**Propagation of plate bending waves in the vicinity of one- and two-dimensional acoustic ‘black holes’**

This item was submitted to Loughborough University's Institutional Repository by the/an author.

**Citation:** KRYLOV, V.V., 2007. Propagation of plate bending waves in the vicinity of one- and two-dimensional acoustic ‘black holes’. IN: Papadrakakis, M. ... et al. (eds.). Proceedings of the First International ECCOMAS Thematic Conference on Computational Methods in Structural Dynamics and Earthquake Engineering (COMPDYN 2007), Rethymno, Crete, Greece, 13-16 June 2007.

**Additional Information:**

- This is a conference paper. Further details of this conference can be found at: [http://194.42.10.251/compdyn2007/index.html](http://194.42.10.251/compdyn2007/index.html)

**Metadata Record:** [https://dspace.lboro.ac.uk/2134/6097](https://dspace.lboro.ac.uk/2134/6097)

**Version:** Accepted for publication

Please cite the published version.
This item was submitted to Loughborough’s Institutional Repository (https://dspace.lboro.ac.uk/) by the author and is made available under the following Creative Commons Licence conditions.

For the full text of this licence, please go to:
http://creativecommons.org/licenses/by-nc-nd/2.5/
PROPAGATION OF PLATE BENDING WAVES IN THE VICINITY OF ONE- AND TWO-DIMENSIONAL ACOUSTIC ‘BLACK HOLES’

Victor V. Krylov

Department of Aeronautical and Automotive Engineering, Loughborough University, Loughborough, Leicestershire LE11 3TU, UK
e-mail: V.V.Krylov@lboro.ac.uk

Keywords: Acoustic black holes, Bending wave propagation, Damping of structural vibrations.

Abstract One of the ways of damping resonant vibrations of engineering structures or their components is to reduce reflections of bending waves from their free edges. A new efficient method of reducing edge reflections proposed by the present author is based on using specially designed plates or bars of variable thickness in combination with strips of thin absorbing layers placed at the edges. Such plates or bars utilise gradual change in their thickness from the value corresponding to the thickness of the basic plate or bar to almost zero. If to use some specific power-law profiles for these plates or bars then they would ideally provide zero reflection of bending waves from their sharp edges even in the absence of the absorbing layers at the edges, thus materialising the so-called ‘acoustic black holes’ for bending waves. In the present paper, this effect is considered for one-dimensional and two-dimensional acoustic black holes. In the latter case black holes are materialised by cylindrically symmetrical pits (cavities) of power-law profile nearly protruding through the bottom of the plate. Geometrical acoustics (ray tracing) theory for plate bending wave propagation in the vicinity of one- and two-dimensional acoustic black holes is considered, including definition of ray trajectories and calculation of the reflection coefficients. Finally, we discuss possible practical applications of the above-mentioned one- and two-dimensional black holes for damping structural vibrations.
1. INTRODUCTION

It is well known that one of the ways of damping resonant vibrations of engineering structures or their components is to reduce reflections of bending (flexural) waves from their free edges [1]. An efficient method of reducing edge reflections of bending waves in plates or bars has been recently considered theoretically [2-5] and confirmed experimentally [6,7] by the present author and his co-workers. The method described in [2-7] utilises a one-dimensional power-law change in thickness of a plate or a bar, partly covered by a thin damping layer, from the value corresponding to the thickness of the basic plate or bar to almost zero. If a profile of such a one-dimensional plate (wedge) is described by certain power-law relationships then flexural waves propagating towards the sharp edge will slow down and become trapped near the edge. In particular, in tapered plates with power-law exponent equal or larger than two, the incident flexural waves become trapped when they approach the sharp edge. Therefore, they do not reflect back from the edge that thus materialises a one-dimensional acoustic ‘black hole’.

In the present paper, the phenomenon of acoustic black holes is considered for both one- and two-dimensional cases. In the latter case the black holes are materialised by cylindrically symmetrical pits (cavities) of power-law profile nearly protruding through the bottom of a plate. If such a pit is described by a power law relationship with the exponent being equal or larger than two, then, combined with a small piece of thin damping layer attached to the bottom of a plate, such a pit materialises a two-dimensional acoustic black hole for plate bending waves. Like in one-dimensional case, two-dimensional black holes can absorb a substantial part of the wave energy, thus resulting in effective damping of incident waves. An important practical advantage of using two-dimensional black holes is that they are relatively easy to manufacture. Indeed, the simplest way to produce a two-dimensional black hole is to drill a pit using a steel cutter of a circular shape until the cutter nearly protrudes. The resulting spherical shape approximates the required quadratic profile near the centre of the pit, which makes it to behave like a black hole.

![Diagram](image)

Figure 1: A non-symmetrical wedge-like damper of power-law profile covered by an absorbing layer on one side (a); such a structure materialises a one-dimensional ‘acoustic black hole’ that can be used for suppression of resonant vibrations in a basic plate (b).
A practical realisation of a one-dimensional non-symmetrical wedge-like damper covered by an absorbing film on one side is illustrated in Figure 1. The thick end of a damper is glued to the edge of a basic plate reflections from which are to be suppressed. To avoid reflections from the boundary between the plate and the wedge that may be caused by a sudden change in geometry, the shape of the wedge is smoothened so that it gradually transforms into the plate with the thickness equal to the thickness of the basic plate.

When a flexural wave propagates towards the sharp edge of an elastic wedge shown in Figure 1, it slows down and its amplitude grows. After reflection from the edge, with the module of the reflection coefficient normally being equal to unity, the whole process repeats in the opposite direction [8-12]. However, if the wedge edge has a cross section described by a power law relationship between the local thickness \( h \) and the distance from the edge \( x \): \( h(x) = \varepsilon x^m \), where \( m \geq 2 \), then the wave never reaches the edge and hence never reflects back, thus materialising an acoustic (or vibration) ‘black hole’. The unusual effect of power-law profile on flexural wave propagation in elastic wedges has attracted some initial attention in respect of their possible applications as vibration dampers [13]. However, because of the deviations of manufactured wedges from the ideal power-law shapes, largely due to ever-present truncations of the edges, the estimated reflection coefficients for such ‘non-ideal’ wedges are as high as 50-70 \% [13]. Therefore, in practice such structures can not be used as vibration dampers.

The situation, however, can be radically improved by depositing thin absorbing layers on the surfaces of the above-mentioned wedges, very close to their sharp edges [2-7]. Thus, the combination of wedges of power-law profiles with thin absorbing layers can result in very efficient damping systems for flexural vibrations.
The same principle of bending wave propagation near the edges of wedge-like structures of power-law profile can be applied also to the case of two-dimensional acoustic black holes. A typical example of such a structure is shown in Figure 2. In this case, the rays of bending waves that propagate through the pit can be trapped at its very bottom, near the curved sharp edges around the point of protrusion. Therefore, these particular rays are not reflected, and the associated part of the wave elastic energy is removed from the system, thus contributing to the overall damping of the plate resonant vibrations.

In what follows we briefly describe the basic theory behind the phenomenon, initially for the case of one-dimensional black holes. Then we turn our attention to two-dimensional black holes. Geometrical acoustics (ray tracing) theory for plate bending wave propagation in the vicinity of two-dimensional acoustic black holes will be considered, including determination of ray trajectories and calculation of ray reflection coefficients. Finally, we consider some possible practical applications of the above-mentioned one- and two-dimensional acoustic black holes and their combinations for damping structural vibrations.

2. BENDING WAVE PROPAGATION IN ONE-DIMENSIONAL WEDGES OF POWER-LAW PROFILE

To understand the phenomenon of acoustic black holes one can start with the simplest one-dimensional case of plane flexural wave propagation in the normal direction towards the edge of a free wedge described by a power-law relationship \( h(x) = \varepsilon x^m \), where \( m \) is a positive rational number and \( \varepsilon \) is a constant. Since flexural wave propagation in such wedges can be described in the geometrical acoustics approximation (see [8-10] for more detail), the integrated wave phase \( \Phi \) resulting from the wave propagation from an arbitrary point \( x \) located in the wedge medium plane to the wedge tip (\( x = 0 \)) can be written in the form:

\[
\Phi = \int_{0}^{x} k(x) \, dx .
\]  

Here \( k(x) \) is a local wavenumber of a flexural wave for a wedge in contact with vacuum: \( k(x) = 12^{1/4} k_p^{1/2} (\varepsilon x^m)^{-1/2} \), where \( k_p = \omega/c_p \) is the wavenumber of a symmetrical plate wave, \( c_p = 2c_l(1-c_l^2/c_s^2)^{1/2} \) is its phase velocity, and \( c_l \) and \( c_s \) are longitudinal and shear wave velocities in a wedge material, and \( \omega = 2\pi f \) is circular frequency. One can easily check that the integral in Eqn. (1) diverges for \( m \geq 2 \). This means that the phase \( \Phi \) becomes infinite under these circumstances and the wave never reaches the edge. Therefore, it never reflects back either, i.e. the wave becomes trapped, thus indicating that the above mentioned ideal wedges represent acoustic ‘black holes’ for incident flexural waves.

Real fabricated wedges, however, always have truncated edges. And this adversely affects their performance as ‘black holes’. If for ideal wedges of power-law shape (with \( m \geq 2 \)) it follows from Eqn. (1) that even an infinitely small material attenuation, described by the imaginary part of \( k(x) \), would be sufficient for the total wave energy to be absorbed, this is not so for truncated wedges. Indeed, for truncated wedges the lower integration limit in (1) must be changed from \( 0 \) to a certain value \( x_0 \) describing the length of truncation, thus resulting in the amplitude of the total reflection coefficient \( R_0 \) being expressed in the form [13]
\[ R_0 = \exp(-2 \int_{x_0}^{\infty} \text{Im} k(x) dx). \]  \hspace{1cm} (2)

According to Eqn. (2), for typical low values of attenuation in such wedge materials as steel, even very small truncations \( x_0 \) result in the reflection coefficients \( R_0 \) becoming as large as 50-70%.

As has been proposed in [2-7], in order to improve the situation for real wedges (with truncations), one should cover wedge surfaces near the edges by thin absorbing layers (films) of thickness \( \delta \), e.g. by polymeric films. Two types of wedge geometry can be considered: a symmetrical wedge and a non-symmetrical wedge bounded by a plain surface at one of the sides, the latter being shown in Figure 1. For each of these cases either two or only one of the sides can be covered by absorbing layers. Note that non-symmetrical wedges are easier to manufacture. They also have the advantage in depositing absorbing layers: the latter can be deposited on a flat surface, which is much easier. From the point of view of theoretical description, there is no difference between symmetrical and non-symmetrical wedges as long as geometrical acoustics approximation is concerned and the wedge local thickness \( h(x) = \varepsilon x^m \) is much less than the flexural wavelength.

To analyse the effect of thin absorbing films on flexural wave propagation in a wedge in the framework of geometrical acoustics approximation one should consider first the effect of such films on flexural wave propagation in plates of constant thickness. The simplest way of approaching this problem is to use the already known solutions for plates covered by absorbing layers of arbitrary thickness obtained by different authors with regard to the description of damped vibrations in such sandwich plates (see e.g. [14, 15]). Using this approach (see [4,5] for more detail), one can derive the corresponding analytical expressions for the reflection coefficients of flexural waves from the edges of truncated wedges covered by thin absorbing layers.

For illustration purposes, let us consider a wedge of quadratic shape, i.e. \( h(x) = \varepsilon x^2 \), covered by absorbing layers on both surfaces. For such a wedge one can derive the following analytical expression for the resulting reflection coefficient \( R_0 \) [4,5]:

\[ R_0 = \exp(-2\mu_1 - 2\mu_2), \]  \hspace{1cm} (3)

where

\[ \mu_1 = \frac{12^{1/4} k_p^{1/2} \eta}{4 \varepsilon^{1/2}} \ln \left( \frac{x}{x_0} \right), \]  \hspace{1cm} (4)

\[ \mu_2 = \frac{3 \cdot 12^{1/4} k_p^{1/2} \nu \delta}{4 \varepsilon^{3/2}} \frac{E_2}{E_1} \frac{1}{x_0^{2}} \left( 1 - \frac{x_0^2}{x^2} \right). \]  \hspace{1cm} (5)

Here \( \nu \) is the loss factor of the material of the absorbing layer and \( \delta \) is its thickness, \( \eta \) is the loss factor of the wedge material, \( x_0 \) is the wedge truncation length, \( E_1 \) and \( E_2 \) are respectively the Young’s moduli of the plate and of the absorbing layer. As expected, in the absence of the absorbing layer (\( \delta = 0 \) or \( \nu = 0 \), and hence \( \mu_2 = 0 \)), Eqns. (3)-(5) reduce to the results obtained in [13] (where a typographical misprint has been noticed). If the absorbing layer is present (\( \delta \neq 0 \) and \( \nu \neq 0 \)), it brings the additional reduction of the reflection...
coefficient that depends on the layer loss factor $\nu$ and on the other geometrical and physical parameters of the wedge and the layer. In the case of a wedge of quadratic shape covered by damping layers on one surface only, Eqns. (3) and (4) remain unchanged, whereas Eqn (5) should be replaced by

$$
\mu_2 = \frac{3 \cdot 12^{1/4} k \nu \delta}{8 \varepsilon^{3/2}} \frac{E_2}{E_1} \frac{1}{x_0^2} \left(1 - \frac{x_0^2}{x^2}\right).
$$

(6)

In deriving Eqns. (3)-(6) the effect of thin absorbing layers on flexural wave velocity has been neglected, which can be done for very thin absorbing layers. The extension of the analysis to the case of absorbing layers of arbitrary thickness has been performed in the paper [5].

Note that geometrical acoustics approximation for the above-mentioned quadratic wedges ($m = 2$) is valid for all $x$ provided that the following applicability condition is satisfied:

$$
\frac{\omega}{c_t \varepsilon} >> 1,
$$

(7)

where $c_t$ is shear wave velocity in the wedge material. For the majority of practical situations this condition can be easily satisfied even at very low frequencies.

Using Eqns. (3)-(6), let us illustrate the effect of the above-mentioned one-dimensional acoustic black hole on bending wave reflection coefficients. Let us choose the following
values of the film parameters: \( \nu = 0.25 \) (i.e., consider the film as being highly absorbing), \( E_2/E_1 = 0.3 \) and \( \delta = 15 \mu m \). Let the parameters of the quadratically shaped wedge be: \( \varepsilon = 0.05 \text{ m}^{-1} \), \( \eta = 0.01 \), \( x_0 = 2 \text{ cm} \), \( x = 50 \text{ cm} \) and \( c_p = 3000 \text{ m/s} \). Then, e.g. for a wedge covered by absorbing films on both sides and at the frequency \( f = 10 \text{ kHz} \) it follows from Eqns. (3)-(5) that in the presence of the damping film \( R_0 = 0.022 \) (i.e. 2.2%), whereas in the absence of the damping film \( R_0 = 0.542 \) (i.e. 54.2%). Thus, in the presence of the damping film the value of the reflection coefficient is much smaller than for a wedge with the same value of truncation, but without a film. Note that almost all absorption of the incident wave energy takes place in the vicinity of the sharp edge of a wedge.

The effect of wedge truncation length \( x_0 \) on the reflection coefficient \( R_0 \) for the above example is shown in Figure 3. For comparison, the curve corresponding to the wedge not covered by absorbing films is shown on the same Figure as well. One can see that the behaviour of the reflection coefficient \( R_0 \) as a function of \( x_0 \) is strongly influenced by the parameter \( \beta_2 = E_2/E_1 \) describing relative stiffness of the absorbing film. The larger the film stiffness the higher values of truncation \( x_0 \) can be allowed to keep the reflection coefficient low.

The above-mentioned discussion of one-dimensional bending wave propagation and reflection in plates of power-law profile is entirely applicable also to bars of power-law profile. Let us consider, for example, a practically very important case of tapered bars of circular cross-section. Such tapered bars are very easy to manufacture, and one can propose to use them for a number of practical purposes, e.g. for damping impact-generated flexural vibrations in the grips of tennis racquets and golf clubs (see Figure 4).

![Figure 4: A tapered bar of circular cross-section with the radius R described by a power-law profile and with the sharp end covered by a thick absorbing layer (a tennis racquet grip) (a); such a device forms a one-dimensional acoustic black hole that can be used for suppression of impact-induced resonant vibrations in the main structure of tennis racquets (only part of which close to the grip is shown) (b).](image)

If the radius \( R \) of a tapered part of the circular bar is described by a power-law relationship \( R = \varepsilon x^n \), then it follows from the well-known expressions for bending waves in bars that the local wavenumber of a bending wave propagating in the tapered part of the bar can be written in the form
where $k_b = \omega/c_b$ is the wavenumber of longitudinal waves in a bar, and $c_b = (E/\rho)^{1/2}$ is the velocity of such longitudinal waves. In writing down Eqn. (8) we have used the fact that the area moment of inertia for a bar of circular cross-section $I = \pi R^4/4$. Obviously, since Eqn. (8) has the same form as the corresponding expression for bending waves in tapered plates, all the above mentioned conclusions obtained for plates are valid for tapered bars as well. In particular, the existence of the acoustic black hole effect in tapered bars for $m \geq 2$ (see Eqn. (1)) and the possibility of obtaining very low reflection coefficients after covering their sharp ends by layers of absorbing materials.

The discussion so far concerned one-dimensional problems of flexural wave interaction with acoustic black holes, including the case of plates of variable thickness for which we analysed only normal incidence of a bending wave on a sharp edge. In the case of oblique incidence, one should take into account the non-zero component of the bending wave vector in the direction parallel to the edge, which is constant during the whole process of wave propagation. Note, however, that because of the normal component’s of the bending wave vector being very large near the edge, the angle of wave approach to the edge will be almost normal (see Figure 5).

\[ k(x) = k_b^{1/2} \frac{\sqrt{2}}{\sqrt{R(x)}} = k_b^{1/2} \frac{\sqrt{2}}{e^{1/2}x^{m/2}} , \quad (8) \]

Figure 5: Reflection of a bending wave from a sharp edge of a quadratic wedge at oblique incidence; both incident and reflected waves are propagating almost in normal direction near the edge.

It is important now to recall that the main contribution to the total accumulated phase of the propagating bending wave is made at a very small distance near the edge. Therefore, in all practical calculations of the reflection coefficients of bending waves propagating at oblique angles to the wedge edges of power-law profiles it is sufficient to use the above-mentioned one-dimensional results obtained for the case of normal incidence of bending waves.
3. **BENDING WAVE PROPAGATION OVER TWO-DIMENSIONAL PITS OF POWER-LAW PROFILE**

Let us now turn to the case of two-dimensional acoustic black holes, such as nearly protruding cylindrically symmetrical pits drilled in a regular thin plate of constant thickness. To consider bending wave propagation over two-dimensional pits of power-law profile it is convenient to use a geometrical acoustics approach in Hamiltonian formulation similar to that earlier applied by the present author for analysing Rayleigh surface wave propagation across smooth large-scale surface irregularities [16, 17]. Application of this approach to the case of bending wave propagation in thin plates of variable thickness, including plates with cylindrically symmetrical pits, is even simpler as in this case there is no geometrically induced anisotropy of bending wave velocity.

It is easy to show that the equations defining ray trajectories for the case of bending wave propagation over cylindrically symmetrical pits can be written in the form

\[
\frac{dr}{d\theta} = r \frac{1}{\tan \alpha}
\]

\[
\frac{d\alpha}{d\theta} = -1 - r \frac{\partial n}{\partial r}.
\]

\[nr \sin \alpha = \text{const}\]

Here \(r\) and \(\theta\) are polar coordinates associated with a pit and measured along the plate middle plane, \(\alpha\) is the angle between \(r\) and \(k\), where \(k\) is the bending wave vector, \(n(r) = c_0/c(r)\) is the index of refraction of a bending wave in the pit area, and \(c_0\) and \(c = c(r)\) are the bending wave phase velocities in the basic plate of constant thickness \(h_0\) and in the pit area respectively. Obviously, \(n(r) = h_0^{1/2}/h(r)^{1/2}\), where \(h(r)\) is the local thickness of a plate in the pit area. In what follows we assume that, like in the above-mentioned case of one-dimensional wedges, \(h(r)\) is described by a power-law relationship, i.e. \(h(r) = \epsilon r^m\).

Note that the last equation in (9) represents Snell’s law in the axisymmetrical case. It is convenient to rewrite this equation in the form linking the initial point of the ray trajectory outside the pit (in particular, this can be an excitation point), characterised by the values \(r_0, \alpha_0\) and \(n_0 = 1\), with the point of observation in the pit area, characterised by the values \(r, \alpha\) and \(n\):

\[nr \sin \alpha = r_0 \sin \alpha_0 = \rho.\]

Here we have introduced the so-called impact parameter \(\rho = r_0 \sin \alpha_0\) which is important for determination of ray trajectories for different values of \(\alpha_0\). Obviously, for the given values of \(r_0\) and \(\alpha_0\), this parameter describes the closest distance between the ray trajectory and the centre of the polar coordinate system in the absence of a pit (in this case the trajectories are straight lines). The presence of a pit influences the rays that propagate through it, and the corresponding ray trajectories are no longer straight lines.

The analysis of Eqns. (9) and/or (10) shows that, in the case of symmetrical pits of power-law profile with \(m \geq 2\), all rays with the values of \(|\rho|\) that are less than a certain critical
value, including a direct ray for which $\rho = 0$, will deflect towards the centre of the pit, approaching it almost in the normal direction (Figure 6). Since the central (nearly protruding) area of the pit makes a major contribution to the reflection coefficients, one can use one-dimensional calculations for the reflection coefficients of all such rays, like in the above-mentioned case of oblique incidence on the edge of a wedge (see Figure 5).

Figure 6: Three typical ray trajectories illustrating propagation of bending waves over a pit of power-law profile; the ray below is trapped by the black hole, which means that its reflection from it can be negligible.

From the practical point of view, the above-mentioned two-dimensional acoustic black hole (a pit) can be placed at any point of a plate or any other plate-like or shell-like structure. The effect of such a black hole will be in eliminating some rays, intersecting with the black hole, from contributing to the overall frequency response function of a structure, which will result in substantial damping of resonant vibrations. To amplify the effect of two-dimensional acoustic black holes one can use combinations of several black holes distributed over the structure, if this does not compromise its main functions.

4. SOME PRACTICAL APPLICATIONS OF ONE- AND TWO-DIMENSIONAL ACOUSTIC BLACK HOLES

One of the most important advantages of the acoustic black holes as dampers of structural vibrations is that they are efficient even for relatively thin and narrow strips of attached absorbing layers. This is in contrast with the traditional techniques employing covering the whole surfaces of structures by relatively thick layers of absorbing materials. And this important feature of the acoustic black holes can be very attractive for many practical applications.

For example, one of such potential practical applications can be damping of unwanted resonant vibrations in turbine blades [7]. It is quite obvious that the classical attachment of relatively thick absorbing layers on the whole surface (see e.g. [14, 15]) is incompatible with
the main function of a blade. If, however, to make one of the blade edges in the shape materialising the acoustic black hole effect, then even a thin absorbing layer deposited at the very edge of the blade would be able to provide an efficient vibration damping in the whole system. In particular, such a thin absorbing layer could be made of one of the specially developed vibration absorptive and temperature-resistant alloys. One could expect that utilising such alloys would make it possible to manufacture integrated turbine blades with much improved vibration damping performance.

As was mentioned in Section 2, one of the promising applications of tapered bars of power-law profile can be their use for damping impact-induced flexural vibrations in tennis racquets and golf clubs. Obviously, they can be used also in numerous structures containing different types of bars.

Two-dimensional acoustic black holes open new possibilities of damping structural vibrations without compromising their main functions. For example, drilling pits of power-law profile in stiffening ribs of different aerospace structures will not compromise their overall rigidity. In the same time, such pits combined with strips of thin absorbing layers will reduce quite substantially the levels of unwanted structural vibrations.

5. CONCLUSIONS

Some theoretical results have been described regarding the acoustic ‘black hole’ effect for bending waves propagating in elastic wedges and bars of power-law profile (one-dimensional black holes) and in cylindrically symmetrical pits of power-law profile (two-dimensional black holes). It has been demonstrated that the presence of thin absorbing layers on the surfaces of such wedges, bars and pits can result in very low reflection coefficients of flexural waves from their edges even in the presence of edge truncations. As a result of this, such wedges, bars and pits can be used as efficient means of damping structural vibrations.

The additional advantage of using the proposed wedge-like, bar-like or pit-like vibration dampers over traditional types of vibration dampers lays in the fact that the former are compact and relatively easy to manufacture. They can be integrated into a basic structure on a design stage without compromising its main functions.

In spite of the encouraging theoretical results described in this paper and their first experimental confirmation [6,7], further investigations are needed to explore the most efficient ways of developing the proposed novel dampers of structural vibrations based on the acoustic black hole effect.

6. REFERENCES


