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Application of dispersion relations for the analysis of surface-wave scattering

V. V. Krylov

Physics Faculty, M. V. Lomonosov State University, Moscow
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Aspects of the applicability of the theory of dispersion relations for the analysis of surface-wave scattering are discussed. A dispersion relation with two subtractions is used to calculate the phase shift of the coefficient of transmission of a Rayleigh wave across an obstacle with vertical walls; this problem is not amenable to direct analytical treatment. The results are compared with existing experimental data.

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The investigation of surface-wave scattering by inhomogeneities of the free boundary of a solid is important in relation to seismology, ultrasonic surface flaw detection, acoustoelectronics, and the physics of surface effects. The greater part of analytical studies of the scattering of surface and bulk waves has been carried out by means of the first approximation of perturbation theory. This fact is attributable to the considerable difficulties that arise in the second approximation, even for smooth periodic inhomogeneities. In the case of isolated obstacles, on the other hand, it is impossible in general to obtain the second approximation analytically, because in coordinate space the standard analytical expressions for the surface Green's tensor are valid only in the radiation far zone. However, certain second-order effects are significant and must be taken into account in the analysis of devices that use surface-wave scattering processes. Foremost among such effects is the phase variation of a surface wave when it is transmitted across an obstacle; this effect plays an important part in surface-wave resonators. Also of definite significance is the second-order effect of surface-wave reflection from a periodic grating that does not contain second Bragg harmonics of the perturbations.

A powerful tool in coping with the stated difficulties is afforded by the general laws of scattering processes that do not depend on the particular physical nature of the obstacles. These laws include, for example, the scattering conservation laws, which lately have received increased attention. In addition to the conservation laws, an important role in scattering theory is also played by dispersion relations, which relate the real and imaginary parts of the functions characterizing scattering. Whereas the conservation laws, which impart unitarity to the S-matrix, usually follow from the temporal homogeneity of the scattering system, the dispersion relations follow from another general principle: causality, which imparts analyticity to the S-matrix or, what are associated one-to-one with the latter, the scattering amplitudes in the domain of complex wave numbers. To the best of our knowledge, these problems have not been discussed in the literature in application to the scattering of surface waves.

We discuss certain aspects of the application of the theory of dispersion relations to the analysis of surface-wave scattering, with emphasis on the most characteristic attributes of this case, and we give an example of the specific calculation, using those attributes, of the phase shift of the coefficient of transmission of a Rayleigh wave across an obstacle with vertical walls.

As in the scattering of bulk waves in unbounded space, the causality condition for surface waves implies that when an unbounded wave packet is incident on an inhomogeneity, the corresponding scattered wave packet cannot appear at an observation point situated at a large distance r from the scatterer in the direction ϕ (we consider the two-dimensional problem for definiteness) before a certain lapse of time t0 required for the packet to travel from the scatterer to the observation point. A consequence of this fact is the analyticity of the scattering amplitude f_RR(ϕ, k_R) as a function of the complex wave number k_R in a certain domain of the complex plane of k_R (Refs. 16 and 17). Here the subscript R signifies that the corresponding quantities refer to a Rayleigh wave.

However, all of the final conclusions are equally valid for other types of surface waves. R is readily shown to be, in the case of bulk-wave scattering in an inhomogeneous medium, the most important quantity from the point of view of application of the dispersion relations is the amplitude of forward scattering of a surface wave, as characterized by the value ϕ = 0. Here the shape and dimensions of the obstacle do not enter into the notational representation of the causality condition, and so the corresponding dispersion relations associated with the analyticity of f_RR(ϕ = 0, k_R) are valid for any scatterers and have the most fundamental significance. For brevity in what follows, in place of f_RR(ϕ = 0, k_R) we shall write simply f_RR(k). It is readily verified by analogy with Ref. 17 that the causality condition implies analyticity of the function f_RR(k) in the lower half-plane of the complex variable k.

To answer the question of the existence of dispersion relations it is also necessary to investigate the asymptotic behavior of f_RR(k) as k → ∞. Let the scattering obstacle represent a recess in the surface of a solid. Then, for large real values of k, the quantity f_RR(k) clearly describes the additional phase lead as the Rayleigh wave bends around the obstacle and thus represents an exponential growth function. For this reason, dispersion relations are not directly applicable to f_R(k). This fact, however, does not pose a major sacrifice, because in practice one is usually concerned not with the forward-scattering am-

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plutide itself, but with the related Rayleigh wave transmission coefficient $K_T = 1 + i\eta(m)(k)$, the modulus of which is easily determined experimentally or can be calculated by means of the optical theorem from known values of $|K_{\text{ref}}|$ and $|K_V|$, where $K_{\text{ref}}$ and $K_V$ are the coefficients of reflection and volume scattering, respectively. In this case it is natural to consider the dispersion relations for the modulus and phase of the analytical function $K_T(k)$ (Ref. 17). Accordingly, we analyze the expression

$$G(k) = \ln K_T(k) = \ln |K_T(k)| + i\theta(k),$$

(1)

in which $\theta(k)$ is the phase of the transmission coefficient. Then

$$K_T(k) = \exp \left(\ln |K_T(k)| + i\theta(k)\right).$$

as much as $K_T(k)$ is analytical in the lower half-plane of the complex variable $k$, the function $G(k)$ is also analytical in this half-plane everywhere except at the possible zeros of $K_T(k)$, which could impart logarithmic singularities to $G(k)$. In accordance with the general theory, $K_T(k)$ can have zeros only on the real axis. It follows from physical considerations, however, that $K_T(k)$ does not vanish anywhere on the real axis either (of course, this is not true of $\eta(m)(k)$, which is equal to zero at $k = 0$). We now consider the asymptotic behavior of $G(k)$ as $k \to \infty$. Clearly, $G(k) = \theta(k)$. Therefore, elementary dispersion relations of the Kramers-Kronig type, whose existence demands quadratic integrability of $G(k)$, are not applicable to the function $G(k)$. Neither are relations with a single subtraction valid. However, dispersion relations with two or more subtractions are applicable. In particular, the dispersion relation with two subtractions has the form

$$G(k) = G(k_0) + (k-k_0)G'(k_0) - \frac{(k-k_0)^2}{\pi} \int_{-\infty}^{\infty} \frac{G(k')-G(k_0)-(k-k_0)G'(k_0)}{(k'-k_0)^2} \, dk'.$$

(2)

where $k_0$ is a certain real value of $k$ and the integral is interpreted in the Cauchy principal-value sense. We note that the sign in front of the integral in (2) differs from the sign in Ref. 17, because the function $G(k)$ is analytic in the lower half-plane of $k$. Taking the real and imaginary parts of expression (2), we at once obtain two relations between $\text{Re}G(k)$, and $\text{Im}G(k)$. In particular, the extraction of the imaginary part with regard for (2) provides us with the dispersion relation we want, expressing $\theta(k)$ in terms of the values of $\ln K_T(k)$ on the real axis:

$$\theta(k) = \theta(k_0) + (k-k_0)\theta'(k_0) + \frac{(k-k_0)^2}{\pi} \int_{-\infty}^{\infty} \frac{\ln|K_T(k')| - \ln|K_T(k_0)| - (k-k_0)\ln|K_T(k_0)|}{(k'-k_0)^2} \, dk'. $$

(3)

We use the dispersion relation (3) to estimate the phase shift of the coefficient of transmission of a Rayleigh wave across an obstacle with vertical walls. We use the well-known function $|K_T(h/\lambda)|$ calculated by numerical methods (Ref. 18) in the interval $0 \leq h/\lambda \leq 1$. Here $\lambda = 2\pi/k$, and $h$ is the depth of the obstacle. Since the phase shifts for such obstacles do not admit calculation by means of perturbation theory and the numerical calculation of the phases is considerably more complicated than the calculation of the amplitudes and is not particularly accurate, the indicated estimation is of major interest.

Taking $k_0 = 0$ and using the fact that $\theta(0) = \ln |K_T(0)| = 0$ and $\theta'(0) = [\ln |K_T(0)|]' = 0$ (the latter equality is a consequence of perturbation theory, which shows that in the first approximation with respect to $k$ forward scattering from an obstacle with vertical walls is absent), we transform from (3) to the simpler notation

$$\theta(k) = \frac{k^2}{\pi} \int_{-\infty}^{\infty} \frac{\ln|K_T(k')|}{(k'-k)^2} \, dk'.$$

(4)

The quantity $\ln K_T(x)$, where $x = h/\lambda$, plotted in accordance with Ref. 18 (see Fig. 1), can be approximated by the simple relation

$$\ln |K_T(x)| = -20x^2/(1+3x^2+4x^2).$$

(5)

Substituting this expression into (4) and transforming to the variable $x$, we obtain

$$\theta(x) = \frac{-20x^2}{\pi} \int_{-\infty}^{\infty} \frac{dx'}{((1+3x^2+4x^2)(x'-x))^2}. $$

(6)

To calculate (6) we go to integration with respect to the complex variable $x'$, using the contour of integration shown in Fig. 2. The integral (6) is equal to the sum of the half-residue at the point $x' = x$ and the residue at the point $x' = \infty = (-3+iv7)/8$, corresponding to the root of the equation $1 + 3x^2 + 4x^2 = 0$, in the upper half-plane. Therefore,

$$\theta(x) = (22.6x^2 + 62x^2)/(1+3x^2+4x^2).$$

(7)

It is readily verified that $\theta(x) \approx 22.6x^2$ for $x \ll 1$. Comparing this expression with the experimental relation for the phase shift $\theta(x) = 25.2x^2$ measured for a Y-Z LiNbO$_3$ crystal, which is characterized by the same equivalent Polisson ratio ($c = 0.31$) as that used in Ref. 18, we see that the agreement is excellent. For $x \gg 1$ the phase shift of the transmission coefficient assumes the value $\theta(x) \approx 5.56 + 15.5x$. Both terms admit a very simple interpretation in this case. Thus, 5.56 rad is obviously the sum of the phase shifts that occur in transmission of the
Rayleigh wave across the four right-angle corners of the rectangular obstacle\(^\text{19,20}\) and \(15.5x\) is the additional linear phase lead due to bending of the surface wave around the obstacle. It follows from simple geometrical considerations that the latter is equal to 2\(\pi k = 4\pi k \approx 12.5x\), which is highly consistent with the value calculated on the basis of the dispersion relation. The given example shows that what appear at first glance to be different parameters characterizing surface-wave scattering are in fact interrelated by equations of the type (2)–(4) ensuing from the causality principle. These relations can be used not only to reconstruct the function \(G(x)\) completely or, equivalently, \(K_T(x)\) and \(G_R(x)\), from the known behavior of the real or imaginary part, as above, but also to apply various limiting cases (in accordance with the present discussion) and qualitative considerations to this same objective.

At this point we mention another fact typical of surface-wave scattering, namely the asymmetry of the functions \(\ln K_T(x)\) and \(\delta(x)\) with respect to the change of sign of \(x\) [see (5) and (7)]. The same is true, of course, of the real and imaginary parts of the scattering amplitude. It is generally known that the real and imaginary parts of the scattering amplitude or their moduli and phases in the case of scattering of homogeneous waves are even and odd functions of the frequency, respectively.\(^\text{17}\) It can be shown that this is not true in general for surface waves, owing to the violation of translational invariance of the corresponding space due to the presence of the boundary. A similar asymmetry occurs in connection with violation of the evenness of the attenuation coefficient for a Rayleigh wave propagating along a three-dimensional isotropic rough surface\(^\text{21}\) (see also Ref. 21). Here the attenuation in the long-wave limit is proportional to \(\omega^4\), where \(\omega\) is the cyclic frequency, and does not satisfy the well-known property of homogeneous waves, namely symmetry of the refractive indices \(n(\omega)\) characterizing attenuation and dispersion, \(n(-\omega) = n^a(\omega)\) or \(\text{Re}n(-\omega) = \text{Re} n(\omega)\) and \(\text{Im}n(-\omega) = -\text{Im}n(\omega)\) (Refs. 16 and 17). We note that the \(-\omega^4\) law for the attenuation of a Rayleigh wave has been observed experimentally by optical methods.\(^\text{22}\) In application to dispersion relations of the type (2), the above-described symmetry violation incurs certain difficulties to the extent that the transition to integration over the physical frequency domain \(\omega > 0\) is impossible (Refs. 16, 17). The simplest way to surmount these difficulties is to approximate the functions used in the dispersion relations by analytical functions for \(\omega > 0\) and then to continue them analytically into the nonphysical domain \(\omega < 0\). This is in fact the procedure we have used in the foregoing example of calculating the phase shift of the Rayleigh-wave transmission coefficient.

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Translated by J. S. Wood

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