

Loughborough University Institutional Repository

On the velocities of localized vibration modes in immersed solid wedges

This item was submitted to Loughborough University's Institutional Repository by the/an author.

Citation: KRYLOV, V.V., 1998. On the velocities of localized vibration modes in immersed solid wedges. *Journal of the Acoustical Society of America*, 103 (2), pp.767-770.

Additional Information:

- This article was published in the journal, *Journal of the Acoustical Society of America* [© Acoustical Society of America] and is also available at: <http://asa.aip.org/jasa.html>

Metadata Record: <https://dspace.lboro.ac.uk/2134/6157>

Version: Published

Publisher: © Acoustical Society of America

Please cite the published version.

This item was submitted to Loughborough's Institutional Repository (<https://dspace.lboro.ac.uk/>) by the author and is made available under the following Creative Commons Licence conditions.



creative commons
COMMONS DEED

Attribution-NonCommercial-NoDerivs 2.5

You are free:

- to copy, distribute, display, and perform the work

Under the following conditions:

 **Attribution.** You must attribute the work in the manner specified by the author or licensor.

 **Noncommercial.** You may not use this work for commercial purposes.

 **No Derivative Works.** You may not alter, transform, or build upon this work.

- For any reuse or distribution, you must make clear to others the license terms of this work.
- Any of these conditions can be waived if you get permission from the copyright holder.

Your fair use and other rights are in no way affected by the above.

This is a human-readable summary of the [Legal Code \(the full license\)](#).

[Disclaimer](#) 

For the full text of this licence, please go to:
<http://creativecommons.org/licenses/by-nc-nd/2.5/>

On the velocities of localized vibration modes in immersed solid wedges

Victor V. Krylov

Centre for Research into the Built Environment, The Nottingham Trent University, Burton Street, Nottingham NG1 4BU, United Kingdom

(Received 10 July 1997; accepted for publication 27 October 1997)

The approximate theory of localized elastic waves in immersed solid wedges earlier developed for wedges with small values of the apex angle [V. V. Krylov, Proc. IEEE Ultrason. Symp., Cat. #94, CHO 793–796 (1994)] has predicted that the effect of water loading results in velocity decrease for wedge modes travelling in the subsonic regime of wave propagation. The results of this theory, in particular the absolute values of wedge wave velocity calculated for slender Plexiglas wedges, agree well with the corresponding experiments. The present study demonstrates that for relative values of wedge wave velocity, as compared with those for wedges in vacuum, this theory provides good quantitative agreement with the experiments on Plexiglas samples also for large values of the apex angle. In addition to this, a generalization of the theory is undertaken to describe the effect of heavier wedge material and a supersonic regime of wave propagation. The corresponding results show good agreement with the existing velocity measurements in immersed brass wedges. © 1998 Acoustical Society of America. [S0001-4966(98)03102-6]

PACS numbers: 43.20.Jr [ANN]

INTRODUCTION

In a recently published work, De Billy¹ carried out measurements of the velocities of localized antisymmetric modes propagating along tips of elastic solid wedges immersed in water. The velocities of such modes (also called wedge acoustic waves) were measured on Plexiglas and brass samples with the apex angle varying from 20 to 90 degrees. For Plexiglas samples, for which a subsonic regime of wedge wave propagation takes place, the experiments showed a noticeable decrease in velocities due to water loading. As was mentioned in Ref. 1, this was in qualitative agreement with the predictions of the approximate analytical theory earlier developed by the present author for slender immersed wedges in subsonic regime of wave propagation.² However, no direct comparison with the theory was given in Ref. 1, probably because it was not expected from the theory to provide accurate results for large values of the wedge apex angle. Note in this connection that measurements of the absolute values of wedge wave velocities carried out by Chamuel³ on Plexiglas wedges with apex angles varying from about 8 to 13 degrees have shown excellent agreement with the theory.² Latest finite element calculations by Hladky-Hennion *et al.*⁴ carried out for Plexiglas wedges with angles in the range from 20 to 90 degrees demonstrated good quantitative agreement with the experiments.¹ However, in the case of brass wedges, which provide both subsonic and supersonic propagation regimes, only subsonic regime has been calculated.⁴

In what follows we demonstrate that (a) for relative values of wedge wave velocity, as compared with those for wedges in vacuum, the approximate analytical theory developed for slender immersed wedges² provides good quantitative agreement with the experiments on Plexiglas samples also for large values of the apex angle; and (b) a generalization of this theory to describe the effect of heavier wedge

material and a supersonic regime of wave propagation explains the results of the velocity measurements for immersed brass wedges.¹

I. THEORY

We remind the reader that the approximate theory of localized elastic waves in immersed solid wedges developed in Ref. 2 is based on the geometrical acoustics approach considering a slender wedge as a plate with a local variable thickness $d = x\Theta$, where Θ is the wedge apex angle and x is the distance from the wedge tip measured in the middle plane (Fig. 1). The velocities c of the localized wedge modes propagating in y direction are determined in the geometrical acoustics approximation as solutions of the Bohr–Sommerfeld type equation^{5–7}

$$\int_0^{x_t} [k^2(x) - \beta^2]^{1/2} dx = \pi n, \quad (1)$$

where $\beta = \omega/c$ is the yet unknown wave number of a wedge mode, $k(x)$ is a current local wave number of a flexural wave in a plate of variable thickness, $n = 1, 2, 3, \dots$ is the mode number, and x_t is the so called ray turning point being determined from the equation $k^2(x) - \beta^2 = 0$.

For example, in the case of wedge in vacuum $k(x) = 12^{1/4} k_p^{1/2} (\Theta x)^{-1/2}$, $x_t = 2\sqrt{3} k_p / \Theta \beta^2$, and $k_p = \omega/c_p$, where ω is circular frequency, $c_p = 2c_l(1 - c_l^2/c_t^2)^{1/2}$ is the so called plate wave velocity, c_l and c_t are propagation velocities of longitudinal and shear acoustic waves in plate material. Then, taking the integral in Eq. (1) and solving the resulting algebraic equation yields the extremely simple analytical expression for wedge wave velocities:^{5–7}

$$c = c_p n \Theta / \sqrt{3}. \quad (2)$$

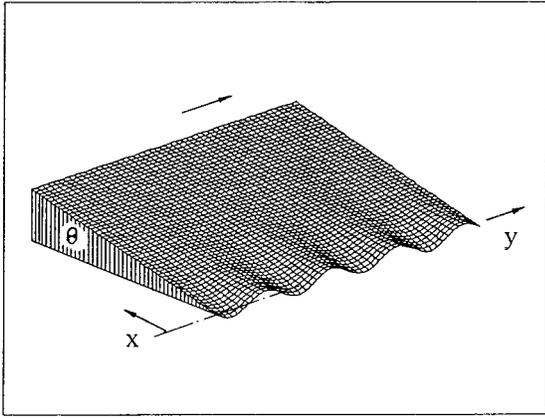


FIG. 1. Antisymmetric wedge waves.

Expression (2) agrees well with the other theories⁸⁻¹⁰ and with the experimental results.⁸ Note that, although, strictly speaking, the geometrical acoustics approach is not valid for the lowest order wedge mode ($n=1$),⁶ in practice it provides quite accurate results for wedge wave velocities in this case as well. The analytical expressions for amplitude distributions of wedge modes are rather cumbersome⁵⁻⁷ and are not displayed here.

To calculate the velocities of wedge waves in a wedge imbedded in liquid one has to make use of the expression for a plate wave local wave number $k(x)$ which takes into account the effect of liquid loading.² The starting point to derive $k(x)$ for this case is the well known dispersion equation for the lowest order flexural mode of a plate imbedded in liquid:

$$\frac{1}{2} \frac{\omega^4}{c_l^4 k^4} - \frac{1}{6} (c_l^2 - c_t^2) \frac{d^2}{c_l^2 c_t^2} \omega^2 + \frac{\rho_f}{\rho_s} \frac{\omega^4}{c_l^4 k^4} \frac{1}{d \sqrt{k^2 - \omega^2/c_f^2}} = 0, \quad (3)$$

where d is the plate thickness, $k = \omega/c$ is the wave number of propagating flexural mode, c is its phase velocity, $k_l = \omega/c_l$ and $k_t = \omega/c_t$ are respectively the wave numbers of longitudinal and shear acoustic waves in plate material, ρ_s and ρ_f are respectively the mass densities of solid and liquid. Note that, using the notation for flexural rigidity $D = (\rho_s c_p^2/12)d^3$, one can easily transform Eq. (3) to the more familiar form which is often used in the literature for flexural waves in thin plates faced to liquid at both sides.^{11,12} In the absence of liquid ($\rho_f = 0$) Eq. (3) reduces to the well known dispersion relation for flexural waves in a thin plate in vacuum.

In further consideration it is convenient to distinguish two characteristic cases of the relation between the mass densities of liquid and wedge material: $\rho_f/\rho_s \approx 1$ and $\rho_f/\rho_s \ll 1$.

For the case $\rho_f/\rho_s \approx 1$ typical for light solid materials in water we limit our analysis by subsonic regime of wave propagation ($k > \omega/c_f$). Moreover, for simplicity, we impose even more severe restriction considering very slow propagating plate flexural modes ($k \gg \omega/c_f$). In such a case one can use the approximation of incompressible liquid, i.e., neglect the ω/c_f term in (3), and the solution of Eq. (3) versus ω yields

$$\omega = \frac{1}{\sqrt{3}} \frac{c_l}{c_l} k^{5/2} \sqrt{\rho_s} \frac{d^{3/2}}{\sqrt{\rho_s k d + 2\rho_f}} \sqrt{c_l^2 - c_t^2}. \quad (4)$$

For $kd \ll 1$ typical for thin plates one can also neglect the term $\rho_s k d$ in Eq. (4), keeping in mind that $\rho_f/\rho_s \approx 1$. Then, taking into account that for a plate of variable thickness representing a wedge $d = d(x) = x\Theta$ and solving (4) versus k , one obtains the following expression for the local flexural wave number $k(x)$ describing the effect of liquid loading:

$$k(x) = \left[\sqrt{6} \frac{c_l}{c_t} \frac{1}{\sqrt{c_l^2 - c_t^2}} \frac{\sqrt{\rho_f}}{\sqrt{\rho_s}} \frac{\omega}{(x\Theta)^{3/2}} \right]^{2/5}. \quad (5)$$

Substituting (5) into (1) and introducing the nondimensional notations $\eta = c/c_t$ and $z = k_l d = x\Theta \omega/c_t$ one can derive the following equation versus η :²

$$\int_0^{(A\eta)^{5/3}} \left[\frac{A^2}{z^{6/5}} - \frac{1}{\eta^2} \right]^{1/2} dz = \pi n \theta, \quad (6)$$

where

$$A = 6^{1/5} (\rho_f/\rho_s)^{1/5} (1 - c_t^2/c_l^2)^{-1/5} \\ = 6^{1/5} (\rho_f/\rho_s)^{1/5} [2(1 - \sigma)]^{1/5} \quad (7)$$

is a nondimensional parameter dependent on the relation between the mass densities of liquid and solid ρ_f/ρ_s [the power of the mass density ratio 1/5 in Eq. (7) corrects the earlier misprinted value of 2/5 in Ref. 2] and on the Poisson ratio σ . Eq. (6) can be easily solved numerically using any appropriate algorithm, e.g., standard Mathcad package.² Note, however, that by change of variable $z = A^{5/3} \eta^{5/3} x$, it may be further simplified as

$$A^{5/3} \eta^{2/3} \int_0^1 (x^{-6/5} - 1)^{1/2} dx = \pi n \theta. \quad (8)$$

[The author's attention to this way of simplification of Eq. (6) has been drawn by A. N. Norris.] After numerical calculation of the integral in Eq. (8), one can easily derive the explicit analytical expression for wedge wave velocities:

$$c = c_l A^{-5/2} D^{-3/2} (\pi n)^{3/2} \Theta^{3/2}, \quad (9)$$

where $D = \int_0^1 (x^{-6/5} - 1)^{1/2} dx = 2.102$. According to Eq. (9), the dependence of c on Θ for immersed slender wedges is proportional to $\Theta^{3/2}$. This agrees well with the empirical power law $\Theta^{1.522}$ established by Chamuel³ by matching the numerical solution of Eq. (6).

In the case $\rho_f/\rho_s \ll 1$ typical for heavy solid materials in water and for any solids in gases, one can solve Eq. (3) approximately, to the first perturbation order versus ρ_f/ρ_s , seeking a solution for k in the form

$$k = k_{(0)} + k_{(1)}, \quad (10)$$

where $k_{(0)}$ is the solution of Eq. (3) for a plate in vacuum ($\rho_f/\rho_s = 0$) and $k_{(1)}$ is yet unknown small correction term which takes into account the effect of liquid loading. Substituting (10) into (3) and retaining terms of only the first order versus ρ_f/ρ_s , one can derive the following approximate expression for k :

$$k = k_{(0)} \left[1 + \frac{\rho_f}{2\rho_s d} \left(k_{(0)}^2 - \frac{\omega^2}{c_f^2} \right)^{-1/2} \right]. \quad (11)$$

Obviously, for $\rho_f/\rho_s=0$ Eq. (11) goes over to the well known expression for a wave number $k_{(0)}$ of a flexural plate wave in vacuum $k_{(0)} = [\sqrt{3}\omega c_l / (c_t \sqrt{c_l^2 - c_t^2} d)]^{1/2}$. The second term, proportional to ρ_f/ρ_s , gives a small correction $k_{(1)}$, describing liquid loading. It is seen that this term becomes imaginary in supersonic regime of wave propagation, when a local wave number of a plate wave in vacuum exceeds the wave number of sound in liquid. Note that Eq. (11) is not valid for very small values of d . It also becomes invalid for those values of d for which the velocity of flexural wave is equal to the velocity of sound in liquid. In both these cases the contribution of the second term in square brackets of (11) is not small and the perturbation solution can no longer be applied. Regarding wedge waves, this means that the above considered perturbation approach is not applicable for wedges with very small apex angles θ and for velocities of wedge waves approaching the velocity of sound in liquid.

To apply the Bohr–Sommerfeld type equation for calculating wedge wave velocities in the case considered, one should follow the well known way of generalization of geometrical acoustics for complex wave numbers¹³ and substitute real part of Eq. (11) into Eq. (1). Doing so and using the nondimensional notations $\eta = c/c_t$ and $z = k_t d = x\Theta \omega/c_t$, one can derive the following equation versus η :

$$\int_0^{B\eta^2} \left[\frac{B}{z} \operatorname{Re} \left(1 + \frac{\rho_f}{\rho_s} \frac{1}{z \sqrt{(B/z) - (c_t^2/c_f^2)}} \right) - \frac{1}{\eta^2} \right]^{1/2} dz = \pi n \theta, \quad (12)$$

where $B = \sqrt{3}c_l/\sqrt{c_l^2 - c_t^2}$. Note that the above described geometrical acoustics approach also allows calculation of the wedge wave energy loss factor 2γ due to the radiation of sound into liquid, so that the wave number of attenuated wedge waves can be written in the form $\beta' = \beta(1 + i\gamma) = (\omega/c)[1 + i\gamma]$. However, calculation of γ is out of the scope of this paper devoted to wedge wave velocities only.

II. RESULTS

The results of numerical calculations of wedge wave velocities for Plexiglas wedges in water obtained according to Eqs. (6) or (9) ($\rho_s = 1180 \text{ kg/m}^3$, $\rho_f = 1000 \text{ kg/m}^3$, $c_f = 1478 \text{ m/s}$, $c_l = 2732 \text{ m/s}$, $c_t = 1363 \text{ m/s}$, $\sigma = 0.334$; these yields $A = 1.466$) are shown in Fig. 2 for different apex angles. Following Ref. 1, they are presented as the ratio $c_{\text{wat}}/c_{\text{vac}}$ between the velocities of the first order localized modes in immersed wedges and in the same wedges in vacuum (solid curve). Note that the slope of the theoretical curve $c_{\text{wat}}/c_{\text{vac}}$ displayed in Fig. 2 decreases as the wedge angle increases. This reflects the fact that, according to Eqs. (2) and (9), the function $c_{\text{wat}}/c_{\text{vac}}$ is proportional to $\Theta^{1/2}$. The corresponding experimental results¹ are displayed in the same picture. It is clearly seen that the agreement between the theory and the experiment is remarkably good, although for values of the wedge apex angle larger than 30 degrees the geometrical acoustics theory is not expected to be

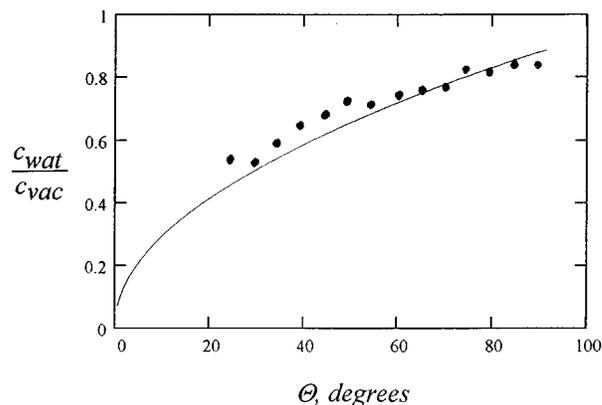


FIG. 2. Theoretically calculated ratio $c_{\text{wat}}/c_{\text{vac}}$ for Plexiglas wedges as a function of the apex angle (solid curve); points indicate the corresponding experimental results Ref. 1.

accurate.^{5,6} The most likely reason for such a good agreement is that, because of the presentation of the results in terms of relative values of wedge wave velocity, as compared with wedges in vacuum, the corresponding systematic errors caused by the limits of applicability of thin plate theory to plates of relatively high local thickness occurring for large apex angles Θ are expected to be the same for immersed wedges and for wedges in vacuum. Therefore, they might cancel each other.

The velocities for brass wedges in water ($\rho_s = 8600 \text{ kg/m}^3$, $\rho_f = 1000 \text{ kg/m}^3$, $c_f = 1478 \text{ m/s}$, $c_l = 4350 \text{ m/s}$, $c_t = 2127 \text{ m/s}$, $\sigma = 0.343$; these yields $B = 1.986$) were calculated in the process of numerical solution of Eq. (12). In doing so, we introduced a small artificial damping under square root in the second term of (12)—to avoid singularity when the value of local flexural wave velocity becomes equal to the velocity of sound in liquid. The results are shown in Fig. 3 as the ratio $c_{\text{wat}}/c_{\text{vac}}$ between the velocities of the first order wedge mode in immersed wedges and in the same wedges in vacuum (solid curve). Note that a small local minimum around $\Theta = 47$ degrees corresponding to the damped singularity indicates the values of the apex angle where the above theory is not accurate. It is not clear whether

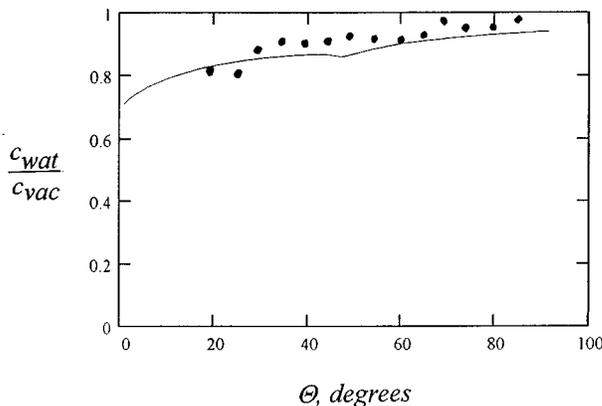


FIG. 3. Theoretically calculated ratio $c_{\text{wat}}/c_{\text{vac}}$ for brass wedges as a function of the apex angle (solid curve); points indicate the corresponding experimental results Ref. 1.

this small minimum is present in reality or it is just a sequence of limitation of the perturbational approach considered. The corresponding experimental results¹ displayed in the same picture fluctuate significantly due to experimental errors and do not clarify this question. Therefore, there is no point on this stage to investigate it in more detail. Regarding the average behavior of experimental points, it is fair to say that the agreement between the theory and the experiment in Fig. 3 is good enough for wedge angles from 20 to 90 degrees used in the experiment. One can expect that the reason for such a good agreement for brass wedges is the same as in the case of Plexiglas wedges. Making more definite conclusions on the limits of applicability of the simple thin plate theory approximation for both these cases would require using more advanced theories of plate flexural vibrations that goes beyond the scope of this paper.

III. CONCLUSIONS

Resuming the above, one can say that the geometrical acoustics approach to the theory of localized vibration modes in immersed solid wedges provides clear understanding of the phenomena involved and gives results which are in good agreement with the existing experiments for both subsonic and supersonic regimes of wave propagation. In particular, it has been demonstrated that for relative values of wedge wave velocity, as compared with those for wedges in vacuum, the theory provides good quantitative agreement with the experiments on Plexiglas samples also for large values of the apex angle. A generalization of the theory to describe the effect of heavier wedge material and a supersonic regime of wave propagation has been carried out. The results show good agreement with the existing velocity measurements for immersed brass wedges.

ACKNOWLEDGMENTS

The author would like to thank the referees of the paper for useful comments and the Associate Editor, Professor A.

N. Norris, for his idea of deriving the analytical expression (9) for velocities of localized vibration modes propagating in wedges made of light materials.

- ¹M. de Billy, "On the influence of loading on the velocity of guided acoustic waves in linear elastic wedges," *J. Acoust. Soc. Am.* **100**, 659–662 (1996).
- ²V. V. Krylov, "Propagation of wedge acoustic waves along wedges imbedded in water," *Proc IEEE Ultrasonics Symposium*, Cannes, France, Cat. #94, CHO 793–796 (1994).
- ³J. R. Chamuel, "Flexural edge waves along free and immersed elastic waveguides," in *Review of Progress in Quantitative Nondestructive Evaluation*, Vol. 16, edited by D. O. Thompson and D. E. Chimenti (Proc. 16th Symp. Quant. Nondestruct. Eval., 28 July–2 August 1996, Brunswick, Maine) (Plenum, New York, 1996), pp. 129–136.
- ⁴A.-C. Hladky-Hennion, P. Langlet, and M. de Billy, "Finite element analysis of the propagation of acoustic waves along waveguides immersed in water," *J. Sound Vib.* **200**, 519–530 (1997).
- ⁵V. V. Krylov, "Conditions for validity of the geometrical-acoustic approximation in application to waves in an acute-angle solid wedge," *Sov. Phys. Acoust.* **35**, 176–180 (1989).
- ⁶V. V. Krylov, "Geometrical-acoustics approach to the description of localized vibrational modes of an elastic solid wedge," *Sov. Phys. Tech. Phys.* **35**, 137–140 (1990).
- ⁷V. V. Krylov and D. F. Parker, "Harmonic generation and parametric mixing in wedge acoustic waves," *Wave Motion* **15**, 185–200 (1992).
- ⁸P. E. Lagasse, I. M. Mason, and E. A. Ash, "Acoustic surface waveguides—analysis and assessment," *IEEE Trans. Sonics Ultrason.* **20**, 143–154 (1973).
- ⁹S. L. Moss, A. A. Maradudin, and S. L. Cunningham, "Vibrational edge modes for wedges of arbitrary interior angles," *Phys. Rev. B* **8**, 2999–3008 (1973).
- ¹⁰J. McKenna, G. D. Boyd, and R. N. Thurston, "Plate theory solution for guided flexural acoustic waves along the tip of a wedge," *IEEE Trans. Sonics Ultrason.* **21**, 178–186 (1974).
- ¹¹M. C. Junger and D. Feit, *Sound, Structures and their Interaction* (Acoustical Society of America, Woodbury, NY, 1993).
- ¹²D. G. Crighton, A. P. Dowling, J. E. Ffowcs Williams, M. Heckl, and F. G. Leppington, *Modern Methods in Analytical Acoustics* (Springer-Verlag, London, 1992).
- ¹³Yu. A. Kravtsov and Yu. I. Orlov, *Geometrical Optics of Inhomogeneous Media*, Springer Ser. Wave Phen., Vol. 6 (Springer-Verlag, Berlin, 1990).