Existence theorems for periodic solutions to partial differential equations with applications in hydrodynamics

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Existence theorems for periodic solutions to partial differential equations with applications in hydrodynamics

Gurjeet Singh Bagri

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Abstract

The thesis looks at a number of existence theorems that prove the existence of small-amplitude periodic solutions to systems of partial differential equations. The existence theorems we consider are the Hopf bifurcation theorem, the Lyapunov centre theorem, the Weinstein-Moser theorem, and extensions of these theorems; the Hopf-Iooss bifurcation theorem, the Lyapunov-Iooss centre theorem and the Weinstein-Moser-Iooss theorem, respectively. The theorems have been derived so that they are applicable to functional analytical problems, and have been represented in a coherent and uniform manner in order to bridge the fundamental structure common to them all. Applications of these theorems, in this standardised form, are then applied in a systematic way to two particular hydrodynamical problems; the water wave problem and the Navier-Stokes equations.

The classic water wave problem concerns the irrotational flow of a perfect fluid of unit density, subject to the forces of gravity and surface tension. We apply the Lyapunov-Iooss centre theorem to prove the existence of doubly-periodic waves; a doubly-periodic wave is a travelling wave that possess spatially periodic profiles in two different horizontal directions. Fundamental to our approach is the spatial dynamics formulation. The spatial dynamics formulation involves formulating a system of partial differential equations, defined on some spatial domain, as a dynamical system where one of the unbounded spatial variables plays the role of time. We catalogue a variety of parameter values for which it is possible to obtain doubly periodic waves, and we conclude with an existence result for doubly periodic waves under specific parameter restrictions.

The Navier-Stokes equations in an exterior domain models the flow of an incompressible, viscous fluid past an obstacle $\mathcal{O}$. We apply the Hopf-Iooss bifurcation theorem to the defining equations to determine the existence of time-periodic waves. Our approach involves a careful examination of the Oseen problem to which we apply a ‘cut-off’ technique. This technique is used to constructs a solution to the Oseen problem using the respective solutions to the Oseen problem on a bounded domain and free space (the existence of which are well established). Time-periodic solutions are established using the Hopf-Iooss bifurcation theorem provided certain spectral conditions are met. The verification of the conditions may only be possible numerically, and so beyond the scope of our investigation.
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“Truth is ever to be found in the simplicity, and not in the multiplicity and confusion of things” - Newton
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Chapter 1

Introduction

We begin by addressing a fundamental question; under what conditions does the dynamical system

\[ \frac{dx}{dt} = F(x) \]  

(1.1)

possess periodic solutions? If \( F \) is some linear function of \( x \), standard theory in dynamical analysis shows that periodic solutions of (1.1) occur whenever \( F \) possess purely imaginary eigenvalues. If \( F \) is nonlinear so that \( F(x) = Lx + N(x) \) (\( Lx \) is the linear part of \( F(x) \) and \( N(x) \) the nonlinear part) the natural procedure would be to study the linearized problem

\[ \frac{dx}{dt} = Lx \]  

(1.2)

and seek purely imaginary eigenvalues of \( L \). Locally, solutions of the linearized problem are qualitatively the same as that of the original nonlinear problem, so that if purely imaginary eigenvalues of \( L \) exist then one naturally assumes that problem (1.1) possesses periodic solutions. However, this is not always the case, as the following simple example shows (taken from Strogatz [48]).

Consider the problem

\[
\begin{align*}
\frac{dx}{dt} &= -y + ax(x^2 + y^2), \\
\frac{dy}{dt} &= x + ay(x^2 + y^2),
\end{align*}
\]

(1.3)

on a two dimensional cartesian domain \((x, y)\), where \( a \) is some nonzero real parameter. By linearizing the above system about \((x^*, y^*) = (0, 0)\) we obtain the linearized problem

\[ \frac{dx}{dt} = Lx; \]

here \( x = (x, y) \) denotes the deviation from the fixed point \((0, 0)\) and \( L \) is given by

\[
L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
It is clear that $L$ has a pair of purely imaginary eigenvalues $\pm i$. However, if we change the variables to polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, the nonlinear system becomes

\[
\begin{align*}
\frac{dr}{dt} &= ar^3, \\
\frac{d\theta}{dt} &= 1,
\end{align*}
\]

which implies that (1.3) has no periodic solutions since $a$ is specifically nonzero.

The purpose of this thesis is to investigate certain existence theorems, that prove existence of periodic solutions whenever the linearized function $L$ possesses purely imaginary eigenvalues, and to consider their applications to particular problems from hydrodynamics. The existence theorems we consider are the Hopf bifurcation theorem, the Lyapunov centre theorem, the Weinstein-Moser theorem, and extensions of these theorems; the Hopf-looss bifurcation theorem, the Lyapunov-looss centre theorem and the Weinstein-Moser-looss theorem, respectively. The theorems have been derived so that they are applicable to functional analytical problems, and have been represented in a coherent and uniform manner in order to bridge the fundamental structure common to them all. Applications of these theorems, in this standardised form, are then applied in a systematic way to two particular hydrodynamical problems; the water wave problem and the Navier-Stokes equations. Literature for the above mentioned theorems is readily available for problems in $\mathbb{R}^n$, see, for example, Marsden & McCracken [37] for the Hopf bifurcation theorem, Ambrosetti & Prodi [2] for the Lyapunov centre theorem, and Bartsch [4] for the Weinstein-Moser theorem. For infinite dimensional problems, see Kielhöfer [26], who provides a thorough analytical study of various existence results of bifurcating solutions, including the Hopf bifurcation theorem and Lyapunov centre theorem (in fact some of the earlier theorems we discuss have been derived using ideas from Kielhöfer [26]). Nevertheless, one fails to find a compendium that brings together these theorems (including the Hopf-looss bifurcation theorem, the Lyapunov-looss centre theorem and the Weinstein-Moser theorem) in a unified way, presenting them as variations of a single theme, and that considers their applications in this standardised form to the hydrodynamical problems we discuss (in particular the water wave problem).

The first chapter begins with a discussion on the Lyapunov-Schmidt reduction; a tool in functional analytical bifurcation theory which forms the basis of all the existence theorems we discuss. By giving (1.1) a functional analytical setting (this is done in the usual way by setting $x$ as a function $x = x(\cdot) : \mathbb{R} \to Z$ and $F$ as a mapping from $Z$ to $Z$ for some Banach space $Z$), the resulting problem is of infinite dimensions. The Lyapunov-Schmidt reduction (originally introduced by Lyapunov [34], [35] and Schmidt [43]) proves useful in that it reduces infinite dimensional problems to finite dimensional ones. We also discuss a variational version of the Lyapunov-Schmidt reduction; the Variational Lyapunov-Schmidt reduction. This version preserves the variational structure of a nonlinear problem when the Lyapunov-Schmidt reduction is applied.
Next we discuss in detail the periodic existence theorems. We begin with the Hopf bifurcation theorem and the Lyapunov centre theorem. These are two structurally similar theorems in that they both incorporate the Lyapunov-Schmidt reduction in the same way to reduce the nonlinear problem to a two dimensional one. This is a result of an assumption both theorems utilize. This assumption, known as the non-resonance condition, requires that $L$ possesses a pair of purely imaginary eigenvalues that are not integer multiples of any other purely imaginary eigenvalue. The two theorems differ in the manner in which the reduced problem is solved. The Hopf bifurcation theorem considers problems which have a dependency on some bifurcation parameter $\lambda$. As this parameter varies through some critical value $\lambda_0$ (known as the bifurcation value) it is required that the eigenvalues of $L$ cross the imaginary axis non-tangentially (see Figure 1.1). This requirement ensures that the reduced problem has nontrivial solutions.

![Figure 1.1: The transversality condition. The solid dots represent eigenvalues of $L$. As $\lambda$ varies through the bifurcation value $\lambda_0$ the eigenvalues cross the imaginary axis non-tangentially.](image)

The transversality condition is not required for the Lyapunov centre theorem, which instead solves the reduced problem using certain structural constraints that the nonlinear problem admits to. We consider two versions of the Lyapunov centre theorem; the Lyapunov centre theorem for conservative systems and the Lyapunov centre theorem for reversible systems. The first version considers problems that are conservative, that is there exists a function $H$, known as a conserved quantity, that is constant for all solutions $x(t)$ to the problem. If the second derivative of this conserved quantity satisfies a certain non-degeneracy condition then the reduced problem can be solved. The second version considers reversible problems. A dynamical system of the form (1.1) is said to be reversible if for some transformation mapping $R : Z \rightarrow Z$ the relationship

$$F(Rx) = -RF(x)$$
holds. This skew-symmetric property can also be expressed as follows: if $x$ is a solution of the dynamical system (1.1) then $Rx$ is a solution to the same problem in reverse time. In turn we find that eigenvalues of the linearized function $L$ demonstrate a distinct behaviour; they are symmetric with respect to the imaginary axis (see Figure 1.2). Not only does this prove to be a sufficient condition in solving the reduced problem, it also implies that the transversality condition never holds for reversible problems.

Figure 1.2: Eigenvalues of $L$ for reversible problems are symmetric with respect to the imaginary axis.

We consider a particular type of conservative system known as a Hamiltonian system. The problem (1.1) is said to be a Hamiltonian system if

$$\frac{dx}{dt} = F(x) = JH'(x),$$

where the function $H$ is a conserved quantity known as the Hamiltonian with gradient $H' : Z \to Z$, and $J : Z \to Z$ is some pseudo-idempotent operator. The third theorem, the Weinstein-Moser theorem (originally formulated by Weinstein [52] and revisited by Moser [39]) considers Hamiltonian systems. In contrast to the previous theorems, the Weinstein-Moser theorem examines a certain non-resonance case, and determines periodic solutions by using a variational argument. It shows that one can define an associated function $g : Z \to \mathbb{R}$, known as a potential function, such that critical points of $g$ on some compact manifold correspond to periodic solutions of the Hamiltonian system. One of the key hypothesis in the Weinstein-Moser theorem is that $\frac{dH'}{dt}(0)$ is strictly positive definite on the eigenspace of the purely imaginary eigenvalues (so that the Hamiltonian possess a local minimum), and a more detailed analysis reveals that the periodic solutions lie on the energy surface of the Hamiltonian and exist near its minimum. One can intuitively appreciate this by considering a Hamiltonian systems on the two dimensional cartesian domain $(x, y)$. Since
Figure 1.3: Solutions of a two dimensional Hamiltonian system which lie on the curve $H = \epsilon$. Near the local minimum of $H$ these correspond to closed orbits in phase space.

the Hamiltonian is a conserved quantity ($H(x(t)) = \epsilon$ for some positive constant $\epsilon$) it is clear that we obtain closed trajectories near the minimum of the Hamiltonian (see Figure 1.3).

The last section of the chapter looks at extensions of the previous theorems. These are respectively the *Hopf-looss bifurcation theorem*, the *Lyapunov-looss centre theorem* and the *Weinstein-Moser-looss theorem*. Unlike the original versions, these theorems can be applied to problems for which zero lies in the essential spectrum of $L$; the condition that integer multiples of the purely imaginary eigenvalues of $L$ lie strictly in its resolvent set is required by all three theorems, which is why they cannot be applied. These extensions were derived using the ideas formulated in looss [21]. looss considered a hydrodynamical problem for which zero was an eigenvalue embedded in the essential spectrum of the linearized operator. He showed that one can still apply the standard Lyapunov centre theorem to obtain periodic solutions provided that for each $x^*$ in the domain of the nonlinear part $N$ there exists an unique $x$ in the domain of $L$ such that

$$L(x) = -N(x^*).$$

The remaining chapters looks at applications of the Lyapunov-looss and Hopf-looss theorem. We apply the Lyapunov-looss theorem to the *water wave problem* and the Hopf-looss theorem to the *Navier-Stokes equations*. Chapter 3 investigates the water wave problem. The classic water wave problem concerns the irrotational flow of a perfect fluid of unit density, subject to the forces of gravity and surface tension. The flow of the fluid is across a three dimensional Cartesian coordinate system $(x, \hat{y}, z)$, where $x$, $z$, denote the horizontal coordinates and $\hat{y}$ the vertical coordinate. The fluid is bounded above by a free surface $\{\hat{y} = \eta(x, z, t)\}$, and below by a rigid horizontal bottom $\{\hat{y} = -h\}$. In terms of an Eulerian
velocity potential $\phi(x, y, z, t)$, the mathematical problem is given by

\begin{align}
\phi_{xx} + \phi_{yy} + \phi_{zz} &= 0, & \text{in } -h < y < \eta(x, z, t) \quad (1.4) \\
\eta_t &= \phi_y - \eta_x \phi_x - \eta_z \phi_z, & \text{on } \hat{y} = \eta, \quad (1.5) \\
\phi_t &= \frac{1}{2} (\phi_x^2 + \phi_y^2 + \phi_z^2) - g\eta \\
&\quad + \tau \left[ \frac{\eta_x}{(1 + \eta_x^2 + \eta_z^2)^{\frac{3}{2}}} \right]_x + \tau \left[ \frac{\eta_z}{(1 + \eta_x^2 + \eta_z^2)^{\frac{3}{2}}} \right]_z + B, & \text{on } \hat{y} = \eta, \quad (1.6) \\
\phi_y &= 0, & \text{on } \hat{y} = -h, \quad (1.7)
\end{align}

where $g$ and $\tau$ are respectively the acceleration due to gravity and the coefficient of surface tension, and $B$ is a constant known as the Bernoulli constant.

We restrict our attention to travelling water waves. These are solutions of the water wave problem that are uniformly translating in the horizontal spatial direction with constant speed $c$. Mathematically, these solutions take the special form

$$
\phi(x, t) = \phi(x - ct, \hat{y}, z, t),
$$

$$
\eta(x, t) = \eta(x - ct, z).
$$

There is a wide range of literature available on travelling water waves. However most of these concentrate on numerical studies or approximations by simpler model equations. Determining explicit solutions for the full set of equations proves to be more difficult, and at present the advances made thus far have involved using functional analytical techniques.

We shall concentrate on a particular type of travelling water wave, namely a double-periodic wave. We apply the Lyapunov-Iooss centre theorem to prove the existence of doubly-periodic waves for the case $h = \infty$ (in this case the condition (1.7) is replaced with $\phi_y \to 0$ as $\hat{y} \to -\infty$). A doubly-periodic wave is a travelling wave that possess spatially periodic profiles in two different horizontal directions. An example of such a wave is given in Figure 1.4. One of the first result regarding the existence of doubly periodic waves was given by Craig & Nicholls [6]. Their existence theory is based on an observation made by Zakharov [55], that the functions $\eta(x, z, t), \xi(x, z, t) = \phi(x, \eta(x, z, t), t)$ satisfy the classic water wave problem, and that these form the variables of a Hamiltonian formulation of the time-dependent water wave problem

$$
\eta_t = \frac{\delta H}{\delta \xi}, \quad \xi_t = \frac{\delta H}{\delta \eta},
$$

where $H(\eta, \xi)$ is the energy functional (the Hamiltonian formulation was expressed in this more convenient form by Craig & Sulem [7]). Craig & Nicholls [6] determine doubly-periodic waves by taking an arbitrary periodic spatial domain, and using the classical variational principle that travelling waves are critical points of $H$ subject to the horizontal momentum $I$ (this variational principle is reduced to a finite-dimensional problem using the same method as the Weinstein-Moser theorem).

We use an alternative approach in proving the existence of doubly-periodic waves, known as the spatial dynamics formulation. The spatial dynamics formulation involves formulating
a system of partial differential equations, defined on some spatial domain, as a dynamical system where one of the unbounded spatial variables plays the role of time ('time-like' variable). The method was introduced by Kirchgässner [27] in the early 1980’s and has become the basis of dynamical analysis for a wide range of problems in the applied sciences, see, for example, Haragus & Scheel [19] who apply the spatial dynamics formulation to a reaction diffusion equation to study existence and stability of corner defects (a special type of travelling wave solution with a certain deformed spatial profile), and F. Dias & G. Iooss [8, pp. 43–99]. The first application of the spatial dynamics formulation to the water wave problem was done by Kirchgässner [28] who determined two dimensional travelling waves (\(z\)-independent travelling wave solutions of (1.4)–(1.7)) that have spatially periodic profiles in the \(x\)-direction. Kirchgässner [28] introduces a stream function \(\psi(x, \hat{y})\), and by setting \(\psi\) as the vertical coordinate, and a careful manipulation of equations, he arrives at a spatial dynamics formulation to the two dimensional water wave problem.

A useful technique that has been used to find solutions of the spatial dynamics formulation of the water wave problem is the centre-manifold reduction. Here the purely imaginary eigenvalues of the linear part of the spatial dynamics system admit an invariant manifold known as the centre manifold which contains all small, bounded solutions (see Vanderbauwhede [50] and Vanderbauwhede & looss [51]). Key to the centre manifold reduction is the fact that certain symmetries of the original nonlinear problem are preserved; nonlinear problems that are conservative, Hamiltonian or reversible are respectively inherited by the reduced problem.

The first application of the spatial dynamics method to the full three dimensional water wave problem was carried out Groves & Mielke [17]. Here they consider waves that are periodic in \(z\) and take \(x\) to be the 'time-like' variable. In fact one has the freedom to take any horizontal spatial direction to be the time-like variable. This was explored in detail by
Groves & Haragus [16], who introduce new oblique horizontal coordinates $x_1, z_1$ that make respective angles $\theta_1, \theta_2$ with the positive $x$-axis (see Figure 1.5). Wave solutions are then assumed to be $2\pi/\nu$-periodic in $z_1$ and $x_1$ is taken to be the time-like variable in the spatial dynamics formulation.

![Diagram of the oblique coordinate system](image)

Figure 1.5: Diagram of the oblique coordinate system $(x_1, z_1)$. The $x_1$- and $z_1$-axes are obtained by rotating the $x$-axis through angles of $\theta_1$ and $\theta_2$, respectively, where $\theta_1, \theta_2 \in (\pi, \pi)$ are chosen with $\theta_1 - \theta_2 \neq 0, \pi, \pi$.

The key step in the spatial dynamics formulation comes from an observation made by Luke [33]. Luke [33] was able to show that solutions to a certain variation problem solved the two dimensional water wave problem. In accordance with Luke [33], a variational principle for the three dimensional water wave problem is given in Groves & Mielke [17]. For the three dimensional water wave problem with the oblique coordinates $(x_1, z_1)$, this variational problem is given by

$$
\delta \int \left\{ \int_{-\infty}^{0} \left\{ \frac{1}{2} \left( \phi_{x_1}^2 + \phi_y^2 + \nu^2 \phi_{z_1}^2 + 2\nu \cos(\theta_1 - \theta_2) \phi_{x_1} \phi_{z_1} \right) \right\} \, d\hat{y} 
+ \sqrt{1 + \eta_{x_1}^2 + \nu^2 \eta_{z_1}^2 + 2\nu \cos(\theta_1 - \theta_2) \eta_{x_1} \eta_{z_1}} 
+ \frac{1}{2} \gamma \eta^2 - 1 + (\eta_{x_1} \sin \theta_2 + \nu \eta_{z_1} \sin \theta_1) \phi|_{\hat{y}=\eta} \right\} 
\right\} \, dz_1 \, dx_1.
$$

The additional benefit of considering the above variation problem in replace of the original set of equations is that one can perform the Legendre transform to derive a Hamiltonian spatial dynamics system. Although this approach is formal, we rigorously show that this Hamiltonian system correctly defines a Hamiltonian system in the functional analytical sense defined on some subset $D_F$ of the Sobolev space $X_{s+1}$, where

$$
X_s := H_{per}^{s+1}(S) \times H_{per}^s(S) \times H_{per}^{s+1}(\Sigma) \times H_{per}^s(\Sigma).
$$
and \( S = (0, 2\pi), \Sigma = (-\infty, 0) \times (0, 2\pi) \). The resulting system is of the form

\[
\frac{d\mathbf{u}}{dx_1} = F(\mathbf{u}) = JH'(\mathbf{u}),
\]

where \( \mathbf{u} = (\eta, \xi, \Phi, U) \in D_F \), to which we apply the Lyapunov-Iooss theorem to determine periodic solutions in \( x_1 \); the main work of the application is to verify that the associated linearized operator satisfies the spectral condition in the theorem. The periodic solutions determined correspond to doubly periodic solutions. Our work differs from that of Groves & Haragus [16] in that they consider a fluid of finite depth. The case \( h = -\infty \) yields the added difficulty in that zero exists as an element of the essential spectrum of the associated linearized operator, and so unlike Groves & Haragus [16] we do not (and cannot) use the centre manifold reduction in determining the periodic solutions. Nevertheless, we effectively reduce to a finite-dimensional problem using the Lyapunov-Iooss theorem. We catalogue a variety of parameter values for which it is possible to obtain doubly periodic waves. These are obtained using a neat geometrical method that looks at the intersections between a certain dispersion curve (which varies according to the parameter \( \gamma \)) and a set of lines (which depend upon the angles \( \theta_1, \theta_2 \) and the frequency \( \nu \)). Our final result is summarized in the following theorem.

**Theorem 1.0.1** Let the parameter values \( \theta_1, \theta_2 \) and \( \nu \) be fixed, and suppose that \( \gamma \) belongs to the curve

\[
\sigma(\gamma + \sigma^2) = (\omega \sin \theta_2 + \nu \sin \theta_1)^2,
\]

were \( \sigma^2 = \omega^2 + \nu^2 + 2\nu \omega \cos(\theta_1 - \theta_2) \), and does not belong to the curves

\[
\sigma_{n,m}(\gamma + \sigma_{n,m}^2) = (n\omega \sin \theta_2 + nm \sin \theta_1)^2, \quad m \in \mathbb{N}, \quad n \in \mathbb{N} \setminus \{1, 0, -1\},
\]

where \( \sigma_{n,m}^2 = n^2 \omega^2 + m^2 \nu^2 + 2nm\nu \omega \cos(\theta_1 - \theta_2) \). Under such parameter restrictions, the Lyapunov-Iooss theorem yields the existence of a family of doubly-periodic waves with frequency \( \nu \) in the \( z \)-direction and approximately \( \omega \) in the \( x \)-direction.

In the last chapter we investigate the *Navier-Stokes problem*. The defining equations model the motion of an incompressible viscous fluid subject to the action a body force \( f \). Defined on some three dimensional Cartesian domain \( \Omega \subset \mathbb{R}^3 \), the non-dimensional Navier-Stokes equations are given by

\[
\frac{\partial \mathbf{v}}{\partial t} + R(\mathbf{v} \cdot \nabla)\mathbf{v} - \Delta \mathbf{v} + \nabla q = f, \tag{1.8}
\]
\[
\text{div} \mathbf{v} = 0, \tag{1.9}
\]

where \( \mathbf{v}(x,t) \) denotes the dimensionless velocity vector and \( q(x,t) \) the dimensionless pressure at the point \( x = (x_1, x_2, x_3) \in \Omega \). The parameter \( R \) is known as the Reynolds number.
Associated with the Navier-Stokes equations are two auxiliary problems which have received many contributions; the Stokes problem given by
\[
\frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v} + \nabla q = \mathbf{f},
\]
\[
\text{div } \mathbf{v} = 0,
\]
and the Oseen problem given by
\[
\frac{\partial \mathbf{v}}{\partial t} + R(\mathbf{v}_\infty \cdot \nabla)\mathbf{v} - \Delta \mathbf{v} + \nabla q = \mathbf{f},
\]
\[
\text{div } \mathbf{v} = 0
\]
(the Stokes and Oseen problem are respectively found by linearizing the Navier-Stokes equations about zero and the constant vector \(\mathbf{v}_\infty \in \mathbb{R}^3\)). A thorough investigation of solutions to the Stokes problem in a bounded domain is presented in Ladyzhenskaya \[30\], and Galdi \[11\] presents a collection of results on the stationary solutions to the Stokes and Oseen problem in bounded and unbounded domains, including exterior domains (the complement in \(\mathbb{R}^3\) of bounded domains).

In order for the Hopf-Iooss bifurcation to be applicable to the Navier-Stokes problem it is important to establish the existence of a non-uniform steady solution. The first result regarding the existence of a steady solution to the full Navier-Stokes problem was given by Leray \[31\], who considered both a bounded and exterior domain. However, for the exterior domain Leray \[31\] did not provide qualitative information about the asymptotic structure of the solution at large distance. This was developed further by Finn \[10\], who showed that the steady solution necessarily tends to a limit as \(x \to \infty\) everywhere. In fact he showed that the stationary solution can be written as \(\mathbf{v}_\infty + \mathbf{u}_0(x)\), where \(\mathbf{v}_\infty\) is the uniform limit, and \(\mathbf{u}_0(x)\) is such that \(|\mathbf{u}_0(x)| \to 0\) as \(|x| \to \infty\).

Other significant developments include Heywood \[20\] who showed that on an exterior domain with homogeneous boundary conditions Finn’s stationary solution forms a limit as \(t \to \infty\) in \(L^2\) of a nonstationary solution to the Navier-Stokes problem. Similar time-decay results in \(L^p\) are given in Iwashita \[23\] (see also Kobayashi & Shibata \[29\] and Sazonov \[42\]).

In contrast to the spatially periodic solutions we determined for the water wave problem, the periodic solutions that we seek for the Navier-Stokes equations are periodic in time. Existence of time-periodic solutions to the Navier-Stokes equations when periodic external forcing \(\mathbf{f}\) is assumed has received many contributions. We mention the work by Serrin \[44\], who proposed a method to construct time-periodic solutions for when \(\Omega\) is bounded. From a heuristic point of view, his idea is that any sufficiently smooth solution to the Navier-Stokes problem with time-periodic external forcing necessarily tends to a time-periodic solution \(\mathbf{u}(x, t)\) as \(t\) tends to infinity. Subsequent work on time-periodic solutions on bounded domains can be found in Kaniel & Shinbrot \[24\], for example. For unbounded domains, in particular the space \(\mathbb{R}^3\) and exterior domains, similar results on the existence of time-periodic solutions can be found respectively in Maremonti \[36\] and Galdi & Sohr \[12\].
Other existence results for periodic solutions involve applications of the Hopf bifurcation theorem, see, for example, Yudovich [54] for an application of the Hopf bifurcation theorem to the Navier-Stokes problem on a bounded domain, Sazonov [41] for the Navier-Stokes problem in an exterior domain, and Melcher, Schneider & Uecker [38] for an application in free space with periodic forcing. Following Sazonov [41], we apply the Hopf-Iooss bifurcation theorem to the Navier-Stokes equations (1.8)–(1.9) in an exterior domain; we assume that $\Omega$ is the exterior to a smooth bounded domain $\mathcal{O}$ centred at the origin of the Cartesian coordinate system $(x_1, x_2, x_3)$ ($x_1, x_2$ denote the horizontal coordinates and $x_3$ the vertical coordinate) and we impose the boundary condition $v = 0$ on $\partial \Omega$ (see Figure 4.1). Considering an exterior domain yields the additional complication that zero belongs to the essential spectrum of the associated linearized operator, and so it is appropriate here that we use the Hopf-Iooss theorem. In fact in our investigation we reproduce the results stated in Sazonov [41]. However, although Sazonov derives clearly the conditions under which a Hopf-bifurcation takes place, he only provides a sketch on how it is applied to the Navier-Stokes equations, and in some places the results he states lack clear justification. Our aim has been to fill in these gaps by providing a coherent, rigorous treatment of these conditions.

![Figure 1.6: Flow of the fluid past an obstacle $\mathcal{O}$ in the horizontal $x_1$-direction.](image)

In order to apply the Hopf-Iooss theorem to our problem we begin by providing it a functional analytical setting. We make note that for our exterior domain $\Omega$ the space $L^p(\Omega)$ admits the Helmholtz-Weyl decomposition

$$L^p(\Omega) = S_p(\Omega) \oplus G_p(\Omega),$$
where \( S_p(\Omega) \) is the completion of the set
\[
\{ v \in C_0^\infty(D) : \text{div} v = 0 \text{ in } D \}
\]
and \( G_p(D) := \{ \nabla q : q \in \hat{W}^{1,p}(D) \} \). This in turn defines a projection \( \Pi_p : L^p(\Omega) \to L^p(\Omega) \) onto \( S_p(\Omega) \) along \( G_p(\Omega) \). Our problem is therefore formulated as
\[
\frac{du}{dt} = F(u, R)
\]
and \( F : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap S_p(\Omega) \times \mathbb{R} \to S_p(\Omega) \) is given by
\[
F(u, R) = \Pi_p \left( \Delta u - R \frac{\partial u}{\partial x_1} - R u \cdot \nabla u_0 + R(u_0 \cdot \nabla)u - R(u \cdot \nabla)u \right).
\]
We check that the above dynamical system satisfies the required conditions of the Hopf-Iooss theorem; the main work here is to show that the linearized operator \( L_R : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap S_p(\Omega) \times \mathbb{R} \to S_p(\Omega) \) given by
\[
L_R u = \Pi_p \left( \Delta u - R \frac{\partial u}{\partial x_1} - R u \cdot \nabla u_0 \right)
\]
satisfies the spectral conditions of the theorem. In fact the spectral properties of \( L_R \) are closely related to that of the Oseen operator \( A_R : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap S_p(\Omega) \times \mathbb{R} \to S_p(\Omega) \) given by
\[
A_R = \Pi_p \left( \Delta u - R \frac{\partial u}{\partial x_1} \right),
\]
and we investigate the spectral properties of \( A_R \) by using the method in Kobayashi & Shibata [29]. Kobayashi & Shibata [29] use a 'cut-off' technique to determine solutions to the Oseen problem. They consider three distinct regions in the exterior domain used in the construction of the solution; a ball centred at the origin of radius \( b \), \( b - 1 \) and \( b - 2 \), respectively (here \( b \) is chosen large enough so that the obstacle is contained inside the ball of radius \( b - 2 \) (see Figure 1.7)). Using the solutions to the Oseen problem on a bounded domain and free space (the existence of which are well established), their idea is to construct a solution that is given in terms of the solution to the Oseen problem in free space outside the ball of radius \( b - 1 \), in terms of the solution to the Oseen problem on a bounded domain inside the ball of radius \( b - 2 \), and by both of these functions inside the annular region \( b - 2 < |x| < b - 1 \). Our final result is summarized in the following theorem.

**Theorem 1.0.2** Suppose that

(i) \( L_R \) has a plus-minus pair of purely imaginary eigenvalues which depend continuously on \( R \) and cross the imaginary axis in a non-tangential and non-resonant fashion as \( R \) passes through a critical value;
(ii) 0 is not an eigenvalue of \( L_R \).

Under these conditions the Hopf-Iooss bifurcation theorem yields the existence of a family of time-periodic waves solutions to the Navier Stokes problem in an exterior domain.

A final note is that although the above assumptions assert the existence of time-periodic solutions, verifying \((i)\) and \((ii)\) may only be possible using numerical method, which is beyond the scope of the thesis.

![Diagram showing three distinct regions in \( \Omega \) used in the construction of the solution to the Oseen problem in an exterior domain.](image)

Figure 1.7: The three distinct regions in \( \Omega \) used in the construction of the solution to the Oseen problem in a exterior domain.
Chapter 2

Local existence theorems for periodic solutions

2.1 Lyapunov-Schmidt reduction

The Lyapunov-Schmidt reduction forms a basis to all of the periodic existence theorems we discuss. The Lyapunov-Schmidt reduction is a procedure that is used to reduce an infinite dimensional nonlinear problem to a finite dimensional one. The reduction holds for values local to a particular solution of the original nonlinear problem. The key part to the Lyapunov-Schmidt reduction is the use of the implicit function theorem. The implicit function theorem is a theorem that locally solves certain nonlinear problems of the form

\[ F(x, y) = z \]

by setting one of the variables as a unique function of the other.

**Theorem 2.1.1 (Implicit function theorem)** Consider the nonlinear problem

\[ F(x, y) = z, \quad (2.1) \]

where \( F : U \times V \to Z \) is a continuous mapping, with open sets \( U \subset X, V \subset Y \), and where \( X, Y \) and \( Z \) are Banach spaces. Assume that the partial derivative \( d_1F : U \times V \to L(X, Z) \) exists, and is continuous at some point \( (x^*, y^*) \) in \( U \times V \). Furthermore, at the point \( (x^*, y^*) \) make the following assumptions:

(i) \( F(x^*, y^*) = z^* \).

(ii) \( d_1F[x^*, y^*] : X \to Z \) is a bijection.

Under these hypotheses, there exist neighbourhoods \( U_1 \subset U, V_1 \subset V \) of \( x^*, y^* \), respectively, and a unique continuous function \( x : V_1 \to U_1 \), such that
(i) $x(y^*) = x^*$;
(ii) $F(x(y), y) = z^*$ for all $y \in V_1$;
(iii) every solution of (2.1) in $U_1 \times V_1$ is of the form $(x(y), y)$.

Furthermore, if the function $F$ is $k$-times continuously differentiable on $U \times V$ then the function $x$ is also $k$-times continuously differentiable on $V_1$.

The proof of the implicit function theorem can be found, for example, in Dieudonné [9, Theorem 10.2.1]. The Lyapunov-Schmidt reduction involves a decomposition of the function space into a finite-dimensional and an infinite-dimensional part; the nonlinear equation admits a corresponding decomposition. The implicit function theorem is used to solve locally the infinite dimensional part of the problem, so that we are left with the task of solving a finite-dimensional problem.

**Theorem 2.1.2 (The Lyapunov-Schmidt reduction)** Consider the problem

$$F(x, \mu) = 0,$$  
(2.2)

where $F : U \times V \rightarrow Z$ is a continuously differentiable operator, with open sets $U \subset X$, $V \subset \mathbb{R}$, and where $X$ and $Z$ are Banach spaces. Suppose that the spaces $X$ and $Z$ can be decomposed into complementary spaces

$$X = X_1 \oplus X_2,$$  
(2.3)

$$Z = Z_1 \oplus Z_2,$$  
(2.4)

where $X_1$ and $Z_1$ are finite dimensional, and $X_2$ and $Z_2$ are closed. Let $P$ and $Q$ be the respective projections on to $X_1$, $Z_1$ along $X_2$, $Z_2$ given by

$$P : X \rightarrow X_1, \quad P(X) = X_1,$$  
(2.5)

$$Q : Z \rightarrow Z_1, \quad Q(Z) = Z_1.$$  
(2.6)

In addition, assume that there exist elements $x^* \in U$, $\mu_0 \in V$ such that

(i) $F(x^*, \mu_0) = 0$;

(ii) $(I - Q)d_1F[x^*, \mu_0]I_{X_2} : X_2 \rightarrow Z_2$ is a bijection.

Setting $x_1^* = Px^*$, $x_2^* = (I - P)x^*$, so that $x^* = x_1^* + x_2^*$, the above assumptions assert the existence of local neighbourhoods $U_1 \subset X_1$, $U_2 \subset X_2$ of $x_1^*$, $x_2^*$, respectively, a neighbourhood $V_1 \subset V$ of $\mu_0$, and a continuously differentiable function $x_2 : U_1 \times V_1 \rightarrow U_2$, such that solutions to the infinite dimensional problem

$$F(x, \mu) = 0,$$  
(2.7)
where $x = x_1 + x_2$ ($x_1 = Px$, $x_2 = (I - P)x$) and $x_1$, $x_2$, $\mu$ belong to the respective neighbourhoods $U_1$, $U_2$, $V_1$, are found by solving the reduced finite dimensional problem

$$QF(x_1 + x_2(x_1, \mu), \mu) = 0$$

(the solutions to (2.7) are then given by $x = x_1 + x_2(x_1, \mu)$).

**Proof.** There is no loss of generality in supposing that $x^* = 0$. The decompositions (2.3), (2.4) and the projections (2.5), (2.6) allow us to write problem (2.2) as

$$QF(x_1 + x_2, \mu) = 0,$$

(2.8)

$$(I - Q)F(x_1 + x_2, \mu) = 0.$$

(2.9)

The first equation is finite dimensional, and the latter infinite dimensional. Consider the infinite dimensional equation. Define $\tilde{F} : \tilde{U}_2 \times (\tilde{U}_1 \times V) \to Z_2$, where $\tilde{U}_1 = P(U)$, $\tilde{U}_2 = (I - P)(U)$, as

$$\tilde{F}(x_2(x_1, \mu)) = (I - Q)F(x_1 + x_2, \mu).$$

It is clear that $\tilde{F}$ is continuously differentiable. In addition

$$\tilde{F}(0, (0, \mu_0)) = (I - Q)F(0, \mu_0) = 0$$

and

$$d_1\tilde{F}[0, (0, \mu_0)] = (I - Q)d_1F[0, \mu_0]I_{X_2}.$$

By the assumption (ii) we see that $d_1\tilde{F}[0, (0, \mu_0)] : X_2 \to Z_2$ is a bijection. In light of the implicit function theorem, there exists a neighbourhood $U_2$ of the origin in $U_2$, a neighbourhood $U_1 \times V_1 \subset \tilde{U}_1 \times V$ of $(0, \mu_0)$, and a continuously differentiable function $x_2 : U_1 \times V_1 \to U_2$ such that

$$\tilde{F}(x_2(x_1, \mu), (x_1, \mu)) = 0$$

for all $(x_1, \mu) \in U_1 \times V_1$. It follows that equation (2.9) is satisfied with $x_2 = x_2(x_1, \mu)$, $x_1 \in U_1$, $\mu \in V_1$. It therefore remains to solve

$$QF(x_1 + x_2(x_1, \mu), \mu) = 0.$$

□

We define a certain type of linear mapping, known as a *Fredholm operator*. In the definition below we use the conventional notation $\ker A$, $\Im A$ to denote the kernel and respectively the image of a linear mapping $A : X \to Z$ (for linear spaces $X$ and $Z$), and the notation $\dim X$, $\text{codim} X$ to denote the dimension of $X$ and the dimension of the complement of $X$, respectively.
Definition 2.1.3 A linear mapping $A : U \to Z$, $U \subset X$, where $X$ and $Z$ are Banach spaces, is a Fredholm operator if

(i) $\dim \ker A$ is finite dimensional;

(ii) $\text{codim } \text{Im } A$ is finite dimensional;

(iii) $\text{Im } A$ is closed in $Z$.

The integer $\dim \ker A - \text{codim } \text{Im } A$ is known as the Fredholm index.

A Fredholm operator is particularly useful in that certain assumptions made for the Lyapunov-Schmidt theorem are automatically satisfied. To be specific, if $d_1 F[x^*, \mu_0]$ (referring to our previous notation) is a Fredholm operator, then the decompositions (2.3), (2.4) hold, and the condition (ii) is automatically satisfied. In particular

$$X = \ker d_1 F[x^*, \mu_0] \oplus X_2,$$
$$Z = Z_1 \oplus \text{Im } d_1 F[x^*, \mu_0].$$

Moreover, if $d_1 F[x^*, \mu_0]$ is Fredholm, the Lyapunov-Schmidt reduction shows that solutions of the nonlinear problem $F(x, \mu) = 0$, local to some particular solution $(x^*, \mu_0)$, will depend upon the solutions of the linearised problem $d_1 F[x^*, \mu_0] x = 0$.

We now discuss the variational Lyapunov-Schmidt theorem. The variational Lyapunov-Schmidt theorem asserts that the gradient of a potential function is preserved during the Lyapunov-Schmidt reduction. In the following $X$ and $Z$ denote Hilbert spaces, where $X$ is continuously embedded in $Z$. The continuous embedding allows one to define an inner product on $Z$ yet carry out differential calculus in $X$. All other notation is as before.

Definition 2.1.4 Let $X$ and $Z$ be Hilbert spaces, where $X$ is continuously embedded in $Z$. Let $(\cdot, \cdot)$ be an inner product defined on $Z$. The continuous mapping $H' : U \subset X \to Z$ is known as the gradient of the potential function $H \in C^2(U, \mathbb{R})$, with respect to the inner product $(\cdot, \cdot)$, if

$$dH[x](h) = (H'(x), h)$$

for all $x \in U$, $h \in X$.

Theorem 2.1.5 (Variational Lyapunov-Schmidt theorem) Let $X$ and $Z$ be a Hilbert spaces, where $X$ is continuously embedded in $Z$. Let $(\cdot, \cdot)$ be an inner product defined on $Z$ and suppose that $F(x, \mu)$ is the gradient of the potential function $f \in C^2(U \times V, \mathbb{R})$ with respect to $(\cdot, \cdot)$.

Assume that $d_1 F[x^*, \mu_0]$ is Fredholm with index zero, so that the decompositions

$$X = \ker d_1 F[x^*, \mu_0] \oplus (\text{Im } d_1 F[x^*, \mu_0] \cap X),$$
$$Z = \ker d_1 F[x^*, \mu_0] \oplus \text{Im } d_1 F[x^*, \mu_0].$$
The reduced function \( QF(x_1 + x_2(x_1, \mu), \mu) \), obtained via the Lyapunov-Schmidt reduction, is the gradient of the potential function \( \tilde{f} : U_1 \times V_1 \rightarrow \mathbb{R} \) given by
\[
\tilde{f}(x_1, \mu) = f(x_1 + x_2(x_1, \mu), \mu).
\]

**Proof.** A direct differentiation of \( \tilde{f} \) with respect to \( x_1 \) gives
\[
d\tilde{f}[x_1, \mu](h) = d_1f[x_1 + x_2(x_1, \mu), \mu](h + d_1x_2[x_1, \mu](h))
\]
\[
= \left( F(x_1 + x_2(x_1, \mu), \mu), h + d_1x_2[x_1, \mu](h) \right),
\]
where \( h \in X_1 \). Decomposing \( F \) via the projection \( Q \) we get
\[
d\tilde{f}[x_1, \mu](h) = (QF(x_1 + x_2(x_1, \mu), \mu), h) + \left( (I - Q)F(x_1 + x_2(x_1, \mu), \mu), d_1x_2[x_1, \mu](h) \right)
\]
\[
= (QF(x_1 + x_2(x_1, \mu), \mu), h),
\]
since \( (I - Q)F(x_1 + x_2(x_1, \mu), \mu) = 0 \). □

Note that \( d_1^2f[x, \mu](h_1, h_2) \) is symmetric with respect to \( h_1, h_2 \in X \). It follows that \( d_1F[x^*, \mu_0] \) is self adjoint. This implies that the kernel of \( d_1F[x^*, \mu_0] \) is orthogonal to its image, and so the projections \( P, Q \) are orthogonal projections.

The variational Lyapunov-Schmidt theorem is particularly useful in that it makes it possible to solve the nonlinear problem \( F(x, \mu) = 0 \), in the neighbourhood of \((x^*, \mu_0)\), by determining the critical points of the potential \( \tilde{f} \) that lie on the finite dimensional subspace \( U_1 \). This is the strategy employed by the Weinstein-Moser theorem, which we discuss in detail in Section 2.4.

### 2.2 Hopf bifurcation theorem

We present some preliminary analysis, in which the problem of existence of periodic solutions to
\[
\frac{dx}{dt} = F(x)
\]
(2.10)
is represented as a functional analytical problem. Let \( F : U \rightarrow Z, U \subset X \), where \( X \) and \( Z \) are real Banach spaces, and \( X \) is continuously and densely embedded in \( Z \). Make the substitution \( \dot{x}(t) = x(t/\omega), \omega \neq 0 \), so that (2.10) becomes
\[
\frac{d\dot{x}}{dt} = \frac{1}{\omega}F(\dot{x}).
\]
(2.11)
It follows that \( \dot{x}(t) \) is a \( 2\pi \)-periodic solution of (2.11) if and only if \( x(t) \) is a \( 2\pi/\omega \)-periodic solution of (2.10). Now define \( G : \tilde{U} \times \tilde{V} \rightarrow \tilde{Z} \) by
\[
G(x, \omega) = \frac{dx}{dt} - \frac{1}{\omega}F(x)
\]
where $\tilde{U} \subset \tilde{X}, \tilde{V} \subset \mathbb{R}$. The spaces $\tilde{X}, \tilde{Z}$ are 2π-periodic Sobelov spaces defined by

\[
\tilde{X} := H^1_{\text{per}}(\mathbb{R}, Z) \cap H^0_{\text{per}}(\mathbb{R}, X), \\
\tilde{Z} := H^0_{\text{per}}(\mathbb{R}, Z).
\]

According to the above analysis, periodic solutions of the dynamical system (2.10) are non-trivial solutions of the autonomous problem

\[
G(x, \omega) = 0.
\]  

(2.12)

To briefly explain our choice of spaces, we seek solutions that are as regular as the problem requires. Since $dx/dt$ is taken to be an element in $Z$, we seek solutions that are well defined on $X$, but with a time-derivative defined on $Z$. We set $x$ as an operator $x(\cdot)$ on $t$, and accordingly introduce the above operator spaces.

In the analysis that follows we make use of the fact that (2.12) is autonomous. Therefore $G(x, \omega)$ will admit the symmetry $S_\theta : x(t) \mapsto x(t + \theta), \theta \in \mathbb{R}$, that is

\[
G(S_\theta x, \omega) = G(x, \omega).
\]

In addition, since we are only interested in real solutions, we may henceforth write $x(t)$ in terms of its Fourier series

\[
x(t) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} c_n e^{int},
\]

where the Fourier coefficients $c_n \in X$ are given by

\[
c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} x(t) e^{-int} dt,
\]

and where we set $c_n = \overline{c_{-n}}$ to ensure real solutions.

**Theorem 2.2.1 (The Hopf bifurcation theorem)** Suppose that $X$ and $Z$ are real Banach Spaces, where $X$ is continuously and densely embedded in $Z$. Consider the autonomous evolutionary equation

\[
\frac{dx}{dt} = F(x, \mu),
\]

(2.13)

where $F \in C^3(U \times V, Z)$ with open sets $U \subset X, V \subset \mathbb{R}$, and where

\[
F(0, \mu) = 0
\]

for all $\mu \in V$.

Define $L_\mu = d_1 F[0, \mu]$ and suppose there exists an element $\mu_0 \in V$, a neighbourhood $\Omega$ of $\mu_0$ in $V$, and a continuously differentiable curve \{($\lambda(\mu), \varphi(\mu))$\}$_{\mu \in \Omega}$ of eigensolutions of $L_\mu$ such that
(i) \((\lambda(\mu_0), \varphi(\mu_0)) = (i\omega_0, \varphi_0), \omega_0 \neq 0\), where the eigenvalue \(\lambda(\mu_0) = i\omega_0\) is simple;

(ii) \(\Re \lambda'(\mu_0) \neq 0\) (the transversality condition).

Furthermore, assume that the non-resonance condition

\[
in\omega_0 \in \rho(L_{\mu_0}), \quad n \in \mathbb{Z} \setminus \{-1, 1\},
\]

holds, and the spectral hypotheses

\[
\| (i\omega_0 I - L_{\mu_0})^{-1} \|_{Z \to Z} \leq \frac{C}{|n|}, \quad \| (i\omega_0 I - L_{\mu_0})^{-1} \|_{Z \to \tilde{X}} \leq C, \quad n \to \pm \infty,
\]

are made (for some positive constant \(C\) independent of \(n\)).

Under these hypotheses, there exist a neighbourhood \(S\) of the origin in \(\mathbb{R}\) and a continuously differentiable curve \(\{(x(s), \mu(s))\}_{s \in S}\) of real periodic solutions to (2.13), such that \((x(0), \mu(0)) = (0, \mu_0)\). There also exists a continuously differentiable curve of frequencies \(\{\omega(s)\}_{s \in S}\), where \(\omega(0) = \omega_0\), whereby the period of the solutions are given by \(2\pi/\omega(s)\).

**Proof.** From our preliminary analysis, we see that the problem at hand is

\[
G(x, \omega, \mu) = 0,
\]

where \(G \in C^2(\tilde{U} \times \tilde{V}, \tilde{Z})\) (for notational simplicity, we have set \((\omega, \mu) \in \tilde{V} \subset \mathbb{R}^2\)). We shall apply the Lyapunov-Schmidt reduction to (2.16) by first showing that \(d_1G[0, \omega_0, \mu_0]\) is a Fredholm operator. Define \(A = d_1G[0, \omega_0, \mu_0]\) and write

\[
x(t) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} c_n e^{int},
\]

where

\[
c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} x(t) e^{-int} \, dt, \quad c_n = \overline{c_{-n}}, \quad c_n \in \tilde{X}.
\]

Therefore

\[
Ax = \frac{dx}{dt} - \frac{1}{\omega_0} L_{\mu_0} x = 0 \iff (i\omega_0 I - L_{\mu_0}) c_n = 0.
\]

Since by assumption \(i\omega_0 \in \rho(L_{\mu_0})\) for \(n \in \mathbb{Z} \setminus \{-1, 1\}\), we have that \(c_n = 0\) for \(n \neq \pm 1\). For \(n = \pm 1\) we get \(c_1 = \alpha \varphi_0, \quad c_{-1} = \overline{\alpha \varphi_0}, \quad \alpha \in \mathbb{C}\), where \(\varphi_0, \overline{\varphi_0}\) are the eigenvectors of \(L_{\mu_0}\) with respective eigenvalues \(i\omega_0, -i\omega_0\). Thus by (2.17) it follows that \(\dim \ker A = 2\), and in particular

\[
\ker A = \{ \beta \varphi_0 e^{it} + \overline{\beta \varphi_0} e^{-it} : \beta \in \mathbb{C} \},
\]

which is well defined in \(\tilde{X}\).
We proceed to show that $\text{Im} A$ is closed in $\tilde{Z}$ with complement $\ker A$, so that $A$ is a Fredholm operator with index zero. We discuss $\text{Im} A$ in a similar fashion to $\ker A$. Notice that for $x \in \tilde{X}$ given by (2.17) we have

$$Ax = z \iff (i\omega_0 I - L_{\mu_0})c_n = a_n,$$

where $a_n/\omega_0$ are the corresponding Fourier coefficients of $z(t)$. By assumption, the operator $(i\omega_0 I - L_{\mu_0}) : X \to Z$ is bijective for all $n \neq \pm 1$. Thus the above Fourier coefficient problem is uniquely solvable for all $c_n \in X$, $a_n \in Z$, $n \neq \pm 1$. For $n = \pm 1$ we have the problem

$$(i\omega_0 I - L_{\mu_0})c_1 = a_1, \quad (2.19)$$
$$(-i\omega_0 I - L_{\mu_0})c_{-1} = a_{-1}. \quad (2.20)$$

Note that since $i\omega_0$ and $-i\omega_0$ are both algebraically simple eigenvalues of $L_{\mu_0}$ then $\varphi_0 \notin \text{Im} (i\omega_0 I - L_{\mu_0})$ and $\overline{\varphi_0} \notin \text{Im} (-i\omega_0 I - L_{\mu_0})$. Thus $Z$ can be decomposed as

$$Z = \text{span} \varphi_0 \oplus \text{Im} (i\omega_0 I - L_{\mu_0}),$$
$$Z = \text{span} \overline{\varphi_0} \oplus \text{Im} (-i\omega_0 I - L_{\mu_0}).$$

Therefore (2.19), (2.20) is solvable for all $c_1, c_{-1} \in X$, $a_1, a_{-1} \in Z$ provided $\Pi a_1 = \Pi a_{-1} = 0$, where $\Pi : Z \to Z$ and $\overline{\Pi} : Z \to Z$ are the spectral projections onto $\varphi_0$ and $\overline{\varphi_0}$, respectively.

Explicitly these projections are given by

$$\Pi a_1 = \langle a_1, \varphi_0^* \rangle \varphi_0, \quad (2.21)$$
$$\overline{\Pi} a_{-1} = \langle a_{-1}, \overline{\varphi_0}^* \rangle \overline{\varphi_0}, \quad (2.22)$$

where $\langle , \rangle$ denotes the bilinear pairing of $Z$ and its dual space $Z^*$. Here the functions $\varphi_0^*, \overline{\varphi_0}^* \in Z^*$ are eigenvectors of $L_{\mu_0}^* : Z^* \to Z^*$ with respective eigenvalues $i\omega_0$ and $-i\omega_0$; the adjoint $L_{\mu_0}^*$ of $L_{\mu_0}$ exists since $X$ is densely defined on $Z$. Using the Hahn-Banach theorem, the dual vector $\varphi_0^*$ is chosen so that

(i) $\langle \varphi_0, \varphi_0^* \rangle = 1$;

(ii) $\langle z, \varphi_0^* \rangle = 0$ for all $z \in \text{Im} (i\omega_0 I - L_{\mu_0})$.

Similarly the function $\overline{\varphi_0}^*$ is chosen so that $\langle \overline{\varphi_0}, \overline{\varphi_0}^* \rangle = 1$ and $\langle z, \overline{\varphi_0}^* \rangle = 0$ for all $z \in \text{Im} (-i\omega_0 I - L_{\mu_0})$.

Our preceding analysis suggests that

$$\text{Im} A = \{ z \in \tilde{Z} : \Pi a_1 = \Pi a_{-1} = 0 \}; \quad (2.23)$$
it still needs to be checked that for \( z \in \text{Im} \ A \), as given above, the problem \( Ax = z \) has a solution \( x \) that is well defined in \( \tilde{X} \). Since the Fourier coefficients of \( x(t) \) are given by
\[
c_n = (i n \omega_0 I - L \mu_0)^{-1} a_n,
\]
it follows that
\[
\|x\|_{H_0^{\text{per}}(R,X)}^2 = \left\| \sum_{n \in \mathbb{Z}} c_n e^{int} \right\|_{H_0^{\text{per}}(R,X)}^2
\]
\[
= \sum_{n \in \mathbb{Z}} \|c_n\|_X^2
\]
\[
= \sum_{n \in \mathbb{Z}} \| (i n \omega_0 I - L \mu_0)^{-1} a_n \|_X^2,
\]
which by our spectral assumption (2.15) implies that
\[
\|x\|_{H_0^{\text{per}}(R,X)}^2 \leq C \sum_{n \in \mathbb{Z}} \|a_n\|_Z^2,
\]
\[
= C \left\| \sum_{n \in \mathbb{Z}} a_n e^{int} \right\|_{H_0^{\text{per}}(R,Z)}^2
\]
\[
= C \|z\|_Z^2.
\]
A similar calculation shows that
\[
\|x\|_{H_1^{\text{per}}(R,Z)}^2 \leq C \|z\|_Z^2,
\]
and so \( x \) is well defined in \( \tilde{X} \).

Because the spectral projections \( \Pi, \overline{\Pi} \) are continuous, it follows from (2.23) that \( \text{Im} \ A \) is a closed subset of \( \tilde{Z} \). Furthermore since \( \ker A \subset Z \) then \( \ker A \) is clearly the compliment of \( \text{Im} \ A \) in \( Z \). We conclude that \( A : \tilde{X} \to \tilde{Z} \) is a Fredholm operator of index zero. As a result, we obtain the decompositions
\[
\tilde{X} = \ker A \oplus (\text{Im} \ A \cap \tilde{X}),
\]
\[
\tilde{Z} = \text{Im} A \oplus \ker A.
\]
The projection \( P : \tilde{Z} \to \tilde{Z} \) onto \( \ker A \) along \( \text{Im} A \) is defined by
\[
(Pz)(t) = \Pi a_1 e^{it} + \overline{\Pi} a_{-1} e^{-it},
\]
which by (2.21), (2.22) implies that
\[
(Pz)(t) = \frac{\omega_0}{2\pi} \int_0^{2\pi} \langle z(t), e^{-it} \varphi_0^* \rangle \ dt \varphi_0 e^{it} + \frac{\omega_0}{2\pi} \int_0^{2\pi} \langle z(t), e^{it} \overline{\varphi_0} \rangle \ dt \overline{\varphi_0} e^{-it}. \tag{2.24}
\]
We set \( x_1 = Px, x_2 = (I - P)x \) and apply the Lyapunov-Schmidt reduction. This results in the two dimensional problem

\[
PG(x_1 + x_2(x_1, \omega, \mu), \omega, \mu) = 0, \tag{2.25}
\]

where \( x_1 \in U_1 \subset \ker A, (\omega, \mu) \in V_1 \subset \tilde{V} \) and \( x_2 : U_1 \times V_1 \to U_2 \subset \text{Im } A \cap \tilde{X} \) is a \( C^2 \)-function. By (2.24), equation (2.25) is equivalent to the complex equation

\[
\frac{\omega_0}{2\pi} \int_0^{2\pi} \langle G(x_1 + x_2(x_1, \omega, \mu), \omega, \mu), e^{-it}\varphi_0^* \rangle \ dt = 0 \tag{2.26}
\]

(the second summand in (2.24) is complex conjugate to the first). Since \( x_1 \in U_1 \) can be written as \( x_1 = \beta \varphi_0 e^{it} + \overline{\beta} \varphi_0 e^{-it} \) for \( \beta \in \hat{U}_1 \subset \mathbb{C} \), we can define the operator \( \phi \in C^2(\hat{U}_1 \times V_1, \mathbb{C}) \) by

\[
\phi(\beta, \omega, \mu) = \frac{\omega_0}{2\pi} \int_0^{2\pi} \langle G(x_1 + x_2(x_1, \omega, \mu), \omega, \mu), e^{-it}\varphi_0^* \rangle \ dt,
\]

and write equation (2.26) as

\[
\phi(\beta, \omega, \mu) = 0. \tag{2.27}
\]

We make use of the crucial symmetrical property of \( G(x, \omega, \mu) \); it admits the symmetry \( S_\theta : x(t) \mapsto x(t+\theta), \theta \in \mathbb{R} \). Since the Lyapunov-Schmidt reduction preserves this structure, the reduced problem (2.25) also admits this symmetry. Thus if \( x_1(t) \) is a solution of (2.25) then \( x_1(t + \theta) \) given by

\[
x_1(t + \theta) = \beta \varphi_0 e^{it\theta} + \overline{\beta} \varphi_0 e^{-it\theta}
\]

is also a solution. In this way, \( x_1(t) \) is identified with \( x_1(t + \theta) \). Writing \( \beta = se^{iw}, w \in \mathbb{R}, \) where \( s = |\beta| \), one can see that any solution of (2.25) is of the form

\[
x_1(t) = s(\varphi_0 e^{it} + \overline{\varphi_0} e^{-it}),
\]

and so \( \phi \) depends only upon \( s = |\alpha| \). With a slight abuse of notation, we write (2.27) as

\[
\phi(s, \omega, \mu) = 0, \tag{2.28}
\]

where \( s \in S \subset \mathbb{R} \). Henceforth, problem (2.28) will be referred to as the reduced complex problem.

Observe that \( \phi(0, \omega, \mu) = 0 \) for all \( (\omega, \mu) \in V_1 \). This presents a complication in applying the implicit function theorem to find nontrivial solutions of (2.28); a direct application of the theorem yields the trivial solution line \((0, \omega, \mu)\). To factor out this trivial solution, we write

\[
\phi(s, \omega, \mu) = \tilde{\phi}(\omega, \mu, s)s,
\]

where \( \tilde{\phi} \in C^1(V_1 \times S, \mathbb{C}) \), and where \((\omega, \mu, 0)\) is not a solution curve of \( \tilde{\phi}(\omega, \mu, s) = 0 \). In doing so, the trivial solution line is segregated and nontrivial solutions of (2.28) are found by solving \( \tilde{\phi}(\omega, \mu, s) = 0 \).
Observe that \( \tilde{\phi}(\omega, \mu, 0) = d_1 \phi[0, \omega, \mu] \). Therefore

\[
\tilde{\phi}(\omega, \mu, 0) = \frac{\omega_0}{2\pi} \int_0^{2\pi} \left< d_1 G[0, \omega, \mu] (\varphi_0 e^{it} + \overline{\varphi_0} e^{-it} + d_1 x_2[0, \omega, \mu] (\varphi_0 e^{it} + \overline{\varphi_0} e^{-it})), e^{-it} \varphi_0^* \right> dt,
\]

which implies that

\[
\tilde{\phi}(\omega_0, \mu_0, 0) = \frac{\omega_0}{2\pi} \int_0^{2\pi} \left< A(\varphi_0 e^{it} + \overline{\varphi_0} e^{-it}), \varphi_0^* e^{-it} \right> dt = 0, \tag{2.29}
\]

where we have used the fact that \( (\varphi_0 e^{it} + \overline{\varphi_0} e^{-it}) \in \ker A \) and \( d_1 x_2[0, \omega, \mu_0] = 0 \); the latter can be seen by differentiating \((I - P)G(x_1 + x_2(\omega, \mu), \omega, \mu) = 0\) with respect to \(x_1\) at \((0, \omega, \mu_0)\).

In the following we show that \( d_1 \tilde{\phi}((\omega_0, \mu_0), 0) : \mathbb{R}^2 \to \mathbb{C} \) is a bijection. We write

\[
d_1 \tilde{\phi}((\omega_0, \mu_0), 0)(\omega, \mu) = d_1 \tilde{\phi}[\omega_0, \mu_0, 0](\omega) + d_2 \tilde{\phi}[\omega_0, \mu_0, 0](\mu), \tag{2.30}
\]

where the Fréchet derivatives \( d_1 \tilde{\phi}[\omega_0, \mu_0, 0] \) and \( d_2 \tilde{\phi}[\omega_0, \mu_0, 0] \) are given by

\[
d_1 \tilde{\phi}[\omega_0, \mu_0, 0] &= \frac{\omega_0}{2\pi} \int_0^{2\pi} \left< d_{1,1} G[0, \omega, \mu_0] (I, \varphi_0 e^{it} + \overline{\varphi_0} e^{-it}), e^{-it} \varphi_0^* \right> dt \\
&= \frac{\omega_0}{2\pi} \int_0^{2\pi} \left< \omega_0^{-2} \left< \omega_0^{-2} L_{\mu_0} (\varphi_0 e^{it} + \overline{\varphi_0} e^{-it}), e^{-it} \varphi_0^* \right> dt \\
&= \frac{\omega_0}{2\pi} \int_0^{2\pi} \left< \omega_0^{-2} (i \omega_0 \varphi_0 e^{it} - i \omega_0 \overline{\varphi_0} e^{-it}), e^{-it} \varphi_0^* \right> dt \\
&= i,
\]

\[
d_2 \tilde{\phi}[\omega_0, \mu_0, 0] &= \frac{\omega_0}{2\pi} \int_0^{2\pi} \left< d_{1,1} G[0, \omega, \mu_0] (I, \varphi_0 e^{it} + \overline{\varphi_0} e^{-it}), e^{-it} \varphi_0^* \right> dt \\
&= \frac{\omega_0}{2\pi} \int_0^{2\pi} \left< -\omega_0^{-1} L_{\mu_0}' (\varphi_0 e^{it} + \overline{\varphi_0} e^{-it}), e^{-it} \varphi_0^* \right> dt \\
&= -\left< L_{\mu_0}' \varphi_0, \varphi_0^* \right>. \tag{2.31}
\]

In the above we have used the fact that \( \varphi_0, \overline{\varphi_0} \) are the eigenvectors of \( L_{\mu_0} \) with eigenvalues \( i \omega_0, -i \omega_0 \). We have also noted that since \( G(x, \omega, \mu) = dx/dt - \omega^{-1} F(x, \mu) \), one finds that \( d_2 G[0, \omega, \mu_0](I, x) = \omega_0^{-2} L_{\mu_0} x \) and \( d_{3,1} G[0, \omega, \mu_0](I, x) = -\omega_0^{-1} L_{\mu_0}' x \). The last equation in (2.31) can be simplified further. From the eigenvalue problem \( L_\mu \varphi(\mu) = \lambda(\mu) \varphi(\mu) \) we find that

\[
L_\mu' \varphi(\mu) + L_\mu \varphi'(\mu) = \lambda'(\mu) \varphi(\mu) + \lambda(\mu) \varphi'(\mu),
\]
which at $\mu = \mu_0$ becomes
\[ L'_{\mu_0} \varphi_0 = (i\omega_0 I - L_{\mu_0}) \varphi'_{\mu_0} + \lambda'_{\mu_0} \varphi_0, \]
and a bilinear pairing with $\varphi_0^*$ results in
\[ \langle L'_{\mu_0} \varphi_0, \varphi_0^* \rangle = \lambda'_{\mu_0}. \]
It follows that
\[ d_1 \bar{\phi}[(\omega_0, \mu_0), 0](\omega, \mu) = \imath \omega - \lambda'_{\mu_0} \mu = \imath (\omega - \imath \lambda'_{\mu_0} \mu) - \Re \lambda'_{\mu_0} \mu, \]
which shows that $d_1 \bar{\phi}[(\omega_0, \mu_0), 0] : \mathbb{R}^2 \rightarrow \mathbb{C}$ is a bijection (since $\Re \mu'_{\mu_0} \neq 0$ by assumption).

Taking $S$ to be smaller if necessary, the implicit function theorem asserts the existence of a $C^1$-branch
\[ \{(\omega(s), \mu(s), s)\}_{s \in S}, \]
such that $(\omega(0), \mu(0), 0) = (\omega_0, \mu_0, 0)$ and
\[ \bar{\phi}(\omega(s), \mu(s), s) = 0. \]
This $C^1$-branch gives nontrivial solutions to the reduced complex problem (2.28); the solutions bifurcate from the trivial solution line. Tracing back the various definitions, one finds that there exists a unique continuously differentiable curve
\[ \{(x(s), \mu(s))\}_{s \in S}, \]
where
\[ x(s) = x_1(s) + x_2(x_1(s), \omega(s), \mu(s)), \quad (x_1(s))(t) = s(\varphi_0 e^{it} + \varphi_0^* e^{it}), \]
of $2\pi/\omega(s)$-periodic solutions to (2.13).

**2.3 Lyapunov centre theorem**

The *Lyapunov centre theorem* makes use of most of the key hypotheses required in the Hopf bifurcation theorem. As a result, the reduction onto the complex plane holds identically. It differs in the manner in which the reduced complex problem is solved. The issue lies in being able to apply the implicit function theorem to determine nontrivial solutions of $\phi(s, \omega, \mu) = 0$. The Hopf bifurcation theorem achieves this by using the transversality condition. The Lyapunov centre theorem on the other hand uses structural properties of the original nonlinear problem to show that $\Re \phi = 0$; a condition which proves sufficient for a successful application of the implicit function theorem. Our discussion begins with the first version of the Lyapunov centre theorem, which looks at conservative systems.
Definition 2.3.1 Let $X$ and $Z$ be Hilbert spaces, where $X$ is continuously embedded in $Z$. Suppose $F : U \subset X \to Z$. The problem
\[ \frac{dx}{dt} = F(x) \] (2.32)
is said to be conservative if there exists a function $H : U \to \mathbb{R}$, known as a conserved quantity, such that for any solution $x \in U$ of (2.32)
\[ H(x(t)) = \text{const}. \]

Theorem 2.3.2 (The Lyapunov centre theorem for conservative systems) Suppose that $X$ and $Z$ are real Hilbert spaces, where $X$ is continuously and densely embedded in $Z$. Consider the evolutionary equation
\[ \frac{dx}{dt} = F(x), \] (2.33)
where $F \in C^3(U, Z)$, $U \subset X$, and where $F(0) = 0$.
Define $L = dF[0]$ and make the spectral assumptions that

(i) $\pm i\omega_0$ ($\omega_0 \neq 0$) are simple eigenvalues of $L$ with respective eigenvectors $\varphi_0, \overline{\varphi_0}$;

(ii) $in\omega_0 \in \rho(L)$ for $n \in \mathbb{Z} \setminus \{-1, 1\}$;

(iii) $\| (in\omega_0 I - L)^{-1} \|_{Z \to Z} \leq C/|n|$, $\| (in\omega_0 I - L)^{-1} \|_{Z \to X} \leq C$, $n \to \pm\infty$.

Suppose the system (2.33) is conservative, and that the corresponding conserved quantity $H : U \subset X \to \mathbb{R}$ is of class $C^2$. Assume that the gradient $H' : U \subset X \to Z$ is defined with respect to an inner product $(\cdot, \cdot)$ on $Z$. Furthermore, assume that $H'(0) = 0$ and that
\[ (dH'[0]\varphi_0, \varphi_0) \neq 0. \]

Under these hypotheses, there exist a neighbourhood $S$ of the origin in $\mathbb{R}$, a continuously differentiable curve $\{ x(s) \}_{s \in S}$ of real periodic solutions to (2.33), where $x(0) = 0$. There also exists a continuously differentiable curve of frequencies $\{ \omega(s) \}_{s \in S}$, $\omega(0) = \omega_0$, whereby the period of the solutions are given by $2\pi/\omega(s)$.

Proof. Following the same procedure as in the proof of the Hopf bifurcation theorem, we arrive at the reduced complex problem
\[ \phi(s, \omega) = 0, \] (2.34)
where $\phi \in C^2(S \times V_1, \mathbb{C})$, $S$ and $V_1$ are respective neighbourhoods of the origin and $\omega_0$ in $\mathbb{R}$, and $\phi(0, \omega) = 0$ for all $\omega \in V_1$. Decomposing $\phi$ into real and imaginary parts, we show that
\[ \text{Im} \phi(s, \omega) = 0 \Rightarrow \text{Re} \phi(s, \omega) = 0 \] (2.35)
for all \((s, \omega)\) near \((0, \omega_0)\).

Since
\[
(I - P)G(x_1 + x_2(x_1, \omega)) = 0
\]
(a consequence of the Lyapunov-Schmidt reduction), it follows that
\[
\frac{d}{dt}(x_1 + x_2(x_1, \omega)) = P \frac{d}{dt}(x_1 + x_2(x_1, \omega)) + (I - P)\frac{1}{\omega}F(x_1 + x_2(x_1, \omega)). \tag{2.36}
\]

Because (2.33) is conservative, we find that
\[
(H'(x), F(x)) = 0
\]
for any solution \(x \in U\).

We shall show that this property results in a certain restriction on the bifurcation problem (2.34). Note that
\[
H(x(t)) \text{ is } 2\pi\text{-periodic. Therefore}
\]
\[
\int_0^{2\pi} \frac{d}{dt}H(x_1(t) + x_2(x_1(t), \omega)) \, dt = 0,
\]
which gives
\[
\int_0^{2\pi} (H'(x_1(t) + x_2(x_1(t), \omega)), \frac{d}{dt}(x_1(t) + x_2(x_1(t), \omega))) \, dt = 0. \tag{2.37}
\]

The identity (2.36) and the condition \((H'(x), F(x)) = 0\) imply that
\[
\int_0^{2\pi} \left( H'(x_1(t) + x_2(x_1(t), \omega)), PG(x_1(t) + x_2(x_1(t), \omega), \omega) \right) \, dt = 0. \tag{2.38}
\]

By definition of \(\phi(s, \omega)\), we see that
\[
PG(x_1(t) + x_2(x_1(t), \omega), \omega) = \phi(s, \omega)\varphi_0 e^{it} + \overline{\phi(s, \omega)}\overline{\varphi_0} e^{-it}. \tag{2.39}
\]

Thus (2.38) can be written as
\[
\int_0^{2\pi} \left( H'(x_1(t) + x_2(x_1(t), \omega)), \varphi_0 e^{it} \right) \, dt \phi(s, \omega)
\]
\[+ \int_0^{2\pi} \left( H'(x_1(t) + x_2(x_1(t), \omega)), \overline{\varphi_0} e^{-it} \right) \, dt \overline{\phi(s, \omega)} = 0.
\]

Since the second summand is conjugate to the first, we obtain a condition for the bifurcation function \(\phi(s, \omega)\), namely
\[
\text{Re } (\phi(s, \omega)h(s, \omega)) = 0, \tag{2.40}
\]
where the \(C^1\)-function \(h : U_3 \times V_1 \to \mathbb{C}\) is defined by
\[
h(s, \omega) = \int_0^{2\pi} \left( H'(x_1(t) + x_2(x_1(t), \omega)), \varphi_0 e^{it} \right) \, dt.
\]
An obvious implication of (2.40) is that

$$\text{Re} \phi(s, \omega) \text{Re} h(s, \omega) = \text{Im} \phi(s, \omega) \text{Im} h(s, \omega).$$

Observe that if $\text{Re} h(s, \omega) \neq 0$ then our claim (2.35) holds. Since $H'(0) = 0$ then

$$h(0, \omega) = \int_0^{2\pi} (H'(0), \varphi_0 e^{it}) \, dt = 0,$$

and so it suffices to show that the derivative of $h$ with respect to $s$ is real and nonzero at $(0, \omega_0)$. Differentiating with respect to $s$ gives

$$d_1 h[0, \omega_0] = \int_0^{2\pi} (dH'[0](\varphi_0 e^{it} + \overline{\varphi_0} e^{-it}), \varphi_0 e^{it}) \, dt
= 2\pi (dH'[0] \varphi_0, \varphi_0)
\neq 0$$

(since $(dH'[0] \varphi_0, \varphi_0) \neq 0$). In addition, since $dH'[0]$ is symmetric with respect to the inner product, the derivative $d_1 h[0, \omega_0]$ is also real. Our claim (2.35) is justified.

With regards to the reduced complex problem (2.34), it suffices to solve

$$\text{Im} \phi(s, \omega) = 0$$

for all $(s, \omega) \in S \times V_1$, where the neighbourhoods $S, V_1$ are made smaller if necessary. We follow the same argument as in the Hopf bifurcation theorem. We segregate the trivial solution line by formulating the above problem as

$$\text{Im} \phi(s, \omega) = \hat{\phi}(\omega, s)s,$$

where $\hat{\phi} \in C^1(V_1 \times S, \mathbb{R})$. In turn we find

$$\hat{\phi}(\omega_0, 0) = 0,$$
$$d_1 \hat{\phi}[\omega_0, 0] = I.$$

The implicit function theorem asserts the exististence of a continuously differentiable branch $\{ (\omega(s), s) \}_{s \in S}$ of solutions to $\hat{\phi}(\omega, s) = 0$. These solutions by (2.35) solve the reduced complex problem (2.34). The statement in the theorem is obtained by tracing back the various definitions.

$\square$

**Definition 2.3.3** Let $X$ and $Z$ be Banach spaces, where $X$ is continuously embedded in $Z$. Suppose $F : U \subset X \to Z$. The problem

$$\frac{dx}{dt} = F(x)$$
is said to be reversible if there exists a linear operator $R \in L(X, X) \cap L(Z, Z)$, known as the reverser, such that
\[ F(Rx) = -RF(x) \] (2.41)
for all $x \in U$.

The next version of the Lyapunov centre theorem, which looks at reversible systems, admits a certain spectral constraint. If a system is reversible then the differential operator $L = dF[0]$ also satisfies $LR = -RL$. Consequently, if $\mu$ is an eigenvalue of $L$ with eigenvector $\varphi$ then $-\mu$ is also an eigenvalue with eigenvector $R\varphi$, and so non-real eigenvalues and non-purely imaginary eigenvalues appear as quadruplets, $\mu, -\mu, \overline{\mu}, -\overline{\mu}$. Since simple eigenvalues can not cross the imaginary axis, the transversality condition never holds.

**Theorem 2.3.4 (The Lyapunov centre theorem for reversible systems)** Suppose that $X$ and $Z$ are real Banach spaces, where $X$ is continuously and densely embedded in $Z$. Consider the evolutionary equation
\[ \frac{dx}{dt} = F(x), \] (2.42)
where $F \in C^3(U, Z)$, $U \subset X$, and where $F(0) = 0$. Define $L = dF[0]$ and make the spectral assumptions that

(i) $\pm i\omega_0$ ($\omega_0 \neq 0$) are simple eigenvalues of $L$ with respective eigenvectors $\varphi_0, \overline{\varphi_0}$;

(ii) $i n \omega_0 \in \rho(L)$ for $n \in \mathbb{Z} \setminus \{-1, 1\}$;

(iii) $\|(in\omega_0 I - L)^{-1}\|_{Z \to Z} \leq C/|n|$, $\|(in\omega_0 I - L)^{-1}\|_{Z \to X} \leq C$ as $n \to \pm \infty$.

In addition, assume that (2.42) is reversible with respect to a reverser $R \in L(X, X) \cap L(Z, Z)$.

Under these hypotheses, there exists a neighbourhood $S$ of the origin in $\mathbb{R}$, a continuously differentiable curve $\{x(s)\}_{s \in S}$ of real periodic solutions to (2.42), where $x(0) = 0$. There also exists a continuously differentiable curve of frequencies $\{\omega(s)\}_{s \in S}$, $\omega(0) = \omega_0$, whereby the period of the solutions are given by $2\pi/\omega(s)$.

**Proof.** Arguing as before, we arrive at the reduced complex problem (2.34). In contrast to our previous theorem, decomposing $\phi$ into real and imaginary parts, we show that
\[ \text{Re} \phi(s, \omega) = 0 \] (2.43)
for all $s \in S$, $\omega \in V_1$.

Due to the reversible property of $F$, $G(x, \omega)$ satisfies
\[ G(RSx, \omega) = -RSG(x, \omega), \] (2.44)
where the time reversal $S$ is defined by $(Sz)(t) = z(-t)$. The Lyapunov-Schmidt reduction preserves the above symmetry, and so we obtain

$$PG(RSx_1 + x_2(RSx_1), \omega) = -RSPG(x_1 + x_2(x_1), \omega). \quad (2.45)$$

Since $LR = -RL$, the eigenvectors $\varphi_0$, $\overline{\varphi_0}$ satisfy $R\varphi_0 = \overline{\varphi_0}$ and $R\overline{\varphi_0} = \varphi_0$, and because $x_1 = s(\varphi_0 e^{it} + \overline{\varphi_0} e^{-it})$, it follows that $RSx_1 = x_1$. Hence (2.45) becomes

$$PG(x_1 + x_2(x_1), \omega) = -RSPG(x_1 + x_2(x_1), \omega). \quad (2.46)$$

Writing $PG(x_1 + x_2(x_1), \omega)$ in terms of the function $\phi(s, \omega)$ (cf. (2.39)), one finds from equation (2.46) that

$$\phi(s, \omega)\varphi_0 e^{it} + \overline{\phi(s, \omega)}\overline{\varphi_0} e^{-it} = -RS\phi(s, \omega)\varphi_0 e^{it} - R\overline{\phi(s, \omega)}\overline{\varphi_0} e^{-it}$$

$$= -\phi(s, \omega)\overline{\varphi_0} e^{-it} - \overline{\phi(s, \omega)}\varphi_0 e^{it},$$

which holds if and only if

$$\phi(s, \omega) = -\overline{\phi(s, \omega)}.$$

Therefore $\text{Re} \phi(s, \omega) = 0$, and so it suffices to solve $\text{Im} \phi(s, \omega) = 0$. As before, we arrive at the existence of a continuously differentiable solution curve $\{(s, \omega(s))\}_{s \in S}$ that solves the reduced complex problem (2.34), and the statement in the theorem is obtained by tracing back the definitions. 

### 2.4 Weinstein-Moser theorem

The *Weinstein-Moser theorem* considers a particular type of conservative system, namely a *Hamiltonian system*.

**Definition 2.4.1** Let $X$ and $Z$ be Hilbert spaces, where $X$ is continuously embedded in $Z$. Suppose $F : U \subset X \rightarrow Z$. The problem

$$\frac{dx}{dt} = F(x)$$

is said to be a Hamiltonian system if there exists an operator $H \in C^2(U, \mathbb{R})$, known as the Hamiltonian, with gradient $H' : U \subset X \rightarrow Z$ (defined with respect to an inner product $(\cdot, \cdot)$ on $Z$) such that

$$F(x) = JH'(x) \quad (2.47)$$

for all $x \in U$; here $J \in L(Z, Z)$ is such that $(Jz, z) = 0$ for all $z \in Z$ and $J^2 = -I$. 


Since \( H' \) satisfies \( (H'(x), JH'(x)) = 0 \) for all \( x \in X \), we see that the Hamiltonian is also the associated conserved quantity. As it was for reversible systems, non-real and non-purely imaginary eigenvalues of \( dF[0] \) appear as quadruplets, and so the transversality assumption never holds. In addition, the non-resonance condition is relinquished. Therefore the reduced problem is not necessarily of two dimensions. For that reason one cannot aim to perform the previous analysis in solving the reduced problem, and so an alternative approach is required.

The Hamiltonian structure of (2.47) implies that \( JG(x, \omega) \) is the gradient of a potential function \( g : \bar{U} \times \tilde{V} \rightarrow \mathbb{R} \) given by

\[
g(x, \omega) = \int_0^{2\pi} \frac{1}{2} (J \frac{dx}{dt}, x(t)) \ dt + \frac{1}{\omega} \int_0^{2\pi} (H(x(t)) - s^2) \ dt. \tag{2.48}
\]

Indeed, since

\[
d_1 g[x, \omega](h) = \int_0^{2\pi} \frac{1}{2} (J \frac{dx}{dt}, h) + \frac{1}{2} (J \frac{dh}{dt}, x) + \frac{1}{\omega} d_1 H[x](h) \ dt,
\]

we find

\[
d_1 g[x, \omega](h) = \int_0^{2\pi} (J(\frac{dx}{dt} - \frac{1}{\omega} JH'(x)), h) \ dt,
\]

where we have integrated by parts and used the antisymmetry of \( J \). Thus

\[
d_1 g[x, \omega](h) = (JG(x, \omega), h)_{\tilde{Z}} \tag{2.49}
\]

for all \( h \in \tilde{Z} \); here we refer to the inner product on \( \tilde{Z} \) defined by

\[
(z_1, z_2)_{\tilde{Z}} = \int_0^{2\pi} (z_1(t), z_2(t)) \ dt, \quad z_1, z_2 \in \tilde{Z}.
\]

The key to the Weinstein-Moser theorem is the fact that solutions of \( G(x, \omega) = 0 \) form critical points of \( g(x, \omega) \) on \( \bar{U} \). In fact, due to the Lagrange multiplier method, we see that critical points of \( g(x, \omega) \) are critical points of the function

\[
B(x) = \int_0^{2\pi} \frac{1}{2} (J \frac{dx}{dt}, x(t)) \ dt
\]

subject to the constraint

\[
C(x) = \int_0^{2\pi} (H(x(t)) - s^2) \ dt = 0,
\]

and \( 1/\omega \) is the Lagrange multiplier. By the variational Lyapunov-Schmidt theorem, property (2.49) holds equivalently for \( PG(x_1 + x_2(x_1, \omega), \omega) \), where the reduced potential function is given by \( \tilde{g}(x_1, \omega) = g(x_1 + x_2(x_1, \omega), \omega) \). The Weinstein-Moser theorem determines critical points of \( \tilde{g}(x_1, \omega) \) on some compact subspace of \( \ker A \), and shows that these solve \( PG(x_1 + x_2(x_1, \omega(x_1)), \omega(x_1)) = 0 \).
Theorem 2.4.2 (The Weinstein-Moser theorem) Suppose that $X$ and $Z$ are real Hilbert spaces, where $X$ is continuously and densely embedded in $Z$. Consider the Hamiltonian system

$$\frac{dx}{dt} = JH'(x), \quad (2.50)$$

where the gradient $H' \in C^3(U, Z)$, $U \subset X$, is defined with respect to an inner product $(\cdot, \cdot)$ on $Z$, and where $H'(0) = 0$.

Define $L = JdH'[0]$ and make the following spectral assumptions:

(i) There exists a finite number $m$ of geometrically simple purely imaginary eigenvalues $\pm i\omega_k$ of $L$ with respective eigenvectors $\varphi_k, \overline{\varphi_k}$, $k = 0, 1, 2, ..., m - 1$.

(ii) The eigenvalues $\pm i\omega_k$ are in resonance with the pair $\pm i\omega_0$, that is

$$\frac{\omega_k}{\omega_0} = n, \quad n \in \mathbb{Z}.$$ 

Furthermore, $\imath n\omega_0 \in \rho(L)$ for all $n \in \mathbb{Z} \setminus K$, where

$$K = \{\pm \omega_k/\omega_0 : k = 0, 1, 2, ..., m - 1\}.$$ 

(iii) The resolvent estimates

$$\|(\imath n\omega_0 I - L)^{-1}\|_{Z \to Z} \leq C, \quad \|(\imath n\omega_0 I - L)^{-1}\|_{Z \to X} \leq C, \quad n \to \pm \infty,$$

hold.

In addition, assume that

$$\sum_{k=0}^{m-1} (dH'[0]\varphi_k, \varphi_k) > 0 \quad (2.51)$$

Under these hypotheses, for each level set $H = s^2$, where $s \in \mathbb{R}$ is sufficiently close to $H(0)$, there exist at least $m$ distinct periodic solutions $x_m(s)$ of (2.70) such that $x_m(s) \to 0$ as $s \to H(0)$. There also exist corresponding frequencies $\omega_m(s)$, whereby the period of each solution is given by $2\pi/\omega_m(s)$, and where $\omega_m(s) \to \omega_0$ as $s \to H(0)$.

Proof. Although the purely imaginary eigenvalues are in resonance, the above assumptions still imply that the operator $A : \tilde{X} \to \tilde{Z}$, given by

$$Ax = \frac{dx}{dt} = \frac{1}{\omega_0} Lx,$$
is Fredholm with index zero. Since the purely imaginary eigenvalues $\pm i\omega_k$ ($k = 0, 1, 2, \ldots, m-1$) resonate with the pair $\pm i\omega_0$, it follows, following the usual argument, that $\ker A$ is finite dimensional and of the form

$$\ker A = \{ \beta_0 \varphi_0 e^{it} + \beta_1 \varphi_1 e^{i\omega_1/\omega_0} + \ldots + \beta_{m-1} \varphi_{m-1} e^{i\omega_{m-1}/\omega_0} + \bar{\beta}_0 \varphi_0 e^{-it} + \bar{\beta}_1 \varphi_1 e^{-i\omega_1/\omega_0} + \ldots + \bar{\beta}_{m-1} \varphi_{m-1} e^{-i\omega_{m-1}/\omega_0} : \beta_k \in \mathbb{C}, k = 0, 1, 2, \ldots, m-1 \}.$$ 

Consider the problem $Ax = z$. This yields a coefficient problem involving the Fourier coefficients $c_n, a_n$, of $x(t), z(t)$, respectively. Since the algebraic multiplicity of each of the eigenvalues $\pm i\omega_k$ is equal to its corresponding geometric multiplicity, the operators $(\pm i\omega_k I - L) : Z \to Z$ are Fredholm operators of index zero. It follows that

$$\text{Im } A = \{ z \in \tilde{Z} : \Pi_k a_n = 0, n = \omega_k/\omega_0, k = 0, 1, 2, \ldots, m-1 \},$$

where the spectral projections $\Pi_k : Z \to Z$ project onto $\varphi_k$, respectively. The resolvent estimates ensure that solutions $x$ of $Ax = z$ have convergent Fourier series in $X$. Thus $A$ is a Fredholm operator of index zero, and we obtain the decompositions

$$\tilde{X} = \ker A \oplus (\text{Im } A \cap \tilde{X}),$$

$$\tilde{Z} = \text{Im } A \oplus \ker A.$$

A direct application of the Lyapunov-Schmidt reduction results in the finite dimensional problem

$$PG(x_1 + x_2(x_1, \omega), \omega) = 0, \quad (2.52)$$

where $P$ is the orthogonal projection onto $\ker A$ along $\text{Im } A$, $x_1$ belongs to the neighbourhood $U_1$ of the origin in $\ker A$, $\omega$ belongs to the neighbourhood $V_1 \subset \tilde{V}$ of $\omega_0$, and $x_2 : U_1 \times V_2 \to U_2 \subset \text{Im } A \cap \tilde{X}$ is a $C^2$-function. Since $JG(x, \omega)$ defines the potential $g : \tilde{U} \times \tilde{V} \to \mathbb{R}$ given by (2.48), then

$$d_1 \tilde{g}[x_1, \omega](h) = (JPG(x_1 + x_2(x_1, \omega), \omega), h)_{\tilde{Z}}, \quad h \in \ker A, \quad (2.53)$$

follows from the variational Lyapunov-Schmidt theorem; here $\tilde{g} : U_1 \times V_1 \to \mathbb{R}$ is defined by $\tilde{g}(x_1, \omega) = g(x_1 + x_2(x_1, \omega), \omega)$. In the following we show that the equation

$$d_1 \tilde{g}[x_1, \omega](x_1) = 0$$

can be solved with the assertion $\omega = \omega(x_1)$ for all $x_1$ in $U_1$, where $U_1$ is taken smaller if necessary.

According to the variational property (2.53), the problem at hand is to solve

$$(J \frac{dx_1}{dt} + \frac{1}{\omega} H'(x_1 + x_2(x_1, \omega)), x_1)_{\tilde{Z}} = 0. \quad (2.54)$$
Observe that $J \frac{dx_1}{dt} = -1/\omega_0 dH'[0] x_1$ (since $x_1 \in \ker A$). Writing

$$H'(x_1 + x_2(x_1, \omega)) = dH'(0) x_1 + N(x_1 + x_2(x_1, \omega)), \quad (2.55)$$

where $N(x_1 + x_2(x_1, \omega)) = H'(x_1 + x_2(x_1, \omega)) - dH'[0] x_1$, equation (2.54) becomes

$$\left(\frac{1}{\omega} - \frac{1}{\omega_0}\right)(dH'[0] x_1, x_1) Z + \frac{1}{\omega} (N(x_1 + x_2(x_1, \omega)), x_1) Z = 0. \quad (2.56)$$

Introduce the function $\phi : U_1 \times V_1 \to \mathbb{R}$ defined by

$$\phi(x_1, \omega) = \frac{1}{\omega} - \frac{1}{\omega_0} + \frac{1}{\omega} \frac{(N(x_1 + x_2(x_1, \omega)), x_1)}{(dH'[0] x_1, x_1) Z}, \quad x_1 \neq 0, \quad (2.57)$$

$$\phi(x_1, \omega) = \frac{1}{\omega} - \frac{1}{\omega_0}, \quad x_1 = 0, \quad (2.58)$$

and consider the auxiliary problem

$$\phi(x_1, \omega) = 0.$$ 

The assumption (2.51) implies that $(dH'[0] x_1, x_1) Z > 0$; therefore $\phi(x_1, \omega)$ is a continuously differentiable for $x_1 \neq 0$. Since by definition $N(0) = 0$, $dN[0] = 0$, it follows that

$$N(x_1 + x_2(x_1, \omega)) = o(\|x_1\| Z),$$

and so $\phi(x_1, \omega)$ is continuous at $x_1 = 0$. Furthermore, since $d_2 x_2[0, \omega] = 0$ (because $x_2(0, \omega) = 0$ for all $\omega \in V_1$), it follows that $d_2 x_2[\omega] = O(\|x_1\| Z)$, which implies that

$$dN[x_1 + x_2(x_1, \omega)] d_2 x_2[\omega] = o(\|x_1\| Z)$$

(since $dN[x_1 + x_2(x_1, \omega)] = O(\|x_1\| Z)$). The partial derivative $d_2 \phi : U_1 \times V_1 \to L(\mathbb{R}, \mathbb{R})$ is therefore continuous at the point $(0, \omega_0)$. Using the fact that $\phi(0, \omega_0) = 0$ and $d_2 \phi[0, \omega_0] = -\omega_0^{-2}$, one finds by the implicit function theorem that there exists a unique continuous function $\omega : U_1 \to V_1$ ($U_1$ is taken smaller if necessary) such that $\omega(0) = \omega_0$ and

$$\phi(x_1, \omega(x_1)) = 0$$

for all $x_1 \in U_1$ (note that $\omega$ is continuously differentiable on $U_1 \setminus \{0\}$). In turn $\omega(x_1)$ solves (2.56), and therefore

$$d_1 \bar{g}[x_1, \omega(x_1)](x_1) = 0 \quad (2.59)$$

for all $x_1 \in U_1$.

Next define $\bar{H} : U_1 \to \mathbb{R}$ by

$$\bar{H}(x_1) = H(x_1 + x_2(x_1, \omega(x_1))).$$
and consider the manifold
\[ S_s := \{ x_1 \in U_1 : \int_0^{2\pi} \tilde{H}(x_1(t)) \, dt = 2\pi s^2 \}. \]

Because \( H'(0) = 0 \) and \( (dH'[0]_1, x_1)_Z > 0 \), it follows that \( d\tilde{H}[0] = 0 \) and \( d^2\tilde{H}[0](x_1, x_1) > 0 \). Therefore \( \tilde{H}(0) \) is a local minimum, and so for \( s \) sufficiently close to \( H(0) \) the space \( S_s \) defines a compact sphere-like manifold. It follows that
\[ \ker A = T_{x_1}S_s \oplus \text{span } x_1 \] (2.60)
for fixed \( x_1 \in S_s \), where \( T_{x_1}S_s \) is the tangent manifold of \( S_s \) at \( x_1 \).

We define the function \( \hat{g} : S_s \to \mathbb{R} \) by
\[ \hat{g}(x_1) = \tilde{g}(x_1, \omega(x_1)), \]
and show that critical points of \( \hat{g} \) on \( S_s \) are critical points of the reduced potential \( \tilde{g} \). A direct differentiation gives
\[
\begin{align*}
d_1\hat{g}[x_1](h) &= d_1\tilde{g}[x_1, \omega(x_1)](h) + d_2\tilde{g}[x_1, \omega(x_1)]d\omega[x_1](h) \\
&= d_1\tilde{g}[x_1, \omega(x_1)](h) + d_1\tilde{g}[x_1 + x_2(x_1, \omega(x_1)), \omega]d_2[x_1, \omega(x_1)]d\omega[x_1](h) \\
&\quad + d_2\tilde{g}[x_1 + x_2(x_1, \omega(x_1)), \omega]d\omega[x_1](h) \\
&= d_1\tilde{g}[x_1, \omega(x_1)](h)
\end{align*}
\]
for all \( h \in \ker A \). The second term disappears since \( d_2[x_1, \omega(x_1)]d\omega[x_1](h) \in \text{Im } A \). The last term disappears by directly differentiating the explicit form (2.48), and noting that \( x_1 \in S_s \). This suggests that if \( x_s \) is a critical point of \( \tilde{g} \) on \( S_s \) then
\[ d_1\tilde{g}[x_s, \omega(x_s)](T_{x_s}S_s) = 0. \]

To conclude, since \( S_s \) is compact, \( \tilde{g} \) possesses at least two critical points \( x_s \), which by the result (2.59) and the decomposition (2.60) imply that \( d_1\tilde{g}[x_s, \omega(x_s)](h) = 0 \) for all \( h \in \ker A \), and hence
\[ PG(x_s + x_2(x_s, \omega(x_s)), \omega(x_s)) = 0 \]
(by (2.53)). Moreover, since for any solution the Hamiltonian is constant, it follows that these critical points belong to the level set \( H = s^2 \).

This asserts the existence of at least two distinct solutions of the reduced problem
\[ PG(x_1 + x_2(x_1, \omega), \omega) = 0. \]
We extend this by showing that there exists \( m \) critical points of \( \hat{g} \). Consider the time shift operator \( S_\theta \) defined by \( (S_\theta x)(t) = x(t + \theta) \), \( \theta \in \mathbb{R} \). It is clear that if \( x(t) \) is a solution of (2.70) then \( (S_\theta x)(t) \) is also a solution. In order to filter out these families of solutions we consider the quotient space
\[ \frac{S_s}{S_\theta}, \]
and instead look for critical points of \( \dot{g} \) acting on \( S_s/S_\theta \). Because elements \( x_1 \) in \( S_s \) have the form

\[
x_1(t) = \beta_0 \varphi_0 e^{it} + \beta_1 \varphi_1 e^{i\theta_1 t/\omega_0} + \ldots + \beta_m e^{i\theta_{m-1} t/\omega_0} + \beta_0 \varphi_0 e^{-it} + \beta_1 \varphi_1 e^{-i\theta_1 t/\omega_0} + \ldots + \beta_m e^{-i\theta_{m-1} t/\omega_0}
\]

we see that

\[
\frac{S_s}{S_\theta} \cong \frac{S^{2m-1}}{\sim},
\]

where the equivalence relation \( \sim \) identifies \( \{\beta_0, \beta_1, ..., \beta_m\} \) with

\[
\{\beta_0 e^{i\theta}, \beta_1 e^{i\theta_1/\omega_0}, ..., \beta_m e^{i\theta_{m-1}/\omega_0}\}.
\]

Since \( |e^{i\theta_k/\omega_0}| = 1 \) for all \( \theta \in \mathbb{R}, k = 0, ..., m-1 \), then \( S^{2m-1}/\sim \) is topologically equivalent to the complex projective space \( \mathbb{C}P^{m-1} \), and thus

\[
\frac{S_s}{S_\theta} \cong \mathbb{C}P^{m-1}.
\]

The complex projective space \( \mathbb{C}P^{m-1} \) is the space of complex lines that go through the origin of \( \mathbb{C}^m \). The Lusternik-Schnirelman theorem asserts that continuously differentiable functions defined on \( \mathbb{C}P^{m-1} \) have at least \( m \) distinct critical points. A generalized version of the Lusternik-Schnirelman theorem can be found in Willem [53, pp. 81–90]. As a result, there exists \( m \) critical points of \( \dot{g} \) on \( S_s/S_\theta \), which by tracing back the various definitions leads to the \( m \) distinct solutions \( x_m(s) \) of (2.70) stated in the theorem.

In the following we reflect on the importance of the positive definite condition (2.51). We illustrate with an example (taken from Ambrosetti & Prodi [2, p. 151]) that shows that if (2.51) was merely non-degenerate \((>0 \text{ replaced with } \neq 0)\), periodic solutions do not generally exist.

Consider the Hamiltonian system

\[
\frac{dx}{dt} = JH'(x),
\]

where \( x = (x_1, x_2, y_1, y_2) \) belongs to \( \mathbb{R}^4 \), and the Hamiltonian \( H : \mathbb{R}^4 \to \mathbb{R} \) is given by

\[
H(x) = \frac{1}{2} (x_1^2 - x_2^2 + y_1^2 - y_2^2) + (x_1^2 + x_2^2 + y_1^2 + y_2^2)(y_1y_2 - x_1x_2).
\]
One finds that $x = 0$ is a fixed point of (2.61), and that the Jacobian $dH'[0] : \mathbb{R}^4 \to \mathbb{R}^4$, given by

$$dH'[0] = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},$$

satisfies the non-degeneracy condition $(dH'[0]h, h) \neq 0$ for all $h \in \mathbb{R}^4$. Observe that

$$JdH'[0] = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},$$

possesses a pair of geometrically simple purely imaginary eigenvalues $\pm i$, and that $\pm in$, $n \in \mathbb{Z} \setminus \{1, -1\}$, belongs to its resolvent set. Furthermore $JdH'(0)$ satisfies the resolvent conditions (the resolvent conditions always hold in finite dimensions). However, if $x = (x_1, x_2, y_1, y_2)$ is a solution of (2.61) then

$$\frac{d}{dt}(x_1y_2 + y_1x_2) = -4(x_1x_2 - y_1y_2)^2 - 2(x_1x_2)^2 - 2(y_1y_2)^2 \leq 0,$$

which implies that other than the trivial solution no periodic solutions exist.

2.5 The Hopf-Iooss theorem, the Lyapunov-Iooss theorem and the Weinstein-Moser-Iooss theorem

The previous theorems cannot be applied to problems for which zero is an element of the essential spectrum of $L$. The issue is that the non-resonance condition is violated (any $n$-multiple of the purely imaginary eigenvalues of $L$ are required to lie strictly in its resolvent set). Nevertheless, it is still possible to determine periodic solutions provided one can show that $L$ is invertible on the range of $N(x) = F(x) - Lx$. The final section of this chapter looks at modifications of the Hopf bifurcation, the Lyapunov centre and the Weinstein-Moser theorem that account for such conditions. These are respectively the Hopf-Iooss theorem, the Lyapunov-Iooss theorem and the Weinstein-Moser-Iooss theorem.

**Theorem 2.5.1 (The Hopf-Iooss theorem)** Suppose that $X$ and $Z$ are real Banach spaces, where $X$ is continuously and densely embedded in $Z$. Consider the autonomous evolutionary equation

$$\frac{dx}{dt} = F(x, \mu), \quad (2.62)$$
where $F \in C^3(U \times V, Z)$ with open set $U \subset X$, $V \subset \mathbb{R}$, and where $F(0, \mu) = 0$ for all $\mu \in V$.

Define $L_\mu = d_1 F[0, \mu]$ and suppose there exists an element $\mu_0 \in V$, a neighbourhood $\Omega$ of $\mu_0$ in $V$, and a continuously differentiable curve $\{(\lambda(\mu), \varphi(\mu))\}_{\mu \in \Omega}$ of eigensolutions of $L_\mu$. Make the assumptions that

(i) $(\lambda(\mu_0), \varphi(\mu_0)) = (i \omega_0, \varphi_0)$, $\omega_0 \neq 0$, where $i \omega_0$ is simple;

(ii) $\Re \lambda'(\mu_0) \neq 0$;

(iii) 0 belongs to the essential spectrum of $L_\mu$ for $\mu \in \Omega$;

(iv) $in \omega_0 \in \rho(L_{\mu_0})$ for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$;

(v) $\|(in\omega_0 I - L_{\mu_0})^{-1}\|_{Z \to Z} \leq C/|n|$, $\|(in\omega_0 I - L_{\mu_0})^{-1}\|_{Z \to X} \leq C$ as $n \to \pm \infty$.

Furthermore, define $N : U \times \Omega \to Z$ by $N(x, \mu) = F(x, \mu) - L_\mu x$, and assume that for each $x^* \in U$ the problem

$$L_\mu x = -N(x^*, \mu) \quad (2.63)$$

has a unique solution $x \in X$ that is a three times continuously differentiable function of $x^*$.

Under these hypotheses, there exist a neighbourhood $S$ of the origin in $\mathbb{R}$ and a continuously differentiable curve $\{(x(s), \mu(s))\}_{s \in S}$ of real periodic solutions to (2.62) such that $(x(0), \mu(0)) = (0, \mu_0)$. There also exists a continuously differentiable curve of frequencies $\{\omega(s)\}_{s \in S}$, where $\omega(0) = \omega_0$, whereby the period of the solutions are given by $2\pi/\omega(s)$.

**Proof.** As usual, the problem is to solve

$$G(x, \omega, \mu) = 0, \quad (2.64)$$

where $G \in C^2(\bar{U} \times \bar{V}, \bar{Z})$, $\bar{U} \subset \bar{X}$, $\bar{V} \subset \mathbb{R}^2$. Following the same analysis as in the Hopf bifurcation theorem, one finds that $A = d_1 G[0, \omega_0, \mu_0]$ is not a Fredholm operator with index zero; the issue is that $(in\omega_0 I - L_{\mu_0}) : X \to Z$ is not a bijection for $n = 0$ (because 0 is contained in the essential spectrum of $L_{\mu_0}$). We therefore decompose $\tilde{X}$ and $\tilde{Z}$ by

$$\tilde{X} = (X_1 \cap \tilde{X}) \oplus (X_2 \cap \tilde{X}),$$

$$\tilde{Z} = X_1 \oplus X_2,$$

where

$$X_1 = \{ \beta_0 + \beta_1 \varphi_0 e^{it} + \overline{\beta_1} \overline{\varphi_0} e^{-it} : \beta_0 \in Z, \beta_1 \in \mathbb{C} \},$$

$$X_2 = \{ z \in \tilde{Z} : [z]_0 = 0, \Pi[z]_1 = \overline{\Pi[z]_{-1}} = 0 \}$$

([z]_n denotes the nth Fourier coefficient of $z(t)$), and introduce the projection $Q : \tilde{Z} \to \tilde{Z}$ onto $X_1$ along $X_2$ defined by

$$(Qz)(t) = [z]_0 + (Pz)(t),$$

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where $P : \tilde{Z} \rightarrow \tilde{Z}$ is the projection given in (2.24). One finds that $(I - Q)A : X_2 \cap \tilde{X} \rightarrow X_2$ is a bijection, and so setting $x_1 = Qx$, $x_2 = (I - Q)x$, it follows by the Lyapunov-Schmidt reduction that (2.64) is reduced to

$$QG(x_1 + x_2(x_1, \omega, \mu), \omega, \mu) = 0,$$

(2.65)

where $x_1 \in U_1 \subset X_1$, $(\omega, \mu) \in V_1 \subset \tilde{V}$. Moreover, setting $x_0 = [x_1]_0$, $\tilde{x}_1 = Px_1$, $\tilde{x}_2(x_0, \tilde{x}_1, \omega, \mu) = x_2(x_1, \omega, \mu)$, and dropping the tildes, (2.65) becomes

$$[G(x_0 + x_1 + x_2(x_0, x_1, \omega, \mu), \omega, \mu)]_0 = 0,$$

(2.66)

$$PG(x_0 + x_1 + x_2(x_0, x_1, \omega, \mu), \omega, \mu) = 0,$$

(2.67)

where $x_0, x_1$ belong to corresponding subsets $Y_0, Y_1$ in $U_1$, respectively.

We write $F(x, \mu) = L_\mu x + N(x, \mu)$ so that (2.66) becomes

$$L_\mu x_0 = -[N(x_0 + x_1 + x_2(x_0, x_1, \omega, \mu), \mu)]_0$$

and therefore

$$L_\mu \int_0^{2\pi} x(t) \, dt = -\int_0^{2\pi} N(x_0 + x_1 + x_2(x_0, x_1, \omega, \mu), \mu) (t) \, dt.$$

By assumption (2.63), there exists a solution $x = x(x_0, x_1, \omega, \mu)$ to the above problem. Moreover, it follows from $N(0, \mu) = 0$ and $d_1 N[0, \mu] = 0$ that $x(0, 0, \omega, \mu) = 0$ and $d_1 x[0, 0, \omega, \mu] = 0$ for $(\omega, \mu) \in V_1$. Introducing the operator $\phi : Y_0 \times Y_1 \times V_1 \rightarrow X$ given by

$$\phi(x_0, x_1, \omega, \mu) = x_0 - x(x_0, x_1, \omega, \mu),$$

we therefore find that $\phi(0, 0, \omega_0, \mu_0) = 0$ and $d_1 \phi(0, 0, \omega_0, \mu_0) = I$. Taking neighbourhoods smaller if necessary, the implicit function theorem asserts the existence of a $C^2$-function $x_0 : Y_1 \times V_1 \rightarrow Y_0$ such that

$$\phi(x_0(x_1, \omega, \mu), x_1, \omega, \mu) = 0,$$

and which has the property $x_0(0, \omega, \mu) = 0$ for $(\omega, \mu) \in V_1$.

It follows that for $x_1 \in Y_1$, $(\omega, \mu) \in V_1$,

$$L_\mu x_0(x_1, \omega, \mu) = -[N(x_0(x_1, \omega, \mu) + x_1 + x_2(x_0(x_1, \omega, \mu), x_1, \omega, \mu), \mu)]_0,$$

and so problem (2.66), (2.67) is reduced to

$$PG(x_1 + x_3(x_1, \omega, \mu), \omega, \mu) = 0,$$

where $x_3(x_1, \omega, \mu) = x_0(x_1, \omega, \mu) + x_2(x_0(x_1, \omega, \mu), x_1, \omega, \mu)$ with $x_3(0, \omega, \mu) = 0$ for $(\omega, \mu) \in V_1$. One now follows the usual steps in the Hopf bifurcation theorem to solve the above reduced problem and arrive at the statement made in the theorem. \qed
The additional conditions, regarding the essential spectrum and the invertibility of $L$, are only used during the reduction process. Since the reduced problem can be also solved using the methods employed by the Lyapunov centre theorem and the Weinstein-Moser theorem, one equivalently derives the Lyapunov-Iooss theorem for conservative and reversible systems, and the Weinstein-Moser-Iooss theorem.

**Theorem 2.5.2 (The Lyapunov-Iooss theorem for conservative systems)** Suppose that $X$ and $Z$ are real Hilbert spaces, where $X$ is continuously and densely embedded in $Z$. Let $(\cdot, \cdot)$ denote an inner product defined on $Z$. Consider the autonomous evolutionary equation

$$\frac{dx}{dt} = F(x), \quad (2.68)$$

where $F \in C^3(U, Z)$, with open set $U \subset X$, and where $F(0) = 0$.

Define $L = d_1F[0]$ and make the assumptions that

1. $\pm i\omega_0 (\omega_0 \neq 0)$ are simple eigenvalues of $L$ with respective eigenvectors $\varphi_0, \overline{\varphi_0}$;
2. $0$ belongs to the essential spectrum of $L$;
3. $in\omega_0 \in \rho(L)$ for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$;
4. $\| (in\omega_0 I - L)^{-1} \|_{Z \to Z} \leq C/|n|$, $\| (in\omega_0 I - L)^{-1} \|_{Z \to X} \leq C$ as $n \to \pm \infty$;
5. the system $(2.68)$ is conservative with conserved quantity $H \in C^2(U, \mathbb{R})$ such that $(H''(0)\varphi_0, \varphi_0) \neq 0$.

Furthermore, define $N : U \to Z$ by $N(x) = F(x) - Lx$, and assume that for each $x^* \in U$ the problem

$$Lx = -N(x^*),$$

has a unique solution $x \in X$ that is a three times continuously differentiable function of $x^*$.

Under these assumptions, there exists a neighbourhood $S$ of the origin in $\mathbb{R}$, a continuously differentiable curve $\{x(s)\}_{s \in S}$, where $x(0) = 0$, of real periodic solutions to $(2.68)$. There also exists a continuously differentiable curve of frequencies $\{\omega(s)\}_{s \in S}$, where $\omega(0) = \omega_0$, and whereby the period of the solutions are given by $2\pi/\omega(s)$.

**Theorem 2.5.3 (The Lyapunov-Iooss theorem for reversible systems)** Suppose that $X$ and $Z$ are real Banach spaces, where $X$ is continuously and densely embedded in $Z$. Consider the autonomous evolutionary equation

$$\frac{dx}{dt} = F(x), \quad (2.69)$$

where $F \in C^3(U, Z)$, with open set $U \subset X$, where $F(0) = 0$.

Define $L = d_1F[0]$ and make the assumptions that
Furthermore, define $N : U \to Z$ by $N(x) = F(x) - Lx$, and assume that for each $x^* \in U$ the problem
\[ Lx = -N(x^*), \]
has a unique solution $x \in X$ that is a three times continuously differentiable function of $x^*$.

Under these assumptions, there exists a neighbourhood $S$ of the origin in $\mathbb{R}$, a continuously differentiable curve $\{x(s)\}_{s \in S}$, where $x(0) = 0$, of real periodic solutions to (2.69). There also exists a continuously differentiable curve of frequencies $\{\omega(s)\}_{s \in S}$, where $\omega(0) = \omega_0$, and whereby the period of the solutions are given by $2\pi/\omega(s)$.

**Theorem 2.5.4 (The Weinstein-Moser-Iooss theorem)** Suppose that $X$ and $Z$ are real Hilbert spaces, where $X$ is continuously and densely embedded in $Z$. Consider the Hamiltonian system
\[ \frac{dx}{dt} = JH'(x), \tag{2.70} \]
where the gradient $H' \in C^3(U, Z)$, $U \subset X$, is defined with respect to an inner product $(\cdot, \cdot)$ on $Z$, and where $H'(0) = 0$.

Define $L = JdH'[0]$ and make the assumptions that

(i) $\pm i\omega_k$, $k = (0, 1, 2, ..., m - 1)$, are geometrically simple purely imaginary eigenvalues of $L$ with respective eigenvectors $\varphi_k$, $\overline{\varphi}_k$, and are in resonance with the pair $\pm i\omega_0$;

(ii) $0$ belongs to the essential spectrum of $L$;

(iii) $i\omega_0 \in \rho(L)$ for $n \in \mathbb{Z} \setminus K$, where $K = \{\pm \omega_k/\omega_0 : k = 0, 1, 2, ..., m - 1\}$;

(iv) $\|(i\omega_0 I - L)^{-1}\|_{Z \to Z} \leq C/|n|$, $\|(i\omega_0 I - L)^{-1}\|_{Z \to X} \leq C$ as $n \to \pm \infty$;

(v) $\sum_{k=0}^{m-1} (dH'[0]\varphi_k, \varphi_k) > 0$.

Furthermore, define $N : U \to Z$ by $N(x) = F(x) - Lx$, and assume that for each $x^* \in U$ the problem
\[ Lx = -N(x^*), \]
has a unique solution $x \in X$ that is a three times continuously differentiable function of $x^*$.

Under these hypotheses, for each level set $H = s^2$, where $s \in \mathbb{R}$ is sufficiently close to $H(0)$, there exist at least $m$ distinct periodic solutions $x_m(s)$ of (2.70) such that $x_m(s) \to 0$ as $s \to H(0)$. There also exist corresponding frequencies $\omega_m(s)$, whereby the period of each solution is given by $2\pi/\omega_m(s)$, and where $\omega_m(s) \to \omega_0$ as $s \to H(0)$. 

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Chapter 3

Three dimensional doubly-periodic water waves

3.1 The mathematical problem

We consider the irrotational flow of a perfect fluid in a domain bounded above by a free surface subject to the forces of gravity and surface tension. In the usual Cartesian coordinate system the fluid domain is $D_\eta = \{(x, \hat{y}, z) \in \mathbb{R}^3 : -\infty < \hat{y} < \eta(x, z, t)\}$, where $x$ and $z$ denote the horizontal coordinates and $\hat{y}$ is the vertical coordinate. In terms of an Eulerian velocity potential $\phi$, the mathematical problem is to solve Laplace’s equation

$$\phi_{xx} + \phi_{\hat{y}\hat{y}} + \phi_{zz} = 0,$$

with boundary conditions

$$\eta_t = \phi_{\hat{y}} - \eta_x \phi_x - \eta_z \phi_z,$$  \quad \text{on } \hat{y} = \eta, \quad (3.2)

$$\phi_t = -\frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) - g\eta$$

$$+ \tau \left[ \frac{\eta_x}{(1 + \eta_x^2 + \eta_z^2)^{3/2}} \right]_x + \tau \left[ \frac{\eta_z}{(1 + \eta_x^2 + \eta_z^2)^{3/2}} \right]_z + B,$$  \quad \text{on } \hat{y} = \eta, \quad (3.3)

$$\phi_{\hat{y}} \to 0,$$  \quad \text{as } \hat{y} \to -\infty. \quad (3.4)

Here $g$ is the gravitational constant, $\tau$ is the coefficient of surface tension, and $B$ is Bernoulli’s number (see, for example, Stoker [47, ch. 1]). The boundary conditions (3.2), (3.3) are known as respectively the \textit{kinematic} and \textit{dynamic boundary conditions} at the free surface.

We restrict our attention to \textit{travelling water waves}, that is solutions of (3.1)–(3.4) that propagate uniformly in the horizontal $x$-direction, so that

$$\eta(x, z, t) = \hat{\eta}(x - ct, z), \quad \phi(x, \hat{y}, z, t) = \hat{\phi}(x - ct, \hat{y}, z), \quad (3.5)$$
where $c$ denotes the wave speed. Substituting (3.5) into (3.1)–(3.4), and scaling the variables and functions via

$$
(x - ct, \hat{y}, z) \mapsto \frac{\tau}{c^2}(x - ct, \hat{y}, z),
$$

$$
\hat{\phi}(x - ct, \hat{y}, z) \mapsto \frac{\tau}{c}\hat{\phi}(x - ct, \hat{y}, z),
$$

$$
\hat{\eta}(x - ct, \hat{y}, z) \mapsto \frac{\tau}{c^2}\hat{\eta}(x - ct, \hat{y}, z),
$$

we arrive at the travelling water-wave problem

\begin{align*}
\phi_{xx} + \phi_{\hat{y}\hat{y}} + \phi_{zz} &= 0, & \text{in } \hat{y} < \eta(x, z), & \text{(3.6)} \\
\phi_{\hat{y}} &= \eta_x \phi_x + \eta_z \phi_z - \eta_x, & \text{on } \hat{y} = \eta(x, z), & \text{(3.7)} \\
-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + \gamma \eta & \quad - \left[ \frac{\eta_x}{(1 + \eta_x^2 + \eta_z^2)^{\frac{3}{2}}} \right]_x - \left[ \frac{\eta_z}{(1 + \eta_x^2 + \eta_z^2)^{\frac{3}{2}}} \right]_z = 0, & \text{on } \hat{y} = \eta(x, z), & \text{(3.8)} \\
\phi_{\hat{y}} & \to 0, & \text{as } \hat{y} \to -\infty, & \text{(3.9)}
\end{align*}

where for notational simplicity we have dropped the hats and written $x$ in place of $x - ct$; Bernoulli’s constant is taken to be zero. The domain of definition $D_\eta$ has been accordingly redefined, and the single dimensionless parameter $\gamma > 0$ is given by

$$
\gamma = \frac{g\tau}{c^4}.
$$

Observe that the travelling water wave problem (3.6)–(3.9) possess certain symmetries. The equations are invariant under translation in the spatial coordinates $x$, $z$, and in the velocity potential $\phi$. In addition they are also invariant under the transformation

$$
x \mapsto -x, \quad z \mapsto -z, \quad \eta \mapsto \eta, \quad \phi \mapsto -\phi.
$$

We introduce an oblique coordinate system $(x_1, z_1)$, where the $x_1$- and $z_1$-axes make angles of $\theta_1$ and $\theta_2$, respectively, with the positive $x$-axis (see Figure 3.1), so that

$$
\begin{align*}
x_1 &= x \sin \theta_2 - z \cos \theta_2, \\
z_1 &= x \sin \theta_1 - z \cos \theta_1,
\end{align*}
$$

where $\theta_1, \theta_2 \in (-\pi, \pi)$ are chosen with $\theta_1 - \theta_2 \neq 0, \pi, -\pi$ (to ensure that the $x_1$- and $z_1$-axes do not coincide).
Figure 3.1: Diagram of the oblique coordinate system \((x_1, z_1)\). The \(x_1\)- and \(z_1\)-axes are obtained by rotating the \(x\)-axis through angles of \(\theta_1\) and \(\theta_2\), respectively, where \(\theta_1, \theta_2 \in (\pi, \pi)\) are chosen with \(\theta_1 - \theta_2 \neq 0, \pi, \pi\).

We seek solutions of (3.6)–(3.9) of the form \(\phi(x, \hat{y}, z) = \tilde{\phi}(x_1, \hat{y}, z_1), \eta(x, z) = \tilde{\eta}(x_1, z_1)\) that are periodic with fixed period \(P\) in the \(z_1\)-direction. These functions satisfy the equations

\[
\phi_{xx} + \phi_{\hat{y}\hat{y}} + \nu^2 \phi_{zz} + 2\nu \cos(\theta_1 - \theta_2) \phi_{xz} = 0, \quad \text{in } \hat{y} < \eta(x, z), \quad (3.10)
\]

\[
\phi_{\hat{y}} = -\sin \theta_2 \eta_x - \nu \sin \theta_1 \eta_z + \eta_x \phi_x + \nu^2 \eta_x \phi_z + \nu \cos(\theta_1 - \theta_2)(\eta_x \phi_z + \eta_z \phi_x), \quad \text{on } \hat{y} = \eta(x, z), \quad (3.11)
\]

\[
\frac{1}{2}(\phi_{\hat{y}}^2 + \nu^2 \phi_{\hat{z}}^2 + 2\nu \cos(\theta_1 - \theta_2) \phi_{x\hat{z}})
- \sin \theta_2 \phi_x - \nu \sin \theta_1 \phi_z + \gamma \eta - \left(\frac{\eta_x}{R}\right)_x - \nu^2 \left(\frac{\eta_z}{R}\right)_z,
-\nu \cos(\theta_1 - \theta_2) \left[\left(\frac{\eta_x}{R}\right)_z + \left(\frac{\eta_z}{R}\right)_x\right] = 0, \quad \text{on } \hat{y} = \eta(x, z), \quad (3.12)
\]

\[
\phi_{\hat{y}} \to 0, \quad \text{as } \hat{y} \to -\infty, \quad (3.13)
\]

where

\[
R = \sqrt{1 + \eta^2_x + \nu^2 \eta^2_z + 2\nu \cos(\theta_1 - \theta_2)\eta_x \eta_z}
\]

and \(\nu = 2\pi/P\). Here we have abused notation by omitting the subscripts and tildes, and normalized the period in \(z\) to \(2\pi\).

We restrict our attention to a particular type of solution of (3.10)–(3.13), namely a solution that is also periodic in the \(x\)-direction. An example of a doubly-periodic wave of this
kind is shown in Figure 3.2. Note that the periodic profiles of these waves are not necessarily aligned with their direction of propagation. **Oblique line waves** are doubly-periodic solutions of (3.10)–(3.13) which are independent of $z$ (see Figure 3.3). Observe that an oblique line wave, which possesses a periodic profile in one horizontal direction, also possesses spatially periodic profiles in all other horizontal directions. Furthermore, oblique line-wave solutions of (3.10)–(3.13) are transformed into solutions of the mathematical problem for two-dimensional travelling waves ($z$-independent solutions with $\theta_1 = 0, \theta_2 = \pi/2$) by the transformation

$$(x, y) \mapsto \frac{1}{\sin^2 \theta_2} (x, y), \quad (\phi, \eta) \mapsto \left( \frac{\phi}{\sin \theta_2}, \frac{\eta}{\sin^2 \theta_2} \right), \quad \gamma \mapsto \gamma \sin^4 \theta_2.$$ 

Figure 3.2: A doubly-periodic wave. The wave has a periodic profile in both the $x$- and $z$-directions, and moves with constant speed and without change in shape in the direction shown by the solid arrow.

Figure 3.3: An oblique line wave. The wave has a periodic profile in the $x$-direction, and is constant in the $z$-direction.
3.2 Spatial dynamics formulation

We proceed by formulating (3.10)–(3.13) as an evolutionary equation in which the unbounded spatial direction $x$ plays the role of the time-like variable, and each point of its infinite-dimensional phase space is a wave which is $2\pi$-periodic in $z$. A solution of the evolutionary equation represented by a periodic trajectory through its phase space therefore corresponds to a water wave which is periodic in both $x$ and $z$, that is a doubly-periodic water wave (see Figure 3.4). The evolutionary system is found using the method given by Groves & Haragus [16]. We use a formal variational principle to obtain, via the Legendre transform, a spatial dynamics formulation of the problem (3.10)–(3.13); the resulting problem is a formal reversible Hamiltonian system. A functional-analytical setting for this system is provided and periodic solutions are then obtained using the Lyapunov-Iooss theorem for reversible systems.

![Figure 3.4: Each point in phase space corresponds to a wave which is periodic in $z$. A periodic trajectory through this phase space (left) therefore generates a doubly periodic water wave (right).](image)

Consider the variational principle

$$
\delta \int \int_{0}^{2\pi} \left\{ \int_{-\infty}^{\eta} \left\{ \frac{1}{2} \left( \phi_{x}^{2} + \phi_{\hat{y}}^{2} + \nu^{2} \phi_{z}^{2} + 2\nu \cos(\theta_{1} - \theta_{2})\phi_{x}\phi_{z} \right) \right\} d\hat{y} \\
+ \frac{1}{2} \gamma \eta^{2} + R - 1 + (\eta_{x} \sin \theta_{2} + \nu \eta_{z} \sin \theta_{1})\phi|_{\hat{y}=\eta} \right\} dz \ dx = 0, \quad (3.14)
$$

in which the variation is taken in $(\eta, \phi)$; the oblique travelling water wave problem (3.10)–(3.13) corresponds formally to the Euler-Lagrange equations for this variational problem.

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(see, for example, Luke [33]). Because of the difficulty in performing analysis on a variable domain, we fix the domain using the transformation

\[ y = \hat{y} - e^y \eta, \]

which maps the variable domain \( \{(x, \hat{y}, z) \in \mathbb{R} : z \in (0, 2\pi), -\infty < \hat{y} < \eta(x, z)\} \) to the fixed domain \( \mathbb{R} \times (0, 2\pi) \times (-\infty, 0) \) (the calculation \( \hat{y}'(y) = 1 + e^y \eta \) shows that this transformation is invertible provided that \( |\eta|_\infty < 1 \)). Introducing the new variable

\[ \Phi(x, y, z) = \phi(x, \hat{y}, z), \]

one finds that the variational principle (3.14) is transformed to

\[ \delta \mathcal{L} = 0, \quad \mathcal{L} = \int L(\eta, \eta_x, \Phi, \Phi_x)dx, \]

where the Lagrangian \( L \) is given by

\[
L(\eta, \eta_x, \Phi, \Phi_x) = \int_{0}^{2\pi} \left\{ \int_{-\infty}^{0} \left\{ \frac{1}{2} (\Phi_x - \eta_x \Phi_y e^y)^2 + \frac{1}{2} \Phi_y^2 + \frac{\nu^2}{2} (\Phi_z - \eta_z \Phi_y e^y)^2 + \nu \cos(\theta_1 - \theta_2)(\Phi_x - \eta_x \Phi_y e^y)(\Phi_z - \eta_z \Phi_y e^y) \right\} (1 + \eta e^y)dy \\
+ \frac{1}{2} \gamma \eta^2 + R - 1 + (\eta_x \sin \theta_2 + \nu \eta_z \sin \theta_1) \Phi\big|_{y=0} \right\} dz. \tag{3.15}
\]

Proceeding formally, we perform the Legendre transform on the Lagrangian (3.15). Introduce new variables \( \xi \) and \( U \) defined by

\[
\xi = \frac{\delta L}{\delta \eta_x} = -\int_{-\infty}^{0} \left\{ (\Phi_x - \eta_x \Phi_y e^y) \Phi_y e^y + \nu \cos(\theta_1 - \theta_2)(\Phi_z - \eta_z \Phi_y e^y) \Phi_y e^y \right\} (1 + \eta e^y)dy \\
+ \frac{1}{R} (\eta_x + \nu \eta_z \cos(\theta_1 - \theta_2)) + \sin \theta_2 \Phi\big|_{y=0},
\]

\[
U = \frac{\delta L}{\delta \Phi_x} = \left( \Phi_x - \eta_x \Phi_y e^y + \nu \cos(\theta_1 - \theta_2)(\Phi_z - \eta_z \Phi_y e^y) \right) (1 + \eta e^y),
\]

where the variational derivatives are taken respectively \( L^2(0, 2\pi) \) and \( L^2((0, 2\pi) \times (-\infty, 0)) \), and define the Hamiltonian \( H(\eta, \xi, \Phi, U) \) by the formula
\[ H(\eta, \xi, \Phi, U) = \int_0^{2\pi} \xi \eta_x \, dz + \int_0^{2\pi} \int_{-\infty}^{0} U \Phi_x \, dy \, dz - L(\eta, \eta_x, \Phi, \Phi_x), \]
\[ = \int_0^{2\pi} \left\{ \int_{-\infty}^{0} \left( \frac{1}{2} \frac{U^2}{1 + \eta \nu y} - \frac{1}{2} \frac{\Phi_y^2}{(1 + \eta \nu y)^2} \right) + \frac{1}{2} \sin^2(\theta_1 - \theta_2)(\Phi_z - \eta \Phi_y \nu y)^2 \right. \]
\[ \left. - \frac{U}{(1 + \eta \nu y)}(\Phi_z - \eta \Phi_y \nu y) \cos(\theta_1 - \theta_2) \right\} (1 + \eta \nu y) \, dy \]
\[ - \frac{1}{2} \eta_\nu^2 + 1 - (1 - W^2)^\frac{1}{2} (1 + \nu^2 \eta_\nu^2 \sin^2(\theta_1 - \theta_2))^{\frac{1}{2}} \]
\[ - W \nu \eta_\nu \cos(\theta_1 - \theta_2) - \nu \eta_\nu \sin \theta_1 \Phi|_{y=0} \right\} \, dz, \quad (3.16) \]

where
\[ W = \xi + \int_{-\infty}^{0} U \Phi_y y \, dy - \sin \theta_2 \Phi|_{y=0}. \quad (3.17) \]

Hamilton's equations
\[ \eta_x = \frac{\delta H}{\delta \xi}, \quad \xi_x = -\frac{\delta H}{\delta \eta}, \quad \Phi_x = \frac{\delta H}{\delta U}, \quad U_x = -\frac{\delta H}{\delta \Phi} \quad (3.18) \]

are given explicitly by
\[ \eta_x = \frac{W}{(1 - W^2)^\frac{1}{2}} (1 + \nu^2 \eta_\nu^2 \sin^2(\theta_1 - \theta_2))^{\frac{1}{2}} - \nu \eta_\nu \cos(\theta_1 - \theta_2), \quad (3.19) \]
\[ \xi_x = \int_{-\infty}^{0} \left\{ \nu^2 \sin^2(\theta_1 - \theta_2) \left[ (\Phi_z - \eta_\nu \Phi_y \nu y) \Phi_y \nu y (1 + \eta \nu y) \right] \right. \]
\[ \left. + \frac{\nu^2}{2} \sin^2(\theta_1 - \theta_2) \left[ (\Phi_z - \eta_\nu \Phi_y \nu y)^2 - \frac{\Phi_y \nu y}{(1 + \eta \nu y)^2} \right] \right\} \, dy \]
\[ - \xi_\nu \cos(\theta_1 - \theta_2) - \nu^2 \sin^2(\theta_1 - \theta_2) \left[ \frac{\nu \eta_\nu (1 - W^2)^\frac{1}{2}}{(1 + \nu^2 \eta_\nu^2 \sin^2(\theta_1 - \theta_2))^{\frac{1}{2}}} \right] \]
\[ + \gamma \eta_\nu + \nu (\cos(\theta_1 - \theta_2) \sin \theta_1 - \sin \theta_1) \Phi_z \big|_{y=0}, \quad (3.20) \]
\[ \Phi_x = U - \Phi_z \nu \cos(\theta_1 - \theta_2) + \frac{\Phi_y \nu W y}{(1 - W^2)^\frac{1}{2}} (1 + \nu^2 \eta_\nu^2 \sin^2(\theta_1 - \theta_2))^{\frac{1}{2}}, \quad (3.21) \]
\[ U_x = -\Phi_y y + \left[ \frac{\eta \Phi_y \nu y}{(1 + \eta \nu y)^2} \right] - \nu^2 \sin^2(\theta_1 - \theta_2) \left[ (\Phi_z - \eta_\nu \Phi_y \nu y) (1 + \eta \nu y) \right] \]
\[ + \nu^2 \sin^2(\theta_1 - \theta_2) \left[ (\Phi_z - \eta_\nu \Phi_y \nu y) \eta_\nu \nu y (1 + \eta \nu y) \right] \]
\[ - U_\nu \cos(\theta_1 - \theta_2) + \frac{[U \nu y]_y W}{(1 - W^2)^\frac{1}{2}} (1 + \nu^2 \eta_\nu^2 \sin^2(\theta_1 - \theta_2))^{\frac{1}{2}}, \quad (3.22) \]

with nonlinear boundary condition.
\[
\Phi_y = \frac{\eta \Phi_y e^y}{1 + \eta e^y} + \nu^2 \sin^2(\theta_1 - \theta_2)(\Phi_z - \eta_z \Phi_y e^y) \eta_z e^y (1 + \eta e^y) \\
+ (U e^y - \sin \theta_2) \left( \frac{W}{(1 - W^2)^{1/2}} (1 + \nu^2 \eta_z^2 \sin^2(\theta_1 - \theta_2))^{1/2} \right) \\
+ \nu \eta_z (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1)
\] (3.23)

at \( y = 0 \), which arises as a result of the integration by parts used to derive (3.22).

We now confirm rigorously that the equations (3.19)–(3.22) subject to the boundary condition (3.23) define a Hamiltonian system in the sense of Definition 2.4.1, where the Hamiltonian is defined by the formula (3.16). Consider the Hilbert spaces

\[
H^s_{\text{per}}(S) := \{ u \in H^s_{\text{loc}}(\mathbb{R}) : u(z + 2\pi) = u(z), \ z \in \mathbb{R} \}, \\
H^s_{\text{per}}(\Sigma) := \{ u \in H^s_{\text{loc}}((-\infty, 0) \times \mathbb{R}) : u(y, z + 2\pi) = u(y, z), \ y \in (-\infty, 0), \ z \in \mathbb{R} \},
\]

where \( S = (0, 2\pi), \ \Sigma = (-\infty, 0) \times (0, 2\pi) \). The following lemma is a collection of results regarding the above spaces, which we shall make use of throughout this investigation.

**Lemma 3.2.1**

(i) The spaces \( H^s_{\text{per}}(S) \) and \( H^t_{\text{per}}(\Sigma) \) are Banach algebras for \( s > 1/2, \ t > 1 \).

(ii) For each \( s \in [0, 1] \) the mapping

\[
(u_1, u_2) \mapsto u_1 u_2
\]

defines a continuous bilinear operator \( H^{s+1}_{\text{per}}(\Sigma) \times H^s_{\text{per}}(\Sigma) \to H^s_{\text{per}}(\Sigma) \). The same result holds when \( \Sigma \) is replaced by \( S \).

(iii) For each \( s \in [0, \infty) \) the mapping

\[
(u_1, u_2) \mapsto \int_{-\infty}^0 u_1 u_2 \, dy
\] (3.24)

defines a continuous bilinear operator \( H^s_{\text{per}}(\Sigma) \times H^s_{\text{per}}(\Sigma) \to H^s_{\text{per}}(S) \).

**Proof.** The proof of the first assertion can be found in Lions & Magenes [32, vol. 1, ch. 1, Theorem 9.8], and the proof of the second assertion follows using the procedure used by Groves & Mielke [17, Lemma A.2]. For the last assertion, choose \( j \in \mathbb{N} \) and use the Cauchy-Schwarz inequality to obtain

\[
\left\| \int_{-\infty}^0 u_1 u_2 \, dy \right\|_{H^s_{\text{per}}(S)} \leq C \left\| u_1 \right\|_{H^s_{\text{per}}(\Sigma)} \left\| u_2 \right\|_{H^s_{\text{per}}(\Sigma)}
\]
The formula for each operators in Triebel [49, §1.19.5].

We introduce the space

\[ X_s := H^s_{\text{per}}(S) \times H^s_{\text{per}}(S) \times H^s_{\text{per}}(\Sigma) \times H^s_{\text{per}}(\Sigma), \]

where \( s \in (0, 1/2) \) and let \( \mathcal{U} \) be a neighbourhood of the origin in \( X_{s+1} \) chosen small enough so that \( |W| < 1/2 \) on \( \mathcal{U} \). Using Lemma 3.2.1 and the fact that \( s > 0 \), we find that the right-hand side of (3.19)–(3.22) defines a smooth function \( F : \mathcal{U} \to X_s \) and hence also \( D_F \to X_s \), where

\[ D_F = \{(\eta, \xi, \Phi, U) \in \mathcal{U} : (3.23) \text{ is satisfied}\}; \]

furthermore \( D_F \) is a dense subset of \( X_s \) (because \( s < 1/2 \)). Equations (3.19)–(3.22) subject to (3.23) therefore define an evolutionary system in the phase space \( X_s \). This system is reversible with the reverser

\[ R(\eta, \xi, \Phi, U) = (\eta(-.), -\xi(-.), -\Phi(-.), U(-.)). \]

It remains to identify the Hamiltonian structure rigorously. Formula (3.16) defines a function \( H \in C^\infty(\mathcal{U}, \mathbb{R}) \), and an explicit calculation yields

\[
\begin{align*}
\frac{\delta H}{\delta \eta}(\eta, \xi, \Phi, U)(\hat{\eta}, \hat{\xi}, \hat{\Phi}, \hat{U}) &= \int_{\Sigma} \left\{ U\hat{U} - \nu^2 \sin^2(\theta_1 - \theta_2)(\hat{\Phi}_z - \eta_2 \Phi_y e^y)(\hat{\Phi}_z - \hat{\eta}_2 \Phi_y e^y - \xi_2 \Phi_y e^y)(1 + \eta e^y) \right\} \, dy \, dz \\
&- \int_{\Sigma} \left\{ \frac{\nu^2}{2} \sin^2(\theta_1 - \theta_2)(\Phi_z - \eta_2 \Phi_y e^y)^2 e^y \hat{\eta} + U(\hat{\Phi}_z - \hat{\eta}_2 \Phi_y e^y) \nu \cos(\theta_1 - \theta_2) \right. \\
&\quad + \frac{\Phi^2 \hat{\eta} e^y}{(1 + \eta e^y)^2} + \frac{\Phi_y \hat{\Phi}_y}{(1 + \eta e^y)} + \hat{U} \Phi_z \nu \cos(\theta_1 - \theta_2) \left. \right\} \, dy \, dz \\
&+ \int_{\mathcal{S}} \left\{ \frac{W(1 + \nu^2 \eta_z^2 \sin^2(\theta_1 - \theta_2))^{1/2}}{(1 - W^2)^{1/2}} \left( \hat{\xi} + \int_{-\infty}^{0} \left\{ U\hat{\Phi}_y e^y + \hat{U} \Phi_y e^y \right\} \, dy - \sin \theta_2 \Phi|_{y=0} \right) \\
&\quad - \nu \cos(\theta_1 - \theta_2) \left( W\hat{\eta}_z + \eta_z \left( \hat{\xi} - \sin \theta_2 \Phi|_{y=0} \right) \right) \\
&\quad - \frac{\eta_z \hat{\eta}_z \nu^2 \sin^2(\theta_1 - \theta_2)(1 + \nu^2 \eta_z^2 \sin^2(\theta_1 - \theta_2))^{1/2}}{(1 + \nu^2 \eta_z^2 \sin^2(\theta_1 - \theta_2))^{1/2}} - \nu \sin \theta_1 \Phi|_{y=0} + \eta_z \Phi|_{y=0} - \gamma \eta \eta \right\} \, dz.
\end{align*}
\]

(3.25)

The formula

\[ H'(\eta, \xi, \Phi, U) = \left( \frac{\delta H}{\delta \eta}, \frac{\delta H}{\delta \xi}, \frac{\delta H}{\delta \Phi}, \frac{\delta H}{\delta U} \right) (\eta, \xi, \Phi, U) \]
defines a smooth function $H' : U \to X_s$, and a straightforward calculation shows that
\[
dH[u](\tilde{u}) = (H'(u), \tilde{u})
\]
for $u = (\eta, \xi, \Phi, U) \in D_F$ and $\tilde{u} = (\tilde{\eta}, \tilde{\xi}, \tilde{\Phi}, \tilde{U}) \in X_{s+1}$, where $(\cdot, \cdot)$ is the $L^2(S) \times L^2(S) \times L^2(\Sigma) \times L^2(\Sigma)$-inner product. It follows that $H' \in C^\infty(D_F, X_s)$ is the gradient of $H$ with respect to this inner product on $X_s$. Defining $J : X_s \to X_s$ by
\[
J(\eta, \xi, \Phi, U) = (\xi, -\eta, U, -\Phi),
\]
we see that (3.19)–(3.22) subject to the boundary condition (3.23) is precisely the Hamiltonian system
\[
\frac{du}{dx} = JH'(u) = F(u)
\]
defined on the domain $D_F$.

### 3.3 The nonlinear boundary condition

An immediate complication arises due to the nonlinear boundary condition (3.23). The Lyapunov-Iooss theorem cannot be applied directly to (3.26) since the domain $D_F$ is not an open subset of a linear space. We resolve this difficulty using a change of variable which converts (3.26) into an equivalent system with a domain that is a subset of a linear space. Define
\[
f(\eta, \xi, \Phi, U) = \frac{\eta \Phi y e^y}{(1 + \eta e^y)} + \nu^2 \sin^2(\theta_1 - \theta_2)(\Phi_z - \eta_z \Phi y e^y)\eta_z e^y(1 + \eta e^y) + e^y \nu \eta_z (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1) + e^y(U - \sin \theta_2)\left(\frac{W}{(1 - W^2)^{\frac{3}{2}}} (1 + \nu^2 \eta_z^2 \sin^2(\theta_1 - \theta_2))^\frac{1}{2}\right),
\]
and write the boundary condition (3.23) as
\[
\Phi_y = f(\eta, \xi, \Phi, U), \quad \text{on } y = 0.
\]

It follows from Lemma 3.2.1 that $f : U \to H^{s+1}_{\text{per}}(\Sigma)$ is a smooth mapping. Define $M : U \to X_{s+1}$ by
\[
M : (\eta, \xi, \Phi, U) \mapsto (\eta, \Lambda, \Gamma, U), \quad \Lambda = \xi - \sin \theta_2 \Phi|_{y=0}, \quad \Gamma = \Phi + \zeta y,
\]

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in which $\zeta \in H_{\text{per}}^{s+3}(\Sigma)$ is the unique solution of the elliptic boundary-value problem
\begin{align*}
-\zeta_{yy} - \zeta_{zz} &= f(\eta, \xi, \Phi, U), & \text{in } \Sigma, \\
\zeta &= 0, & \text{on } y = 0.
\end{align*}
(3.28)

Clearly
\begin{align*}
\Gamma_y &= \Phi_y + \zeta_{yy} \\
&= \Phi_y - \zeta_{zz} - f(\eta, \xi, \Phi, U),
\end{align*}
and since $\zeta_{zz} = 0$ on $y = 0$, the boundary condition (3.23) becomes
\begin{align*}
\Gamma_y &= 0, & \text{on } y = 0.
\end{align*}
(3.29)

This transformation therefore converts the nonlinear boundary condition (3.23) into the linear boundary condition (3.29). The next result confirms that this is a valid change of variable.

**Lemma 3.3.1**

(i) The mapping $M$ is a smooth diffeomorphism from the neighbourhood $U$ of the origin in $X_{s+1}$ onto a neighbourhood $V$ of the origin in $X_{s+1}$.

(ii) For each $u \in U$ the operator $dM[u] : X_{s+1} \to X_{s+1}$ extends to an isomorphism $\hat{d}M[u] : X_s \to X_s$. The operators $\hat{d}M[u], \hat{d}M[u]^{-1} \in L(X_s, X_s)$ depend smoothly upon $u \in U$.

**Proof.** The mapping $M : U \to X_{s+1}$ is smooth since it depends linearly upon the smooth function $f : U \to H_{\text{per}}^{s+1}(\Sigma)$. Note that $M(0) = 0$ and
\begin{align*}
dM[\eta, \xi, \Phi, U](\eta^*, \xi^*, \Phi^*, U^*) &= (\eta^*, \xi^* - \sin \theta_2 \Phi^*|_{y=0}, \Phi^* + \zeta^*_y, U^*),
\end{align*}
where $\zeta^* \in H_{\text{per}}^{s+3}(\Sigma)$ is the unique solution of the elliptic boundary-value problem
\begin{align*}
-\zeta^*_{yy} - \zeta^*_{zz} &= df[\eta, \xi, \Phi, U](\eta^*, \xi^*, \Phi^*, U^*), & \text{in } \Sigma, \\
\zeta^* &= 0, & \text{on } y = 0.
\end{align*}

A direct calculation shows that

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\[ \text{df}[\eta, \xi, \Phi, U](\eta^*, \xi^*, \Phi^*, U^*) = \]
\[ \frac{\Phi_y e^{y^*}}{(1 + \eta e^y)} + e^y \nu \eta^*_y (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1) \]
\[ + \frac{\Phi^* e^{y^*} \eta^*_y}{(1 + \eta e^y)} - \frac{\Phi_y e^{y^*} \eta^*_y}{(1 + \eta e^y)^2} + e^{y^*} \nu^2 \sin^2(\theta_1 - \theta_2) (\Phi_y - \eta e^{y^*}) \eta^*_y \eta^*_y \]
\[ + (1 + \eta e^y) e^{y^*} \nu^2 \sin^2(\theta_1 - \theta_2) (\eta^*_y (\Phi_y - \eta e^{y^*}) + \eta_y (\Phi^*_y - \eta e^{y^*} e^y) - \eta^*_y \eta^*_y e^y) \]
\[ + \frac{e^y W (U - \sin \theta_2)}{(1 - W^2)^{\frac{3}{2}}} \nu^2 \eta^*_y \sin^2(\theta_1 - \theta_2) + \frac{e^y W U^*}{(1 - W^2)^{\frac{3}{2}}} (1 + \nu^2 \eta^*_y \sin^2(\theta_1 - \theta_2))^{\frac{3}{2}} \]
\[ + \frac{e^y (U - \sin \theta_2)}{(1 - W^2)^{\frac{3}{2}}} (1 + \nu^2 \eta^*_y \sin^2(\theta_1 - \theta_2))^{\frac{1}{2}} \left( \xi^* + \int_{-\infty}^0 U^* \Phi_y e^y + U \Phi^* e^y \, dy \right) \]
\[ - \sin \theta_2 \Phi^* |_{y=0} \]  
\[ (3.30) \]

which in particular implies that
\[ \text{df}[0](\eta^*, \xi^*, \Phi^*, U^*) = e^y \nu \eta^*_y (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1) \]
\[ - e^y \xi^* \sin \theta_2 + e^{y^*} \nu^2 \theta^*_2 \Phi^* |_{y=0}. \]  
\[ (3.31) \]

The inverse of \( \text{d}M[0] : X_{s+1} \to X_{s+1} \) is therefore given by
\[ \text{d}M[0]^{-1}(\eta^*, \Lambda^*, \Gamma^*, U^*) = (\eta^*, \Lambda^* + \sin \theta_2 \Phi^* |_{y=0}, \Phi^*, U^*), \]

where \( \Phi^* = \Gamma^* - \chi_y \) and \( \chi \in H^{s+3}_{\text{per}}(\Sigma) \) is the unique solution of the elliptic boundary-value problem
\[ -\chi_y y - \chi_{zz} = e^y \nu \eta^*_y (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1) - \Lambda^* e^y \sin \theta_2, \]
\[ \chi = 0, \]
\[ \text{in } \Sigma, \]
\[ \chi \text{ on } y = 0. \]

Replacing \( U \) with a smaller neighbourhood of the origin in \( X_{s+1} \) if necessary, we find by the inverse function theorem that \( M \) is a diffeomorphism of \( U \) onto a neighbourhood \( V \) of the origin in \( X_{s+1} \).

Examining the right hand side of (3.30), we find by the results of Lemma 3.2.1 that \( \text{d}f[\mathbf{u}] : X_{s+1} \to H^{s+1}_{\text{per}}(\Sigma) \) extends to a linear continuous operator \( \hat{\text{d}}f[\mathbf{u}] : X_s \to H^s_{\text{per}}(\Sigma) \), which depends smoothly on \( \mathbf{u} \in U \). It follows that the same is true of \( \text{d}M[\mathbf{u}] \); there exists an extension \( \hat{\text{d}}M[\mathbf{u}] : X_s \to X_s \) which depends smoothly upon \( \mathbf{u} \in U \). The extension is unique since \( X_{s+1} \) is dense in \( X_s \).

Now define the map \( B : L(X_s, X_s) \times \mathcal{U} \to L(X_s, X_s) \) by the formula
\[ B(T, \mathbf{u}) = \hat{\text{d}}M[\mathbf{u}](T) - I, \]

where \( I \) denotes the identity operator in \( X_s \). Observe that
\[ B(\hat{\text{d}}M[0]^{-1}, 0) = 0, \quad \text{d}_1 B[\hat{\text{d}}M[0]^{-1}, 0] = I. \]
Using the implicit function theorem, we find that there exists a unique smooth function 
\( T : \mathcal{U} \to L(X_s, X_s) \) which satisfies \( B(T(u), u) = 0 \) for each \( u \in \mathcal{U} \) (we take \( \mathcal{U} \) smaller if necessary). It follows from the uniqueness of \( T \) that \( T(u) = \tilde{d}M[u]^{-1}, u \in \mathcal{U} \); in particular we conclude that \( \tilde{d}M[u]^{-1} : X_s \to X_s \) depends smoothly upon \( u \in \mathcal{U} \).

□

The diffeomorphism \( M \) transforms (3.26) to
\[
\frac{dv}{dx} = G(v) = Lv + N(v),
\]
where \( v = (\eta, \Lambda, \Gamma, U), G : D_G \to X_s \) is the smooth vector field given by
\[
G(v) = \tilde{d}M[M^{-1}(v)](F(M^{-1}(v))),
\]
\( L = dG[0], N(v) = G(v) - dG[0](v) \) and
\[
D_G = \{ (\eta, \Lambda, \Gamma, U) \in \mathcal{V} : \Gamma_y|_{y=0} = 0 \}.
\]
Observe that reversibility is preserved under the above transformation, that is
\[
G(Rv) = -RG(v),
\]
where the reverser \( R : X_s \to X_s \) is again defined by
\[
R(\eta, \Lambda, \Gamma, U) = (\eta(-.), -\Lambda(-.), -\Gamma(-.), U(-.)).
\]

3.4 The formal linearization

In this section we begin our study of the linear operator \( L : D(L) \subset X_s \to X_s \) on the right-hand side of (3.32), where
\[
D(L) = \{ (\eta, \Lambda, \Gamma, U) \in X_{s+1} : \Gamma_y|_{y=0} = 0 \}.
\]
We proceed by examining the linear operator \( A : D(A) \subset X_s \to X_s \) defined by the explicit formulae
\[
A(\eta, \xi, \Phi, U) = \begin{pmatrix}
-\nu\eta_z \cos(\theta_1 - \theta_2) + \xi - \sin \theta_2 \Phi|_{y=0} \\
\gamma\eta - \nu^2 \eta_{zz} \sin^2(\theta_1 - \theta_2) - \xi \nu \cos(\theta_1 - \theta_2) + \nu (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1) \Phi_z|_{y=0} \\
U - \Phi_z \nu \cos(\theta_1 - \theta_2) \\
-\Phi_{yy} - \nu^2 \sin^2(\theta_1 - \theta_2) \Phi_{zz} - U \nu \cos(\theta_1 - \theta_2)
\end{pmatrix},
\]
(3.33)
where

\[
D(A) = \{ (\eta, \xi, \Phi, U) \in X_{s+1} : \Phi_y|_{y=0} = \nu \eta \sin \theta_2 \cos(\theta_1 - \theta_2) - \xi \sin \theta_2 + \sin^2 \theta_2 \Phi|_{y=0} \}.
\]  

(3.34)

The operator \( A \) is the formal linearization of the vector field \( F : D_F \to X_s \) and is related to \( L \) by the identity

\[
L = \hat{d}M[0](AdM[0]^{-1}).
\]  

(3.35)

Since \( dM[0] : D(A) \to D(L) \) is a homeomorphism the topological properties of \( A \) and \( L \) are identical, and in particular their spectra coincide.

We discuss certain spectral properties of \( A \) by considering the resolvent equation

\[
(A - i\omega I)u = u^*.
\]

Writing \( u = (\eta, \xi, \Phi, U) \) and \( u^* = (\eta^*, \xi^*, \Phi^*, U^*) \) as Fourier series

\[
u = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} u_n e^{inz}, \quad u^* = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} u^*_n e^{inz},
\]

we find that

\[
\xi_n - \sin \theta_2 \Phi_n|_{y=0} - (i\omega + i\nu \cos(\theta_1 - \theta_2))\eta_n = \eta^*_n, \quad (\gamma + n^2 \nu^2 \sin^2(\theta_1 - \theta_2))\eta_n - (i\omega + i\nu \cos(\theta_1 - \theta_2))\xi_n + i\nu (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1)\Phi_n|_{y=0} = \xi^*_n, \quad U_n - (i\omega + i\nu \cos(\theta_1 - \theta_2))\Phi_n = \Phi^*_n, \quad -\Phi_{yy} + n^2 \nu^2 \sin^2(\theta_1 - \theta_2)\Phi_n - (i\omega + i\nu \cos(\theta_1 - \theta_2))U_n = U^*_n,
\]

(3.36) (3.37) (3.38) (3.39)

with the boundary condition

\[
\Phi_{yy}|_{y=0} = i\nu \eta_n (\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1) - \xi_n \sin \theta_2 + \sin^2 \theta_2 \Phi_n|_{y=0}.
\]  

(3.40)

The following proposition is used to give a detailed description of the purely imaginary eigenvalues of \( A \) (and hence of \( L \)) in the next section.

**Proposition 3.4.1** The purely imaginary number \( i\omega, \omega \in \mathbb{R} \setminus \{0\} \), is an eigenvalue of \( A \) if and only if there exists an integer \( n \in \mathbb{Z} \) such that

\[
\sigma(\gamma + \sigma^2) = (\omega \sin \theta_2 + n\nu \sin \theta_1)^2.
\]  

(3.41)

Here \( \sigma \geq 0 \) is given by \( \sigma^2 = (\omega^2 + n^2 \nu^2 + 2n\nu \omega \cos(\theta_1 - \theta_2)) \).
Proof. Choosing \((\eta^*, \xi^*, \Phi^*, U^*) = (0, 0, 0, 0)\), one finds by elementary calculus that (3.36)–(3.40) has a nonzero solution if and only if (3.41) is satisfied. In this case, these equations have a one-parameter family of solutions given by the explicit formulae

\[
\eta_n = \frac{\beta(i\omega \sin \theta_2 + i\nu \sin \theta_1)}{(\gamma + \sigma^2)},
\]
\[
\xi_n = \beta \sin \theta_2 + \frac{\beta(i\omega \sin \theta_2 + i\nu \sin \theta_1)(i\omega + i\nu \cos(\theta_1 - \theta_2))}{(\gamma + \sigma^2)},
\]
\[
U_n(y) = (i\omega + i\nu \cos(\theta_1 - \theta_2))\beta e^{\sigma y},
\]
\[
\Phi_n(y) = \beta e^{\sigma y},
\]

where \(\beta\) is a complex constant, so that \(i\omega\) is an eigenvalue of \(A\) with (finite-dimensional) eigenspace

\[
\left\{ (\eta, \xi, \Phi, U) e^{inz} : (\omega, n) \text{ satisfies (3.41)} \right\}.
\]

\[\square\]

Our next result shows that zero is a point of the continuous spectrum of \(A\).

**Lemma 3.4.2** The equation

\[Au = u^*\]

has a unique solution \(u \in D(A)\) for all \(u^* \in X_s\) with the property that

\[
\int_{-\infty}^{y} \int_{-\infty}^{\tilde{t}} U_0^*(t) \, dt \, d\tilde{t}, \int_{-\infty}^{y} U_0^*(t) \, dt \in L^2(-\infty, 0),
\]

and which satisfies the compatibility condition

\[
\int_{-\infty}^{0} U_0^*(y) \, dy - \eta_0^* \sin \theta_2 = 0;
\]

here \([u^*]_0 = (\eta_0^*, \xi_0^*, \Phi_0^*, U_0^*)\) denotes the 0\(^{th}\) Fourier coefficient of \(u^*\). This solution satisfies the estimate

\[
\|u\|_{Y_{s+1}} \leq C \|u^*\|_{Y_s},
\]

where \(Y_s = \{ u \in X_s : [u]_0 = 0 \}\) and \([u]_0 \in \mathbb{R} \times \mathbb{R} \times H^{s+1}(-\infty, 0) \times H^s(-\infty, 0)\) is given by the explicit formulae

\[
\eta_0 = \frac{\xi_0}{\gamma},
\]
\[
\xi_0 = \eta_0^* + \sin \theta_2 \int_{-\infty}^{0} \int_{-\infty}^{\tilde{t}} U_0^*(t) \, dt \, d\tilde{t},
\]
\[
\Phi_0 = -\int_{-\infty}^{y} \int_{-\infty}^{\tilde{t}} U_0^*(t) \, dt \, d\tilde{t},
\]
\[
U_0 = \Phi_0^*.
\]

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Proof. For $\omega = 0$ equations (3.36)–(3.40) have the explicit solution

$$
\eta_n = \frac{\text{in} \nu \cos(\theta_1 - \theta_2)n_2 + \xi_n^*}{(\gamma + n^2\nu^2)} + \frac{\text{in} \nu \sin \theta_1}{n\nu(\gamma + n^2\nu^2) - n^2\nu^2 \sin^2 \theta_1} \int_{-\infty}^{0} e^{n\nu t} f_n^*(t) \, dt
$$

$$
\xi_n = \eta_n^* + \text{in} \nu \cos(\theta_1 - \theta_2) \eta_n + \frac{\sin \theta_2(\gamma + n^2\nu^2)}{n\nu(\gamma + n^2\nu^2) - n^2\nu^2 \sin^2 \theta_1} \int_{-\infty}^{0} e^{n\nu (t+y)} f_n^*(t) \, dt
$$

$$
U_n(y) = \Phi_n^*(y) + \text{in} \nu \cos(\theta_1 - \theta_2) \frac{2n}{2n} \int_{-\infty}^{0} e^{-n\nu|y-t|} f_n^*(t) \, dt
$$

$$
\Phi_n(y) = \frac{1}{2n} \int_{-\infty}^{0} e^{-n\nu|y-t|} f_n^*(t) \, dt
$$

where

$$
f_n^* = U_n^* + \text{in} \nu \cos(\theta_1 - \theta_2) \Phi_n^*,
$$

$$
g_n^* = -\text{in} \nu \sin \theta_1 (\xi_n^* + \text{in} \nu \cos(\theta_1 - \theta_2) \eta_n^*) - (\gamma + n^2\nu^2) n^2 \sin \theta_2,
$$

for $n \neq 0$. Using the estimate

$$
\int_{-\infty}^{0} \left( \int_{-\infty}^{0} e^{-n\nu|y-t|} f_n^*(t) \, dt \right) \, dy, \int_{-\infty}^{0} \left( \int_{-\infty}^{0} e^{n\nu (t+y)} f_n^*(t) \, dt \right) \, dy \leq \frac{C}{n^2} \int_{-\infty}^{0} |f_n^*(t)|^2 \, dt,
$$

we find from these formulae that

$$
|n|\int_{-\infty}^{0} |\eta_n|^2 + |n|\int_{-\infty}^{0} |\xi_n|^2 + |\Phi_{nyy}|_0^2 + |n|\int_{-\infty}^{0} |\Phi_{nyy}|_0^2 + |n|\int_{-\infty}^{0} |\Phi_n|^2 + |n|\int_{-\infty}^{0} \|U_n\|^2 + |n|\int_{-\infty}^{0} \|U_n\|^2
$$

$$
\leq C \left( |n|\int_{-\infty}^{0} |\eta_n|^2 + |\xi_n|^2 + |\Phi_{nyy}|_0^2 + |n|\int_{-\infty}^{0} |\Phi_{nyy}|_0^2 + |n|\int_{-\infty}^{0} |\Phi_n|^2 + |n|\int_{-\infty}^{0} \|U_n\|^2 \right),
$$

$$
|n|\int_{-\infty}^{0} |\eta_n|^2 + |n|\int_{-\infty}^{0} |\xi_n|^2 + |n|\int_{-\infty}^{0} |\Phi_{nyy}|_0^2 + |n|\int_{-\infty}^{0} |\Phi_{nyy}|_0^2 + |n|\int_{-\infty}^{0} |\Phi_n|^2 + |n|\int_{-\infty}^{0} \|U_n\|^2 + |n|\int_{-\infty}^{0} \|U_n\|^2
$$

$$
\leq C \left( |n|\int_{-\infty}^{0} |\eta_n|^2 + |\xi_n|^2 + |n|\int_{-\infty}^{0} |\Phi_{nyy}|_0^2 + |n|\int_{-\infty}^{0} |\Phi_{nyy}|_0^2 + |n|\int_{-\infty}^{0} |\Phi_n|^2 + |n|\int_{-\infty}^{0} \|U_n\|^2 \right)
$$

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for sufficiently large \(|n|\), so that
\[
\|(\eta, \xi, \Phi, U)\|_{Y_1} \leq C \|(\eta^*, \xi^*, \Phi^*, U^*)\|_{Y_0},
\]
\[
\|(\eta, \xi, \Phi, U)\|_{Y_2} \leq C \|(\eta^*, \xi^*, \Phi^*, U^*)\|_{Y_1}.
\]

It follows from a result given by Triebel \cite[§1.17.1, Theorem 1]{Triebel} that \([Y_0, Y_1]_t = Y_t, [Y_1, Y_2]_t = Y_{t+1}\) for \(t \in [0, 1]\), and so by interpolation we conclude that
\[
\|(\eta, \xi, \Phi, U)\|_{Y_{s+1}} \leq C \|(\eta^*, \xi^*, \Phi^*, U^*)\|_{Y_s}. \tag{3.44}
\]

For \(n = 0\) the corresponding equations
\[
\xi_0 - \sin \theta \Phi_0|_{y=0} = \eta_0^*,
\]
\[
\gamma \eta_0 = \xi_0^*,
\]
\[
U_0 = \Phi_0^*,
\]
\[
-\Phi_{0yy} = U_0^*,
\]
with boundary condition
\[
\Phi_{0y}|_{y=0} = \sin^2 \theta \Phi_0|_{y=0} - \xi_0 \sin \theta,
\]
where \((\eta_0^*, \xi_0^*, \Phi_0^*, U_0^*) \in \mathbb{R} \times \mathbb{R} \times H^{s+1}(-\infty, 0) \times H^s(-\infty, 0)\), admit the explicit solution \((\eta_0^*, \xi_0^*, \Phi_0^*, U_0^*) \in \mathbb{R} \times \mathbb{R} \times H^{s+2}(-\infty, 0) \times H^{s+1}(-\infty, 0)\) given in the statement of the lemma provided that
\[
\int_0^y \int_{-\infty}^t U_0^*(t) \, dt \, d\tilde{t}, \quad \int_0^y U_0^*(t) \, dt \in L^2(-\infty, 0)
\]
and the compatibility condition
\[
\int_{-\infty}^0 U_0^*(y) \, dy - \eta_0^* \sin \theta = 0
\]
is satisfied. \(\square\)

We conclude this section with some resolvent estimates for \(A\).

**Lemma 3.4.3** Suppose that \(\omega \in \mathbb{R} \setminus \{0\}\) does not satisfy the equation (3.41) for any \(n \in \mathbb{Z}\). The operator \(A - i\omega I\) defines isomorphisms \(D_1(A) \to X_0\) and \(D_2(A) \to X_1\), where
\[
D_j(A) = \{ (\eta, \xi, \Phi, U) \in X_j : \Phi_y|_{y=0} = \nu \eta \sin \theta_2 \cos \theta_1 - \sin \theta_1 - \xi \sin \theta_2 + \sin^2 \theta_2 \Phi|_{y=0} \}, \quad j = 1, 2.
\]
Moreover,
\[
\|(A - i\omega I)^{-1}\|_{X_0 \to X_1} \leq C, \tag{3.45}
\]
\[
\|(A - i\omega I)^{-1}\|_{X_0 \to X_0} \leq \frac{C}{|\omega|} \tag{3.46}
\]
for all sufficiently large values of \(|\omega|\).
Proof. Using elementary calculus, one finds that (3.36)–(3.40) has the unique solution \((\eta_n, \xi_n, \Phi_n, U_n)\) given by the explicit formulae

\[
\eta_n = \frac{1}{(\gamma + \sigma^2)} \left( \frac{\omega \sin \theta_2 + \nu \sin \theta_1}{\sigma(\gamma + \sigma^2) - (\omega \sin \theta_2 + \nu \sin \theta_1)^2} \right) \int_{-\infty}^{0} e^{\sigma t} f_n^*(t) \, dt
\]

\[
+ \frac{i \omega \sin \theta_2 + i \nu \sin \theta_1}{(\omega \sin \theta_2 + i \nu \sin \theta_1)} \int_{-\infty}^{0} e^{\sigma t} f_n^*(t) \, dt
\]

\[
\xi_n = \eta_n^* + (i \omega + \nu \cos(\theta_1 - \theta_2)) \eta_n
\]

\[
+ \frac{\sin \theta_2 (\gamma + \sigma^2)}{\sigma(\gamma + \sigma^2) - (\omega \sin \theta_2 + \nu \sin \theta_1)^2} \int_{-\infty}^{0} e^{\sigma t} f_n^*(t) \, dt
\]

\[
+ \frac{\sin \theta_2}{\sigma(\gamma + \sigma^2) - (\omega \sin \theta_2 + \nu \sin \theta_1)^2} \eta_n^*
\]

\[
U_n(y) = \Phi_n^*(y) + (i \omega + \nu \cos(\theta_1 - \theta_2)) \int_{-\infty}^{0} e^{-\sigma |y - t|} f_n^*(t) \, dt
\]

\[
+ \frac{\omega \sin \theta_2 + \nu \sin \theta_1}{(\omega \sin \theta_2 + \nu \sin \theta_1)} \int_{-\infty}^{0} e^{\sigma (t + y)} f_n^*(t) \, dt
\]

\[
+ \frac{2 \sigma (\sigma \gamma + \sigma^2) - (\omega \sin \theta_2 + \nu \sin \theta_1)^2}{\sigma(\gamma + \sigma^2) - (\omega \sin \theta_2 + \nu \sin \theta_1)^2} \int_{-\infty}^{0} e^{\sigma (t + y)} f_n^*(t) \, dt
\]

\[
\Phi_n(y) = \int_{-\infty}^{0} e^{-\sigma |y - t|} f_n^*(t) \, dt + \frac{1}{\sigma(\gamma + \sigma^2) - (\omega \sin \theta_2 + \nu \sin \theta_1)^2} e^{\sigma y} g_n^*
\]

\[
+ \frac{1}{2\sigma} \left( \frac{\sigma(\gamma + \sigma^2) + (\omega \sin \theta_2 + \nu \sin \theta_1)^2}{\sigma(\gamma + \sigma^2) - (\omega \sin \theta_2 + \nu \sin \theta_1)^2} \right) \int_{-\infty}^{0} e^{\sigma (t + y)} f_n^*(t) \, dt
\]

where

\[
f_n^* = U_n^* + (i \omega + \nu \cos(\theta_1 - \theta_2)) \Phi_n^*,
\]

\[
g_n^* = -(\omega \sin \theta_2 + \nu \sin \theta_1)(\xi_n^* + (i \omega + \nu \cos(\theta_1 - \theta_2)) \eta_n^*) - (\gamma + \sigma^2) \eta_n^* \sin \theta_2.
\]
Applying the inequality
\[
\int_{-\infty}^{0} \left( \int_{-\infty}^{0} e^{-\sigma|y-t|} f_n^*(t) \, dt \right)^2 \, dy, \int_{-\infty}^{0} \left( \int_{-\infty}^{0} e^{\sigma(y+t)} f_n^*(t) \, dt \right)^2 \, dy \leq \frac{C}{\sigma^2} \int_{-\infty}^{0} |f_n^*(t)|^2 \, dt
\]
and the estimates
\[
\frac{1}{C}(|\omega|^2 + |n|^2) \leq \sigma^2 \leq C(|\omega|^2 + |n|^2),
\]
\[
|\sigma(\gamma + \sigma^2) - (\omega \sin \theta_2 + n\nu \sin \theta_1)^2| \geq C\sigma^3,
\]
which hold for sufficiently large \(|(\omega, n)|\), to formula (3.47), and proceeding as in the previous lemma, we find that
\[
\| (\eta, \xi, \Phi, U) \|_{X_1} \leq C_{\omega} \| (\eta^*, \xi^*, \Phi^*, U^*) \|_{X_0},
\]
\[
\| (\eta, \xi, \Phi, U) \|_{X_2} \leq C_{\omega} \| (\eta^*, \xi^*, \Phi^*, U^*) \|_{X_1}
\]
for each \(\omega\) which does not satisfy (3.41) for some \(n \in \mathbb{Z}\).

A more careful analysis is required to obtain the estimates (3.45) and (3.46). Combining the estimates
\[
\| \Phi_n \|_0^2 \leq C \left( \frac{1}{\sigma^4} \| f_n^* \|_0^2 + \frac{1}{\sigma^2} \| g_n^* \|_0^2 \right),
\]
and
\[
\| f_n^* \|_0^2 \leq C \left( \| U_n^* \|_0^2 + \sigma^2 \| \Phi_n^* \|_0^2 \right),
\]
\[
\| g_n^* \|_0^2 \leq C \left( \sigma^4 \| \eta_n^* \|_0^2 + \sigma^2 \| \xi_n^* \|_0^2 \right)
\]
yields
\[
\| \Phi_n \|_0^2 \leq C \left( \frac{1}{\sigma^4} \| U_n^* \|_0^2 + \frac{1}{\sigma^2} \| \Phi_n^* \|_0^2 + \frac{1}{\sigma^2} \| \eta_n^* \|_0^2 + \frac{1}{\sigma^2} \| \xi_n^* \|_0^2 \right),
\]
and therefore
\[
|n|^4 \| \Phi_n \|_0^2 + |\omega|^2 |n|^2 \| \Phi_n \|_0^2 \leq C \left( \| U_n^* \|_0^2 + |n|^2 \| \Phi_n^* \|_0^2 + |n| \| \eta_n^* \|_0^2 + \frac{1}{|n|} \| \xi_n^* \|_0^2 \right) \tag{3.48}
\]
for sufficiently large \(|(\omega, n)|\).

Equations (3.36) and (3.37) show that
\[
|n\nu \sin(\theta_1 - \theta_2)\eta_n^* + n\nu \sin(\theta_1 - \theta_2) \sin \theta_2 \Phi_n|_{y=0}^2 + |\xi_n^* - \nu\nu(\sin \theta_2 \cos(\theta_1 - \theta_2) - \sin \theta_1) \Phi_n|_{y=0}^2
\]
\[
= |n\nu \sin(\theta_1 - \theta_2)|^2 |\xi_n|^2 + |n\nu \sin(\theta_1 - \theta_2)(\omega + n\nu \cos(\theta_1 - \theta_2))|^2 |\eta_n|^2
\]
\[
+ |\gamma + n^2 \nu^2 \sin^2(\theta_1 - \theta_2)|^2 |\eta_n|^2 + |\omega + n\nu \cos(\theta_1 - \theta_2)|^2 |\xi_n|^2
\]
\[
- 2\gamma(\omega + n\nu \cos(\theta_1 - \theta_2)) \Im \eta_n \xi_n,
\]
in which the identity

\[ |a + ib| = |a|^2 + |b|^2 + 2\text{Im}(ab) \quad a, b \in \mathbb{C}, \]

has been used. It follows that

\[
|n|^4|\eta_n|^2 + |n|^2|\xi_n|^2 + |\omega|^2 \left( |n|^2|\eta_n|^2 + |\xi_n|^2 \right) \\
\leq C \left( |n|^2|\eta_n^*|^2 + |\xi_n^*|^2 + |n|^2|\Phi_n|_{y=0}^2 + (|\omega| + |n|) \left( |\eta_n|^2 + |\xi_n|^2 \right) \right),
\]

and therefore that

\[
|n|^4|\eta_n|^2 + |n|^2|\xi_n|^2 + |\omega|^2 \left( |n|^2|\eta_n|^2 + |\xi_n|^2 \right) \leq C \left( |n|^2|\eta_n^*|^2 + |\xi_n^*|^2 + |n|^2|\Phi_n|_{y=0}^2 \right) \tag{3.49}
\]

for sufficiently large \(|(\omega, n)|\).

It follows from equations (3.38) and (3.39) that

\[
\|\Phi_{ny}^*\|_0^2 + \|U_n^* - n^2\nu^2 \sin^2(\theta_1 - \theta_2)\Phi_n\|_0^2 \\
= \|U_{ny}\|_0^2 + |\omega + n\nu \cos(\theta_1 - \theta_2)|^2 \|\Phi_{ny}\|_0^2 + |\omega + n\nu \cos(\theta_1 - \theta_2)|^2 \|U_n\|_0^2 \\
+ \|\Phi_{nyy}\|_0^2 + 2(\omega + n\nu \cos(\theta_1 - \theta_2)) \text{Im} \int_{-\infty}^0 (\Phi_{nyy}U_n - U_{ny}\Phi_{ny}) \, dy.
\]

Using the fact that

\[
\text{Im} \int_{-\infty}^0 (\Phi_{nyy}U_n - U_{ny}\Phi_{ny}) \, dy = \text{Im} \Phi_{ny}|_{y=0} U_n|_{y=0}
\]

and the estimate

\[
|2\text{Im} \Phi_{ny}|_{y=0} U_n|_{y=0}| \leq \frac{1}{\epsilon_1} \|\Phi_{ny}|_{y=0}\|_0^2 + \epsilon_1 \|U_n|_{y=0}\|_0^2 \\
\leq \frac{1}{\epsilon_1} \|\Phi_{ny}|_{y=0}\|_0^2 + \epsilon_1 \|U_n\|_0 \|U_n\|_1,
\]

one finds that

\[
\|\Phi_{nyy}\|_0^2 + |n|^2 \|\Phi_{ny}\|_0^2 + \|U_{ny}\|_0^2 + |n|^2 \|U_n\|_0^2 + |\omega|^2 \left( \|\Phi_{ny}\|_0^2 + \|U_n\|_0^2 \right) \\
\leq C \left( \|\Phi_{ny}^*\|_0^2 + \|U_n^*\|_0^2 + |n|^4 \|\Phi_n\|_0^2 \\
+ (|\omega| + |n|) \left( \frac{1}{\epsilon_1} \|\Phi_{ny}|_{y=0}\|_0^2 + \epsilon_1 \|U_n\|_0 \|U_n\|_1 \right) \right). \tag{3.50}
\]

Combining (3.49) and (3.50) and using the inequality

\[
(|\omega| + |n|) \|U_n\|_0 \|U_n\|_1 \leq \frac{1}{2} \left( |\omega|^2 + |n|^2 \right) \|U_n\|_0^2 + \|U_n\|_1^2
\]

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and the estimate
\[ |\Phi_{ny}|^2_{y=0}^2 \leq C \left( |n|^2 |\eta|^2 + |\xi|^2 + |\Phi_n|_{y=0}^2 \right), \]
which is obtained from the boundary condition (3.40), one finds that
\[
\left| n^4 |\eta|^2 + n^2 |\xi|^2 + \|\Phi_{nyy}\|_0^2 + |n|^2 \|\Phi_{ny}\|_0^2 + |U_{nyy}\|_0^2 + |n|^2 \|U_n\|_0^2 \right. \\
\left. + |\omega|^2 \left( |n|^2 |\eta|^2 + |\xi|^2 + \|\Phi_{ny}\|_0^2 + |n|^2 \|\Phi_n\|_0^2 + |U_n\|_0^2 \right) \right) \\
\leq C \left( |n|^2 |\eta|^2 + |\xi|^2 + \|\Phi_{ny}\|_0^2 + |n|^2 \|\Phi_n\|_0^2 + |U_n\|_0^2 \right) \\
+ \left( |\omega| + |n| \right) |\Phi_n|_{y=0}^2 + |n| |\Phi_n|_{y=0}^2 \right) .
\]

Using the estimate
\[ |\Phi_n|_{y=0}^2 \leq \frac{1}{2\epsilon_2} |\Phi_n|_0^2 + \frac{\epsilon_2}{2} |\Phi_n|_1^2, \]
one concludes that
\[
\left| n^4 |\eta|^2 + n^2 |\xi|^2 + \|\Phi_{nyy}\|_0^2 + |n|^2 \|\Phi_{ny}\|_0^2 + |U_{nyy}\|_0^2 + |n|^2 \|U_n\|_0^2 \right. \\
\left. + |\omega|^2 \left( |n|^2 |\eta|^2 + |\xi|^2 + \|\Phi_{ny}\|_0^2 + |n|^2 \|\Phi_n\|_0^2 + |U_n\|_0^2 \right) \right) \\
\leq C \left( |n|^2 |\eta|^2 + |\xi|^2 + \|\Phi_{ny}\|_0^2 + |n|^2 \|\Phi_n\|_0^2 + |U_n\|_0^2 \right)
\]
for sufficiently small \( \epsilon_2 \) and sufficiently large \( |(\omega, n)| \), so that
\[
\| (\eta, \xi, \Phi, U) \|_{X_1}^2 + |\omega|^2 \| (\eta, \xi, \Phi, U) \|_{X_0}^2 \leq C \| (\eta^*, \xi^*, \Phi^*, U^*) \|_{X_0}^2
\]
for all sufficiently large values of \(|\omega|\).

\[ \square \]

### 3.5 Purely imaginary eigenvalues

This section presents a geometric method of identifying the nonzero purely imaginary eigenvalues \( \pm i \omega \) of the operator \( L : D(L) \subset X_s \rightarrow X_s \) which correspond to solutions of the dispersion relation
\[
\sigma (\gamma + \sigma^2) = (\omega \sin \theta_2 + n\nu \sin \theta_1)^2.
\]

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Setting
\[ k = \omega \sin \theta_2 + n \nu \sin \theta_1, \quad l = -\omega \cos \theta_2 - n \nu \cos \theta_1, \]

one can write the dispersion relation as
\[ D(k, l) = -k^2 + (\gamma + (k^2 + l^2))\sqrt{k^2 + l^2} = 0. \]

Purely imaginary eigenvalues of \( L \) therefore correspond to points of intersection between the real branches \( C_{dr} \) of the dispersion relation \( D(k, l) = 0 \) and the lines \( K_n \), where
\[ K_n = \{(k, l) \in \mathbb{R}^2 : k = \omega \sin \theta_2 + n \nu \sin \theta_1, \quad l = -\omega \cos \theta_2 - n \nu \cos \theta_1, \quad \omega \in \mathbb{R}\}, \quad n \in \mathbb{Z}, \]

and \( C_{dr} \) is given parametrically by
\[ C_{dr} = \{(k, l) \in \mathbb{R}^2 : k = \pm \sqrt{(\gamma + r^2)r}, \quad l = \pm \sqrt{r^2 - (\gamma + r^2)r}, \quad r \in \mathbb{R}\}. \]

Since we are only concerned with nonzero eigenvalues, the intersection between \( C_{dr} \) and \( K_0 \) at the origin is ignored. The shape of \( C_{dr} \) is shown in Figure 3.5 for a range of positive values of the parameter \( \gamma \). We notice that \( C_{dr} \) is symmetric with respect to the \( k \)- and \( l \)-axis. In the case \( \gamma = 0 \) the real branches \( C_{dr} \) become unbounded, so there are infinitely many purely imaginary eigenvalues and an application of the Lyapunov-Iooss theorem is not possible. The existence of doubly-periodic waves in this case has been discussed by Iooss & Plotnikov [22] using a Nash-Moser method.

The intersection between \( C_{dr} \) and the \( k \)-axis correspond to roots of the dispersion equation \( D(k, 0) = 0 \). Observe that \( k = 0 \) is a root, and that nonzero roots \( k = k_c \) satisfy the equation
\[ k_c^2 - |k_c| + \gamma = 0 \quad (3.53) \]
(these roots correspond to purely imaginary eigenvalues \( ik_c \) for the two-dimensional travelling water wave problem) looss [21, problem 0]). It follows that for \( \gamma > 1/4 \) no nonzero real roots exist, and for \( 0 < \gamma < 1/4 \) there exist two pairs of nonzero simple real roots \( \pm k_c^1, \pm k_c^2 \), as shown in Figure 3.5. These simple roots converge to form a double root \( \pm 1/2 \) at \( \gamma = 1/4 \), and as \( \gamma \to 0 \) the 'inner' roots \( \pm k_c^1 \to 0 \) and the 'outer' roots \( \pm k_c^2 \to \pm 1 \).

The real branches have the limiting behaviour
\[ l^2 \sim \frac{2k_c(1 - 2|k_c|)}{(1 + 2|k_c|)}(k - k_c) \quad (3.54) \]
as \( k \to k_c^j \). Notice in Figure 3.5 the qualitative difference in the shape of \( C_{dr} \) as \( \gamma \) is varied.

There exists a critical value \( \gamma^* \) of \( \gamma \) such that for \( 0 < \gamma < \gamma^* \) the portion \( C_{dr}^+ \) of \( C_{dr} \) in the positive quadrant has two points of inflection. We see that \( C_{dr}^+ \) is concave to the left of the first point of inflection, convex between the first point and the second point of inflection, and concave thereafter. As \( \gamma \) increases through the critical value \( \gamma = \gamma^* \) these two points of inflection merge and disappear, and the real branches at the two pairs of simple roots have the limiting behaviour described in (3.54).
Figure 3.5: *The three distinct graphs of the real branches of the dispersion curve D(k, l) = 0 for different ranges of γ.*

Turning to the lines $K_n$, we see that they are parallel, equidistant and form an angle $\theta_2$ with the positive $l$-axis, for given values of $\nu$, $\theta_1$ and $\theta_2$. They also pass through the points $P_n = (n\nu \sin \theta_1, -n\nu \cos \theta_1), \ n \in \mathbb{Z}$, on the line

$$L = \{(k, l) \in \mathbb{R} : k = c \sin \theta_1, \ l = -c \cos \theta_1, \ c \in \mathbb{R}\},$$

which passes through the origin and makes an angle $\theta_1$ with the positive $l$-axis (see Figure 3.6). Since $\nu$ determines the distance between the lines $K_n$, it is clear that the number of intersections of $K_n$ and the real branches $C_{dr}$ depends only upon $\theta_2$, $\gamma$ and $\nu$. The remaining parameter $\theta_1$ determines the values of the corresponding eigenvalues: an intersection of $K_n$ with $C_{dr}$ corresponds to an eigenvalue $i\omega$, where $\omega$ is the signed distance from the intersection to the point $P_n$. Furthermore, notice that the sets $K_n \cap C_{dr}$ and $K_{-n} \cap C_{dr}$ have the same cardinality, that is if the intersection of $K_n$ and $C_{dr}$ corresponds to the purely imaginary eigenvalue $i\omega$ of $L$ then the intersection $K_{-n} \cap C_{dr}$ refers to $-i\omega$.

We consider different scenarios of intersection for different values of the parameters $\theta_2$, $\gamma$ and $\nu$. An isolated intersection between $K_n$ and $C_{dr}$ refers to an algebraically simple purely imaginary eigenvalue of $L$, and algebraically double eigenvalues are indicated by points of tangential contact. Since no intersections occur for $\gamma > 1/4$, we restrict our attention to $\gamma < 1/4$ only. We begin by choosing $\nu$ sufficiently large so that only $K_0$ intersects $C_{dr}$. We then vary the angle $\theta_2$ and catalogue possible intersection scenarios. The results are
Figure 3.6: Intersections of the lines $\mathcal{K}_n$ with $C_{dr}$ correspond to purely imaginary eigenvalues.

shown in Figure 3.7. Observe that a plus-minus pair of purely imaginary eigenvalues emerge from a point of tangency between $C_{dr}$ and $\mathcal{K}_0$. Such occurrences shall be referred to as eigenvalue resonances.

It is also instructive to examine the complementary point of view, in which one fixes $\theta_2$ and looks at the intersection scenarios as $\gamma$ varies. These results are presented in Figure 3.8.

We now consider the affect of varying $\nu$ for fixed values of $\gamma$ and $\theta_2$. The parameter $\nu$ is decreased from an initially large value to a critical value $\nu_c$. As $\nu_c$ reaches its critical value, the lines $\mathcal{K}_{\pm1}$ become tangent to $C_{dr}$, and two pairs of algebraically simple purely imaginary eigenvalues emerge through an eigenvalue resonance. Figure 3.9 shows three intersection scenarios for $\gamma^* < \gamma < 1/4$, and analogous scenarios occur for $0 < \gamma < \gamma^*$. Further eigenvalue resonances occur at $\nu_c/n$, when the curves $\mathcal{K}_{\pm n}$ touch $C_{dr}$ tangentially.

There are in fact additional points of tangential contact between $\mathcal{K}_{\pm1}$ and $C_{dr}$ that occur when $\nu$ is increased to a second critical value $\nu_{c,2}$. These are shown in Figure 3.10 for $\gamma$ fixed in $\gamma^* < \gamma < 1/4$, where we have assumed that $\nu_{c,2} > \nu_c/2$, so that $\mathcal{K}_{\pm1}$ and $\mathcal{K}_0$ are still the only intersecting lines under consideration. Analogous intersection scenario occur in the case $0 < \gamma < \gamma^*$.

We now turn to degenerate cases that occur when intersections of $\mathcal{K}_{n_1}, \mathcal{K}_{n_2}$ with $C_{dr}$ correspond to purely imaginary eigenvalues that have the same value. This degeneracy
Figure 3.7: Intersection scenarios involving $K_0$ for fixed $\gamma$. The solid dots represent simple purely imaginary eigenvalues and the hollow dots represent double purely imaginary eigenvalues. For the two cases $0 < \gamma < \gamma^*$, $\gamma^* < \gamma < 1/4$, two pairs of geometrically simple purely imaginary eigenvalues emerge from zero through eigenvalue resonances as $K_0$ becomes tangent to $C_{dr}$. 
Figure 3.8: Intersection scenarios involving $\mathcal{K}_0$ for fixed $\theta_2$. Four purely imaginary eigenvalues through eigenvalue resonances.

occurs when the signed distances of the points of intersection of $\mathcal{K}_{n_1}, \mathcal{K}_{n_2}$ with $C_{dr}$ to $P_{n_1}, P_{n_2}$ are equal. We illustrate with the case $\theta_1 = \pm \pi/2$, $\theta_2 = 0$, and the intersection of $\mathcal{K}_{\pm 1}$ and $C_{dr}$ (see Figure 3.11). The resulting purely imaginary eigenvalues are geometrically double. This special case has been studied in detail by Groves & Sandstede [15] and Haragus & Kirchgässner [18] in the context of water waves with finite depth.

Finally, we consider the case where $\nu, \theta_1, \theta_2$ are fixed, and we identify the values of $\gamma$ for which we obtain tangential intersections between the lines $\mathcal{K}_n$ and the branches $C_{dr}$. This case is illustrated in Figure 3.12 for a particular choice of parameters $\nu, \theta_1, \theta_2$. In general, we find that reducing the value of $\gamma$ increases the number of purely imaginary eigenvalues that arise through eigenvalue resonances.
Figure 3.9: Intersection scenarios involving $K_{-1}$, $K_0$ and $K_1$ as $\nu$ is varied from $\nu > \nu_c$ to $\nu < \nu_c$ ($\gamma$ is fixed in $\gamma^* < \gamma < 1/4$). The diagrams on the left show the lines $K_0$, $K_1$, $K_{-1}$ and the curve $C_{dr}$ in the $(k,l)$ plane. The diagram on the right show the eigenvalues of the linear operator $\mathcal{L}$. 
Figure 3.10: Further intersection scenarios for involving $K_{-1}$, $K_0$ and $K_1$ as $\nu$ is varied from $\nu > \nu_{c,2}$ to $\nu < \nu_{c,2}$. 
Figure 3.11: Degenerate intersection scenario of $\mathcal{K}_{\pm 1}$ and $C_{dr}$. The parameter $\gamma$ is fixed in $\gamma^* < \gamma < 1/4$ and $\theta_1 = \pm \pi/2$, $\theta_2 = 0$. The solid dots represent geometrically double purely imaginary eigenvalues.
Figure 3.12: Spectral picture in $\gamma$-parameter space for fixed $\nu$, $\theta_1$ and $\theta_2$. The points $\gamma_n$, $n \in \mathbb{N}$, are the points when $K_n$ make tangential contact with $C_{dr}$, respectively. Each such tangential intersection point gives rise to two pairs of purely imaginary eigenvalues through eigenvalue resonances as $\gamma$ is reduced.
3.6 Application of the Lyapunov-Iooss theorem

It remains to confirm that the spatial dynamics formulation (3.32) of the hydrodynamic problem satisfies the spectral hypotheses of the reversible Lyapunov-Iooss theorem (Theorem 2.5.3).

Lemma 3.6.1

(i) The purely imaginary number \( i \omega, \omega \in \mathbb{R} \setminus \{0\}, \) is an eigenvalue of \( L \) if and only if there exists \( n \in \mathbb{Z} \) such that

\[
\sigma(\gamma + \sigma^2) = (\omega \sin \theta_2 + n \nu \sin \theta_1)^2.
\]

All other nonzero purely imaginary numbers belong to the resolvent set of \( L \).

(ii) There exists a constant \( C \) such that

\[
\| (L - i \omega I)^{-1} \|_{X_s \rightarrow X_{s+1}} \leq C,
\]

\[
\| (L - i \omega I)^{-1} \|_{X_s \rightarrow X_s} \leq \frac{C}{|\omega|}
\]

for all sufficiently large values of \( |\omega| \).

Proof. (i) The first statement follows from Proposition 3.4.1. Suppose that \( \omega \neq 0 \) does not satisfy the given equation for any \( n \in \mathbb{Z} \). Using Lemma 3.4.3 and formula (3.35) (with appropriate extensions of \( \hat{d}M[0] \) and \( dM[0]^{-1} \)), we find that \( L - i \omega I \) defines isomorphisms \( D_1(L) \rightarrow X_0 \) and \( D_2(L) \rightarrow X_1 \), where

\[
D_j(L) = \{ (\eta, A, \Gamma, U) \in X_j : \Gamma_y|_{y=0} = 0 \}, \quad j = 1, 2.
\]

It follows by interpolation that \( (L - i \omega I)^{-1} : X_s \rightarrow X_{s+1} \) is a continuous operator.

(ii) Choose \( \omega_1 \in \rho(L) \) and observe that

\[
\| (L - i \omega I)^{-1} v \|_{X_2} \leq C_{\omega_1} \| (L - i \omega_1 I)(L - i \omega I)^{-1} v \|_{X_1}
\]

\[
\leq C_{\omega_1} \| (L - i \omega I)^{-1} (L - i \omega_1 I) v \|_{X_1}
\]

\[
\leq C_{\omega_1} \| (L - i \omega I)^{-1} \|_{X_0 \rightarrow X_1} \| (L - i \omega I) v \|_{X_0}
\]

\[
\leq C_{\omega_1} \| v \|_{X_1} \tag{3.55}
\]

and similarly

\[
\| (L - i \omega I)^{-1} v \|_{X_1} \leq \frac{C_{\omega_1}}{|\omega|} \| v \|_{X_1} \tag{3.56}
\]
for each \( v \in D_1(L) \) and sufficiently large \( |\omega| \); here we have used the estimates

\[
\| (L - i\omega I)^{-1} \|_{X_0 \rightarrow X_1} \leq C, \\
\| (L - i\omega I)^{-1} \|_{X_0 \rightarrow X_0} \leq \frac{C}{|\omega|}
\]

for sufficiently large \( |\omega| \) (see Lemma 3.4.3).

Interpolating between (3.55) and (3.57), we find that \((L - i\omega I)^{-1}\) defines a continuous operator \([X_0, D_1(L)]_s \rightarrow [D_1(L), D_2(L)]_s\). The boundary condition \( \Gamma_y[y=0] \) defining \( D_j(L) \) is normal in the sense of Grisvard [14], and the interpolation results in this reference assert that \([X_0, D_1(L)]_s = X_s\) for \( s \in (0, 1/2) \). On the other hand, it follows from a result by Triebel [49, §1.17.1, Theorem 1] that \([D_1(L), D_2(L)]_s = D(L)\). We conclude that \((L - i\omega I)^{-1} : X_s \rightarrow D(L)\) is a continuous operator with

\[
\| (L - i\omega I)^{-1} \|_{X_s \rightarrow X_{s+1}} \leq C
\]

for sufficiently large \( |\omega| \). Similarly, interpolating between (3.56) and (3.58), we find that \((L - i\omega I)^{-1}\) defines a continuous operator \([X_0, D_1(L)]_s \rightarrow [X_0, D_1(L)]_s\), that is \( X_s \rightarrow X_s \) with

\[
\| (L - i\omega I)^{-1} \|_{X_s \rightarrow X_s} \leq \frac{C}{|\omega|}
\]

for sufficiently large \( |\omega| \).

Now we turn to the spectral condition at the origin.

**Lemma 3.6.2** The equation

\[
L v = -N(v^*)
\]

has a unique solution \( v \in D(L) \) for each \( v^* \in D_G \). The solution \( v \in X_{s+1}\) depends smoothly upon \( v^* \in X_{s+1}\).

**Proof.** Let us write (3.59) as

\[
A u = -d\hat{M}[0]^{-1}(N(v^*)),
\]

where \( u = dM[0]^{-1}(v) \). Since the linear operators \( d\hat{M}[M^{-1}(v^*)] \) and \( d\hat{M}[0]^{-1} \) do not alter the first and fourth components of their arguments, it follows that the first and fourth components of the right hand side of (3.60) are precisely the first and fourth components of the nonlinear part of \( F(M^{-1}(v^*)) \), that is the nonlinear parts of
\[
(F(M^{-1}(v^*)))_1 = \frac{W^*}{(1 - W^{*2})^{1/2}} (1 + \nu^2 \eta^*_z \sin^2(\theta_1 - \theta_2))^{1/2} - \nu \eta^*_z \cos(\theta_1 - \theta_2), \quad (3.61)
\]

\[
(F(M^{-1}(v^*)))_4 = \left[-\Phi_y + \frac{U^* e^y W^*}{(1 - W^{*2})^{1/2}} (1 + \nu^2 \eta^*_z \sin^2(\theta_1 - \theta_2))^{1/2} + \frac{\eta^* \Phi_y e^y}{(1 + \eta^* e^y)}
\right.
\]
\[
\quad + \nu^2 \sin^2(\theta_1 - \theta_2) (\Phi_z^* - \eta^*_z \Phi_y e^y) \eta^*_z e^y (1 + \eta^* e^y)
\]
\[
\left. - \nu^2 \sin^2(\theta_1 - \theta_2) (\Phi_z^* - \eta^*_z \Phi_y e^y) (1 + \eta^* e^y)
\right]_z,
\]

where

\[
W^* = \Lambda^* + \int_{-\infty}^0 U^* \Phi_y e^y \, dy
\]

and \( \Phi^* = \Phi^*(v^*) \).

Observe that

\[
[(F(M^{-1}(v^*)))_1]_0 = \left[\frac{W^*}{(1 - W^{*2})^{1/2}} (1 + \nu^2 \eta^*_z \sin^2(\theta_1 - \theta_2))^{1/2}\right]_0
\]

\[
[(F(M^{-1}(v^*)))_4]_0 = -\partial^2_y [\Phi^*]_0 + \partial_y \left( [g(\eta^*, \Lambda^*, \Gamma^*, U^*)]_0 e^y \right),
\]

where

\[
g(\eta^*, \Lambda^*, \Gamma^*, U^*) = \frac{U^* W^*}{(1 - W^{*2})^{1/2}} (1 + \nu^2 \eta^*_z \sin^2(\theta_1 - \theta_2))^{1/2} + \frac{\eta^* \Phi_y}{(1 + \eta^* e^y)}
\]
\[
+ \nu^2 \sin^2(\theta_1 - \theta_2) (\Phi_z^* - \eta^*_z \Phi_y e^y) \eta^*_z (1 + \eta^* e^y), \quad (3.63)
\]

so that \( \int_{-\infty}^y [(F(M^{-1}(v^*)))_4]_0 \, dt = -\partial_y [\Phi^*]_0 + [g(\eta^*, \Lambda^*, \Gamma^*, U^*)]_0 e^y \in L^2(-\infty, 0) \). Furthermore, the calculation

\[
\int_{-\infty}^0 \left( \int_{-\infty}^y \mathcal{H} e^{\tilde{t}} \, d\tilde{t} \right)^2 \, dy \leq \int_{-\infty}^0 \left( \int_{-\infty}^0 \mathcal{H} e^{\tilde{t}} \, d\tilde{t} \right)^2 \, dy \leq C \int_{-\infty}^0 |\mathcal{H}|^2 \, dy
\]

shows that \( \int_{-\infty}^y \mathcal{H} e^{\tilde{t}} \, d\tilde{t} \in L^2(-\infty, 0) \) whenever \( \mathcal{H} \in L^2(-\infty, 0) \). We conclude that

\[
\int_{-\infty}^y \int_{-\infty}^\tilde{t} [(F(M^{-1}(v^*)))_4]_0 \, dt \, d\tilde{t} = -[\Phi^*]_0 + \int_{-\infty}^y [g(\eta^*, \Lambda^*, \Gamma^*, U^*)]_0 e^y \, d\tilde{t} \quad \in L^2(-\infty, 0)
\]

\[
\in L^2(-\infty, 0).
\]
Notice that

\[
\int_{-\infty}^{0} [(F(M^{-1}(v^*)))_4]_0 \, dy = \left[ -\Phi_y^* + \frac{U^* e^y W^*}{(1 - W^*^2)\frac{1}{2}} (1 + \nu^2 \eta_2^* \sin^2(\theta_1 - \theta_2) ) + \frac{\eta^* \Phi_y e^y}{(1 + \eta^* e^y)} 
+ \nu^2 \sin^2(\theta_1 - \theta_2)(\Phi_x^* - \eta_2^* \Phi_y e^y) \eta_x e^y (1 + \eta^* e^y) \right]_0^{y=0}
\]

and

\[
\left[ -\Phi_y^* + \frac{U^* - \sin(\theta_2) e^y W^*}{(1 - W^*^2)\frac{1}{2}} (1 + \nu^2 \eta_2^* \sin^2(\theta_1 - \theta_2) ) + \frac{\eta^* \Phi_y e^y}{(1 + \eta^* e^y)} 
+ \nu^2 \sin^2(\theta_1 - \theta_2)(\Phi_x^* - \eta_2^* \Phi_y e^y) \eta_x e^y (1 + \eta^* e^y) \right]_0^{y=0} = 0
\]

because \((\eta^*, \xi^*, \Phi^*, U^*) \in D_F\), where \(\xi^* = \xi^*(v^*)\). It follows that

\[
\int_{-\infty}^{0} [(F(M^{-1}(v^*)))_4]_0 \, dy = \sin \theta_2 \left[ \frac{W^*}{(1 - W^*^2)\frac{1}{2}} (1 + \nu^2 \eta_2^* \sin^2(\theta_1 - \theta_2) ) \right]_0^{y=0}
\]

Since the nonlinear parts of \(F(M^{-1}(v^*)))_1\) and \(F(M^{-1}(v^*)))_4\) satisfy the same constraints as \(F(M^{-1}(v^*)))_1\) and \(F(M^{-1}(v^*)))_4\), we conclude that \(u^* = -dM[0]^{-1}(N(v^*))\) satisfies the hypothesis of Lemma 3.4.2. The unique solution \(u\) of equation (3.60) satisfies

\[
\|u\|_{Y_{s+1}} \leq C \|N(v^*)\|_{Y_s}
\]

so that \(u - [u]_0\) depends smoothly upon \(v^*\), and since \(\eta_0^*, \xi_0^* \in \mathbb{R}, \Phi_0^* \in H^{s+1}(-\infty, 0), \int_{-\infty}^{y} \int_{-\infty}^{1} U_0^*(t) \, dt \, d\bar{t} \in H^{s+2}(-\infty, 0)\) depend smoothly upon \(v^*\) (see above) we conclude from the formulae given in Lemma 3.4.2 that \([u]_0\) also depends smoothly upon \(v^*\). Finally, the relation \(v = dM[0]^{-1}u\) implies that \(v\) depends smoothly upon \(v^*\). \(\square\)

Section 3.5 above provides a catalogue of the simple purely imaginary eigenvalues of \(K_s\). The Lyapunov-Iooss theorem asserts the existence of a family of periodic solutions to the spatial dynamics systems (3.32) for each plus-minus pair of purely imaginary eigenvalues, provided they are not in resonance with any other pair. Such periodic solutions correspond to doubly-periodic waves (with periodic spatial profiles in the \(x\) and \(z\)-directions).

Consider first the intersection scenarios involving \(K_0\) and the dispersion curve \(C_{dR}\) (Figures 3.7 and 3.8). A periodic solution in this case corresponds to an oblique line wave (a wave that is periodic in \(x\) and independent of \(z\) (see Figure 3.3)). Since the 'outer' pair
of eigenvalues is always non-resonant, an application of the Lyapunov-Iooss theorem yields the existence of a family of oblique line waves. Furthermore, suppose the parameter $\theta_2$ is fixed, and that $\gamma$ does not belong to the family of curves

$$n^2\omega^2 - n\omega \sin^2 \theta_2 + \gamma = 0, \quad n \in \mathbb{Z} \setminus \{1, -1\}.$$ 

In this case, the 'inner' pair of eigenvalues remain out of resonance with all other pairs, and so the Lyapunov-Iooss theorem asserts the existence of two families of oblique line waves.

Let us now turn to the general case, in which intersections of $K_n, n \neq 0$, with $C_{dr}$ are involved. Figures 3.9 and 3.10 look at the case when $\nu$ is varied and intersections involving $K_0, K_{-1}$ and $K_1$ arise. For such cases, non-resonant pairs of eigenvalues correspond to doubly periodic travelling waves (see Figure 3.2). Consider the particular case for when $\pm i\omega$ is a pair of simple purely imaginary eigenvalues associated with the Fourier modes $n = \pm 1$, that is not in resonance with any other pair: let the parameter values $\theta_1, \theta_2$ and $\nu$ be fixed, and suppose that $\gamma$ belongs to the curve

$$\sigma(\gamma + \sigma^2) = (\omega \sin \theta_2 + \nu \sin \theta_1)^2,$$

were $\sigma^2 = \omega^2 + \nu^2 + 2\nu \omega \cos(\theta_1 - \theta_2)$, and does not belong to the curves

$$\sigma_{n,m}(\gamma + \sigma_{n,m}^2) = (n\omega \sin \theta_2 + \nu m \sin \theta_1)^2, \quad m \in \mathbb{N}, \quad n \in \mathbb{N} \setminus \{1, 0, -1\},$$

where $\sigma_{n,m}^2 = n^2\omega^2 + m^2\nu^2 + 2nm\nu \omega \cos(\theta_1 - \theta_2)$. Under such parameter restrictions, the Lyapunov-Iooss theorem yields the existence of a family of doubly-periodic waves with frequency $\nu$ in the $z$-direction and approximately $\omega$ in the $x$-direction. This proves Theorem 1.0.1 stated in the introduction.
Incompressible viscous flow past an obstacle

4.1 The mathematical problem

We consider the flow of an incompressible, viscous fluid past a three dimensional obstacle $\mathcal{O}$ centred at the origin of the Cartesian coordinate system $(x_1, x_2, x_3)$ (see Figure 4.1). Here $x_1, x_2$ denote the horizontal coordinates and $x_3$ the vertical coordinate. The boundary of the obstacle $\mathcal{O}$ is taken to be smooth. Assuming no external forces are present, the flow...
is described by the non-dimensional **Navier-Stokes equations**

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \frac{1}{R} \Delta v + \nabla q = 0, \quad (4.1)
\]

\[
\text{div } v = 0 \quad (4.2)
\]

defined on the exterior domain \( \Omega = \mathbb{R}^3 \setminus \mathcal{O} \). Here \( v(x, t), \, x = (x_1, x_2, x_3) \in \Omega \), denotes the dimensionless velocity vector and \( q(x, t) \) denotes the dimensionless pressure. The parameter \( R \) (known as the Reynolds number) is assumed to be positive. On the boundary \( \partial \Omega \) the no-slip condition

\[
v(x)|_{x \in \partial \Omega} = 0 \quad (4.3)
\]

is made, and in addition it is assumed that

\[
v \to v_\infty = (1, 0, 0), \quad |x| \to \infty. \quad (4.4)
\]

The following result concerns the nontrivial stationary solution of the above Navier-Stokes problem, the existence of which was first established by Leray [31]. The additional properties were confirmed by Finn [10] and Babenko [3].

**Lemma 4.1.1** Consider the stationary Navier-Stokes problem

\[
(v \cdot \nabla)v + \nabla q - \frac{1}{R} \Delta v = 0, \quad x \in \Omega,
\]

\[
\text{div } v = 0, \quad \text{in } \Omega,
\]

\[
v = 0, \quad \text{on } \partial \Omega,
\]

\[
v \to v_\infty, \quad \text{as } |x| \to \infty,
\]

on the exterior domain \( \Omega \). This problem has a solution \( \tilde{v} \in C^2(\Omega), \tilde{q} \in C^1(\Omega) \), where \( \tilde{v}(x) \) satisfies

\[
\int_\Omega |\nabla \tilde{v}|^2 \, dx < \infty,
\]

and is of the form

\[
\tilde{v}(x) = v_\infty + u_0(x).
\]

The function \( u_0(x) \) has the following properties:

(i) \( u_0 \in L^p(\Omega) \) for all \( p > 2 \).

(ii) \( u_0, \nabla u_0 \in L^\infty(\Omega) \).

(iii) \( \lim_{r \to \infty} \sup_{|x| > r} (|u_0(x)|, |\nabla u_0(x)|) = 0 \).
Consider the perturbation $u = v - \tilde{v}$, $\hat{q}(x) = q(x) - \tilde{q}(x)$ from the stationary solution $\bar{v}, \tilde{q}(x)$. Substituting into the problem (4.1)–(4.4), and scaling $t \mapsto R t$ and $\hat{q} \mapsto \hat{q}/R$, we arrive at the system

$$\frac{\partial u}{\partial t} + R \frac{\partial u}{\partial x_1} + R (u \cdot \nabla)u_0 + R (u_0 \cdot \nabla)u = 0,$$

in $\Omega$, (4.5)

$$(u \cdot \nabla)u - \Delta u + \nabla q = 0,$$

in $\Omega$, (4.6)

$$\text{div } u = 0,$$

on $\partial \Omega$, (4.7)

$$u \rightarrow 0,$$

as $|x| \rightarrow \infty$, (4.8)

where we have abused notation by dropping the hat.

We formulate the boundary value problem (4.5)–(4.8) as a dynamical system defined on a suitably-chosen Banach space. We make note of the fact that for any bounded or exterior domain $D$ with smooth boundary, or for $D = \mathbb{R}^3$, the space $L^p(D)$ admits the Helmholtz-Weyl decomposition

$$L^p(D) = S_p(D) \oplus G_p(D)$$

(4.9)

for all $1 < p < \infty$, where $S_p(D)$ is the completion of the set

$$\{ w \in C_0^\infty(D) : \text{div } w = 0 \text{ in } D \}$$

with respect to the $L^p(D)$ norm,

$$G_p(D) := \left\{ \nabla q : q \in \hat{W}^{1,p}(D) \right\}$$

(see Galdi [11, ch. 3, Theorem 1.2]), and

$$\hat{W}^{m,p}(D) := \left\{ u \in W^{m,p}_{\text{loc}}(\overline{D}) : \| \partial^\alpha u \|_{L^p(D)} < \infty, |\alpha| = m \right\}.$$  

Introducing the equivalence relation $\sim$ on $\hat{W}^{m,p}(D)$ defined by $u_1 \sim u_2$ if and only if $\| \partial^\alpha (u_1 - u_2) \|_{L^p(D)} = 0$, $|\alpha| = m$, observe that the space $\hat{W}^{m,p}(D)/\sim$ is a Banach space with norm

$$\| u \|_{\hat{W}^{m,p}(D)/\sim} = \sum_{|\alpha| = m} \| \partial^\alpha u \|_{L^p(D)}.$$  

We denote the equivalence class in $\hat{W}^{m,p}(D)/\sim$ represented by $u \in \hat{W}^{m,p}(D)$ by $[u]$. In the case $D = \mathbb{R}^3$ the Helmholtz-Weyl decomposition takes the explicit form

$$u = w + \nabla q,$$
where
\[ q = F^{-1} \left[ \frac{ik \cdot \hat{u}}{|k|^2} \right] \in \dot{W}^{1,p}(D), \]
\[ w = u - \nabla q \in S_p(D) \]
and \( \hat{u}(k) \) denotes the Fourier transform of \( u(x) \) (see Galdi [11, pp. 103-105]).

Denote by \( \Pi_p : L^p(\Omega) \to L^p(\Omega) \) the continuous projection onto \( S_p(\Omega) \) along \( G_p(\Omega) \), and consider the dynamical system
\[
\frac{du}{dt} = F(u, R),
\]
(4.10)
where \( F : X_p(\Omega) \times \mathbb{R} \to S_p(\Omega) \) is given by
\[
F(u, R) = L_R u + N(u, R)
\]
with
\[
L_R u = A_R u + B_R u,
\]
\[
A_R u = \Pi_p \left( \Delta u - R \frac{\partial u}{\partial x_1} \right),
\]
\[
B_R u = -\Pi_p R \left( (u \cdot \nabla)u_0 + (u_0 \cdot \nabla)u \right),
\]
\[
N(u, R) = -\Pi_p R \left( (u \cdot \nabla)u \right),
\]
and
\[
X_p(\Omega) := W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap S_p(\Omega).
\]

In the following we apply the Hopf-Iooss theorem to the system (4.10) to find periodic solutions. Such solutions correspond to time-periodic solutions \( v(x) \) of the Navier-Stokes problem (4.1)–(4.4) (the term \( q(x) \) can be reconstructed from the decomposition (4.9)). Throughout the remainder of this investigation, when concerning Sobelov spaces, we shall always assume \( p \geq 2 \), unless otherwise stated.

### 4.2 The Oseen operator

Consider the Oseen operator \( A_R : X_p(\Omega) \to S_p(\Omega) \) given by
\[
A_R u = \Pi_p \left( \Delta u - R \frac{\partial u}{\partial x_1} \right).
\]
In the following we determine certain spectral properties of $A_R$ by considering the Oseen boundary-value problem

$$\Delta u - R \frac{\partial u}{\partial x_1} - \nabla q - \lambda u = f, \quad \text{in } \Omega,$$
$$\text{div } u = 0, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \partial \Omega,$$
$$u \to 0, \quad \text{as } |x| \to \infty.$$

In particular, we show that the set $\Sigma$, given by

$$\Sigma = \{ \lambda \in \mathbb{C} : R^2 \text{Re } \lambda + (\text{Im } \lambda)^2 > 0 \},$$

is contained in the resolvent set of $A_R$ (see Figure 4.2).

![Figure 4.2: The spectrum of $A_R$ is contained in the shaded region.](image)

We solve the above boundary-value problem using a four-step procedure developed by Kobayashi & Shibata [29], who use it to solve a more generalized form of the Oseen boundary-value problem.

1. The solution operator for the corresponding Oseen problem in a bounded domain is deduced from that for a related boundary-value problem (the Stokes problem) in Section 4.2.1.
2. The solution operator for the corresponding Oseen problem in free space is obtained using Fourier analysis in Section 4.2.2.

3. The next step is to consider inhomogeneities $f$ with compact support (Section 4.2.3). Here we seek a solution which resembles the solution in step (i) close to the obstacle and the solution in step (ii) at large distances; the main tool in the construction is Fredholm alternative theory.

4. Finally, a judicious choice of Ansatz for the solution reduces the general case to the special case considered in step (iii) (Section 4.2.4).

4.2.1 The Oseen problem in a bounded domain

We begin by determining the solution of the Oseen problem in a bounded domain. The first step is the following result for an associated boundary-value problem, the proof of which is given, for example, in Solonnikov [45].

**Lemma 4.2.1** Let $D$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary. Suppose $f \in W^{m,p}(D)$ for some non-negative integer $m$. Consider the Stokes problem

\[
\begin{align*}
\Delta u - \nabla q &= f, & \text{in } D, \\
\text{div } u &= 0, & \text{in } D, \\
 u &= 0, & \text{on } \partial D, \\
\int_D q \, dx &= 0.
\end{align*}
\]

There exist unique functions $u \in W^{m+2,p}(D)$ and $q \in W^{m+1,p}(D)$ which solve the above boundary value problem and satisfy the estimate

\[
\|u\|_{W^{m+2,p}(D)} + \|q\|_{W^{m+1,p}(D)} \leq C \|f\|_{W^{m,p}(D)}. 
\]

The result of Lemma 4.2.1 for the Stokes problem can be used to obtain an existence result for the Oseen problem in a bounded domain. This result is presented in Lemma 4.2.3 below; the following proposition is used in its proof.

**Proposition 4.2.2** Let $D$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary. Suppose $u \in W^{m,p}(D)$ for some positive integer $m$. The Neumann boundary value problem

\[
\begin{align*}
\Delta q &= \text{div } u, & \text{in } D, \\
\frac{\partial q}{\partial n} &= n \cdot u, & \text{on } \partial D, \\
\int_D q \, dx &= 0,
\end{align*}
\]

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where \( n \) is the outward pointing unit normal to \( \partial D \), has a unique solution \( q \in W^{m+1,p}(D) \) which satisfies the estimate

\[
\|q\|_{W^{m+1,p}(D)} \leq C \|u\|_{W^{m,p}(D)}.
\]

Moreover, the projection \( \Pi_p \) is a continuous operator \( W^{m,p}(D) \rightarrow W^{m,p}(D) \cap S_p(D) \), and is given by the formula

\[
\Pi_p u = u - \nabla q.
\]

**Proof.** The existence of a unique solution \( q \in W^{m+1,p}(D) \) to (4.13)–(4.15) which satisfies the given estimates follows by standard elliptic theory.

According to the Helmholtz-Weyl decomposition each \( u \in W^{m,p}(D) \) admits a unique representation

\[
u = w + \nabla \tilde{q}\]

for some \( w \in S_p(D) \) and \( \tilde{q} \in W^{1,p}(D) \) with

\[
\int_D \tilde{q} \, dx = 0.
\]

Clearly

\[
\int_D (\nabla \tilde{q} - u) \cdot \nabla \tau \, dx = - \int_D w \cdot \nabla \tau \, dx
\]

for all \( \tau \in W^{1,r}(D) \), where \( 1/p + 1/r = 1 \); furthermore

\[
\int_D w \cdot \nabla \tau \, dx = 0
\]

for each \( w \in C_0^\infty(D) \) with \( \text{div } w = 0 \) (by the divergence theorem), and by density for each \( w \in S_p(D) \). It follows that

\[
\int_D (\nabla \tilde{q} - u) \cdot \nabla \tau \, dx = 0
\]

for each \( \tau \in W^{1,r}(D) \), so that \( \tilde{q} = q \) (by the uniqueness of weak solutions to (4.13)–(4.15)).

Finally, note that \( \Pi_p u = w = u - \nabla q \) and

\[
\|\Pi_p u\|_{W^{m,p}(D)} \leq \|u\|_{W^{m,p}(D)} + \|\nabla q\|_{W^{m,p}(D)} \leq \|u\|_{W^{m,p}(D)}.
\]
Lemma 4.2.3 Let $D$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary. Suppose $f \in W^{m,p}(D)$ for some non-negative integer $m$, $c \in \mathbb{C}$ and $\lambda \in \Sigma \cup \{0\}$, where $\Sigma$ is given in (4.12). Consider the Oseen problem

$$\Delta u - R\frac{\partial u}{\partial x_1} - \nabla q - \lambda u = f, \quad \text{in } D,$$

$$\text{div } u = 0, \quad \text{in } D,$$

$$u = 0, \quad \text{on } \partial D,$$

and

$$\int_D q(x) \, dx = c. \quad (4.19)$$

There exist continuous linear operators

$$U_{\lambda,D} \in L(W^{m,p}(D) \times \mathbb{C}, W^{m+2,p}(D) \cap W^{1,p}_0(D) \cap S_p(D)),$$

$$P_{\lambda,D} \in L(W^{m,p}(D) \times \mathbb{C}, W^{m+1,p}(D))$$

such that $u = U_{\lambda,D}[f,c]$ and $q = P_{\lambda,D}[f,c]$ is the unique solution of the boundary value problem (4.16)–(4.19) in the class $\{(u,q) \in W^{m+2,p}(D) \cap W^{1,p}_0(D) \cap S_p(D) \times W^{m+1,p}(D)\}$.

Proof. Let us write

$$T w = \Pi_p \Delta w,$$

so that $T : W^{m+2,p}(D) \cap W^{1,p}_0(D) \cap S_p(D) \to W^{m,p}(D) \cap S_p(D)$ has a continuous inverse (see Lemma 4.2.1). Writing

$$Q_\lambda w = -T^{-1} \Pi_p \left( R \frac{\partial w}{\partial x_1} + \lambda w \right),$$

observe that the solution operator for the equation

$$\Pi_p \left( \Delta u - \frac{\partial u}{\partial x_1} - \lambda u - f \right) = 0 \quad (4.20)$$

is formally given by $(I + Q_\lambda)^{-1} T^{-1} \Pi_p f$. We claim that $(I + Q_\lambda)^{-1} T^{-1} \Pi_p$ is a continuous operator $W^{m,p}(D) \to W^{m+2,p}(D) \cap W^{1,p}_0(D) \cap S_p(D)$ for $\lambda \in \Sigma \cup \{0\}$.

Our claim is justified by showing that $(I + Q_\lambda)^{-1} : W^{m+2,p}(D) \cap W^{1,p}_0(D) \cap S_p(D) \to W^{m+2,p}(D) \cap W^{1,p}_0(D) \cap S_p(D)$. Since $T : W^{m+1,p}(D) \cap S_p(D) \to W^{m+3,p}(D) \cap W^{1,p}_0(D) \cap S_p(D)$ is continuous, using the compact embedding $W^{m+3,p}(D) \hookrightarrow W^{m+2,p}(D)$ one finds that $Q_\lambda$ is compact as an operator from $W^{m+2,p}(D) \cap W^{1,p}_0(D) \cap S_p(D)$ into itself. In view of Fredholm alternative theory, it therefore suffices to show that $(I + Q_\lambda)$ is injective.

Suppose that $g \in W^{m+2,p}(D) \cap W^{1,p}_0(D) \cap S_p(D)$ satisfies

$$(I + Q_\lambda)g = 0. \quad (4.21)$$
Equation (4.21) holds if and only if \( T(I + Q_\lambda)g = 0 \). Using the definitions of \( T \) and \( Q_\lambda \), we find that

\[
\Pi_p \left( \Delta g - \frac{\partial g}{\partial x_1} - \lambda g \right) = 0,
\]

which by Proposition 4.2.2 implies that

\[
\Delta g - \frac{\partial g}{\partial x_1} - \nabla \tilde{q} - \lambda g = 0 \tag{4.22}
\]

in \( D \) for some \( \tilde{q} \in W^{m+1,p}(D) \). Since \( D \) is bounded and \( p \geq 2 \), the spaces \( W^{s,p}(D) \) are continuously embedded in \( W^{s,2}(D) \) for all non-negative integers \( s \). In particular we find \( g \in W^{2,2}(D) \cap W^{1,2}_0(D) \cap S_2(D), \tilde{q} \in W^{1,2}(D) \).

We take the scalar product of (4.22) with the complex conjugate \( g \) and integrate over \( \Omega \) to obtain

\[
0 = \| \nabla g \|^2_{L^2(D)} + \text{Re} \lambda \| g \|^2_{L^2(D)} + i \left( \text{Im} \lambda \| g \|^2_{L^2(D)} + \text{Im} R \int_D \frac{\partial g}{\partial x_1} \cdot \bar{g} \, dx \right), \tag{4.23}
\]

where we have used the fact that

\[
\int_D \nabla \tilde{q} \cdot \bar{g} \, dx = 0
\]

(\( \Pi_2 \) is an orthogonal projection). Equating the real parts of equation (4.23), we find \( \nabla g = 0 \) in \( D \) for \( \lambda = 0 \), which implies that \( g = 0 \) since \( g = 0 \) on \( \partial D \). Now suppose \( \lambda \neq 0 \). Equating the imaginary parts, one finds that

\[
|\text{Im} \lambda|^2 \| g \|^4_{L^2(D)} = \left| \text{Im} R \int_D \frac{\partial g}{\partial x_1} \cdot \bar{g} \, dx \right|^2 \leq R^2 \| \nabla g \|^2_{L^2(D)} \| g \|^2_{L^2(D)} \leq -R^2 \text{Re} \lambda \| g \|^4_{L^2(D)},
\]

so that \( g = 0 \) for \( \lambda \in \Sigma \).

Since \( (I + Q_\lambda)^{-1}T^{-1}\Pi_p f \) solves (4.20), it follows from Proposition 4.2.2 that there exists a function \( \tilde{q} \in W^{m+1,p}(D) \) such that \( (I + Q_\lambda)^{-1}T^{-1}\Pi_p f \) and \( \tilde{q} \) satisfy the equations (4.16)–(4.18). For \( c \in \mathbb{C} \) we set

\[
d = \frac{c}{|\text{vol}(D)|}.
\]

The solution operators \( U_{\lambda,D} : W^{m,p}(D) \times \mathbb{C} \to W^{m+2,p}(D) \cap W^{1,p}_0(D) \cap S_p(D) \) and \( P_{\lambda,D} : W^{m,p}(D) \times \mathbb{C} \to W^{m+1,p}(D) \) are then given by \( U_{\lambda,D}[f,c] = (I+Q_\lambda)^{-1}T^{-1}\Pi_p f \) and \( P_{\lambda,D}[f,c] = \tilde{q} + d \).

\[\blacksquare\]
4.2.2 The Oseen problem in free space

Next the Oseen problem in free space is considered. For this we draw upon a classical result on the $L^p$ boundedness of Fourier-multiplier operators (the proof of which can be found in Stein [46, ch. 4, Theorem 3] for example).

**Theorem 4.2.4** Suppose there exists a function $m \in C^2(\mathbb{R}^3 \setminus \{0\})$ and a constant $C > 0$ such that

$$|k|^{|\alpha|} |\partial^\alpha m(k)| \leq C$$

for all multi-indices $|\alpha| \leq 2$ and $k = (k_1, k_2, k_3) \in \mathbb{R}^3$. The Fourier-multiplier operator $L$, given by $L[f] = F^{-1}[m(k)F[f]]$, maps $L^p(\mathbb{R}^3)$ continuously into itself.

**Lemma 4.2.5** Suppose $\lambda \in \Sigma$ and $f \in L^p(\mathbb{R}^3)$. The Oseen problem

$$\Delta u - R \frac{\partial u}{\partial x_1} - \nabla q - \lambda u = f, \quad \text{in } \mathbb{R}^3, \quad (4.24)$$

$$\text{div } u = 0, \quad \text{in } \mathbb{R}^3, \quad (4.25)$$

has a solution $u = U_\lambda[f] \in W^{2,p}(\mathbb{R}^3) \cap S_p(\mathbb{R}^3)$, $q = P[f] \in \hat{W}^{1,p}(\mathbb{R}^3)$ (unique up to additive constants in $q$), where $U_\lambda : L^p(\mathbb{R}^3) \rightarrow W^{2,p}(\mathbb{R}^3) \cap S_p(\mathbb{R}^3)$ and $[P] : L^p(\mathbb{R}^3) \rightarrow \hat{W}^{1,p}(\mathbb{R}^3)/\sim$ are continuous operators defined by

$$U_\lambda[f] = F^{-1}\left[ \frac{k(k \cdot \hat{f}(k))}{(\lambda + |k|^2 + ik_1 R)|k|^2} - \frac{\hat{f}(k)}{\lambda + |k|^2 + ik_1 R} \right], \quad (4.26)$$

$$P[f] = F^{-1}\left[ \frac{i k \cdot \hat{f}(k)}{|k|^2} \right]. \quad (4.27)$$

Here $\hat{f}(k)$ denotes the Fourier transform of $f(x)$.

**Proof.** Applying the Fourier transform to (4.24), (4.25) we obtain

$$-|k|^2 \hat{u} - ik_1 R \hat{u} - i k \hat{q} - \lambda \hat{u} = \hat{f}, \quad \text{in } \mathbb{R}^3,$$

$$i k \cdot \hat{u} = 0,$$

where $\hat{u}$, $\hat{q}$, denote the Fourier transforms of $u$ and $q$, respectively. It follows that $\hat{u}$, $\hat{q}$ are given by the formulae

$$\hat{u} = \frac{k(k \cdot \hat{f}(k))}{(\lambda + |k|^2 + ik_1 R)|k|^2} - \frac{\hat{f}(k)}{\lambda + |k|^2 + ik_1 R}, \quad \hat{q} = \frac{i k \cdot \hat{f}(k)}{|k|^2}. \quad (4.28)$$

Calculating the necessary derivatives and using the estimate

$$|\lambda + |k|^2 + ik_1 R| \geq C(1 + |k|^2), \quad k \in \mathbb{R}^3,$$
we find that the Fourier multipliers associated with \( U_\lambda[f] \) satisfy the conditions of Theorem 4.2.4, so that
\[
\|U_\lambda[f]\|_{W^{2,p}(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)}
\]
(clearly \( U_\lambda[f] \in S_p(\mathbb{R}^3) \) also). Finally, observe that the Helmholtz-Weyl decomposition of \( f \) is precisely
\[
f = (f - \nabla \mathcal{P}[f]) + \nabla \mathcal{P}[f]
\in S_p(\mathbb{R}^3) \quad G_p(\mathbb{R}^3)
\]
(see the paragraph below (4.9)), so that \( \mathcal{P}[f] \in \dot{W}^{1,p}(\mathbb{R}^3) \) with
\[
\|\mathcal{P}[f]\|_{\dot{W}^{1,p}(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)}.
\]

\[\square\]

In the sections that follow we make use of the following result. It concerns a particular type of function space on which the function \( \mathcal{P} \) defined in (4.27) defines a continuous operator.

**Lemma 4.2.6** Consider the space \( L^p_0(D) \) defined by
\[
L^p_0(D) := \{ f \in L^p(\mathbb{R}^3) : f(x) = 0 \text{ for all } x \notin D \},
\]
where \( D \) is a fixed bounded domain in \( \mathbb{R}^3 \). The operator \( \mathcal{P} \) defined by (4.27) maps \( L^p_0(D) \) continuously into \( W^{1,p}(\mathbb{R}^3) \).

**Proof.** It has already been established that \( \nabla \mathcal{P}[f] \in L^p(\mathbb{R}^3) \). It remains to show that \( \mathcal{P}[f] \) belongs to \( L^p(\mathbb{R}^3) \). Here the main difficulty is the fact that the Fourier transform of \( \mathcal{P}[f] \) contains a singularity at the origin. This singularity can be isolated by introducing a monotone decreasing smooth cut-off function \( \psi(k) \) such that
\[
\psi(k) = \begin{cases} 
1, & |x| \leq 1, \\
0, & |x| \geq 2;
\end{cases} \quad (4.29)
\]
we write \( \mathcal{P}[f] = \mathcal{P}^1[f] + \mathcal{P}^2[f] \), where
\[
\mathcal{P}^1[f] = \mathcal{F}^{-1} \left[ \frac{\psi(k)ik \cdot \hat{f}(k)}{|k|^2} \right], \quad \mathcal{P}^2[f] = \mathcal{F}^{-1} \left[ \frac{(1 - \psi(k))ik \cdot \hat{f}(k)}{|k|^2} \right].
\]

Since the function \( \mathcal{P}^2[f] \) satisfies the conditions of Theorem 4.2.4, it follows that \( \|\mathcal{P}^2[f]\|_{L^p(\mathbb{R}^3)} \leq \|f\|_{L^p(\mathbb{R}^3)} \). Turning to \( \mathcal{P}^1[f] \), note that it has the equivalent form
\[
\mathcal{P}^1[f] = \sum_{n=1}^{3} \Pi_n \ast f_n, \quad (4.30)
\]
where
\[ \Pi_n(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y_n}{|y|^3} \tilde{\psi}(x-y) \, dy, \]
and \( \tilde{\psi}(x) = \mathcal{F}^{-1}[\psi(k)] \). We show that \( \Pi_n \) belongs to the space \( L^{\tilde{p}}(\mathbb{R}^3) \) for \( \tilde{p} > 3/2 \).

Note that \( \tilde{\psi} \) is a function of Schwartz class, so that
\[ |\Pi_n(x)| \leq C \int_{\mathbb{R}^3} \frac{1}{|y|^2 (1 + |x-y|)^4} \, dy. \]

A bound for the integral on the right hand side is obtained by observing that \( 1 + |x-y| \geq (1 + |x|)/2 \) when \( |y| \leq (1 + |x|)/2 \). Splitting the integral, one finds that
\[ \int_{|y| \leq (|x|+1)/2} \frac{1}{|y|^2(1 + |x-y|)^4} \, dy \leq \frac{C}{(1 + |x|)^3}, \]
\[ \int_{|y| \geq (|x|+1)/2} \frac{1}{|y|^2(1 + |x-y|)^4} \, dy \leq \left( \frac{2}{(1 + |x|)} \right)^2 \int_{\mathbb{R}^3} \frac{1}{(1 + |y|)^4} \, dy. \]

It follows that
\[ |\Pi_n(x)| \leq \frac{C}{(1 + |x|)^2}, \]
from which we obtain
\[ \int_{\mathbb{R}^3} |\Pi_n(x)|^{\tilde{p}} \, dx \leq C \int_{\mathbb{R}^3} \frac{1}{(1 + |x|)^{2\tilde{p}}} \, dx \]
\[ = C \int_{0}^{\infty} \frac{r^2}{(1 + r)^{2\tilde{p}}} \, dr \]
\[ < \infty \]
for \( \tilde{p} > 3/2 \).

An application of Young's inequality for convolutions (see for example Adams & Fournier [1, Corollary 2.25]) now shows that
\[ \| P^1[f] \|_{L^p(\mathbb{R}^3)} \leq C \| \Pi_n \|_{L^p(\mathbb{R}^3)} \| f \|_{L^1(\mathbb{R}^3)} \]
\[ \leq C \| f \|_{L^1(\mathbb{R}^3)} \]
\[ \leq C \| f \|_{L^p(\mathbb{R}^3)}, \]
where we have used the fact that the support of \( f \) lies in \( D \). \qed
4.2.3 The exterior Oseen problem for inhomogeneities with compact support

Consider the exterior problem

\[ \Delta u - R \frac{\partial u}{\partial x_1} - \nabla q - \lambda u = f, \quad \text{in } \Omega, \quad (4.31) \]
\[ \text{div } u = 0, \quad \text{in } \Omega, \quad (4.32) \]
\[ u = 0, \quad \text{on } \partial \Omega, \quad (4.33) \]

where \( \lambda \in \Sigma \), \( f \in L^p_0(\Omega_b) \), \( \Omega_b = B_b(0) \cap \Omega \), and where the positive constant \( b \) is chosen large enough so that the obstacle \( \mathcal{O} \) is contained within the ball of radius \( b - 2 \) (see Figure 4.3).

We aim to construct a solution to (4.31)–(4.33) given in terms of the solution operators.
\[ U_\lambda : L^p_0(\Omega_b) \rightarrow W^{2,p}(\mathbb{R}^3) \cap S_p(\mathbb{R}^3), \quad \mathcal{P} : L^p_0(\Omega_b) \rightarrow W^{1,p}(\mathbb{R}^3) \] (see Lemma 4.2.5 and Lemma 4.2.6) outside the ball of radius \( b - 1 \), and in terms of the operators \( U_{\lambda,\Omega_b} : L^p_0(\Omega_b) \rightarrow W^{2,p}(\Omega_b) \cap W^{1,p} \cap S_p(\Omega_b) \), \( \mathcal{P}_{\lambda,\Omega_b} : L^p_0(\Omega_b) \rightarrow W^{1,p}(\Omega_b) \) defined by

\[
U_{\lambda,\Omega_b}[f] = U_{\lambda,\Omega_b} \left[ f, \int_{B_b(0)} \mathcal{P}[f] \, dx \right], \\
\mathcal{P}_{\lambda,\Omega_b} = \mathcal{P}_{\lambda,\Omega_b} \left[ f, \int_{B_b(0)} \mathcal{P}[f] \, dx \right]
\]

(see Lemma 4.2.3) inside the ball of radius \( b - 2 \). Our procedure involves using the smooth monotone decreasing cut-off function \( \phi(x) \) such that

\[
\phi(x) = \begin{cases} 
1, & |x| \leq b - 2, \\
0, & |x| \geq b - 1.
\end{cases} \tag{4.34}
\]

To ensure that the constructed function satisfies the divergence free condition (4.32), we make use of the following results (the proof of Lemma 4.2.7 can be found in Giga & Sohr [13] for example).

**Lemma 4.2.7** Let \( D \) be a bounded domain with smooth boundary and \( m \) be a non-negative integer. Consider the space

\[
\mathbb{W}^{m,p}_0(D) := \left\{ u \in W^{m,p}_0(D) : \int_D u(x) \, dx = 0 \right\}.
\]

There exists a continuous linear operator \( B : \mathbb{W}^{m,p}_0(D) \rightarrow \mathbb{W}^{m+1,p}_0(D) \) such that

\[
\text{div } B[u] = u.
\]

**Lemma 4.2.8** Consider the cut-off function \( \phi \) given by (4.34), and the annular region

\[
D_b = \left\{ x \in \mathbb{R}^3 : b - 2 \leq |x| \leq b - 1 \right\}.
\]

(i) If \( u \in W^{2,p}(\Omega_b) \) is solenoidal and \( u = 0 \) on \( \partial \Omega_b \), then \( \nabla \phi \cdot u \in \mathbb{W}^{2,p}_0(D_b) \).

(ii) If \( u \in W^{2,p}(\mathbb{R}^3) \) is solenoidal, then \( \nabla \phi \cdot u \) belongs to the space \( \mathbb{W}^{2,p}_0(D_b) \).

**Proof.** Consider the case (i). Observe that \( \nabla \phi \cdot u \in \mathbb{W}^{2,p}_0(D_b) \). Since

\[
\text{div } (\phi u) = \nabla \phi \cdot u + \phi \text{div } u,
\]

one finds that

\[
\int_{D_b} \nabla \phi \cdot u \, dx = \int_{B_{b-1}(0) \cap \Omega_b} \text{div } (\phi u) \, dx = \int_{\partial B_{b-1}(0) \cup \partial \Omega_b} \phi u \cdot n \, dS = 0,
\]

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and therefore $\nabla \phi \cdot u \in W^{2,p}_{0}(D_0)$. Part (ii) is established using the same argument. 

Write

$$V_\lambda[f] = (1-\phi)U_\lambda[f] + \phi U_{\lambda,\Omega_b}[f] + G_\lambda[f], \quad (4.35)$$
$$Q_\lambda[f] = (1-\phi)P[f] + \phi P_{\lambda,\Omega_b}[f], \quad (4.36)$$

where

$$G_\lambda[f] = E(B[\nabla \phi \cdot U_\lambda[f]] - B[\nabla \phi \cdot U_{\lambda,\Omega_b}[f]]),$$

and $E : W^{3,p}_0(D_0) \to W^{3,p}_0(\Omega)$ is the extension by zero operator. The above analysis shows that the functions (4.35), (4.36) define continuous operators $L^p_0(\Omega_b) \to W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap S_p(\Omega)$ and $L^p_0(\Omega_b) \to W^{1,p}(\Omega)$, respectively. Furthermore, they satisfy

$$\Delta V_\lambda[f] - R\frac{\partial V_\lambda[f]}{\partial x_1} - \nabla Q_\lambda[f] - \lambda V_\lambda[f] = f + S_\lambda f, \quad (4.37)$$

in $\Omega$, where $S_\lambda : L^p_0(\Omega_b) \to W^{1,p}_0(\Omega_b)$ is defined by

$$S_\lambda f = -2\nabla \phi \cdot \nabla U_\lambda[f] - \Delta \phi U_\lambda[f] + R\frac{\partial \phi}{\partial x_1} U_\lambda[f]$$
$$+ 2\nabla \phi \cdot \nabla U_{\lambda,\Omega_b}[\phi f] + \Delta \phi U_{\lambda,\Omega_b}[f] - R\frac{\partial \phi}{\partial x_1} U_{\lambda,\Omega_b}[f]$$
$$+ \Delta G_\lambda[f] - R\frac{\partial G_\lambda[f]}{\partial x_1} - \lambda G_\lambda[f] + \nabla \phi P[f] - \nabla \phi P_{\lambda,\Omega_b}[f]. \quad (4.38)$$

Suppose $(I+S_\lambda)$ has a continuous inverse from $L^p_0(\Omega_b)$ to itself. Then $u = V^*_\lambda[f]$, $q = Q^*_\lambda[f]$ solve the exterior problem (4.31)–(4.33), where $V^*_\lambda : L^p_0(\Omega_b) \to W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap S_p(\Omega)$, $Q^*_\lambda : L^p_0(\Omega_b) \to W^{1,p}(\Omega)$ are the modified operators given by

$$V^*_\lambda[f] = V_\lambda[(I+S_\lambda)^{-1}f], \quad (4.39)$$
$$Q^*_\lambda[f] = Q_\lambda[(I+S_\lambda)^{-1}f]. \quad (4.40)$$

In the remaining analysis we confirm that $(I+S_\lambda) : L^p_0(\Omega_b) \to L^p_0(\Omega_b)$ has a continuous inverse. Since $W^{1,p}_0(\Omega_b)$ is compactly embedded in $L^p_0(\Omega_b)$, the operator $S_\lambda$ defines a compact mapping on $L^p_0(\Omega_b)$. In view of Fredholm alternative theory, it therefore suffices to show that $(I+S_\lambda)$ is injective.

We make use of following lemma. It is concerned with extending divergence free vectors in $\Omega$ into the obstacle $O$ to create divergence free vectors in $\mathbb{R}^3$.

**Proposition 4.2.9** Let $m$ be a non-negative integer. Suppose that $u \in W^{m,p}(\Omega)$ satisfies $\text{div} \, u = 0$ in $\Omega$ and $u = 0$ on $\partial \Omega$. There exists an extension $v \in W^{m,p}(\mathbb{R}^3)$ of $u$ into the obstacle with the properties that $u = v$ in $\Omega$, and $\text{div} \, v = 0$ in $O$.
Proof. The existence of $u \in W^{m,p}(\mathbb{R}^3)$ such that $u = \overline{u}$ in $\Omega$ follows by a standard construction (see, for example, Adams & Fournier [1, Theorem 5.22]). Since $\text{div} \overline{u} = \text{div} u = 0$ in $\Omega$, we find that $\text{div} \overline{u} \in W^{m-1,p}_0(\mathcal{O})$. Furthermore

$$\int_{\mathcal{O}} \text{div} \overline{u} \, dx = -\int_{\partial \mathcal{O}} n \cdot \overline{u} \, dS = 0,$$

where the unit normal $n$ points in the outward direction to $\partial \Omega$, so that $\text{div} \overline{u} \in W^{m-1,p}_0(\mathcal{O})$.

The function $v$ is defined by $v = u - B[\text{div} u]$. □

The following lemma discusses the solution to the homogeneous exterior Oseen problem.

**Lemma 4.2.10** Suppose that $\lambda \in \Sigma$, and that $u \in W^{2,p}(\Omega)$, $q \in \hat{W}^{1,p}(\Omega)$ satisfy the homogeneous exterior Oseen problem

$$\Delta u - R \frac{\partial u}{\partial x_1} - \nabla q - \lambda u = 0, \quad \text{in } \Omega,$$

$$\text{div} u = 0, \quad \text{in } \Omega,$$

$$u = 0, \quad \text{on } \partial \Omega. \quad (4.41)$$

The functions $u$ and $q$ are respectively zero and zero up to additive constants.

**Proof.** Proposition 4.2.9 asserts the existence of an extension $v \in W^{2,p}(\mathbb{R}^3)$ of $u$ into the obstacle such that $v = u$ in $\Omega$ and $\text{div} v = 0$ in $\mathcal{O}$. Because $q$ belongs to $W^{1,p}(\Omega_b)$, and a standard construction yields an extension $\tilde{q} \in W^{1,p}(\mathbb{R}^3)$ of $q|_{\Omega_b}$ (Adams & Fournier [1, Theorem 5.22]). The formula

$$Q(x) = \begin{cases} \tilde{q}(x), & x \in \overline{\Omega}, \\ q(x), & x \in \Omega \end{cases}$$

defines an extension $Q \in \hat{W}^{1,p}(\mathbb{R}^3)$ of $q$ into the obstacle. Define

$$f = \Delta v - R \frac{\partial v}{\partial x_1} - \nabla Q - \lambda v \quad (4.44)$$

and note that $\text{supp} f \subset \overline{\Omega} \subset B_b(0)$, so that $f \in L^p(B_b(0))$, and hence $f \in L^2(B_b(0))$ (because $L^p(B_b(0))$ is continuously embedded in $L^2(B_b(0))$).

Because $v$ and $Q$ satisfy (4.44) and $\text{div} v = 0$ in free space, they are given by the formulae $v = U_\lambda[f]$ and $[Q] = [P[f]]$ (see Lemma 4.2.5). It follows that $u = U_\lambda[f]$, $[q] = [P[f]]$ in $\Omega$, and in particular $u \in W^{2,2}(\Omega)$ and $\nabla q \in L^2(\Omega)$. Taking the scalar product of (4.41) with $\overline{u}$ and integrating over $\Omega$ yields

$$0 = \|\nabla u\|_{L^2(\Omega)}^2 + \text{Re} \lambda \|u\|_{L^2(\Omega)}^2 + i \left\{ \text{Im} \lambda \|u\|_{L^2(\Omega)}^2 + \text{Im} R \int_{\partial \Omega} \frac{\partial u}{\partial x_1} \cdot \overline{u} \, dx \right\},$$

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and the argument given below equation (4.23) shows that $\mathbf{u} = \mathbf{0}$ for $\lambda \in \Sigma$. By equation (4.41) we then have $\nabla q = 0$. □

Finally, we discuss the injectivity of the operator $(I + S_\lambda)$.

**Lemma 4.2.11** For $\lambda \in \Sigma$ the operator $(I + S_\lambda) : L^p_0(\Omega_b) \to L^p_0(\Omega_b)$ is injective.

**Proof.** Suppose that $f \in L^p_0(\Omega_b)$ satisfies

$$(I + S_\lambda)f = 0. \tag{4.45}$$

It follows from equation (4.37) and Lemma 4.2.10 that $V_\lambda[f] = 0$, $Q_\lambda[f] = 0$ (the additive constant is zero in this case since $Q_\lambda[f] \in W^{1,p}(\Omega)$), that is

$$(1 - \phi)U_\lambda[f] + \phi U_{\lambda,\Omega_b}[f] + G_\lambda[f] = 0, \tag{4.45}$$

$$(1 - \phi)P[f] + \phi P_{\lambda,\Omega_b}[f] = 0, \tag{4.46}$$

in $\Omega$. Since $\text{supp} \ G_\lambda[f] \subset D_b$ the above equations imply that

$$U_\lambda[f](x) = 0, \quad P[f](x) = 0, \quad |x| \geq b - 1, \tag{4.47}$$

$$U_{\lambda,\Omega_b}[f](x) = 0, \quad P_{\lambda,\Omega_b}[f](x) = 0, \quad |x| \leq b - 2. \tag{4.48}$$

It follows from (4.47) that $\mathbf{u}_1 = U_\lambda[f]$ and $q_1 = P[f]$ belong to respectively $W^{2,p}(B_b(0))$ and $W^{1,p}(B_b(0))$, and solve the Oseen boundary value problem

$$\Delta \mathbf{u} - R \frac{\partial \mathbf{u}}{\partial x_1} - \nabla q - \lambda \mathbf{u} = f \quad \text{in } B_b(0),$$

$$\text{div} \mathbf{u} = 0 \quad \text{in } B_b(0),$$

$$\mathbf{u} = 0 \quad \text{on } \partial B_b(0),$$

$$\int_{B_b(0)} q \, d\mathbf{x} = \int_{B_b(0)} P[f] \, d\mathbf{x}. \tag{4.49}$$

Set $\mathbf{u}_2 = U_{\lambda,\Omega_b}[f]$ in $\Omega_b$, $\mathbf{u}_2 = 0$ in $\mathcal{O}$ and $q_2 = P_{\lambda,\Omega_b}[f]$ in $\Omega_b$, $q_2 = 0$ in $\mathcal{O}$. The condition (4.48) ensures that $\mathbf{u}_2 \in W^{2,p}(B_b(0))$ and $q_2 \in W^{1,p}(B_b(0))$. Since $\mathbf{u}_2$, $q_2$ also solve the above boundary value problem, it follows from the uniqueness of solutions to the Oseen problem in a bounded domain (see Lemma 4.2.3) that $\mathbf{u}_1 = \mathbf{u}_2$ and $q_1 = q_2$ in $B_b(0)$.

It follows from equations (4.47), (4.48) that $U_\lambda[f](x) = 0$, $P[f](x) = 0$ for $|x| \geq b - 1$ and $|x| \leq b - 2$, and that

$$G_\lambda[f] = B[\nabla \phi \cdot U_\lambda[f]] - B[\nabla \phi \cdot U_{\lambda,\Omega_b}[f]] = 0 \quad \text{in } D_b,$$

which, along with the equations (4.45), (4.46), implies that $U_\lambda[f](x) = 0$, $P[f](x) = 0$ for all $x$ in $\mathbb{R}^3$. Finally, the relation

$$\Delta U_\lambda[f] - R \frac{\partial U_\lambda[f]}{\partial x_1} - \nabla P[f] - \lambda U_\lambda[f] = f$$

implies that $f = 0$. □
4.2.4 Spectral properties of the Oseen operator

In this section we establish that \( \Sigma \) is a subset of the resolvent set of \( A_R \). This result follows from the solvability of the exterior Oseen problem

\[
\Delta u - R \frac{\partial u}{\partial x_1} - \nabla q - \lambda u = f, \quad \text{in } \Omega, \tag{4.49}
\]

\[
\text{div } u = 0, \quad \text{in } \Omega, \tag{4.50}
\]

\[
u = 0, \quad \text{on } \partial \Omega, \tag{4.51}
\]

for general \( f \in L^p(\Omega) \).

**Theorem 4.2.12** Suppose that \( f \in L^p(\Omega) \) and \( \lambda \in \Sigma \). The boundary-value problem (4.49)–(4.51) has a solution \( u \in W^{2,p}(\Omega) \), \( q \in \hat{W}^{1,p}(\Omega) \) (unique up to additive constants in \( q \)) such that

\[
\|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)},
\]

\[
\|q\|_{\hat{W}^{1,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}. \tag{4.52}
\]

**Proof.** Consider the smooth cut-off function \( \phi \) given by (4.34), and define

\[
v = (1 - \phi)U_{\lambda}[f] + B[\nabla \phi \cdot U_{\lambda}[f]],
\]

\[
\tilde{q} = (1 - \phi) \left( P[f] - \frac{1}{|\text{vol}(B_b(0))|} \int_{B_b(0)} P[f] \, dx \right),
\]

where \( f \) has been extended by zero inside \( \Omega \). Observe that \( v, \tilde{q} \) belong to the spaces \( W^{2,p}(\Omega), \hat{W}^{1,p}(\Omega) \), respectively, and satisfy

\[
\Delta v - R \frac{\partial v}{\partial x_1} - \nabla \tilde{q} - \lambda v = f + R_{\lambda} f, \quad \text{in } \Omega,
\]

\[
\text{div } v = 0, \quad \text{in } \Omega,
\]

\[
v = 0, \quad \text{on } \partial \Omega,
\]

where \( R_{\lambda} : L^p(\Omega) \to L^p_0(\Omega_b) \) is given by

\[
R_{\lambda}f = -\phi f - 2\nabla \phi \cdot \nabla U_{\lambda}[f] - \Delta \phi U_{\lambda}[f] + R \frac{\partial \phi}{\partial x_1} U_{\lambda}[f]
+ \Delta B[\nabla \phi \cdot U_{\lambda}[f]] - R \frac{\partial}{\partial x_1} B[\nabla \phi \cdot U_{\lambda}[f]] - \lambda B[\nabla \phi \cdot U_{\lambda}[f]]
+ \nabla \phi \left( P[f] - \frac{1}{|\text{vol}(B_b(0))|} \int_{B_b(0)} P[f] \, dx \right).
\]

Note that each term on the right hand side defines a continuous linear mapping from \( L^p(\Omega) \) to \( L^p_0(\Omega_b) \) (the continuity of the last term follows from Poincaré's inequality), so that

\[
\|R_{\lambda}f\|_{L^p_0(\Omega_b)} \leq C \|f\|_{L^p(\Omega)}. \tag{4.52}
\]

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Now set
\[
\tilde{V}_\lambda[f] = v - V_\lambda^*[R\lambda f], \\
\tilde{Q}_\lambda[f] = q - Q_\lambda^*[R\lambda f],
\]
where \( V_\lambda^* : L^p_0(\Omega_b) \to W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap S_p(\Omega) \) and \( Q_\lambda^* : L^p_0(\Omega_b) \to W^{1,p}(\Omega) \) are the continuous operators given in (4.39), (4.40), respectively. Observe that \( u = \tilde{V}_\lambda[f], \ q = \tilde{Q}_\lambda[f] \) satisfy (4.49)–(4.51). Furthermore, by (4.52) it follows that \( u = \tilde{V}_\lambda[f], \ q = \tilde{Q}_\lambda[f] \) belong to \( W^{2,p}(\Omega), \tilde{W}^{1,p}(\Omega) \), respectively, where
\[
\| \tilde{V}_\lambda[f] \|_{W^{2,p}(\Omega)} \leq C \| f \|_{L^p(\Omega)}, \\
\| [\tilde{Q}_\lambda[f]] \|_{\tilde{W}^{1,p}(\Omega/\sim)} \leq C \| f \|_{L^p(\Omega)}.
\]

The uniqueness of the solution follows from Lemma 4.2.10.

\[ \square \]

### 4.3 Inversion of the Oseen operator

The goal of this section is the following result concerning the invertibility of the Oseen operator. This result is used in Section 4.4 below to verify hypothesis (ii) in the Hopf-Iooss theorem (Theorem 2.5.1).

**Theorem 4.3.1** Let \( 2 < p < 4 \) and consider the linear mapping \( B_R : X_p(\Omega) \to S_p(\Omega) \) and the nonlinear operator \( N(\cdot, R) : X_p(\Omega) \to S_p(\Omega) \) given respectively by

\[
B_R v = -\Pi_p R \left( (v \cdot \nabla) u_0 + (u_0 \cdot \nabla)v \right), \\
N(v, R) = -\Pi_p R \left( (v \cdot \nabla)v \right).
\]

The Oseen operator \( A_R : X_p(\Omega) \to S_p(\Omega) \) is injective, invertible on the range of \( B_R \) and \( N(\cdot, R) \), and satisfies the estimates

\[
\| A_R^{-1} B_R v \|_{X_p(\Omega)} \leq C \| v \|_{W^{1,p}(\Omega)}, \\
\| A_R^{-1} N(v, R) \|_{X_p(\Omega)} \leq C \| v \|_{W^{2,p}(\Omega)}^2
\]

for every \( v \in X_p(\Omega) \).

In order to prove Theorem 4.3.1 we consider the Oseen boundary-value problem

\[
\Delta u - R \frac{\partial u}{\partial x_1} - \nabla q = f, \quad \text{in } \Omega, \\
\text{div } u = 0, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial \Omega,
\]

where \( u = \tilde{V}_\lambda[f], \ q = \tilde{Q}_\lambda[f] \) satisfy (4.49)–(4.51). Furthermore, by (4.52) it follows that \( u = \tilde{V}_\lambda[f], \ q = \tilde{Q}_\lambda[f] \) belong to \( W^{2,p}(\Omega), \tilde{W}^{1,p}(\Omega) \), respectively, where
\[
\| \tilde{V}_\lambda[f] \|_{W^{2,p}(\Omega)} \leq C \| f \|_{L^p(\Omega)}, \\
\| [\tilde{Q}_\lambda[f]] \|_{\tilde{W}^{1,p}(\Omega/\sim)} \leq C \| f \|_{L^p(\Omega)}.
\]

The uniqueness of the solution follows from Lemma 4.2.10.

\[ \square \]
where \( f \in L^p(\Omega) \), and seek a solution \( u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap S_{\rho}(\Omega) \), \( q \in W^{1,p}(\Omega) \). The calculations in Section 4.2 suggest that a solution is given by \( u = \tilde{V}_0[f] \), \( q = \tilde{Q}_0[f] \), where

\[
\tilde{V}_0[f] = (1 - \phi)U_0[f] + B[\nabla \phi \cdot U_0[f]] - (1 - \phi)U_0[ (I + S_0)^{-1} R_0 f ] \nonumber \\
- \phi U_{0, \Omega_b}[ (I + S_0)^{-1} R_0 f ] - G_0[ (I + S_0)^{-1} R_0 f ], \tag{4.56}
\]

\[
\tilde{Q}_0[f] = (1 - \phi)P[f] - (1 - \phi)P[ (I + S_0)^{-1} R_0 f ] - \phi P_{0, \Omega_b}[ (I + S_0)^{-1} R_0 f ] \tag{4.57}
\]

and

\[
G_0[f] = E \left( B[\nabla \phi \cdot U_0[f]] - B[\nabla \phi \cdot U_{0, \Omega_b}[f]] \right), \nonumber \\
S_0 f = -2\nabla \phi \cdot \nabla U_0[f] - \Delta \phi U_0[f] + R \frac{\partial \phi}{\partial x_1} U_0[f] \nonumber \\
+ 2\nabla \phi \cdot \nabla U_{0, \Omega_b}[f] + \Delta \phi U_{0, \Omega_b}[f] - R \frac{\partial \phi}{\partial x_1} U_{0, \Omega_b}[f] \nonumber \\
+ \Delta G_0[f] - R \frac{\partial G_0[f]}{\partial x_1} + \nabla \phi P[f] - \nabla \phi P_{0, \Omega_b}[f], \tag{4.58}
\]

\[
R_0 f = -\phi f - 2\nabla \phi \cdot \nabla U_0[f] - \Delta \phi U_0[f] + R \frac{\partial \phi}{\partial x_1} U_0[f] \nonumber \\
+ \Delta B[\nabla \phi \cdot U_0[f]] - R \frac{\partial}{\partial x_1} B[\nabla \phi \cdot U_0[f]] \nonumber \\
+ \nabla \phi \left( P[f] - \frac{1}{\text{vol}(B_b(0))} \int_{B_b(0)} P[f] \, dx \right); \tag{4.59}
\]

here

\[
U_0[f] = F^{-1} \left[ \frac{k(k \cdot \hat{f}(k))}{(|k|^2 + ik_1 R)|k|^2} - \frac{\hat{f}(k)}{|k|^2 + ik_1 R} \right], \tag{4.60}
\]

\[
P[f] = F^{-1} \left[ \frac{ik \cdot \hat{f}(k)}{|k|^2} \right] \tag{4.61}
\]

formally solve the corresponding problem in free space, while

\[
U_{0, \Omega_b} = U_{0, \Omega_b} \left[ \cdot, \int_{B_b(0)} P[\cdot] \, dx \right], \tag{4.62}
\]

\[
P_{0, \Omega_b} = P_{0, \Omega_b} \left[ \cdot, \int_{B_b(0)} P[\cdot] \, dx \right] \tag{4.63}
\]

are the solution operators for the corresponding problem in the bounded domain \( \Omega_b \). We prove Theorem 4.3.1 by verifying this suggestion for certain choices of \( f \), namely

\[
f_1 = (v \cdot \nabla)v, \quad f_2 = (v \cdot \nabla)u_0, \quad f_3 = (u_0 \cdot \nabla)v,
\]

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where \( \mathbf{v} \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap S_p(\Omega) \). We proceed by showing that the functions \( \tilde{V}_0[f^j], \tilde{Q}_0[f^j] \) belong to the respective spaces \( W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap S_p(\Omega), \tilde{W}^{1,p}(\Omega) \), and satisfy

\[
\left\| \tilde{V}_0[f^j] \right\|_{W^{2,p}(\Omega)} + \left\| \tilde{Q}_0[f^j] \right\|_{\tilde{W}^{1,p}(\Omega)} \lesssim C \left\| \mathbf{v} \right\|_{W^{2,p}(\Omega)}, \quad j = 1, \tag{4.64}
\]

\[
\left\| \tilde{V}_0[f^j] \right\|_{W^{2,p}(\Omega)} + \left\| \tilde{Q}_0[f^j] \right\|_{\tilde{W}^{1,p}(\Omega)} \lesssim C \left\| \mathbf{v} \right\|_{W^{1,p}(\Omega)}, \quad j = 2, 3, \tag{4.65}
\]

for \( 2 < p < 4 \); a separate argument is used to demonstrate uniqueness of the solution.

To establish (4.64), (4.65) we re-examine the theory presented in Section 4.2 for \( \lambda = 0 \). First note that the analysis of the Oseen problem in bounded domains (Section 4.2.1) is valid for \( \lambda = 0 \). In particular it follows from Lemma 4.2.3 that (4.62), (4.63) define continuous operators \( \mathcal{U}_{0,\Omega_b} : L^p(\Omega_b) \to W^{2,p}(\Omega_b) \cap W^{1,p}_0(\Omega_b) \cap S_p(\Omega_b), \mathcal{P}_{0,\Omega_b} : L^p(\Omega_b) \to \tilde{W}^{1,p}(\Omega_b) \) such that \( \mathbf{u} = \mathcal{U}_{0,\Omega_b}[f], q = \mathcal{P}_{0,\Omega_b}[f] \) uniquely solve the boundary-value problem

\[
\Delta \mathbf{u} - R \frac{\partial \mathbf{u}}{\partial x_1} - \nabla q = f, \quad \text{in } \Omega_b,
\]

\[
\text{div } \mathbf{u} = 0, \quad \text{in } \Omega_b,
\]

\[
\mathbf{u} = 0, \quad \text{on } \partial \Omega_b,
\]

and

\[
\int_{\Omega} q \, dx = \int_{\Omega_b} \mathcal{P}[f] \, dx
\]

for any \( f \in L^p(\Omega_b) \). In Section 4.3.1 and 4.3.2 below we generalise Sections 4.2.2 (free space) and 4.2.3 (inhomogeneities in \( L^p(\Omega_b) \)) to the case \( \lambda = 0 \). Note that \( \mathcal{P} \) does not depend upon \( \lambda \), and recall that \( [\mathcal{P}] : L^p(\mathbb{R}^3) \to \tilde{W}^{1,p}(\mathbb{R}^3)/ \sim \) and \( \mathcal{P} : L^p(\Omega_b) \to W^{1,p}(\mathbb{R}^3) \) are continuous operators for all \( p \geq 2 \) (see Lemma 4.2.5 and 4.2.6). Our discussion therefore involves obtaining information about the properties of \( \mathcal{U}_0 \) only. These strands are brought together in the proof of Theorem 4.3.1 in Section 4.3.3.

### 4.3.1 The Oseen problem in free space (\( \lambda = 0 \))

In this section we prove the following result.

**Theorem 4.3.2** Consider the Oseen boundary-value problem

\[
\Delta \mathbf{u} - R \frac{\partial \mathbf{u}}{\partial x_1} - \nabla q = f, \quad \text{in } \mathbb{R}^3, \tag{4.66}
\]

\[
\text{div } \mathbf{u} = 0, \quad \text{in } \mathbb{R}^3, \tag{4.67}
\]

where \( f \in L^p_0(B_0(0)) \).
\[(i)\] For \(p > 2\) equations (4.60), (4.61) define continuous operators \(U_0 : L^p_0(B_0(0)) \to W^{2,p}(\mathbb{R}^3) \cap S_p(\mathbb{R}^3), \) \(P : L^p_0(B_0(0)) \to W^{1,p}(\mathbb{R}^3)\) such that \(u = U_0[f], q = P[f]\) is the unique solution to the above boundary-value problem in the class \(\{(u, q) \in W^{2,p}(\mathbb{R}^3) \cap S_p(\mathbb{R}^3) \times W^{1,p}(\mathbb{R}^3)\}\).

\[(ii)\] Equations (4.60), (4.61) define continuous operators

\[
[U_0] : L^2_0(B_0(0)) \to \left(\dot{W}^{2,2}(\mathbb{R}^3)/\sim\right) \cap S_2(\mathbb{R}^3),
\]

\[
P : L^2_0(B_0(0)) \to W^{1,2}(\mathbb{R}^3)
\]

such that

\[
\nabla U_0[f] \in L^2(\mathbb{R}^3), \quad \lim_{\kappa \to \infty} \frac{1}{\kappa} \int_{\kappa \leq |x| \leq 2\kappa} |U_0[f]|^2 \, dx = 0,
\]

and \(u = U_0[f], q = P[f]\) is the unique solution to the above boundary-value problem in the class

\[
\left\{(u, q) \in \dot{W}^{2,2}(\mathbb{R}^3) \cap S_2(\mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3), \lim_{\kappa \to \infty} \frac{1}{\kappa} \int_{\kappa \leq |x| \leq 2\kappa} |U_0[f]|^2 \, dx = 0 \right\}.
\]

The assertions concerning uniqueness follows from the observation that any solution of (4.66), (4.67) is given by the formulae (4.60), (4.61). Since the mapping properties of \(P\) have already been established, it remains to discuss \(U_0\). Because the Fourier multiplier associated with \(U_0[f]\) possess a singularity at the origin, it is convenient to use the cut-off function \(\psi\) given in (4.29) and write \(U_0[f] = U_0^1[f] + U_0^2[f]\), where

\[
U_0^1[f] = \mathcal{F}^{-1} \left[ \psi(k) k(k \cdot \hat{f}(k)) \frac{\psi(k) \hat{f}(k)}{|k|^2 + i k_1 R} \right], \quad (4.68)
\]

\[
U_0^2[f] = \mathcal{F}^{-1} \left[ \frac{(1 - \psi(k)) k(k \cdot \hat{f}(k))}{(|k|^2 + i k_1 R)|k|^2} - \frac{(1 - \psi(k)) \hat{f}(k)}{|k|^2 + i k_1 R} \right]. \quad (4.69)
\]

An application of Theorem 4.2.4 yields the following result.

**Lemma 4.3.3** For each \(p \geq 2\) the operator \(U_0^2\) given by (4.69) maps \(L^p(\mathbb{R}^3)\) continuously into \(W^{2,p}(\mathbb{R}^3)\).

We now discuss the function \(U_0^1[f]\) for \(f \in L^p_0(B_0(0))\). Observe that in component form it can be written as

\[
(U_0^1[f])_m = \sum_{n=1}^{3} \chi_{m,n} * f_n, \quad m = 1, 2, 3, \quad (4.70)
\]

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where \((U_0^1[\mathbf{f}])_n, f_n, n = 1, 2, 3\), denote the components of \(U_0^1[\mathbf{f}]\), \(\mathbf{f}\), respectively, and the functions \(\chi_{m,n}\) are given by

\[
\chi_{m,n} = \mathcal{F}^{-1} \left[ \frac{\psi(k)(\delta_{mn} - k_m k_n |k|^{-2})}{|k|^2 + i k_1 R} \right]
\]

(\(\delta_{m,n}\) denotes the Kronecker delta). We show that the functions \(\chi_{m,n}\) belong to the space \(L^{\tilde{p}}(\mathbb{R}^3)\) for all \(\tilde{p} > 2\), and that its derivatives \(\partial^\alpha \chi_{m,n}, |\alpha| = 1\), belong to \(L^{\tilde{p}}(\mathbb{R}^3)\) for \(4/3 < \tilde{p} < 2\) (see Lemma 4.3.6 below). The following two propositions state some preliminary results required for this purpose.

**Proposition 4.3.4** The inverse Fourier transform

\[
\mathcal{F}^{-1} \left[ \frac{\delta_{mn} - k_m k_n |k|^{-2}}{|k|^2 + ik_1 R} \right]
\]

is given explicitly by

\[
(\delta_{mn}\Delta - \partial_m \partial_n) E(x),
\]

where

\[
E(x) = \frac{1}{4\pi R} \int_0^{\tilde{R}(|x| - x_1)} \frac{1 - e^{-t}}{t} dt.
\]

Furthermore, for each \(0 \leq \delta \leq 1\) there exists a constant \(C_\delta\) such that

\[
|\partial^\alpha E(x)| \leq \frac{C_\delta}{(1 + (|x| - x_1)^4 |x|)^\delta}, \quad |\alpha| = 2,
\]

\[
|\partial^\alpha E(x)| \leq \frac{C_\delta}{(1 + (|x| - x_1)^4 |x|^2)^\delta} + \frac{C_\delta}{(1 + (|x| - x_1)^4 + 1/2 |x|^{3/2})^\delta}, \quad |\alpha| = 3.
\]

**Proof.** The first assertion follows by a straightforward calculation. With regards to the estimates, we set

\[
h(x) = \frac{R}{2}(|x| - x_1), \quad f(t) = \frac{1 - e^{-t}}{t},
\]

so that

\[
\partial_m h(x) = \frac{R}{2} \left( \frac{x_1}{|x|} - \delta_{1m} \right),
\]

\[
\partial_m \partial_n h(x) = \frac{R}{2|x|} \left( \delta_{mn} - \frac{x_m x_n}{|x|^2} \right),
\]

\[
\partial_m \partial_n \partial_r h(x) = -\frac{R}{2|x|^2} \left( \delta_{mn} \frac{x_r}{|x|} + \delta_{nr} \frac{x_m}{|x|} + \delta_{rm} \frac{x_n}{|x|} - 3 \frac{x_m x_n x_r}{|x|^3} \right),
\]

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and
\[
\left| \frac{d^k f}{dt^k}(t) \right| \leq \frac{C_{\delta_k}}{(1 + |t|)^{\delta_k}}
\]
for \(0 \leq \delta_k \leq k + 1, k \geq 0\), where \(C_{\delta_k}\) is a constant that depends on \(\delta_k\).

Using the relation
\[
|\partial_m h(x)| \leq \sqrt{R} \left( \frac{h(x)}{|x|} \right)^{\frac{1}{2}},
\]
for \(m = 1, 2, 3\), and the fact that
\[
\partial_m \partial_n E(x) = \frac{1}{4\pi R} \left( f(h(x))\partial_m \partial_n h(x) + f'(h(x))\partial_m h(x)\partial_n h(x) \right),
\]
we find that
\[
|\partial_m \partial_n E(x)| \leq \frac{C_{\delta}}{(1 + h(x))^{\delta}|x|},
\]
where we have chosen \(\delta_0 = \delta\) and \(\delta_1 = \delta + 1\). Turning to the second estimate, we use the formula
\[
\partial_m \partial_n E(x) = \frac{1}{4\pi R} \left[ f(h(x))\partial_m \partial_n \partial_r h(x) + f'(h(x))\partial_m h(x)\partial_n h(x) + \partial_r h(x)\partial_m \partial_n h(x) \right.
\]
\[
+ f''(h(x))\partial_m h(x)\partial_n h(x)\partial_r h(x) \bigg],
\]
and proceed as above, choosing \(\delta_0 = \delta\), \(\delta_1 = \delta + 1\) and \(\delta_2 = \delta + 2\).

**Proposition 4.3.5** For each \(0 \leq \delta \leq 1\) there exists a constant \(C_{\delta}\) such that
\[
|\partial^{\alpha} \chi_{m,n}(x)| \leq \frac{C_{\delta}}{(1 + h(x))^{\delta + |\alpha|/2}(1 + |x|)^{1+|\alpha|/2}}, \quad |\alpha| \leq 1.
\]

**Proof.** Write \(\tilde{\psi} = \mathcal{F}^{-1}[\psi]\), so that
\[
\chi_{m,n}(x) = \int_{\mathbb{R}^3} \left\{ (\delta_{m,n}\Delta - \partial_m \partial_n) E(y)\tilde{\psi}(x - y) \right\} dy.
\]

The results of Proposition 4.3.4 imply that
\[
|\partial^{\alpha} \chi_{m,n}(x)| \leq \int_{\mathbb{R}^3} \frac{C_{\delta}}{(1 + h(y))^{\delta + |\alpha|/2}} \frac{1}{|y|^{2|\alpha|}} |\tilde{\psi}(x - y)| \, dy
\]
\[
+ \int_{\mathbb{R}^3} \frac{C_{\delta}}{(1 + h(y))^{\delta + |\alpha|/2}} \frac{1}{|y|^{1+|\alpha|/2}} |\tilde{\psi}(x - y)| \, dy.
\]
The observation \( h(x) \leq h(x - y) + h(y) \) yields

\[
\frac{1}{1 + h(y)} \leq \frac{1 + h(x - y)}{1 + h(x)},
\]

and using this estimate and the fact that \( \tilde{\psi} \) is a function of Schwartz class, we find that

\[
|\partial^\alpha \chi_{m,n}(x)| \leq \frac{C_\delta}{(1 + h(x))^{\delta|\alpha|}} \int_{\mathbb{R}^3} \frac{1}{1 + |x - y|} \frac{1}{|y|^{2|\alpha|}} dy + \frac{C_\delta}{(1 + h(x))^{\delta + |\alpha|/2}} \int_{\mathbb{R}^3} \frac{1}{1 + |x - y|} \frac{1}{|y|^{1 + |\alpha|/2}} dy. \tag{4.71}
\]

Turning to the integrals on the right-hand side, observe that

\[
\int_{|y| \leq (|x| + 1)/2} \frac{1}{|y|^{\tilde{p}}(1 + |x - y|)^4} dy \leq \frac{C}{(1 + |x|)^{1 + \tilde{p}}},
\]

\[
\int_{|y| \geq (|x| + 1)/2} \frac{1}{|y|^{\tilde{p}}(1 + |x - y|)^4} dy \leq \left( \frac{2}{(1 + |x|)} \right) \tilde{p} \int_{\mathbb{R}^3} \frac{1}{1 + |y|} dy, \tag{4.72}
\]

for \( 0 < \tilde{p} < 3 \) (because \( 1 + |x - y| \geq (1 + |x|)/2 \) when \( |y| \leq (1 + |x|)/2 \). The assertion follows by combining inequalities (4.71) and (4.72). \( \square \)

**Lemma 4.3.6**

(i) The functions \( \chi_{m,n} \) belong to the space \( L^{\tilde{p}}(\mathbb{R}^3) \), where \( \tilde{p} > 2 \).

(ii) The functions \( \partial^\alpha \chi_{m,n} \), \( |\alpha| = 1 \), belong to the space \( L^{\tilde{p}}(\mathbb{R}^3) \), where \( 4/3 < \tilde{p} < 2 \).

**Proof.** (i) A direct calculation yields

\[
\int_{\mathbb{R}^3} |\chi_{m,n}(x)|^{\tilde{p}} dx \leq C \int_{\mathbb{R}^3} \frac{1}{(1 + h(x))^{\delta(\delta + 1)}(1 + |x|)^{\delta}} dx \\
\leq C \int_{\mathbb{R}^3} \frac{1}{h(x)^{\delta(\delta + 1)}(1 + |x|)^{\tilde{p}}} dx \\
= C \int_0^\infty \frac{r^2}{(1 + r)^{\tilde{p}\delta}} dr \int_0^\pi \frac{\sin \theta}{(1 - \cos \theta)^{\tilde{p}\delta}} d\theta.
\]

It follows that \( \chi_{m,n} \in L^{\tilde{p}}(\mathbb{R}^3) \) for \( \tilde{p}(\delta + 1) > 3 \) and \( \tilde{p}\delta < 1 \), that is when \( \tilde{p} \) and \( \delta \) satisfy

\[
\frac{3}{\delta + 1} < \tilde{p} < \frac{1}{\delta}, \quad \delta \in [0, 1/2).
\]

Considering all admissible values of \( \tilde{p} \) and \( \delta \), one finds that \( \chi_{m,n}(x) \in L^{\tilde{p}}(\mathbb{R}^3) \) for all \( \tilde{p} > 2 \).
The same method yields
\[
\int_{\mathbb{R}^3} |\partial^\alpha \chi_{m,n}(x)| \hat{p} \, dx \leq C \int_0^\infty \frac{r^2}{(1 + r)^{3\hat{p}/2 + \hat{p}(\delta + 1/2)}} \, dr \int_0^{\pi} \frac{\sin \theta}{(1 - \cos \theta)^{\hat{p}(\delta + 1/2)}} \, d\theta,
\]
so that \( \partial^\alpha \chi_{m,n}(x) \in L^\hat{p}(\mathbb{R}^3) \) for \( 3\hat{p}/2 + \hat{p}(\delta + 1/2) > 3 \) and \( \hat{p}(\delta + 1/2) < 1 \), that is when \( \hat{p} \) and \( \delta \) satisfy
\[
\frac{3}{\delta + 2} < \hat{p} < \frac{1}{\delta + 1/2}, \quad \delta \in [0, 1/4).
\]
It follows that \( \partial^\alpha \chi_{m,n}(x) \in L^\hat{p}(\mathbb{R}^3) \) for all \( 4/3 < \hat{p} < 2 \). \qed

The continuity of \( U_0 : L^p_0(B_b(0)) \to W^{2,p}(\mathbb{R}^3) \) for \( p > 2 \) now follows from the following result and Lemma 4.3.3.

**Lemma 4.3.7** Suppose that \( p > 2 \). The operator \( U_0^1 \) maps \( L^p_0(B_b(0)) \) continuously into \( W^{2,p}(\mathbb{R}^3) \).

**Proof.** Using the representation (4.70) and Young’s inequality for convolutions, we find that
\[
\| U_0^1[f] \|_{L^p(\mathbb{R}^3)} \leq C \sum_{m,n=1}^3 \| \chi_{m,n} \|_{L^p(\mathbb{R}^3)} \| f \|_{L^1_0(B_b(0))}
\]
for \( p > 2 \), where we have used Lemma 4.3.6 and the embedding \( L^p_0(B_b(0)) \hookrightarrow L^1_0(B_b(0)) \). An application of Theorem 4.2.4 yields
\[
\| \partial^\alpha U_0^1[f] \|_{L^p(\mathbb{R}^3)} \leq C \| f \|_{L^p_0(B_b(0))}, \quad |\alpha| = 2;
\]
we conclude that \( U_0^1 \) maps \( L^p_0(B_b(0)) \) continuously into \( W^{2,p}(\mathbb{R}^3) \). \qed

Finally, we present the corresponding result for \( p = 2 \)

**Lemma 4.3.8** The operator \([U_0]\) maps \( L^2_0(B_b(0)) \) continuously into \( \tilde{W}^{2,2}(\mathbb{R}^3)/\sim \). Furthermore, for each \( f \in L^2_0(B_b(0)) \) the function \( U_0[f] \) satisfies \( \nabla U_0[f] \in L^2(\mathbb{R}^3) \) and
\[
\lim_{\kappa \to \infty} \frac{1}{\kappa} \int_{\kappa \leq |x| \leq 2\kappa} |U_0[f]|^2 \, dx = 0.
\]
Proof. In view of Lemma 4.3.3 it suffices to consider the operator $\mathcal{U}_0^1$. Using the representation (4.70), Young’s inequality and the embedding $L^2_0(B_0(0)) \hookrightarrow L^3_0(B_0(0))$, one finds that
\[
\left\| \mathcal{U}_0^1[f] \right\|_{L^\infty(\mathbb{R}^3)} \leq C \sum_{m,n=1}^3 \|\chi_{m,n}\|_{L^\infty(\mathbb{R}^3)} \|f\|_{L^2_0(B_0(0))},
\]
and since
\[
\left\| \partial^\alpha \mathcal{U}_0^1[f] \right\|_{L^\infty(\mathbb{R}^3)} \leq C \sum_{m,n=1}^3 \left\| \partial^\alpha \chi_{m,n} \right\|_{L^\infty(\mathbb{R}^3)} \|f\|_{L^2_0(B_0(0))}, \quad |\alpha| = 1,
\]
and since
\[
\left\| \partial^\alpha \mathcal{U}_0^1[f] \right\|_{L^2(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{R}^3)} = C' \|f\|_{L^2_0(B_0(0))}, \quad |\alpha| = 2
\]
(a direct application of Theorem 4.2.4), it follows that $\mathcal{U}_0^1[f] \in \tilde{W}^{2,2}(\mathbb{R}^3)$. Similarly, we find that
\[
\left\| \nabla \mathcal{U}_0^1[f] \right\|_{L^2(\mathbb{R}^3)} \leq C \sum_{m,n=1}^3 \left\| \nabla \chi_{m,n} \right\|_{L^s(\mathbb{R}^3)} \|f\|_{L^2_0(B_0(0))},
\]
where $1/2 = 1/q - (1-1/s)$. Choosing $1 < q < 4/3$, so that $4/3 < s < 2$, and using Lemma 4.3.6 and the embedding $L^2_0(B_0(0)) \hookrightarrow L^3_0(B_0(0))$, one concludes that $\nabla \mathcal{U}_0^1[f] \in L^2(\mathbb{R}^3)$.

With regards to the growth condition, we use Hölder’s inequality to obtain
\[
\frac{1}{\kappa} \int_{\kappa \leq |x| \leq 2\kappa} \left| \mathcal{U}^1_0[f] \right|^2 \, dx \leq \frac{1}{\kappa} \left( \int_{\kappa \leq |x| \leq 2\kappa} \, dx \right)^{1/3} \left( \int_{\kappa \leq |x| \leq 2\kappa} \left| \mathcal{U}^1_0[f] \right|^3 \, dx \right)^{2/3}
\leq \frac{C}{\kappa} \left( \int_{\kappa \leq |x| \leq 2\kappa} r^2 \, dr \right)^{1/3} \left( \int_{\kappa \leq |x| \leq 2\kappa} \left| \mathcal{U}^1_0[f] \right|^3 \, dx \right)^{2/3}
\leq C \left( \int_{\kappa \leq |x| \leq 2\kappa} \left| \mathcal{U}^1_0[f] \right|^3 \, dx \right)^{4/3}.
\]
The limit (4.74) follows from this calculation and the estimate
\[
\left\| \mathcal{U}^1_0[f] \right\|_{L^3(\mathbb{R}^3)} \leq C \|\chi_{m,n}\|_{L^3(\mathbb{R}^3)} \|f\|_{L^2_0(B_0(0))} \leq C \|f\|_{L^2_0(B_0(0))},
\]
which shows that $\mathcal{U}^1_0[f]$ belongs to $L^3(\mathbb{R}^3)$ for $f \in L^2_0(B_0(0))$. \hfill \Box

### 4.3.2 The Oseen problem for inhomogeneities in $L^p_0(\Omega_b)$ ($\lambda = 0$)

The next step is to consider the Oseen boundary-value problem (4.53)–(4.55) for inhomogeneities $f \in L^p_0(\Omega_b)$ using the method developed in Section 4.2.3: we show that
\((I + S_0)^{-1}\), where \(S_0\) is given by (4.58), defines a continuous mapping from \(L^p_0(\Omega_b)\) to itself. Lemma 4.3.7 and 4.3.8 imply that \(S_0\) defines a mapping from \(L^p_0(\Omega_b)\) to \(W^{1,p}_0(\Omega_b)\), while the compactness of the embedding \(W^{1,p}_0(\Omega_b) \hookrightarrow L^p_0(\Omega_b)\) implies that \(S_0\) as a mapping \(L^p_0(\Omega_b) \rightarrow L^p_0(\Omega_b)\) is compact. Using the usual Fredholm argument, one deduces that \((I + S_0) : L^p_0(\Omega_b) \rightarrow L^p_0(\Omega_b)\) has a continuous inverse provided it is injective. To prove injectivity, we show that the homogenous Oseen problem corresponding to (4.53)–(4.55) has only the trivial solution.

**Lemma 4.3.9** Consider the homogeneous Oseen problem

\[
\begin{align*}
\Delta u - R \frac{\partial u}{\partial x_1} - \nabla q &= 0, & \text{in } \Omega, \\
\text{div } u &= 0, & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega.
\end{align*}
\]  

(i) Suppose \(u \in \hat{W}^{2,2}(\Omega), q \in \hat{W}^{1,2}(\Omega)\) satisfy the boundary-value problem (4.75)–(4.77). Furthermore, suppose that \(\nabla u \in L^2(\Omega)\) and

\[
\lim_{\kappa \to \infty} \frac{1}{\kappa} \int_{\kappa \leq |x| \leq 2\kappa} |u|^2 \, dx = 0,
\]  

and that \([q] = [\bar{q}]\) for some \(\bar{q} \in W^{1,2}(\Omega)\).

The functions \(u\) and \(\bar{q}\) are identically zero.

(ii) Suppose \(p > 2\) and \(u \in W^{2,p}(\Omega), q \in \hat{W}^{1,p}(\Omega)\) satisfy the boundary-value problem (4.75)–(4.77). The functions \(u\) and \(q\) are respectively zero and zero up to additive constants.

**Proof.** (i) Consider the cut-off function \(\psi\) given in (4.29) and set \(\psi_\kappa(x) = \psi(x/\kappa)\). Taking the scalar product of (4.75) with \(\psi_\kappa u\) and integrating by parts, we find that

\[
- \frac{1}{\kappa} \int_\Omega \sum_{j=1}^3 \frac{\partial \psi(x/\kappa)}{\partial x_j} \frac{\partial u}{\partial x_j} \cdot \bar{u} \, dx - \int_\Omega \nabla u \cdot \psi_\kappa \nabla \bar{u} \, dx - i \text{Im} \int_\Omega \frac{\partial u}{\partial x_1} \cdot \psi_\kappa \bar{u} \, dx
\]

\[
+ \frac{1}{\kappa} \int_\Omega \frac{\partial \psi(x/\kappa)}{\partial x_1} u \cdot \bar{u} \, dx + \frac{1}{\kappa} \int_\Omega \nabla \psi(x/\kappa) \cdot \bar{q} u \, dx = 0.
\]  

Taking the limit \(\kappa \to \infty\) and considering the real part of (4.79), we find that

\[
\|\nabla u\|_{L^2(\Omega)} = 0,
\]  

which in view of (4.78) implies that \(u = 0\). Equation (4.75) then shows that \(\nabla q = 0\), so that \(\bar{q} = 0\).
The arguments given above equation (4.44) assert the existence of \( v \in W^{2,p}(\mathbb{R}^3) \), \( Q \in \hat{W}^{1,p}(\mathbb{R}^3) \) such that \( v = u, Q = q \) in \( \Omega \) and \( \text{div} \, v = 0 \) in \( \mathbb{R}^3 \). Define
\[
f = \Delta v - R \frac{\partial v}{\partial x_1} - \nabla Q \tag{4.80}
\]
and note that \( \text{supp} \, f \subset \overline{\Omega} \), so that in particular \( f \in L^2_0(B_6(0)) \). Since \( v \) and \( Q \) solve (4.80) and \( \text{div} \, v = 0 \) in free space, they are given by the formulae \( v = U_0[f] \) and \( [Q] = [P[f]] \), where \( P[f] \in W^{1,2}(\mathbb{R}^3) \) and \( v \in W^{2,2}(\mathbb{R}^3) \) with \( \nabla v \in L^2(\mathbb{R}^3) \),
\[
\lim_{\kappa \to \infty} \frac{1}{\kappa} \int_{\kappa \leq |x| \leq 2\kappa} |v|^2 \, dx = 0
\]
(Theorem 4.3.2 (ii)). Because \( v = u, Q = q \) in \( \Omega \), part (i) asserts that \( u \) and \( q \) are respectively zero and zero up to additive constants. \( \square \)

**Lemma 4.3.10** The operator \((I + S_0) : L^p_0(\Omega_b) \to L^p_0(\Omega_b)\) is injective for \( p \geq 2 \).

**Proof.** Observe that \((I + S_0)f, f \in L^p_0(\Omega_b)\), satisfies
\[
\Delta V_0[f] - R \frac{\partial V_0[f]}{\partial x_1} - \nabla Q_0[f] = (I + S_0)f,
\]
where \( V_0[f], Q_0[f] \) are given by
\[
V_0[f] = (1 - \phi)U_0[f] + \phi U_{0,\Omega_b}[f] + G_0[f],
\]
\[
Q_0[f] = (1 - \phi)P[f] + \phi P_{0,\Omega_b}[f].
\]
Using Theorem 4.3.2 one finds that for \( f \in L^p_0(\Omega_b), p > 2 \), the functions \( Q_0[f], V_0[f] \) belong to the respective spaces \( W^{1,p}(\Omega), W^{2,p}(\Omega) \), while for \( f \in L^2_0(\Omega_b) \) these functions belong to the respective spaces \( W^{1,2}(\Omega), W^{2,2}(\Omega) \) with \( \nabla V_0[f] \in L^2(\Omega) \) and
\[
\lim_{\kappa \to \infty} \frac{1}{\kappa} \int_{\kappa \leq |x| \leq 2\kappa} |V_0[f]|^2 \, dx = 0.
\]
Suppose that \((I + S_0)f = 0\), so that \( V_0[f], Q_0[f] \) satisfy the criteria of Lemma 4.3.9; it follows that \( V_0[f] = 0, [Q_0[f]] = [0] \) (and hence \( Q_0[f] = 0 \)). The argument given in the proof of Lemma 4.2.11 now shows that \( f = 0 \), and hence \((I + S_0) : L^p_0(\Omega_b) \to L^p_0(\Omega_b)\) is injective. \( \square \)
4.3.3 Inversion of the Oseen operator on the range of $B_R$ and $N(\cdot, R)$

Finally, we consider the Oseen boundary-value problem (4.53)–(4.55) with inhomogeneities $f^1 = (v \cdot \nabla)v$, $f^2 = (v \cdot \nabla)u_0$ and $f^3 = (u_0 \cdot \nabla)v$, where $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap S_p(\Omega)$. Note that

$$\|f^1\|_{L^p(\Omega)} \leq C \|v\|_{L^\infty(\Omega)} \|\nabla v\|_{L^p(\Omega)}$$

which follows from the embedding $W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, and

$$\|f^2\|_{L^p(\Omega)} \leq C \|\nabla u_0\|_{L^\infty(\Omega)} \|v\|_{L^p(\Omega)},$$

$$\|f^3\|_{L^p(\Omega)} \leq C \|u_0\|_{L^\infty(\Omega)} \|\nabla v\|_{L^p(\Omega)}.$$  

Using the above estimates and the fact that $[\mathcal{P}] : L^p(\mathbb{R}^3) \to \dot{W}^{1,p(\mathbb{R}^3)}/\sim$ is continuous, we find that

$$\|\mathcal{P}[f^1]\|_{\dot{W}^{1,p(\mathbb{R}^3)}/\sim} \leq C \|v\|_{W^{2,p}(\Omega)}, \quad \|\mathcal{P}[f^j]\|_{\dot{W}^{1,p(\mathbb{R}^3)}/\sim} \leq C \|v\|_{W^{1,p}(\mathbb{R}^3)}, \quad j = 2, 3.$$  

**Lemma 4.3.11** For $2 < p < 4$ the functions $U_0[f^j]$, $j = 1, 2, 3$, belong to the space $W^{2,p}(\mathbb{R}^3)$ and satisfy the estimates

$$\|U_0[f^1]\|_{W^{2,p}(\mathbb{R}^3)} \leq C \|v\|_{W^{2,p}(\Omega)},$$

$$\|U_0[f^j]\|_{W^{2,p}(\mathbb{R}^3)} \leq C \|v\|_{W^{1,p}(\mathbb{R}^3)}, \quad j = 2, 3.$$  

**Proof.** It follows from Lemma 4.3.3 and the estimates (4.81)–(4.83) that $U_0^2[f^j]$, $j = 1, 2, 3$, satisfies the given estimates. We now derive the corresponding results for $U_0^1[f^j]$, $j = 1, 2, 3$.

We write $U_0^1[f^j]$ in component form as

$$(U_0^1[f^j])_m = \sum_{n=1}^3 \chi_{m,n} \ast f_n^j, \quad m = 1, 2, 3,$$  

where $f_n^j$ denote the components of $f^j$, $j = 1, 2, 3$. Consider the function $f^1 = (v \cdot \nabla)v$. Applying Young’s inequality for convolutions to (4.85), we find that

$$\left\|\sum_{n=1}^3 \chi_{m,n} \ast f_n^1\right\|_{L^p(\mathbb{R}^3)} = \left\|\sum_{k,n=1}^3 \chi_{m,n} \ast v_k \partial_k v_n\right\|_{L^p(\mathbb{R}^3)}$$

$$= \left\|\sum_{k,n=1}^3 \chi_{m,n} \ast \partial_k (v_k v_n)\right\|_{L^p(\mathbb{R}^3)}.$$  

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Lemma 4.3.6 that
\[ v \] where for the last term on the right-hand side of (4.59) we have used Poincaré’s inequality and the estimates (4.84). Since \( U_0 : L^p(\Omega_b) \to W^{1,p}(\mathbb{R}^3) \) is a continuous mapping (see Section 4.3.1), and \( (I + S_0) : L^p(\Omega_b) \to L^p(\Omega_b) \) has a continuous inverse (see Section 4.3.2), we conclude from formulae (4.56), (4.57) that \( \tilde{V}_0[\tilde{f}] \in W^{2,p}(\Omega), \tilde{Q}_0[\tilde{f}] \in W^{1,p}(\Omega), \) for \( 2 < p < 4 \), and satisfy the estimates (4.64), (4.65). Finally, Lemma 4.3.9 (ii) implies that (4.53)–(4.55) has only the trivial solution for \( \tilde{f} = 0 \). These remarks complete the proof of Theorem 4.3.1.
4.4 Application of the Hopf-Iooss theorem

It remains to confirm that the evolution equation (4.10) satisfies the spectral hypotheses of the Hopf-Iooss theorem (Theorem 2.5.1). These hypotheses are concerned with the linearized operator \( L_R : X_p(\Omega) \to S_p(\Omega) \) given by

\[
L_R u = A_R u + B_R u, \tag{4.86}
\]

where \( p \in (2, 4) \), \( A_R : X_p(\Omega) \to S_p(\Omega) \) is the Oseen operator given in (4.11), and \( B_R : X_p(\Omega) \to S_p(\Omega) \) is given by

\[
B_R v = -\Pi_p R \left( (v \cdot \nabla)u_0 + (u_0 \cdot \nabla)v \right). \tag{4.87}
\]

We begin with a compactness result.

**Lemma 4.4.1** The operator \( B_R : X_p(\Omega) \to S_p(\Omega) \) given by (4.87) is compact.

**Proof.** Introduce a monotone decreasing smooth cut-off function \( \phi_b \) such that

\[
\phi_b(x) = \begin{cases} 
1, & |x| \leq b - 1, \\
0, & |x| \geq b,
\end{cases}
\]

and define the continuous operator \( P^b : X_p(\Omega) \to X_p(\Omega_b) \) by

\[
(P^b v)(x) = \phi_b(x)v(x),
\]

and the operator \( B^b_R : W^{1,p}(\Omega_b) \cap W_0^{1,p}(\Omega_b) \cap S_p(\Omega_b) \to S_p(\Omega) \) by

\[
B^b_R v = -\Pi_p R \left( (v \cdot \nabla)u_0 + (u_0 \cdot \nabla)v \right).
\]

Using the fact that \( u_0, \nabla u_0 \in L^\infty(\Omega) \) (see Lemma 4.1.1), one finds that \( B^b_R \) is a continuous mapping from \( W^{1,p}(\Omega_b) \cap W_0^{1,p}(\Omega_b) \cap S_p(\Omega_b) \) into \( S_p(\Omega) \); it follows that the composition \( B_R^b I^b P^b : X_p(\Omega) \to S_p(\Omega) \) is compact, where \( I^b : X_p(\Omega_b) \to W^{1,p}(\Omega_b) \cap W_0^{1,p}(\Omega_b) \cap S_p(\Omega_b) \) is the compact embedding operator.

Observe that

\[
\left\| (B_R^b I^b P^b - B_R I) v \right\|_{L_p(\Omega)} \leq C \|v\|_{W^{1,p}(\Omega)} \sum_{|\alpha| \leq 1} \sup_{x \in \Omega} |\partial^\alpha ((\phi_b(x) - 1)u_0(x))|,
\]

where \( I : X_p(\Omega) \to W^{1,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap S_p(\Omega) \) is the continuous embedding operator. It follows that

\[
\left\| (B_R^b I^b P^b - B_R I) \right\|_{X_p(\Omega) \to S_p(\Omega)} \leq C \sup_{|x| > b - 1} \left( |u_0(x)| + |\nabla u_0(x)| \right) \to 0
\]

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as $b \to \infty$ (see Lemma 4.1.1). The assertion in the lemma follows from the fact that the limit of a sequence of compact operators is itself a compact operator (see, for example, Renardy and Rogers [40, Theorem 8.82]). □

It is now possible to establish certain spectral properties of the linear operator $L_R$ using the results obtained in Sections 4.2 and 4.3.

**Lemma 4.4.2** Consider the linear operator $L_R : X_p(\Omega) \to S_p(\Omega)$ given in (4.86).

(i) The spectrum of $L_R$ consists of the essential spectrum of $A_R : X_p(\Omega) \to S_p(\Omega)$ and a countable set of isolated eigenvalues.

(ii) Each non-zero purely imaginary number is either an isolated eigenvalue of $L_R$ or belongs to its resolvent set.

(iii) For each $i\omega, \omega \in \mathbb{R}$, with $|\omega|$ sufficiently large, the resolvent operator $(L_R - i\omega I)^{-1} : S_p(\Omega) \to X_p(\Omega)$ exists and satisfies the estimates

$$
\|(L_R - i\omega I)^{-1}\|_{S_p(\Omega) \to X_p(\Omega)} \leq C, \\
\|(L_R - i\omega I)^{-1}\|_{S_p(\Omega) \to S_p(\Omega)} \leq \frac{C}{|\omega|}.
$$

**Proof.** Part (i) is a consequence of the fact that $B_R$ is a compact perturbation of $A_R$ (see Lemma 4.4.1 and Kato [25, ch. 4, Theorem 5.35]), while part (ii) follows from Lemma 4.2.12, which asserts in particular that $i\mathbb{R} \setminus \{0\} \subset \rho(A_R)$.

It was shown by Borchers & Sohr [5, Lemma 2.2] that the Stokes operator $T : X_p(\Omega) \to S_p(\Omega)$ given by $Tu = \Pi_p \Delta u$ satisfies the estimate

$$
|\omega| \|u\|_{L^p(\Omega)} + \|u\|_{W^2,p(\Omega)} \leq C \|(T - i\omega I)u\|_{L^p(\Omega)}
$$

for each $\omega \in \mathbb{R}$ with $|\omega|$ sufficiently large. Observe that

$$(T - i\omega I)u = (L_R - i\omega I)u + R \Pi_p \left( \frac{\partial u}{\partial x_1} + (u \cdot \nabla)u_0 + (u_0 \cdot \nabla)u \right),$$

so that

$$
\|(T - i\omega I)u\|_{L^p(\Omega)} \leq \|(L_R - i\omega I)u\|_{L^p(\Omega)} + R \Pi_p \left( 1 + \|u_0\|_{L^\infty(\Omega)} \right) \|u\|_{W^1,p(\Omega)} + \|u_0\|_{W^{1,\infty}(\Omega)} \|u\|_{L^p(\Omega)}.
$$
Estimating
\[ \| u \|_{W^{1,p}(\Omega)} \leq \| u \|_{W^{2,p}(\Omega)}^{1/2} \| u \|_{L^p(\Omega)}^{1/2}, \]
where \( 0 < \epsilon < 1 \), we arrive at the bound
\[ |\omega| \| u \|_{L^p(\Omega)} + \| u \|_{W^{2,p}(\Omega)} \leq C \left( \| (L_R - i\omega I) u \|_{L^p(\Omega)} + \| u \|_{W^{2,p}(\Omega)} + \frac{1}{\epsilon} \| u \|_{L^p(\Omega)} \right), \]
and choosing \( \epsilon \) sufficiently small and \( |\omega| \) sufficiently large we find that
\[ |\omega| \| u \|_{L^p(\Omega)} + \| u \|_{W^{2,p}(\Omega)} \leq C \| (L - i\omega I) u \|_{L^p(\Omega)}. \] (4.88)
In particular, this estimate implies that \((L_R - i\omega I)^{-1} : X_p(\Omega) \to S_p(\Omega)\) is injective, so that \(i\omega\) is not an eigenvalue of \(L_R\). It follows from part (ii) that \(i\omega \in \rho(L_R)\), and the resolvent estimates in part (iii) are now a consequence of (4.88).

\[ \square \]

**Lemma 4.4.3** Suppose that \( \lambda = 0 \) is not an eigenvalue of \( L_R \) and consider the equation
\[ L_R u = -N(v, R), \] (4.89)
where \( N(\cdot, R) : X_p(\Omega) \to S_p(\Omega) \) is the nonlinear operator given by
\[ N(v, R) = -\Pi_p R \left( (v \cdot \nabla)v \right). \]
For each \( v \in X_p(\Omega) \) there exists a unique solution \( u \in X_p(\Omega) \) to (4.89) which depends smoothly upon \( v \).

**Proof.** It follows from Theorem 4.3.1 and Lemma 4.4.1 that \( A_R^{-1}B_R \) is a compact mapping from \( X_p(\Omega) \) into itself. According to the usual Fredholm theory, the operator \((I + A_R^{-1}B_R) : X_p(\Omega) \to X_p(\Omega)\) is therefore invertible if and only if it is injective. To demonstrate its injectivity, observe that
\[ (I + A_R^{-1}B_R)f = 0 \]
if and only if
\[ (A_R + B_R) \underbrace{f}_{L_R} = 0, \]
and this equation has the unique solution \( f = 0 \), since \( 0 \) is not an eigenvalue of \( L_R \).
Using the formula
\[ L_R^{-1} = (I + A_R^{-1}B_R)^{-1}A_R^{-1} \]
and Theorem 4.3.1 we conclude that \( L_R \) is continuously invertible on the range of \( N(\cdot, R) \).
\[ \square \]

An existence theory for periodic solutions is completed by showing that
(i) $L_R$ has a plus-minus pair of purely imaginary eigenvalues which depend continuously on $R$ and cross the imaginary axis in a non-tangential and non-resonant fashion as $R$ passes through a critical value;

(ii) 0 is not an eigenvalue of $L_R$.

For a given obstacle $O$ these conditions appear to be verifiable only numerically.
References


