^p T^*N) \}. $$
and

\[ \Omega^p_{0,\delta,d}(M_1 \setminus \partial M_1) := \{ \omega_\delta + dx \wedge \tilde{\omega}_d : \omega_\delta \in \delta_N C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^{p+1} T^* N), \tilde{\omega}_d \in d_N C^\infty_0(M_1 \setminus \partial M_1, \pi^* \wedge^{p-2} T^* N) \}. \]

Also, denote by \( \Delta_H, \Delta_{d,\delta} \) and \( \Delta_{\delta,d} \) the Friedrichs extension of \( d\delta + \delta d \) restricted to the above three spaces respectively, and by

\[ L^2_H(M_1, \wedge^p T^* M_1), \quad L^2_{d,\delta}(M_1, \wedge^p T^* M_1) \quad \text{and} \quad L^2_{\delta,d}(M_1, \wedge^p T^* M_1) \]

their corresponding completions with respect to the \( L^2 \) inner product.

From (3.6) and Lemma 3.2, we get that

\[ \Omega^p_0(M_1 \setminus \partial M_1) = \Omega^p_{0,H}(M_1 \setminus \partial M_1) \oplus \Omega^p_{0,d,\delta}(M_1 \setminus \partial M_1) \oplus \Omega^p_{0,\delta,d}(M_1 \setminus \partial M_1), \]

and by (3.4), \( \Delta \) maps each of the spaces \( \Omega^p_{0,H}(M_1 \setminus \partial M_1), \Omega^p_{0,d,\delta}(M_1 \setminus \partial M_1) \) and \( \Omega^p_{0,\delta,d}(M_1 \setminus \partial M_1) \) to itself. A similar statement holds for their corresponding \( L^2 \) spaces. Hence, we get the following

**Theorem 3.5.** According to the decomposition

\[ L^2(M_1, \wedge^p T^* M_1) = L^2_H(M_1, \wedge^p T^* M_1) \oplus L^2_{d,\delta}(M_1, \wedge^p T^* M_1) \oplus L^2_{\delta,d}(M_1, \wedge^p T^* M_1), \]

the Laplacian \( \Delta = \Delta_H \oplus \Delta_{d,\delta} \oplus \Delta_{\delta,d} \) leaves each of these three subspaces invariant, in the sense that they are invariant under any spectral projection of \( \Delta \).

**Proof.** Let \( \hat{q}, \hat{q}_H, \hat{q}_{d,\delta} \) and \( \hat{q}_{\delta,d} \) be the quadratic forms associated to \( \Delta \) when it is restricted to the spaces \( \Omega^p_0(M_1 \setminus \partial M_1), \Omega^p_{0,H}(M_1 \setminus \partial M_1), \Omega^p_{0,d,\delta}(M_1 \setminus \partial M_1) \) and \( \Omega^p_{0,\delta,d}(M_1 \setminus \partial M_1) \) respectively, and let \( q, q_H, q_{d,\delta} \) and \( q_{\delta,d} \) be their corresponding closures. We have that \( \hat{q} = \hat{q}_H \oplus \hat{q}_{d,\delta} \oplus \hat{q}_{\delta,d}, \) and by the orthogonality of the above spaces we get similarly for the closures that \( q = q_H \oplus q_{d,\delta} \oplus q_{\delta,d} \). Now the proof follows by the next two general facts. \( \square \)

**Lemma 3.6.** Let \( H_1, H_2 \) be two Hilbert spaces and \( q_1, q_2 \) be two closed quadratic forms on \( H_1 \) and \( H_2 \) respectively. Let \( B_1 \) and \( B_2 \) be the corresponding self-adjoint operators of \( q_1 \) and \( q_2 \) respectively. Then \( B_1 \oplus B_2 \) on \( \text{Dom}(B_1) \oplus \text{Dom}(B_2) \) is the corresponding self-adjoint operator of the form \( q_1 \oplus q_2 \) on \( H_1 \oplus H_2 \).

**Proof.** Let \( H_1^+ = \text{Dom}(q_1) \) in \( H_1 \), and \( H_1^- \) be the space of the conjugate linear functionals on \( H_1^+ \). If \( \langle \cdot, \cdot \rangle_1 \) is the inner product in \( H_1 \), then we define the map \( j_1 : H_1 \rightarrow H_1^- \) by \( \psi \mapsto \langle \cdot, \psi \rangle_1 \), which is bounded. Let the operator \( \hat{B}_1 : H_1^+ \rightarrow H_1^- \), such that \( \langle \hat{B}_1 \alpha \phi \rangle = q_1(\phi, \alpha) + \langle \phi, \alpha \rangle_1, \forall \alpha, \phi \in H_1^+, \) which is an isometric
isomorphism by the Riesz lemma (Theorem II.4 in [39] vol.I). Consider the operator $B_1$ with \( \text{Dom}(B_1) = \{ \psi \in H_1^+ : \hat{B}_1 \psi \in \text{Ran} j_1 \} \) and $B_1 = j_1^{-1} \hat{B}_1$. Then, $B_1$ is the unique self-adjoint operator associated to $q_1$ (Theorem VIII.15 in [39] vol.I). Similarly, we can do the analogous for $q_2$ in $H_2$.

Now let the space $H = H_1 \oplus H_2$ and the quadratic form $q = q_1 \oplus q_2$ on $\text{Dom}(q) = \text{Dom}(q_1) \oplus \text{Dom}(q_2)$, such that for every $\phi, \omega \in \text{Dom}(q_1)$ and $\phi', \omega' \in \text{Dom}(q_2)$, there is $q(\omega \oplus \omega', \phi \oplus \phi') = q_1(\omega, \phi) + q_2(\omega', \phi')$. If $\text{Dom}(q) = H^+$, then $H^+ = H_1^+ \oplus H_2^+$. Let $H^-$ be the space of the conjugate linear functionals in $H^+$.

There is $H^- = H_1^- \oplus H_2^-$. Let $j : H^+ \to H^-$ such that $j(\phi \oplus \phi') = (\cdot, \phi \oplus \phi')$, for all $\phi \in H_1$ and $\phi' \in H_2$, where $(\cdot, \cdot) = (\cdot, \cdot)_1 + (\cdot, \cdot)_2$ is the inner product in $H$. There is $j(\phi \oplus \phi') = (\cdot, \phi \oplus \phi') = (\cdot, \phi)_1 + (\cdot, \phi')_2$, thus $j = j_1 \oplus j_2$. Let the operator $\hat{B} : H^+ \to H^-$ defined by

$$ (\hat{B}(\phi \oplus \phi'))(\phi \oplus \phi') = q(\phi \oplus \phi', \alpha \oplus \alpha') + (\phi \oplus \phi', \alpha \oplus \alpha'), $$

where $\phi, \alpha \in H_1^+$ and $\phi', \alpha' \in H_2^+$. There is

$$ (\hat{B}(\phi \oplus \phi'))(\phi \oplus \phi') = q_1(\phi, \alpha) + q_2(\phi', \alpha') + (\phi, \alpha)_1 + (\phi', \alpha')_2, $$

thus $\hat{B} = \hat{B}_1 \oplus \hat{B}_2$. Consider the operator $B$, such that $\text{Dom}(B) = \{ \psi \oplus \psi' \in H^+ : \hat{B}(\psi \oplus \psi') \in \text{Ran} j \}$, where $\psi \in H_1^+$ and $\psi' \in H_2^+$, and $B = j^{-1} \hat{B}$. Then $B$ is the unique self-adjoint operator associated to $q$ (Theorem VIII.15 in [39] vol.I). There is

$$ \text{Dom}(B) = \{ \psi \oplus \psi' \in H_1^+ \oplus H_2^+ : \hat{B}_1(\psi) \oplus \hat{B}_2(\psi') \in \text{Ran} (j_1 \oplus j_2) \} = \{ \psi \in H_1^+ : \hat{B}_1(\psi) \in \text{Ran} j_1 \} \oplus \{ \psi' \in H_2^+ : \hat{B}_2(\psi') \in \text{Ran} j_2 \}. $$

Hence, $\text{Dom}(B) = \text{Dom}(B_1) \oplus \text{Dom}(B_2)$. Also,

$$ B = j^{-1} \hat{B} = (j_1 \oplus j_2)^{-1}(\hat{B}_1 \oplus \hat{B}_2) = (j_1^{-1} \hat{B}_1) \oplus (j_2^{-1} \hat{B}_2), $$

thus $B = B_1 \oplus B_2$, and the lemma is proved.

\[\Box\]

**Lemma 3.7.** Let $B_1$ and $B_2$ be two self-adjoint operators defined in $\text{Dom}(B_1) \subset H_1$ and $\text{Dom}(B_2) \subset H_2$ respectively, for some Hilbert spaces $H_1$ and $H_2$. Let the operator $B_1 \oplus B_2$ defined in $\text{Dom}(B_1) \oplus \text{Dom}(B_2) \subset H_1 \oplus H_2$. If $dE_{B_1}(\lambda)$, $dE_{B_2}(\lambda)$ and $dE_{B_1 \oplus B_2}(\lambda)$ are the spectral families of $B_1$, $B_2$ and $B_1 \oplus B_2$ respectively, then $dE_{B_1 \oplus B_2}(\lambda) = dE_{B_1}(\lambda) \oplus dE_{B_2}(\lambda)$. 
By the spectral theorem for a self-adjoint operator, we have that

\[ \int_R \lambda dE_{B_1 \oplus B_2}(\lambda) = B_1 \oplus B_2 = (\int_R \lambda dE_{B_1}(\lambda)) \oplus (\int_R \lambda dE_{B_2}(\lambda)) = \int_R \lambda dE_{B_1}(\lambda) \oplus dE_{B_2}(\lambda), \]

and the proof follows by the uniqueness of the spectral measure. \(\Box\)

In order to find the equivalent system of differential equations corresponding to the spectral equation of the Laplacian, when it is restricted to each of the invariant spaces of the decomposition of Theorem 3.5, we will restrict \(p\)-forms to the boundary and then we will expand them, by using an appropriate basis. These basis are obtained by the following standard fact.

**Proposition 3.8.** Let the spaces

\[ L^2_\mathcal{H}(N, \wedge^p T^*N), \ L^2_\mathcal{D}(N, \wedge^p T^*N), \text{ and } L^2_\delta(N, \wedge^p T^*N) \]

be the completion of \(\text{Ker}\Delta_N|_{\Omega^p(N)}\), \(d_N\Omega^{p-1}(N)\) and \(\delta_N\Omega^{p+1}(N)\) with respect to the \(L^2\) inner product. We can choose an orthonormal basis \(\{\theta^i\}_{i \in \mathbb{N}}, \{\phi^j\}_{i \in \mathbb{N}}\) and \(\{\psi^i\}_{i \in \mathbb{N}}\) of the spaces \(L^2_\mathcal{H}(N, \wedge^p T^*N), \ L^2_\mathcal{D}(N, \wedge^p T^*N)\) and \(L^2_\delta(N, \wedge^p T^*N)\) respectively, such that

\[ \theta^i \in \text{Ker}\Delta_N|_{\Omega^p(N)}, \ \phi^j \in d_N\Omega^{p-1}(N) \text{ and } \psi^i \in \delta_N\Omega^{p+1}(N), \]

and also \(\Delta_N\phi^j = \mu^2 \phi^j, \ \Delta_N\psi^i = \mu^2 \psi^i, \ d_N\psi^i = \mu_i \phi^j\) and \(\delta_N\phi^j = \mu_i \psi^i\), where \(\mu \in \mathbb{R}\) and \(\{\theta^i\}_{i \in \mathbb{N}}, \{\phi^j\}_{i \in \mathbb{N}}\) and \(\{\psi^i\}_{i \in \mathbb{N}}\) are the corresponding basis for the \(p-1\) case. We also have the decomposition

\[ L^2(N, \wedge^p T^*N) = L^2_\mathcal{H}(N, \wedge^p T^*N) \oplus L^2_\mathcal{D}(N, \wedge^p T^*N) \oplus L^2_\delta(N, \wedge^p T^*N) \]

**Proof.** According to Corollary 2.41, if \(\{\tilde{\psi}^i\}_{i \in \mathbb{N}}\) is an orthonormal basis in the space \(L^2_\delta(N, \wedge^p T^*N)\), with \(\Delta_N\tilde{\psi}^i = \mu^2 \tilde{\psi}^i\) and \(\tilde{\psi}^i \in \delta_N\Omega^p(N)\), then, since by the Hodge decomposition \(d_N\tilde{\psi}^i\) is an orthogonal basis in \(L^2_\delta(N, \wedge^p T^*N)\), choose \(\phi^i\) to be \(d_N\tilde{\psi}^i / \mu_i\) for any \(i\). Then, \(\delta_N\phi_i = \delta_N d_N\tilde{\psi}^i / \mu_i = \Delta_N\tilde{\psi}^i / \mu_i = \mu_i \tilde{\psi}_i\). \(\Box\)

By using these basis, Lemma 3.2 and Definition 3.4, we get the following
Lemma 3.9. Consider the space

\[ \mathcal{U}_p = L^2([1, \infty), x^{-\gamma_p} dx) \oplus L^2([1, \infty), x^{-\gamma_{p-1}} dx). \]

Then,

\[ L^2_H(M_1, \wedge^p T^* M_1) = \bigoplus_i L^2_{H,i}(M_1 \wedge^p T^* M_1) \]

\[ L^2_{d,H}(M_1, \wedge^p T^* M_1) = \bigoplus_i L^2_{d,H,i}(M_1 \wedge^p T^* M_1) \]

and

\[ L^2_{\delta,d}(M_1, \wedge^p T^* M_1) = \bigoplus_i L^2_{\delta,d,i}(M_1 \wedge^p T^* M_1) \]

where

\[ L^2_{H,i}(M_1, \wedge^p T^* M_1) = \{ \alpha \theta^i + dx \wedge \beta \tilde{\theta}^i : \alpha \oplus \beta \in \mathcal{U}_p \}, \]

\[ L^2_{d,H,i}(M_1, \wedge^p T^* M_1) = \{ \alpha \phi^i + dx \wedge \beta \tilde{\phi}^i : \alpha \oplus \beta \in \mathcal{U}_p \}, \]

and

\[ L^2_{\delta,d,i}(M_1, \wedge^p T^* M_1) = \{ \alpha \psi^i + dx \wedge \beta \tilde{\psi}^i : \alpha \oplus \beta \in \mathcal{U}_p \}. \]

We can derive now a system of ordinary differentially equations equivalent to the spectral equation of the Laplacian \( \Delta \omega = \lambda^2 \omega \), when the Laplacian is restricted to each of the three spaces of the decomposition of Theorem 3.5. We define first the following notation

\[ \mathcal{D}_p = \begin{pmatrix} -\partial_x^2 + \frac{\gamma_p}{x} \partial_x & 0 \\ 0 & -\partial_x^2 + \frac{\gamma_{p-1}}{x} \partial_x - \frac{\gamma_{p-1}}{x^2} \end{pmatrix} \text{ and } \mathcal{A} = \begin{pmatrix} 0 & 1 \\ x^{2a} & 0 \end{pmatrix}. \]

Then by Lemma 3.9, the equation \( \Delta \omega = \lambda^2 \omega \) becomes respectively

The space \( L^2_H(M_1, \wedge^p T^* M_1) \)

\[ (\mathcal{D}_p - \lambda^2 I) \omega_i = 0, \quad i \in \mathbb{N} \quad (3.7) \]

where

\[ \omega_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \text{ and } \omega = \sum_i \alpha_i \theta^i + dx \wedge \beta_i \tilde{\theta}^i. \]
The space $L^2_{d,δ}(M_1, \wedge^p T^* M_1)$

\[(D_p + \mu_1^2 x^{2a} I + \mu_1 \frac{2a}{x} \mathcal{X} - \lambda^2 I) \omega_i = 0, \; i \in \mathbb{N} \tag{3.8}\]

where

$$\omega_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \quad \text{and} \quad \omega = \sum_i \alpha_i \phi^i + dx \wedge \beta_i \tilde{\psi}^i.$$ 

The space $L^2_{\tilde{d},\tilde{d}}(M_1, \wedge^p T^* M_1)$

\[(D_p + \mu_1^2 x^{2a} I - \lambda^2 I) \omega_i = 0, \; i \in \mathbb{N} \tag{3.9}\]

where

$$\omega_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \quad \text{and} \quad \omega = \sum_i \alpha_i \psi^i + dx \wedge \beta_i \tilde{\phi}^i.$$
3.4 The discrete spectrum

In this section we will show that the Laplacian $\Delta$ restricted to each of the spaces $L_{d,\delta}^2(M_1, \wedge^p T^* M_1)$ and $L_{\delta,d}^2(M_1, \wedge^p T^* M_1)$ has discrete spectrum. Thus, only the space with harmonic components will contribute to the continuous spectrum. We deal with each of the spaces separately.

$L_{d,\delta}^2(M_1, \wedge^p T^* M_1)$

According to the Definition 3.4, take

$$\omega = \sum_i \alpha_i \phi^i + dx \wedge \beta_i \tilde{\psi}^i \in \Omega_{0,d,\delta}^p(M_1 \setminus \partial M_1).$$

By (3.4), we have

$$\Delta \omega = \sum_i \left( (-\partial_x^2 + \frac{\gamma p}{x} \partial_x) \alpha_i + x^{2a} \mu_i^2 \alpha_i + \frac{2a}{x} \mu_i \beta_i \right) \phi^i$$

$$+ dx \wedge \left( (-\partial_x^2 + \frac{\gamma p-1}{x} \partial_x) \beta_i + (x^{2a} \mu_i^2 - \frac{\gamma p-1}{x^2}) \beta_i + \frac{2a}{x} x^{2a} \mu_i \alpha_i \right) \tilde{\psi}^i.$$  (3.10)

By Lemma 3.2, an alternative way to define the spaces $L_{d,\delta,i}^2(M_1, \wedge^p T^* M_1)$ of Lemma 3.9, is by the closure of the spaces

$$\Omega_{0,d,\delta,i}^p(M_1 \setminus \partial M_1) = \{ \alpha_i \phi^i + dx \wedge \beta_i \tilde{\psi}^i : \alpha_i, \beta_i \in C_0^\infty([1, \infty)) \},$$

with respect to the $L^2$ inner product. From Lemma 3.9 we have

$$L_{d,\delta}^2(M_1, \wedge^p T^* M_1) = \bigoplus_i L_{d,\delta,i}^2(M_1, \wedge^p T^* M_1).$$

Let the operators $A_i$ and $V_i$ to be the Friedrichs extensions of the operators $\tilde{A}_i, \tilde{V}_i$, acting on the space $\Omega_{0,d,\delta,i}^p(M_1 \setminus \partial M_1)$, where

$$\tilde{A}_i \omega = (-\partial_x^2 + \frac{\gamma p}{x} \partial_x) \alpha_i \phi^i + dx \wedge (-\partial_x^2 + \frac{\gamma p-1}{x} \partial_x) \beta_i \tilde{\psi}^i$$

is the term with derivatives and

$$\tilde{V}_i \omega = (x^{2a} \mu_i^2 \alpha_i + \frac{2a}{x} \mu_i \beta_i) \phi^i + dx \wedge \left( \frac{2a}{x} x^{2a} \mu_i \alpha_i + (x^{2a} \mu_i^2 - \frac{\gamma p-1}{x^2}) \beta_i \right) \tilde{\psi}^i$$

is like a potential term. Each of the $\tilde{A}_i, \tilde{V}_i$ leaves the space $\Omega_{0,d,\delta,i}^p(M_1 \setminus \partial M_1)$ invariant. If we apply Lemma 3.6 to the spaces

$$L_{d,\delta,j}^2(M_1, \wedge^p T^* M_1)$$

and

$$\bigoplus_{i \neq j} L_{d,\delta,i}^2(M_1, \wedge^p T^* M_1),$$
with the quadratic forms of $\tilde{A}_j \oplus \tilde{V}_j$ and $\bigoplus_{i \neq j}(\tilde{A}_i \oplus \tilde{V}_i)$ respectively, and Lemma 3.7 to the same spaces with the operators $A_j \oplus V_j$ and $\bigoplus_{i \neq j}(A_i \oplus V_i)$ respectively, we get that

$$\Delta|_{L^2_{d,\delta,i}(M_1, \wedge^{p} T^* M_1)} = A = \bigoplus_i (A_i \oplus V_i),$$

and $A$ leaves each of the subspaces invariant under any spectral projection. We will show that each of the operators $A_i \oplus V_i$, with domain in the space $L^2_{d,\delta,i}(M_1, \wedge^{p} T^* M_1)$, has discrete spectrum with a lower bound, and that these lower bounds form an increasing sequence in $i$. Then, from the previous equation, the discreteness of $\Delta|_{L^2_{d,\delta,i}(M_1, \wedge^{p} T^* M_1)}$ will follow.

Take $\omega_i \in \Omega^0_{d,\delta,i}(M_1 \setminus \partial M_1)$. After integrating by parts, we find that

$$(\omega_i, A_i \omega_i) = (\partial_x \alpha_i, \partial_x \alpha_i)_{L^2(1, \infty, x^{-\gamma_p} dx)} + (\partial_x \beta_i, \partial_x \beta_i)_{L^2(1, \infty, x^{-\gamma_p+1} dx)}.$$ 

Thus $A_i$ is self-adjoint and $A_i \geq 0$. Also, the eigenvalue equation $V_i \omega_i = \lambda_i \omega_i$ for the self-adjoint operator $V_i$, has two solutions for $\lambda_i$, namely

$$\lambda_i^\pm = \mu_i^2 x^{2a} - \frac{1}{2} \left( \frac{\gamma_{p-1}}{x^2} \pm \sqrt{\frac{\gamma_{p-1}^2}{x^4} + \frac{16a^2}{\mu_i^2 x^{2a}}} \right). \quad (3.11)$$

We have

$$\lambda_i^\pm = |\mu_i| x^a \left( |\mu_i| x^a - \frac{1}{2} \left( \frac{\gamma_{p-1}}{|\mu_i| x^{a+2}} \pm \sqrt{\frac{\gamma_{p-1}^2}{\mu_i^2 x^{2(a+2)}} + \frac{16a^2}{x^2}} \right) \right)$$

$$\geq |\mu_i| x^a \left( |\mu_i| x^a - \frac{1}{2} \left( \frac{|\gamma_{p-1}|}{|\mu_i| x^{a+2}} + \sqrt{\frac{\gamma_{p-1}^2}{\mu_i^2 x^{2(a+2)}} + \frac{16a^2}{x^2}} \right) \right).$$

From this we get

$$\lambda_i^\pm \geq |\mu_i| x^a \left( |\mu_i| x^a - \frac{1}{2} \left( \frac{|\gamma_{p-1}|}{|\mu_i|} + \sqrt{\frac{\gamma_{p-1}^2}{\mu_i^2} + 16a^2} \right) \right). \quad (3.12)$$

Also

$$\lambda_i^\pm \geq \mu_i^2 x^{2a} - \frac{x^a}{2} \left( \frac{|\gamma_{p-1}|}{x^{a+2}} + \sqrt{\frac{\gamma_{p-1}^2}{x^{2(a+2)}} + \frac{16a^2}{\mu_i^2 x^{2a}}} \right)$$

$$\geq \mu_i^2 x^{2a} - \frac{x^a}{2} \left( |\gamma_{p-1}| + \sqrt{\gamma_{p-1}^2 + 16a^2 \mu_i^2} \right).$$
which gives
\[ \lambda_i^\pm \geq -\left( \frac{|p-1|}{4|\mu|} + \sqrt{\frac{|p-1|}{16\mu^2} + a^2} \right)^2. \tag{3.13} \]

We will prove first the following

**Lemma 3.10.** Each of the operators \( A_i \oplus V_i \) has discrete spectrum.

**Proof.** By the spectral theorem, the positivity of an operator is equivalent to the positivity of its spectrum. From (3.13), we get that \( \lambda_i^\pm \) are uniformly bounded from below. Thus, there exists a constant \( \tilde{c} \) and an operator \( \tilde{C} \), acting on \( L^2_{d,d,i}(M_1, \wedge^p T^* M_1) \) by \( \tilde{C} \omega = \tilde{c} \omega \), such that \( V_i + \tilde{C} > 0 \) for all \( i \). Since by (3.11) we have that \( \lambda_i^\pm \to +\infty \) when \( x \to \infty \), for any \( c > 0 \) there exists a \( x_0 \) such that in the space \( (1 - \chi_{[1,x_0]} \tilde{C} \omega = \omega \), Let the operator \( U_i \) act by \( \rho_i \omega = -c\chi_{[1,x_0]} \omega \) for any \( \omega \in L^2_{d,d,i}(M_1, \wedge^p T^* M_1) \). We have
\[ A_i + V_i > A_i + U_i + C - \tilde{C}. \tag{3.14} \]

By the min-max principle (Theorem 2.18), and by following the proof of Theorem XIII.16 of [39], it is enough to show that \( \rho_m(A_i + V_i) \to \infty \) when \( m \to \infty \), where \( \rho_m \) is defined in Theorem 2.18. From (3.14), we get
\[ \rho_m(A_i + V_i) > \rho_m(A_i + U_i) + c - \tilde{c}. \tag{3.15} \]

At this point, since \( A_i \) and \( U_i \) are self-adjoint, we will show that \( U_i \) is a relatively compact perturbation of \( A_i \). Then, from Theorem 4.9, \( A_i + U_i \) and \( A_i \) will have the same essential spectrum. Thus, since \( \sigma_{ess}(A_i) \geq 0 \), there exists some \( \tilde{m} > 0 \) such that \( \rho_m(A_i + U_i) > -1 \) when \( m > \tilde{m} \). Since \( c \) was arbitrary, from (3.15) we have that \( \rho_m(\Delta) \to \infty \) when \( m \to \infty \). Hence, we need to show that \( U_i(A_i + i)^{-1} \) is a compact operator. The operator \( U_i(A_i + i)^{-1} \) acts by
\[ U_i(A_i + i)^{-1}(\alpha_i \phi^i + \beta_i \psi^i) = -c\chi_{[1,x_0]}(x)(-\partial_x^2 + \gamma p \partial_x + i)^{-1}\alpha_i \phi^i + (-\partial_x^2 + \gamma p - 1 \partial_x + i)^{-1}\beta_i \psi^i. \]

By Lemma 3.2, it is enough to show that each of the operators
\[ -c\chi_{[1,x_0]}(x)(-\partial_x^2 + \gamma p \partial_x + i)^{-1} \quad \text{and} \quad -c\chi_{[1,x_0]}(x)(-\partial_x^2 + \gamma p - 1 \partial_x + i)^{-1} \]
with domains in \( L^2([1, \infty), x^p \, dx) \) and \( L^2([1, \infty), x^{p-1} \, dx) \) respectively, are compact. Since they are similar, we will do it for the first operator. Let us denote the
operator $-\partial_x^2 + \frac{2}{x} \partial_x$ subject to Dirichlet boundary conditions at $x = 1$ by $A_{\partial,p}$.

Since an operator $T$ is compact if and only if $TT^*$ is compact (Proposition 2.17), we will show that $\chi_{[1,x_0]}(A_{\partial,p}^2 + 1)^{-1} \chi_{[1,x_0]}$ is compact. For any $k, l \in \mathbb{R}$ with $l > 0$, we will denote by $f_{k,l}(x)$ to be some smooth function which is one if $x \leq k$, zero if $x \geq k + l$ and decreasing in $[k, k+l]$. Take some $\varepsilon > 0$. Since the multiplication by the function $\chi_{[1,x_0]}$ is a bounded operation in the space $L^2([1, \infty), x^{-\gamma}dx)$, from the fact that

$$\chi_{[1,x_0]}(A_{\partial,p}^2 + 1)^{-1} \chi_{[1,x_0]} = \chi_{[1,x_0]} f_{x_0+2\varepsilon,\varepsilon}(A_{\partial,p}^2 + 1)^{-1} f_{x_0,\varepsilon} \chi_{[1,x_0]},$$

it is enough to show that the operator $f_{x_0+2\varepsilon,\varepsilon}(A_{\partial,p}^2 + 1)^{-1} f_{x_0,\varepsilon}$ is compact in the space $L^2([1, \infty), x^{-\gamma}dx)$. Since $f_{x_0,\varepsilon}(x)$ is zero for $x \geq x_0 + \varepsilon$, it is enough to show compactness of the above operator in the space $L^2([1, x_0 + 3\varepsilon], x^{-\gamma}dx)$. If we put also Dirichlet boundary conditions for the operator $-\partial_x^2 + \frac{2}{x} \partial_x$ at the points $x = 1$ and $x = x_0 + 3\varepsilon$, and denote the resulting operator by $\tilde{A}_{\partial,p}$. This operator is a Sturm Liouville operator of order 2 on a compact interval and it therefore has compact resolvent by standard Sturm Liouville theory (compare Theorem 2.36). Since it is also self-adjoint, we have that $\pm i \in \rho(\tilde{A}_{\partial,p})$. Thus, $(\tilde{A}_{\partial,p} \pm i)^{-1}$ are compact. Since $f_{x_0+2\varepsilon,\varepsilon}(A_{\partial,p}^2 + 1)^{-1} f_{x_0,\varepsilon}$ maps to the domain of $\tilde{A}_{\partial,p}$, by using the Leibniz rule we get

$$(\tilde{A}_{\partial,p}^2 + 1) f_{x_0+2\varepsilon,\varepsilon}(A_{\partial,p}^2 + 1)^{-1} f_{x_0,\varepsilon} =$$

$$(-\partial_x^2 + \frac{2}{x} \partial_x)^2 + 1) f_{x_0+2\varepsilon,\varepsilon}(A_{\partial,p}^2 + 1)^{-1} f_{x_0,\varepsilon} = f_{x_0+2\varepsilon,\varepsilon} f_{x_0,\varepsilon} + Q.$$

Since $\text{dist}(\text{supp} \partial_x f_{x_0+2\varepsilon,\varepsilon}, \text{supp} f_{x_0,\varepsilon}) > 0$, the remaining term $Q$ is an operator with smooth compactly supported Schwartz kernel, hence it is compact. Also, multiplication by $f_{x_0+2\varepsilon,\varepsilon} f_{x_0,\varepsilon}$ is a bounded operation. So, by the last equation we find that

$$f_{x_0+2\varepsilon,\varepsilon}(A_{\partial,p}^2 + 1)^{-1} f_{x_0,\varepsilon} =$$

$$(\tilde{A}_{\partial,p} + i)^{-1}(\tilde{A}_{\partial,p} - i)^{-1}(f_{x_0+2\varepsilon,\varepsilon} f_{x_0,\varepsilon} + Q),$$

which proves the compactness of $f_{x_0+2\varepsilon,\varepsilon}(A_{\partial,p}^2 + 1)^{-1} f_{x_0,\varepsilon}$. \hfill \Box

To complete the proof, it remains to show that each of $\sigma(A_i + V_i)$ has a lower bound, and that these lower bounds form an increasing sequence. It is enough to prove this fact for all, except possible finitely many $i$. From (3.12), we get

$$\lambda^+ \geq |\mu_i| x^\alpha \left( \frac{|\mu_i|}{2} \frac{|\gamma_{p-1}|}{|\mu_1|} + \sqrt{\frac{\gamma_{p-1}^2}{\mu_1^2} + 16a^2} \right).$$
We can find some \( \tilde{i} \), such that for \( i > \tilde{i} \), we have
\[
|\mu_i| - \frac{1}{2} \left( \frac{|\gamma_{p-1}|}{|\mu_1|} + \sqrt{\frac{|\gamma_{p-1}|^2}{\mu_1^2} + 16a^2} \right) > 1.
\]
Hence, for \( i > \tilde{i} \) we have \( \lambda_i^\pm > |\mu_i| \). Thus, since \( A_i \) is positive, by the min-max principle, we find that \( \rho_1(A_i + V_i) = \rho_1(A_i) + \rho_1(V_i) \geq |\mu_i| \), and since \( |\mu_i| \) form an increasing sequence, we are done.

The proof of the discreteness of the spectrum of the restriction of \( \Delta \) to this space is similar to that one for the previous space. The only difference is the corresponding operators \( V_i' \) of \( V_i \), which are the Friedrichs extensions of \( \tilde{V}_i' \) defined by

\[
\tilde{V}_i' \omega = x^{2a} \mu_i^2 \alpha_i \psi^i + dx \land (x^{2a} \mu_i^2 - \frac{\gamma_{p-1}}{x^2}) \beta_i \tilde{\varphi}_i,
\]
where
\[
\omega = \alpha_i \psi^i + dx \land \beta_i \tilde{\varphi}_i \in \Omega_{0,\delta,d}^p(M_1 \setminus \partial M_1),
\]
which we need to check that their eigenvalues are bounded from below and tend to infinity when \( x \to \infty \). The equation \( V_i' \omega = \tilde{\lambda}_i \omega \) gives
\[
\tilde{\lambda}_i^+ = x^{2a} \mu_i^2 \text{ and } \tilde{\lambda}_i^- = x^{2a} \mu_i^2 - \frac{\gamma_{p-1}}{x^2},
\]
which guarantee the above conditions. We can summarize the two results proved in this section, in the following theorem

**Theorem 3.11.** The restriction of Laplace operator \( \Delta \) to each of the spaces \( L^2_{\delta,d}(M_1, \wedge^p T^* M_1) \) and \( L^2_{\delta,d}(M_1, \wedge^p T^* M_1) \) of the decomposition of Theorem 3.5, has discrete spectrum.
3.5 The continuous spectrum

In order to find the continuous spectrum of the Laplacian and the continuous part of its spectral decomposition, we deal now with the space $L^2_{\mathcal{H}}(M_1, \wedge^p T^*M_1)$. We have that $\Delta|_{L^2_{\mathcal{H}}(M_1, \wedge^p T^*M_1)} \omega = \lambda^2 \omega$ is given by the system (3.7). After the transformation

$$\alpha_i(x) = x^{\gamma_p + \frac{1}{2}} f_i(x)$$

and

$$\beta_i(x) = x^{\gamma_p - 1 + \frac{1}{2}} g_i(x),$$

this system becomes

$$x^2 f''_i(x) + x f'_i(x) + [\lambda^2 x^2 - (\gamma_p + \frac{1}{2})^2] f_i(x) = 0$$

and

$$x^2 g''_i(x) + x g'_i(x) + [\lambda^2 x^2 - (\gamma_p - 1 - \frac{1}{2})^2] g_i(x) = 0.$$ 

If we let $t = x\lambda$, $w_i(t) = f_i(t/\lambda)$ and $h_i(t) = g_i(t/\lambda)$, we find

$$t^2 w''_i(t) + tw'_i(t) + [t^2 - (\gamma_p + \frac{1}{2})^2] w_i(t) = 0$$

and

$$t^2 h''_i(t) + th'_i(t) + [t^2 - (\gamma_p - 1 - \frac{1}{2})^2] h_i(t) = 0.$$ 

The above equations are of Bessel type, of order $\frac{\gamma_p + 1}{2}$ and $\frac{\gamma_p - 1 - 1}{2}$ respectively. Hence, for the Laplacian restricted to the space of the harmonic components we have the following.

**Theorem 3.12.** Let $\mathcal{H}^p(N) = L^2_{\mathcal{H}}(N, \wedge^p T^*N)$ be the space of $L^2$ harmonic $p$-forms on the boundary $N$. The equation $\Delta|_{L^2_{\mathcal{H}}(M_1, \wedge^p T^*M_1)} \omega = \lambda^2 \omega$ has general solution

$$\omega = x^{b_p} B_{b_p}(\lambda x) \theta + dx \wedge x^{b_p - 1} \tilde{B}_{b_p - 1}(\lambda x) \tilde{\theta},$$

where $B_b$ and $\tilde{B}_b$ are any solutions of the Bessel equation of order $b$, $\theta \in \mathcal{H}^p(N)$ and $\tilde{\theta} \in \mathcal{H}^{p-1}(N)$ and

$$b_p = \frac{\gamma_p + 1}{2}.$$ 

Thus, the $p$-forms

$$x^{b_p} G_{b_p}(\lambda, x) \theta$$

and

$$dx \wedge x^{b_p - 1} G_{b_p - 1}(\lambda, x) \tilde{\theta}$$

are generalized $\lambda^2$-eigenforms of the Dirichlet Laplacian restricted to the space $L^2_{\mathcal{H}}(M_1, \wedge^p T^*M_1)$, where

$$G_b(\lambda, x) = Y_b(\lambda) J_b(\lambda x) - J_b(\lambda) Y_b(\lambda x),$$

with $J_b$ and $Y_b$ to be the Bessel functions of order $b$, of the first and the second kind respectively.
The function $G_b(\lambda, x)$ is called the cylinder function of order $b$. Properties of Bessel and cylinder functions are collected for reference in the appendix. In the remainder of this section, we will prove the following theorem, which together with the results of the last section give the spectral theorem for $\Delta$.

**Theorem 3.13.** *(Spectral theorem, continuous part)* The Laplacian, $\Delta$, restricted to the space $L^2_H(M_1, \wedge^p T^* M_1)$, which comes from the decomposition in Theorem 3.5, has

$$\text{Dom}(\Delta) = \left( L_p \otimes H^p(N) \right) \oplus dx \wedge \left( \hat{\mathcal{L}}_{p-1} \otimes H^{p-1}(N) \right),$$

where

$$\mathcal{L}_p = \{ \alpha(x) \in L^2([1, \infty), x^{-\gamma_p} dx) : \lambda^2 \mathbb{W}_{bp}^{-1}(x^{-bp} \alpha(x)) \in L^2([0, \infty), d\mu_{bp}(\lambda)) \},$$

$$\hat{\mathcal{L}}_p = \{ \beta(x) \in L^2([1, \infty), x^{-\gamma_p} dx) : \lambda^2 \mathbb{W}_{bp-1}^{-1}(x^{-bp} \beta(x)) \in L^2([0, \infty), d\mu_{bp-1}(\lambda)) \},$$

$\mathbb{W}_b$ is the Weber transform of order $b$ and

$$d\mu_b(\lambda) = (J^2_b(\lambda) + Y^2_b(\lambda)) \lambda d\lambda.$$

For the spectrum of $\Delta$ we have that $\sigma_{\text{sing}}(\Delta) = \emptyset$, $\sigma_{\text{ac}}(\Delta) = [0, \infty)$ and $\sigma_{\text{cont}}(\Delta) = [0, \infty)$. Also, $\Delta$ in the above domain has the following spectral decomposition

$$\Delta(\alpha \theta + dx \wedge \beta \tilde{\theta}) = x^{bp} \mathbb{W}_b \left( \lambda^2 \mathbb{W}_{bp}^{-1}(t^{-bp} \alpha(t)) \right) \theta + dx \wedge x^{bp-1} \mathbb{W}_{bp-1} \left( \lambda^2 \mathbb{W}_{bp-1}^{-1}(t^{-bp-1} \beta(t)) \right) \tilde{\theta}.$$

To prove this theorem, we first have to define the transform $\mathbb{W}_b$ and prove its properties. Then we can interpret those properties in terms of $\Delta |_{L^2_H(M_1, \wedge^p T^* M_1)}$ and its spectral decomposition. The following result permits us to begin by defining $\mathbb{W}_b$ on smooth compactly supported functions.

**Theorem 3.14.** *(Weber)* If for some function $f$ of real variable the following integral $\int_0^\infty f(\lambda) \sqrt{\lambda} d\lambda$ exists and is absolutely convergent, then for any real $b$ we have

$$\int_1^\infty \left( \int_0^\infty f(\lambda) G_b(\lambda, x) G_b(y, x) \lambda d\lambda \right) x dx = \frac{J^2_b(y) + Y^2_b(y)}{2} \left( f(y + 0) + f(y - 0) \right),$$

provided that the positive number $y$ lies inside an interval in which $f$ has finite total variation.
Proof. See 14.52 in [46].

Thus we can make the following definition

**Definition 3.15.** (Weber transform) Let \( f \in C_0^\infty([0, \infty)) \). For any real \( b \), define the transform \( \mathbb{W}_b(f) \) of \( f \) to be the function

\[
\mathbb{W}_b(f)(x) = \int_0^\infty f(\lambda) G_b(\lambda, x) \lambda d\lambda.
\]

Now we need to check that this transform extends to a bijective isometry

\[
\mathbb{W}_b : L^2([0, \infty), d\mu_b(\lambda)) \to L^2([1, \infty), x \, dx).
\]

First we check that there is some extension. We have the following result

**Proposition 3.16.** The transform \( \mathbb{W}_b \) extends to an isometry from the space \( L^2([0, \infty), d\mu_b(\lambda)) \) onto its image \( \subseteq L^2([1, \infty), x \, dx) \).

**Proof.** First note that for \( f, g \in C_0^\infty([0, \infty)) \), the Weber transform preserves the \( L^2 \) inner product:

\[
(\mathbb{W}_b f, \mathbb{W}_b g)_{L^2([1, \infty), x \, dx)} = \int_0^\infty \left( \int_0^\infty f(\lambda) G_b(\lambda, x) \lambda d\lambda \right) \left( \int_1^\infty g(t) G_b(t, x) t \, dt \right) x \, dx = \\
\int_0^\infty g(t) \left( \int_0^\infty \left( \int_0^\infty f(\lambda) G_b(\lambda, x) G_b(t, x) \lambda d\lambda \right) x \, dx \right) t \, dt = \\
\int_0^\infty g(t) f(t) (J_b^2(t) + Y_b^2(t)) t \, dt = (f, g)_{L^2([0, \infty), d\mu_b(t))}
\]

where we have used Fubini’s theorem in the second step and Weber’s theorem in the third step. Thus \( \mathbb{W}_b \) maps from \( C_0^\infty([0, \infty)) \) to \( L^2([1, \infty), x \, dx) \).

Take any \( f \in L^2([0, \infty), d\mu_b(\lambda)) \). Then there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \), with \( f_n \in C_0^\infty[0, \infty) \), such that \( \|f - f_n\|_{L^2([0, \infty), d\mu_b(\lambda))} \) goes to zero, as \( n \) goes to \( \infty \). \( \{f_n\}_{n \in \mathbb{N}} \) is Cauchy in \( L^2([0, \infty), d\mu_b(\lambda)) \), so \( \{\mathbb{W}_b f_n\}_{n \in \mathbb{N}} \) is Cauchy in \( L^2([1, \infty), x \, dx) \), and we define the \( \mathbb{W}_b f \) to be the limit of this sequence, which
does not depend on the choice of \( \{f_n\}_{n \in \mathbb{N}} \). Then we have
\[
\left| \|f\|_{L^2([0,\infty),d\mu_b(\lambda))} - \|W_b f\|_{L^2([1,\infty),xdx)} \right| =
\left| \|f\|_{L^2([0,\infty),d\mu_b(\lambda))} - \|f_n\|_{L^2([0,\infty),d\mu_b(\lambda))} + \|f_n\|_{L^2([0,\infty),d\mu_b(\lambda))} - \|W_b f\|_{L^2([1,\infty),xdx)} \right| \leq
\left| \|f - f_n\|_{L^2([0,\infty),d\mu_b(\lambda))} + \|W_b f_n - W_b f\|_{L^2([1,\infty),xdx)} \right| \leq
\left| \|f\|_{L^2([0,\infty),d\mu_b(\lambda))} - \|f_n\|_{L^2([0,\infty),d\mu_b(\lambda))} + \|W_b f_n - W_b f\|_{L^2([1,\infty),xdx)} \right|,
\]
and the last sum goes to zero when \( n \) goes to \( \infty \).

Since the transform extends as an isometry, it is automatically injective. To show surjectivity, we use Weber’s inversion formula from [45].

**Theorem 3.17.** (Weber’s inversion formula) If for some function \( f \) of real variable the integral \( \int_1^\infty f(x)\sqrt{x}dx \) exists and is absolutely convergent, then for any real \( b \) we have
\[
\int_0^\infty \left( \int_1^\infty f(x)G_b(\lambda,x)G_b(\lambda,y)J_2^b(\lambda) + Y_2^b(\lambda) \right) xdx \lambda d\lambda = \frac{f(y+0) + f(y-0)}{2},
\]
provided that the positive number \( y \) lies inside an interval in which \( f \) has finite total variation.

Now we have the full transform we prove the following

**Proposition 3.18.** The extension
\[
W_b : L^2([0,\infty),d\mu_b(\lambda)) \rightarrow L^2([1,\infty),xdx).
\]
is a bijective isometry.

**Proof.** From the Weber’s inversion formula we have that if \( g \in C_0^\infty([1,\infty)) \), then \( g = W_b f \), where
\[
f(\lambda) = \int_1^\infty \frac{g(x)G_b(\lambda,x)}{J_2^b(\lambda) + Y_2^b(\lambda)} xdx \in L^2([0,\infty),d\mu_b(\lambda)).
\]
Consider now any element \( g \in L^2([1,\infty),xdx) \). Take a sequence \( \{g_i\}_{i \in \mathbb{N}} \) of elements in \( C_0^\infty([1,\infty)) \) such that \( \|g - g_i\|_{L^2([1,\infty),xdx)} \rightarrow 0 \) as \( i \rightarrow \infty \). From the above proposition, there exists a sequence of elements \( \{f_i\}_{i \in \mathbb{N}} \) in \( L^2([0,\infty),d\mu_b(\lambda)) \) such
that \( g_i = \mathbb{W}_b f_i \). Since \( \mathbb{W}_b \) is an isometry from the space \( L^2([0, \infty), d\mu_b(\lambda)) \) onto its image, the sequence \( \{f_i\}_{i \in \mathbb{N}} \) is Cauchy. If we denote its limit by \( f \) we get

\[
\|\mathbb{W}_b f - g\|_{L^2([1, \infty), xd\lambda)} = \|\mathbb{W}_b f - \mathbb{W}_b f_i + \mathbb{W}_b f_i - g\|_{L^2([1, \infty), xd\lambda)} \leq \|\mathbb{W}_b f - \mathbb{W}_b f_i\|_{L^2([1, \infty), xd\lambda)} + \|g_i - g\|_{L^2([1, \infty), xd\lambda)} = \|f - f_i\|_{L^2([0, \infty), d\mu_b(\lambda))} + \|g_i - g\|_{L^2([1, \infty), xd\lambda)}.
\]

Since both terms in the last sum tend to zero as \( i \to \infty \), we find that \( g = \mathbb{W}_b f \). \( \square \)

We also get that, \( f \) such that \( g = \mathbb{W}_b f \) is unique, which allows us to define the inverse of the Weber transform

\[
\mathbb{W}_b^{-1} : L^2([1, \infty), xd\lambda) \to L^2([0, \infty), d\mu_b(\lambda)).
\]

Since \( \mathbb{W}_b \) is an isometry, its inverse is also an isometry. Equivalently, from Weber’s theorem and its inversion formula, \( \mathbb{W}_b^{-1} \) can be defined for every \( g \in C_0^\infty([1, \infty)) \) by the integral

\[
\mathbb{W}_b^{-1}(g)(\lambda) = \int_1^\infty g(x)G_b(\lambda, x) \frac{dx}{J^2(\lambda) + Y^2(\lambda)},
\]

and extended then continuously to the whole space \( L^2([1, \infty), xd\lambda) \).

For any \( p \)-form \( \omega = \omega + dx \land \tilde{\omega} \), we denote by \( A_p^\pm \omega \) the \( p \)-form \( x^{bp}\omega + dx \land x^{bp-1}\tilde{\omega} \). Take some \( \theta \oplus \tilde{\theta} \in \mathcal{H}^p(N) \oplus \mathcal{H}^{p-1}(N) \) and \( \alpha(x), \beta(x) \in L^2([1, \infty), xd\lambda) \) such that \( \alpha = \mathbb{W}_b \tilde{\alpha} \) and \( \beta = \mathbb{W}_{b_{p-1}} \tilde{\beta} \) for some \( \tilde{\alpha}(\lambda), \tilde{\beta}(\lambda) \in C_0^\infty([0, \infty)) \). From Theorem (3.12) we have that

\[
A_p^- \Delta A_p^+ (\alpha \theta + dx \land \beta \tilde{\theta}) = A_p^- \Delta \left( \left( \int_0^\infty \tilde{\alpha}(\lambda)G_{bp}(\lambda, x) \lambda d\lambda \right) \theta + dx \land \left( \int_0^\infty \tilde{\beta}(\lambda)G_{bp-1}(\lambda, x) \lambda d\lambda \right) \tilde{\theta} \right) = A_p^- \Delta \left( \int_0^\infty \tilde{\alpha}(\lambda)x^{bp}G_{bp}(\lambda, x) \lambda d\lambda \theta \right)
\]

\[
+ dx \land \left( \int_0^\infty \tilde{\beta}(\lambda)x^{bp-1}G_{bp-1}(\lambda, x) \lambda d\lambda \right) \tilde{\theta}
\]

\[
= A_p^- \left( \left( \int_0^\infty \tilde{\alpha}(\lambda)x^{bp}\lambda^2G_{bp}(\lambda, x) \lambda d\lambda \right) \theta + dx \land \left( \int_0^\infty \tilde{\beta}(\lambda)x^{bp-1}\lambda^2G_{bp-1}(\lambda, x) \lambda d\lambda \right) \tilde{\theta} \right)
\]

\[
= \mathbb{W}_{bp}(\lambda^2 \tilde{\alpha}) \theta + dx \land \mathbb{W}_{bp-1}(\lambda^2 \tilde{\beta}) \tilde{\theta}.
\]

(3.16)
Since the operator
\[ x^b : f \in L^2([1, \infty), xdx) \rightarrow f x^b \in L^2([1, \infty), x^{-\gamma}dx) \]
is an isometry, from (3.16) and Lemma 3.2 we get that
\[ x^b \alpha \theta + dx \wedge x^b \beta \tilde{\theta} \in L^2_H(M_1, \wedge^p T^* M_1) \]
and
\[ \Delta (x^b \alpha \theta + dx \wedge x^b \beta \tilde{\theta}) = x^b \mathbb{W}_b (\lambda^2 \tilde{\alpha}) \theta + dx \wedge x^b \mathbb{W}_{b-1} (\lambda^2 \tilde{\beta}) \tilde{\theta} \in L^2_H(M_1, \wedge^p T^* M_1). \]
Theorem 3.13 now follows.
3.6 Meromorphic continuation of the resolvent

The resolvent $(\Delta - \lambda^2)^{-1}$ is a bounded operator on $L^2$ for $\lambda^2$ in the resolvent set $\mathbb{C} \setminus \sigma(\Delta)$. In this section, we show that the resolvent can be continued meromorphically to a Riemann surface, as a bounded operator between some weighted $L^2$ spaces. By the separation into boundary harmonic and boundary perpendicular forms from before, we can do this independently for the Laplacian restricted to $L^2_H(M_1, \wedge^p T^* M_1)$ and to $L^2_{q,d}(M_1, \wedge^p T^* M_1) \oplus L^2_{q,d}(M_1, \wedge^p T^* M_1)$, according to Theorem 3.5. Since the spectrum of $\Delta|_{L^2_{q,d}(M_1, \wedge^p T^* M_1) \oplus L^2_{q,d}(M_1, \wedge^p T^* M_1)}$ is discrete, the resolvent is already a meromorphic family of operators on the space $L^2_{q,d}(M_1, \wedge^p T^* M_1) \oplus L^2_{q,d}(M_1, \wedge^p T^* M_1)$ on $\mathbb{C}$ with poles at the eigenvalues. So in particular, it is also a meromorphic family of operators on any smaller space lifted to any cover of $\mathbb{C}$. Thus it suffices to study the extension of the resolvent of $\Delta|_{L^2_H(M_1, \wedge^p T^* M_1)}$. We will do this in three steps. First, using the spectral theorem from the previous section we will get a formal way to extend the resolvent. Then, we will determine the spaces of forms between which the extended family of operators will map. Finally, we will study the relationship between the operators on different leaves of the logarithmic cover of $\mathbb{C}$ branched at the origin to determine the minimal cover we can use for the extension.

Since we chose our spectral parameter to be a square, $\lambda^2 \geq 0$, the parameter $\lambda$ lives on the double cover of $\mathbb{C}$. Then, the physical sheet, $\lambda^2 \notin [0, \infty)$, on which the resolvent is originally defined, corresponds to the lower half-plane in the double cover. We will extend the resolvent from the lower half-plane up or down to the logarithmic cover branched at the origin. We will denote the parameter on this cover by $z$. We say that $z$ lies on the $k$-leaf of the logarithmic cover when

$$(2k - 1)\pi \leq \arg z < (2k + 1)\pi.$$ 

Let us denote by $\exp(i\theta)(z)$ the action of $\mathbb{R}$ on the logarithmic cover defined by rotation of $z$ by radians $\theta$ up to the logarithmic cover if $\theta > 0$ and down if $\theta < 0$. We denote $\exp(-i\pi)(z)$ by $-z$.

To get a formal extension of the resolvent, note that by the spectral theorem for $\Delta$ restricted to $L^2_H(M_1, \wedge^p T^* M_1)$, Theorem 3.13, we have that for $z$ in the lower half-plane of the zero leaf, the resolvent $R_z(\Delta) = (\Delta - z^2)^{-1}$ of the Laplacian on the cusp, acting on $L^2_H(M_1, \wedge^p T^* M_1)$ is given by

$$R_z(\Delta)(\alpha \theta + dx \wedge \beta \tilde{\theta}) = x^{b_p} \mathbb{W}_{b_p} \left( \frac{1}{\chi^2 - z^2} \mathbb{W}^{-1}_{b_p} (t^{-b_p} \alpha(t)) \right) \theta$$

$$+ dx \wedge x^{b_p-1} \mathbb{W}_{b_p-1}^{-1} \left( \frac{1}{\chi^2 - z^2} \mathbb{W}_{b_p-1}^{-1} (t^{-b_p-1} \beta(t)) \right) \tilde{\theta}.$$
When we specialize further to \( \alpha, \beta \in C_0^\infty([1, \infty)) \), this can be written in terms of the double integrals

\[
R_z(\Delta)(\alpha \theta + dx \land \beta \tilde{\theta}) = \left( x^{b_p} \int_0^\infty \frac{\lambda G_{b_p}(\lambda, x)}{\lambda^2 - z^2} \left( \int_1^\infty t^{1-b_p} \alpha(t) \frac{G_b(\lambda, t)}{J_{b_p}^2(\lambda) + Y_{b_p}^2(\lambda)} \, dt \right) \, d\lambda \right) \theta + dx \land \left( x^{b_p-1} \int_0^\infty \frac{\lambda G_{b_p-1}(\lambda, x)}{\lambda^2 - z^2} \left( \int_1^\infty t^{1-b_p-1} \beta(t) \frac{G_{b_p-1}(\lambda, t)}{J_{b_p-1}^2(\lambda) + Y_{b_p-1}^2(\lambda)} \, dt \right) \, d\lambda \right) \tilde{\theta}.
\] (3.17)

Define for any \( b \in \mathbb{R} \) the function

\[
m_b(\lambda, z, x, t) = \frac{\lambda}{\lambda^2 - z^2} \frac{G_b(\lambda, x)G_b(\lambda, t)}{J_b^2(\lambda) + Y_b^2(\lambda)}.
\]

Now for any \( b \in \mathbb{R} \), define the (singular) integral kernel \( r_b(x, t) \) formally by the integral

\[
r_b(z, x, t) = x^{b-1} \int_0^\infty m_b(\lambda, z, x, t) \, d\lambda,
\]

where this kernel is understood in terms of its action on smooth compactly supported functions by the integrals on the right hand of the Equation (3.17), which we express formally as

\[
R_z(\Delta)(\alpha \theta + dx \land \beta \tilde{\theta}) = \left( \int_1^\infty \alpha(t) r_b(z, x, t) \, dt \right) \theta + dx \land \left( \int_1^\infty \beta(t) x^{t-b-1} r_{b-1}(z, x, t) \, dt \right) \tilde{\theta}.
\] (3.18)

We will consider the above two summands separately, starting with the first one. By using the facts discussed in the appendix that make

\[
\frac{G_b(\lambda, x)G_b(\lambda, t)}{J_b^2(\lambda) + Y_b^2(\lambda)}
\]

a meromorphic function in \( \lambda \) on the logarithmic cover of \( \mathbb{C} \), we will define a family of operators parametrized over the logarithmic cover of \( \mathbb{C} \) as follows. Let \( z \) and \( \tilde{z} \) be two nearby points on the \( k \)-leaf of the logarithmic cover over \( \mathbb{C} \) (we use both \( z \) and \( \tilde{z} \) when we take derivatives of the extension later). Let \( k = (k_1, k_2) \in \mathbb{Z}^2 \), \( k \neq (0, 0) \), and let \( P(\lambda, z, \tilde{z}) \) be any polynomial in \( \lambda, z, \) and \( \tilde{z} \), whose degree in \( \lambda \) is \( \leq 2(k_1 + k_2 - 1) \). Let \( \Gamma \) be a curve in the logarithmic cover which starts at 0, is identified with the ray \([R, \infty)\) along the real axis of the zero leaf for some large \( R \),
and such that $z$ is contained in the region $(\Gamma)$ between $\Gamma$ and $[0, \infty)$ of the zero leaf and $-z$ is not. Further, assume that $\Gamma$ does not intersect the path obtained by rotating $\Gamma$ down to the logarithmic cover by $\pi$ radians (this condition is not essential) and assume that $J_{bp}^2(\lambda) + Y_{bp}^2(\lambda) \neq 0$ along $\Gamma$. Then define
\[
\tilde{m}_{bp,k,P}(\lambda, z, \tilde{z}, x, t) = \frac{P(\lambda, z, \tilde{z})}{(\lambda^2 - z^2)^k_1(\lambda^2 - \tilde{z}^2)^k_2} \frac{G_{bp}(\lambda, x)G_{bp}(\lambda, t)}{J_{bp}^2(\lambda) + Y_{bp}^2(\lambda)} \lambda,
\]
and define the singular integral kernel
\[
\tilde{r}_{bp,k,P,\Gamma}(z, \tilde{z}, x, t) = x^{b_p t^{1-b_p}} \int_{\Gamma} \tilde{m}_{bp,k,P}(\lambda, z, \tilde{z}, x, t) d\lambda,
\]
where this kernel is understood formally in terms of its action on an element of $C_0^\infty([0, \infty))$. Let $\{q\}$ be the set of poles of $\tilde{m}_{bp,k,P}(z, \tilde{z}, \lambda, t, x)$ in $\lambda$ between the path $\Gamma$ and the positive real semi-axis in the zero leaf (i.e. $q \in (\Gamma)$). We will show that the associated operator
\[
Q_{q}^p_{bp,k,P,\Gamma}(\alpha \theta)(x) = \left( \int_{1}^{\infty} \alpha(t) \tilde{r}_{bp,k,P,\Gamma}(z, \tilde{z}, x, t) dt \right) \theta
\]
extends to a family of bounded operators between weighted $L^2$ spaces.

For the second summand of the right hand of Equation (3.18), the setup is similar, but where instead of $\tilde{r}_{bp,k,P,\Gamma}(z, \tilde{z}, x, t)$ we use the kernel
\[
\tilde{\rho}_{bp-1,k,P,\Gamma}(z, \tilde{z}, x, t) = x^{b_{p-1} t^{1-b_{p-1}}} \int_{\Gamma'} \tilde{m}_{bp-1,k,P}(\lambda, z, \tilde{z}, x, t) d\lambda,
\]
where $\Gamma'$ must avoid the poles of $\tilde{m}_{bp-1,k,P}(\lambda, z, \tilde{z}, x, t)$. Then we define the operator
\[
Q''_{bp-1,k,P,\Gamma}(dx \wedge \beta(t) \tilde{\theta})(x) = dx \wedge \left( \int_{1}^{\infty} \beta(t) \tilde{\rho}_{bp-1,k,P,\Gamma}(z, \tilde{z}, x, t) dt \right) \tilde{\theta}.
\]

Finally, let the operator $Q_{bp,bp-1,k,P,\Gamma}$ acting on $p$-forms $\alpha \theta + dx \wedge \beta \tilde{\theta} \in Q^p_{0,\mathcal{H}}(M_1 \setminus \partial M_1)$, given by
\[
Q_{bp,bp-1,k,P,\Gamma}(\alpha \theta + dx \wedge \beta \tilde{\theta}) = Q'_{bp,k,P,\Gamma}(\alpha \theta) + Q''_{bp-1,k,P,\Gamma}(dx \wedge \beta \tilde{\theta}).
\]
By the residue theorem, the definition of $Q_{b_p,b_{p-1},k,P,z,\tilde{z}}$ is independent of the choice of the paths $\Gamma$ and $\Gamma'$. Also, when $k = (1,0)$ and $P = 1$, by the residue theorem, the operator $Q_{b_p,b_{p-1},k,P,z,\tilde{z}}$ for $z$ in the lower half plane of the zero leaf coincides with $R_2(\Delta)$ in a dense subset of $L^2$, and this will be the desired meromorphic continuation of our resolvent family. We consider this more general family because it will also contain the derivatives of the resolvent family. Define the weighted $L^2$ spaces on which we will show $Q_{b_p,b_{p-1},k,P,z,\tilde{z}}$ acts by

$$H_{\mathcal{H},-} = e^{\frac{x^2}{2}} L^2_{\mathcal{H}}(M_1, \wedge^p T^* M_1) =$$

$$\left(L^2([1, \infty), e^{-x^2} x^{-\gamma} dx) \otimes L^2_{\mathcal{H}}(N, \wedge^p T^* N)\right) \oplus$$

$$\left(dx \wedge \left(L^2([1, \infty), e^{-x^2} x^{-\gamma} dx) \otimes L^2_{\mathcal{H}}(N, \wedge^{p-1} T^* N)\right)\right)$$

and

$$H_{\mathcal{H},+} = e^{-\frac{x^2}{2}} L^2_{\mathcal{H}}(M_1, \wedge^p T^* M_1) =$$

$$\left(L^2([1, \infty), e^{x^2} x^{-\gamma} dx) \otimes L^2_{\mathcal{H}}(N, \wedge^p T^* N)\right) \oplus$$

$$\left(dx \wedge \left(L^2([1, \infty), e^{x^2} x^{-\gamma} dx) \otimes L^2_{\mathcal{H}}(N, \wedge^{p-1} T^* N)\right)\right).$$

**Lemma 3.19.** If $z$, $\tilde{z}$ are two points on the logarithmic cover over $\mathbb{C}$ branched at the origin, then

$$Q_{b_p,b_{p-1},k,P,z,\tilde{z}} \in \mathcal{L}(H_{\mathcal{H},+}, H_{\mathcal{H},-}).$$

**Proof.** Take $\omega = \alpha \theta + dx \wedge \beta \eta \in H_{\mathcal{H},+}$ and $\tilde{\omega} = \tilde{\alpha} \tilde{\theta} + dx \wedge \tilde{\beta} \tilde{\eta} \in H_{\mathcal{H},-}$, where $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in C^\infty_0((1, \infty))$, $\theta, \tilde{\theta} \in \mathcal{H}^p(N)$ and $\eta, \tilde{\eta} \in \mathcal{H}^{p-1}(N)$, i.e.

$$\omega, \tilde{\omega} \in \Omega^p_{0,\mathcal{H}}(M_1 \setminus \partial M_1).$$

If we denote the inner products in the spaces $H_{\mathcal{H},+}$ and $H_{\mathcal{H},-}$ by $(\cdot, \cdot)_+$ and $(\cdot, \cdot)_-$ respectively, then by (3.20), (3.21), (3.22) and Lemma 3.2, we have

$$|\langle \tilde{\omega}, Q_{b_p,b_{p-1},k,P,z,\tilde{z}} \omega \rangle_+| =$$

$$\left|\left(\int_1^\infty \tilde{\alpha}(x) e^{-x^2} x^{-\gamma} \left(\int_1^\infty \alpha(t) \tilde{\rho}_{b_p,k,P}(z, \tilde{z}, x, t) dt\right) dx\right)(\tilde{\theta}, \theta)_N + \right.$$
\[ \left( 2\pi i \sum_{q \in (\Gamma), q \neq z} \int_{1}^{\infty} \alpha(t)x^{b_{p}t^{1-b_{p}}} \text{Res}_{\lambda=q} \tilde{m}_{b_{p}, k, P}(\lambda, z, \tilde{z}, x, t) \, dt \right) \, dx \right) + \left( \int_{1}^{\infty} \beta(x)e^{-x}x^{-\gamma_{p-1}} \left( \int_{1}^{\infty} \beta(t)\tilde{p}_{b_{p-1}, k, P, \Gamma_{p-1}}(z, \tilde{z}, x, t) \, dt \right) \, dx \right) \right) (\bar{\eta}, \eta)_{N} - (\bar{\eta}, \eta)_{N} \left( \int_{1}^{\infty} \beta(x)e^{-x}x^{-\gamma_{p-1}} \right) \right] (3.23) \]

We choose the path \( \Gamma \) to coincide with the real axis of the zero leaf at \([0, x_{1}] \) and \([x_{2}, +\infty) \), for some fixed \( 0 < x_{1} < x_{2} \), and let \( \tilde{\Gamma} = \Gamma \setminus ([0, x_{1}] \cup [x_{2}, +\infty]) \). Do similarly for the path \( \Gamma' \). We split the proof into several steps, according to the terms of the Equation (3.23).

1st step.

For the first term on the right in (3.23), we have that

\[
\left| \int_{1}^{\infty} \alpha(x)e^{-x}x^{-\gamma_{p}} \left( \int_{1}^{\infty} \alpha(t)\tilde{r}_{b_{p}, k, P, \Gamma}(z, \tilde{z}, x, t) \, dt \right) \, dx \right| \leq \int_{1}^{\infty} |\tilde{\alpha}(x)|x^{1-b_{p}}e^{-x} \left( \int_{1}^{\infty} |\alpha(t)|t^{1-b_{p}} \int_{0}^{x_{1}} |\tilde{m}_{b_{p}, k, P}(\lambda, z, \tilde{z}, x, t)| \, d\lambda dt \right) \, dx + \int_{1}^{\infty} |\tilde{\alpha}(x)|x^{1-b_{p}}e^{-x} \left( \int_{1}^{\infty} |\alpha(t)|t^{1-b_{p}} \int_{x_{2}}^{\infty} |\tilde{m}_{b_{p}, k, P}(\lambda, z, \tilde{z}, x, t)| \, d\lambda dt \right) \, dx. 
\]

(3.24)

There is

\[
|\tilde{m}_{b_{p}, k, P}(\lambda, z, \tilde{z}, x, t)| = \frac{|P(\lambda, z, \tilde{z})|}{|\lambda^{2} - z^{2}|^{k_{1}}|\lambda^{2} - \tilde{z}^{2}|^{k_{2}}} |\kappa_{b}(\lambda, x, t)|,
\]

where

\[
\kappa_{b}(\lambda, x, t) = \frac{G_{b}(\lambda, x)G_{b}(\lambda, t)}{J_{b}^{2}(\lambda) + Y_{b}^{2}(\lambda)} \lambda,
\]

or

\[
\kappa_{b}(\lambda, x, t) = \left( Y_{b}^{2}(\lambda)J_{b}(\lambda)xJ_{b}(\lambda t) - Y_{b}(\lambda)J_{b}(\lambda)Y_{b}(\lambda t)J_{b}(\lambda x) - Y_{b}(\lambda)J_{b}(\lambda)Y_{b}(\lambda x)J_{b}(\lambda t) + J_{b}^{2}(\lambda)Y_{b}(\lambda x)Y_{b}(\lambda t) \right) \lambda^{2} / (J_{b}^{2}(\lambda) + Y_{b}^{2}(\lambda)).
\]

(3.25)
i) We deal first with the third term on the right in (3.24). By fixing $x$ and $t$, if we use the asymptotic behavior of the Bessel functions when $\lambda \to +\infty$ (see (5.4) in the appendix), we have that

$$
\kappa_b(\lambda, x, t) = \frac{2}{\pi \sqrt{xt}} \left( \sin^2(\lambda + \theta_b) \cos(\lambda x + \theta_b) \cos(\lambda t + \theta_b) - \sin(\lambda + \theta_b) \cos(\lambda + \theta_b) \sin(\lambda x + t) + 2\theta_b \right) + \cos^2(\lambda + \theta_b) \sin(\lambda x + \theta_b) \sin(\lambda t + \theta_b) (1 + O(\lambda^{-1})),
$$

where $\theta_b = -\left( \frac{\lambda}{2} + \frac{\pi}{4} \right)$. Thus, $|\kappa_b(\lambda, x, t)| \leq \frac{c(x, t)}{\sqrt{xt}}$, for some continuous function $c(x, t)$. Since

$$
\lim_{x,t\to+\infty} \kappa_b(\lambda, x, t) \sqrt{xt}
$$

is bounded as a function of $\lambda \in [x_2, +\infty)$, we can assume that $c(x, t) \leq c_1$, for some constant $c_1 > 0$. Hence, we have

$$
1 \left| \hat{\alpha}(x) \right| x^{1-b_p} e^{-x^2} \left( \int_1^\infty |\alpha(t)| t^{1-b_p} \int_1^\infty |\tilde{m}_{b_p,k,P}(\lambda, z, \hat{z}, x, t)| d\lambda dt \right) dx 
\leq c_1 \int_1^\infty |\hat{\alpha}(x)| x^{1-b_p} e^{-x^2} \left( \int_1^\infty |\alpha(t)| t^{1-b_p} \int_1^\infty \frac{|P(\lambda, z, \hat{z})|}{|\lambda^2 - z^2| |\lambda - \hat{z}|^2} d\lambda dt \right) dx 
= c_1 c_2 \int_1^\infty |\hat{\alpha}(x)| x^{1-b_p} e^{-x^2} (\int_1^\infty |\alpha(t)| t^{1-b_p} dt) 
\leq c_1 c_2 \int_1^\infty |\hat{\alpha}(x)| x^{-\gamma} e^{-x^2} (\int_1^\infty |\alpha(t)| t^{-\gamma} e^{\frac{t^2}{2}} dt) 
\leq c_1 c_2 \int_1^\infty |\hat{\alpha}(x)| x^{-\gamma} e^{-x^2} dx \int_1^\infty |\alpha(t)| t^{-\gamma} e^{\frac{t^2}{2}} dt 
\leq c_1 c_2 c_3 \int_1^\infty |\hat{\alpha}(x)| x^{-\gamma} e^{-x^2} dx \left( \int_1^\infty |\alpha(t)|^2 t^{-\gamma} e^\frac{t^2}{2} dt \right)^{1/2} \left( \int_1^\infty e^{-x^2} dx \right)^{1/2},
$$

(3.26)

where

$$
c_2 = \int_{x_2}^\infty \frac{|P(\lambda, z, \hat{z})|}{|\lambda^2 - z^2| |\lambda - \hat{z}|^2} d\lambda, \quad c_3 = \int_1^\infty e^{-x^2} dx,
$$

and we have used the Schwarz inequality.

ii) We regard now the second term on the right in (3.24). Take $\lambda \in \bar{\Gamma}$, such that $(2k - 1)\pi \leq \arg \lambda < (2k + 1)\pi$. Thus, when $x, t \to \infty$, we have that $\lambda x, \lambda t \to \infty$ in the $k$-leaf. If we use the asymptotic behavior of the Bessel function on the $k$-leaf, given by (5.5) in the appendix, we can find that when $x, t \to \infty$
the numerator in (3.25) is of
\[
\left( c_4 Y_b(\lambda) - c_Y J_b(\lambda) \right) e^{i\lambda_0(x+t)} + \left( \hat{c}_4 Y_b(\lambda) - \hat{c}_Y J_b(\lambda) \right) e^{-i\lambda_0(x+t)}
\]
\[
+ \left( c_4 Y_b(\lambda) - c_Y J_b(\lambda) \right) \left( \hat{c}_4 Y_b(\lambda) - \hat{c}_Y J_b(\lambda) \right) \left( e^{i\lambda_0(x-t)} + e^{-i\lambda_0(x-t)} \right) \Bigg) / \sqrt{xt}
\]
modulo a \((1 + O(\lambda_0^{-1}))\) term, where \(\lambda_0\) is the projection of \(\lambda\) onto the zero leaf. If we denote \(\hat{\lambda} = \max\{|\Im\lambda| : \lambda \in \hat{\Gamma}\}\), then there exists a constant \(c_4 > 0\) such that
\[
|\kappa_b(\lambda, x, t)| \leq c_4 e^{\hat{\lambda}(x+t)} / \sqrt{xt}, \quad \text{for all } x, t \geq 1 \text{ and } \lambda \in \hat{\Gamma}.
\]
Hence, we have
\[
\int_1^\infty |\hat{\alpha}(x)| x^{1-b} e^{-x^2} \left( \int_1^\infty |\alpha(t)| t^{1-b} \int_{\hat{\Gamma}} |\tilde{n}_{b,p, k} P(\lambda, z, \tilde{z}, x, t)| d\lambda dt \right) dx \leq c_4 \left( \int_1^\infty |\hat{\alpha}(x)| x^{1-b} e^{-x^2} e^{\hat{\lambda} x} dx \right) \left( \int_1^\infty |\alpha(t)| t^{1-b} e^{\hat{\lambda} t} dt \right) \leq c_4 c_5 c_6 \left( \int_1^\infty |\hat{\alpha}(x)|^2 x^{-\gamma} e^{-x^2} dx \right)^{1/2} \left( \int_1^\infty |\alpha(t)|^2 t^{-\gamma} e^{2t} dt \right)^{1/2},
\]
where
\[
c_5 = \int_{\hat{\Gamma}} \frac{|P(\lambda, z, \tilde{z})|}{|\lambda^2 - z|^{\frac{3}{2}} |\lambda^2 - \tilde{z}|^{\frac{3}{2}}}, \quad c_6 = \int_1^\infty e^{-x^2 + 2\lambda x} dx,
\]
and we have used the Schwarz inequality.

iii) Now we regard the first term on the right part in (3.24). Assume first that \(b > 0\) in (3.25). If we fix \(x, t\) and take \(\lambda \to 0\), by (5.7) in the appendix we find that
\[
\kappa_b(\lambda, x, t) = \lambda \left( \frac{(xt)^b}{\Gamma(b)} - \frac{(\frac{x}{t})^b}{\Gamma(b)} + \frac{(xt)^{-b}}{\Gamma(b)} \right) + \frac{\pi^2}{\Gamma(\frac{3}{2})^2} \left( \frac{3}{2} \right)^{-2b} + \frac{\Gamma^2(b + 1)}{\Gamma(\frac{3}{2})^2} \left( \frac{3}{2} \right)^{-2b} O(1).
\]
Hence, there exists some continuous function \(\tilde{c}_1(x, t)\), such that
\[
|\kappa_b(\lambda, x, t)| \leq \tilde{c}_1(x, t) \left( (xt)^b - \frac{x^b}{t} - \frac{t^b}{x} + (xt)^{-b} \right).
\]
Since for fixed \(\lambda \in [0, x_1]\) we have that
\[
\lim_{x, t \to \infty} \frac{\kappa_b(\lambda, x, t)}{(xt)^b - \frac{x^b}{t} - \frac{t^b}{x} + (xt)^{-b}} = 0,
\]
we can assume that \(\tilde{c}_1(x, t)\) is bounded in \(x\) and \(t\).
Let now $b < 0$, and take again the limit of (3.25) as $\lambda \to 0$, for fixed $x, t$. By using the definition of the Bessel function of the second kind

$$Y_b(z) = \frac{\cos \pi b J_b(z) - J_{-b}(z)}{\sin \pi b},$$

where the above relation is considered as a limit when $b$ becomes an integer, we can see that all the increasing terms in the numerator of (3.25) cancel out, and the remaining term is of $O(1)$ in $\lambda$. Thus, by (5.7) we get

$$\kappa_b(\lambda, x, t) = \frac{\lambda}{\pi^2(\frac{\lambda}{2})^{2b}} O(1) \text{ when } \lambda \to 0.$$ 

Hence, $|\kappa_b(\lambda, x, t)|$ is bounded in $\lambda \in [0, x_1]$, and since for fixed $\lambda$ we have

$$\lim_{x, t \to \infty} \kappa_b(\lambda, x, t) = 0,$$

we find again that $|\kappa_b(\lambda, x, t)|$ is bounded for all $x, t$ and $\lambda \in [0, x_1]$.

Finally assume that $b = 0$. In this case, if we take $\lambda \to 0$ in (3.25), for $x, t$ fixed, and use again (5.7), we find

$$\kappa_b(\lambda, x, t) = \frac{\lambda \ln x \ln t}{\frac{\pi^2}{4} + (\ln \frac{\lambda}{2} + \gamma)^2} O(1),$$

where $\gamma$ is the Euler-Mascheroni constant. So, there exists a continuous function $\tilde{c}_2(x, t)$, such that

$$|\kappa_b(\lambda, x, t)| \leq \tilde{c}_2(x, t)|\ln x||\ln t|,$$

and by observing that

$$\lim_{x, t \to \infty} \frac{|\kappa_b(\lambda, x, t)|}{|\ln x||\ln t|} = 0,$$

we can again assume that $\tilde{c}_2(x, t)$ is bounded for all $x, t$ and $\lambda \in [0, x_1]$.

We can summarize the above arguments to the following estimation

$$|\kappa_b(\lambda, x, t)| \leq \begin{cases} c_b |(xt)^b - (\frac{\lambda}{x})^b| & \text{if } b > 0 \\ c_b |\ln x||\ln t| & \text{if } b = 0 \\ c_b & \text{if } b < 0 \end{cases},$$

for an appropriate constant $c_b > 0$, $x, t \geq 1$ and $\lambda \in [0, x_1]$. Hence, if we denote by

$$c_7 = \int_{0}^{x_1} \left| \frac{P(\lambda, z, \tilde{z})}{|\lambda^2 - z^2|^{k_1} |\lambda^2 - \tilde{z}^2|^{k_2}} \right| d\lambda,$$
for $b_p > 0$, we have

$$\int_{1}^{\infty} |\tilde{a}(x)| x^{-b_p} e^{-x^2} \left( \int_{1}^{\infty} |\alpha(t)| t^{1-b_p} \int_{0}^{x_1} |\tilde{m}_{b_p,k,P}(\lambda, z, \tilde{z}, x, t)| d\lambda dt \right) dx \leq$$

$$c_6 \int_{1}^{\infty} |\tilde{a}(x)| x^{-b_p} e^{-x^2} \left( \int_{1}^{\infty} |\alpha(t)| t^{1-b_p} \int_{0}^{x_1} |P(\lambda, z, \tilde{z})| \frac{1}{|\lambda^2 - z^2|^{k_1} |\lambda^2 - \tilde{z}^2|^{k_2}} (xt)^{b_p} \right.$$

$$\left. - \left( \frac{x}{t} \right)^{b_p} - \frac{t}{x} b_p + (xt)^{-b_p} \right) d\lambda dt \right) dx \leq$$

$$c_6 c_7 \left( \int_{1}^{\infty} |\tilde{a}(x)| xe^{-x^2} dx \right) \left( \int_{1}^{\infty} |\alpha(t)| t^{1-2b_p} dt \right)$$

$$+ \left( \int_{1}^{\infty} |\tilde{a}(x)| xe^{-x^2} dx \right) \left( \int_{1}^{\infty} |\alpha(t)| t^{1-2b_p} dt \right)$$

$$+ \left( \int_{1}^{\infty} |\tilde{a}(x)| x^{-2b_p} e^{-x^2} dx \right) \left( \int_{1}^{\infty} |\alpha(t)| t^{1-2b_p} dt \right)$$

$$\leq c_8 \left( \int_{1}^{\infty} |\tilde{a}(x)|^2 x^{-\gamma_p} e^{-x^2} dx \right)^{1/2} \left( \int_{1}^{\infty} |\alpha(t)|^2 t^{-\gamma_p} e^{t^2} dt \right)^{1/2},$$

for some constant $c_8 > 0$, where at the last step we worked as before, by using the Schwarz inequality. Similar estimations can be found when $b_p = 0$ or $b_p < 0$.

Thus, generally we have

$$\int_{1}^{\infty} |\tilde{a}(x)| x^{-b_p} e^{-x^2} \left( \int_{1}^{\infty} |\alpha(t)| t^{1-b_p} \int_{0}^{x_1} |\tilde{m}_{b_p,k,P}(\lambda, z, \tilde{z}, x, t)| d\lambda dt \right) dx \leq$$

$$c_9 \left( \int_{1}^{\infty} |\tilde{a}(x)|^2 x^{-\gamma_p} e^{-x^2} dx \right)^{1/2} \left( \int_{1}^{\infty} |\alpha(t)|^2 t^{-\gamma_p} e^{t^2} dt \right)^{1/2}, \quad (3.28)$$

for some constant $c_9 > 0$.

2nd step.

We deal now with the second term on the right part in (3.23). From the discussion in the appendix, for any $b \in \mathbb{R}$, $G_b(\lambda, x)$ is holomorphic as a function of $\lambda \in \mathbb{C}$. So, any pole of $\tilde{m}_{b_p,k,P}(z, \tilde{z}, \lambda, x, t)$, over $\lambda$, is a zero of $J^2_{b_p}(\lambda) + Y^2_{b_p}(\lambda)$.

Note that since

$$J^2_{b_p}(\lambda) + Y^2_{b_p}(\lambda) = (J_b(\lambda) + iY_b(\lambda))(J_b(\lambda) - iY_b(\lambda)) = H^{(1)}_b(\lambda)H^{(2)}_b(\lambda),$$

where $H^{(1)}_b(\lambda)$ and $H^{(2)}_b(\lambda)$ are the Hankel functions of the first and second kind respectively, the poles of $\tilde{m}_{b_p,k,P}(\lambda, z, \tilde{z}, x, t)$, over $\lambda$, are the zeros of the Hankel
functions of order $b_p$. Also, $J_b^2(\lambda) + Y_b^2(\lambda)$ has simple zeros only, since if there is any zero $q$ of order $k \geq 2$, we have that $J_b^2(\lambda) + Y_b^2(\lambda) = (z - q)^k \hat{f}(\lambda)$. Now if we differentiate the last equation over $\lambda$ and put $\lambda = q$, we find that $J_b'(q) = -i Y_b'(q)$, which together with $J_b^2(q) = \pm i Y_b^2(q)$, contradicts with the relation $W(J_b(\lambda), Y_b(\lambda)) = 2/\pi \lambda$ of the Wronskian of the Bessel equation. Thus, $\kappa_b(\lambda, x, t)$ has only simple poles. Take any such a pole $q \in (\Gamma)$. Then

$$\text{Res}_{\lambda=q} \tilde{m}_{b_p,k,p}(\lambda, z, \tilde{z}, x, t) = \lim_{\lambda \to q} \tilde{m}_{b_p,k,p}(\lambda, z, \tilde{z}, x, t)(\lambda - q) = \frac{qP(q, z, \tilde{z})}{(q^2 - z^2)k_1 (q^2 - \tilde{z}^2)^k} G_{b_p}(q, x)G_{b_p}(q, t) \lim_{\lambda \to q} \frac{1}{J_{b_p}^2(\lambda) + Y_{b_p}^2(\lambda)}.$$

By the part (ii) of the first step, we have that

$$|G_{b_p}(q, x)G_{b_p}(q, t)| \leq \hat{c}_q \frac{e^{\tilde{q}(x+t)}}{\sqrt{xt}} \text{ for all } x, t,$$

for some constant $\hat{c}_q > 0$, where $\tilde{q} = \max_q |\Im q|$. Thus, there exists some constant $c_q > 0$ such that the following holds

$$|\text{Res}_{\lambda=q} \tilde{m}_{b_p}(\lambda, x, t)| \leq c_q \frac{e^{\tilde{q}(x+t)}}{\sqrt{xt}}.$$

Hence,

$$\left| \int_1^\infty \hat{\alpha}(x)e^{-x^2}x^{-\gamma_p} \right| \leq (2\pi i) \sum_{q \in (\Gamma), q \neq z} \int_1^\infty \alpha(t)x^{b_p}t^{1-b_p} \text{Res}_{\lambda=q} \tilde{m}_{b_p,k,p}(\lambda, z, \tilde{z}, x, t)dt \right) dx \right| \leq c_10 \left( \int_1^\infty |\hat{\alpha}(x)|^{1/2}x^{-\gamma_p}e^{-x^2}dx \right)^{1/2} \left( \int_1^\infty |\alpha(t)|^{1/2}t^{-\gamma}e^{\tilde{q}t}dt \right)^{1/2}, \quad (3.29)$$

for some constant $c_{10} > 0$, where at the last step we used the Schwarz inequality.

If we do with the third and the last term on the right part of (3.23) similar work to the steps 1 and 2, we can find similar inequalities to (3.26), (3.27), (3.28)
and (3.29). Hence, from (3.23), there exists some constant $c > 0$ such that

$$\left| (\tilde{\omega}, Q_{b_p,b_{p-1},k,P_z,\tilde{z} \tilde{\omega}}) \right| \leq c \left( \|	ilde{\alpha}\|_{L^2((1,\infty),e^{-x^2 - \gamma x} \, dx)} \|	ilde{\theta}\|_N \right)^{3/4} + 
\left( \|	ilde{\alpha}\|_{L^2((1,\infty),e^{-x^2 - \gamma x} \, dx)} \|	ilde{\theta}\|_N \right)^{3/4} \leq 
\left( c \left( \|	ilde{\alpha}\|_{L^2((1,\infty),e^{-x^2 - \gamma x} \, dx)} \|	ilde{\theta}\|_N \right)^{3/4} + 
\left( \|	ilde{\alpha}\|_{L^2((1,\infty),e^{-x^2 - \gamma x} \, dx)} \|	ilde{\theta}\|_N \right)^{3/4} \right).
$$

\begin{equation}
\left( \|	ilde{\alpha}\|_{L^2((1,\infty),e^{-x^2 - \gamma x} \, dx)} \|	ilde{\theta}\|_N \right)^{3/4} + 
\left( \|	ilde{\alpha}\|_{L^2((1,\infty),e^{-x^2 - \gamma x} \, dx)} \|	ilde{\theta}\|_N \right)^{3/4} = 
\left( c \left( \|	ilde{\alpha}\|_{L^2((1,\infty),e^{-x^2 - \gamma x} \, dx)} \|	ilde{\theta}\|_N \right)^{3/4} + 
\left( \|	ilde{\alpha}\|_{L^2((1,\infty),e^{-x^2 - \gamma x} \, dx)} \|	ilde{\theta}\|_N \right)^{3/4} \right).
\end{equation}

We now denote the operator $Q_{b_p,b_{p-1},k,P_z,\tilde{z}}$ with $k_1 = 1, k_2 = 0$ and $P(\lambda, z, \tilde{z}) = 1$ by $R_z(\Delta)$, which is an extension of $R_z(\Delta)$. Since any derivative of $1/(\lambda^2 - z^2)$ over $z$ is of the form $P(\lambda, z)/(\lambda^2 - z^2)^{k_1}$ with the order of the polynomial $P(\lambda, z)$ over $\lambda$ not to be greater than $2(k_1 - 1)$, we can define the $k$-th derivative of $R_z(\Delta)$ over $z$, denoted by $R_z^{(k)}(\Delta)$, to be the operator $Q_{b_p,b_{p-1},k,k,P_z,\tilde{z}}$ for $k_1 = k+1, k_2 = 0$ and

$$P(\lambda, z, \tilde{z}) = (\lambda^2 - z^2)^{k+1} \frac{d^k}{d\tilde{z}^k} \left( \frac{1}{\lambda^2 - z^2} \right),$$

which is bounded from $H_{\mathcal{H}_+}$ to $H_{\mathcal{H}_-}$ by Lemma 3.19. Also, for any $k$, the term

$$\left( \frac{d^k}{d\tilde{z}^k} \left( \frac{1}{\lambda^2 - z^2} \right) - \frac{d^{k+1}}{d\tilde{z}^{k+1}} \left( \frac{1}{\lambda^2 - z^2} \right) \right)$$

is of the form

$$\frac{P(\lambda, z, \tilde{z})}{(\lambda^2 - z^2)^{k_1} (\lambda^2 - \tilde{z}^2)^{k_2}},$$

where $P(\lambda, z, \tilde{z})$ is some polynomial over $\lambda, z$ and $\tilde{z}$ with order over $\lambda$ not greater than $2(k_1 + k_2 - 1)$. Hence, by Lemma 3.19 for this case, we get the inequality (3.30), where the constant $c$ goes to zero when $\tilde{z}$ goes to $z$. The last follows by the fact that the term in (3.31), and hence the numerator $P(\lambda, z, \tilde{z})$ in (3.32), goes to zero when $\tilde{z}$ goes to $z$. Hence, we have that $(R_z^{(k)}(\Delta) - R_z^{(k)}(\Delta)) / (\tilde{z} - z)$ converges strongly to $R_z^{(k+1)}(\Delta)$, when $\tilde{z}$ goes to $z$. So, we first define
Definition 3.20. Let $\mathcal{S}$ be the Riemann surface of the function $\log z$. Denote by $H_\nu$, the set of the zeros of the Hankel functions of the first and second kind, of order $\nu$, on the logarithmic cover of $\mathbb{C}$. Assume that $z \neq 0$ is on $\mathcal{S}$ such that $z \notin H_{b_\nu} \cup H_{b_{\nu-1}}$. Take two curves $\Gamma$ and $\Gamma'$ around $z$, joining $0$ and $+\infty$ of the zero leaf, such that $\Gamma \cap -\Gamma = \{0\}$, $\Gamma' \cap -\Gamma' = \{0\}$, $\Gamma \cap H_{b_\nu} = \emptyset$ and $\Gamma' \cap H_{b_{\nu-1}} = \emptyset$. Define the operator $\mathcal{R}_z(\Delta)$ acting on $\Omega_{0,H}^p(M_1 \setminus \partial M_1)$ by

$$\mathcal{R}_z(\Delta)(\omega) = \left( \int_1^\infty \alpha(t) K_{b_\nu,\Gamma}(z, x, t) dt \right) \theta + dx \wedge \left( \int_1^\infty \beta(t) \frac{x}{t} K_{b_{\nu-1},\Gamma'}(z, x, t) dt \right) \tilde{\theta}$$

$$- \left( 2\pi i \sum_{q \in (\Gamma) \cap H_{b_\nu}} \int_1^\infty \alpha(t) x^{b_{\nu-1}} \frac{t^{1-b}}{t^{1-b_\nu}} \text{Res}_{q=m_{b_\nu}}(\lambda, z, x, t) dt \right) \theta$$

$$- dx \wedge \left( 2\pi i \sum_{q \in (\Gamma') \cap H_{b_{\nu-1}}} \int_1^\infty \beta(t) x^{b_{\nu-1}} \frac{t^{1-b}}{t^{1-b_{\nu-1}}} \text{Res}_{q=m_{b_{\nu-1}}}(\lambda, z, x, t) dt \right) \tilde{\theta},$$

for any $\omega = \alpha \theta + dx \wedge \beta \tilde{\theta} \in \Omega_{0,H}^p(M_1 \setminus \partial M_1)$, with integral kernel

$$K_{b,\Gamma}(z, x, t) = x^{b t^{1-b}} \int_\Gamma m_b(\lambda, z, x, t) d\lambda,$$

where

$$m_b(\lambda, z, x, t) = \frac{1}{\lambda^2 - z^2} \frac{G_b(\lambda, x) G_b(\lambda, t)}{J_b^2(\lambda) + Y_b^2(\lambda)} \lambda,$$

with

$$G_b(r, l) = Y_b(r) J_b(l) - J_b(r) Y_b(l),$$

and $(\Gamma)$, $(\Gamma')$ are the areas between the semi axis $[0, \infty)$ of the zero leaf and the curves $\Gamma$ and $\Gamma'$ respectively.

Then, we can state the following

Theorem 3.21. Consider the weighted $L^2$ spaces

$$H_{\mathcal{H}^+} = e^{-\frac{\pi^2}{4}} L^2_{\mathcal{H}}(M_1, \Lambda^p T^* M_1) \text{ and } H_{\mathcal{H}^-} = e^{-\frac{\pi^2}{4}} L^2_{\mathcal{H}}(M_1, \Lambda^p T^* M_1).$$

The operator $\mathcal{R}_z(\Delta)$, from Definition 3.20, can be extended to a function from the Riemann surface $\mathcal{S}$ to the space $\mathcal{L}(H_{\mathcal{H}^+}, H_{\mathcal{H}^-})$. Also, $\mathcal{R}_z(\Delta)$ is a meromorphic continuation of the resolvent $R_z(\Delta|_{L^2_{\mathcal{H}}(M_1, \Lambda^p T^* M_1)})$ of $\Delta|_{L^2_{\mathcal{H}}(M_1, \Lambda^p T^* M_1)}$ from the lower half plane to $\mathcal{S}$, with simple poles only, which coincide with the set $H_{b_\nu} \cup H_{b_{\nu-1}}$. 
We now take any point \( z \) to be on the lower half plane of the \( k \)-leaf. Then, \(-z\) is in the \((k-1)\)-leaf. Let \( z_0 \) be its projection on the zero leaf and
\[
\omega = \alpha \theta + dx \land \beta \tilde{\theta} \in \Omega_{0,\mathcal{H}}^p(M_1 \setminus \partial M_1).
\]
We use the residue theorem to compute the following difference, where \( b_p \) and \( b_{p-1} - 1 \) are taken not to be integers and the cases \( b_p \in \mathbb{Z} \) or \( b_{p-1} - 1 \in \mathbb{Z} \) are assumed to be limit cases,
\[
\mathcal{R}_z(\Delta)(\omega) - \mathcal{R}_{z_0}(\Delta)(\omega) = \\
\left(2\pi i \sum_{q=\pm z} \int_1^\infty \alpha(t)x^{b_p}t^{1-b_p}\text{Res}_{\lambda=q}m_{b_p}(\lambda, z, x, t)dt\right)\theta \\
+dx \land \left(2\pi i \sum_{q=\pm z} \int_1^\infty \beta(t)x^{b_{p-1}-1}t^{1-b_{p-1}}\text{Res}_{\lambda=q}m_{b_{p-1}-1}(\lambda, z, x, t)dt\right)\tilde{\theta}
\]

\[
= \left( \int_1^\infty \alpha(t)x^{b_p}t^{1-b_p}\tau_{b_p}(x, t)dt \right)\theta \\
+dx \land \left( \int_1^\infty \beta(t)x^{b_{p-1}-1}t^{1-b_{p-1}}\tau_{b_{p-1}-1}(x, t)dt \right)\tilde{\theta},
\]

where
\[
\tau_{b}(x, t) = i\pi G_b(z_0, x)G_b(z_0, t)\left(\frac{1}{J_b^2(z)} + \frac{1}{J_b^2(-z)}\right).
\]

By (5.3) in the appendix, we have that the last kernel is never zero. Therefore, the full logarithmic cover is needed to find the meromorphic extension of the resolvent.

In order to obtain the behavior of the resolvent at the point zero, we take the limit as \( z \to 0 \), with \( z \) to be on the \( k \)-leaf, at the formula for \( \mathcal{R}_z(\Delta) \) given by Definition 3.20. We choose the path \( \Gamma \) to coincide with the real axis of the zero leaf at \([0, x_1]\) and \([x_2, +\infty)\), for some fixed \( 0 < x_1 < x_2 \), and do similarly for the path \( \Gamma' \). For \( z \) sufficiently small, there exists some paths \( \Gamma \) and \( \Gamma' \) such that there are no zeros of the Hankel functions of order \( b_p \) and \( b_{p-1} \) in the areas \( \Gamma \) and \( \Gamma' \), since otherwise we will have accumulation of the zeros at the origin, which is not possible. Thus, if we omit the last two terms in the expression of \( \mathcal{R}_z(\Delta) \), for \( \omega \) as in Definition 3.20, we have
\[
\mathcal{R}_z(\Delta)(\omega) = \\
\left( \int_1^\infty \alpha(t)x^{b_p}t^{1-b_p}\left( \int_{\Gamma} \frac{1}{\lambda^2 - z^2} \frac{G_{b_p}(\lambda, x)G_{b_p}(\lambda, t)}{J_{b_p}(\lambda) + Y_{b_p}(\lambda)} \lambda d\lambda \right)dt \right)\theta \\
+dx \land \left( \int_1^\infty \beta(t)x^{b_{p-1}-1}t^{1-b_{p-1}}\left( \int_{\Gamma'} \frac{1}{\lambda^2 - z^2} \frac{G_{b_{p-1}-1}(\lambda, x)G_{b_{p-1}-1}(\lambda, t)}{J_{b_{p-1}-1}(\lambda) + Y_{b_{p-1}-1}(\lambda)} \lambda d\lambda \right)dt \right)\tilde{\theta}.
\]
From the first step of the proof of Lemma 3.19, we have that the term
\[
\frac{G_b(\lambda, x)G_b(\lambda, t)}{J_b^2(\lambda) + Y_b^2(\lambda)} \lambda
\]
is uniformly bounded in \(\lambda\). Thus, by replacing \(\lambda\) by \(z\mu\) in the integral
\[
\int_{\Gamma} \frac{1}{|\lambda^2 - z^2|} d\lambda,
\]
we obtain
\[
\frac{1}{|z|} \int_{\Gamma_1} \frac{1}{|\mu^2 - 1|} d\mu,
\]
for some path \(\Gamma_1\) depending on \(z\), which goes around of the point 1 on the zero leaf, by joining 0 and \(\infty\) on the \(-k\)-leaf. For fixed ray of approaching zero, the above integral converges and does not depend on the path \(\Gamma_1\). Thus we have that
\[
\mathcal{R}_z(\Delta) = \mathcal{O}(1/|z|), \text{ as } z \to 0,
\]
with the constant depending on \(k\).
Chapter 4

The Laplacian on a manifold with a generalized cusp

In this chapter, we lay out the results we have obtained regarding the spectral theory for the Laplacian on a manifold with a generalized cusp.

4.1 Compact perturbation

Consider now any compact $n$-dimensional manifold $M_0$ which has common boundary $\partial M_0 = N$ with the cusp $M_1 = [1, \infty) \times N$. Then, $M = M_0 \cup M_1$ is an $n$-dimensional noncompact manifold with cusp. Let $g$ be a metric on $M$, such that the restriction $g_1$ of $g$ on $M_1$ is given by (3.1). Let $\Omega^p_0(M), \Omega^p_0(M_0 \setminus \partial M_0)$ and $\Omega^p_0(M_1 \setminus \partial M_1)$ be the space of smooth compactly supported $p$-forms on $M$, $M_0 \setminus \partial M_0$ and $M_1 \setminus \partial M_1$ respectively, and $L^2(M, \wedge^p T^* M), L^2(M_0, \wedge^p T^* M_0)$ and $L^2(M_1, \wedge^p T^* M_1)$ their completions with respect to the Riemannian inner product induced by $g$, when it is restricted to each of the manifolds $M, M_0$ and $M_1$ respectively. Since any section in $\wedge^p T^* M$ can be uniquely written as a sum of sections with support in $M_0$ and $M_1$ respectively, we have that

$$L^2(M, \wedge^p T^* M) = L^2(M_0, \wedge^p T^* M_0) \oplus L^2(M_1, \wedge^p T^* M_1).$$

Hence, by Theorem 3.5, we have

$$L^2(M, \wedge^p T^* M) = L^2(M_0, \wedge^p T^* M_0)$$

$$\oplus L^2_{d\delta_1}(M_1, \wedge^p T^* M_1) \oplus L^2_{d\delta}(M_1, \wedge^p T^* M_1) \oplus L^2_{\delta d}(M_1, \wedge^p T^* M_1). \quad (4.1)$$

Let us denote by $\Delta_0$ the Friedrichs extension of $d\delta + \delta d$ defined on $\Omega^p_0(M \setminus N_{x=1})$ which, for the same reason as in Lemma 3.3, satisfies Dirichlet boundary
conditions at \( x = 1 \). Then, according to the notation in Definition 3.4, we have
\[
\Delta_0 = \Delta_0|_{L^2(M_0, \wedge^p T^* M_0)} \oplus \Delta_H \oplus \Delta_{d,\delta} \oplus \Delta_{\delta,\delta},
\]
and for the corresponding resolvents
\[
R_z(\Delta_0) = R_z(\Delta_0|_{L^2(M_0, \wedge^p T^* M_0)}) \oplus R_z(\Delta_H) \oplus R_z(\Delta_{d,\delta}) \oplus R_z(\Delta_{\delta,\delta}). \tag{4.2}
\]
From the Corollary 2.41 and Theorem 3.11, we have that each of the sets \( \sigma(\Delta_{\delta,\delta}) \), \( \sigma(\Delta_{d,\delta}) \) and \( \sigma(\Delta_0|_{L^2(M_0, \wedge^p T^* M_0)}) \) are discrete. Hence,
\[
R_z(\Delta_0|_{L^2(M_0, \wedge^p T^* M_0)}) \oplus R_z(\Delta_{d,\delta}) \oplus R_z(\Delta_{\delta,\delta})
\]
is meromorphic as a function of \( z \) from \( \mathbb{C} \) to
\[
\mathcal{L}
\left(
L^2(M_0, \wedge^p T^* M_0) \oplus L^2_{d,\delta}(M_1, \wedge^p T^* M_1) \oplus L^2_{\delta,\delta}(M_1, \wedge^p T^* M_1)\right).
\]
Let us define the spaces
\[
H_- = e^{-\frac{z^2}{\tau}} L^2(M, \wedge^p T^* M) \quad \text{and} \quad H_+ = e^{\frac{z^2}{\tau}} L^2(M, \wedge^p T^* M).
\]
Since we have the inclusion \( H_+ \subset L^2(M, \wedge^p T^* M) \subset H_- \), from (4.2) and Theorem 3.21 we get that \( R_z(\Delta_0) \) can be continued meromorphically as a function of \( z \) from \( S \) to \( \mathcal{L}(H_+, H_-) \). Let us denote again by \( \Delta \) the unique extension of \( d\delta + \delta d \) defined on \( \Omega^n_0(M) \). Then, we have the following result.

**Theorem 4.1.** The resolvent of \( \Delta \) has a meromorphic continuation from the physical sheet to the Riemann surface \( S \), given by Definition 3.20, as a function from \( S \) to \( \mathcal{L}(H_+, H_-) \), where \( H_\pm = e^{\mp \frac{z^2}{\tau}} L^2(M, \wedge^p T^* M) \). Also, the essential spectra of \( \Delta \) and \( \Delta_0 \) coincide.

**Proof.** Take a tubular neighborhood \( Y \) of \( N \), which we identify with \( N \times [0, 2] \). Assume that \( N \times [0, 1] \subset M_0 \) and \( N \times [1, 2] \subset M_1 \). Let \( \Delta_* \) be the Friedrichs extension of \( d\delta + \delta d \) on \( \Omega^n_0(M_0 \cup (N \times [1, 2])) \), which as before, corresponds to the Laplacian with Dirichlet boundary conditions, and let \( R_z(\Delta_*) \) be its resolvent. Also, let \( R_z(\Delta_0) \) be the meromorphic continuation of the resolvent of \( \Delta_0 \), to the Riemann surface \( S \). Let \( r(\alpha, \beta)(x) \in C^\infty(\mathbb{R}) \) be a decreasing function with \( r(\alpha, \beta) = 0 \) if \( x \geq \beta \) and \( r(\alpha, \beta) = 1 \) if \( x \leq \alpha \). Consider the functions on \( N \times [1, 2] \)
\[
\chi_1(x, \tilde{x}) = r(1 + 3/4, 1 + 7/8)(x), \quad \chi_2(x, \tilde{x}) = 1 - r(1 + 1/8, 1 + 1/4)(x)
\]
and
\[
\upsilon_1(x, \tilde{x}) = r(1 + 1/2, 1 + 5/8)(x), \quad \upsilon_2(x, \tilde{x}) = 1 - \upsilon_1(x, \tilde{x}),
\]
which depend only on the variable $x \in [1, 2]$, and $\tilde{x} \in N$. There is
\[ \chi_1 u_1 + \chi_2 u_2 = 1 \text{ and } \text{dist}(\text{supp} \chi_1, \text{supp} u_i) > 0. \]

We can extend canonically the above functions to smooth functions on $M$, such that they are constant on $M \setminus (N \times [1, 2])$. Let the operator
\[ Q_z = u_1 R_z(\Delta_+ + \chi_2). \]

$Q_z$ is a meromorphic function from $S$ to $L(H_+, H_-)$. We have that
\[ Q_z(\Delta - z^2 I) = I + u_1 R_z(\Delta_+)(2\nabla_{\text{grad} \chi_1} - \Delta \chi_1) + u_2 R_z(\Delta_0)(2\nabla_{\text{grad} \chi_2} - \Delta \chi_2), \]
where from [40], we have used the fact that there exists a hermitian connection $\nabla$ such that for any function $\phi$ and a $p$-form $\omega$ we have
\[ \Delta(\phi \omega) = \phi \Delta \omega + \Delta \phi \omega - 2\nabla_{\text{grad} \phi} \omega. \quad (4.3) \]

The integral kernels of the operators
\[ u_1 R_z(\Delta_+)(2\nabla_{\text{grad} \chi_1} - \Delta \chi_1) \in L(H_-) \text{ and } u_2 R_z(\Delta_0)(2\nabla_{\text{grad} \chi_2} - \Delta \chi_2) \in L(H_-) \]
are smooth (they are smooth off the diagonal) and have compact support in the second variable. By the polar decomposition, a bounded operator $A$ with the above properties is compact, since $\sqrt{AA^*}$ is compact (since its integral kernel is smoothing with compact support in both variables). Hence, the family of operators
\[ T_z = u_1 R_z(\Delta_+)(2\nabla_{\text{grad} \chi_1} - \Delta \chi_1) + u_2 R_z(\Delta_0)(2\nabla_{\text{grad} \chi_2} - \Delta \chi_2) \]
is a meromorphic family of compact operators in $H_-$. The residues of $R_z(\Delta_+)$, and hence of
\[ u_1 R_z(\Delta_+)(2\nabla_{\text{grad} \chi_1} - \Delta \chi_1) \]
are finite rank operators. Also, according to Definition 3.20, if $q$ is a zero of $J_{b_p}^2(\lambda) + Y_{b_p}^2(\lambda)$ in the area $(\Gamma)$, and
\[ \omega = \alpha \theta + dx \wedge \beta \tilde{\theta} \in \Omega^p_0(M_1 \setminus \partial M_1), \]
we have that
\[ \text{Res}_z = q R_z(\Delta N)(\omega) = \left( \frac{x_b G_{b_p}(q, x)}{2} \text{Res}_\lambda = q \left( \frac{1}{J_{b_p}^2(\lambda) + Y_{b_p}^2(\lambda)} \right) \int_1^\infty \alpha(t) t^{1-b_p} G_{b_p}(q, t) dt \right) \theta \]
4.1 Compact perturbation

if \( J_{b-1}^2(q) + Y_{b-1}^2(q) \neq 0 \), or

\[
\left( \frac{x^{b_p} G_{b_p}(q, x)}{2} \right) \text{Res}_{\lambda=q} \left( \frac{1}{J_{b_p}^2(\lambda) + Y_{b_p}^2(\lambda)} \int_1^{\infty} \alpha(t) t^{1-b_p} G_{b_p}(q, t) dt \right) \theta \\
+ \left( \frac{x^{b_p-1} G_{b_p-1}(q, x)}{2} \right) \text{Res}_{\lambda=q} \left( \frac{1}{J_{b_p-1}^2(\lambda) + Y_{b_p-1}^2(\lambda)} \int_1^{\infty} \beta(t) t^{1-b_p-1} G_{1-b_p-1}(q, t) dt \right) \tilde{\theta}
\]

if \( J_{b-1}^2(q) + Y_{b-1}^2(q) = 0 \). Similarly, we can calculate the residues corresponding to the poles in the area \((\Gamma')\), to see that they are all finite rank operators. Thus from (4.2), we get that the residues of \( R_z(\Delta_0) \), and hence of \( v_2 R_z(\Delta_0)(2\nabla_{\text{grad} \chi_2} - \Delta \chi_2) \), are also of finite rank. If we denote by \( \| \cdot \| \) the norm in \( \mathcal{L}(H_-) \), then if we use the spectral theorem for compact operators we have the following estimation

\[
\| T_z \omega \| \leq \| v_1 R_z(\Delta_+)(2\nabla_{\text{grad} \chi_1} - \Delta \chi_1) \omega \| + \| v_2 R_z(\Delta_0)(2\nabla_{\text{grad} \chi_2} - \Delta \chi_2) \omega \| \leq \frac{c}{|\Im z|^2} \| \omega \|,
\]

for some constant \( c > 0 \). Hence, the operator \( I + T_z \) is invertible for large values of \( \Im z \), and since the residues of \( T_z \) are of finite rank, by the meromorphic Fredholm theory (Theorem 2.19) it is invertible for any \( z \) except a discrete set of points, where the family of operators \( (I + T_z)^{-1} \) has finite rank residues. So, we have that

\[
(I + T_z)^{-1} Q_z(\Delta - z^2 I) = I,
\]

and \( (I + T_z)^{-1} Q_z \) is a meromorphic continuation of the resolvent of \( \Delta \).

We state next the following result related to the scattering properties of the Laplacians \( \Delta \) and \( \Delta_0 \).

**Lemma 4.2.** The wave operators \( W^\pm(\Delta, \Delta_0) \) exist and are complete.

**Proof.** We have that \( \Delta \) and \( \Delta_0 \) are positive operators. Thus, from Theorem 2.22, it suffices to prove that \( (\Delta + I)^{-k} - (\Delta_0 + I)^{-k} \) is trace class, for some \( k \in \mathbb{R} \). According to the proof of the previous theorem, let the operator \( Q'_k \) defined as

\[
Q'_k = v_1 R^k_\Delta(\Delta_+^k) \chi_1 + v_2 R^k_\Delta(\Delta_0) \chi_2,
\]

where \( k \) is a positive integer. If we use the formula (4.3) \( k \) times, we get that

\[
Q'_k(\Delta + I)^k = I + K,
\]

for some constant \( c > 0 \). Hence, the operator \( I + T_z \) is invertible for large values of \( \Im z \), and since the residues of \( T_z \) are of finite rank, by the meromorphic Fredholm theory (Theorem 2.19) it is invertible for any \( z \) except a discrete set of points, where the family of operators \( (I + T_z)^{-1} \) has finite rank residues. So, we have that

\[
(I + T_z)^{-1} Q_z(\Delta - z^2 I) = I,
\]

and \( (I + T_z)^{-1} Q_z \) is a meromorphic continuation of the resolvent of \( \Delta \).
where $K$ is a compact operator, since its integral kernel is smooth and has compact support in the second variable. Hence,

$$Q_k' = (I + K)(\Delta + I)^{-k}. \quad (4.4)$$

Now consider that operator $Q''_k$ defined as

$$Q''_k = \nu_1 R^k_{-i}(\Delta)\chi_1 + \nu_2 R^k_{-i}(\Delta_0)\chi_2.$$

By the same argument as before, there is

$$Q''_k(\Delta + I)^k = I + \tilde{K}, \quad (4.5)$$

where $\tilde{K}$ is a again compact operator, since its integral kernel is smooth and has compact support in the second variable as well. If we subtract (4.4) and (4.5), we get

$$\nu_2 \left( R^k_{-i}(\Delta) - R^k_{-i}(\Delta_0) \right)\chi_2 = (K - \tilde{K})(\Delta + I)^{-k}.$$

Since $\nu_2 = 1 - r$ and $\chi_2 = 1 - \tilde{r}$, for some smooth compactly supported functions $r$ and $\tilde{r}$, the above equation becomes

$$R^k_{-i}(\Delta) - R^k_{-i}(\Delta_0) = (K - \tilde{K})(\Delta + I)^{-k} - r \left( R^k_{-i}(\Delta) - R^k_{-i}(\Delta_0) \right)\tilde{r} + r \left( R^k_{-i}(\Delta) - R^k_{-i}(\Delta_0) \right)\tilde{r}.$$

Since the family of trace class operators forms an ideal, and since $k$ can be chosen to be arbitrary large, the proof follows from the following standard statement.

**Proposition 4.3.** Consider an operator $A \in \Psi^{-m}(X, E)$ mapping from $C^\infty_0(X, E)$ to $L^2(X, E)$, for some $n$ dimensional manifold $X$, having possibly a boundary, and a smooth complex vector bundle $E$ on $X$. If $m > n$ and the kernel of $A$ has compact support either in the first or in the second variable, then $A$ is trace class.

**Proof.** Assume first that the kernel of $A$ has compact support in the second variable. Then, the kernel of the operator $A^* \in \Psi^{-m}(X, E)$ has compact support in the first variable, so $A^*$ is densely defined and hence $A$ is closeable (Theorem VIII.1 in [39]). Slightly abusing notation we denote its closure by the same symbol. Since the kernel of the operator $A^*A \in \Psi^{-2m}(X, E)$ has compact support in both variables, we can assume that $A^*A \in \Psi^{-2m}(\tilde{X}, E)$, for some compact manifold $\tilde{X} \subset X$, having possibly a boundary. Consider any elliptic invertible
self-adjoint operator $C \in \Psi^1(\tilde{X}, E)$ (e.g. $(\Delta_E + I)^{1/2}$, where $\Delta_E = \nabla_E^* \nabla_E$ is a Laplace operator on $E$), and take an orthonormal $L^2(\tilde{X}, E)$ eigenbasis $\{\phi_i\}$ of $C$, with corresponding eigenvalues $\{\lambda_i\}$. By the Schwarz inequality we have

$$(\phi_i, \sqrt{A^*A}\phi_i) \leq \sqrt{(\phi_i, \phi_i)} \cdot \sqrt{(\sqrt{A^*A}\phi_i, \sqrt{A^*A}\phi_i)}.$$ 

So,

$$(\phi_i, \sqrt{A^*A}\phi_i) \leq \sqrt{(\phi_i, A^*A\phi_i)} = \sqrt{|\lambda_i|^{-2m}(\phi_i, C^mA^*AC^m\phi_i)} \leq c|\lambda_i|^{-m},$$

for some constant $c$, where at the last step we used the fact that $C^mA^*AC^m$ is a pseudodifferential operator of zero order, and hence bounded, in $\tilde{X}$. Thus, by the last inequality we obtain

$$\text{tr}|A| = \sum_i (\phi_i, \sqrt{A^*A}\phi_i) \leq c \sum_i |\lambda_i|^{-m} < \infty,$$

where the last bound follows from the fact that since $C^{-m} \in \Psi^{-m}(\tilde{X}, E)$ with $m > n$, it follows by [34] (Proposition 2.1) if $\tilde{X}$ has no boundary and by [1] (page 20, (4)) if $\tilde{X}$ has boundary that $C^{-m}$ is trace class. Hence, $|A|$ is trace class. By the polar decomposition (Theorem VIII.32 in [39]), there exists a partial isometry $U$ in $L^2(X, E)$ such that $A = U|A|$, from which it follows that $A$ is trace class.

If the kernel of $A$ has compact support in the first variable, then the closed operator $A^* \in \Psi^{-m}(X, E)$ has kernel compactly supported in the second variable. By the above argument, $|A^*|$ is trace class, and hence by the polar decomposition of $A^*$, we get that $A$ is trace class as well. $\square$
4.2 Generalized eigenforms and scattering matrix

We can construct now a generalized eigenform of the extension $\Delta$ of the Laplacian $d\delta + \delta d$ defined on $\Omega^0_M$, satisfying some properties. Since we have a general solution of the equation $(\Delta_0|_{M_1} - \lambda^2 I)\omega = 0$ on the continuous subspace (Theorem 3.12), we can follow some standard construction technique to get a generalized eigenform of $\Delta$ on $M$. By expanding the last one on the cusp $M_1$, the scattering matrix will appear. We start by the following

**Definition 4.4.** Let $\mathcal{H}^p(N)$ denote the space of $L^2$ harmonic $p$-forms on $N$, and let $\mathcal{R}_L(\Delta)$ be the meromorphic continuation of the resolvent of $\Delta$ to the Riemann surface $\mathcal{S}$, obtained by Theorem 4.1. Take $\theta \oplus \bar{\theta} \in \mathcal{H}^p(N) \oplus \mathcal{H}^{p-1}(N)$ and some $\chi(x) \in C^\infty(\mathbb{R})$, such that $\chi(x) = 0$ if $x < 1$ and $\chi(x) = 1$ if $x > 1 + \varepsilon$, for some $\varepsilon > 0$ sufficiently small. Let

$$\omega_\lambda = \chi(x) \left( x^{bp} H^{(1)}_{bp}(\lambda x) \theta + dx \wedge x^{bp-1} H^{(1)}_{bp-1-1}(\lambda x) \bar{\theta} \right)$$

and $\tilde{\omega}_\lambda = (\Delta - \lambda^2 I)\omega_\lambda$,

where $H^{(1)}_b$ denotes the Hankel function of the first kind of order $b$. We define for $\lambda \in \mathcal{S}$ and $y \in M$ the $p$-form $E_\lambda(y, \theta \oplus \bar{\theta}) = \omega_\lambda - \mathcal{R}_L(\Delta)\tilde{\omega}_\lambda$.

By its construction, $E_\lambda(y, \theta \oplus \bar{\theta})$ is smooth over $y \in M$ and meromorphic over $\lambda \in \mathcal{S}$. Note that for $\lambda^2$ in the resolvent set, i.e. $\Re \lambda < 0$, we have that $\omega_\lambda/\chi \notin L^2(M_1, \wedge^p T^* M_1)$. Thus, for $\Re \lambda < 0$, we have that $\omega_\lambda \notin L^2(M, \wedge^p T^* M)$, which implies that $\omega_\lambda \notin \text{Dom}(\Delta)$. So, $E_\lambda(y, \theta \oplus \bar{\theta})$ is not identically zero. Also, by Theorem 3.12, $\tilde{\omega}_\lambda = 0$ for $x > 1 + \varepsilon$. Hence, $\tilde{\omega}_\lambda$ is compactly supported, so it is in the domain of the resolvent $R_L(\Delta)$ of $\Delta$, and $E_\lambda(y, \theta \oplus \bar{\theta})$ is well defined. Since $R_L(\Delta)$ and $\mathcal{R}_L(\Delta)$ coincide for $\Re \lambda < 0$, we have $(\Delta - \lambda^2 I)E_\lambda(y, \theta \oplus \bar{\theta}) = 0$ when $\Re \lambda < 0$, and by the meromorphicity of $E_\lambda(y, \theta \oplus \bar{\theta})$ over $\lambda$, we have that $(\Delta - \lambda^2 I)E_\lambda(y, \theta \oplus \bar{\theta}) = 0$ for all $\lambda \in \mathcal{S}$. Thus, $E_\lambda(y, \theta \oplus \bar{\theta})$ is a generalized eigenform of the Laplacian, and when $\Re \lambda < 0$, we have that $E_\lambda(y, \theta \oplus \bar{\theta}) = \omega_\lambda \in L^2(M, \wedge^p T^* M)$. Finally, note that in the above definition, $E_\lambda(y, \theta \oplus \bar{\theta})$ does not depend on the cut-off function $\chi$. To see this, assume that $E_\lambda(y, \theta \oplus \bar{\theta})$ depends on $\chi$, and take the difference $\mathcal{E} = E_\lambda(y, \theta \oplus \bar{\theta}, \chi) - E_\lambda(y, \theta \oplus \bar{\theta}, \chi)$. When $\lambda^2$ is in the resolvent set, i.e. $\Re \lambda < 0$, we have that $\Delta \mathcal{E} = \lambda^2 \mathcal{E}$ and that $\mathcal{E} \in L^2(M, \wedge^p T^* M)$, which implies that $\mathcal{E} = 0$. By the meromorphic dependence of $\mathcal{E}$ over $\lambda$, we get that $\mathcal{E} = 0$ everywhere in $\mathcal{S}$.

Let us now restrict $E_\lambda(y, \theta \oplus \bar{\theta})$ to the cusp $M_1$ and expand it for $x > 1 + \varepsilon$ and $\Re \lambda < 0$, according to the decomposition of Theorem 3.5. For any $\theta \oplus \bar{\theta} \in \mathcal{S}$, the scattering matrix appears.
\( \mathcal{H}^p(N) \oplus \mathcal{H}^{p-1}(N) \), we have that
\[
\chi(x) \left( x^{\beta_p} H^{(2)}_b(\lambda x) \theta + dx \wedge x^{\beta_{p-1}} H^{(2)}_{b_{p-1}-1}(\lambda x) \tilde{\theta} \right) \in L^2(M, \wedge T^* M),
\]
for \( \Im \lambda < 0 \). Thus, since the Hankel functions \( H^{(1)}_b(z) \) and \( H^{(2)}_b(z) \) form a fundamental system for the Bessel equation of order \( b \), according to Theorem 3.12, we have the following expansion for \( x > 1 + \varepsilon \)
\[
E_\lambda(y, \theta \oplus \tilde{\theta}) = x^{\beta_p} H^{(1)}_b(\lambda x) \theta + dx \wedge x^{\beta_{p-1}} H^{(1)}_{b_{p-1}-1}(\lambda x) \tilde{\theta} + x^{\beta_p} H^{(2)}_b(\lambda x) C_{p,\lambda}(\theta) + dx \wedge x^{\beta_{p-1}} H^{(2)}_{b_{p-1}-1}(\lambda x) \tilde{C}_{p-1,\lambda}(\tilde{\theta}) + \Psi_\lambda(y, \theta \oplus \tilde{\theta}),
\]
(4.6)
where
\[
\Psi_\lambda(y, \theta \oplus \tilde{\theta}) \in L^2_{d,\delta}(M_1, \wedge^p T^* M_1) \oplus L^2_{\delta,d}(M_1, \wedge^p T^* M_1), \quad \text{when } \Im \lambda < 0,
\]
and \( C_{p,\lambda} \), \( \tilde{C}_{p-1,\lambda} \) are some endomorphisms on the spaces \( \mathcal{H}^p(N) \) and \( \mathcal{H}^{p-1}(N) \) respectively. According to Theorem 3.12, \( C_{p,\lambda} \) is uniquely determined by the choice of the solution \( H^{(1)}_b(z) \) of the Bessel equation in the Definition 4.4. For some \( z \in \mathbb{C}, z \neq 0 \), if we compare \( E_\lambda(y, z\theta \oplus \tilde{\theta})/z \) and \( E_\lambda(y, \theta \oplus \tilde{\theta}) \), by using the above uniqueness, we will find that \( C_{p,\lambda} \) is linear. Also, if we consider the inner product \( (E_\lambda(y, \theta \oplus \tilde{\theta}) - \omega_\lambda, e^{-x^2}\theta)_M \) and make use of Lemma 3.2 and Theorem 3.5, we can see that \( (C_{p,\lambda}(\theta), \theta)_N \) is a meromorphic function of \( \lambda \in S \). Thus, \( C_{p,\lambda} \) is meromorphic for \( \lambda \in S \). Similar properties hold for \( \tilde{C}_{p-1,\lambda} \). Also, note that since \( \varepsilon \) in definition 4.4 can be arbitrary small, and since \( E_\lambda(y, \theta \oplus \tilde{\theta}) \) is independent of \( \chi \), the expansion (4.6) holds for \( x > 1 \).

We will prove now that the tail term \( \Psi_\lambda(y, \theta \oplus \tilde{\theta}) \) in the expansion of \( E_\lambda(y, \theta \oplus \tilde{\theta}) \) decays exponentially for any \( \lambda \in S \). Since we have that
\[
\Psi_\lambda(y, \theta \oplus \tilde{\theta}) = \Psi_{\lambda,d,\delta}(y, \theta \oplus \tilde{\theta}) + \Psi_{\lambda,\delta,d}(y, \theta \oplus \tilde{\theta}),
\]
with
\[
\Psi_{\lambda,d,\delta}(y, \theta \oplus \tilde{\theta}) \in L^2_{d,\delta}(M_1, \wedge^p T^* M_1) \quad \text{and} \quad \Psi_{\lambda,\delta,d}(y, \theta \oplus \tilde{\theta}) \in L^2_{\delta,d}(M_1, \wedge^p T^* M_1)
\]
when \( \Im \lambda < 0 \), we will show that each of \( \Psi_{\lambda,d,\delta}(y, \theta \oplus \tilde{\theta}) \), \( \Psi_{\lambda,\delta,d}(y, \theta \oplus \tilde{\theta}) \) has this property. We will do it for the term \( \Psi_{\lambda,d,\delta}(y, \theta \oplus \tilde{\theta}) \), since the proof for the other term turns out to be a special case of the proof we will give. Since \( \Psi_{\lambda,d,\delta}(y, \theta \oplus \tilde{\theta}) \) satisfies the eigenvalue equation for \( \Delta|_{M_1} \), we have that if according to Lemma 3.9, \( \Psi_{\lambda,d,\delta}(y, \theta \oplus \tilde{\theta}) = \sum_i \alpha_i \phi_i^d + dx \wedge \beta_i \tilde{\phi}_i^d \), then the coefficients \( \alpha_i, \beta_i \) satisfy the system
where the system becomes

\[ Y \]

and denoting by \( \lambda_i \) the eigenvalues of \( A \).

By putting \( y_{i,1}(x) = w_i(x), \ y_{i,2}(x) = h_i(x), \ y_{i,3}(x) = w_i'(x) \) and \( y_{i,4}(x) = h_i'(x) \), and denoting by \( Y_i \) the vector \( (y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4})^T \), we get the equivalent first order system

\[ Y_i' = A_i Y_i, \quad \text{where} \quad A_i = \begin{pmatrix} 0 & I \\ V_i & 0 \end{pmatrix} \quad \text{and} \quad V_i = \begin{pmatrix} V_i & f_i \\ f_i & U_i \end{pmatrix}. \]

The eigenvalues of \( A_i \), indexed by \( j \), and their corresponding eigenvectors are given by

\[ \lambda_j = \pm \sqrt{\frac{V_i + U_i \pm \sqrt{(V_i - U_i)^2 + 4f_i^2}}{2}} \quad \text{and} \quad s_{i,j} = (1, \frac{V_i - \lambda_j^2}{f_i}, \lambda_j, \frac{V_i - \lambda_j^2}{f_i})^T \]

respectively. Let the matrix \( S_i = (s_{i,1}, s_{i,2}, s_{i,3}, s_{i,4}) \), which diagonalizes \( A_i \). If we apply to the system (4.9) the transformation \( Y_i = S_iQ_i \), for some \( Q_i = (q_{i,1}(x), q_{i,2}(x), q_{i,3}(x), q_{i,4}(x))^T \), it becomes

\[ Q_i' = (S_i^{-1}A_iS_i - S_i^{-1}S_i')Q_i. \]

It is easy to see that the diagonal matrix \( S_i^{-1}A_iS_i \) can be written in the form \((B_i + J_i)x^a\), where \( B_i \) is diagonal with diagonal elements \( \mu_i, \mu_i, -\mu_i \) and \(-\mu_i\), with \( \mu_i > 0 \), and the Hilbert-Schmidt norm of \( J_i \) goes to zero when \( x \to \infty \). Also, if we explicitly calculate the Hilbert-Schmidt norm of the matrix \( S_i^{-1}S_i' \), we can see that it is of \( o(x^a) \). Hence, if we apply the transformation \( t = x^{a+1/2} + 1 \) to (4.11), it becomes

\[ \frac{dQ_i}{dt} = (B_i + K_i)Q_i, \]

where the Hilbert-Schmidt norm of \( K_i \) goes to zero when \( t \to \infty \). We can now use the following theorem from [38]
Theorem 4.5. (Perron) Consider the first order $n$-dimensional system

$$
\frac{dY(t)}{dt} = (B + K(t))Y(t),
$$

where $Y(t)$ is a column vector and $B, K(t)$ are (possible complex valued) matrices such that $B$ is independent of $t$, and the Hilbert-Schmidt norm of $K(t)$ goes to zero when $t \to \infty$ (almost diagonal system). Then, the system has $n$ independent solutions $Y_i, i \in \{1, ..., n\}$ such that if $|Y_i|$ is the length of the vector $Y_i$, then

$$
\lim_{t \to \infty} t^{-1} \log |Y_i| = \rho_i,
$$

where $\rho_i = \Re \lambda_i$, and $\lambda_i$ are the $n$ eigenvalues of $B$.

From the above theorem, we get that any solution $h_i(x), w_i(x)$ satisfies

$$
\sqrt{|w_i(x)|^2 + |h_i(x)|^2 + |w_i'(x)|^2 + |h_i'(x)|^2} \sim e^{\frac{\mu_i}{a+1}x^{a+1}}.
$$

If we assume a solution with $+ \in$ in the above relation, and assume without loss of generality that $|w'_i| \sim e^{\frac{\mu_i}{a+1}x^{a+1}}$, then we have that $\Re w'_i \sim e^{\frac{\mu_i}{a+1}x^{a+1}}$ or $\Im w'_i \sim e^{\frac{\mu_i}{a+1}x^{a+1}}$. After an integration we get that $w_i \sim \int e^{\frac{\mu_i}{a+1}x^{a+1}}$. Hence, we have that in any solution of (4.7) either both $w_i, h_i$ decay exponentially, or at least one of them increases exponentially. Hence, without loss of generality, we can assume that the general solution for $w_i$ is a linear combination of an exponentially decreasing term and an exponentially increasing one. By taking the inner product $(E\lambda(y, \theta \oplus \tilde{\theta}) - \omega \lambda_i e^{-\frac{\mu_i}{a+1}x^{a+1}})M_1$ and using Lemma 3.2, we can see that $\alpha_i$, and hence $w_i$, depends meromorphically on $\lambda$. When $\Im \lambda < 0$ we have that $\Psi_{\lambda, \delta, d}(y, \theta \oplus \tilde{\theta}) \in L^2_\delta(M_1, \Lambda^\ast T^*M_1)$, which implies $\alpha_i(x) \in L^2([1, \infty), e^{-\gamma_0}dx)$, or that $w_i(x) \in L^2([1, \infty), dx)$. Hence, the exponentially increasing term of $w_i$ is zero and the exponentially decreasing term is meromorphic in $\lambda$, when $\Im \lambda < 0$. Thus, by the meromorphicity of $w_i$, we get that the exponentially increasing term is zero for any $\lambda \in \mathcal{S}$. So, we have that any solution for $h_i(x), w_i(x)$ must be of $O(e^{-\frac{\mu_i}{a+1}x^{a+1}})$, which gives $\alpha_i(x) = O(x^{a/2}e^{-\frac{\mu_i}{a+1}x^{a+1}})$ and $\beta_i(x) = O(x^{a/2}e^{-\frac{\mu_i}{a+1}x^{a+1}})$ when $\lambda \in \mathcal{S}$. If we do similar work with the term $\Psi_{\lambda, \delta, d}(y, \theta \oplus \tilde{\theta})$, we get the behavior of the tail term to be

$$
\Psi_{\lambda}(y, \theta \oplus \tilde{\theta}) = O(x^{a/2}e^{-\frac{\mu_i}{a+1}x^{a+1}}), \text{ when } \lambda \in \mathcal{S}.
$$

We can summarize the results up to this point to the following theorem
Theorem 4.6. The p-form $E_\lambda(y, \theta \oplus \tilde{\theta})$ from Definition 4.4 has the following properties

1) $E_\lambda(y, \theta \oplus \tilde{\theta})$ is smooth in $y \in M$ and meromorphic in $\lambda \in S$.
2) $(\Delta - \lambda^2 I)E_\lambda(y, \theta \oplus \tilde{\theta}) = 0$, for any $y \in M$ and $\lambda \in S$.
3) For $x > 1$ and $\lambda \in S$, we have the following expansion

$$E_\lambda(y, \theta \oplus \tilde{\theta}) = x^{b_p} H^{(1)}_{b_p}(\lambda x) \theta + dx \wedge x^{b_p-1} H^{(1)}_{b_p-1}(\lambda x) \tilde{\theta} + x^{b_p} H^{(2)}_{b_p}(\lambda x) C_{p,\lambda}(\theta) + dx \wedge x^{b_p-1} H^{(2)}_{b_p-1}(\lambda x) \tilde{C}_{p,\lambda}(\tilde{\theta}) + \Psi_\lambda(y, \theta \oplus \tilde{\theta}),$$

where

$$C_{p,\lambda} = \begin{pmatrix} C_{p,\lambda} & 0 \\ 0 & \tilde{C}_{p-1,\lambda} \end{pmatrix} \in \text{End}(H^p(N) \oplus H^{p-1}(N))$$

is linear, meromorphic in $\lambda \in S$, and is called the (stationary) scattering matrix associated to $E_\lambda(y, \theta \oplus \tilde{\theta})$. For the tail term we have that

$$\Psi_\lambda(y, \theta \oplus \tilde{\theta}) = O(x^{b_p-1/2} e^{-\frac{\mu}{b_p+1} x^{b_p+1}}), \quad \forall \lambda \in S,$$

where $\mu > 0$ is the square root of the smallest nonzero eigenvalue of the p-form Laplacian of the boundary $N$. Also, $E_\lambda(y, \theta \oplus \tilde{\theta})$, $C_{p,\lambda}$ and $\Psi_\lambda(y, \theta \oplus \tilde{\theta})$ are uniquely determined by the above properties.

We will give now the relation between the stationary scattering matrix defined in the previous theorem and the dynamical scattering matrix associated to the Laplacians $\Delta$ and $\Delta_0$.

Lemma 4.7. If $S_{p,\lambda} \in \text{End}(H^p(N) \oplus H^{p-1}(N))$ is the (dynamical) scattering matrix associated to the Laplacians $\Delta$ and $\Delta_0$, then following relation holds

$$S_{p,\lambda} = \begin{pmatrix} -H^{(2)}_{b_p}(\lambda) & 0 \\ H^{(2)}_{b_p}(\lambda) & -H^{(2)}_{b_p-1}(\lambda) \\ 0 & -H^{(1)}_{b_p-1}(\lambda) \end{pmatrix} C_{p,\lambda},$$

for any $\lambda \in S$.

Proof. The existence and the completeness of the wave operators associated to the Laplacians $\Delta$ and $\Delta_0$, are guaranteed by Lemma 4.2. The wave operators, given by

$$W^\pm(\Delta, \Delta_0) = s - \lim_{t \to \mp \infty} e^{i \Delta t} e^{-i \Delta_0 t} P_{ac}(\Delta_0),$$

and acting on the absolutely continuous subspace $H_{ac}(\Delta_0) = L^2(H(M_1, \wedge^p T^* M_1)$, of the decomposition (4.1), allow us to define the scattering operator associated to the Laplacians $\Delta$ and $\Delta_0$, by

$$S(\Delta, \Delta_0) = (W^-(\Delta, \Delta_0))^* W^+(\Delta, \Delta_0),$$
which is a unitary operator in $L^2(M_4, \wedge^p T^* M_4)$ and commutes with $\Delta_0$. Let us consider the generalized eigenform of $\Delta_{0|M_4}$ given by

$$E_{0,\lambda}(y, \theta + \tilde{\theta}) = x^{bp} \left( H_{bp}^{(2)}(\lambda) H_{bp}^{(1)}(\lambda x) - H_{bp}^{(1)}(\lambda) H_{bp}^{(2)}(\lambda x) \right) \theta$$

$$+ dx \wedge x^{bp-1} \left( H_{bp-1}^{(2)}(\lambda) H_{bp-1}^{(1)}(\lambda x) - H_{bp-1}^{(1)}(\lambda) H_{bp-1}^{(2)}(\lambda x) \right) \tilde{\theta}.$$

Since

$$\Delta W^+(\Delta, \Delta_0) P_{ac}(\Delta_0) = W^+(\Delta, \Delta_0) \Delta_0 P_{ac}(\Delta_0),$$

we have that

$$W^+(\Delta, \Delta_0) E_{0,\lambda}(y, \theta + \tilde{\theta}) = E_{\lambda}(y, \theta^\pm + \tilde{\theta}^\pm) \text{ for any } \lambda \in \mathbb{R},$$

and for some $\theta^\pm + \tilde{\theta}^\pm \in \mathcal{H}^p(N) \oplus \mathcal{H}^{p-1}(N)$ depending on $\lambda$, where in the above relation $E_{0,\lambda}(y, \theta + \tilde{\theta})$ and $E_{\lambda}(y, \theta^\pm + \tilde{\theta}^\pm)$ are to be understood distributional in $\lambda$. This means that these relations become relations between $L^2$ functions if smeared out with smooth compactly supported functions in $\lambda$, with respect to the spectral measure. In the same way as in [17] (Theorem 6.2), one has

$$W^+(\Delta, \Delta_0) E_{0,\lambda}(y, \theta + \tilde{\theta}) =$$

$$\chi(x) E_{0,\lambda}(y, \theta + \tilde{\theta}) - \lim_{\epsilon \to 0} R_{\pm (\lambda - \text{sgn}(\lambda)i\epsilon)}(\Delta)(\Delta - \lambda^2 I) \chi(x) E_{0,\lambda}(y, \theta + \tilde{\theta}), \quad (4.12)$$

for any $\lambda \in \mathbb{R}$. Hence, by considering the case with $+$, and assuming that $\lambda > 0$, which implies that we approach $\lambda$ from the lower half-plane, if we compare the non $L^2$ components in (4.12) for $x > 1$, we obtain that

$$\theta^+ + \tilde{\theta}^+ = \left( H_{bp}^{(2)}(\lambda) \theta \right) \oplus \left( H_{bp-1}^{(2)}(\lambda) \tilde{\theta} \right), \quad (4.13)$$

for any $\lambda \in \mathcal{S}$. Since $W^-(\Delta, \Delta_0) = \overline{W^+(\Delta, \Delta_0)}$, for $\lambda \in \mathbb{R}$ and $x > 1$ we have

$$\overline{W^+(\Delta, \Delta_0) E_{0,\lambda}(y, \theta + \tilde{\theta})} = W^-(\Delta, \Delta_0) E_{0,\lambda}(y, \theta + \tilde{\theta}) =$$

$$x^{bp} H_{bp}^{(1)}(\lambda) H_{bp}^{(2)}(\lambda x) \tilde{\theta} + dx \wedge x^{bp-1} H_{bp-1}^{(1)}(\lambda) H_{bp-1}^{(2)}(\lambda x) \tilde{\theta}$$

$$+ x^{bp} H_{bp}^{(1)}(\lambda) H_{bp}^{(1)}(\lambda x) \overline{c_{p,\lambda}(\theta)} + dx \wedge x^{bp-1} H_{bp-1}^{(1)}(\lambda) H_{bp-1}^{(1)}(\lambda x) \overline{c_{p-1,\lambda}(\tilde{\theta})}$$

$$+ \Psi_{\lambda}(y, \theta + \tilde{\theta}).$$

Since for $\lambda \in \mathbb{R}$ we also have that $\overline{E_{0,\lambda}(y, \theta + \tilde{\theta})} = -E_{0,\lambda}(y, \theta + \tilde{\theta})$, from the uniqueness from Theorem 4.6 and last equation we obtain

$$\theta^- + \tilde{\theta}^- = \left( \begin{array}{cc} -H_{bp}^{(1)}(\lambda) & 0 \\ 0 & -H_{bp-1}^{(1)}(\lambda) \end{array} \right) \overline{c_{p,\lambda}(\theta + \tilde{\theta})},$$

for any $\lambda \in \mathcal{S}$.
for any $\lambda \in \mathcal{S}$. If we use the fact that $\tilde{C}_{p,\lambda} \circ C_{p,\lambda} = I$ for any $\lambda \in \mathbb{R}$, which comes from Theorem 4.9 of the next section, we get also that

$$\theta \oplus \tilde{\theta} = \begin{pmatrix} \frac{-1}{H_{bp}(\lambda)} & 0 \\ 0 & \frac{-1}{H_{bp-1}(\lambda)} \end{pmatrix} C_{p,\lambda}(\theta \oplus \tilde{\theta}^-), \quad (4.14)$$

valid for any $\lambda \in \mathcal{S}$. Since $W_-(\Delta, \Delta_0)$ is a partial isometry, we have that $(W_-(\Delta, \Delta_0))^* = (W_-(\Delta, \Delta_0))^{-1}$. Hence,

$$(W_-(\Delta, \Delta_0))^* E_{\lambda}(y, \theta^- \oplus \tilde{\theta}^-) = E_{0,\lambda}(y, \theta \oplus \tilde{\theta}).$$

By combining the above relation together with (4.13) and (4.14), we find that the scattering operator satisfies the following

$$S(\Delta, \Delta_0) E_{0,\lambda}(y, \theta \oplus \tilde{\theta}) = E_{0,\lambda}(y, \eta \oplus \tilde{\eta}),$$

for any $\lambda \in \mathcal{S}$, where

$$\eta \oplus \tilde{\eta} = \begin{pmatrix} \frac{-1}{H_{bp}(\lambda)} & 0 \\ 0 & \frac{-1}{H_{bp-1}(\lambda)} \end{pmatrix} C_{p,\lambda}(\theta \oplus \tilde{\theta}).$$

Hence, if $S_{p,\lambda} \in \text{End}(\mathcal{H}^p(N) \oplus \mathcal{H}^{p-1}(N))$ is the (dynamical) scattering matrix, we have the following relation

$$S_{p,\lambda} = \begin{pmatrix} \frac{-1}{H_{bp}(\lambda)} & 0 \\ 0 & \frac{-1}{H_{bp-1}(\lambda)} \end{pmatrix} C_{p,\lambda},$$

for any $\lambda \in \mathcal{S}$. \qed
4.3 Properties of the scattering matrix

In this section, we use Theorem 4.6 in order to find the properties of the scattering matrix. We can prove that $C_{p,\lambda}$ is a unitary endomorphism and also we can find its functional equation. The main idea for the proof of the unitarity is the well known behavior of $E_\lambda(y, \theta \oplus \tilde{\theta})$ at infinity, on the noncompact part of $M$. For the functional equation, the proof is based on the uniqueness from Theorem 4.6. Also, we can use this uniqueness to find the commutation relation between $C_{p,\lambda}$ and the Hodge star operator. Next, we define the conjugate $\bar{\lambda}$ of $\lambda$ in such a way that if $\lambda \in S$ lies in the $k$-leaf, then $\bar{\lambda}$ lies in the $-k$-leaf.

**Theorem 4.8.** (unitarity) $C_{p,\lambda}^* \circ C_{p,\lambda} = I$ for all $\lambda \in S$.

**Proof.** Consider the manifold $M_t = M_0 \cup ([1, t) \times N)$, for some $t > 1$, together with the inner product $(\cdot, \cdot)_{M_t}$ induced by the metric $g$ when it is restricted to $M_t$, namely $(v, w)_{M_t} = \int_{M_t} v \wedge w$, for any $v, w \in \Omega^p(M_t)$. Since $E_\lambda(y, \theta \oplus \tilde{\theta})$ is in the kernel of $\Delta - \lambda^2 I$, we have the following equality

$$
\left( (\Delta - \bar{\lambda}^2 I)E_\lambda(y, \theta \oplus \tilde{\theta}), E_\lambda(y\theta \oplus \tilde{\theta}) \right)_{M_t} = \left( E_\lambda(y, \theta \oplus \tilde{\theta}), (\Delta - \lambda^2 I)E_\lambda(y, \theta \oplus \tilde{\theta}) \right)_{M_t},
$$

which gives

$$
\left( \Delta E_\lambda(y, \theta \oplus \tilde{\theta}), E_\lambda(y, \theta \oplus \tilde{\theta}) \right)_{M_t} - \left( E_\lambda(y, \theta \oplus \tilde{\theta}), \Delta E_\lambda(y, \theta \oplus \tilde{\theta}) \right)_{M_t} = 0.
$$

For any $v, w \in \Omega^p(M_t)$ we have the following Green’s formula (see [8])

$$
(\Delta u, w)_{M_t} - (u, \Delta w)_{M_t} = \int_{\partial M_t} \vec{u} \wedge *dw - \vec{w} \wedge *du + \delta \vec{u} \wedge *w - \delta \vec{w} \wedge *u.
$$

If we apply this to the previous equation we get

$$
\int_{\partial M_t} \vec{E}_\lambda(y, \theta \oplus \tilde{\theta}) \wedge *dE_\lambda(y, \theta \oplus \tilde{\theta}) - E_\lambda(y, \theta \oplus \tilde{\theta}) \wedge *d\vec{E}_\lambda(y, \theta \oplus \tilde{\theta})
$$

$$
+ \delta \vec{E}_\lambda(y, \theta \oplus \tilde{\theta}) \wedge *E_\lambda(y, \theta \oplus \tilde{\theta}) - \delta E_\lambda(y, \theta \oplus \tilde{\theta}) \wedge *\vec{E}_\lambda(y, \theta \oplus \tilde{\theta}) = 0. \quad (4.15)
$$

Let us use the notations

$$
f_{1,\lambda}(x) = x^{bp}H_{bp}^{(1)}(\lambda x), \quad f_{2,\lambda}(x) = x^{bp}H_{bp}^{(2)}(\lambda x),
$$

$$
g_{1,\lambda}(x) = x^{bp-1}H_{bp-1}^{(1)}(\lambda x) \quad \text{and} \quad g_{2,\lambda}(x) = x^{bp-1}H_{bp-1}^{(2)}(\lambda x).
$$

Then,

$$
E_\lambda(y, \theta \oplus \tilde{\theta}) = f_{1,\lambda}\theta + f_{2,\lambda}C_{p,\lambda}(\tilde{\theta}) + g_{1,\lambda}dx \wedge \tilde{\theta} + g_{2,\lambda}dx \wedge \tilde{C}_{p-1,\lambda}(\tilde{\theta}) + \Psi_p(y, \theta \oplus \tilde{\theta}),
$$
for $x > 1$. By Equations (3.2) and (3.3), Lemma 3.1 and Theorem 4.6, we have

$$dE_\lambda(y, \theta + \tilde{\theta}) = dx \wedge \left( \partial_x f_{1, \lambda} \theta + \partial_x f_{2, \lambda} C_{p, \lambda}(\theta) \right) + \mathcal{O}(x^{\gamma_p - 1/2}e^{-\frac{\mu}{\lambda + \mu + 1}}x^{n+1}),$$

$$*dE_\lambda(y, \theta + \tilde{\theta}) = x^{-\gamma_p - 1} * N \left( \partial_x f_{1, \lambda} \theta + \partial_x f_{2, \lambda} C_{p, \lambda}(\theta) \right) + \mathcal{O}(x^{\gamma_p - 1/2}e^{-\frac{\mu}{\lambda + \mu + 1}}x^{n+1}),$$

$$*E_\lambda(y, \theta + \tilde{\theta}) = x^{-\gamma_p - 1} * N \left( g_{1, \lambda} \tilde{\theta} + g_{2, \lambda} \tilde{C}_{p, \lambda}(\tilde{\theta}) \right) + (-1)^p x^{-\gamma_p} dx \wedge * N \left( f_{1, \lambda} \theta + f_{2, \lambda} C_{p, \lambda}(\theta) \right) + \mathcal{O}(x^{\gamma_p - 1/2}e^{-\frac{\mu}{\lambda + \mu + 1}}x^{n+1})$$

and

$$\delta E_\lambda(y, \theta + \tilde{\theta}) = \gamma_p \frac{1}{x} \left( g_{1, \lambda} \tilde{\theta} + g_{2, \lambda} \tilde{C}_{p, \lambda}(\tilde{\theta}) \right) - \left( \partial_x g_{1, \lambda} \tilde{\theta} + \partial_x g_{2, \lambda} \tilde{C}_{p, \lambda}(\tilde{\theta}) \right) + \mathcal{O}(x^{\gamma_p - 1/2}e^{-\frac{\mu}{\lambda + \mu + 1}}x^{n+1}),$$

where $\gamma_p$ and $b_p$ are defined in Lemma 3.2 and Theorem 3.12. Hence, Equation (4.15) becomes

$$\int_{N, x=1} \left( \bar{f}_{1, \lambda} \tilde{\theta} + \bar{f}_{2, \lambda} \bar{C}_{p, \lambda}(\theta) \right) \wedge x^{-\gamma_p - 1} * N \left( \partial_x f_{1, \lambda} \theta + \partial_x f_{2, \lambda} C_{p, \lambda}(\theta) \right)$$

$$\left( f_{1, \lambda} \theta + f_{2, \lambda} C_{p, \lambda}(\theta) \right) \wedge x^{-\gamma_p - 1} * N \left( \partial_x \bar{f}_{1, \lambda} \tilde{\theta} + \partial_x \bar{f}_{2, \lambda} \bar{C}_{p, \lambda}(\tilde{\theta}) \right)$$

$$+ \left( \frac{\gamma_p - 1}{x} \left( \bar{g}_{1, \lambda} \tilde{\theta} + \bar{g}_{2, \lambda} \bar{C}_{p, \lambda}(\tilde{\theta}) \right) - \left( \partial_x \bar{g}_{1, \lambda} \tilde{\theta} + \partial_x \bar{g}_{2, \lambda} \bar{C}_{p, \lambda}(\tilde{\theta}) \right) \right)$$

$$\wedge x^{-\gamma_p - 1} * N \left( \bar{g}_{1, \lambda} \bar{\tilde{\theta}} + \bar{g}_{2, \lambda} \bar{C}_{p, \lambda}(\bar{\tilde{\theta}}) \right)$$

$$\left( \partial_x g_{1, \lambda} \tilde{\theta} + \partial_x g_{2, \lambda} \tilde{C}_{p, \lambda}(\tilde{\theta}) \right)$$

$$\wedge x^{-\gamma_p - 1} * N \left( \bar{g}_{1, \lambda} \bar{\tilde{\theta}} + \bar{g}_{2, \lambda} \bar{C}_{p, \lambda}(\bar{\tilde{\theta}}) \right) + \mathcal{O}(t^{\gamma_p - 1/2}e^{-\frac{\mu}{\lambda + \mu + 1}}t^{n+1}) = 0 \tag{4.16}$$

It is enough to prove the unitarity property when $\lambda$ is in the zero leaf, since then, by the meromorphicity of $C_{p, \lambda}$ over $\lambda$, and the holomorphicity of the function $z \to \bar{z}$, it will hold everywhere in $\mathcal{S}$. Hence, if we take $\lambda$ to be in the zero leaf, then by the Equations (5.5) and (5.6) of the appendix we can obtain the following asymptotic behaviors

$$f_{1, \lambda} = \sqrt{\frac{2}{\pi}} e^{ib_p x} e^{i\lambda x} \sqrt{\lambda x} (1 + \mathcal{O}(x^{-1})), \quad f_{2, \lambda} = \sqrt{\frac{2}{\pi}} e^{-ib_p x} e^{-i\lambda x} \sqrt{\lambda x} (1 + \mathcal{O}(x^{-1})), $$

$$\partial_x f_{1, \lambda} = \lambda \sqrt{\frac{2}{\pi}} e^{ib_{p-1} x} e^{i\lambda x} \sqrt{\lambda x} (1 + \mathcal{O}(x^{-1}))$$
and
\[ \partial_x f_{2,\lambda} = \lambda \sqrt{\frac{2}{\pi}} e^{-i\theta_1} x^{b_p} e^{-i\lambda x} \left(1 + \mathcal{O}(x^{-1})\right), \]
and similarly for \( g_{1,\lambda}, g_{2,\lambda} \) and their derivatives. If we take now the limit of the Equation (4.16) when \( t \) goes to infinity and use the previous asymptotic expansions, we notice that all the mixed terms in Equation (4.16) cancel out. The non mixed terms remain only, and they give the following relation
\[ \frac{4i}{\pi} \left( (\hat{\theta}, \bar{\theta})_N - (\bar{C}_p_{1,\lambda}(\hat{\theta}), \bar{C}_p_{1,\lambda}(\bar{\theta}))_N \right) + \frac{4i}{\pi} \left( (\theta, \theta)_N - (C_{p,\lambda}(\theta), C_{p,\lambda}(\theta))_N \right) t^{-2a} \]
\[ + \mathcal{O}(t^{|p-1|} e^{-2\pi \frac{1}{t} t^{1+1}}) = 0 \]
when \( t \to \infty \), where \( a \) is the parameter of the generalized cusp. Since the first two coefficients in the above expansion must be zero, we get the unitarity of \( C_{p,\lambda} \).

Next, we use uniqueness of \( E(\lambda, y, \phi, \psi) \) from Theorem 4.6, in order to prove the functional equation for \( C_{p,\lambda} \) and some other properties as well. For \( \lambda \in \mathcal{S} \) in the \( k \)-leaf, by \(-\lambda\) we mean rotation of the point \( \lambda \) down to the logarithmic cover by angle \( \pi \).

**Theorem 4.9.** For any \( \lambda \in \mathcal{S} \), we have \( \bar{C}_{p,\lambda} \circ C_{p,\lambda} = I \). Also, the scattering matrix satisfies the following functional equation
\[ (C_{p,-\lambda} - I) \circ C_{p,\lambda} = \left( \begin{array}{cc} e^{-i\pi \gamma_p} & 0 \\ 0 & e^{-i\pi \gamma_{p-1}} \end{array} \right) (I - C_{p,\lambda}). \]

**Proof.** Since \( \bar{H}_{b_{1}}(\lambda x) = H_{b_{2}}(\bar{\lambda}x) \) and \( \bar{H}_{b_{2}}(\lambda x) = H_{b_{1}}(\bar{\lambda}x) \), by Theorem 4.6, for \( x > 1 \) we get
\[ E_\lambda(y, \theta \oplus \bar{\theta}) = x^{b_p} H_{b_{2}}(\bar{\lambda}x) \bar{\theta} + dx \wedge x^{b_{p-1}} H_{b_{p-1}}(\bar{\lambda}x) \bar{\theta} \]
\[ + x^{b_p} H_{b_{2}}^{(1)}(\bar{\lambda}x) C_{p,\lambda}(\theta) + dx \wedge x^{b_{p-1}} H_{b_{p-1}}^{(1)}(\bar{\lambda}x) C_{p,\lambda}^{(-1,\lambda)}(\bar{\theta}) + \Psi_\lambda(y, \theta \oplus \bar{\theta}), \]
and
\[ E_\lambda(y, \hat{C}_{p,\lambda}(\theta) \oplus C_{p,\lambda}(\theta)) = \]
\[ x^{b_p} H_{b_{2}}^{(1)}(\bar{\lambda}x) C_{p,\lambda}(\theta) + dx \wedge x^{b_{p-1}} H_{b_{p-1}}^{(1)}(\bar{\lambda}x) C_{p,\lambda}(\theta) \]
\[ + x^{b_p} H_{b_{2}}^{(1)}(\bar{\lambda}x) C_{p,\lambda} \circ C_{p,\lambda}(\theta) + dx \wedge x^{b_{p-1}} H_{b_{p-1}}^{(2)}(\bar{\lambda}x) \hat{C}_{p,\lambda} \circ C_{p,\lambda}^{(-1,\lambda)}(\theta) \]
\[ + \Psi_\lambda(y, \hat{C}_{p,\lambda}(\theta) \oplus C_{p,\lambda}(\theta)). \]
Comparing the above equations, by using uniqueness from Theorem 4.6, we get that \( C_{p,\lambda} \circ C_{p,\lambda}(\theta) = \hat{\theta} \) and \( \hat{C}_{p,\lambda} \circ C_{p,\lambda}(\theta) = \hat{\theta} \).
For the proof of the functional equation, we make use of the Equations (5.3) of the appendix. By Theorem 4.6, we find

\[
E_{-\lambda}(y, \theta \oplus \tilde{\theta}) = \frac{1}{2} \left( 2 \cos \pi b_p \cdot \theta - e^{i\pi b_p} \cdot C_{p, -\lambda}(\theta) \right) + x_{b_p}^{y, \lambda} \mathcal{H}_{b_p}^{(1)}(\lambda x) \left( 2 \cos \pi b_p \cdot \tilde{\theta} - e^{i\pi b_p} \cdot \tilde{C}_{p-1, -\lambda}(\tilde{\theta}) \right) + d x \wedge x_{b_p-1}^{y, \lambda} \mathcal{H}_{b_p-1}^{(2)}(\lambda x) e^{-i\pi b_p \tilde{\theta}} + d x \wedge x_{b_p-1}^{y, \lambda} \mathcal{H}_{b_p-1}^{(2)}(\lambda x) e^{-i\pi b_p \tilde{\theta}} + \Psi_{-\lambda}(y, \theta \oplus \tilde{\theta}).
\]

We also have

\[
E_{\lambda}(y, e^{-i\pi b_p} C_{p, \lambda}(\theta) \oplus e^{-i\pi b_p} \tilde{C}_{p-1, \lambda}(\tilde{\theta})) = \frac{1}{2} \left( 2 \cos \pi b_p \cdot \theta - e^{i\pi b_p} \cdot C_{p, \lambda}(\theta) \right) + x_{b_p}^{y, \lambda} \mathcal{H}_{b_p}^{(1)}(\lambda x) e^{-i\pi b_p \tilde{\theta}} + d x \wedge x_{b_p-1}^{y, \lambda} \mathcal{H}_{b_p-1}^{(2)}(\lambda x) e^{-i\pi b_p \tilde{\theta}} + d x \wedge x_{b_p-1}^{y, \lambda} \mathcal{H}_{b_p-1}^{(2)}(\lambda x) e^{-i\pi b_p \tilde{\theta}} + \Psi_{\lambda}(y, e^{-i\pi b_p} C_{p, \lambda}(\theta) \oplus e^{-i\pi b_p} \tilde{C}_{p-1, \lambda}(\tilde{\theta})).
\]

By comparing the last two equations, we get by uniqueness that

\[
2 \cos \pi b_p \cdot e^{i\pi b_p} C_{p, \lambda} - e^{i2\pi b_p} C_{p, -\lambda} \circ C_{p, \lambda} = I
\]

and

\[
2 \cos \pi b_p - e^{i\pi b_p-1} \tilde{C}_{p-1, \lambda} - e^{i2\pi b_p-1} \tilde{C}_{p-1, -\lambda} \circ \tilde{C}_{p-1, \lambda} = I
\]

By using the fact that \(2 \cos \pi b \cdot e^{i\pi b} = 1 + e^{i2\pi b}\) and \(b_p = \gamma \frac{p+1}{2}\), we get the final result.

Next, we use the uniqueness from Theorem 4.6, in order to prove some commutation relation between the scattering matrix and the Hodge star operator.

**Theorem 4.10.** Let \(*_N\) be the Hodge star operator on the manifold \(N\). Then, for any \(\lambda \in S\), the following commutation relation holds

\[
*_N C_{p, \lambda} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & e^{i\pi \gamma_n-p-1} \\ e^{i\pi \gamma_n-p} & 0 \end{pmatrix} C_{n-p, \lambda} *_N = 0.
\]

**Proof.** We use Theorem 4.6 applied to the case of \(n-p\) forms, i.e. \((\theta \oplus \tilde{\theta}) \in \mathcal{H}^{n-p}(N) \oplus \mathcal{H}^{n-p-1}(N)\). Since the Hodge star operator commutes with the Laplacian, \(*E_{\lambda}(y, \theta \oplus \tilde{\theta})\) is a \(\lambda^2\) eigenform of the \(\Delta\) acting on \(p\)-forms. If we note that
Let \( b_p - \gamma_p = 1 - b_p \) and \( 1 - b_{n-p-1} = b_p \), then by Lemma 3.1 and Theorem 4.6, when \( x > 1 \) we have

\[
*E_\lambda(y, \theta \oplus \tilde{\theta}) = x^{b_p} H_{b_p}^{(1)}(\lambda x) *_N \tilde{\theta} + x^{b_p} H_{b_p}^{(2)}(\lambda x) *_N \hat{C}_{n-p-1,\lambda}(\tilde{\theta}) + (-1)^{n-p} dx \wedge x^{b_p-1} H_{b_p-1}^{(1)}(\lambda x) *_N \theta
\]

\[
+ (-1)^{n-p} dx \wedge x^{b_p-1} H_{b_p-1}^{(2)}(\lambda x) *_N \hat{C}_{n-p,\lambda}(\theta) + *_N \Psi_\lambda(y, \theta \oplus \tilde{\theta}).
\]

By the fact that \( H_{b_p}^{(1)}(z) = e^{i\pi b} H_{b_p}^{(1)}(z) \) and \( H_{b_p}^{(2)}(z) = e^{-i\pi b} H_{b_p}^{(2)}(z) \), the above equation becomes

\[
*E_\lambda(y, \theta \oplus \tilde{\theta}) = x^{b_p} H_{b_p}^{(1)}(\lambda x) e^{i\pi b} *_N \tilde{\theta} + x^{b_p} H_{b_p}^{(2)}(\lambda x) e^{-i\pi b} *_N \hat{C}_{n-p-1,\lambda}(\tilde{\theta}) + (-1)^{n-p+1} dx \wedge x^{b_p-1} H_{b_p-1}^{(1)}(\lambda x) e^{i\pi b} *_N \theta
\]

\[
+ (-1)^{n-p+1} dx \wedge x^{b_p-1} H_{b_p-1}^{(2)}(\lambda x) e^{-i\pi b} *_N \hat{C}_{n-p,\lambda}(\theta) + *_N \Psi_\lambda(y, \theta \oplus \tilde{\theta}).
\]

By Theorem 4.6, we also have for \( x > 1 \) that

\[
E_\lambda(y, e^{i\pi b} *_N \tilde{\theta} \oplus (-1)^{n-p+1} e^{i\pi b} *_N \theta)
\]

\[
x^{b_p} H_{b_p}^{(1)}(\lambda x) e^{i\pi b} *_N \tilde{\theta} + (-1)^{n-p+1} dx \wedge x^{b_p-1} H_{b_p-1}^{(1)}(\lambda x) e^{i\pi b} *_N \theta
\]

\[
+ x^{b_p} H_{b_p}^{(2)}(\lambda x) C_{p,\lambda}(e^{i\pi b} *_N \tilde{\theta})
\]

\[
+ (-1)^{n-p+1} dx \wedge x^{b_p-1} H_{b_p-1}^{(2)}(\lambda x) \hat{C}_{p-1,\lambda}(e^{i\pi b} *_N \theta)
\]

\[
+ *_N \Psi \lambda(y, e^{i\pi b} *_N \tilde{\theta} \oplus (-1)^{n-p+1} e^{i\pi b} *_N \theta)
\]

Comparing the last two equations, and using uniqueness from Theorem 4.6, we get

\[
*_N \hat{C}_{n-p-1,\lambda}(\tilde{\theta}) = e^{i\pi b} C_{p,\lambda}(*_N \tilde{\theta}) \quad \text{and} \quad *_N C_{n-p,\lambda}(\theta) = e^{i\pi b} \hat{C}_{p-1,\lambda}(*_N \theta).
\]

If we use the relation \( b_p = \frac{\gamma}{2} \), and replace back \( n - p \) with \( p \), we get the final result.

Finally, we use uniqueness from Theorem 4.6 again and the fact that \( \Delta \) commutes with \( d \), in order to find some relation between the components \( C_{p,\lambda} \) and \( \hat{C}_{p,\lambda} \) of the scattering matrix.

**Theorem 4.11.** \( C_{p,\lambda} = \hat{C}_{p,\lambda} \), for any \( \lambda \in \mathcal{S} \).

**Proof.** Let \( E_\lambda(y, \theta \oplus \tilde{\theta}) \) be as in Theorem 4.6. Since \( \Delta \) commutes with \( d \), \( dE_\lambda(y, \theta \oplus \tilde{\theta}) \) is a \( \lambda^2 \) generalized eigenform of the Laplacian. If the tail term \( \Psi_\lambda(y, \theta \oplus \tilde{\theta}) \) is decomposed by

\[
\Psi_\lambda(y, \theta \oplus \tilde{\theta}) = \Psi_{1,\lambda}(y, \theta \oplus \tilde{\theta}) + dx \wedge \Psi_{2,\lambda}(y, \theta \oplus \tilde{\theta}),
\]
then by (3.2), for $x > 1$ we have
\[
dE_\lambda(y, \theta \oplus \tilde{\theta}) = d_N\Psi_{1,\lambda}(y, \theta \oplus \tilde{\theta}) + dx \wedge \left( \partial_x \left( x^b H^{(1)}_{b_p}(\lambda x) \theta \right) + x^b H^{(2)}_{b_p}(\lambda x) C_{p,\lambda}(\theta) \right.
\]

\[
+ \partial_x \Psi_{1,\lambda}(y, \theta \oplus \tilde{\theta}) - d_N\Psi_{2,\lambda}(y, \theta \oplus \tilde{\theta}) \right)
\]

\[
= dx \wedge \left( \lambda x^b H^{(1)}_{b_{p-1}}(\lambda x) \theta + \lambda x^b H^{(2)}_{b_{p-1}}(\lambda x) C_{p,\lambda}(\theta) \right)
\]

\[
+ d_N\Psi_{1,\lambda}(y, \theta \oplus \tilde{\theta}) + dx \wedge \left( \partial_x \Psi_{1,\lambda}(y, \theta \oplus \tilde{\theta}) - d_N\Psi_{2,\lambda}(y, \theta \oplus \tilde{\theta}) \right),
\]

where we have used the following relations between the Hankel functions (see the appendix)

\[
z\partial_z H^{(1)}_b(z) = zH^{(1)}_{b_{p-1}}(z) - bH^{(1)}_b(z) \quad \text{and} \quad z\partial_z H^{(2)}_b(z) = zH^{(2)}_{b_{p-1}}(z) - bH^{(2)}_b(z).
\]

By Theorem 4.6 in the case of $p + 1$, we also have
\[
E_\lambda(y, 0 \oplus \lambda \theta) = dx \wedge \left( \lambda x^b H^{(1)}_{b_{p-1}}(\lambda x) \theta + \lambda x^b H^{(2)}_{b_{p-1}}(\lambda x) C_{p,\lambda}(\theta) \right) + \Psi_\lambda(y, 0 \oplus \lambda \theta).
\]

By comparing the last two equations, and using uniqueness from Theorem 4.6, we get the final result. \hfill \qed

Note that by the functional equation of the scattering matrix, we get that the eigenvalues of $C_{p,0}$ coincide with the set $\{1, -e^{-i\pi p}\}$, and by the commutation relation with the Hodge star operator, we see that $*_N$ interchanges the above two eigenspaces.
Chapter 5

Appendix: Properties of Bessel functions

In this chapter, we will collect definitions and properties about Bessel and related functions that we use elsewhere in the document. Further discussion about these special functions, and proofs about the relations we state here can be found in [46].

The Bessel function are defined by the solutions of the Bessel equation of order $b \in \mathbb{C}$ given by

$$z^2 f''(z) + zf'(z) + (z^2 - b^2)f(z) = 0.$$  

If $\Gamma(z)$ denotes the gamma function, the Bessel function $J_b(z)$ of the first kind is defined by the solution of the Bessel equation which is given by

$$J_b(z) = z^b \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{z}{2})^{2n}}{n!\Gamma(n + b + 1)},$$

where the above series converges absolutely and uniformly for any $z, b \in \mathbb{C}$. The Bessel function $Y_b(z)$ of the second kind is defined for any $z \in \mathbb{C}$ by the equations

$$Y_b(z) = \frac{J_{b}(z) \cos \pi b - J_{-b}(z)}{\sin \pi b}$$

if $b \in \mathbb{C}$ is not an integer, and by the limit

$$\lim_{\nu \to b} \frac{J_{\nu}(z) \cos \pi \nu - J_{-\nu}(z)}{\sin \pi \nu}$$

if $b$ is an integer. The functions $J_b(z)$ and $Y_b(z)$ form a fundamental system of solutions of the Bessel equation and their Wronskian is given by

$$\mathcal{W}(J_b(z), Y_b(z)) = \frac{2}{\pi z}.$$
Another fundamental system of solutions of the Bessel equation is given by the Hankel functions \( H_b^{(1)}(z) \) and \( H_b^{(2)}(z) \) of order \( b \), of the first and the second kind respectively, which are defined by the equations

\[
H_b^{(1)}(z) = J_b(z) + iY_b(z) \quad \text{and} \quad H_b^{(2)}(z) = J_b(z) - iY_b(z). \tag{5.1}
\]

The cylinder function \( G_b(y, x) \) of order \( b \) is defined by the equation

\[
G_b(y, x) = Y_b(y)J_b(yx) - J_b(y)Y_b(yx).
\]

If we consider the case that \( b \) is an integer as a limit case, then by the definition of the cylinder function we have that

\[
G_b(y, x) = \frac{J_b(y)\cos \pi b - J_{-b}(y)J_b(yx) - J_b(y)J_{-b}(yx)\cos \pi b}{\sin \pi b} = \frac{J_b(y)J_{-b}(yx) - J_{-b}(y)J_b(yx)}{\sin \pi b} = \frac{1}{\sin \pi b} \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}(y^2)^{2(m+n)}}{m!n!\Gamma(m+b+1)\Gamma(n-b+1)}(2\cos \pi b - x^{2m+b}).
\]

From the above we have that \( G_b(y, x) \) is holomorphic as a function of \( y \in \mathbb{C} \) for any \( b \in \mathbb{C} \).

The function \( J_b \) can be continued holomorphically to some Riemann surface depending on the parameter \( b \). The same can be done with the rest of the functions we have defined. Consider now \( z \) to lie on the \( k \)-leaf of the logarithmic cover (i.e. \((2k-1)\pi \leq \arg z < (2k+1)\pi\)), and denote by \( z_0 \) its projection onto the zero leaf. Also, if by \(-z\) we denote the rotation of \( z \) down to the logarithmic cover by \( \pi \), then by the definition of \( J_b \) and \( Y_b \) and by (5.1) we have that

\[
J_b(z) = e^{2k\pi bi}J_b(z_0), \quad Y_b(z) = e^{-2k\pi bi}Y_b(z_0) + 2i\sin 2k\pi b \frac{\cos \pi b}{\sin \pi b} J_b(z_0),
\]

\[
H_b^{(1)}(z) = \frac{\sin(1-2k)\pi b}{\sin \pi b} H_b^{(1)}(z_0) - e^{-\pi bi} \frac{\sin 2k\pi b}{\sin \pi b} H_b^{(2)}(z_0),
\]

\[
H_b^{(2)}(z) = \frac{\sin(1+2k)\pi b}{\sin \pi b} H_b^{(2)}(z_0) + e^{\pi bi} \frac{\sin 2k\pi b}{\sin \pi b} H_b^{(1)}(z_0), \tag{5.2}
\]

and

\[
J_b(-z) = e^{-\pi b}J_b(z), \quad Y_b(-z) = e^{\pi b}Y_b(z) - 2i\cos \pi b \cdot J_b(z),
\]

\[
H_b^{(1)}(-z) = 2\cos \pi b \cdot H_b^{(1)}(z) + e^{-\pi b}H_b^{(2)}(z), \quad H_b^{(2)}(-z) = -e^{\pi b}H_b^{(1)}(z). \tag{5.3}
\]
which hold for any \( b \in \mathbb{C} \), by taking the limit when \( b \) is an integer.

Let us assume that \( b \in \mathbb{R} \). We have the following asymptotic behavior of \( J_b \) and \( Y_b \) as \( z_0 \to \infty \) on the zero leaf

\[
J_b(z_0) = \sqrt{\frac{2}{\pi z_0}} \cos(z_0 + \theta_b)(1 + \mathcal{O}(z_0^{-1}))
\]

and

\[
Y_b(z_0) = \sqrt{\frac{2}{\pi z_0}} \sin(z_0 + \theta_b)(1 + \mathcal{O}(z_0^{-1})),
\]

where \( \theta_b = -\left(\frac{\pi b}{2} + \frac{\pi}{4}\right) \). If we combine the above relations with the Equations (5.2), we can obtain the following asymptotic expansions when \( z \to \infty \) lies on the \( k \)-leaf

\[
J_b(z) = (c_J \frac{e^{iz_0}}{\sqrt{z_0}} + \tilde{c}_J \frac{e^{-iz_0}}{\sqrt{z_0}})(1 + \mathcal{O}(z^{-1}))
\]

\[
Y_b(z) = (c_Y \frac{e^{iz_0}}{\sqrt{z_0}} + \tilde{c}_Y \frac{e^{-iz_0}}{\sqrt{z_0}})(1 + \mathcal{O}(z^{-1}))
\]

\[
H^{(1)}_b(z) = (c_{H^{(1)}} \frac{e^{iz_0}}{\sqrt{z_0}} + \tilde{c}_{H^{(1)}} \frac{e^{-iz_0}}{\sqrt{z_0}})(1 + \mathcal{O}(z^{-1}))
\]

and

\[
H^{(2)}_b(z) = (c_{H^{(2)}} \frac{e^{iz_0}}{\sqrt{z_0}} + \tilde{c}_{H^{(2)}} \frac{e^{-iz_0}}{\sqrt{z_0}})(1 + \mathcal{O}(z^{-1}))
\]

where

\[
c_J = \frac{e^{(2k\pi b + \theta_b)i}}{\sqrt{2\pi}}, \quad \tilde{c}_J = \frac{e^{(2k\pi b - \theta_b)i}}{\sqrt{2\pi}},
\]

\[
c_Y = \frac{ie^{i\theta_b}}{\sqrt{2\pi} \sin \pi b} \left( e^{\pi b \sin 2k \pi b} + \sin(2k - 1)\pi b \right),
\]

\[
\tilde{c}_Y = \frac{ie^{i\theta_b}}{\sqrt{2\pi} \sin \pi b} \left( e^{-\pi b \sin 2k \pi b} + \sin(2k + 1)\pi b \right),
\]

\[
c_{H^{(1)}} = \frac{\sin(1 - 2k)\pi b}{\sin \pi b} \sqrt{\frac{2}{\pi} e^{i\theta_b}}, \quad \tilde{c}_{H^{(1)}} = -\frac{\sin 2k \pi b}{\sin \pi b} \sqrt{\frac{2}{\pi} e^{-(\pi b + \theta_b)i}},
\]

\[
c_{H^{(2)}} = \frac{\sin 2k \pi b}{\sin \pi b} \sqrt{\frac{2}{\pi} e^{(\pi b + \theta_b)i}} \quad \text{and} \quad \tilde{c}_{H^{(2)}} = -\frac{\sin(1 + 2k)\pi b}{\sin \pi b} \sqrt{\frac{2}{\pi} e^{-i\theta_b}}.
\]

Also, by using the following identities

\[
z_0J'_b(z_0) + bJ_b(z_0) = z_0J_{b-1}(z_0) \quad \text{and} \quad z_0Y'_b(z_0) + bY_b(z_0) = z_0Y_{b-1}(z_0),
\]
satisfied by the Bessel functions on the zero leaf, we can get the asymptotic expansion for the following derivatives over $z$, as $z \to \infty$ on the $k$-leaf

\[
J_b'(z) = (C_J e^{iz_0} + \tilde{C}_J e^{-iz_0})\left(1 + \mathcal{O}(z_0^{-1})\right),
\]

\[
Y_b'(z) = (C_Y e^{iz_0} + \tilde{C}_Y e^{-iz_0})\left(1 + \mathcal{O}(z_0^{-1})\right),
\]

\[
H^{(1)}_b'(z) = (C_{H^{(1)}} e^{iz_0} + \tilde{C}_{H^{(1)}} e^{-iz_0})\left(1 + \mathcal{O}(z_0^{-1})\right)
\]

and

\[
H^{(2)}_b'(z) = (C_{H^{(2)}} e^{iz_0} + \tilde{C}_{H^{(2)}} e^{-iz_0})\left(1 + \mathcal{O}(z_0^{-1})\right),
\]

where

\[
C_J = \frac{e^{2\pi b - \theta_{b-1}}i}{\sqrt{2\pi}}, \quad \tilde{C}_J = \frac{e^{2\pi b - \theta_{b-1}}i}{\sqrt{2\pi}},
\]

\[
C_Y = \frac{ie^{ib_{b-1}}}{\sqrt{2\pi} \sin \pi b} \left(e^{i\pi b} \sin 2\pi b + \sin(2k - 1)\pi b\right),
\]

\[
\tilde{C}_Y = \frac{ie^{ib_{b-1}}}{\sqrt{2\pi} \sin \pi b} \left(e^{-i\pi b} \sin 2\pi b + \sin(2k + 1)\pi b\right),
\]

\[
C_{H^{(1)}} = \frac{\sin(1 - 2k)\pi b}{\sin \pi b} \sqrt{\frac{2}{\pi}} e^{i\theta_{b-1}}, \quad \tilde{C}_{H^{(1)}} = -\frac{\sin 2k\pi b}{\sin \pi b} \sqrt{\frac{2}{\pi}} e^{-(\pi b + \theta_{b-1})i},
\]

\[
C_{H^{(2)}} = \frac{\sin 2k\pi b}{\sin \pi b} \sqrt{\frac{2}{\pi}} e^{\pi b - \theta_{b-1}} \quad \text{and} \quad \tilde{C}_{H^{(2)}} = \frac{\sin(1 + 2k)\pi b}{\sin \pi b} \sqrt{\frac{2}{\pi}} e^{-i\theta_{b-1}}.
\]

Finally, when $z_0 \to 0$ on the zero leaf, we have the asymptotic expansions of the Bessel functions of real order $b$

\[
J_b(z_0) = \frac{1}{\Gamma(b + 1)} \left(\frac{z_0}{2}\right)^b \left(1 + \mathcal{O}(z_0^2)\right), \quad \forall b \in \mathbb{R},
\]

\[
Y_b(z_0) = -\frac{\Gamma(b)}{\pi} \left(\frac{2}{z_0}\right)^b \left(1 + \mathcal{O}(z_0^2)\right), \quad \text{if } b > 0,
\]

\[
Y_b(z_0) = -\frac{\cos \pi b \Gamma(-b)}{\pi} \left(\frac{z_0}{2}\right)^b \left(1 + \mathcal{O}(z_0^2)\right), \quad \text{if } b < 0
\]

and

\[
Y_b(z_0) = \frac{2}{\pi} \left(\ln \frac{z_0}{2} + \gamma\right) \left(1 + \mathcal{O}(z_0^2)\right), \quad \text{if } b = 0,
\]

where $\gamma$ is the Euler-Mascheroni constant.
Bibliography


