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Measurement of the mechanical properties granular packs by wavelength scanning interferometry

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Abstract

A wavelength scanning interferometer has been constructed to observe both the normal and in-plane displacements particle by particle at the base of a model granular pack. The pack comprised 25,000 of steel beads supported by a thick glass substrate and subjected to local disturbing forces on its upper surface. The system allows measurement of normal displacements of the beads to a precision of ca. 0.1 nm, thereby providing highly accurate determination of contact forces whilst minimizing artefacts due to substrate and grain compliance. The probability distribution of the normalized contact force was found to be approximately independent of the applied load on the upper surface of the granular pack and has an exponential tail. The probability distribution of the normalized response force and lateral displacement have similar power law tails. The interactions between contact forces and lateral displacements suggest that significant internal rearrangement occurs in the granular pack as the load is increased, and particle displacement plays an important role in the mechanics of the granular material.
I. INTRODUCTION

Granular materials are conglomerations of discrete macroscopic particles, which behave very differently from the ordinary solids, liquids and gases we know from physics textbooks [1-4] since their complex force transmission leads to highly nonlinear behaviour and presents considerable challenges to producing quantitative, predictive models [1-4]. A common approach to developing an understanding of the mechanical properties of granular media is to study the system at the single particle level (microscopic behaviour), and connect this to the readily observable bulk properties (macroscopic behaviour) [5-6].

Continuum models have been successful in describing the mechanical behaviour of granular media [5-6]. These models, however, rely on appropriate constitutive relations to provide closure and it has not been possible (to date) to derive these from first principles. A number of researchers have posited constitutive relations for granular packs based on observations [5-10] and the search for suitable constitutive descriptions has led to a multiplicity of descriptions of the force propagation, commonly in the form of PDEs that can be categorised as hyperbolic, parabolic, or elliptic [1-6], and through numerical analysis of isostaticity or the marginally rigid state [11-17]. At their core, though, these descriptions are not complementary [1-4] which is compounded by the relatively small number of experiments that have so far been used to test these theoretical results [16-26].

Two recently reported experiments present contrasting results, highlighting some of the outstanding issues in the static mechanics of granular materials [27]. The first experiment created jammed and unjammed states by rotating a plunger on the beads. The tails of the force probability distribution were found to change when the system goes from the jammed to the unjammed state. In the case of the second experiment the tails of the force probability distribution were found to decay faster than exponential when the system undergoes volume changes [28]. One interpretation is that granular packs change from the isostatic state to the hyperstatic state when moving along the volume fraction axis in a phase diagram of shear force, volume fraction and temperature [29]. The particle rearrangements that must occur lead to more contacts and therefore to more contact forces around the mean value (normalized force \( f = F/\langle F \rangle = 1 \), where \( F \) is the normal force component and \( \langle ... \rangle \) denotes ensemble
average). The end result is that the contact force probability distribution in the region of $f > 1$ decays faster than the exponential tail and a peak around $f = 1$ (similar to solid behaviour) appears. A direct analogy of this behaviour can be observed in the system of compressible particles and emulsions etc. [41-43]. By contrast, for the case of non-compressible particles, when the volume was not changed significantly, no peak in the probability distribution was observed [20-21].

Because of the contradictory experimental results above, there is a need to examine further the mechanical behaviour of granular media at the single particle level. It is experiments of this kind that are required to provide benchmark data and validate theoretical predictions.

In this paper, we will present the results of a new experimental technique that examines the response of a granular pack to perturbations in the applied stress state. Our technique is based on phase-shifting interferometry, which in addition to fine displacement resolution has two important advantages over previously used techniques. Firstly, it is capable of a wide range of load measurements from the self-weight of a small fraction of a single bead up to hundreds of beads, and crucially can observe both increases and decreases in the contact forces at the base of a deep granular pack. Secondly, in addition to the normal forces it is able to determine lateral displacements in the contact positions down to less than 0.1 $\mu$m. These advantages, when taken together, give the technique particular power when investigating at the microscopic level the transmission of forces and the associated particle rearrangements.

In the following sections, we firstly introduce the system configuration and our phase-shifting algorithm. The system calibration results are shown at the end of section II. In section III, the spatial and probability distributions of contact forces, and the experimentally-determined Green’s functions for both contact forces and lateral displacements, are presented in detail.

II. EXPERIMENTAL DETAILS

We have developed an experimental technique based on the principle of interferometry that enables us to measure deformations much smaller than that produced by the self-weight of a single particle on a stiff glass surface. Our broad
approach is as follows. A granular pack is placed within a container that has a glass viewing port cut into the bottom surface. This pack is then placed above a Fizeau interferometer. By imaging the bottom surface of the glass viewing port, which also serves as a stiff, but deformable substrate, we are able to determine the magnitude of the small deformations of the upper surface of the glass substrate produced by the steel-glass contacts. The technical details of this process are described below.

A. Fizeau interferometer

The technique is based on a Fizeau interferometer design and works on the principle that light incident onto a glass plate is split into a wavefront that is reflected by the lower glass-air interface of the plate and a part that is transmitted through the glass. This transmitted light is then reflected by the upper mirrored surface and subsequently interferes with the light reflected at the first surface. In our system, the glass plate or substrate is wedge shaped, which results in a characteristic pattern of parallel fringes when interference occurs. Any modification of the surfaces results in a deviation from the parallel fringe pattern, and it is these differences that we measure in our experiments.

The experimental arrangement is shown in Figure 1. The linearly polarized light beam emitted from the laser is enlarged by the objective lens, OL, collimated by the field lens, FL and directed onto the optical wedge by means of the mirrors M1, M2 and M3. The enlarged parallel beam is reflected by the two surfaces of the optical wedge, forming the interference fringes. The light source is an external cavity diode laser (New Focus Vortex 6005) with a wavelength centred at 635.05 nm that can be scanned by about 0.162 nm. In our experiment the wavelength of the laser is scanned in order to alter the phase differences between the interfering beams in a controlled fashion. The fringe patterns are imaged onto a CCD camera (VDS Vosskühler GmbH CCD-1300QFB), and then captured and transferred to a PC for analysis.

The optical wedge is made from optical glass BK7 (refractive index $n \approx 1.51$, elastic modulus 82 GPa, with a rectangular geometry of area $160 \times 10$ mm$^2$ and wedge angle $0.006^\circ$) and provides a substrate which both supports the granular pack and provides the means by which we can determine the particle normal forces. On the top surface of the wedge a reflective Ni80/Cr20 layer is applied of sufficient thickness to ensure negligible transmission of light to the granular pack. When beads are put into the
container, the contact forces at the bottom lead to a localized deformation of the upper surface of the optical wedge whilst the lower (glass-air) boundary is largely unaffected, resulting in a small change of optical path difference, which is, in turn, detected by the interferometer.

B. Phase-shifting interferometry

The magnitude of the deformations caused by the indentation of beads on the substrate surface are potentially down to the order of a few nanometres, and we therefore require high resolution from the technique. This is achieved by making small changes in the wavelength of the laser which as a result of the optical path difference between the two interfering waves causes small phase shifts in the interferogram. Phase-shifting in this way allows one to eliminate ambiguities in the sign of the phase signal and to reduce the effects of systematic and random errors, thereby enabling the measurement at each camera pixel of deformations of $\lambda/1000$ or better [30-33]. In order to increase the signal to noise ratio, we have designed a new sampling phase-shifting algorithm with improved performance over an earlier published algorithm [31-32]. This uses up to 114 frames and includes an iterative process to overcome non-linearity and mode hops in the laser output, resulting in a noise level in the deformation detection of the order of 0.1 nm [31]. Further details of phase-shifting techniques can be found in a wide number of sources in the literature [30-33].

C. Contact location and deformation peak

Having determined the change in deformation field for an experiment, our next step is to find the location of the contact points and determine the deformation magnitude for each. First of all, the deformation map is correlated with an adaptable 3D Lorentzian filter to find an approximate measure of the contact point locations. Following this step, we perform a nonlinear least-square regression in the neighbourhood of each indentation [33-35], to find the exact location and magnitude of the deformation. The model deformation profile is given by

$$I(x, y) = I_0 + I_1 \cdot x + I_2 \cdot y - \frac{I_3}{1 + I_4 \cdot (x - l_x)^2 + I_5 \cdot (y - l_y)^2},$$

(1)
where the coefficients $l_0$, $l_1$ and $l_2$ account for the offset and gradient of the background plane, whilst $l_3$ is the peak deformation, $l_4$, $l_5$ are parameters defining the width of the deformation and $l_6$, $l_7$ are the coordinates of the peak deformation in a local Cartesian coordinate system $(x, y)$ lying in the plane of the glass surface. Although classical Hertzian contact theory should provide a reasonable description of the deformation state near the contact region, the use of an empirical Lorentizan fit side-steps any questions about the validity of the Hertzian assumptions, and the influence of the finite point spread function of the imaging system. Clearly there is no underlying theoretical reason to choose the Lorentzian form of Eqn. (1), however in practice it has been found to give much closer fits to the experimental data than other functions such as a Gaussian distribution etc [33]. In Figure 2(a) we show the raw data for a sample deformation map of the glass substrate due to a load of two 8-mm-diameter steel beads (40.02 mN), and the fitted form of the deformation map obtained by fitting Eqn. (1) to the raw data (Figure 2(b)). Using this method, we are able to locate the position of a contact point to a precision of about ± 0.1 μm and the deformation to a precision of about ± 0.1 nm.

D. Calibration of substrate deformation vs. load

In order to determine the normal contact forces, it is necessary to calibrate the observed deformations against known forces. This was achieved by applying known loads to a single 8-mm-diameter bead placed on the surface of the substrate. A range of loads was applied, up to a maximum of 1045.8 mN. For each load the contact point and deformation were determined using the method outlined in Section C. The calibration curve for a bead of 8 mm is shown in Figure 3.

The calibration curve was characterised by performing a 2nd order linear regression on the data based on the equation,

$$F = S_1 \cdot l_3^2 + S_2 \cdot l_3 + S_3,$$  \hspace{1cm} (2)

where $F$ is the bead load (8 mm steel ball, 20.01 mN) and $S_1$, $S_2$ and $S_3$ are calibration coefficients. For the data shown in Figure 3, these were calculated to be $S_1 = 0.000425 \text{ bead/nm}^2$, $S_2 = 0.215812 \text{ bead/nm}$ and $S_3 = 0.0091633 \text{ bead}$. The results show that the standard deviation of contact force measurement is $\sim \pm 2.7\%$. 

MechanicalPropertiesGranularPack_repository.docx
E. Experimental procedure

A key test of available theories of force propagation in granular materials is the system response to ‘point’ loads. When a load is applied to the top surface of the granular pack, the particles transmit their loads to particles below them. The particles resting on the bottom surface transmit this extra load to the support, and it is this change in contact force and position that were measured in the experiments. We used three applied forces, of magnitude $L_1 = 249.4$ beads, $L_2 = 394.3$ beads and $L_3 = 872.6$ beads (equivalent to 4.99 N, 7.89 N and 17.46 N respectively), placed onto the centre of the top surface of the granular packs, using the punches and disc shown in Figure 1(b). The 48 mm diameter disc was used to spread the load over a few beads to reduce the effect of penetration into the pack. We employed steel balls (AISI 52100 Low Alloy Chrome Steel) of diameter $d_B = 8$ mm and mass $m_g = 20.01$ mN as our granular medium. These high precision bearings have a spherical error of 2.5 µm and a maximum lot diameter variation of 5.0 µm. In our experiment, the beads are placed within a cylinder of diameter 424 mm, and height 130 mm, equivalent to a diameter of 53 beads and height 16 beads respectively. A viewing slot, in which the optical wedge resides, was cut into the bottom surface. The whole container is supported over the interferometer, allowing access to the illuminating light via the mirror M3 and the collimating field lens FL (Figure 1(a)). The whole interferometer sits on a translation stage and is thereby able to scan the length of the glass window underneath the container.

The packs were created by pouring beads through a hopper, whose outlet was about 20 mm above the packs to limit impact damage during deposition. The pack surface was then “flattened” by removing the excess material without disturbing the rest. The final height of the granular pack was equivalent to 11 layers of HCP packed beads.

We consider two datasets denoted here Experiment 1 and Experiment 2. Each dataset corresponds to a combination of up to 15 groups of trials. At each trial, a new granular pack was rebuilt to randomize the bead positions relative to those in the previous trial. Contact forces were measured for both the self-weight of the pack, and after applying the different local disturbing forces to the centre of the top surface of the packs. For each state, the interferometer was scanned across the field of view (160×10 mm$^2$) of
the optical wedge in 15 steps. The loads and the notation for each set of experiments are shown for clarity in Table 1.

III. ANALYSIS

A. Contact force distributions

Figure 4(a) shows the contact deformation distribution and contact point map from one trial with one loading state (Experiment 1C in Table 1). Figure 4(b) shows the measured locations of the contact points in the \((x, y)\) plane, with the particle cross-sections shown to guide the eye. The system is able to pick out the majority of the particles within the field of view, but there are some regions where they are unable to be located, primarily due to damage to the chrome film of the optical wedge. These regions covered approximately 10% of the field of view.

B. Force correlations

In this section, we consider the correlations between the contact forces before the local disturbing force is applied, \(F_s\), and those after, \(F_a\). Figure 5(a) shows \(F_s\) in the unloaded state versus \(F_a\) for the applied load of \(L_1 = 249.4\) beads. We can see from the clustering of points around a gradient close to 1 that the contact forces are not modified significantly. When we increase the load to \(L_3 = 872.6\) beads (E, see Table 1), however, we see some changes (Figure 5(b)). The low magnitude contact forces are still, in general, clustered about a gradient \(\sim 1\), suggesting that in this range of contact forces, the addition of the localized forces at the surface results in no major change in behaviour. For larger contact forces, however, the contact forces are now clustered about a gradient noticeably larger than 1. Additionally, we see that there are a few points that are close to the axes, suggesting a switching from low force to high force and vice versa, reminiscent of what might be expected to occur if a particle suddenly became incorporated into a force chain, or alternatively, removed from one.

C. Contact force probability distribution

A contact force probability distribution was built up by combining the 15 trials for each experiment and binning the measured contact forces. Figure 6(a) shows the contact force probability distribution \(P(f)\), where the contact force \(F\) has been normalized by the global mean value \(<F>\) for 5 sets of data (A to E, see Table 1), respectively. The distributions are similar and show very little deviation from each other, within the scatter of the data. This similarity suggests that we are justified in
combining all the data into one figure to determine an average behaviour (see Figure 6(b)).

Previous work in this area has suggested that we should expect an exponentially decaying tail with a decay constant close to 1 [18-24]. By examining the region where \( f > 1 \), we determined that this was broadly true for our system. A fit of the form

\[
P(f) \propto \exp(-\beta \cdot f)
\]

was found to agree closely with the data, where the decay constant was found to be \( \beta = 1.16 \pm 0.11 \), and the intercept with the coordinate axis to be \( P(0) \approx 0.69 \). In Figure 6(b) we compare our experimental results with the form predicted by Mueth [21] (i.e., \( P(f) = a \cdot (1 - b \cdot e^{-f}) \cdot e^{-\beta f} \), where \( a = 3, b = 0.75 \) and \( \beta = 1.5 \)).

Although there is some consensus on what the form of the tail of the contact force probability distribution should look like, there is still considerable debate regarding its behaviour at small forces. Some theoretical predictions show that a peak is expected to appear in the region around \( f = 1 \) [36-39]. There are other suggestions, however, that the peak disappears even when the pack of hard particles is in the jammed state [40]. Our results show no peak but rather that the force plateaus as \( f \) tends to zero (Figure 6(b)).

**D. Fraction of load borne by intervals of contact force**

In granular media, the load is supported by the propagation of stresses through the contacts. On a microscopic or single particle scale, this propagation is often thought of as being through chains, where a substantial fraction of the load is channelled through the chain. If one considers that the large contact forces observed correspond to the end points of force chains, and the small contact forces correspond to grains in the matrix, one can gain insight into the fraction of load borne by the chains and that borne by the matrix by considering the distribution of the fraction of load, \( \Phi(f) \), calculated using

\[
\Phi(f) = f \cdot P(f),
\]

Since \( P(f) \) does not appear to be strongly dependent on the loading, we combine the data for all the experiments in Table 1, and show \( \Phi(f) \) in Figure 7. The modal value of
$\phi(f)$ is around $f = 1$ and the mean value, as should be expected, is at $f = 1$. This means that most of the load is sustained by particles around the mean contact force, and that the larger contact forces, potentially force chains, support a proportionately low amount of the load.

E. **Coarse grained response forces and displacements in the spatial domain**

A granular pack can be observed to respond to a localized force on the upper surface by changes in the contact force distribution, and by changes in the location of the contact forces.

In our experiments, we measured the change in load at each grain visible through the wedge when a force was applied to the upper surface. These changes were then coarse grained to determine the average behaviour of the particles at the base as a function of position relative to the loading point. Figure 8(a) shows the response forces $\Delta F_{BA} = F_B - F_A$, $\Delta F_{CA} = F_C - F_A$ and $\Delta F_{ED} = F_E - F_D$ to the three perturbing forces, $L_1$, $L_2$ and $L_3$ respectively. Figure 8(b) shows the coarse grained absolute values of the responses $|\Delta F_{BA}|$, $|\Delta F_{CA}|$ and $|\Delta F_{ED}|$. For small local disturbing forces we see a small increase in the contact force distribution. As we increase the load to $L_3$, we see signs of a new behaviour emerging; there is a suggestion of a dip at the centre, in line with the predictions of double peaks [5-10]. This suggestion that there is a shift from a single peak to a double peak in the average contact force distribution, lends some support to the ideas of Goldenberg and Goldhirsch, who have shown in their simulations that there is a crossover from a single-peaked to a two-peaked response forces as the applied load is increased [5-6]. Other studies have shown similar results, e.g., Geng reported that as the disorder in a packing increased, the mean force propagation direction shifted from one peak to two peaks [20-24].

Although granular packs are often modelled as being infinitely stiff, in reality, the application of a load to the upper surface results in small displacements in the particle positions. Our experimental facility is able to detect lateral displacement of the particle contact points down to a resolution of ± 0.1 μm. The coarse-grained responses of the lateral displacements of the beads in the basal layer of the granular packs are shown in Figure 8(c), for different local disturbing forces, respectively. The lateral displacements due to the relevant applied loads are calculated using the equations
\[ D_{AB} = \sqrt{(l_6^B - l_6^A)^2 + (l_7^B - l_7^A)^2} \]
\[ D_{CA} = \sqrt{(l_6^C - l_6^A)^2 + (l_7^C - l_7^A)^2} \]
\[ D_{ED} = \sqrt{(l_6^E - l_6^D)^2 + (l_7^E - l_7^D)^2} \],

(5)

where \( l_6 \) and \( l_7 \) are given in Eqn. (1).

It is interesting to compare the spatial variations of the displacements with that of the response forces. Figure 8(d) to (f) show the coarse grained lateral displacements and response forces as a function of position. Generally we observe that the changes in force and displacement are anti-correlated: an increase in normal force tends to correspond to a small lateral displacement, whilst a reduction in the observed force tends to correspond to a large displacement.

F. Response force probability distribution

The response force probability distributions can be calculated by binning the changes in contact force following the application of the point load at the upper surface [36-39]. Figure 9(a) shows the response distributions for all three applied loads, where the change in contact force has been normalised by the applied load \( f_L = \Delta F/L \). Using this normalization procedure we see that the distributions lie close to one another, suggesting that the forces in the granular media are scaling with the applied load. From these distributions we can see clearly the positive and negative responses to the applied load; some contact forces have been reduced even though the applied load is compressive. The curves are asymmetric: the probability of finding a positive response is greater than that of finding a negative one, with the modal value being close to zero.

The response probability distribution, combined for all three applied loads, is shown in Figure 9(b). The peak of the distribution is located at \( f_L \sim 0 \). Its shape is a rather distinctive sharp peak with a slower fall off in the tails. We use the form,

\[ P \propto \exp(-\beta_L \cdot f_L), \]

(6)
where $\beta_\ell = 125 \pm 16$ in the region $0 < f_\ell < 0.04$, and $\beta_\ell = -258 \pm 18$ in the region $-0.02 < f_\ell < 0$.

The probability distribution of the magnitude of the changes in the response forces is shown in the Figure 9(c). This distribution obeys a power law, such that

$$P = P_L \cdot |f_\ell|^{\alpha_L}, \quad (7)$$

where $P_L = 8.34 \times 10^{-4}$ and $\alpha_L = -2.22$.

**G. Displacement probability distributions**

The probability distributions of the lateral displacements of the beads are shown in Figure 10(a), after normalization by the applied force $(\zeta = (D/d_a)/(L/mg))$. In the region $0 < \zeta < 2 \times 10^{-3}$, we see very little variation in the distributions with the applied load. After recombining all of the data into one set, the normalized lateral displacement probability distribution is shown in Figure 10(b) and appears to be described in the following way,

$$P = P_\delta \cdot \zeta^{\alpha_\delta}, \quad (8)$$

where the coefficients were found to be $P_\delta \approx 2.59 \times 10^{-9}$ and $\alpha_\delta \approx -2.48$.

**H. Relationship between response forces and displacements**

The lateral displacements can be considered as signatures of micro-deformations or rearrangements of the granular media at the microscopic level. In order to investigate in more detail the relationship between the normal response forces and the lateral displacements, we renormalize the absolute value of the normalized response forces $(f_\ell = |\Delta F/L|)$ and lateral displacements $(\zeta = [(D/d_a)/(L/mg)])$, by their maximum values respectively, and show their probability distributions in Figure 11(a). We found that their behaviour is quite similar, and each obeys a power law,

$$P = P_\circ \cdot (\zeta_{L,D})^\alpha, \quad (9)$$
where $P_0 \approx 0.013$, $\alpha \approx -2.22$ and $\xi_i = |f_i|/\max(|f_i|)$ for response forces, $\xi_d = \zeta / \max(\zeta)$ for displacements, respectively.

Although the main features of the probability distribution for the normalized displacement (Figure 10(a)) were found to be approximately independent of load, subtle load-dependent effects are revealed if the data is sorted first into a vector of monotonically-increasing values and then plotted as a function of the index of the vector. In the case of the normalized response forces (Figure 11(b)), very little difference is seen between the three load cases. However, the sequences of normalized displacements are quite different. Figure 11(c) shows that most normalized particle displacements decrease with the applied load but that the tails at large displacement increase rapidly with the applied loads. The maximum values of the normalized displacements, $\nu = \max[\zeta]$, are shown in Figure 11(d). Instead of an approximately constant value that might intuitively be expected, these are found to increase approximately linearly with the normalized applied loads. The absolute displacements of the most mobile grains are therefore found to increase approximately quadratically with the applied load. That the maximum displacement increases faster than the maximum response force when the applied load increases suggests that particle displacements may be responsible for the unjamming of the granular pack under a relatively large local disturbing force and thus potentially play a more direct role than the response forces in this process.

**IV. DISCUSSION**

Our experimental results show clearly that when a granular pack is subjected to a local disturbing force, there is a redistribution of contact forces and contact points, with some forces increasing, and some decreasing. At low forces, we observe no evidence of significant rearrangement, the pre and post loads are broadly correlated regardless of the size of the initial contact force. Consistent with this, the average spatial distribution of forces suggests a single peak. When we increase the perturbing load sufficiently, there is evidence that the physics of the system changes; the largest particle displacements become proportionally larger, suggesting grain rearrangements, there are indications of sharp fluctuations in the magnitude of the contact forces, and
there is some evidence pointing towards the forces being propagated along the surface of a cone, rather than via a diffusive mechanism.

One area of variation in the literature is the small force behaviour of the probability distribution [11]. Our results show that there is a plateau, that is to say the distribution becomes broadly constant, as we approach $f = 0$. Mueth, Makse and others [20-24, 36-40] have made similar observations, both theoretically and experimentally.

A network of non-cohesive grains with the minimum number required to be rigid is called isostatic or marginally rigid. Some of these ideas support theories published in the literature [16-17]. For example, Moukarzel [11-15] reported theoretical results for the case when an infinitesimal change in the length is applied to randomly chosen bonds in a 2D granular pack. The responses in a region many layers away from the perturbation were found to have positive and negative values, and to decay symmetrically around zero with exponential tails, in a fashion similar to our observations. The probability distribution of their displacements were also predicted to be a power law. Although Moukarzel’s predictions conform to our results well, there are some differences, such as the observation that negative response forces decay faster than positive forces, whereas theory predicts that they should be symmetrical around zero.

As a final comment on the importance of being able to measure both response forces and particle displacements, the Liu phase transition diagram [29] relating macroscopic stress and strain in a granular medium is often difficult to investigate experimentally, because of the small changes in volume fraction. This is especially true in the case of isostatic granular media since the global displacements of hard particles are small. In these situations, it is better to use the differential form of Liu’s diagram with two new microscopic parameters: the responses to the local disturbing force and the particle displacements. Both of these quantities can be measured with the technique presented here.

V. CONCLUSIONS

A granular pack, consisting of 25,000 nominally identical spherical beads within a cylindrical container, was the subject of an investigation to better understand the mechanisms by which forces propagate in such packs. The indentations of a
representative subset of the beads on a substrate at the bottom of the pack were measured by a phase-shifting Fizeau interferometer which, after data processing, allowed the normal force distributions and lateral bead displacements to be determined simultaneously to high accuracy. A significant improvement of our technique over those previously published is the capacity to accurately determine both increases and decreases in contact forces, recognized as a signature of a ‘fragile material’ [7], and lateral displacement particle by particle. Due to this, we were able to obtain the force probability distributions and the distributions of the changes in the forces, which showed characteristic exponential law forms. There is also a strong correlation between the magnitudes of the forces transmitted to the bottom of the pack, and the movement of the particles. The displacements of the most mobile grains were found to increase approximately quadratically with the applied load, a result which points to particle rearrangement being a feature of the changing physics that is observed as the disturbing forces grow in magnitude.

VI. ACKNOWLEDGEMENTS

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REFERENCE

Figure Captions

![Schematic diagram](image1)

![Photo](image2)

Figure 1: (a) Schematic diagram and (b) photo of the experimental system for measurement of the contact forces at the bottom of a granular pack.
Figure 2: Deformation map following the application of a 2-bead load to the substrate surface. (a) Raw data deformation map, (b) model form of the deformation map obtained by fitting Eqn. (1) to the data shown in (a).
Figure 3: Calibration curve for substrate deformation vs. normalized contact force; $I_3$ is the peak of the substrate deformation (nm) (Eqn. (1)), $F$ is the contact force (mN) and $mg$ is the weight of an 8 mm steel bead, 20.01 mN.
Figure 4: Contact deformation distribution in the whole field of view of the optical wedge, showing (a) contact deformation distribution in one of trial C (see Table 1), and (b) map of contact points. The central points represent measured contact centres, whilst the circles show the projection of the bead circumference onto the bottom of the granular pack.
Figure 5. Contact force correlations for before and after the local disturbing forces have been applied to the top surface. (a) $L_1 = 4.99N$, and (b) $L_3 = 17.46N$. $f_A, f_B, f_D, f_E$ are the normalized contact forces $f=F/<F>$ for experiments A, B, D and E, respectively.
Figure 6: Probability distributions of the normalized contact forces. (a) normalized contact force probability distributions for experimental runs A to E (see Table 1), and (b) the combined normalized contact force probability distribution compared against results from the carbon paper method by Mueth [21]. $f = F/\langle F \rangle$ is the normalized contact force, $P$ is the probability distribution.
Figure 7: Combined normalized contact force fraction, $\Phi$ (Eqn. (4)).
Figure 8: Coarse grained mean response forces and lateral displacements as a function of the radial position. (a) mean response forces for the applied loads $L_1$, $L_2$, $L_3$, (b) mean absolute values of response forces for the applied loads $L_1$, $L_2$, $L_3$, (c) mean lateral displacements for the applied loads $L_1$, $L_2$, $L_3$, (d), (e) and (f) the coarse grained mean absolute values of response forces and lateral displacements for the applied loads $L_1$, $L_2$ and $L_3$, respectively, where $\Delta F$ is the response force, $D$ is the lateral displacement, $y$ is the coordinate along the principal axis of the optical wedge, shown in the Fig.4(b) and $d_B$ is the bead diameter (8mm).
Figure 9. Probability distributions for the response forces, normalized by the applied loads $L_1$, $L_2$, and $L_3$, shown separately in (a) and combined in (b). (c) probability distribution of the magnitude of the normalized response forces (all datasets combined).
Figure 10: Probability distributions for the lateral displacements, normalized by the applied loads $L_1$, $L_2$, and $L_3$, shown separately in (a) and combined in (b).
Figure 11: Comparison between the response forces and lateral displacements as a function of applied load. (a) probability distributions $\xi_F$ and $\xi_D$, for response forces and displacements, respectively. (b) and (c), normalized sequences of normalized response forces, $\Delta F/L$, and normalized displacements $(D/d_b)/(L/mg)$, respectively, sorted in ascending order, where $n$ is the index of the sequence, and $N$ is the total number of contact points. (d) relationship between the applied forces, $(L/mg)$, and maximum lateral displacements, $v = \max[(D/d_b)/(L/mg)]$. 
## Table 1: Notation and system details for experiments A, B, C, D and E.

<table>
<thead>
<tr>
<th></th>
<th>Experiment 1</th>
<th>Experiment 2</th>
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<td><strong>Sequence of Experiment</strong></td>
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