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Numerical Computation of Gas Flow Through an Exhaust Duct: An Investigation of the Exit Boundary Conditions

by

Manases Menang TitahMboh

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of

Doctor of Philosophy of Loughborough University

1997

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Dedication

To Ngong, Akembom Ngong, Nih, Sessie Nikieh, Mamawa, Nikieh, Ninyoh, Aghoh, Sama, Vigny, Ngogne, Vizhu, Viyoff, Vidzeng, Tefungwa, Abong, Nkumeh, ... and all other Mbohnyangs
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Abstract

A numerical investigation of the exit plane boundary conditions of an engine exhaust duct is presented.

The conventional boundary condition which is used in non-linear analysis is the so-called zero-pressure condition. Various forms of implementation of this condition are used to investigate the relative effects upon the error which arises from numerical approximation of the zero-pressure condition. The computational domain is then extended downstream of the exit boundary, to model acoustic radiation into a free or half space without the need for any boundary condition at the duct exit plane.

The Sommerfeld radiation condition is used to set the boundary conditions at a finite far-field location, making it possible for the computational domain to be set at a finite size. Calculations on the extended domain are used to determine the error in the radiated sound levels which is caused by the fundamental inadequacy of the zero-pressure boundary condition in representing the actual conditions at the exit plane.

A modification of the conventional zero-pressure exit boundary condition is used, which gives improved results in the non-linear flow regime, without the need to extend the flow domain downstream of the exit boundary. For calculations on the simple duct domain, the flux-split scheme of Radespiel and Kroll is used to reduce spurious modes of the numerical scheme, which are convected to the exit boundary, so that the solution is improved.

For the different flow domains considered, examples of small-amplitude single-frequency and multiple-frequency disturbances are presented, followed by higher amplitude multiple-frequency engine source examples. The results for small-amplitude disturbances are compared to those from linearised frequency-domain acoustic analysis. Exit plane velocity profiles and far-field noise spectra corresponding to the computed flows are pre-
sented and discussed. Finally, two sets of experimental data, one for a Wankel Rotary Engine and one for a piston engine, are examined against computed data.
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Part I

Problem Specification
Chapter 1

Introduction
1.1 The Internal Combustion Engine

The internal combustion engine works by a cyclic process that results in fuel being burnt and its energy converted into motion. The procedure is to inject a fuel-oxygen mixture into the combustion chamber, compress the mixture, burn the compressed mixture and then emit the exhaust gases. The burning expands the gases, producing a high pressure which drives a piston or rotor as the gases move into the exhaust duct.

The task undertaken here is concerned with the behaviour and flow properties of the exhaust gas during the fourth stage and it is assumed that the exhaust gas is emitted through a circular cylindrical duct attached to the engine’s exhaust port. Thus the gas properties are considered in the exhaust duct and out through its open end.

In the motor industry considerable effort is put into exhaust noise reduction measures. The commonest method is the use of the muffler, usually attached to the exhaust duct. Mufflers reduce noise either reactivity -by reflecting most of the incident sound energy back up the duct- or absorptively. Absorptive mufflers are lined with acoustic materials which absorb most of the incident sound energy. In either case, some of the energy is transmitted and/or reflected. The performance of a muffler, variously referred to as the ‘insertion loss’, ‘transmission loss’, ‘attenuation’ or ‘noise reduction’, can only be accurately determined by testing it. To this end several experiments can be conducted to determine a satisfactory design which meets given requirements. Two key drawbacks to this method are that it is time consuming, hence costly, and the ‘optimum’ design might be missed with such an approach. The experiments cannot test the whole range of the design parameters since they have to be finite in number, see Sridhana and Crocker [1]. A more viable alternative would be to design a scheme whereby the radiated sound pressure at some given distance from
the exit plane is predicted, so that computerising the scheme eliminates most of the measurements otherwise required in the experimental method.

A schematic representation of an exhaust system on an engine source is shown in figure 1.1a. A downpipe connects the engine manifold to the silencer and a tailpipe issues from the last element of the silencer. The downpipe and tailpipe are both of uniform cross-section and the exhaust gas is discharged into the atmosphere from the open end of the tailpipe. The silencer box, as shown in figure 1.1a, is in reality a very complicated system, usually consisting of one or more mufflers and a catalytic converter, together with interconnecting pipes. The separate mufflers generally have complex internal flow patterns through a series of perforated pipes and baffles. Since this thesis is restricted to modelling the process of noise radiation from the exit plane of the tailpipe, it is sufficient to consider the much simpler system of figure 1.1b, namely a tailpipe with a modified source. The noise source of figure 1.1b is effectively the engine source, downpipe and silencer of figure 1.1a. Alternatively, if the source of figure 1.1b is to be the engine source, then it follows that the engine is unsilenced.

Conventionally, modelling of the exhaust process is done by non-linear time domain analysis when used in engine performance calculations, and by linearised frequency domain analysis when used for acoustic calculations. Acoustic analysis requires more accuracy in the prediction pressure and velocity fluctuations than does performance analysis. Non-linear time domain calculations have so far not provided sufficient accuracy for good acoustic analysis. On the other hand, frequency domain calculations provide adequate results only if modelling of the source is not required. In this case, the flow domain is restricted to the duct, and the analytical radiation impedance is used for the boundary condition at the exit plane for a given frequency of sound. The radiation impedance properly accounts
for radiation of waves from the tailpipe exit into free space or half-space, depending on its form, and allows one to determine the far-field noise for each mode. Since the analysis is linear, the sum over all modes gives the total radiated noise.

In time domain analysis of engine exhaust flows a complete model of the nonlinear source can be included in the analysis, with the potential to predict absolute values of noise radiated from the tailpipe. In addition, nonlinear effects due to high sound levels are accurately represented, which can be of significant benefit, particularly for diesel engine applications with long downpipes. If the analysis is restricted to the source and exhaust system, then a boundary condition is required at the exit plane of the tailpipe, and conventionally the zero-pressure fluctuation model of the flow at the exit plane is used, setting the pressure there equal to atmospheric pressure. This is clearly inaccurate, as the source generates high amplitude pressure fluctuations in the duct, but it has proved adequate for accurate predictions of engine performance.

1.2 Problem Description

In this thesis various implementations are considered for the exit plane boundary conditions for nonlinear time domain analysis, including extending the flow domain past the exit plane. The goal is to improve the accuracy of the time domain analysis and hence to provide a unified time domain calculation, useful for the prediction of both engine performance and radiated noise.

The initial goal is to determine the level of noise in free space at some radius $R$ from the centre of the exit plane of the duct, as depicted in figure 1.1. If there is an infinite flange at the duct exit then noise radiates into half-space
whereas for an unflanged duct noise radiates into the full space external to the duct. To analyse this process, the system of Euler equations is integrated in time at a finite set of points in the duct and its surroundings via a suitable set of difference equations.

The model used initially is a cylindrical duct closed at one end by an oscillating, non-yielding piston, representing the source. The waves generated by the piston's sweeps propagate through the duct and radiate from the open end of the duct in a manner approximated by the time integration of the Euler system of equations for gas dynamics. Once the system has settled to a steady oscillation, the amplitude of the waves at points outside the duct but within the hemisphere/sphere are used to calculate numerical values for the perceived noise. For convenience, points on the locus of the hemisphere/sphere centered at the centre of the exit plane and with radius $R$ are averaged, based on the assumption that points at an equal distance from the radiating source, here assumed to be the centre of the exit plane of the duct, receive the same sound intensity. With this assumption, the exhaust outlet is considered as a monopole source.

1.3 The Flow Domain

As mentioned in section 1.2, the physical domain consists of a cylindrical circular duct, one end of which opens into an infinite sphere, or hemisphere in the case where the duct terminates on the outside of an infinite wall. In numerical solutions to this problem, a boundary of the computational domain has to be set at a finite distance, denoted here as the far-field boundary. A boundary condition must then be imposed to ensure that the acoustic waves all travel outward without reflection. The boundary condition at the inlet to the duct is determined by the engine and silencer system under consideration, and it may be noted that the discussion so far
could be applied to any input condition.

1.4 Literature Review

A large body of research has been carried out on various topics in close relation to the exhaust gas flow from an internal combustion engine. However, there are not many instances of targeted study in which engine exhaust flow is used to determine the exhaust noise. The instances of such integrated research include the various works of Benson et al.[2,3], and Jones et al.[4,5], using characteristic-based computations and of Blair et al.[6,7,8] and Dwyer et al.[9], using finite difference schemes. These are all one-dimensional system solutions of the unsteady Euler equations of inviscid gas flow. The bulk of the rest of research in this field seems to be concentrated on the linearised form of the equations.

1.4.1 Flow Models

Non-linear Unsteady Flow

Consider a one-dimensional homentropic unsteady flow in a uniform duct. The flow satisfies the conservation system for mass, momentum and energy see Benson et al.[2]:

\[
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} \rho u = 0 \tag{1.1}
\]

\[
\frac{\partial}{\partial t} \rho u + \frac{\partial}{\partial x} (\rho u^2 + p) = 0 \tag{1.2}
\]

\[
\frac{\partial}{\partial t} \rho e + \frac{\partial}{\partial x} \rho u(e + p/\rho) = 0 \tag{1.3}
\]
where $p$ is the density, $u$ is the velocity, $p = (\gamma - 1)\rho(e - u^2/2)$ is the pressure for a perfect gas, $\gamma$ is the ratio of the specific heat capacities at constant pressure and constant volume, and $e$ is the specific internal energy.

With specified boundary conditions the flow may be computed in the time domain, whence the radiated noise may be deduced directly. It is worth mentioning that this route implicitly takes account of, and deals with, any non-linearities such as shocks that may occur in the flow.

The use of unsteady flow analysis involves the solution of the unsteady gas flow equations (1.1-1.3) subject to the boundary conditions of the exhaust system. The analysis provides, in particular, a time-solution of the conditions at the exit plane, which can be used to predict the radiated noise. Dwyer et al.[9], Blair et al.[6,10,11] and Jones [4,5] have all used this approach to analyse engine exhaust systems, their common feature being the use of the one-dimensional equations. These computational schemes use the computed effects of the engine exhaust wave action at the pipe exit to find the radiated acoustic pressure. The calculations are performed in a stepwise fashion in the time domain and the flow properties in the entire duct are found at successive times. This process is carried through for a few engine cycles. The computations involve matching the flow from the engine cylinder to the flow in the pipe, at the inlet of the pipe, and the flow from the pipe to flow in the free field at the outlet end of the pipe.

At the inlet boundary the simplest form of boundary condition is that the flow velocity matches that of a rigid piston, say

$$u(t) = U_0 e^{i\omega t}$$

or

$$u(t) = \sum_{j=1}^{N} U_{0j} e^{i\omega_j t}$$
if the motion of the piston consists of the sum of \( N \) harmonic components of amplitudes \( U_{0k} \) and radian frequency \( \omega_j, j = 1, ..., N \).

At a true engine-pipe boundary, the flow is governed by the mass reaction in the engine cylinder. The boundary properties are deduced from the variation of gas mass in the engine cylinder in terms of the rate of port opening. The mass flow rate into the exhaust, \( m(t) \), see Jones et al.\[5\], is given as

\[
\frac{dm(t)}{dt} = \lambda(t)M_c(t)
\]  

(1.4)

where \( \lambda(t) = C(t)c_c(t)A_p(t)/V_c(t) \) and \( M_c(t) \) is the mass in the cylinder. It can be shown that \( m(t) = \lambda(t)M_c(0)e^{-\int_0^t \lambda(t)dt} \) is the general solution for the mass flow rate, where \( t = 0 \) is the time of the exhaust port opening.

The other properties of gas in the cylinder are expressed similarly, in terms of \( \lambda(t) \) and the various initial conditions. Thus the cylinder pressure, \( p_c \), is given by

\[
p_c(t) = p_c(0) \frac{V_c(0)}{V_c(t)} e^{-\gamma\int_0^t \lambda(t)dt} e^{-(\gamma-1)[s_c(0)-s_c(t)]/R_0}.
\]  

(1.5)

Thus, to obtain in-cylinder conditions the integral of \( \lambda \) is evaluated numerically for the time \( t \). A separate integral is evaluated for the crankcase in terms of \( \lambda_{cr}(t) \).

The variables in the mass flow rate are:

- \( C(t) \) a throat parameter, constant for choked flow and related to the ratio of reservoir pressure and the throat pressure.
- \( \rho_c(t) \) the density of gas in the cylinder.
- \( c_c(t) \) the sound speed in the cylinder.
- \( A_p(t) \) the open port area at time \( t \).
$s_c(t)$ the entropy along the characteristic line $c$.

$V_c(t)$ the volume of the cylinder at time, $t$.

$R_0$ the gas constant.

At the exit plane of the exhaust pipe, a variety of boundary conditions have been considered:

- **Velocity Node:** This consideration implies the assumption that the pipe terminates in a totally reflecting wall with the velocity set to zero. The implementation of this condition is relatively straightforward, but the condition is limited in its practical usefulness. This also means that there cannot be mean flow in the medium and hence no engine source can be considered.

- **Zero acoustic pressure.** This time the pressure is assumed to be the same as that of the surroundings. This is a better assumption than that of a velocity node and does allow for the possibility of a mean flow and hence an engine source. However, in reality, the pressure fluctuation does not disappear at the exit plane. This condition is also relatively easy to implement, but there are obvious sources of error to be considered, namely the generation of spurious reflected waves at the exit boundary, see Hirsch [12], due solely to this boundary condition. An implementation which limits this source of error is based on the method of characteristics with the incoming characteristic identified and suppressed, see Hirsch [12].

**Computations**

Calculation of unsteady one-dimensional flow in an engine exhaust involves the numerical solution of equations (1.1-1.3), for which a large number
of methods of solution exists. Benson et al. [2], Jones et al. [4], Blair et al. [6,7,8] and Dwyer et al. [9] have developed numerical schemes specifically for analysing engine exhaust flow.

Many of the methods are for the general analysis of unsteady compressible flows, rather than specifically for exhaust flows. In theory, the numerical solution of the unsteady system can be obtained exactly using the method of characteristics, Jones et al. [5], but in practice it is impossible to achieve this with practical numerical schemes.

Apart from the method of characteristics, the methods of finite-differences and/or finite volumes can be used. In recent times very accurate and straightforward flux splitting (first designed by Steger and Warming, [13]) or vector splitting methods (based on Gudonov and simplified by Roe, [14]) as described in Hirsch [12] have been designed for solving one-dimensional compressible flow models. These are equivalent whether cast as finite-volume schemes or as finite-difference schemes and are examples of the general methods for solving one-dimensional compressible flow problems. In the one-dimensional analysis, these splitting methods achieve their best and simplest forms and results.

A characteristic form of the one-dimensional compressible Euler equations (1.1-1.2) for a uniform duct, where the flow is assumed to be isentropic, is given by:

\[
\frac{1}{\rho c} \frac{dp}{dt} \pm \frac{d}{dt} u = 0 \tag{1.6}
\]

on \(C_+, C_-\) characteristics such that \(dx/dt = u \pm c\), where \(c\) is the speed of sound. Also, for a perfect gas,

\[
\frac{d}{dt} p = \left( \frac{2}{\gamma - 1} \right) \rho c \frac{d}{dt} u. \tag{1.7}
\]
The Reimann variables are given, using equations (1.6, 1.7), as

\[ P_1 = \left( \frac{2}{\gamma - 1} \right) c + u = \text{constant} \]  

(1.8)
on $C_+$ and

\[ Q_1 = \left( \frac{2}{\gamma - 1} \right) c - u = \text{constant} \]  

(1.9)
on $C_-$. The flow solution can be obtained by plotting the $C_+$ and $C_-$ characteristics on an $x - t$ diagram, see Rudinger [15] and Jones et al.[4,5]. Benson et al.[2] developed a computer programme based on characteristics, following path lines on a fixed mesh of constant $x$-lines. After the path lines have been extended due to a given time increment, $P_1$ and $Q_1$ are evaluated at fixed points in the solution domain by interpolation. In Benson et al.[3] the work is extended to include reacting flows with up to twelve reactants. The method was designed for evaluating engine performance.

Blair's wave action computations are very similar to Benson's with the additional assumption of homentropic flow (constant entropy throughout the cycle) and he has considered many more boundary conditions. In Dwyer et al.[9] and Blair [11], computations also include mufflers at the inlet or outlet, but with simple boundary assumptions and no comparison with experiment.

Finite-difference schemes have also been used, see Walter and Chapman [16] and Lakshminarayanan et al.[17], but mostly for engine performance calculations. Margolis et al.[18] used a finite-difference scheme to calculate a wave-action system and to predict radiated noise as well as engine performance. Their scheme is based on that of Dwyer et al.[9] for a duct with slowly-varying area, but also allows for friction and heat transfer and is capable of capturing shocks, see Blair [11].
Other Exit plane analysis

Davies and Yaseen [19] argued that, since

\[ P_1 = u + \frac{2}{\gamma - 1} c \]  

and

\[ Q_1 = u - \frac{2}{\gamma - 1} c \]

are the forward and backward travelling waves respectively, and it is clear that they are both functions of \( u \), it would be reasonable to expect their ratio at the exit plane, namely the reflection coefficient, to also be a function of \( u \). They ventured that this function could be cast as a polynomial of the form

\[ \phi(t) = \frac{Q_1}{P_1} = A_0 + A_1 u + A_2 u^2 + ... \]  

Then, using some experimental data, Davies et al.[19], they derived values for the constants \( A_0, A_1 \) and \( A_2 \), namely

\[ A_0 = -1, \quad A_1 = 0.40, \quad A_2 = -0.086 \]

although in practice it was found that \( A_2 \) varied in the range \(-0.088 < A_2 < -0.084\). Noting that this expression would reduce to a linear form for small \( u \), they plotted experimental values showing very small variations from the linear form. This method would be useful for flows near \( u = 0 \), the linear range. In a later extension, Davies and Jiajin [20] derived the reflection coefficient as a polynomial in the time-varying Mach number, \( M(t) \), namely the ratio of the flow velocity to the stagnation sound speed, and found the coefficients

\[ A_0 = -1, \quad A_1 = 0.4, \quad A_2 = -0.08, \quad A_3 = 0.016. \]

The later extension of the polynomial (1.12) was quite general, allowing for the case of mean flow with Mach number, \( M_0 \), though the coefficients
were derived only for the case $M_0 = 0$. However, the relation (1.12) is only valid if the sound speed is constant. These polynomial methods seem to be borne out by the plots presented in Davies et al. [19,20], but worrying features in both equivalent methods include the need for measured data and the lack of an obvious way of incorporating the findings in unsteady time-domain analysis.

**Linearised Acoustic Flow**

Most of the available work on acoustics is based on a one-dimensional linearised flow model, possibly due to the relative ease of solution of the resulting system of equations and its fair degree of agreement with similar experiments.

Using perturbation theory in line with Stewart [21], the system of equations (1.1) to (1.3) may be linearised leading to the acoustic wave equation for the acoustic pressure. Munjal [22] gives a more general form of the acoustic pressure wave equation for a medium with mean flow speed $U$,

$$\frac{\partial^2 p}{\partial t^2} + 2U \frac{\partial^2 p}{\partial x \partial t} + (U^2 - c_0^2) \frac{\partial^2 p}{\partial x^2} = 0$$

(1.13)

where the acoustic perturbation is assumed to be adiabatic and isentropic. Here $c_0$ is the constant sound speed in the medium.

The general solution to equation (1.13) is given by

$$p = \left[ p_+ e^{-i \frac{kx}{(1+M)}} + p_- e^{i \frac{kx}{(1-M)}} \right] e^{i \omega t}$$

(1.14)

with a mean flow of Mach number $M$ and wavenumber, $k = \omega/c$, for harmonic time variation of angular frequency $\omega$. With this solution, it is easy to determine the radiated acoustic power for each frequency; given a radiation impedance at the exit plane.
In the linearised model, the engine is assumed to be an acoustical source of given strength and impedance, the exhaust system is regarded as a combination of passive acoustical elements, and the surroundings are represented as having a particular acoustic impedance at the tail pipe exit, Embleton et al.[23]. Acoustic analysis of such a system is then analogous to analysis of electrical circuits, as presented in Munjal [22]. The sound source is shown as a constant volume velocity source of strength $v_e'$ with a parallel source impedance $Z_e$. The surroundings are represented by a radiation impedance $Z_r$ and the rest of the exhaust system is shown generally as a four terminal network. This procedure has been used by Jones et al.[5] and Lakshminarayanan et al.[17] to model a complete engine and exhaust system.

**Distributed Impedance**

In ducts with radii much smaller than a wavelength, the acoustic propagation is essentially one-dimensional, satisfying the plane wave equation (1.13). The solution is the general form given earlier in equation (1.14). Provided that there is no mean flow, the duct can be assumed to consist of a distributed impedance of a certain inertance and compliance per unit length, Morse et al.[24]. Where there is a change in the cross-section the transmitted and reflected wave components are resolved by assuming continuity of volume velocity and acoustic pressure across the jump. Davis et al.[25,26] used this approach to analyse a variety of mufflers. Alfredson and Davies [27] used the same procedure to analyse exhaust systems on an operating engine, but comparisons with experiment were poor until they modified the model to account for the mean flow. Under their modifications, the wavenumber is split into two components, namely:

$$k_+ = \frac{k}{1 + M}, \quad k_- = \frac{k}{1 - M}$$
for the forward and backward travelling waves respectively, see equation (1.14). It may be noted that this reduces to a single wave number for zero-mean flow.

Sridhara and Crocker [1] reviewed the representation of the passive exhaust system acoustical components by means of four terminal transmission matrices. In this representation the pressure and velocity downstream are related to the pressure and velocity upstream by

$$\begin{pmatrix} p_1 \\ v_1 \end{pmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{pmatrix} p_2 \\ v_2 \end{pmatrix}$$  \hspace{1cm} (1.15)

where $p_1$, $p_2$, $v_1$ and $v_2$ are the acoustic pressures and velocities respectively, including the positive and negative components at given frequencies. The four parameters, $A_1$, $B_1$, $C_1$ and $D_1$ characterise the device generally and are unrelated to the upstream or downstream impedances.

By repeating this for each of the junctions, the transmission matrix for a muffler system is the product of the transmission matrices of its components, since if

$$\begin{pmatrix} p_2 \\ v_2 \end{pmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{pmatrix} p_3 \\ v_3 \end{pmatrix}$$  \hspace{1cm} (1.16)

it follows from equation (1.15) that

$$\begin{pmatrix} p_1 \\ v_1 \end{pmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{pmatrix} p_3 \\ v_3 \end{pmatrix}$$  \hspace{1cm} (1.17)

and so on. This is useful when analysing exhaust systems with discontinuities, such as sudden expansions and/or sudden contractions as would occur where mufflers or catalytic converters are installed.

Munjal [22] has derived a modified transmission matrix to account for a mean flow.
Impedance and Radiated Noise

At the tail pipe outlet the component of acoustic pressure from the reflected wave, $p_{r-}$, may be expressed in terms of the component from the incident wave $p_{r+}$ with or without flow, as $p_{r-} = p_{r+} R(M) e^{i\theta(M)}$, where $R(M)$ is the reflection coefficient and $\theta$ is the phase difference between the pressure and the velocity. The total acoustic pressure is given by $p_r = p_{r-} + p_{r+}$.

Under the linearised model, the outlet conditions are accounted for by assuming a radiation impedance, $Z_r = p_r/u_r$, at the exit plane given by

$$Z_r = \frac{\rho c}{S} \left( \frac{1 + R(M) e^{i\theta(M)}}{1 - R(M) e^{i\theta(M)}} \right)$$

(1.18)

where $\rho$ and $c$ are the average static values of density and sound speed inside the duct for mean flow of Mach number, $M$ and $S$ is the cross sectional area. With these values obtained by measurement, where needed, the radiated acoustic energy is given by $\nu T Z_r$, see Okda [28].

Alternatively, an expression involving the pipe end correction, $l(M)$ is used with $R(M) e^{i\theta(M)}$ replaced by $-R(M) e^{-i2k l(M)}$. This implies that the pipe outlet reflects the incident wave with a phase change of $\pi$ radians and an amplitude ratio $R(M)$ at a distance $l(M)$ from the exit plane.

Levine and Schwinger [29] obtained exact relations for $R(0)$ and $l(0)/a$ for the case of radiation from an unflanged circular duct of radius $a$, with the gas within the pipe the same as that on the outside, and with zero mean flow. For $ka < 1$ the exact relations are approximated by:

$$R(0) = \exp \left[ -\frac{1}{2} (ka)^2 \right] \left( 1 + \frac{1}{6} (ka)^4 \left[ \log_e (1/\gamma_1 ka) + \frac{19}{12} \right] \right)$$

(1.19)

$$l(0)/a = 0.6133$$

(1.20)

where $\log_e(\gamma_1) = 0.5772$ is Euler’s constant. Equation (1.19) further simplifies as $ka \rightarrow 0$ to $R(0) = 1 - \frac{1}{2} (ka)^2$. 

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Using these approximations we have

$$Z_r = \frac{\rho c}{S} \left( \frac{(ka)^2}{4} + i 0.6133ka \right). \quad (1.21)$$

Alfredson and Davies[30] derived the radiated noise as

$$W_t = \frac{|p_r|^2 S}{2 \rho c} \left[ (1 + M)^2 - R^2(M)(1 - M)^2 \right] \quad (1.22)$$

for mean flow, $|p_r|$ being the amplitude of the incident pressure wave. These authors obtained measurements for $R(M)$ and $\theta(M)$ for flows up to $M = 0.17$, with $0.15 \leq ka \leq 0.8$, observing that the reflection coefficient increased with mean flow speed while the phase changed little. Measurements showed good agreement with calculated values. Later measurements of $R(M)$, by Davies et al.[31], Ross and Crocker [32], and Panicker and Munjal [33], use an artificial sound source consisting of a pipe into which sound and airflow are introduced. Davies et al.[31] also report Munt’s theoretical derivation of the reflection coefficient $R(M)$ as a function of the parameter $ka$ for cold flow. All of these authors report an increased $R(M)$ by comparison to Levine and Schwinger’s zero flow results. With measured and/or calculated values of $R(M)$ and $M$, the radiated noise is calculated using Alfredson and Davies’ relation, equation (1.22). Various other simplifications have been used by Munjal et al.[22] and Ingard and Davis [34] but these only further reduce the possible accuracy of the prediction in more applicable flows.

It is worth remembering that in real duct flows the linear theory cannot account for all of the flow behaviour, even under the amplitude and frequency restrictions. Also, apart from Munjal [22] and Davis [34], there is a marked presence of empirical data in all of the analysis.
End Correction and Reflection Coefficient

The vital importance of the end correction \( l \) and the reflection coefficient \( R \), in determining radiation boundary properties of ducts is evident by the prevalence of these properties in the analytical methods mentioned so far. The determination of the end correction for cylindrical pipes was first investigated in 1896 by Lord Rayleigh [35] in his extensively referenced 'The theory of sound'. The problem of determining the end correction is analytically soluble if the open end is assumed to terminate flushed with an infinite flange. Wendolowski and McPhedran [36] discuss the effects of a flange.

Using the classical theory of acoustic radiation from an unflanged duct of radius \( a \), Levine and Schwinger [29] derived expressions for both \( R(0) \) and the end correction \( l(0) \). They also suggested approximate relations for \( |R(0)| \), but their results did not take the possible presence of mean flow into account. Carrier [37] extended the analysis of Levine and Schwinger to the case of a semi-infinite pipe with mean flow, taking into account the convection but neglecting flow separation at the mouth of the pipe. He showed that the results of Levine and Schwinger [29] could be extended to the case of a moving medium with the parameter \( ka \) replaced by \( ka\sqrt{1-M^2} \). Carrier's analysis was further extended to the case of finite length ducts both by the author himself and also in the work by Ogimoto and Johnston [38-40], though the effect of flow separation was neglected.

Ingard and Singhal [41] studied the effect of mean flow on the reflection coefficient and its phase angle for a pipe with a small flange at the end, but limited their experiments to cases where \( ka < 0.5 \), assuming \( |R| = 1 \) as a basis for comparing their results. They reported instances of \( |R(M)| > 1 \) and observed that the end correction increased to \( l/(1-M^2) \) and the phase
angle, $\theta(M)$ of the reflection coefficient was given as

$$\theta(M) = \pi - 2k l/(1 - M^2).$$

Theoretical analysis of exit boundary conditions has been carried out by Rayleigh [35], Daniell [42], King [43], Nomura et al.[44] and Norris and Sheng [45], who all neglect viscous effects. In Rayleigh's [35] first approximation it is assumed that the axial component of the particle velocity at the pipe mouth is fixed with respect to radius, an assumption whose inaccuracy Rayleigh himself acknowledged. More rigorous solutions by Rayleigh[35], Daniell[42] and King[44] show that it is necessary for the particle velocity to assume a radial profile which increases from the centre of the pipe to the wall. King [43] derived two series for the end correction for $ka = 0$, one converging with the analytical value 0.82166 as its lower bound, and the other converging with the analytical value 0.82159 as its upper bound, but the solution was not in closed form. At the time, 1936, a numerical solution would have been difficult to obtain to any degree of accuracy.

Norris and Sheng [45] extended King's upper bound series to non-zero $ka$ and gave numerical evaluations of the solution in the range $0 \leq ka \leq 3.83171$, but did not report on the particle velocity across the mouth. Nor did Nomura et al.[44].

Wendolowski et al.[36] extended the analysis of Morse and Ingard [46] to provide highly accurate numerical results based on the linear non-viscous wave model. From this solution the upper bound solution of King [43] and the formulations of Daniell [42] and Norris and Sheng [45] may be derived. Wendolowski et al. [36] showed that the end correction for a flanged duct lies in the narrower range $0.821664 \leq l(0)/a \leq 0.821676$ by comparison to Norris and Sheng's apparent upper bound of $l(0)/a = 0.82159$. Using King's analysis Wendolowski et al.[36] report a lower bound $l(0) \geq 0.82166$. 

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Despite the importance of pressure and velocity profiles in the theory of radiation, only Wendolowski et al. [36], of all the authors so far mentioned in this section, offer a detailed investigation of these profiles. Rayleigh [36] and Daniell [42] did report explicit approximations for the velocity profile but neither cited any empirical evidence for their solutions. Nor does there appear to have been an experimental verification of the profile.

Other related work includes Cho's [47] analysis. He considers the problem of a duct terminating in either a hyperbolic horn or an infinite flange. In the analysis he defines a class of 'hyperboloidal wave functions' which are eigensolutions of the wave equation in oblate spherical-co-ordinates. His numerical results include complex reflection coefficients and radiation directivity for various incident wave modes, including spinning and axisymmetric modes. He concludes that the solutions are valid for all frequency ranges including those above and below the cut-off frequencies of the duct modes.

Astley and Eversman [48] present a finite element method for pipe flows with emphasis on numerical implementation of the Sommerfeld far-field boundary condition, but their flow domain is a segment of cylindrical pipe which gradually flares downstream. They introduce the concept of infinite elements for overcoming the problem of deciding how far downstream to go before fixing the far-field boundary so that the Sommerfeld condition is acceptable. They produce results for various test cases, one of which is a flanged duct flow, but their results mainly consist of sinusoidal time plots of the flow velocity, making it difficult to judge the ability of the scheme to model the exit plane velocity profiles.

The usual method for determining the radiated sound pressure is to find the amplitude of pressure waves incident at the outlet using a linear model of the entire exhaust system and then, with the reflection coefficient $R(M)$
and flow Mach number $M$, equate the net acoustic power transmitted with
that radiated from a monopole source as in the last section, see Sridhara et
al.[1]. This is not a wholly analytical method and its main disadvantage is
its reliance on the measurement of the Mach number and either measure-
ment or calculation of the reflection coefficient based on further possibly
inappropriate analysis. A further problem arises from the fact that $R(M)$
values are likely to be inaccurate for lower frequencies where the pipe is
almost totally reflecting. Values of $R(M)$ for $ka < 0.1$ differ from 1.0 by
such a small amount that a correct determination would be ruled out by
the merest error in an analysis or measurement. The relevance of this prob-
lem is underlined by the fact that pipes of radius between 0.03 and 0.06m
are in common use in automobiles and for these $ka$ is sufficiently small
for $R(M)$ to be close to unity over the low frequency range (0–200Hz),
which is of prime interest as it is the frequency range accounting for most
of the exhaust noise. Furthermore, this method is already limited by the
assumptions of linearity and one-dimensional flow in the pipe.

**Monopole Radiation**

A monopole point source is a uniform source radiating energy spherically
outward in all directions. The open end of an unflanged duct can be
modelled as a monopole point source for small values of the ratio of the
duct radius to the wavelength. Under these conditions, spherical waves
propagate outside of the duct while plane waves propagate back into the
duct and the sound power transmitted through the open end is equal to
the sound radiated from the open end modeled as a monopole, if there is
no energy loss.

The point source is at the centre of the duct exit plane and has strength
dependent on the amplitude of the mean velocity over the exit plane area.
Jones and Brown [5] give the far-field acoustic pressure due to such a source as

\[ p(t) = \frac{\rho A_{tp}}{2\pi r_0} \left[ \frac{d}{dt} u(t - r/c) \right] \]

where \( c \) and \( \rho \) are the atmospheric sound speed and mass density, respectively, \( A_{tp} \) is the cross-sectional area of the duct, \( u \) is the acoustic area-mean acoustic velocity and \( r_0 \) is the distance of the far-field spherical locus from the source. The relation was presented for a source in close proximity to the ground. It holds also for any source radiating into a half sphere, such as a flanged duct exit. For an unflanged duct the divisor \( 2\pi r_0 \) is replaced by \( 4\pi r_0 \).

From this expression, the far-field pressure in decibels is given by

\[ P_{db} = 20\log_{10} \left( \frac{|p(t)|}{2 \times 10^{-5}} \right) = 20\log_{10} \left( \frac{\rho A_{tp}}{2\pi r_0} \left| \frac{d}{dt} u \right| \right) - 4.7. \quad (1.23) \]

For single-frequency disturbances the exit velocity profile is sinusoidal, \( u = u_e e^{i\omega t} \) so

\[ \left| \frac{d}{dt} u \right| = |i\omega u_e| = \omega u_e, \]

where \( u_e \) is the amplitude of the acoustic velocity. Alternatively, the far-field noise is given, see Munjal [22], by

\[ P_{db} = 20.0\log_{10}(c_0\rho_0 p_f/2 \times 10^{-5}) \quad (1.24) \]

with

\[ p_f = \sqrt{u_e^2|Z_r|a_0^2 \over 4r_0^2\rho_0c_0} \]

where the acoustic impedance, \( Z_r \), is given by equation (1.21).

This shows that far-field noise can be determined on the basis of the mean exit velocity if the monopole source conditions are satisfied or if the acoustic...
impedance is known. The quantity $P_{db}$ depends on the amplitude of the exit velocity rather than the whole time series profile. It is quite clear that the far-field noise spectra for more complex disturbances, such as multiple-harmonic or engine source flows, can be determined once the time series profiles are resolved into constituent single-frequency modes.

In the frequency domain, the expression for radiated sound pressure amplitude $|p_{rad}|$ is given by

$$|p_{rad}| = \frac{\rho c_0 \omega |v_r|}{4\pi r_0}.$$

(1.25)

This does not strictly imply that transmitted and radiated acoustic power are equated, but it can be shown that for $ka \to 0$ and with the gas within the tail pipe the same as that in the surroundings, this relation implies that the radiated sound power is the same as the transmitted power and given by $Re\{v_r^2 Z_r\}$. This method of determining the radiated acoustic pressure is similar to those used in unsteady flow models.

### 1.5 Objectives of the Research

Having reviewed the work in this field, the aim of this research is to investigate ways of improving the exit plane boundary conditions for non-linear, time domain analysis in order to improve the accuracy of predicted radiated noise levels. To this end both the physical specification of the exit plane boundary conditions and their numerical implementation, when the domain is restricted to the duct, will be investigated. This should show whether there are any accurate ways of implementing the $p = p_0$ condition at the exit plane.

The effect of the $p = p_0$ condition will be investigated by extending the flow domain downstream of the exit plane. This should show to what extent
this boundary condition affects the accuracy of the unsteady time-domain analysis.

In chapter two, the gas-dynamic equations are set out and various forms of simplifications of both the equations and the flow domain are pointed out. Analytic solutions of a spherical wave propagation are also presented, which are later used in the validation of certain aspects of the finite volume scheme.

In part two - chapters three and four - the numerical grids are generated and the numerical schemes are formulated on the grids. The grids are generated for the extended domains, both for the flanged and unflanged ducts, and are useful in computations when the \( p = p_0 \) condition is suppressed. Extended grids are generated based on various transforms of the physical flow domain. The difference in transform functions allows the far-field cell sizes to be controlled.

In chapter four, a discussion of the effects of the exit boundary condition on the numerical accuracy of the finite volume scheme is given. A flux-split scheme is presented which, by suppressing the spurious modes of the Finite Volume scheme, allows an investigation of the reflection of the spurious modes at the exit plane. This leads towards a more accurate implementation of the \( p = p_0 \) condition at the exit plane. Boundary implementations, based on both the conservation equations and on the characteristic equations, are discussed for both the flux-split scheme and the finite volume scheme.

Also, a modification of the exit boundary condition, which allows the pressure at the exit plane to fluctuate, is introduced. This allows a further investigation of an alternative to the \( p = p_0 \) condition without the need for an extended domain.

The numerical implementation of far-field boundary conditions is pre-
sented. This is used in extended domain computations where the exit plane boundary conditions are suppressed.

Finally, in part three – chapters five, six and seven – results are given and discussed. In chapter five results of single frequency computations are presented. Where possible, the plots compare analytic solutions with numerical solutions. In chapter six, results of multiple-harmonic and engine source computations are shown.

Chapter seven is the conclusion and contains suggestions for advancing this research.
Figure 1.1: A schematic representation of the exhaust system.
Chapter 2

General Theory and
Background
2.1 The Flow Equations

The most general equations of fluid motion are the Navier-Stokes equations given, in three-dimensional space as, see Hirsch [12]

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho \mathbf{v} \\ \rho e \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p\mathbf{I} - \mathbf{T} \\ \rho \mathbf{v} H_e - \mathbf{T} \cdot \mathbf{v} - \kappa \nabla T \end{bmatrix} = \begin{bmatrix} 0 \\ \rho f_e \\ \rho f_e \cdot \mathbf{v} + q_H \end{bmatrix}
\]

(2.1)

(2.2)

together with the equation of state,

\[ p = (\gamma - 1) \rho \left( e - \frac{|\mathbf{v}|^2}{2} \right). \]

Here \( \mathbf{T} \) is the viscous shear stress tensor, \( \mathbf{I} \) is the 3 × 3 identity matrix, \( \kappa \) is the conductivity of the gas, \( T \) is the absolute temperature, \( H_e \) is the enthalpy, \( f_e \) are external forces, \( q_H \) represents heat sources, \( \otimes \) denotes the tensor product of two vectors and \( p \) denotes the pressure. The quantities \( \rho, \mathbf{v}, e \) and \( \gamma \) represent density, velocity, specific energy and the ratio \( c_p/c_v \) of the specific heat capacities of the gas at constant pressure to that at constant volume, respectively.

Equation (2.1) can be written in condensed form as

\[
\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}_T = \mathbf{Q},
\]

(2.3)

where

\[ \mathbf{U} = [\rho, \rho \mathbf{v}, \rho e]^T \]

consists of the conservation variables and \( \mathbf{Q} \) is the source term.

The quantity \( \mathbf{F}_T \) consists of fluxes in the three space directions which can be separated into inviscid and viscous parts as

\[ \mathbf{F}_T = \mathbf{F}_I - \mathbf{F}_V. \]
For the inviscid compressible flows considered here it is assumed that $F_i$ is negligible. In the absence of external forces and heat transfer, equations (2.3) then further reduces to the Euler equations

$$\frac{\partial}{\partial t} U + \nabla F_l = 0$$  \hspace{1cm} (2.4)

### 2.2 Boundary Conditions

The boundary conditions of the flow fall into four groups:

- **rigid impervious walls** - where the normal component of the flow velocity is set to zero, i.e. there is no flow through the surface.

- **inlet** - where the flow velocity is assumed to match that of the piston or engine exhaust.

- **the exit plane boundary** - when the flow domain is limited to the duct. For an open exit the pressure is generally assumed to be atmospheric at the exit plane, i.e. $p = p_0$. Alternatively, if the duct terminates in a wall, the axial velocity is set to zero at this boundary.

- **the spherical or hemispherical boundary**, when the domain is continued into the free field, where it is assumed that acoustic waves incident on the boundary are all totally transmitted.

Sufficiently far away from the exit plane, in the free field, the acoustic waves behave like waves originating from a point source located at the centre of the exit plane. They are one-dimensional spherical waves and travel normal to the wavefronts. The far-field boundary is designed to coincide with the shape of the wave front, a surface on which the radiation condition is most conveniently based. This leads to the linearised relation, the Sommerfeld radiation [65] condition, namely
\[ \frac{\partial p}{\partial t} + c \frac{\partial p}{\partial R} = 0, \quad as \ R \to \infty \] (2.5)

where \( c \) is the local sound speed and \( R \) denotes the radius of the spherical wavefront.

### 2.2.1 Co-ordinate System

It is seen from figures 2.1 to 2.3 that the flow domains are axially symmetric. It is assumed that the duct is parallel to the \( z \)-axis of an \((r, \theta, z)\) cylindrical polar co-ordinate system and that there is no angular flow or angular variation in the flow properties.

It follows from equation (2.4) that

\[ \frac{\partial}{\partial t}(rU) + \frac{\partial}{\partial r}(rF) + \frac{\partial}{\partial z}(rG) = H \] (2.6)

where

\[ U = [\rho, \rho u, \rho v, \rho e]^T, \]

\[ F = \left[ \rho u, (p + \rho u^2), \rho u v, \rho u \left( e + \frac{p}{\rho} \right) \right]^T, \]

\[ G = \left[ \rho v, \rho u v, (p + \rho v^2), \rho v \left( e + \frac{p}{\rho} \right) \right]^T, \]

\[ H = [0, p, 0, 0]^T, \]

and \( u \) and \( v \) are the components of velocity in the \( r \)- and \( z \)- directions, respectively.
The pressure is given by the equation of state

\[ p = (\gamma - 1) \rho \left( e - \frac{u^2 + v^2}{2} \right). \tag{2.7} \]

The one drawback for this system of governing equations is that it is undefined on the line of symmetry, \( r = 0 \). One suggestion, see Roache [49], for dealing with this discontinuity is to revert to a Cartesian system. From a physical point of view, since \( r = 0 \) is an axis of symmetry, it behaves like a wall, across which there is no flow. The normal component of velocity on \( r = 0 \) is set to zero and for the rest of the conservation variables and pressure, the normal derivative is set to zero.

### 2.2.2 Inviscid Isentropic Flow

For low Mach number inviscid flow, it is possible to simplify the system of equations (2.6) and (2.7) still further by assuming that the perturbation is isentropic. This works in the limit that time-variations in total energy are negligible. The system of governing equations then reduces to the conservation of mass and momentum, together with

\[ p = p_0 \left[ \frac{\rho}{\rho_0} \right]^\gamma, \tag{2.8} \]

where \( p_0 \) and \( \rho_0 \) are the stagnation values of the pressure and density, respectively. It is this isentropic system of equations that is solved initially for each of the three stages in the modelling process.

In the present work the isentropic system is used mainly as a preliminary test of the numerical scheme before the full non-isentropic system is considered. This is because the system under consideration, being time-varying, cannot strictly have a time-invariant total energy at each point in the flow space. In the isentropic computations it is assumed that the relative amplitude of the energy oscillation is small, which cannot continue to be safe a-
the amplitude of the piston is increased, much less when engine conditions are applied at the inlet boundary.

2.3 The Flow Domain

As described in section 1.3, the physical domain consists of a cylindrical circular duct, one end of which opens into an infinite sphere, or hemisphere in the case where the duct terminates on the outside of an infinite wall.

For the numerical solution, the domain is truncated by imposing a far-field boundary at a fixed radial distance from the centre of the exit plane, see figures 2.2 and 2.3.

2.3.1 Domain Simplifications

Further pre-solution simplifications are applicable to the flow domain but only as part of tests of the numerical schemes. As such they do not reduce the complexity of the system of equations (2.6)-(2.8) - they reduce the complexity of the computational domain. This is the case in this thesis when computations are carried out on a source-in-wall domain, specified later in section 3.2.1. Other more significant domain simplifications, such as the use of symmetry to reduce the domain to two spatial dimensions, do also reduce the complexity of the system of partial differential equations.

One of the aims of domain simplification is to limit the eventual number of grid points in the computational domain and, thus, the number of discrete systems of equations to be solved. There are, however, penalties for each of the possible domain simplifications.
2.3.2 Closed Duct

The simplest model of the exhaust pipe is that of a closed duct, where it is assumed that the system behaves like a duct terminating in a rigid wall at which all incident waves are totally reflected. In this case the first two boundary conditions in section 2.2 are valid and at the exit plane the velocity \( u(L, t) = 0 \) \( \forall \ t \) is assigned, where the duct is of length \( L \) and the disturbance is one-dimensional.

2.3.3 Open Duct

A more realistic model of an exhaust pipe is an open duct, for which the boundary conditions for the calculation are set at the exit plane of the duct. The first three boundary conditions of section 2.2 are now valid.

2.3.4 Conical Horn Domain

The model of a conical horn of angle \( \alpha \), see figure 2.1, is also used for tests on a simplified domain prior to the main solution of the system of equations on the full physical domain. Acoustic waves generated by a pulsating sphere, part of which closes the throat of the horn, radiate out of the throat in a spherical one-dimensional pattern. This is particularly useful since an analytic solution, for linear acoustics, is available for this problem. In this case, the inlet boundary condition is that the gas next to the surface has the same velocity as the surface of the sphere. For the free-field problem the far field waves decay to zero at infinity. An alternative solution for a finite domain can be found if the acoustic velocity is set to zero at some far-field boundary, i.e. a hard wall, or if the Sommerfeld radiation condition is used at a finite boundary to approximate radiation into free space.
Some Analytical Solutions For Linearised Acoustics

Analytic solutions of conical horn acoustics are easily generated. Numerical results generated later are compared to the analytical solutions in order to gauge the accuracy of the numerical scheme.

Two solutions for the decaying spherical wave generated by the pulsating surface of a sphere are presented. In the first solution a hard wall condition is used at a far-field boundary, as stated in the last section. The far-field boundary is set at a distance $x_0 + L$ from the source where $x_0$ is the radius of the sphere and $L$ is the distance of the boundary from the surface.

In the second solution the non-reflective boundary condition of Sommerfeld is imposed.

**Rigid Wall at the Far-field Boundary**

The one-dimensional wave equation for a decaying linear spherical wave is given by \[24\]

$$\frac{1}{(x + x_0)^2} \frac{\partial}{\partial x} \left[(x + x_0)^2 \frac{\partial}{\partial x} p \right] = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p$$

where $x_0$ is the radius of the pulsating sphere, $x$ is the distance from the throat of the horn and $p$ is the acoustic pressure.

The general solution for this equation is

$$p = \frac{1}{x + x_0} \left[p^+ e^{ikx} + p^- e^{-ikx}\right] e^{-i\omega t}$$

for the combination of forward and backward travelling waves.

If the velocity field is given by

$$u = \left[u^+ e^{ikx} + u^- e^{-ikx}\right] e^{-i\omega t}$$

(2.11)
then for linear acoustic waves, the radial momentum equation satisfies.

\[
\frac{\partial}{\partial x} p = -\rho_0 \frac{\partial}{\partial t} u. \tag{2.12}
\]

Substituting equations (2.10) and (2.11) into (2.12) and simplifying gives the relations

\[
\frac{i}{\omega \rho_0 (x + x_0)} \left[ \frac{1}{x + x_0} - ik \right] p^+ = u^+ \tag{2.13}
\]

and

\[
\frac{i}{\omega \rho_0 (x + x_0)} \left[ \frac{1}{x + x_0} + ik \right] p^- = u^- \tag{2.14}
\]

for the amplitudes of the forward and backward characteristics of the velocity in terms of those of the acoustic pressure.

Substituting these relations into equation (2.11) yields

\[
u(x, t) = \frac{i}{\omega \rho_0 (x + x_0)} \left[ a_x e^{ikx} + b_x e^{-ikx} \right] e^{-i\omega t} \tag{2.15}
\]

where

\[
a_x = \left( \frac{1}{x + x_0} - ik \right) p^+, \quad b_x = \left( \frac{1}{x + x_0} + ik \right) p^-.
\]

On a rigid wall positioned at \(x = L\) say, the velocity is zero giving the relation

\[
u(L, t) = \frac{i}{\omega \rho_0 (L + x_0)} \left[ a_L e^{ikL} + b_L e^{-ikL} \right] = 0, \tag{2.16}
\]

whence

\[
p^- = \frac{-a_L e^{2ikL}}{b_L} p^+, \tag{2.17}
\]

where

\[
a_L = \left( \frac{1}{L + x_0} - ik \right), \quad b_L = \left( \frac{1}{L + x_0} + ik \right).
\]
At the inlet boundary, \( x = 0 \), the flow velocity matches that of the spherical surface - a wave of amplitude \( u_0 \) and angular frequency \( \omega \):

\[
  u(0, t) = \frac{i}{\omega \rho_0 x_0} \left[ a_0 p^+ + b_0 p^- \right] e^{-i\omega t} = u_0 e^{-i\omega t}.
\]  

(2.18)

Substitution for \( p^- \) from equation (2.17) then yields

\[
  p^+ = -i\omega \rho_0 x_0 u_0,
\]

where

\[
  a_0 = \frac{1}{x_0} - ik, \quad b_0 = \frac{1}{x_0} + ik.
\]

This gives the values for \( p^+ \) and \( p^- \) as:

\[
  p^+ = -\chi i\omega \rho_0 x_0 u_0
\]

(2.19)

and

\[
  p^- = \left( \frac{a_L}{b_L} \right) \chi i\omega \rho_0 x_0 u_0 e^{2ikL},
\]

(2.20)

where

\[
  \chi = \left[ \frac{i}{x_0} - ik + \left( \frac{i}{x_0} + ik \right) \left( \frac{a_L e^{2ikL}}{b_L} \right) \right]^{-1}.
\]

From here, the values for \( u^+ \) and \( u^- \) follow by substitution in equations (2.13 -2.14). More directly, the function \( u(x, t) \) derived in terms of \( p^+ \) and \( p^- \), equation (2.15), can be computed straight away.

**Sommerfeld Radiation Condition at the Far-field Boundary**

A further useful analytical solution can be obtained using the Sommerfeld radiation condition for linear waves for this domain. It is useful in showing both that the Sommerfeld condition can be applied numerically, and how accurately the computational scheme models it.
Here the analytic solution is determined such that the Sommerfeld condition, given by equation (2.5), is satisfied at the far-field boundary. This process is essentially the same as in the last section for the zero-velocity case except that the condition

\[
\left. \frac{\partial p}{\partial t} \right|_{x=L} + c \left. \frac{\partial p}{\partial x} \right|_{x=L} = 0
\]

is used in place of \( u(L, t) = 0 \).

Expanding and simplifying the above relation yields

\[
e^{2ikL} \frac{p^+}{(L + x_0)} + \left( \frac{1}{L + x_0} + 2ik \right) p^- = 0. \tag{2.21}
\]

From equation (2.21) it is seen that as \( L \to \infty \), \( p^- \to 0 \). This shows that when applied at infinity, the Sommerfeld condition gives zero reflection. When it is applied at a finite boundary, \( L \), the condition results in an ‘error’ given by an inward-travelling wave component of

\[
p^- = \chi_1 e^{2ikL} p^+. \tag{2.22}
\]

where

\[
\chi_1 = \frac{-1}{1 + 2ik(L + x_0)}.
\]

Equation (2.22) is substituted in the inlet condition, equation (2.18), as before, giving

\[
\begin{align*}
\left\{ \frac{1}{x_0} - ik + \left( \frac{1}{x_0} + ik \right) \chi_1 e^{2ikL} \right\} p^+ &= -i\omega_0 \rho_0 x_0 u_0 \\
p^+ &= -i \left\{ \frac{1}{x_0} - ik + \left( \frac{1}{x_0} + ik \right) \chi_1 e^{2ikL} \right\}^{-1} \omega_0 \rho_0 x_0 u_0 \tag{2.24}
\end{align*}
\]

As before, \( p^- \), and hence \( u^+(x) \), \( u^-(x) \) and \( u(x, t) \). follow from equations (2.22), (2.13), (2.14) and (2.11). An alternative to (2.11) for \( u(x, t) \) is (2.15).
The exact solution of the acoustic pressure which has zero reflection follows from equation (2.24) with \( L \to \infty \), or directly from equation (2.10), namely
\[
p_e = \frac{p_e^+}{x + x_0} e^{i(kx - \omega t)}. \tag{2.25}
\]

The exact acoustic velocity can be derived by determining the values of \( u_e^+ \) and \( u_e^- \) corresponding to \( p_e^+ \) and \( p_e^- \).

The percentage error between the exact solution of the acoustic velocity and the solution based on applying the Sommerfeld radiation condition on a finite boundary can be deduced from the ratio
\[
\left| \frac{u_s}{u_e} \right| = \frac{p_s^+}{p_e^+} \left[ 1 + \frac{\left( \frac{1}{x + x_0} + ik \right) e^{-2ikx}}{\left( \frac{1}{x + x_0} - ik \right)} \right], \tag{2.26}
\]
where the subscripts \( e \) and \( s \) correspond to the exact solution and the finite domain solution, respectively, with
\[
\left| \frac{p_s^+}{p_e^+} \right| \approx 1 + \epsilon \left| e^{-2ikx} - \frac{1 + ikx_0}{1 - ikx_0} \right| \leq 1 + 2\epsilon \tag{2.27}
\]
and
\[
\epsilon = \frac{e^{-2ikL}}{1 + 2ik(L + x_0)}. \tag{2.28}
\]
It follows that the ratio \( |u_s/u_e| \) is bounded by
\[
\left| \frac{u_s}{u_e} \right| \leq [1 + 2\epsilon] \left[ 1 + \frac{\left( \frac{1}{x + x_0} + ik \right) e^{-2ikx}}{\left( \frac{1}{x + x_0} - ik \right)} \right] \approx 1 + 3\epsilon. \tag{2.29}
\]
For large \( kL \), \( \epsilon \) decreases linearly with increasing \( kL \). In order to maintain a given error level, \( L \) must therefore be increased proportionately with reducing wavenumber \( k \).

### 2.3.5 Source-in-Wall Flow

In the last subsection, the analytical solution was derived for a spherical wave using the Sommerfeld far-field condition in a conical horn flow. A
still more useful procedure is to apply that solution to a flow modelling the behaviour of an oscillating source in a wall. This is an approximation to exhaust flow from a flanged duct where the source is assumed to behave like the disturbance in the exit plane of the duct.

The procedure for comparing analytical and numerical results here relies on the fact that the analytical solution has no dependence on the horn angle. It is, therefore, valid to use the special shape of the horn of angle $\pi/2$, namely a wall surrounding the inlet. For the purpose of the comparison, it is noted that waves are radiated from the source in the wall radially in all forward directions and there is no angular cross flow. Since the directional velocity is the same in every radial direction, it is valid to compare the 1-D analytical solution to the numerically calculated values along the line of symmetry, $\theta = 0$.

2.3.6 Composite Domain

The 'source-in-wall' domain is a segment of the flanged duct domain. Having verified that a solution can be obtained for the reduced domain, the next step is to replace the source with a duct. In this extension, initially plane waves generated by the oscillating inlet piston of the duct travel down the duct where they are then radiated out into the half sphere.

The unflanged duct domain can also be modelled as a further extension of the above, by translation and rotation of the 'wall' so that its resultant coincides with the outer surface of the duct. In the quasi two-dimensional case, this corresponds to rotating the top segment of the wall by $\pi/2$ and the lower segment by $-\pi/2$.

For each of these extended domains, the far-field computational boundary is set at a finite distance.
Figure 2.1: Finite Domain for Conical Horn Computations
Figure 2.2: Finite Domain for Flanged Duct Computations
Figure 2.3: Finite Domain for Open Duct Computations
Part II

Numerical Solution Procedure
Chapter 3

Numerical Grid Generation
In numerical computations, it is usually not possible to compute numerical values for all the points in the physical domain. Instead, the distribution of the flow properties in the domain is approximated by values computed in a small, suitably distributed, finite subset of the points in the domain. Furthermore, it is rare for a physical domain to coincide with a rectangular transformation of the form

\[ X \leftarrow \alpha x \]  
\[ Y \leftarrow \beta y \]  

that is, a simple dilation of the computational rectangle. Here, \( \alpha \) and \( \beta \) are scalars chosen so that the transforms given in equations (3.1) and (3.2) map the ranges \([x_{\min}, x_{\max}]\) and \([y_{\min}, y_{\max}]\), in the physical domain to the ranges \([\xi_{\min}, \xi_{\max}]\) and \([\eta_{\min}, \eta_{\max}]\), in the computational domain. Certain computational schemes involve taking a large rectangle that includes all of the physical domain, labelling the points as either inside or outside of the physical domain and then carrying out the solution on the ‘inside’ points only. The main pitfall here is that this invariably leads to situations where the physical boundary falls between sampled points, requiring interpolations or extrapolations. Interpolations and extrapolations usually involve truncated series, thereby adding to the inaccuracy of the numerical solution.

In view of these difficulties, it is desirable to construct a computational domain including the boundaries of the physical domain as well as points in the interior.

The choice of these points depends on the flow in question and it may sometimes be necessary to concentrate more computational points in certain areas of the physical domain, where the flow is expected to be changing rapidly with spatial co-ordinates, in order to see more of the details of the flow at such locations.
There are two ways of choosing the finite computational domain, namely algebraic and iterative numerical grid generation.

### 3.1 Numerical Grid Generation

In an iterative numerical grid generation, the physical domain boundary is mapped onto a polygon, usually a rectangle, and the regularly spaced points in the rectangle are then used to map back inversely to points in the interior of the physical domain. A mathematical formulation of this process, in two-dimensional space, goes as follows. The points \((\xi, \eta)\) in the computational polygon satisfy the Laplace equation, as functions of the physical space variables \((x, y)\), subject to the boundary conditions imposed using the boundary mapping, see Thompson [50]. Thus

\[
\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0, \quad (3.3)
\]

\[
\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0. \quad (3.4)
\]

To get a set of points \((x(\xi, \eta), y(\xi, \eta))\) in the physical domain, these relations are inverted to give a set of partial differential equations for \((x, y)\) in terms of \((\xi, \eta)\),

\[
\alpha \frac{\partial^2 x}{\partial \xi^2} - 2\beta \frac{\partial^2 x}{\partial \xi \partial \eta} + \gamma \frac{\partial^2 x}{\partial \eta^2} = 0, \quad (3.5)
\]

\[
\alpha \frac{\partial^2 y}{\partial \xi^2} - 2\beta \frac{\partial^2 y}{\partial \xi \partial \eta} + \gamma \frac{\partial^2 y}{\partial \eta^2} = 0, \quad (3.6)
\]

where

\[
\alpha = \left( \frac{\partial x}{\partial \eta} \right)^2 + \left( \frac{\partial y}{\partial \eta} \right)^2. \quad (3.7)
\]
\[
\beta = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} + \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \tag{3.8}
\]

and

\[
\gamma = \left( \frac{\partial x}{\partial \xi} \right)^2 + \left( \frac{\partial y}{\partial \xi} \right)^2. \tag{3.9}
\]

Equations (3.5) and (3.6) are discretised by central differences to give difference expressions for the points in the interior of the domain.

Setting \( \Delta \xi = \Delta \eta = 1 \), the difference equations for the first-order derivatives of \((x, y)\) are given by:

\[
X_\xi(i, j) = \frac{X(i + 1, j) - X(i - 1, j)}{2}, \tag{3.10}
\]

\[
Y_\xi(i, j) = \frac{Y(i + 1, j) - Y(i - 1, j)}{2}, \tag{3.11}
\]

\[
X_\eta(i, j) = \frac{X(i, j + 1) - X(i, j - 1)}{2} \tag{3.12}
\]

and

\[
Y_\eta(i, j) = \frac{Y(i, j + 1) - Y(i, j - 1)}{2}, \tag{3.13}
\]

where uppercase letters denote the numerical approximations of the corresponding lower case analytical quantities.

From these relations, \( \alpha, \beta \) and \( \gamma \) are approximated by

\[
\alpha(i, j) \approx A(i, j) = X_\eta^2 + Y_\eta^2, \tag{3.14}
\]

\[
\beta(i, j) \approx B(i, j) = X_\xi Y_\eta + X_\eta Y_\xi \tag{3.15}
\]

and

\[
\gamma(i, j) \approx \Gamma(i, j) = X_\xi^2 + Y_\xi^2. \tag{3.16}
\]
The second-order difference equations for $X$ are

$$X_{\xi}(i, j) = X(i + 1, j) - 2X(i, j) + X(i - 1, j), \quad (3.17)$$

$$X_{\eta}(i, j) = X(i, j + 1) - 2X(i, j) + X(i, j - 1), \quad (3.18)$$

and

$$X_{\xi}(i, j) = X_{\eta}(i, j) = \frac{X_{\xi}(i + 1, j) - X_{\xi}(i - 1, j)}{2}. \quad (3.19)$$

Substituting the difference equations in the differential equations yields the point equations for the $X$ and $Y$ co-ordinates as

$$X(i, j) = \left( \frac{A}{2[A + 1]} (X_{\xi} + 2X) - BX_{\eta} + \Gamma (X_{\eta} + 2X) \right)_{i,j}. \quad (3.20)$$

The equations for the $Y$ co-ordinates are of the same form with $Y$ in place of $X$.

Once the boundary values $X(\xi_{\text{min}}, j), X(\xi_{\text{max}}, j), X(i, \eta_{\text{min}})$ and $X(i, \eta_{\text{max}})$ for $X$, and similar conditions for $Y$, are set the system can be solved iteratively by Gauss-Seidel or by SOR, for the distribution of the physical co-ordinates as functions of the computational co-ordinates. For convenience, only integer values of the computational co-ordinates are sampled. This method gives the sets:

$$X(i, j) : i \in \{\xi_{\text{min}} = 0, ..., \xi_{\text{max}} = nx\}, j \in \{\eta_{\text{min}} = 0, ..., \eta_{\text{max}} = ny\}$$

and

$$Y(i, j) : i \in \{\xi_{\text{min}} = 0, ..., \xi_{\text{max}} = nx\}, j \in \{\eta_{\text{min}} = 0, ..., \eta_{\text{max}} = ny\}$$

where

$$i, j \in \mathcal{N}.$$
domain. These factors are usually chosen to concentrate grid points in sensitive parts of the flow domain. In advanced cases, the grid solution is designed to depend on the actual solution process so that $P$ and $Q$ are determined by the behaviour of the flow properties, usually pressure gradients. In such circumstances, the grid is no longer a fixed set of points chosen prior to solution but ‘floats’ within the boundaries of the domain. This class of grid has been labeled ‘adaptive grid generation’ for obvious reasons.

### 3.2 Algebraic Grid Generation

The relatively simpler alternative is to generate the grid algebraically. This involves much the same transformations on the boundaries but is less expensive. Once the physical boundary has been assigned on the computational grid, the interior points can be determined by a once-only interpolation. This will become clearer when the grids for this thesis are generated later on, but it should be noted that the resulting grids are then far less versatile than the so-called adaptive grid.

In the case of the closed duct a straightforward rectangular grid based on a dilation of the duct to a rectangle in the computational domain is chosen. The mapping is given by

\[
X(i, j) \leftarrow x_{\text{min}} + \frac{\xi}{\xi_{\text{max}}} (x_{\text{max}} - x_{\text{min}}) \tag{3.21}
\]

and

\[
Y(i, j) \leftarrow y_{\text{min}} + \frac{\eta}{\eta_{\text{max}}} (y_{\text{max}} - y_{\text{min}}), \tag{3.22}
\]

where integer values of the computational space co-ordinates, $\xi$ and $\eta$, are selected.
3.2.1 Horn and Wall Domain Grids

In the cases of a general conical horn, figure 2.1 and the special case of a sphere-in-wall domain, figure 3.1, the grid is generated as a rectangular grid in the \((r, \theta)\) space through the relations \(x = r \cos \theta\) and \(y = r \sin \theta\). With this transformation, a rectangle in \((r, \theta)\) space is mapped onto a segment of a disc in \((x, y)\). On the computational rectangle \(abcd\), figure 3.1, lines parallel to \(ad\) correspond to arcs originating from the wall \(AB\) and ending on the wall \(CD\). On each of these arcs the distance component \(r\) in the computational domain is fixed and only the angle \(\theta\) varies. Lines parallel to \(ab\) are maps of lines radiating for the semi-circular arc \(AD\) to the arc \(BC\) in the physical domain. In this class of grid lines, the angle \(\theta\) is fixed on each of the lines while the distance, \(r\), varies.

With the above in mind, it is easily seen that the grid generation only involves a single iteration. The grid points are generated by the simple interpolation: \(X(i, j) = r_i \cos \theta_j\) and \(Y(i, j) = r_i \sin \theta_j\), where \(r_i = \frac{iR}{nx}\), \(R\) being the distance from the centre of the small sphere to the surface of the larger sphere, that is \(R = \frac{|AD|}{2} + |AB|\) and \(\theta_j = \frac{j\pi}{nr} - \frac{\pi}{2}\). For a 'true' conical horn, the angle is less than \(\pi/2\) so the values \(\theta_j\) would be given in general by \(\theta_j = 2j\alpha/nr - \alpha\) if the angle is \(\alpha\) in radians.

On closer inspection it may be noted that this is a two stage grid generation in which stage one is to apply a dilation mapping \([r, R]\) to \([0, nx]\) and \([-\pi/2, \pi/2]\) to \([0, nr]\), while stage two involves transforming the resultant set

\[
\{\{i, j\} \in \{[0, nx] \times [0, nr]\}\}
\]

to a set

\[
\{\{x(i, j), y(i, j)\} : (i, j) \in \{[0, nx] \times [0, nr]\}\}
\]

of points in the physical domain.
A grid generated this way for the conical horn domain is shown in figure 3.2. The piston-in-wall domain grid is illustrated in figure 3.3.

### 3.2.2 Flanged Duct Domain Grid

As discussed earlier, the flanged duct is a composite of the duct and the hemisphere of the 'wall' domain. The grid generation for this domain consists of mapping the physical domain to a rectangle which is subdivided into two smaller rectangles, one for the duct and the other for the hemisphere, figure 3.1. The grids for these subdomains are generated in similar ways to those for the simpler domains. The duct is largely rectangular, as before, but the closed end now terminates in a semi-circular arc. In the radial direction the duct is divided into $n_y$ equal intervals, as before, but in the tangential direction it is now necessary to interpolate with different values for $x_{\text{min}}$ and $x_{\text{max}}$ for each of the radial intervals. This gives a grid for the duct section of the form

$$X(i, j) \leftarrow x_{\text{min}}(j) + \frac{\xi}{\xi_{\text{max}}} (x_{\text{max}}(j) - x_{\text{min}}(j)),$$

i.e. the interval of interpolation is now dependent on the radial position in the physical domain. The distribution of $Y(i, j)$ stays the same. The rest of the grid for the hemisphere is generated as in the 'wall' domain case, figure 3.4. The resulting grid is shown in figure 3.6.

### The T-Transform

This grid is generated using a transform that looks like a 'T' laid on its side. This consists of a long rectangle - the 'trunk' of the 'T' with two further smaller rectangles attached either side on the right end of the 'trunk'. The 'trunk' is mapped to a cylindrical duct and a conical horn shape while the two smaller rectangles are mapped to smaller regions, namely $BCD$ and
GEF, to complete the hemisphere, see figure 3.5. The grid within these regions is generated from the smaller rectangles using the $(x, y) \sim (r, \theta)$ transform.

For the 'trunk', the section of grid is generated as in the case of the 'flanged duct' domain with the hemisphere now replaced by the conical horn.

This procedure is quite obvious on inspecting the transform diagrams and resulting grids, see figures 3.5 and 3.7.

The key advantage of the 'T'-transform comes from its usefulness in reducing the sizes of the cells at the far boundary in either the hemispherical or spherical domains. In addition to this, when the spherical domain is discretised, this method is vital in avoiding intersection of grid lines on and about the exit plane. The main potential disadvantage is that some of the grid cells immediately outside the exit plane are triangular rather than quadrilateral in shape. However, this is a minor defect by comparison to the large far field cells and intersecting grid lines that would occur in the rectangular transformation.

One more possibility for the flanged duct domain is what might be called an 'L-Transformation' in which the hemisphere is mapped onto a rectangle so that the line of symmetry goes to $\eta = 0$ and the wall goes to $\xi = 0$. The arc forming the hemispherical boundary is then split between $\xi = \xi_{max}$ and $\eta = \eta_{max}$. If the rectangle is a square, i.e. $\xi_{max} = \eta_{max}$, a convenient transform is one in which the points on the lower half of the far-field boundary, namely, $(r \cos(\theta), r \sin(\theta)), \forall \theta \leq \pi/4$ are mapped onto the set $\{(\xi_{max}, [\eta_{min} - \eta_{max}])\}$ and the rest of the arc of the far-field boundary is mapped onto the set $\{([\xi_{min} - \xi_{max}], \eta_{max})\}$. The interior points are interpolated from the boundary conditions and then smoothed according to the rules specified in section (3.1). This gives mostly quadrilateral cells, except on the far boundary where one cell is triangular, but the main pitfall
in this grid system is the high density of the grid required to produce a relatively uniform cell variation in the neighbourhood of the duct exit.

**Figure 3.8** shows a sample grid under this transform. The main disadvantage of this grid is that it cannot be extended in a straightforward manner to a totally open duct domain grid. It is also less clear how to set the far-field boundary conditions for the flow properties and, to avoid large cells in the neighbourhood of the exit plane, a larger number of cells is needed.

### 3.2.3 The Unflanged Duct Domain Grid

Generation of a grid for the case of an unflanged duct follows simply from the two forms of transforms for generating the ‘flanged duct’ grid, **figures 3.4 and 3.5**, by rotating the wall segments so that they coincide with the outer surface of the duct. However, this simple approach is problematic in that the cells near the far-field boundary become too large, and are not easily controllable with the transform in **figure 3.4**. Also, under this transform, the cells near the exit plane are very skewed.

This leaves the possibility of the ‘T-grid’ or the ‘L-grid’, but no straightforward way has been found to establish an ‘L-transform’ for the open duct domain.

**Figures 3.9 and 3.10** show the radial grid and the ‘T’ grid generated for a finite open duct domain.

Finally it is worth pointing out that due to the axial symmetry of all the domains considered here, only half of each of the grids needs to be generated and the numerical solution is only carried out on those halves.
Figure 3.1: Mapping for the Finite Conical Horn/Wall Domain
Figure 3.2: A Radial Grid for the Finite Conical Horn Domain
Figure 3.3: A Radial Grid for the Finite 'Source-in-wall' Domain
Figure 3.4: Radial Mapping for the Finite Flanged Duct Domain
Figure 3.5: A 'T' Mapping for the Finite Flanged Duct Domain
Figure 3.6: A Radial Grid for the Finite Flanged Duct Domain
Figure 3.7: A 'T' Grid for the Finite Flanged Duct Domain
Figure 3.8: An L-Transform Grid for the Finite Flanged Duct Domain
Figure 3.9: A Radial Grid for the Finite Unflanged Duct Domain
Figure 3.10: A 'T' Grid for the Finite Unflanged Duct Domain
Chapter 4

Numerical Computation
4.1 The Finite Volume Formulation

In section 2.2.1 the axi-symmetric conservation system was given, equation (2.6). This can be rewritten as:

\[
\frac{1}{r^2} \left[ \frac{\partial}{\partial t} (rU) + \frac{\partial}{\partial r} (rF) + \frac{\partial}{\partial z} (rG) \right] = \frac{1}{r^2} H
\]  

(4.1)

Let \( \Delta v \) be a subset of the flow domain. Then the system of governing equations, (4.1), must be satisfied at every point in \( \Delta v \) and summing the system over all the points in the subset, bearing in mind axial symmetry, gives

\[
\int \int_{\Delta v} \left[ \left( \frac{\partial}{\partial t} (U) + \frac{1}{r} \frac{\partial}{\partial r} (rF) + \frac{\partial}{\partial z} (rG) \right) - \frac{1}{r} H \right] drdz = 0
\]

(4.2)

or

\[
\int \int_{\Delta v} \left[ \frac{\partial U}{\partial t} + \left( \frac{1}{r} (F - H) \right) \right] drdz = - \int \int_{\Delta v} \left( \frac{\partial F}{\partial r} + \frac{\partial G}{\partial z} \right) drdz.
\]

(4.3)

By the divergence theorem the integral on the right hand side of equation (4.3) can be cast in the form of a boundary integral giving

\[
\int \int_{\Delta v} \frac{\partial}{\partial t} U drdz = \int \int_{\Delta v} h drdz - \int S (F dr - G dz)
\]

(4.4)

where

\[
h = \frac{1}{r} (H - F).
\]

The finite volume scheme is well known as is discussed by Morton and Paisley [51].

Equation (4.4) says that the rate of change of the conservation variables (mass, momentum and energy) in the space \( \Delta v \) is the flux of the conservation variables into the space minus the flux out of the space in a given time interval.
For a difference system, the volume integrals are approximated by the product of the volume of $\Delta v$ and the average of the integrand in $\Delta v$. For the numerical approximation of the boundary integrals, the boundary, $S$, of $\Delta v$ is approximated by a set of line segments so that it is a polygon. Then using values of $u$ and $p$ at its vertices, the integral along each of the sides of the polygon is approximated by the flux of the conservation variables across that side. The total boundary integral is the sum of the segment integrals in the anti-clockwise direction,

$$\Delta v \frac{\partial U}{\partial t} = \Delta v h - \sum_{k=1}^{n} \int_{S_k} [Fdr - Gdz]$$

(4.5)

where $n$ is the number of edges of the polygon.

By observation

$$\frac{\partial U}{\partial t} = h - \frac{1}{\Delta v} \sum_{k=1}^{n} \int_{S_k} [Fdr - Gdz]$$

so this process should work as long as the volume $\Delta v$ is not null, regardless of how many sides the polygon has. For computational purposes it is convenient to limit the polygons to triangles or quadrilaterals. In either case, the first thing to watch is that the vertices must not be all collinear. When quadrilaterals are used, it is possible for the system to still work when some of the cells degenerate into triangles. Quadrilaterals are used in this thesis.

### 4.1.1 Cell-centred Schemes

Under this scheme, see Morton and Paisley [51], the flow properties are stored at the cell-centroids, such as $(i, j)$. The flux values required at the cell boundaries are approximated by averages of their values at the cell centres. With reference to figure 4.1, the value of flux function $F$ on the
boundary $AB$, for instance, is given by

$$E_{AB} = \frac{E_{i,j} + E_{i+1,j}}{2}.$$  

The values on the other cell edges are calculated similarly.

Based on this approximation of the flux functions, the spatially differenced system of the Euler equations on a rectangular grid domain is given by

$$\left( \frac{\partial}{\partial t} U - \frac{h}{2} \right)_{i,j} \Delta r \Delta z + (E_{i+1/2,j} - E_{i-1/2,j}) \Delta r$$

$$+ (G_{i,j+1/2} - G_{i,j-1/2}) \Delta z = 0$$

which reduces, on division by $\Delta r \Delta z$, to

$$\frac{\partial}{\partial t} U_{i,j} + \frac{E_{i+1,j} - E_{i-1,j}}{2\Delta z} + \frac{G_{i,j+1} - G_{i,j-1}}{2\Delta r} = h_{i,j}.$$  

It is seen that this equation does not involve the flux values $E_{i,j}$ or $G_{i,j}$ and that for nodes where $i + j$ is even the system depends only on points with $i + j$ odd, and vice versa. This property of the scheme leads to odd-even decoupling, creating spurious solutions, see Hirsch[12]. Morton and Paisley[51] carried out a Fourier analysis of this scheme, showing three distinct spurious modes that could be supported by the scheme. The effect of the spurious modes is to cause two-dimensional error waves in the flow solution of the form as in figures 4.2a,b,c.

In each of these figures, a '+' denotes the peak of a spurious wave while a '-' denotes its trough. **Figure 4a** consists of waves of alternate peaks and troughs on neighbouring grid lines, both in the $\xi$- and $\eta$- directions, roughly the structure of a chequer board. **Figure 4.2b** illustrates peaks and troughs on alternate $\xi$ lines, i.e. a saw-tooth cross-section in the $\eta$- direction with each of the $\xi$ lines either a continuous peak or a continuous trough. **Figure 4.2c** has the same structure as 4.2b but with the features rotated through $\pi/2$ radians.
These waves, unless aggravated by an unsuitable boundary condition, a shock or a grid discontinuity, will stay constant in amplitude, preventing the flow solution from converging satisfactorily beyond a certain accuracy. otherwise they could cause the solution to diverge.

4.1.2 Cell-vertex Schemes

With reference to figure 4.1, when using the cell-vertex scheme, see Morton and Paisley [51], all flow properties are stored at the cell vertices, such as A, B, C etc. The difference form of equation (4.5) at the centroid \((i, j)\) is given by

\[
\frac{\partial U_{i,j}}{\partial t} = h_{i,j} - \frac{1}{\Delta v} \sum_{k=1}^{4} \int_{S_k} [F dr - G dz]_k
\]

\[
= h_{i,j} - \frac{1}{\Delta v} \left[ \int_{A}^{B} (F dr - G dz) + \int_{B}^{C} (F dr - G dz) + \int_{C}^{D} (F dr - G dz) + \int_{D}^{A} (F dr - G dz) \right]
\]

\[
\approx h_{i,j} - \frac{1}{\Delta v} \left[ E_{AB} (r_B - r_A) - G_{AB} (z_B - z_A) + E_{BC} (r_C - r_B) - G_{BC} (z_C - z_B) + E_{CD} (r_D - r_C) - G_{CD} (z_D - z_C) + E_{DA} (r_A - r_D) - G_{AD} (z_A - z_D) \right]
\]

For a cell-vertex scheme as used here

\[
E_{AB} = \alpha E_A + (1 - \alpha) E_B
\]

where \(0 \leq \alpha \leq 1\) for a generalised upwinded scheme. The choice of parameter made here is \(\alpha = 1/2\) for a simple averaged scheme. The flux functions at the rest of the cell edges are derived similarly.

It has been shown, see Morton and Paisley [51], that the trapezoidal cell vertex scheme supports a 'chequer-board' spurious mode, figure 4.2a. This is better than the cell-centred scheme which has at least two more
spurious modes, see figures 4.2b, c, in shock-free regions of the flow. In either case, artificial viscosity is needed to damp the spurious modes. This damping will be discussed in a separate subsection.

4.2 A Finite-difference Equivalence

A finite-difference equivalent of the finite volume scheme is derived in the transformed domain, namely \((\xi, \eta)\), in line with Thompson et al. [50]. This procedure gives the finite volume equation (4.5) in a rectangular grid, as required later in section 4.3.2, in a discussion of flux-splitting schemes.

Consider the conservation system in equation (4.1). Then substituting

\[ \frac{\partial F}{\partial r} = \frac{(r_\eta F_\xi - r_\xi F_\eta)}{J} \]

and

\[ \frac{\partial G}{\partial z} = \frac{(-z_\eta G_\xi + z_\xi G_\eta)}{J}, \]

where \(J\) is the modulus of the Jacobian of the transformation, it can be seen that

\[ \frac{\partial}{\partial t} \mathbf{U} + \left( \frac{F_\xi r_\eta - F_\eta r_\xi}{J} \right) + \left( \frac{G_\eta z_\xi - G_\xi z_\eta}{J} \right) = h. \]

Also, it can be verified that

\[ (E r_\eta - G z_\eta)\xi + (G z_\xi - F r_\xi)\eta = (F_\xi r_\eta - G_\xi z_\eta) + (G_\eta z_\xi - E_\eta r_\xi) \]

\[ + (E r_\eta z_\xi - G r_\eta z_\xi) + (G r_\eta z_\xi - F r_\eta z_\xi) \]

with the last two terms cancelling. Applying this simplification leaves the vector system of equations as

\[ J \frac{\partial}{\partial t} \mathbf{U} + (E r_\eta - G z_\eta)\xi + (G z_\xi - F r_\xi)\eta = J h, \quad (4.7) \]

for the conservation of mass, momentum and energy.
As in section 4.1, equation (4.7) is integrated over a finite volume, but this time in the $\xi - \eta$ domain, giving

$$
\int_{\Delta v'} J \left( \frac{\partial}{\partial t} U - h \right) d\xi d\eta = - \int_S \left( (F_{r\eta} - G_{\zeta\eta}) d\eta - (G_{\zeta \xi} - Fr_{\xi}) d\xi \right)
$$

(4.8)

where $\Delta v'$ is the volume in the transformed domain, $(\xi, \eta)$.

Equation (4.8) can be approximated by

$$
\int_{\Delta v'} J \left( \frac{\partial}{\partial t} U - h \right)_{i,j} d\xi d\eta = - \left[ (F_{r\eta} - G_{\zeta\eta})_{i+1/2,j}(\Delta \eta) + (G_{\zeta \xi} - Fr_{\xi})_{i,j+1/2}(-\Delta \xi) + (F_{r\eta} - G_{\zeta\eta})_{i-1/2,j}(-\Delta \eta) + (G_{\zeta \xi} - Fr_{\xi})_{i,j-1/2}(\Delta \xi) \right]
$$

for the cell centred on $(i, j)$. With $\Delta \xi = \Delta \eta = 1$ the volume $\Delta v'$ is of size 1 and the integral terms can be simplified giving the finite-difference relation

$$
J \left. \frac{dU}{dt} \right|_{i,j} = J H_{i,j} - \left\{ (F_{r\eta} - G_{\zeta\eta})_{i+1/2,j} + (G_{\zeta \xi} - Fr_{\xi})_{i,j+1/2} - (F_{r\eta} - G_{\zeta\eta})_{i-1/2,j} + (G_{\zeta \xi} - Fr_{\xi})_{i,j-1/2} \right\}
$$

(4.9)

where $\left. dU/dt \right|_{i,j}$ and $H_{i,j}$ refer to average values in the cell $i, j$.

4.2.1 Artificial Viscosity

As mentioned earlier, sections 4.1.1 and 4.1.2, the cell-vertex finite-volume, and its curvilinear finite-difference equivalent formulation, support a spurious chequer-board mode, see figure 4.2a.

This means that the solution will be contaminated by a small-amplitude regular two-dimensional wave. It has been observed by Morton and Paisley[51] that this does not depend on the grid spacing and cannot be removed by refining the grid. Also, on a fairly regular grid in flows with 'well-behaved boundary conditions', its effect may go unnoticed unless particularly small-scale details of the flow are sought.
The problems posed by the spurious modes are compounded when the flow encounters 'obstacles' such as discontinuities in the computational domain or 'unfavourable' boundary conditions.

An obstacle in the computational domain may come in the form of a sudden change in grid spacing or direction, such as would occur around the exit plane of a duct flow, or quite commonly due to shocks in the flow itself. In aeronautical simulations, flows about aerofoils or wings commonly generate shocks.

An unfavourable boundary condition is usually one where the numerical implementation causes numerical error waves to be reflected back into the flow domain, or around corners of the computational domain where a certain ambiguity may occur in the implementation of the boundary conditions. In either of these cases, the reflected waves may linger in the computation at more or less fixed amplitude, preventing the computation from converging, but not necessarily causing divergence if the computational domain does not also contain obstacles of the form mentioned above. If it does, the wave interference is magnified leading to divergence.

One way of dealing with reflected waves in flow computations is to use characteristic based non-reflective boundary conditions. The idea is to split the flow incident on the boundary into its characteristic components travelling in and those travelling out of the flow domain. The inward-travelling waves are then neglected, in the simplest cases. This procedure, on its own, will not work with flows where there should be reflected waves due to the physical conditions of the flow itself, as there is no way to separate spurious reflections from physical reflections. This is not a problem faced in a vast majority of flow computations but it does come into consideration in this research.

The more popular technique is the addition of artificial dissipation to (4.5).
which can be written as
\[ \frac{\partial U}{\partial t} \bigg|_{i,j} = R_{i,j} \]  
(4.10)
where \( R \) denotes the residual in the cell \((i, j)\). Equation (4.10) is modified to
\[ \frac{\partial U}{\partial t} \bigg|_{i,j} = R_{i,j} + D_{i,j} \]
where \( D_{i,j} \) denotes the dissipation terms, given by
\[ D_{i,j} = \mu(U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}), \]
the sum of the two second-order differences in the \( i \)- and \( j \)- directions, with \( \mu \) a constant chosen to ensure smooth convergence for small-amplitude sources. For the numerical solutions given in this thesis, values of \( \mu \) were found by trial-and-error and were in the range \( 0.004 \leq \mu \leq 0.0065 \). In general it is necessary, unless a shock-fitting scheme is used, to also subtract a term proportional to the fourth-order differences in regions of sharp change, such as in the neighbourhood of shocks, before a solution will converge, see Morton and Paisley [51]. These regions are detected usually by monitoring the pressure gradient. In this thesis, the flow is always subsonic so the fourth-order terms are not needed.

The function of artificial viscosity is to dampen the spurious waves. The objective in this procedure is to strike a balance between removing the spurious modes and introducing too much damping, i.e., avoiding adding so much damping that some of the flow properties are substantially contaminated. This is still not a strictly formal procedure, as can be deduced by the various forms of damping have been used, see Hirsch[12], Morton and Paisley[51] and Turkel[52].

Early forms of artificial viscosity involved adding a constant level of damping at every point in the computational domain. Recently grid dependence,
which amounts to scaling the damping according to the skewness of the local grid, has been introduced.

Recent work by Turkel [52] takes this a step further, scaling by local skewness and also by flow speed. This introduces better control into the solution process and careful choice of the parameters gives much improved results. The down side of these improvements is that they are still not standard, and cannot be applied in an 'off-the-shelf' form for different computations. With Turkel's scheme there are new scaling parameters for each of the extra factors introduced together with base levels designed to avoid the dissipation vanishing anywhere during the computation. All these parameters are derived by trial-and-error. However, the choices are guided by the simpler forms of artificial viscosity parameters. These decisions are to some extent less important in steady-state flow computations, where the damping terms vanish by design as the residuals tend to zero.

At the other extreme, they are particularly important in time-dependent periodic flows involving the propagation of waves as part of the properties of the flow. The residuals do not vanish and, due to the inherent waves of the flow, might 'deceive' the damping scheme into contaminating the flow properties. This is particularly the case in this thesis.

A better remedy is the possible use of characteristic-based flux/vector split schemes where the spurious modes are minimised by the design of the schemes.

4.3 Characteristic-based Schemes

In the last section it was stated that spurious modes arise, contaminating the numerical solutions where the finite volume scheme is used. Various forms of artificial viscosity were introduced for dealing with these modes.
but it was concluded that their influence was not sufficiently clear-cut for computations of unsteady flows.

4.3.1 Upwind Schemes

Central-difference schemes, or their equivalents, do not always follow the proper path of propagation of the flow variables as they tend to use information from outside the domain of dependence of the given grid point. This appears to be relatively unimportant in computations of continuous smoothly-varying flow fields, but in flow fields with discontinuities, such as shocks and/or non-smooth domains, this property of the central-difference schemes is to blame for spurious oscillations that occur in the neighborhood of the discontinuity. The addition of artificial viscosity reduces, but does not eliminate, these oscillations.

By contrast, upwind schemes are designed to take proper account of the domain of dependence of each grid point by properly simulating the direction of propagation of flow information along the characteristic curves, Anderson[53].

Consider the partial differential equation

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial z} = 0.
\]

For positive c, this describes the propagation of a wave in the positive direction of the z-axis. Considering a disturbance traveling from left to right, the flow at grid point i should only depend on points to the left of that point, namely \(i - 1, i - 2, \ldots\) where \(i\) denotes a grid node. To first-order, the partial differential is approximated by the difference equation

\[
\frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t} + c \frac{u_{i}^{n} - u_{i-1}^{n}}{\Delta z} = 0
\]

for marching the variable \(u\) from time \(n\) to time \(n + 1\).
This scheme is known to be stable, but will smear the discontinuity monotonically in time, so that the solution becomes less accurate in the neighbourhood of discontinuities.

If a central-difference scheme is used, the point \( i + 1 \) is made to influence the flow at \( i \), and the computation may diverge.

Upwinding is the basis of a series of advanced CFD schemes of the last twenty years. Schemes such as the flux-splitting method of Van Leer [54], flux-limiters, Godunov, approximate Riemann solvers of Roe [14] and total variation diminishing, TVD, schemes of Harten [54] have been designed to maintain the advantages of upwinding, namely the proper propagation of waves, while minimising its main disadvantage, diffusivity.

### 4.3.2 Flux Splitting

Consider the conservation equation

\[
\frac{\partial u}{\partial t} + \frac{\partial f}{\partial z} = 0
\]

for a compressible flow. If

\[
A = \frac{\partial f}{\partial u}
\]

and \( f \) is a homogeneous function of \( u \), then

\[
f = Au,
\]

see Hirsch[12].

The characteristic speeds of the flow are given by the eigenvalues of \( A \), see Wang et al.[56]. Also, it can be shown that

\[
\Lambda = T^{-1}AT
\]

where \( \Lambda = \{\lambda_1, \lambda_2, \lambda_3\} \) is a diagonal matrix of eigenvalues of \( A \) and \( T \) is a matrix of corresponding normalised eigenvectors.
If \( \Lambda \) is split into non-negative and negative components, \( \Lambda^+ \) and \( \Lambda^- \), then the corresponding matrices
\[
A^+ = T\Lambda^+T^{-1}, \quad A^- = T\Lambda^-T^{-1}
\]
are defined such that
\[
f = f^+ + f^- = A^+u + A^-u.
\]
Hence \( A^+u \) and \( A^-u \) are the components of the flux propagated in the non-negative, and negative directions, respectively. The flow equation can thus be rendered in the split form
\[
\frac{\partial u}{\partial t} + \frac{\partial f^+}{\partial z} + \frac{\partial f^-}{\partial z}
\]
and the components can be differenced appropriately.

A class of schemes based on the method of characteristics has been developed over the past twenty years, whose strong point is reducing the need for artificial viscosity by, in theory, eliminating the spurious modes from the solution process. These schemes come in two sub-classes: - vector splitting and flux difference splitting, which both rely on splitting the convected flow properties into forward and backward travelling components. The differencing is then chosen depending on the direction of the characteristic speed.

A well known class of flux splitting schemes is presented in Hirsch [12] for one-dimensional and two-dimensional flows. The two-dimensional splitting schemes require rectangular grids. The finite-difference relation derived above, equation (4.9), enables some of these schemes to be employed.

A note of caution is sounded, see Hirsch [12], about the influence of the transform metrics, e.g. \( r_\eta \), on the residuals of flux in each finite volume. Some methods of calculating the metrics result in numerical sources being created in the flow residuals.
The flux splitting scheme used in some of the simple duct computations in this work is due to Steffen and Liou [57] with modifications by Radespiel and Kroll [58]. Unlike most other flux-splitting schemes, these two are based around splitting the pressure and convected components of the flux functions separately. The split of the pressure component is based on local acoustic velocity, while the split of the convected conservation properties is based on the local flow velocity. Both Steffen and Liou [57] and Radespiel and Kroll [58] present results which come close to the best of the Roe solver class of schemes but their schemes avoid the costly matrix arithmetic involved in this latter scheme.

The flux split equations are illustrated for the axial direction, thus with reference to equation (4.1), the split for the flux $F$ is

$$ F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(\rho e + p) \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ (\rho e + p) \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.11) $$

At an interface surface perpendicular to the line segment connecting two nodes $L$ and $R$ the convected flux is given by

$$ F^c_{L/R} = u_{L/R} \begin{pmatrix} \rho u \\ \rho u^2 \\ \rho uv \\ u(\rho e + p) \end{pmatrix}_{L/R} = M_{L/R} \begin{pmatrix} \rho c \\ \rho cu \\ \rho cv \\ c(\rho e + p) \end{pmatrix}_{L/R}. \quad (4.12) $$

where

$$ (*)_{L/R} = \begin{cases} (*)_L & \text{if } M_{L/R} \geq 0 \\
(*)_R & \text{otherwise}. \end{cases} $$
and $c$ is the local sound speed. This corresponds to a whole class of schemes based on the choice of function $M_{L/R}$ of the Mach number. It is chosen as a combination of wave speeds, $M \pm 1$ travelling towards the interface surface from the nodes, $L$ and $R$,

$$M_{L/R} = M^+_L + M^-_R,$$

the right hand side being the aforementioned wave speeds.

The Van Lee [57] splitting states these speeds as

$$M^\pm = \begin{cases} 
\pm(M \pm 1)^2/4, & \text{if } |M| \leq 1 \\
(M \pm |M|)/M, & \text{otherwise.}
\end{cases} \quad (4.13)$$

The acoustic flux, i.e. the pressure, is split similarly to the wave speeds using polynomial functions of the Mach speed.

$$p^\pm = \begin{cases} 
\pm \frac{3}{4}(M \pm 1)^2(2 \mp M), & \text{if } |M| \leq 1 \\
\frac{5}{2}(M \pm |M|)/M, & \text{otherwise.}
\end{cases} \quad (4.14)$$

A first-order expansion in $M$ is also possible, namely

$$p^\pm = \begin{cases} 
\pm \frac{5}{2}(1 \mp M), & \text{if } |M| \leq 1 \\
\frac{5}{2}(M \pm |M|)/M, & \text{otherwise.}
\end{cases} \quad (4.15)$$

Putting together the acoustic and convective components of the flux, it can be seen that
\[ F_{L/R} = \frac{1}{2} M_{L/R} \left[ \begin{array}{c}
\rho c \\
\rho c u \\
\rho c v \\
\rho c v \\
c(\rho e + p) \\
c(\rho e + p)
\end{array} \right]_{L} + \left[ \begin{array}{c}
\rho c \\
\rho c u \\
\rho c v \\
\rho c v \\
c(\rho e + p) \\
c(\rho e + p)
\end{array} \right]_{R} \\
- \frac{1}{2} |M_{L/R}| \left[ \begin{array}{c}
\rho c \\
\rho c u \\
\rho c v \\
\rho c v \\
c(\rho e + p) \\
c(\rho e + p)
\end{array} \right]_{R} - \left[ \begin{array}{c}
\rho c \\
\rho c u \\
\rho c v \\
\rho c v \\
c(\rho e + p) \\
c(\rho e + p)
\end{array} \right]_{L}
\]

\[ \frac{1}{2} \frac{p_{L}^{+} + p_{R}^{-}}{0}
\]

In the radial direction, the flux \( g \) is split in the same way.

On a general grid, the velocity components may be replaced by the contravariant velocity components for the flux splitting to stay as simple as the scheme outlined here.

This splitting scheme is known as the ‘Advection Upstream Splitting Method’ (AUSM). A modified version of it, derived by Radespiel and Kroll [58], is based on a finite volume differencing and is a hybrid between the AUSM, which was found to introduce too little dissipation in some regions of the flow, and the van Leer scheme, switching between the two as and when the dissipation requirements dictate.

The dissipation term:

\[ - \frac{1}{2} |M_{L/R}| \left[ \begin{array}{c}
\rho c \\
\rho c u \\
\rho c v \\
\rho c v \\
c(\rho e + p) \\
c(\rho e + p)
\end{array} \right]_{R} - \left[ \begin{array}{c}
\rho c \\
\rho c u \\
\rho c v \\
\rho c v \\
c(\rho e + p) \\
c(\rho e + p)
\end{array} \right]_{L} \]
in expression (4.16) is replaced by
\[
-\frac{1}{2} \phi_{L/R} \left[ \begin{pmatrix} \rho c \\ \rho c u \\ \rho c v \\ c(\rho e + p) \end{pmatrix}_R - \begin{pmatrix} \rho c \\ \rho c u \\ \rho c v \\ c(\rho e + p) \end{pmatrix}_L \right],
\]
where
\[
\phi = (1 - \alpha) \phi^{VL} + \alpha \phi^{modAUSM},
\]
\(\phi^{VL}\) and \(\phi^{modAUSM}\) being the dissipation scales of the Van Leer and the AUSM schemes, respectively, and \(0 \leq \alpha \leq 1\) is a weighting factor to switch between the two scales.

\[
\phi^{VL}_{L/R} = \begin{cases} 
|M_{L/R}|, & \text{if } |M_{L/R}| \geq 1 \\
|M_{L/R}| + \frac{1}{2} (M_R - 1)^2 & \text{if } 0 \leq M_{L/R} < 1 \\
|M_{L/R}| + \frac{1}{2} (M_L + 1)^2 & \text{otherwise}
\end{cases}
\]

\[
\phi^{modAUSM}_{L/R} = \begin{cases} 
|M_{L/R}|, & \text{if } |M_{L/R}| > \delta \\
\frac{M_{L/R}^2 + \delta^2}{2\delta} & \text{otherwise}
\end{cases}
\]
with \(0 < \delta < 0.5\).

4.4 Numerical Boundary Conditions

4.4.1 Inlet

At the inlet boundary there are two cases to consider. For the piston input, the inlet velocity is set to the velocity of the piston, via

\[
n'' = u_0 \cos(\omega t_n) = u_0 \cos(n \omega \Delta t)
\]
where the input is assumed to consist of a single harmonic of frequency $\omega$ and amplitude $u_0$. For a piston oscillation with more than one harmonic, say $N$ harmonics, the source velocity is set to the sum of the contributions from each of the individual harmonics and becomes

$$u^n = \sum_{j=1}^{N} u_{0j} \cos(\omega_j t_n) = \sum_{j=1}^{N} u_{0j} \cos(n \omega_j \Delta t)$$

(4.17)

The density is set through a linear interpolation from the interior locations 1 and 2 as shown in figure 4.3, which for uniform grid spacing implies that

$$\rho_w = 2\rho_1 - \rho_2.$$  

(4.18)

If the flow is isentropic the pressure if set using equation (2.8). For non-isentropic flow, the pressure is also interpolated and the energy is set through the equation of state.

For engine input, the thermodynamic cycle of the combustion chamber is used to derive the properties based on the filling-and-emptying model of Benson et al. [2].

4.4.2 Wall Boundaries

At wall boundaries and on the line of symmetry in the duct, the normal component of the velocity is set to be zero.

For isentropic flow, the density is interpolated according to equation (4.18) and the pressure determined via equation (2.8). For other flows the pressure is also interpolated and the energy is determined through the equation of state according to a relation given later in this chapter.

In extended domain computations the wall boundary conditions outside the duct are set in the same way. On these boundary segments the interpolation
relies on the fact that the flow properties are radiated in spherical wave fronts on which they are instantaneously constant. For computations with radial or 'T' grids, the wavefronts coincide with spherical grid lines. For L-grid computations it is necessary to set the normal derivatives for the density and, where needed, the pressure as discussed later under far-field boundary conditions. Once these are determined, the rest of the conditions are set as stated.

4.4.3 The Zero-Pressure Condition at the Exit Plane

For one-dimensional time-domain analysis the generally used procedure is to terminate the computational domain at the exit plane of the duct using the condition that the pressure at that boundary is fixed at the same value as the atmospheric pressure outside of the duct. This condition has the numerical disadvantage that it induces a reflected numerical error of amplitude $\Delta v_3 = \Delta u$ from the exit plane over each time step $\Delta t$, see Hirsch[12]. The quantity $v_3$ represents the third characteristic variable given by

$$v_3 = u - \frac{1}{\rho c_0} p.$$

In most flow computations where artificial boundaries are imposed boundary reflection occurs. If unchecked, it reflects numerical errors, such as spurious modes, back into the computational domain and could stop the process from converging.

The generally adopted procedure is to apply non-reflective boundary conditions at the computational boundary. This is done by use of characteristic variables at the boundary. The characteristic variables are split into component waves travelling in perpendicular directions in and out of the flow domain. To determine the components travelling outward, only the knowl-
edge of properties in the interior of the computational domain and on the boundary is required. On the other hand, the inward travelling waves involve properties convected into the flow domain, thus requiring knowledge of properties outside the computational domain. It is rare for this knowledge to be available and in its absence the practice is to impose the no reflection condition, i.e. to set the incoming wave amplitudes to zero.

In the present thesis this is not a viable option as there are real wave reflections of large magnitude from the exit plane of the duct.

Later, in Chapter 6, a comparative analysis of results from the simple duct computation with the ‘zero-pressure’ boundary condition is presented. It shows that for low amplitude waves, the best boundary conditions for the conservation variables is an extrapolated characteristic boundary condition.

A modification of the non-reflective characteristic boundary condition is proposed and used for higher amplitude and multiple-harmonic disturbances.

Setting the Conservation Variables in the Exit Plane

With $p = p_0$ imposed, it follows that for isentropic flow

$$\rho = \rho_0$$

which implies that

$$\frac{\partial \rho}{\partial t} \equiv 0.$$ 

Substituting in the continuity equation (1.1) gives

$$\frac{\partial u}{\partial z} = \frac{-u \frac{\partial \rho}{\partial z}}{\rho_0}.$$  \hspace{1cm} (4.19)
The momentum equation (1.2) can be rewritten as
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} \]
which, on substituting for \( \frac{\partial u}{\partial z} \), gives
\[ \frac{\partial u}{\partial t} = \frac{u^2}{\rho_0} \frac{\partial \rho}{\partial z} - \frac{1}{\rho_0} \frac{\partial p}{\partial z}. \]  

(4.20)

Also, for isentropic perturbations,
\[ p = \frac{p_0}{\rho_0^\gamma} \]
\[ \Rightarrow \frac{\partial p}{\partial z} = \frac{\gamma p_0}{\rho_0^\gamma-1} \frac{\partial p}{\partial z} \]
\[ \Rightarrow \frac{\partial p}{\partial z} = \gamma p_0 \frac{\partial \rho}{\rho_0} \]
or
\[ \frac{\partial p}{\partial z} = c_s^2 \frac{\partial \rho}{\partial z} \]
where \( c \) is the local sound speed. Substituting this into equation (4.20) yields
\[ \frac{\partial u}{\partial t} = \frac{1}{\rho_0} \left( u^2 - c_s^2 \right) \frac{\partial \rho}{\partial z}. \]  

(4.21)

Alternatively, equation (4.20) is used to advance the exit plane velocity in time, with no need for the assumption of isentropic flow. The energy at the boundary is determined through the equation of state as
\[ \rho e = \frac{p}{\gamma - 1} + \frac{\rho u^2}{2}. \]  

(4.22)

In each of these cases, knowledge of the axial velocity, \( u \), outside the computational domain is needed to advance the momentum equation accurately at the boundary. In its absence, the backward-difference alone is used in the analysis.
Characteristic Boundary Conditions

At the exit plane boundary, the flow is assumed to be one-dimensional out of the duct, setting $v = 0$. Thus equations (1.1)...(1.3) can be written in condensed form as

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial z} = \frac{\partial u}{\partial t} + A \frac{\partial u}{\partial z} = 0 \quad (4.23)$$

where

$$u = [\rho, \rho u, \rho e]^T,$$

$$f = [\rho u, \rho u^2 + p, u(\rho e + p)]^T$$

and $A = \frac{\partial f}{\partial u}$ is the flux Jacobian. To determine the characteristic variables, the system is aligned with the eigenvectors of $A$. With $S$, the matrix whose rows are the left eigen-vectors of $A$, the relation

$$SAS^{-1} = \Lambda$$

holds where $\Lambda$ is the diagonal matrix of eigenvalues of $A$, so

$$S \frac{\partial u}{\partial t} + \Lambda S \frac{\partial u}{\partial z} = 0.$$

Defining

$$v = Su \quad (4.24)$$

this reduces to the sequence

$$\frac{\partial v_i}{\partial t} + \lambda_i \frac{\partial v_i}{\partial z} = 0, \quad (4.25)$$

a set of wave equations for waves with characteristic speeds $\lambda_i$ equal to the eigenvalues of $A$. Each wave is constant along the curves satisfying $dz/dt = \lambda_i$. 

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Each of the characteristic waves can be treated independently based on the sign of its wave speed, and once the characteristic variables are advanced in time the conservation variables can be recovered via equation (4.24).

Under the extrapolated characteristic boundary condition, the characteristic variables at the boundary are taken as linear extrapolations of the characteristic values at the two neighbouring interior cell vertices.

Under a non-reflective boundary condition, waves travelling into the flow domain, i.e. those with negative characteristic speeds, are suppressed, Thompson [59].

For each of the characteristic cases, knowledge of the Jacobian matrix is not required as the conservation variables can be constructed intuitively from the characteristic variables:

\[
\left. \frac{\partial \rho}{\partial t} \right|_{\text{cons}} = \left. \frac{\partial \rho}{\partial t} \right|_{\text{char}},
\]

\[
\left. \frac{\partial \rho}{\partial t} \right|_{\text{cons}} = \left. \frac{\partial \rho}{\partial t} \right|_{\text{char}},
\]

\[
\left. \frac{\partial \rho u}{\partial t} \right|_{\text{cons}} = \left[ \frac{\partial u}{\partial t} \left( \frac{\rho}{\gamma - 1} + u \frac{\partial \rho}{\partial t} \right) \right]_{\text{char}},
\]

and

\[
\left. \frac{\partial p e}{\partial t} \right|_{\text{cons}} = \left[ \frac{u^2}{2} \frac{\partial \rho}{\partial t} + \rho u \frac{\partial u}{\partial t} + \frac{1}{(\gamma - 1)} \frac{\partial p}{\partial t} \right]_{\text{char}}.
\]

The last equation follows by differentiating the equation (4.22) with respect to time and separating the time derivative of the energy.

**Modified non-reflective characteristic boundary conditions**

Waves travelling into the flow domain pose the difficulty that knowledge of flow properties outside the flow domain is required to advance them in
time. This is generally not available information, but in this problem some of the characteristic variables are, to some extent, known outside the flow domain.

If \( \mathbf{u} = [\rho, u, p]^T \) is used then \( p = p_0 \) and \( \rho = \rho_0 \) are known, \( \rho \) because that is the chosen boundary condition, and \( \rho \), because of the assumption of isentropic flow at the exit boundary. The value of \( u \) is not known, but a linear extrapolation from the interior values is used. This ensures that whatever the sign of each of the \( \lambda_i \), a reasonable difference approximation of \( \partial v / \partial z \) can be made and that

\[
\frac{v_i^{n+1} - v_i^n}{\Delta t} + \lambda_i \frac{\Delta v_i^n}{\Delta z} = 0
\]

can be solved. Under this procedure, the pressure and density at the boundary can be varied – only the values outside the boundary are set to atmospheric conditions. It may be noted that when a forward travelling characteristic is encountered, only property values in the interior of the flow domain or at the boundary are needed and when an inward travelling characteristic is encountered, requiring values downstream of the boundary, the extrapolated values are used, so the atmospheric conditions do not come into consideration.

### 4.4.4 Far-field Boundaries

At an infinite distance from the duct exit in a free field, the disturbance from the exhaust duct would consist solely of outward-travelling spherical waves, with amplitude tending to zero.

In this numerical scheme a finite computational domain is considered. It is assumed that the finite radius of the far-field boundary is sufficiently large that the wavefronts are approximately spherical and that the Sommerfeld [60] radiation condition, which exactly gives reflection-free spherical waves
at infinity, can be implemented at this location.

In section 2.3.4 it was shown that use of the Sommerfeld radiation condition at a finite far-field boundary results in a small reflected wave component which is inversely proportional to the distance of the far-field boundary from the exit plane. Other factors, such as the number of grid points and the acceptable level of accuracy, must be considered when choosing the location of the boundary.

The extended computational domain removes the need for special treatment of the exit plane boundary. It is also now possible to set up experiments to measure noise at specific locations for comparison with directly simulated values from the computation.

To set the conservation variables, it is assumed that the density varies linearly in the local neighbourhood, whence the boundary density is a linear extrapolation of the values in the immediate interior of the computational domain. If the boundary flow is assumed to be isentropic, the density can be set more physically via equation (2.8) once the pressure is determined. The momentum equation in the normal direction to the spherical boundary is linearised to

$$\rho \frac{\partial u_n}{\partial t} = -\frac{\partial p}{\partial n}.$$  

At the far-field boundary, the acoustic pressure satisfies the non-reflecting Sommerfeld condition

$$\frac{\partial p}{\partial t} + c_0 \frac{\partial p}{\partial n} = 0. \quad (4.26)$$

Using this relation and a Taylor series expansion about time \( t = t^n \) a second order time step march for the pressure equation is derived as

$$p^{n+1} = p^n + \Delta t \frac{\partial p}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 p}{\partial t^2} + O(\Delta t^3) \quad (4.27)$$
\[ p^n = c_0 \Delta t \frac{\partial p}{\partial n} - c_0 \frac{\Delta t^2}{2} \frac{\partial}{\partial t} \left( \frac{\partial p}{\partial n} \right) + O(\Delta t^3) \]  
\[ (4.28) \]

\[ p^n = c_0 \Delta t \frac{\partial p}{\partial n} + c_0^2 \Delta t^2 \frac{\partial^2 p}{\partial n^2} + O(\Delta t^3) \]  
\[ (4.29) \]

where \( p^n \) denotes the point value of pressure at the \( n^{th} \) time step. Now

\[ \frac{\partial p}{\partial n} = \left( \frac{\partial p}{\partial z}, \frac{\partial p}{\partial r} \right) \cdot (\bar{z}, \bar{r}) \]  
\[ (4.30) \]

where \((\bar{z}, \bar{r})\) is the unit vector in the required direction which, in this case, is the unit outward normal vector to the spherical boundary surface.

Using relation (4.30) recursively, we obtain

\[ \frac{\partial^2 p}{\partial n^2} = \frac{\partial}{\partial n} \left\{ \left( \frac{\partial p}{\partial z}, \frac{\partial p}{\partial r} \right) \cdot (\bar{z}, \bar{r}) \right\} \]

\[ = \left\{ \frac{\partial}{\partial z} \left[ \left( \frac{\partial p}{\partial z}, \frac{\partial p}{\partial r} \right) \cdot (\bar{z}, \bar{r}) \right], \frac{\partial}{\partial r} \left[ \left( \frac{\partial p}{\partial z}, \frac{\partial p}{\partial r} \right) \cdot (\bar{z}, \bar{r}) \right] \right\} \cdot (\bar{z}, \bar{r}) \]

\[ \Rightarrow \frac{\partial^2 p}{\partial n^2} = \left\{ \left[ \left( \frac{\partial^2 p}{\partial z^2}, \frac{\partial^2 p}{\partial r \partial z} \right) \cdot (\bar{z}, \bar{r}) \right], \left[ \left( \frac{\partial^2 p}{\partial z \partial r}, \frac{\partial^2 p}{\partial r^2} \right) \cdot (\bar{z}, \bar{r}) \right] \right\} \cdot (\bar{z}, \bar{r}) \]

From this, it is seen that knowledge of the first and second \( z \)- and \( r \)-derivatives of \( p \) is sufficient for a second-order time step at the far-field boundary. The next question is how to approximate the \( z \)- and \( r \)-derivatives given that the grid is not aligned with either of these co-ordinate directions.

A Taylor series expansions of \( p \) about a general point \((z_{nz,j}, r_{nz,j})\) for the \( p \) values at two neighbouring points allows the first derivatives at \( P \equiv (z_{nz,j}, r_{nz,j}) \) to be derived, see figure 4.4.

If \( Q \equiv (z_{nz,j-1}, r_{nz,j-1}) \) and \( R \equiv (z_{nz-1,j}, r_{nz-1,j}) \) then

\[ p(Q) = p(P) + (z_Q - z_P) \frac{\partial p}{\partial z} + (r_Q - r_P) \frac{\partial p}{\partial r} + O(h^2) \]
and

\[ p(R) = p(P) + (z_R - z_P) \frac{\partial p}{\partial z} + (r_R - r_P) \frac{\partial p}{\partial r} + O(\Delta h^2) \]

where \( \Delta h = \max(\Delta z, \Delta r) \). Using some algebraic manipulation it can be seen that

\[ \frac{\partial p}{\partial r}(P) = \alpha_1 p(R) + \alpha_2 p(Q) + \alpha_3 p(P) \]

and

\[ \frac{\partial p}{\partial r}(P) = \beta_1 p(R) + \beta_2 p(Q) + \beta_3 p(P) \]

where

\[ \alpha_1 = \frac{z_R - z_P}{(r_Q - r_P)(z_R - z_P) - (z_R - z_P)(r_Q - r_P)}, \]

\[ \alpha_2 = \frac{z_P - z_Q}{(r_Q - r_P)(z_R - z_P) - (z_R - z_P)(r_Q - r_P)}, \]

\[ \alpha_3 = \frac{z_Q - z_R}{(r_Q - r_P)(z_R - z_P) - (z_R - z_P)(r_Q - r_P)}, \]

\[ \beta_1 = \frac{r_R - r_P}{(z_Q - z_P)(r_R - r_P) - (r_R - r_P)(z_Q - z_P)}, \]

\[ \beta_2 = \frac{r_P - r_Q}{(z_Q - z_P)(r_R - r_P) - (r_R - r_P)(z_Q - z_P)}, \]

and

\[ \beta_3 = \frac{r_Q - r_R}{(z_Q - z_P)(r_R - r_P) - (r_R - r_P)(z_Q - z_P)}. \]

This is a first-order approximation of the spatial derivatives which would reduce to backward difference equations in a rectangular aligned grid. A small improvement can be made to this approximation by taking the averages of the values derived above and those of the values obtained by
setting \( Q \equiv (z_{n_z,j+1}, r_{n_z,j+1}) \) with the same point for \( R \). This is a mixed backward- and forward-difference equation not necessarily equivalent to a second-order difference equation. It can be seen that in both sets of equations, the derivatives in the \( i \)-direction are backward-difference based.

At interior points of the domain, second-order differences are derived by moving both \( Q \) and \( R \) in the second step and then averaging the corresponding sets of derivatives. The first step would involve the points \( P \equiv (z_{i,j}, r_{i,j}) \), \( Q \equiv (z_{i-1,j}, r_{i-1,j}) \) and \( R \equiv (z_{i,j-1}, r_{i,j-1}) \) for the backward differences and \( P \equiv (z_{i,j}, r_{i,j}) \), \( Q \equiv (z_{i+1,j}, r_{i+1,j}) \) and \( R \equiv (z_{i,j+1}, r_{i,j+1}) \) for the forward differences.

This method is used to approximate the first derivatives at \( R \equiv (z_{n_z-1,j}, r_{n_z-1,j}) \) prior to calculating the second derivatives at the boundary points \( P \).

The two momentum equations in the axial and radial directions are given by

\[
\frac{\partial}{\partial t} \rho u + \frac{\partial}{\partial z} (\rho u^2 + p) + \frac{\partial}{\partial r} \rho uv = -\frac{\rho uv}{r} \tag{4.31}
\]

and

\[
\frac{\partial}{\partial t} \rho v + \frac{\partial}{\partial z} \rho uv + \frac{\partial}{\partial r} (\rho v^2 + p) = -\frac{\rho v^2}{r} \tag{4.32}
\]

Multiplying equation (4.31) by \( z \) and adding to equation (4.32) multiplied by \( r \) we have, after some simplification and linearisation with respect to the velocity,

\[
\left( \frac{\partial}{\partial t} \rho u_n + \frac{\partial}{\partial t} \rho v \right) \cdot (\bar{z}, \bar{r}) + \left( \frac{\partial}{\partial z} \rho , \frac{\partial}{\partial r} \rho \right) \cdot (\bar{z}, \bar{r}) \approx 0 \tag{4.33}
\]

\[
\equiv \frac{\partial}{\partial t} (\rho u_n) + \frac{\partial}{\partial n} p = 0 \tag{4.34}
\]

where \( \rho u_n \) denotes the momentum normal to the far-field boundary. This relation approximates the far-field momentum as being linear and quasi-one-dimensional. Once \( \rho u_n \) is updated \( \rho u \) and \( \rho v \) are recovered by resolving in the axial- and radial-directions.
The far-field density is extrapolated linearly from the interior of the flow domain. A more logical route is to use the conservation equation with local approximations of the flux derivatives, but its implementation has not shown any significant differences in the linear flow cases.

Another approximation for the far-field density consists of using the isentropic equation (2.8) to derive the density, once the far-field pressure has been derived. It goes without saying that this last route requires the assumption that the far-field disturbance is isentropic.

The energy equation is updated via the state equation, once the other conservation properties and the pressure have been set.

4.4.5 Boundary Ambiguity for Flanged Duct Computations

On the flanged duct the numerical setting of momentum on the wall-duct corner does not follow easily. It is not clear whether to consider the corner as part of the duct, setting the radial velocity component to zero, or as part of the wall, setting the axial component of velocity to zero. Either of these settings 'work' in the sense that they do not diverge, but it cannot be assumed that the computation is totally independent of the corner settings.

4.5 Stepping in the Time Direction

Given an initial distribution of \( u \) and \( p \) in the flow domain, together with the boundary conditions for the flow, a time history of the flow properties can be built by integrating in the time direction, once a suitable method has been chosen for setting the boundary quantities such as \( F_{AB} \) for each cell.
Two time-marching methods are used in this thesis. The Lax-Wendroff method is used with finite volume scheme whereas the Runge-Kutta method is used with the flux splitting scheme. For the former scheme, all of the various grids generated in Chapter 3 are used whereas for the latter, the duct-only grid is used.

4.5.1 Lax-Wendroff

The stepping routine discussed here is the well known Lax-Wendroff scheme. It involves expanding the solution as a Taylor series in the time-direction up to the second-order and then using the original system to transform this time derivative into a spatial derivative. The effect of this process is to reduce the amplitude of the spurious modes where they occur and also to provide a more accurate time integration.

For the present system

\[
\Delta u_{n+1} = u_{n+1} - u_n = \Delta t \frac{\partial}{\partial t} u + \frac{\Delta t^2}{2} \frac{\partial^2}{\partial t^2} u + O(\Delta t^3)
\]

From the unsteady system of equations (4.1)

\[
\frac{\partial}{\partial t} u + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = h
\]

\[
\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right) = -\frac{\partial}{\partial t} \left( \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} - h \right)
\]

\[
= -\frac{\partial}{\partial z} \left( A \frac{\partial u}{\partial t} \right) - \frac{\partial}{\partial y} \left( B \frac{\partial u}{\partial t} \right) + \frac{\partial h}{\partial t}
\]

\[
\frac{\partial}{\partial z} \left[ A \left( \frac{\partial f}{\partial z} + \frac{\partial g}{\partial y} - h \right) \right] + \frac{\partial}{\partial y} \left[ B \left( \frac{\partial f}{\partial z} + \frac{\partial g}{\partial y} - h \right) \right] + \frac{\partial h}{\partial t}
\]
where
\[ A = \frac{\partial}{\partial u} f \]
and
\[ B = \frac{\partial}{\partial u} g \]
are the Jacobian matrices. Noting that \( h \) is the average source in the cell volume and is constant during the time step, the time- and spatial-derivatives of \( h \) can be omitted.

This gives the difference equation in the time direction
\[
\Delta u_{n+1} = -\Delta t \left[ \frac{\partial}{\partial z} f + \frac{\partial}{\partial y} g \right]_{n} + \Delta t \left[ \frac{\partial}{\partial z} \left( A \left( \frac{\partial}{\partial z} f + \frac{\partial}{\partial y} g \right) \right) + \frac{\partial}{\partial y} \left( B \left( \frac{\partial}{\partial z} f + \frac{\partial}{\partial y} g \right) \right) \right] + O(\Delta t^3). 
\]

This equation can be solved in a single step procedure by calculating values for \( A \) and \( B \) and the matrix-vector products. This is a better procedure when a steady solution is sought, as it can be shown that all of the cell residuals do vanish at the steady state, see Morton and Paisley [51]. In this thesis, a one step solution has not proved to give a significant enhancement to the results. Also, only a quasi-steady oscillation can be attained so the need for vanished residuals is not so important here. The relatively simpler two- step procedure is, therefore, preferred. With reference to figure 4.2, this process involves updating the flow properties at the central points \((i,j)\) of the four cells surrounding \( B \) by integrating round each of them and stepping midway into the time step. Then, using these new values, a second integration is carried out round the quadrilateral formed by the four centroids and the flow properties updated at \( B \).

\[
U_{*,n+1} = U^n + \frac{\Delta t}{2} R[U^n]
\]
\[
U_{n+1} = U^n + \Delta t R[U_{*,n+1}]
\]

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Using this method the question of deriving residuals for the vertices from cell-residuals does not arise. Using a single-step routine would require a method for distributing the cell-residuals from the neighbouring cells to the vertex. Some early schemes used a simple average, but Ni [61] used a weighted average by cell size and observed it to be more stable than the simple average. It has been observed that the weighted average results in a volume integral over the surrounding cells with the vertex in question as the cell-centre, resulting in a more conservative routine.

4.5.2 Runge-Kutta

An alternative time stepping method is the use of one of the class of Runge-Kutta integrators. They can be based on the cell-vertex boundary values, but a cell-centred method is described here in view of its eventual use with transformed equations and in flux splitting schemes.

Under this procedure, the volume integrations are carried out around an auxiliary grid connecting the centres of the original grid. This allows residuals and flow properties to be located at the grid vertices directly. With reference to figure 4.2 the fluxes of the conservation variables on the auxiliary line joining centres \((i, j)\) and \((i, j + 1)\) are given by the average of values at points \(B\) and \(C\). Similarly defining fluxes for the other edges, the residual at \(B\) is established.

\[
R_* = R(U^n)
\]

\[
U_* = U^n + \frac{\Delta t}{2} R_*
\]

\[
R = R(U_*)
\]

\[
U^{n+1} = U^n + \frac{\Delta t}{2} (R_* + R)
\]
4.5.3 Accuracy

At the interior of the computational domain, the numerical schemes are second-order in both time and space, see Morton and Paisley[51]. At the far-field boundary, however, the numerical conditions are only first-order in $h$, the local grid size. As $h$ increases the further the far-field boundary is set from the duct exit, the boundary conditions may cause an increase in numerical reflection. However the amplitude of the incident waves at this boundary is relatively small by comparison to the amplitudes at the exit plane, so the numerical reflection has little effect on the computation. Also, the reflection due to the finite far-field boundary has been shown to decrease as the far-field boundary is moved further from the exit plane, see section 2.3.4. A more important source of error is the variation of grid cell size from the exit plane to the far-field boundary. The ideal situation, for which the numerical scheme is second order, is that the cells should be uniform. Accuracy decays as the ratio of the size of neighbouring cells increases. This effect, reported by Morton and Paisley[51], is illustrated in this thesis when the attempt is made to concentrate grid lines near the source, in the 'source-in-wall' computation. The grid cells downstream of the exit plane get larger in all the computations due to the angular placement of the grid lines, and a logarithmic stretching of the grid in the direction downstream of the duct exit only worsens the grid cell ratios. Logarithmic concentration of the grid in the radial direction ensures that the cells increase at least exponentially towards the far-field boundary. The effect, reported in Chapter 5, is to lose accuracy over the whole computational domain.
Figure 4.1: Illustration of referencing in a typical grid cell.

+  -  +  +  +  +  +  +  +  +  
-  +  -  -  -  -  +  -  +  
+  -  +  +  +  +  -  +  +  

(a)  (b)  (c)

Figure 4.2: Illustration of the spurious modes supported by the finite-volume scheme
Figure 4.3: An illustration of grid locations used for boundary interpolation

Figure 4.4: A typical far-field boundary segment
Part III

Results and Conclusions
Chapter 5

Single-Frequency Harmonic Sources
5.1 Overview

This chapter covers results of single-frequency computations. The results follow the sequence of domain simplifications presented in section 2.3. In the closed duct, the open duct and the conical horn domains, analytical solutions in the linear regime are compared with numerical results obtained through the solution of the finite-volume difference equation (4.5). For some of the open duct computations the difference equation is derived through the AUSM scheme of equation (4.16) for comparison.

The solution of the finite-volume equation is through time-marching, using the two-step Lax/Wendroff scheme, see section 4.5.1. Where the AUSM scheme is used it is time-marched through the second-order Runge-Kutta scheme given in section 4.5.2.

5.2 Time Series Plots for Simple Duct Computations

The results presented in this section all refer to a simple cylindrical duct of length 0.55m and radius 0.015m. The computational domain is restricted to the duct alone and is divided into rectangular segments. The inlet boundary is closed by a piston oscillating at a frequency of 100Hz. The piston's velocity amplitude is set at 1.0m/s and, starting from stagnation conditions in the duct, the computation is driven until a quasi-steady wave is generated throughout the flow domain. The piston amplitude is chosen to be small enough such that nonlinear acoustic effects are insignificant, and hence comparisons between the analytical solutions and numerical results give a fair evaluation of the numerical scheme.

The time step for numerical integration is given through the usual Courant-
Frederich-Lewis stability condition [51] as

$$\Delta t = \alpha \Delta l_m / (a_0 + U_0)$$

where $0 < \alpha \leq 0.5$, $\Delta l_m$ is the minimum length of any cell edge, $a_0$ is the stagnation speed of sound and $U_0$ is the velocity amplitude of the input piston.

5.2.1 The Closed Duct

The first case considered is that of a closed duct, as discussed in section 2.3.2. The boundary conditions, given in sections 4.4.1 and 4.4.2, are known precisely and this test case provides the simplest way of testing the basic numerical scheme.

At the start of the computation the flow speed in the duct is $0.0 m/s$, hence the momentum is zero and the pressure is set to stagnation pressure, $p_0 = 10^5 N$. With the density set at $1.21 kg/m^3$, it is possible to set the energy through the equation of state (2.2).

The computational domain consists of a rectangular grid with 34 axial segments and 4 radial segments, equivalent to 210.3 grid points per wavelength for a source frequency of 100Hz. Investigations have been made to show that the grid density here is sufficient. Doubling the axial density of the grid did not give any significant change in the results, see Table 5.1. Furthermore, since the waves are one-dimensional and axial, no advantage is gained by increasing the radial density of the grid.

Figure 5.1 is a plot of the time series variation of the mid-point acoustic pressure, $p' = p - p_0$. Two computed results are presented based on the finite-volume and flux-splitting schemes, see sections 4.1.2 and 4.3.2 respectively. Over the first eight complete cycles of period $T$, $p'/p_0$ is compared to the linearised, quasi-steady analytical solution. The analytical solution
Table 5.1: Maximum error between computed and analytical mid-point pressure for a closed duct computation after convergence.

is obtained in a similar way to the spherical wave solutions outlined in section 2.3.4 but the partial differential equation solved is equation (1.13) without mean flow, i.e. setting $U = 0$. Both numerical schemes are seen to converge in the second cycle.

It is seen that the numerical solution of the finite-volume scheme matches the analytical solution very well whereas the numerical solution of the flux-splitting scheme has noticeable differences both in phase and amplitude. Table 5.1 shows the difference in accuracy between the finite volume scheme and the flux-splitting scheme. The error is measured by the difference in amplitude of the computed wave and the analytical solution. It is seen that the error settles in the neighbourhood of 6% for the flux-splitting scheme and 2.6% for the finite volume scheme.
The analytical solution is for the standing wave which develops in the quasi-steady state. In contrast, the computations start with no disturbance in the duct and illustrate the full time-varying response which ultimately results in the standing wave pattern. Thus, for small $t$, the computed results should be different from the analytical results in both magnitude and phase, as observed in figure 5.1. However, the flux-splitting computation remains out of phase after the quasi-steady state has been reached, in contrast to the finite volume computation. It is unclear whether this is a property of the flux-splitting scheme or whether it is induced by the setup of the computation. However, it is seen in the next section, in the case of the open duct computation, that the flux-splitting result is almost in phase with the corresponding analytical solution. This suggests the present observation could be due largely to the physical setup.

5.2.2 The Open Duct

The next example considered is that of an open duct, with the domain of computation still restricted to the duct alone.

The exit boundary conditions are based on the assumption that the pressure at the exit plane is the same as that of the atmosphere downstream, see section 4.4.3. The conservation variables are then set according to the conservation equations, simplified by the assumption of an isentropic disturbance at the exit plane.

On this basis, the boundary density is fixed, see equation (2.7), and the momentum and energy approximations follow from equation (4.20) and (4.21) respectively and are referred to in the results as isentropic boundary conditions.

Alternatively, the density is set through a linear extrapolation of interior
grid points and the momentum and energy are set through equations (1.2) and (4.21) respectively. This set of boundary conditions is referred to as non-isentropic in the results.

A characteristic boundary implementation is also carried out based on equations (4.23) and (4.24).

The results given are for computations on a rectangular grid with 34 axial segments and 4 radial segments, as for the closed duct computations. A doubling of the grid density in either the axial or the radial directions gives no significant benefit, as seen in Table 5.2. Once again, the waves are one-dimensional and axial, thus the radial grid density should be of no significance.

<table>
<thead>
<tr>
<th>Finite Volume Scheme</th>
<th>Flux-splitting Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>nr</td>
<td>nx</td>
</tr>
<tr>
<td>4</td>
<td>34</td>
</tr>
<tr>
<td>4</td>
<td>68</td>
</tr>
<tr>
<td>8</td>
<td>34</td>
</tr>
<tr>
<td>8</td>
<td>68</td>
</tr>
</tbody>
</table>

Table 5.2: Maximum error between computed and analytical exit-plane velocity for an open duct computation with a zero-pressure boundary condition after convergence.

The errors in velocity amplitude are seen to be stable to increases in grid density, at a level of approximately 2% for the finite volume scheme and 11% for the flux-splitting scheme.

In figure 5.2 plots of the non-dimensional exit velocity, $U_e/U_0$, against the time for eight complete cycles are presented for the finite-volume scheme, based on the three sets of boundary conditions outlined above.

Figure
5.3 shows similar results for the flux-splitting AUSM scheme over five complete cycles. Both figures give comparisons of numerical results with the linearised quasi-steady analytical solution. The computations were started from stagnation conditions, as in section 5.2.1, and the figure legends indicate the cycles to which the results refer. The error in phase which results from use of the flux-splitting scheme is seen to be much less than was the case for the case of the closed duct computations.

Once again the results corresponding to the finite-volume scheme are much closer to the analytical solution than those corresponding to the flux-splitting scheme. However, for both schemes, the convergence is much slower, apart from the case of the flux-splitting scheme with a characteristic boundary implementation, where convergence is as good as with the closed duct computations.

It can be surmised from the results so far that the finite-volume scheme provides a reasonable solution of the flow equations in the simple duct domains, both closed and open.

The open-duct computations are much slower to converge because of numerical reflection of error waves from the exit plane back into the computational domain. Even after settling down, the finite-volume scheme retains a small variation in the output wave amplitude. Using the flux-splitting scheme reduces the spurious modes incident at the exit boundary but numerical reflection is unabated. Adding the characteristic boundary conditions to the scheme seems to also limit numerical reflection, accounting for a much faster convergence rate. When accuracy is considered, it is clear that the finite-volume scheme out-performs the flux-splitting scheme in both duct-only computations, even though the flux-splitting scheme gives smoother convergence.
Figure 5.1: Time Variation of the Mid-point Acoustic Pressure in a Closed Duct

- Finite Volume
- Flux-splitting
- Analytical
Figure 5.2: The Exit Velocity for an Open Duct, Using the Finite-Volume Scheme
Figure 5.3: The Exit Velocity Profile for an Open Duct, Using the Flux-splitting Scheme
5.3 Comparison of Analytical Conical Horn Solutions to Computed Source-in-Wall Results

In section 2.3.4 two analytical solutions are presented for spherical wave disturbances travelling into a conical horn. As stated in section 2.3.5, this setup also models the case of a source-in-wall disturbance.

In this section analytical and finite volume numerical solutions are compared for two different far-field boundary conditions, namely a solid-wall boundary on which the velocity is zero, and an open boundary where the boundary condition is approximated by the Sommerfeld radiation condition. The former case validates the implementation of a non-rectangular, non-uniform grid in the numerical scheme, and the latter indicates the resultant error from the use of the Sommerfeld radiation condition at a finite boundary, and validates its numerical implementation.

In this section, plots of the analytical solutions and the computed results of axial variation of acoustic velocity on the line of symmetry, see figure 3.1, are presented. The computations are performed on the top half of the grid given in figure 3.3 for the specified boundary conditions. At specified instances in the computation, after convergence, the velocity field on the line of symmetry is saved for comparison with the corresponding analytical solution.

The source consists of a sphere of radius 0.025m and the far-field boundary is set at 0.5m from the surface of the sphere. The input disturbance is then determined through the oscillation of the surface of the source-sphere.

For linearised disturbances the stability of the finite-volume scheme is less dependent on the angular grid density than on radial grid density. It was observed that 7 grid points in each quadrant in the angular direction sufficed for numerical stability. The analytical solution to this problem is
one-dimensional, with waves travelling in the radial direction only, hence the angular density of the grid is expected to be of little importance.

On the other hand, grid dependence in the radial direction is significant. It was observed that numerical stability required no fewer than 113 grid points in the axial direction for the range of frequencies examined, namely $100 - 1kHz$. However, it is possible to achieve grid stability with fewer axial grid points, even as few as 29, for lower frequencies such as $100Hz$ where the disturbance has relatively gentler slopes throughout the cycle. The ratio of grid points to wavelengths for a $100Hz$ source simulated on a grid with 29 radial points is 197.3.

Even for the higher frequencies, the instabilities tend to concentrate around areas of the disturbances with steep gradients, not affecting the stability in the rest of the computational domain or the disturbance cycle. With a computational domain consisting of a $13 \times 113$ rectangle for $abcd$ in figure 3.1, all computations with the low amplitude source converged readily. The ratio of grid points to wavelength ranges from 768.74, for a source frequency of $100Hz$, to 76.87, for a source frequency of $1kHz$ in the radial direction. In fitting sufficient grid points for the higher frequency, the lower frequency computations are assigned a much higher grid density than necessary.

Computations for sources with frequencies of $500Hz$ or less converge with few significant deviations from the analytical solutions once grid independence and a stable time step are achieved. Generally, the states of the computation are examined at the time instants $0$, $0.2T$ and $0.4T$, for a cycle of period $T$, with $t = 0$ corresponding to the maximum velocity amplitude of the disturbance after convergence has occurred. For states after $0.5T$ the disturbances change sign relative to those before this state, but have the same computational behaviour.

For disturbances approaching the nonlinear regime with frequencies of
1kHz or higher, steep gradients are developed at the 0.2T (or 0.8T) state. Other than these occurrences of steep disturbances, the other states of the computation converged smoothly both in the linear and nonlinear regimes. It is, therefore, reasonable to measure the performance of the scheme on each setup by these states.

5.3.1 A Solid Far-field Boundary

In the first analytical solution, the case of a conical horn terminating in a wall is considered. In this setup, it may be noted, the numerical boundary conditions are determined exactly, thus the numerical scheme can be assessed easily for effects not accounted for by the boundary conditions.

Figure 5.4 shows a comparison of analytical and finite volume numerical results for a source disturbance of amplitude 1m/s and frequency 1kHz. This figure shows the results at 0.2T of a cycle after convergence. Slight discrepancies between the analytical and the numerical results are observed at axial distances of 0.05 to 0.2m. Neither an increase in grid density nor a reduction in the piston velocity amplitude to 0.01m/s has any effect upon this discrepancy. Errors of this size were only observed for the highest frequency tested, 1kHz, at the time instant of 0.2T (and 0.8T).

Figure 5.5 illustrates a similar comparison between analytical results from linear acoustic theory and finite volume numerical results, for a source of velocity amplitude 20m/s. It is observed that the discrepancies are greater than those in figure 5.4 and this increase is attributable to non-linear effects. Generally, it was found that nonlinear effects were more pronounced when the solid-wall boundary was imposed than was the case for an open boundary. Furthermore, convergence of solid boundary computations was achieved after twenty cycles, typically, compared to five for the open boundary computations.
5.3.2 Sommerfeld Radiation Far-field Boundary

In the second analytical solution, the case of an infinite conical horn approximated by use of the Sommerfeld radiation condition at a finite boundary is considered. The procedure for setting the numerical boundary conditions is outlined in section 4.4.4. The same states as outlined above for the solid-wall case are assessed. The case of this open far-field setup is far more useful with a view to the later computations on the both the flanged duct and unflanged duct setups where the same numerical boundary conditions are needed.

For each value of $kL$, the values of $\epsilon$, and the maximum percentage error between the exact analytical acoustic velocity and the analytical solution based on the Sommerfeld radiation condition at a finite boundary can be determined, see equations (2.28) and (2.29). The quantities $kL$ and $\epsilon$ are the product of the wave number of the source and the distance of the far-field boundary from the source, and the relative magnitude of the backward-travelling component of the acoustic pressure as a fraction of the magnitude of the forward-travelling component, respectively.

It has been observed in section 5.3.1 that the maximum error for the source frequency of 1kHz occurs at the 0.02T state and in the axial range 0.05m to 0.2m, thus examining this state gives a good representation of the error. In table 5.3 the values of the maximum percentage error at the 0.2T state are given for various positions of the far-field boundary, $L$. 'Analytical error' refers to the difference between the exact analytical solution and the analytical solution obtained by use of the Sommerfeld radiation condition at a finite boundary. 'Computed error' refers the difference between the computed solution and the analytical solution obtained over the same finite domain, with the Sommerfeld radiation condition imposed at a finite boundary.
Table 5.3: The error which results from the use of the Sommerfeld radiation condition at a finite distance, $L$ from the source for analytical solutions, and the extra error from using the numerical analysis.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$kL$</th>
<th>Analytical Errors</th>
<th>Computed Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5m</td>
<td>9.24</td>
<td>16.22 %</td>
<td>13.22 %</td>
</tr>
<tr>
<td>1.0m</td>
<td>18.47</td>
<td>8.12 %</td>
<td>3.6 %</td>
</tr>
<tr>
<td>2.0m</td>
<td>36.94</td>
<td>4.06 %</td>
<td>1.56 %</td>
</tr>
<tr>
<td>4.0m</td>
<td>73.89</td>
<td>2.03 %</td>
<td>0.71 %</td>
</tr>
</tbody>
</table>

From the table 5.3, it is seen that the error incurred with a boundary at a finite distance from the source is within the range established in equations (2.28) and (2.29). Also, as predicted, the error falls as the distance of the far-field boundary from the source is increased, approximately linearly with the $kL$ values. It is also to be noted that the error between the finite analytical solution and the finite computed result is much smaller than the errors between the exact solution and the finite domain analytical solution, as seen in figure 5.6, which implies that the numerical implementation of the Sommerfeld radiation condition is correct. This is expected as both finite domain solutions include comparisons between analytical and computed solutions in this section are between finite domain solutions only.

Figures 5.7 -5.12 show a comparison of finite volume computed solutions with analytical solutions for the two frequencies 500Hz and 1kHz at the specified stages in a converged cycle, for a source of velocity amplitude 0.01m/s. Once again, the only observable discrepancies between the analytical and the numerical results occurs over a short axial range at the time instant of 0.2T for a frequency of 1kHz, see figure 5.11.

Using fewer axial grid spaces and clustering close to the wall, where the velocity gradient is steepest, gave less accurate results, not only in the
steep-gradient regions, but also downstream in the neighbourhood of the far-field boundary. Also, grid clustering considerably reduces the CFL number, which, given that the computation is based on the global minimum, could result in as much computation time per cycle, but with less accuracy further downstream. **Figure 5.13** illustrates a finite volume computation on a computational domain stretched logarithmically in the axial direction. It may be noted that, as the computation is first-order in the neighbourhood of the far-field boundary, accuracy should drop as the cell-variation gets larger. Morton and Paisley[51] set out conditions of grid variation for accuracy, namely that grid variation should be as close to uniform as possible. With a logarithmically compressed grid the cells are expanding at least exponentially outward from the source. This may explain why in **figure 5.13** the numerical solution is less accurate than for uniform grid spacing even in the region of high grid density.

It was also observed that the solid-wall boundary computations for higher frequencies are much slower to converge than with a five-times smaller CFL number, requiring five times more cycles of computation to converge. A possible explanation is that spurious modes are totally reflected from the boundary, making the errors persist longer. This observation bodes well for the later computations on composite computational domains for the flanged duct and the unflanged duct, where only free-field boundary conditions are used.

**Figure 5.14** shows the effects of non-linearities when a source disturbance of amplitude $70m/s$ and frequency $1kHz$ is introduced. These effects, which are first noticeable for source amplitudes of $20m/s$ or higher, occur in two states in each cycle and do not appear to affect the accuracy of subsequent states. The increased deviation between the analytical solution and the finite volume computed results in these states may be attributed to non-linear effects, which are ignored in the analytical solution.
Figure 5.4: The Axial Variation of Acoustic Velocity Along a Closed Conical Horn, for a 1kHz Source Disturbance of Amplitude 1m/s at Time 0.2T.
Figure 5.5: The Axial Variation of Acoustic Velocity Along a Closed Conical Horn, for a 1kHz Source Disturbance of Amplitude 20.0m/s at Time 0.2T.
Figure 5.6: A Comparison of the Finite Analytical Solution, the Exact Analytical Solution and the Finite Computed Solution of the Axial Variation of Acoustic Velocity Along an Open Conical Horn, for a 1kHz Source Disturbance of Amplitude 0.01 m/s at Time 0.2T.
Figure 5.7: The Axial Variation of Acoustic Velocity Along an Open Conical Horn, for a 500 Hz Source Disturbance of Amplitude 0.01m/s at Time 0.0T.
Figure 5.8: The Axial Variation of Acoustic Velocity Along an Open Conical Horn, for a 500Hz Source Disturbance of Amplitude 0.01m/s at Time 0.2T.
Figure 5.9: The Axial Variation of Acoustic Velocity Along an Open Conical Horn, for a 500Hz Source Disturbance of Amplitude 0.01m/s at 0.4T.
Figure 5.10: The Axial Variation of Acoustic Velocity Along an Open Conical Horn, for a 1kHz Source Disturbance of Amplitude 0.01m/s at 0.0T.
Figure 5.11: The Axial Variation of Acoustic Velocity Along an Open Conical Horn, for a 1kHz Source Disturbance of Amplitude 0.01m/s at 0.2T.
Figure 5.12: The Axial Variation of Acoustic Velocity Along an Open Conical Horn, for a 1kHz Source Disturbance of Amplitude 0.01m/s at 0.4T.
Figure 5.13: The Axial Variation of Acoustic Velocity Along an Open Conical Horn, for a 1kHz Source Disturbance of Amplitude 0.01m/s at 0.2T with Logarithmic Grid Variation
Figure 5.14: The Axial Variation of Acoustic Velocity Along an Open Conical Horn, for a 1kHz Source Disturbance of Amplitude 70m/s at 0.2T Showing Non-linear Effects.
5.4 Time Series Plots of Exit Plane Velocity for Extended Computational Domains

Figures 5.15 to 5.17 depict the time-variation of the exit plane velocity for extended duct domains. The results are derived through the numerical solution of the finite-volume Euler equations (4.5) on the computational grid of the flanged duct domain, figure 2.2, and the unflanged duct domain, figure 2.3. The various computational grids are given by the point distributions in figures 3.6 - 3.8, for the flanged duct, and figures 3.9-3.10, for the unflanged duct.

Using these extended domains the problem of numerical reflection from the exit plane is removed. More significantly the physically inaccurate $p = p_0$ boundary condition at the exit plane is removed. Provided that the far-field boundary is sufficiently far downstream, the errors due to reflections, i.e. the approximate nature of the far-field boundary conditions, are negligible by comparison.

The difference equations (4.5) are solved by time-marching with the two-step Lax/Wendroff scheme given in section 4.5.1. The boundary conditions are as derived in sections 4.4.1, 4.4.2 and 4.4.4, respectively, for the inlet boundary, the wall boundaries and the far-field boundary. The source consists of a plane piston oscillating harmonically with frequency of $100\, Hz$ and amplitude $1m/s$.

Apart from the computation on an ‘L’ grid, where the exit plane coincides exactly with a grid line, the exit plane velocity profile is derived by a weighted mean of velocities on grid points either side of the exit plane.

Suppose that grid points $A$ and $B$ lie close to a nearby point, $C$, on the intersection of the exit plane with the grid line connecting them. Then, with $AB$ and $CB$ defined as the signed distances between $A$ and $B$, and $C$ and
respectively, the weighted average of a function $g$ on the point $C$, based on its known values at $A$ and $B$, is given through a linear approximation by equating suitable approximations of the local slope of $g$ as

$$g'(C) \approx \frac{g(A) - g(B)}{AB} \approx \frac{g(C) - g(B)}{CB}$$

whence

$$g(C) \approx g(B) - \frac{g(A) - g(B)}{AB} CB.$$

Using this function, the exit plane velocity values on points along all the axial grid lines crossing the exit plane are determined at selected states of the source cycle. The mean exit plane velocity at each of these states is then stored for assessment after the computation. The overall mean exit plane velocity is then evaluated by calculating the total volume flow rate and dividing by the cross-sectional area.

In the computations on the flanged radial grid, there were fourteen grid spaces in the radial direction, sixty on the duct in the lengthwise direction and seventy more on the hemispheric extension. With reference to figure 3.4, the computational domain consists of the $14 \times 130$ rectangle, $adeh$, made up of the smaller rectangles of $14 \times 60$ for the segment $abgh$ and $14 \times 70$ for the segment $bdeg$.

On ‘T’ grids, there were six spaces in the radial direction in the duct and sixty spaces length-wise on the duct and sixty spaces length-wise on the extension. On the flanged duct there were eighteen additional radial spaces in the hemispheric extension while in the unflanged grid there were twenty-nine more. With reference to figure 3.5, the computational rectangles under consideration are $abgh$, $beleg$, $bcd$ and $gef$ with dimensions $12 \times 60$, $12 \times 60$, $18 \times 60$ and $18 \times 60$ respectively for the flanged duct. It is to be noted that $bcd$ and $gef$ are rectangles in the computational domain.
For the unflanged duct the last two rectangles have dimensions 29 \times 60 each. The grid distribution is chosen to be stable for a source disturbance of frequency 2kHz, with a ratio of grid points per wavelength of 40. The distribution is automatically stable for lower frequency disturbances with a grid-point-to-wavelength ratio of 800 for a source of frequency 100Hz.

For the ‘L’ grid the computational rectangle for the duct section had dimensions 6 \times 30 while the rectangle for the hemisphere had dimensions 30 \times 30.

When the ‘T’ grid of the unflanged duct domain contains an even number of radial grid lines in the spherical extension, it is found that one of these lines is vertical. This occurrence was found to be unfavourable, resulting in an infinite slope, when the far-field boundary was derived using trigonometric slopes of the radial grid lines. To avoid the infinite slope, it is necessary to used an odd number of radial grid lines. With reference to equation (4.29), the unit vector \((\bar{x}, \bar{r})\) is approximated by \((\cos \alpha, \sin \alpha)\) where the slope of the local axial grid line is \(tan \alpha\). Using this approximation on a vertical grid line gives an infinite slope. This case can be dealt with as an exception, or more simply by ensuring that there is no vertical grid line.

Figure 5.15 depicts the evolution of the average exit velocity for a flanged duct. The results are all derived from a finite volume Lax Wendroff computation, with an extended flow domain - namely the duct and the hemisphere as discussed in section 2.4.5. The grid distributions are based on the simple radial grid, figure 3.6, the ‘T’ grid, figure 3.7, and the elliptic grid, figure 3.8, respectively for a flanged duct.

For an unflanged duct, figure 5.16 shows a comparison of the evolution of the average exit velocity based on the computational grids of figures 3.9 and 3.10 respectively.

Using extended domains, it is seen that the computations converge in six
cycles, which is generally much faster than for open duct-only domains with an open exit plane boundary condition. As seen in figures 5.2 -5.3, computations for the latter case require at least 20 cycles for convergence, except for the flux-split computation with a characteristic exit boundary implementation, which converges in three cycles. Even after convergence, the duct-only computations retain an error, due to numerical reflection, which persists. This observation illustrates the extent to which numerical reflection affects the convergence rate.

From figure 5.15 it is seen that, for the flanged duct, the computed exit velocity profiles for each of the grid types are all in phase, but there are noticeable differences in magnitude. The differences are more pronounced between the radial grid case and the ‘L’- and ‘T’- grid computations. Figure 5.16 shows that for the unflanged duct, the computed exit velocity profiles for the radial and the ‘T’- grid computations are of very similar magnitude, although they are slightly out of phase. For determining the noise, the phase difference is less important than a difference in magnitude. Of the three grid types considered for either the flanged or the unflanged duct computations, the best in terms of computational cost would be the radial grid, since it contains fewer points than each of the other alternatives. In the computations on the unflanged duct, the choice is clearer since the exit velocity profiles are very close. It can be seen, also, that the result of the radial grid computation is closer to the analytical solution.

Considering the flanged duct computations, it is seen that the results of the elliptic grid are closest to the analytical solution in general, but it is also observed that its convergence does not appear to be complete. The ‘T’ grid computation looks furthest from the analytical solution, so the choice is between the radial grid and the elliptic grid. Considering the grid densities, and the ease of application of the far-field boundary conditions, it would be reasonable to also opt for the radial grid in the flanged duct.
In many of the test cases so far, it is difficult to strike a balance between introducing enough artificial dissipation to ensure smooth convergence and so much as to damp out some of the physical oscillations. This is further complicated by various factors such as discontinuities in the computational domain in the neighbourhood of the exit plane, when extended domains are used, and the fact that the solution does not settle to an absolute steady state, but rather to a quasi-steady state, where the residuals do not fall to zero.

The former occurrence necessitates the use of a grid-variable scaling factor for artificial viscosity. The latter necessitates the use of a factor based on the local flow properties. A popular form of this factor devised by Turkel [56, 58] uses a factor proportional to the local contravariant velocity, adjusted to ensure that the dissipation does not reduce to zero where the velocity falls to zero. This form of artificial viscosity leads to additional dependence on the grid metrics, such as the local gradients of grid lines, which introduces further complexities where the grid metrics are discontinuous. In this example it fails to eliminate the spurious modes in the free-field and produces a visibly different smooth exit plane velocity profile compared to the fixed viscosity scheme, see figure 5.17. By comparison to the far-field pressure, the exit velocity profile produced by the computation is relatively better and could be explained by the fact that the grid metrics on the duct section of the computational domain are relatively smoother. In the far-field there are two discontinuities, one caused by the transition from the duct to the free-field and the other, by the transition from adeh to gef, as depicted in figure 3.5.

Another possible explanation for the inaccuracy of the variable-viscosity computation could be the fact that the grid variations together with the
flow variations during the cycle caused the amount of dissipation to fall to too-low a level at some stages of the computation.

A less probable explanation could be that it is due to boundary reflection, as even the far-field boundary condition is only an approximation. This last explanation, if applicable, would have a very small influence as the computation does converge to smoother time series throughout the domain when simpler forms of dissipation are used.
Figure 5.15: A Comparison of the Cyclic Variation of Acoustic Velocity at the Exit Plane for Different Grid Transforms of The Flanged Duct.
Figure 5.16: A Comparison of the Cyclic Variation of Acoustic Velocity at the Exit Plane for Different Grid Transforms of The Unflanged Duct.
Figure 5.17: A Comparison of the Effects of Constant and Grid-variable Dissipation on Computations on a Flanged 'T' Grid
5.5 Far-field Noise Spectra

Far-field noise spectra are shown for the open duct computations and compared to the corresponding analytically derived spectra for linearised disturbances.

For the duct-only computations, far-field noise is derived through equating the energy flux across the exit plane to the flux across the surface of a sphere centered at the center of the exit plane. This procedure is also carried out for extended domain computations using the the time series of the exit plane velocity fluctuation. The noise on each point of the spherical surface is assumed to have the same intensity.

For the extended domain computations, spectra of far-field noise are also derived directly using the fluctuations of velocity or acoustic pressure on the far-field spherical boundary. For this purpose, the mean boundary velocity is calculated and used to determine the sound intensity on the boundary surface.

In each case, a range of discrete frequency computations for $100 \, Hz, 200 \, Hz, \ldots, 2k \, Hz$ with source amplitude $1m/s$ has been carried out for comparison with analytical far-field noise spectra. The source amplitude is chosen to be sufficiently small that nonlinear effects are negligible.

For each of the discrete frequencies, the computation is carried out to determine the amplitude of the exit plane velocity, from which the far-field noise at the downstream point is deduced according to equation (1.23) for a flanged duct, or an appropriately modified version of the same equation for an unflanged duct.

Far-field noise values for single-frequency waves can be obtained from analytical solutions of the linearised acoustic equations, either by the assumption of monopole radiation, see equation (1.23), or by use of the radiation
impedance at the exit plane, see equation (1.24). The latter is preferable, since it correctly accounts for the radiation into the free field. Results from the point source model are given because they correspond to the $p = p_0$ condition at the exit plane, as used in the duct-only computations.

The far-field location is generally set at a distance equal to the length of the duct from the exit plane for duct-only computations. The effect of the location of the far-field boundary is assessed, in figure 5.22, by examining the noise at a given hemispherical surface in the free-field for various locations of the far-field boundary surface.

5.5.1 Simple Duct Noise Analysis

The duct has length 0.55m and radius 0.015m and for each of the sets of results, the computational grid had seven nodes in the radial direction and a range from forty nodes, for a 100Hz source, to one hundred and eighty nodes for a 2kHz source, in the axial direction. This corresponds to 247 grid points per wavelength for a source of frequency 100Hz and 56 grid points per wavelength for frequency 2kHz.

For the simple duct computational domain, terminating at the exit plane, three sets of results are obtained for the three exit boundary implementations outlined in section 4.4.3, using the Finite Volume Lax/Wendroff scheme and the modified AUSM scheme in turn. Two analytical results, one based on the assumption of monopole radiation from the centre of the exit plane and the other based on the analytical value of the exit plane acoustic impedance, respectively, and two corresponding computed results are compared in each plot.

Figures 5.18 and 5.19 depict the far-field noise spectra derived through the Lax/Wendroff finite volume and the AUSM flux-splitting schemes, re-
spectively. The results corresponding to the different numerical boundary implementations are presented together for comparison.

From these plots, it is seen that the far-field noise is predicted most accurately by use of the flux-split scheme. Although the values corresponding to the Finite Volume Lax/Wendroff scheme are almost as good, they were derived by averaging the exit velocity amplitudes for a number of successive cycles after the calculations had settled down, whereas those for the flux-split scheme were from single cycles after convergence. It may be recalled that in section 5.2 sample time series plots for these two scheme exhibited a dependence on the combination of boundary implementation and the damping of spurious modes associated with each scheme. It was seen that the flux-split AUSM scheme had better dissipative control in those computations.

In each of the computations the \( p = p_0 \) condition is used at the exit boundary. This explains the fact that the phases of the computed noise spectra coincide with that of the analytical spectrum based on the monopole-source condition, see figures 5.18–5.19, which is based on the same exit plane condition. The better analytical solution is based on the acoustic impedance at the the exit plane, an analysis that includes the effects of radiation into the free-field. In both sets of computed results the noise amplitudes appear much closer to the impedance-based analytical solution than to the monopole solution. This is probably fortuitous, in that the artificial viscosity properties of the numerical schemes give a similar effect to the real dissipation as accounted for in the resistive part of the radiation impedance.

In later results, based on the exit velocity profiles derived from extended domain computations, it is seen that the computed results and the impedance-based analytical solutions are close both in phase and magnitude, as one
would expect, see figures 5.24–5.25.

5.5.2 Extended Domain Noise Analysis Based On Far-field Velocity Fluctuations

The next setup consists of a duct of radius 0.025m and length 0.5m. For the analysis in this subsection, a radial flanged grid of the duct with computational dimensions $136 \times 12$ is used. There are 56 radial points along the duct and 80 in the free-field. The far-field boundary is located at a distance 0.5m from the center of the exit plane.

Noise spectra are computed at the far-field boundary using sources of amplitude 0.05m/s, 1.0m/s and 10m/s, respectively. The computed noise values are based on the boundary variation of acoustic velocity and given by

$$p = 20 \log_{10} \left( \frac{\rho_0 u_0 u_f}{2 \times 10^{-5}} \right) \text{dB}$$

for each source, where $u_f$ is the mean amplitude of the velocity on the far-field boundary. This removes the need to model the acoustic behaviour of the exit plane explicitly.

For comparison, the analytical values of the far-field noise are determined using equations (1.23) and (1.24) and the analytical solution of an open duct-only setup.

Figures 5.20–5.22 depict the far-field noise for sources of velocity amplitude, 0.05m/s, 1.0m/s and 10m/s, respectively. When the coarseness of the grid in the far-field (figure 3.6) is considered, the results may be considered as fairly accurate. It is seen that the numerical results tend generally towards the impedance-based analytic solutions. There does not appear to be a noticeable change in comparative accuracy with linear analytical results as the source amplitude is increased. This observation confirms
the assumption that nonlinear effects are negligible, for a source velocity amplitude of $1\text{m/s}$, in computations over extended domains. In particular, it is observed that the numerical results are now in phase with the impedance-based analytical results, as one would expect, since radiation into the free-field is properly accounted for in both sets. This is in contrast to the duct-only computations, figures 5.18 and 5.19 where the numerical results were in phase with the analytical results based on a monopole source.

The main discrepancy between the numerical and analytical results in figures 5.19 to 5.21 is in the noise magnitude at low frequencies, and hence long wavelengths. The errors in the numerical solutions may possibly be explained by the far-field boundary being too near to the exit plane, which would cause the Sommerfeld radiation condition to be inaccurate. The solution to this would be to move the far-field boundary further away, but this solution would incur the penalty of either requiring a denser grid, and hence more computation time, or much larger grid cells in the neighbourhood of that boundary.

Figure 5.23 compares mean noise spectra, on the hemispherical surface of radius $0.5\text{m}$ from the exit plane, for far-field boundaries located at $0.5\text{m}$, $1.0\text{m}$ and $1.5\text{m}$, respectively, in the frequency range $100\text{Hz}$ to $2\text{kHz}$. The source amplitude is set at $10\text{m/s}$. The grid spacing in the axial direction is maintained by adding sufficient grid points as the domain is expanded. This maintains the grid-points to wavelength ratio of the computation.

The computed results appear to be independent of the location of the far-field boundary beyond $0.5\text{m}$, even for a relatively high-amplitude source. This observation resolves the question of extending the computational free-field, saving on the potential number of points in the computational domain. It, however, also excludes the hypothesis that the discrepancies
in the lower frequency computations are due to the location of the far-field boundary being too close to the exit plane. It would appear that a more logical explanation of the discrepancies is the inappropriateness of the comparison, since the analytical values are based on the exit velocity fluctuations whereas the computed values are based on far-field velocity fluctuations.

At higher frequencies, errors could be due to too large grid spaces. The wavelengths are smaller meaning that for the same grid fewer grid cells cover the wavelength. This problem does not appear to be significant since the results generally appear more accurate at the higher frequencies.

5.5.3 Extended Domain Analysis Based On The Exit Plane Velocity Fluctuations

An alternative way to evaluate the accuracy of the extended domain computations is to use numerical values of the exit plane velocity, together with analytical values of radiation impedance, to derive the far-field noise. This is essentially a check on the accuracy of the numerically evaluated velocity profile in the exit plane.

For the extended computational domains grid densities are as given in section 5.4. The plots are a comparison of the computed noise and the analytical noise, where the latter is based on knowledge of the analytical values of acoustic impedance at the exit plane.

The far-field boundary is located at a distance \( R \), equal to the length of the duct, from the exit plane. The duct has a length of 0.55 \( m \) and a radius of 0.015 \( m \).

Figures 5.24 -5.25 show the computed far-field noise for ‘T’ grid distributions of a flanged duct and an unflanged duct, respectively, whereas
Figure 5.26 depicts the results for an unflanged duct based on a radial grid distribution.

It is seen that for each of the extended domain results, the accuracy is good in terms of both magnitude and phase, even at low frequencies. This includes the case of the simple radial grid for the unflanged duct domain, see figure 3.9 where the far-field cells are very large and could be a source of some inaccuracy. The grid also contains very skew grid cells in the neighbourhood of the exit plane.

By comparison to the results based on the computed fluctuations at the far-field boundary, namely figures 5.20 - 5.22, figures 5.24-5.26 show a better level of accuracy. Even at the lower frequencies the accuracy is very good. Thus the extension of the grid into the free-field seems to give greater accuracy to the computed values in the exit plane than to those on the far-field boundary. This is not surprising, since the boundary conditions on the far-field boundary are approximate and errors will have greater influence there than back at the exit plane.

The far-field noise spectra in figures 5.24-5.26 confirm the earlier finding that results from computations with the radial grid transform are as accurate as those obtained using more complicated grid transforms. Thus the radial grid transform should be used on the basis of its simplicity.
Figure 5.18: A Comparison of the Far-field Noise Levels For Different Implementations of the Exit-plane Boundary Conditions With the Lax Wendroff Scheme on a Simple Duct Domain.
Figure 5.19: A Comparison of the Far-field Noise Levels For Different Implementations of the Exit-plane Boundary Conditions With the AUSM Flux-splitting Scheme on a Simple Duct Domain.
Figure 5.20: Noise Output From a Flanged Duct With a Source Velocity Amplitude of 0.05m/s. A Comparison of Computed Noise Levels, Based on the Far-field Results, With Analytical Results.
Figure 5.21: Noise Output From a Flanged Duct With A Source Velocity Amplitude of 1.0 m/s. A Comparison of Computed Noise Levels, Based on the Far-field Results, With Analytical Results.
Figure 5.22: Noise Output From a Flanged Duct With A Source Velocity Amplitude of 10.0m/s. A Comparison of Computed Noise Levels, Based on the Far-field Results, With Analytical Results.
Figure 5.23: Noise Output From a Flanged Duct With A Source Velocity Amplitude of 10.0m/s. A Comparison of the Noise Levels at A Radius of 0.5m From the Exit Plane, For Different Locations of the Far-field Boundary.
Figure 5.24: Noise Output From a Flanged Duct With A Source Velocity Amplitude of 1.0m/s. A Comparison of the Noise Levels Based on the Exit Plane Velocity Profile Computed on a 'T'-grid To Analytical Results.
Figure 5.25: Noise Output From an Unflanged Duct With A Source Velocity Amplitude of 1.0m/s. A Comparison of the Noise Levels Based on the Exit Plane Velocity Profile Computed on a ‘T’-grid To Analytical Results.
Figure 5.26: Noise Output From an Unflanged Duct With A Source Velocity Amplitude of 1.0m/s. A Comparison of the Noise Levels Based on the Exit Plane Velocity Profile Computed on a Radial Grid To Analytical Results.
5.6 Summary of Results Presented in Chapter 5

The results presented in this chapter were computed using either the AUSM flux-splitting scheme or the Lax/Wendroff finite volume scheme.

For duct-only computations both schemes were used and their results compared with analytical solutions. For the closed duct it was found that the finite volume scheme produced better results than the AUSM flux-splitting scheme in terms of both amplitude and phase. In particular, the flux-splitting scheme produced a significant difference in phase to the analytical solution. For the open duct computation, the trend was similar though the flux-splitting scheme produced results almost in phase. These observations dictate a preference for the finite volume scheme in both setups.

Using the source-in-wall setup and the finite volume scheme, spherical radiated waves were simulated for the two cases of a wall at the far-field and Sommerfeld radiation through the far-boundary. For each of these cases the computed results compared favourably with the analytical solutions, validating the implementation of the numerical scheme on a non-rectangular grid. The numerical implementation of the Sommerfeld radiation condition on the finite domain boundary was also validated by these results. The theoretical error analysis, which indicated that the level of the maximum error depends approximately linearly on the distance of the far-field boundary from the source, was confirmed by these results. Hence the location of the far-field boundary can be chosen to give a desired level of accuracy for a given frequency of the source.

For more complex domains, namely the flanged and unflanged duct terminating on a spherical boundary, the exit plane velocity profile was determined and compared favourably to the analytical impedance-based solution. The computations converged in fewer cycles that did the duct-only
finite volume computations.

For the case of the flanged duct, there was no significant difference between results of computations on the radial transform grid and those of a ‘T’ transform grid, nor those of the ‘L’ grid. Accordingly, the grid with the fewest points, namely the radial grid should be preferred.

Considering the unflanged duct, similar observations as for the flanged duct were made. In this case, however, when the grid in the neighbourhood of the exit plane is considered, the ‘T’ transform gives less skew cells and offers better control of the grid size variation from the exit plane into the free-field. The ‘T’ grid is therefore preferred for computations on the unflanged duct domain.
Chapter 6

Multiple-Harmonic and Engine Sources
6.1 More Complex Sources

In this chapter computations with more complex sources are presented. Initially computations with prescribed multiple harmonic sources are assessed for the open duct-only domain and for the flanged and unflanged extended domains. The exit plane velocity profiles are presented, followed by a comparison of the computed far-field noise to the analytically derived values.

These are followed by results of engine source computations, including tables of mass fluxes across various cross-sections of the duct, exit plane velocity profiles for various computational domains and a Fourier analysis of the far-field noise radiated by the engine disturbances.

The computations carried out for the results in this chapter are all similar to those discussed in section 5.1 except for the source boundary conditions. The computational grids used throughout are those found to be stable for single-frequency computations with sources of frequency $2kHz$. These grid densities are well above those found to be stable for sources of frequency $500Hz$, the highest frequency component in the multiple-harmonic sources. The engine sources create significant sound at much higher frequencies, although the dominant sound is that at the engine firing frequency, which is a low frequency sound.

Further, by resolving the exit velocity into its single-frequency components, the far-field noise for each component is derived through equation (1.24). These computed results are plotted alongside the analytical solutions for comparison.
6.2 Multiple-Harmonic Sources

This section is dedicated to results from the modelling of duct flows with a multiple-harmonic source input as described in section 4.4. This input consists of a sum of five harmonic modes with a fundamental mode of frequency $100Hz$. This is used to test the numerical schemes for any dependence on the shape of the source function. This knowledge would help to determine the schemes ability to handle the non-smooth source function likely to appear in a true engine exhaust.

The amplitude of each of the harmonic components was set to $1m/s$ for an acceptable level of convergence. Lower amplitudes were found to be less accurate as the numerical schemes could not resolve the true solution from the spurious modes. The inlet boundary conditions are set according to section 4.4.1 and correspond to equation (4.17) with $u_{ok} = 1.0m/s \ \forall \ k$, $N = 5$ and $\omega_k = k \times 100Hz$.

$$u(0,t_n) = \sum_{k=1}^{5} \cos(k \times 100t_n) = \sum_{k=1}^{5} \cos(k \times 100 \times n \times \Delta t)$$

At this amplitude, the maximum acoustic pressure due to the total disturbance in the duct is $165.4dB$, based on analytical radiation impedance at the exit plane, or $165.9dB$ based on the condition $p = p_0$. These values are on the fringes of the generally accepted range of about $140dB - 160dB$ for linear disturbances and were chosen as the lowest amplitude with acceptable levels of convergence. This allows the exit velocity profiles to be compared to the analytically derived profiles with a reasonable knowledge that non-linear effects are not causing any major discrepancies.

In the linear analytical derivation, the exit profile of the sum of the modes is the sum of the exit profiles of the components. In this way the solution is derived by solving a simple wave equation for each of the modes and then summing them, a relatively easier process than deriving the solution.
for the whole source directly. Also, the analytical solution may be based on the analytic radiation impedance of each component, which cannot be done for the total disturbance.

6.2.1 Open Duct-only Domain Computations

For the simple duct, the number of cells in the radial direction was set at six while in the axial direction, there were one hundred and eighty cells.

Figure 6.1 shows the exit velocity after convergence for a simple duct domain using the Finite Volume Lax/Wendroff scheme with $p = p_0$ and isentropic, non-isentropic and the modified non-reflective characteristic exit boundary implementations, respectively. The computation converged within five cycles for all the cases. The two results due to the isentropic and the non-isentropic boundary conditions coincide throughout the cycle.

Figure 6.2 depicts plots derived from a Flux-splitting computation with conservative and characteristic exit boundary conditions, respectively. The analytical result is compared to that obtained from two conservation-law based boundary implementations, using isentropic and non-isentropic assumptions as in equations 4.18 and 4.19 respectively, as well as those obtained using both the extrapolated and modified non-reflective characteristic boundary conditions as described in section 4.4.3. The computations converged at a similar rate to the Finite Volume scheme.

Computations with conservation-based boundary conditions give identical time series for the exit plane velocity when applied with the Flux-splitting scheme and give a similar order of accuracy as the modified non-reflective characteristic boundary method. Figure 6.1 shows that the Finite Volume scheme has a similar accuracy for the various boundary conditions, but it should be noted that the Flux-splitting scheme does not require explicit
addition of artificial viscosity and is thus better controllable.

When used with the Flux-splitting scheme the internally extrapolated characteristic boundary implementation gives less accurate results for the multiple-harmonic source input. The model for the exit boundary conditions is based around determining the three characteristic curves and setting the boundary properties as equal to values at known locations on the curves. For subsonic flow, the three characteristic speeds show that one of these curves has a negative gradient, meaning that it originates from downstream of the exit plane. This means that the third characteristic variable is convected into the flow domain, see Hirsch [12]. Hence the extrapolated characteristic boundary conditions with all three characteristic variables derived from interior properties is not an ideal choice. However, when applied to the small amplitude single harmonic disturbance, this method converged more smoothly than the modified non-reflective method. This observation leads to the conclusion that the extrapolated method is suitable for small-amplitude single harmonic disturbances but less so for multiple-harmonic source disturbances, whereas the modified non-reflective characteristic method has the reverse properties, giving better accuracy with both the Flux-splitting and the Finite Volume schemes.

6.2.2 Extended Domain Computations

The computational domains are as given in section 5.4 and are chosen to be stable for single-frequency computations with a source frequency of 2kHz. In the computations considered in this section the highest frequency component is 500Hz, which is well within the grid stability for low-amplitude sources. The grid stability was checked for the multiple-harmonic source.

Figures 6.3 and 6.4 depict the mean exit velocity derived from the radial grids for a flanged and unflanged domain, respectively. The profiles are
further from the analytical solutions than most of those derived from the zero-pressure cases. In both the figures, the numerical results are generally closer to the impedance-based analytical solution than to the zero-pressure solution, as one would expect. However, this observation is not uniform and the discrepancy between the numerical results and the impedance-based analytical solution is of the same order as the difference between the two analytical solutions.
Figure 6.1: A Comparison of Exit Plane Velocity Profiles for the Finite Volume Scheme With Different Boundary Conditions, Applied to a Duct-only Domain.
Figure 6.2: A Comparison of Exit Plane Velocity Profiles for The Flux-splitting Scheme With Different Boundary Conditions Applied to a Duct-only Domain.
Figure 6.3: A Comparison of the Exit Plane Velocity Profile of a Radial Flanged Duct Computation, With A Multiple-harmonic Source, to the Analytical Solution
Figure 6.4: A Comparison of the Exit Plane Velocity Profile of a Radial Unflanged Duct Computation, With A Multiple-harmonic Source, to the Analytical Solution
6.2.3 Outline of Discrete Fourier Transforms

The discrete Fourier transform is used to resolve a finite sequence of data, $z_j$,

$$
\hat{z}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} z_j e^{i \frac{2\pi j k}{n}}
$$

(6.1)

for $k = 0, 1, \ldots, n-1$ and $z = x + iy$, see Brigham [62]. Setting $\hat{z}_k = a_k + ib_k$, equation (6.1) may be written as

$$
a_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left( x_j \cos \left( -i \frac{2\pi j k}{n} \right) + y_j \sin \left( -i \frac{2\pi j k}{n} \right) \right)
$$

$$
b_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left( x_j \cos \left( -i \frac{2\pi j k}{n} \right) - y_j \sin \left( -i \frac{2\pi j k}{n} \right) \right)
$$

Each of the $\hat{z}_k$ represent a harmonic of frequency $2\pi k/n$ and amplitude $|a_k + ib_k|$. For real input, $z_j = x_j$, the sequence $\hat{z}_k$ is a Hermitian sequence with the components satisfying $a_{n-k} = a_k, b_{n-k} = b_k, b_0 = 0$ and for even $n$, $b_{n/2} = 0$.

6.2.4 Far-field Noise Due to Multiple-harmonic Source

To determine the far-field noise, the time series of the exit velocity is resolved with a discrete Fourier transform into its main component harmonics. Each of these is then used to derive the far-field noise for its particular frequency.

The time series of the exit plane velocity derived from computations with multiple-harmonic or engine sources are split into the different harmonics by use of a discrete Fourier transform analysis. This gives a spectrum of velocity amplitude values, corresponding to each frequency component of
the source, from which the far-field acoustic pressure is determined for each of the harmonic modes.

Figures 6.5 and 6.6 depict the resolved far-field noise against the mode derived from a simple duct computation with the Flux-Splitting Scheme. The exit boundary pressure is set for each of the cases and then the conservation variables are derived on the basis of assumed isentropic flow, characteristic interpolation and the modified non-reflective characteristic boundary, respectively. It has already been stated that the extrapolated characteristic implementation of the exit boundary conditions results in an underestimated exit plane velocity. This occurrence carries over into the resolved far-field noise values, which are not presented here for that reason. The analytical values are based on the boundary condition $p = p_0$.

Figure 6.5 shows a comparison of the analytical solution with both the isentropic and non-isentropic results. As in the case of the time series plots the two computed results are very close. Either of these boundary implementations may be used to achieve the same level of accuracy.

Figure 6.6 depicts a comparison of the analytical solution with the results of the modified non-reflective characteristic boundary implementation. Apart from the fifth mode, the level of accuracy is comparable to that achieved with the conservation-based boundary implementations.

In figure 6.7 the computed results for the modified non-reflective and non-isentropic exit boundary implementation are compared with the analytical solution for the Finite Volume Lax/Wendroff scheme. As in the Flux-split scheme, the two conservation-based exit boundary conditions produced similar results and only one of them is presented. It can be seen that the different boundary implementations produce the same level of accuracy.

Figures 6.8 and 6.9 depict the far-field noise derived from a flanged open duct and an unflanged open duct respectively. The Finite Volume
Lax/Wendroff scheme is used and the analytical values are based on the analytical exit impedance. For the computed results, the exit plane velocity is used to derive the far-field noise.

Looking at the results in this section, it can be concluded that both the Finite Volume scheme and the Flux-Splitting Scheme produce a similar level of accuracy for each different boundary implementation, and further that these schemes’ accuracy in the linear regime is maintained for more complex waveforms than the single harmonic case as considered in Chapter 5. This gives some confidence for the computations with an engine source input, where the waveforms are now still more irregular.
Figure 6.5: A Comparison of the Analytical Values of the Far-field Noise Radiated From a Simple Duct, Excited by a Multiple-Harmonic Source, With Numerical Values Using a Flux-Splitting Scheme and Conservation-Based Boundary Conditions
Figure 6.6: A Comparison of the Analytical Values of the Far-field Noise Radiated From a Simple Duct, Excited by a Multiple-harmonic Source, With Numerical Values Using a Flux-Splitting Scheme and Modified Non-reflective Boundary Conditions
Figure 6.7: A Comparison of the Analytical Values of the Far-field Noise Radiated From a Simple Duct, Excited by a Multiple-harmonic Source, With Numerical Values Using a Finite Volume Scheme and Modified Non-reflective Boundary Conditions
Figure 6.8: A Comparison of the Analytical Values of the Far-field Noise Radiated From a Simple Duct, Excited by a Multiple-harmonic, With Numerical Values Computed on a Radial Flanged Extended Duct Grid
Figure 6.9: A Comparison of the Analytical Values of the Far-field Noise Radiated From a Simple Duct, Excited by a Multiple-harmonic, With Numerical Values Computed on an Radial Unflanged Extended Duct Grid
6.3 Engine Sources

For engine-source computations the inlet boundary conditions are set using the thermodynamic cycle of the internal combustion engine as outlined in section 1.4.1. The method, due to Benson [63], is known as the ‘filling-and-emptying’ method. The engine geometry used is that of the Wankel Rotary Engine, see Norbye [64]. Once again, only the most basic exhaust system is considered, namely a uniform duct between the engine and the tailpipe exit, see figure 1.1b. The duct dimension as given in Chapter 5, namely a duct length of 0.55\(m\) and a radius of 0.015\(m\).

For comparison with experimental data, computations based on a Norton version of the Wankel Rotary Engine, see Middleton [65], are also presented. The duct dimensions and other details are given in a later section. A further experimental comparison is presented for a piston engine, also detailed later.

The computational results are derived by solving the difference forms, equations (4.5) and (4.16) for the Finite Volume and the Flux-Splitting schemes, respectively, of the axi-symmetric Euler system of conservation equations (2.6).

The accuracy of the computations is measured by comparing the total mass fluxes over a complete cycle across the middle of the duct and the exit plane, to the mass flux crossing into the duct from the engine cavity. Tables of the calculated flux rates are attached for each of the time series plots. These tables consist of the mass change in the engine chamber, denoted by \(\Delta m\), the mass flux across the inlet boundary, denoted by \(Flux_i\), the mass flux across the middle cross-section of the duct, denoted by \(Flux_m\), and the mass flux across the exit plane of the duct, \(Flux_e\).
6.3.1 Open Duct-only Computations

Two sets of computations are carried out for the duct-only domain, using the Finite Volume Scheme and then the AUSM Flux-Splitting Scheme.

For each of these schemes duct calculations are based on the exit boundary condition $p = p_0$ with a non-isentropic implementation, an extrapolated characteristic implementation and a modified 'non-reflective' characteristic condition respectively.

For the Flux-Splitting scheme time-marching is carried out using a second order Runge-Kutta integration. For the Finite Volume Scheme, time marching is based on the two-step Lax/Wendroff integration.

Figure 6.10 depicts exit velocity profiles derived from a simple duct calculation based on the exit boundary condition $p = p_0$ with the non-isentropic implementation, the extrapolated characteristic implementation and the modified non-reflective characteristic condition respectively. The scheme used for these is the Flux-Splitting scheme of Radespiel and Kroll [58].

It is seen that the results from the extrapolated characteristic boundary implementation differ widely from the other results and may be presumed to be inaccurate. This is expected following similar findings when considering the multiple-harmonic results, see figure 6.2. The results based on the non-isentropic boundary implementation and the modified non-reflective boundary implementation are generally similar, except at the highest peak where a significant difference is seen. Comparison of the results in tables 6.1 and 6.3 indicates that the modified non-reflective characteristic condition gives a much better conservation of flux through the duct than the non-isentropic condition. Furthermore, table 6.2 confirms the inaccuracy of the extrapolated characteristic boundary implementation, with $Flux_e$ values very different to the mass flux evaluated other than at the exit.
plane. In summary, these results indicate that the modified non-reflective boundary implementation is to be preferred.

**Figure 6.11** depicts the results of a finite volume Lax Wendroff computation using a non-isentropic and then a modified non-reflective characteristic boundary implementation respectively. The profiles are similar in shape, the most significant discrepancy being that the characteristic-based computation result has a lower global peak.

The trend noticed with the Flux-Split Scheme carries over to the Finite Volume Lax/Wendroff Scheme, as can be seen in **figure 6.11**, but more clearly from the listing of mass fluxes in tables 6.4 and 6.5. The modified non-reflective characteristic boundary condition again gives the best conservation of mass, while the exit velocity profile is very similar to that obtained using the non-isentropic boundary condition.

In all of the results, there is a good match between the evaluated mass flux at the source and at the middle plane of the duct, but much less of a match with the flux across the exit plane. This may again be tentatively explained by difficulties in the implementation of the boundary conditions for the conservation variables at the exit plane. This explanation is supported by the following observations. As mentioned in the discussion of the multiple-harmonic results, the extrapolated characteristic boundary implementation is the least accurate due to the inaccuracy of the third characteristic variable, and is seen here to give the most inaccurate values of mass flux at the exit plane.

Tables 6.6 to 6.9 for computations over extended domains show a better mass flux comparison between the inlet, the middle section and the exit plane. Although these tables show some discrepancies, it can be seen that the differences are generally reduced, by comparison to those produced with the zero-pressure boundary condition, as given in tables 6.1 to 6.5.
These results again support the explanation that the poor results for mass flux across the exit plane in simple duct computations are due to the implementation of the $p = p_0$ boundary condition. Further discussion on the smaller errors in mass flux at the exit plane for computations over extended domains is deferred to Section 6.3.2.

6.3.2 Extended Domain Computations

Computations are carried out for a flanged duct and for an unflanged duct domain, respectively. For each of these domains computational grids based on the radial transform and the ‘T’ transforms are considered. The boundary conditions are set as discussed in Chapter 4, for all of the computational boundaries apart from the inlet, which is set as stated in section 6.3.

For a flanged duct, figure 6.12 depicts results using a simple radial grid and a ‘T’ transform grid respectively. Figure 6.13 depicts similar results for an unflanged duct. From figure 6.12 it is seen that the two grid types for the flanged duct produce a certain degree of disagreement on the exit velocity profiles. The same is true when the unflanged duct is considered, figure 6.13, but the extent of the discrepancies is reduced.

Reverting to the mass flux tables 6.6 and 6.7 for the flanged duct, it is seen that the more acceptable solution for the extended domains is based on a simple radial grid. This conclusion is counter-intuitive since the ‘T’ grids provide a better grid density in the far-field. It would be very interesting to pursue this finding, but suffice it to note that the ‘T’ grid does contain a discontinuity in the neighbourhood of the exit plane where these results are derived. For the unflanged duct, the mass fluxes in tables 6.8 and 6.9 suggest that the ‘T’-grid gives the better mass conservation.
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Table 6.1: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary, Mass Flux Across the Middle Cross-section of the Duct and Mass Flux Across the Exit Plane when the AUSM Scheme and Non-isentropic Boundary Conditions Are Applied to A Duct-only Grid With Engine Source
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Table 6.2: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary, Mass Flux Across the Middle Cross-section of the Duct and Mass Flux Across the Exit Plane when the AUSM Scheme and Extrapolated Characteristic Boundary Conditions are Used

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Table 6.3: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary, Mass Flux Across the Middle Cross-section of the Duct and Mass Flux Across the Exit Plane when the AUSM Scheme and Modified Characteristic Boundary Conditions are Used
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Table 6.4: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary, Mass Flux Across the Middle Cross-section of the Duct and Mass Flux Across the Exit Plane when the Finite Volume Scheme and Non-isentropic Boundary Conditions are Used

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Table 6.5: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary, Mass Flux Across the Middle Cross-section of the Duct and Mass Flux Across the Exit Plane when the Finite Volume Scheme and Modified Characteristic Boundary Conditions are Used
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Table 6.6: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary and Mass Flux Across the Exit Plane when the Finite Volume Scheme is Applied on a Flanged Radial Grid

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Table 6.7: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary and Mass Flux Across the Exit Plane when the Finite Volume Scheme is Applied on a Flanged ‘T’ Grid
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Table 6.8: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary and Mass Flux Across the Exit Plane when the Finite Volume Scheme is Applied on an Unflanged Radial Grid

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Table 6.9: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary and Mass Flux Across the Exit Plane when the Finite Volume Scheme is Applied on an Unflanged 'T' Grid
Figure 6.10: A Comparison of Exit Plane Velocity Profiles for Different Boundary Conditions With The AUSM Scheme Applied to a Duct-only Grid With an Engine Source
Figure 6.11: A Comparison of Exit Plane Velocity Profiles for Different Boundary Conditions With The Finite Volume Scheme Applied to a Duct-only Grid With an Engine Source
Figure 6.12: A Comparison of Exit Plane Velocity Profiles for Different Grid Transforms Applied to a Flanged Duct Grid With an Engine Source
Figure 6.13: A Comparison of Exit Plane Velocity Profiles for Different Grid Transforms Applied to an Unflanged Duct Grid With an Engine Source
6.3.3 Far-field Noise Due to Engine Sources

In this section results for the far-field noise radiated from the duct, when a version of the Wankel Rotary engine is used as the source, are presented. Results are presented for the case of a simple duct, a flanged duct and then an unflanged duct. The spectra are derived by resolving the exit velocity time history into its Fourier modes and then calculating the energy for each of the modes.

For the simple duct, the Finite Volume scheme with Lax Wendroff time stepping is used with the modified non-reflective boundary conditions and also with the isentropic boundary condition. With similar boundary conditions, the AUSM scheme is also used to calculate the exit plane velocity profile. A comparison of output from these two sets of computations is presented in figure 6.14.

It is seen that the two spectra due to the AUSM Flux-splitting scheme are much closer than are those due to the Finite Volume scheme. Furthermore the spectrum due to the Finite Volume Scheme with the modified characteristic boundary condition is also much close to the spectra due to the Flux-splitting scheme. It is also noticeable that both the spectra due to the Finite Volume scheme exhibit a smooth variation in the noise level with the frequency in the lower harmonics. This suggest that the artificial viscosity of the scheme is adversely affecting the refinement of the amplitudes of the harmonic components.

It could be reasonably concluded from these results that while the Flux-splitting scheme controllably damps the spurious modes, reducing any errors that are reflected from the downstream boundary, as per section 5.1.2, the Finite Volume scheme depends on artificial viscosity, which is much less controllable against the spurious modes. This is shown by the fact that the spectra due to the Finite-volume scheme are unrealistically smooth in
the first twenty harmonics. However, it is positive to note that the general levels of the spectra due to the modified characteristic boundary condition, for both schemes, are closer than any other combination in the range of lower harmonics where the highest noise levels are found. It may also be recalled that the modified non-reflective characteristic boundary conditions are based on a relaxation of the non-reflective property.

Figures 6.15 and 6.16 depict spectra corresponding to computations on extended flanged and unflanged duct domains, respectively. In each case a comparison of the 'T' grid and the simple radial grid is made. In both case the results remain close together for the low order modes with high noise levels, then diverge at the higher modes where the noise levels are low and hence not as important. The discrepancy in the amplitude of the high order harmonics is caused by relatively minor differences in the time history of the exit velocity.

Given that the firing frequency is 117Hz, it can be seen that the discrepancies start at frequency ranges above 2.9kHz for which the grid-point per wavelength stability may not be sufficient.
Figure 6.14: Far-field Noise Levels for Different Harmonics of the Engine Firing Frequency, for an Engine Source Computation on a Simple Duct, Comparing two Boundary Conditions and two Schemes.
Figure 6.15: Far-field Noise Levels for Different Harmonics of the Engine Firing Frequency, for an Engine Source Computation on a Flanged Duct, Compared for Two Grid Transforms.
Figure 6.16: Far-field Noise Levels for Different Harmonics of the Engine Firing Frequency, for an Engine Source Computation on an Unflanged Duct. Compared for Two Grid Transforms.
6.3.4 Norton Wankel Rotary Engine Measurements

The experimental results presented here were measured and reported by Middleton [65] at AB Dynamics Ltd. in October 1990. The experiments were designed to examine the effect of a silencer by measuring radiated noise, in turn with and without the silencer, at a distance of 30 ft. (10.8m) from the exit plane of the exhaust duct. Two duct lengths of 1.1m and 0.505m, respectively, both of radius 0.028m were examined. For each of the setups three engine power levels, namely 50%, 75% and 100% were examined as the engine drove a propeller. The engine, propeller and exhaust system were all contained in a single, large reverberant chamber, thus the test conditions were far from ideal for isolating exhaust noise.

Here the data corresponding to the experiments for 100% power, without the silencer, are compared with computed results. For each of the duct lengths two computed results are examined, namely results based on a duct-only domain with a modified characteristic exit boundary condition, and results based on an unflanged duct domain, both with the Finite Volume scheme. For the unflanged duct computation, the 'T' grid transform is chosen, based on its better conservation properties as discussed in section 6.3. Tables of mass flux, similar to those presented in section 6.3, are presented for each of the computed results. The computed values of the radiated noise are based on the exit plane velocity profiles.

Figures 6.17 and 6.18 depict the comparisons of computed and measured noise values for the 0.505m and the 1.1m ducts, respectively. In both plots, the noise values due to the simple duct computation appear closer to the measured values than do the values due to the extended domain computations. In figure 6.18, the simple duct results fit the measured data remarkably well apart from the lower harmonics, where the computed results are closer to each other in both cases. One would expect the dominant
noise to be at the firing frequency and its first few harmonics, as predicted by computed results. Thus the experimental results appear to be in error at low frequencies, where problems of reverberation would be most significant.

Tables 6.10 to 6.13 are lists of fluxes across various cross-sections of the ducts measured for the different computations. From these tables it is seen that the extended domain computations maintains better mass conservation along the duct than the duct-only computations. This observation would suggest that the exit velocity profiles derived from the extended domain computations are more accurate than those derived from duct-only computations. Hence for the far-field noise, the values derived from an extended domain computation would be more accurate than those derived from a duct-only computation. Furthermore, the extended domain computations converge in half as many cycles as the duct-only computations.

The differences between the computed results and the measured data could be attributed to experimental errors and other effects likely to arise from the modelling of the inlet boundary. In particular, the computations are based on a very approximate model of the engine in-cylinder conditions. Although this form of model has proved acceptable for calculations of engine performance, accurate prediction of noise generation requires a much higher order of accuracy.

The differences due to experimental error could be in the form of ground reflection and propeller blade noise, see Middleton [65]. In the experimental data the frequencies of the measured noise did not coincide precisely with the harmonics of the firing frequency. Also, in the set of measured noise values for the 1.1m duct the values corresponding to the fundamental mode and the third harmonic appear further out of line with the rest of the data than would be expected, in particular being too low, as noted earlier.
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Table 6.10: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary, Mass Flux Across the Middle Cross-section of the Duct and Mass Flux Across the Exit Plane when the Finite Volume Scheme And Modified Characteristic Boundary Conditions Are Applied to the 0.505m Experimental Duct With an Engine Source

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Table 6.11: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary, Mass Flux Across the Middle Cross-section of the Duct and Mass Flux Across the Exit Plane when the Finite Volume Scheme And Modified Characteristic Boundary Conditions Are Applied to the 1.1m Experimental Duct With an Engine Source
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Table 6.12: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary and Mass Flux Across the Exit Plane when the Finite Volume Scheme is Applied to the 0.505m Experimental Duct With an Unflanged Radial Grid

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Table 6.13: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary and Mass Flux Across the Exit Plane when the Finite Volume Scheme is Applied to the 1.1m Experimental Duct With an Unflanged ‘T’ Grid
Figure 6.17: Measured Far-field Noise Levels Compared to Computed Values due to the Modified Characteristic B.C. and a Finite Volume Scheme on a Radial Unflanged Grid, With a 0.505m Duct for Different Harmonics of the Engine Firing Frequency.
Figure 6.18: Measured Far-field Noise Levels Compared to Computed Values due to the Modified Characteristic B.C. and a Finite Volume Scheme on a Radial Unflanged Grid, With a 1.1m Duct for Different Harmonics of the Engine Firing Frequency.
6.3.5 Piston Engine Results

The piston engine results are based on data reported by Payri et al. [66]. The experiments were designed to test the effect of a muffler on radiated noise. The authors measured the pressure profile at 155 mm inside of the exit plane of a 6.9 m duct connected to the exhaust port of a half-litre single cylinder spark-ignition engine running at 1500 rpm and full load. The authors also measured the radiated noise at a distance of 0.5 m with the exit plane 1.5 m above the ground.

The computed results are based on the homogeneous ‘filling-and-emptying’ model of Benson. Coupled with the excessive length of the exhaust duct, this introduces large discrepancies between the computed results and the experimental measurements. The computed pressure is found to steepen progressively through the duct resulting in a very narrow pulse at the exit, as one would expect due to non-linear effects, but unlike the computed results of Payri et al. [66]. The in-cylinder pressure and temperature at the instant the exhaust valve opens were taken to be 50 kPa and 1200 K respectively, chosen as credible values which give the correct amplitude of the pressure pulse as measured. The computation made use of a flanged radial grid on an extended domain.

The computed results are assessed by examining the conservation of mass through the duct. To this end, tables of mass fluxes across various cross-sections of the duct are presented. From table 6.14 it is seen that the computation preserves conservation through the duct reasonably well. This suggests that the discrepancies between the computed results and the measurements, which are very large, are mostly due to the engine model, which is beyond the remit of this thesis, but is discussed further in the conclusion.

In figure 6.19 the exit pressure and radiated noise computed using the schemes developed in this thesis are compared with the computed and
measured data of Payri et al.[66]. On the top diagram, it is seen that the amplitude of the pressure pulse as computed by Payri et al.[66] is too high, although the pulse width matches the experimental measurements.

The lower plot depicts the radiated noise at a distance of half a metre from the exit plane. The computed values are averaged over the hemisphere whereas the measured data refers to a single measurement at an angle of 45° to the flow direction.

Payri’s numerical model includes considerations of viscosity and temperature variations whereas in this thesis, the temperature is assumed to be constant and viscosity is neglected. The source models also differ in that Payri considers in-cylinder conditions dependent on measured values whereas the model in this thesis relies on a guess of the initial in-cylinder temperature and pressure. The modelling of temperature and viscous effects could account for the difference in pulse width. The importance of these effects is exaggerated in Payri’s study due to the unrealistically long exhaust duct.

In this thesis a quasi-three dimensional model of the duct and flow is used and the extended computational domain eliminates the problems associated with modelling the exit plane boundary conditions.

The error in the amplitude of the pulse of Payri’s model could then be accounted for by the fact that Payri uses a simplified exit plane boundary condition.

Given the above differences, the discrepancies in the far-field noise are to be expected.
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Table 6.14: Change in Mass in the Engine Cylinder, Mass Flux Across the Inlet Boundary and Mass Flux Across the Exit Plane When the Finite Volume Scheme is Applied on a Flanged Duct With Piston Engine Source as Inlet Boundary
Figure 6.19: Top: Exit Pressure Pulse; bold dashed line (measured), bold solid line (computed by Payri), dashed line (computed with reference in-cylinder pressure 50kPa and temperature 1200K) Lower: Noise at 0.5m from pipe exit; triangles (measured) circles (calculated by Payri) octagons (present).
Chapter 7

Conclusions and Recommendations
This thesis has involved numerically modelling the flow and acoustic properties at the exit plane of an engine exhaust duct in the time domain. In the past, the usual condition used to model these boundary properties has been the zero-pressure condition, whereby the fluctuation of the exit plane pressure from atmospheric pressure was neglected. As mentioned in Chapter 1, numerical predictions of the noise radiated from the car exhaust systems on this basis have been largely inaccurate and hence the aim of this thesis was to investigate ways of improving such predictions by modifying and improving the above pressure boundary condition.

To this end, various forms of computation have been carried out. These have included various implementations of the pressure condition on a rectangular domain, but also, by use of extended computational domains, avoidance of the need to set boundary conditions at the exit plane. A modified implementation based on the method of characteristics has been presented, under which the acoustic pressure at the exit plane was allowed to vary.

7.1 Rectangular Grid

Using a simple duct domain terminating at the duct exit plane, various forms of the numerical boundary implementation, based on the $p = p_0$ condition, have been tried.

Based on the conservation of momentum, two possible implementations were devised, one of which involved simplifying the condition by assuming isentropic flow and the other was a straight use of the conservation of momentum. In every computation, from single-frequency through multiple-harmonic to engine source flows, these two boundary implementations were practically indistinguishable in their results. This applies to both the Finite
Volume Lax/Wendroff scheme and the AUSM flux-splitting scheme.

Two further boundary implementations based on the method of characteristics, were also used, namely simple extrapolation of the characteristic conditions from the interior onto the boundary, and a modified non-reflective condition involving reasonable deductions of flow properties just outside the flow domain, i.e. downstream of the exit plane. For low amplitude single frequency flows, the extrapolated characteristic implementation of the boundary conditions proved more useful, converging better and faster not only than the modified non-reflective characteristic condition, but also than the conservation-based implementations.

On the multiple-harmonic and engine source flow computations, the modified non-reflective characteristic boundary implementation proved more accurate than the extrapolated characteristic implementation.

7.2 Extended Domain Grids

Extended grid domains were used to avoid the \( p = p_0 \) condition at the exit plane. Two physical domains, namely flanged and unflanged ducts, were considered.

For the flanged duct three possible grid configurations were proposed and discussed – the radial grid, the ‘T’ grid and the ‘L’ grid. The ‘L’ grid was seen to result in a larger number of grid cells, and therefore required more computation time, but with no significant advantage in terms of accuracy. The radial grid and the ‘T’ grid were compared in various flow configurations and tended to produce similar levels of accuracy for the properties examined. The ‘T’ grid would logically be favoured if the properties are examined at the far-field boundary, as it would allow better control of the cell sizes.
For the unflanged duct, the radial grid and the ‘T’ grid were tried. As with the flanged duct, the two grid types provided a similar level of accuracy for the properties examined.

In each of these extended domain cases, the Sommerfeld radiation condition was used at the far-field boundary. Prior to implementation on the composite extended grid, the condition was used to compute acoustic propagations in horns, for which linearised analytic solutions are known. The special case of the conical horn used was a half space bounded by a wall with the acoustic disturbance originating from the pulsating surface of a sphere embedded into the wall.

The Finite Volume Lax/Wendroff scheme was then used to compute the flow properties on the extended domains.

In the single-frequency computations the results on the extended domains gave similar levels of accuracy to those of the extrapolated characteristic implementation on the rectangular grid using the AUSM flux-splitting scheme. The flux-splitting scheme allows the spurious modes of the numerical solution to be subdued, lessening the harmful effects of numerical reflection from the exit plane, whereas the extended domains avoids use of the incorrect $p = p_0$ condition, but supports spurious modes.

In the cases of multiple-harmonic and engine source computations, the modified non-reflective boundary implementation on a simple duct grid proved to have the best accuracy. This suggests that the zero pressure fluctuation condition would give reasonable accuracy provided the numerical implementation of the boundary is well chosen and spurious modes in the numerical solution are reduced.

It can therefore be deduced that the $p = p_0$ boundary condition causes inaccuracy by reflecting numerical errors back into the computational domain. Reducing these errors, say by suppressing the spurious modes, improves
the convergence rate. By allowing the exit plane pressure to fluctuate, as is done with the modified non-reflective characteristic boundary condition, it was found that the engine source computations had an accuracy, based on the through flux tables, nearing that of the extended domain computations.

Correlation of computed values of radiated noise levels with experimental results was very poor. The two sets of experimental results which were used were far from ideal for such comparison for different reasons, but much of the discrepancy can be accounted for by inadequacies in the modelling of the source. The influence of the source model on the calculated levels of radiated noise was found to be much greater than that of the radiation model at the exit plane, which was the focus of investigation in this thesis. The inability of time domain models to evaluate radiated noise levels accurately is thus chiefly attributable to deficiencies in the source modelling, and the benefit of improved radiation modelling, as discussed in this thesis, will have no significant effect until these deficiencies are resolved.

### 7.3 Recommendation for Further Investigations

It has been found that reducing spurious modes in the numerical solution, e.g. by use of the AUSM flux-splitting scheme, or reducing numerical reflection from the exit boundary, e.g. by extending the computational domain downstream of the exit plane, both give a similar level of accuracy. A useful investigation would be to find the effect of implementing both improvements. This can be done by implementing a AUSM flux-splitting scheme for computations on the extended grids. Given that the two improvements independently have proved to be useful in improving the accuracy of the numerical solution, it would be reasonable to expect that, in conjunction, they should provide a significantly better accuracy.
The best possible route for future investigation would be to use Direct Numerical Simulation (DNS), methods to extend the far-field boundary and the flow equations. This would reduce numerical errors due to grid size variations and would allow the errors due to reflection at the far-field boundary to be reduced. Also, since DNS allows turbulence and viscous effects to be modelled, it would be possible to take account of the flow separation at the exit plane. One further advantage of DNS methods would be the removal of the need for artificial viscosity, since the method includes viscous effects in the difference equations. But DNS methods are very expensive and it would be necessary to have good models of every aspect of the system to justify their application.

7.3.1 Flow Equations

It has been assumed that the flow in the exhaust duct is axisymmetric. In a real engine exhaust flow, the exhaust port geometry is usually quite complex and certainly non axisymmetric. Thus the exhaust flow and the acoustic waves would be non axisymmetric at the source end of the duct. Furthermore, real exhaust silencer systems generally have some non axisymmetric features. These effects all imply that the flow encountered at the exit plane is likely to contain some asymmetry and hence to be of a more complicated form than that assumed in this thesis.

With these possibilities, it becomes more difficult to specify a reasonable boundary condition at the exit plane. To attempt to account for these eventualities, it may be necessary to use the full 3-D Navier Stokes equations, possibly with localised turbulence modelling near the exit plane. It would then also be necessary to use an extended domain where, due to the expansion of the flow area outside the duct and downstream of the exit plane, the flow becomes more linear in the neighbourhood of the far-field.
boundary. Use of Navier-Stokes equations would also model flow separation, vortex generation and jet expansion near the exit plane.

Such a model would conceivably provide enough resolution to predict such subtle effects as the exit plane radiation impedance more accurately.

7.3.2 Inlet Boundary

At the inlet boundary, a simplistic model of the engine source was used to determine the inlet boundary conditions which did not account for detailed cylinder geometry or in-cylinder variation of the thermodynamic properties. It may be worth improving the model of the source by using an actual geometric time-variation of the flow in the engine cavity in place of the inlet boundary conditions. This would involve a moving boundary model of the cavity, requiring a time-varying grid of the cavity, but this need would be localised and confined to the cavity alone.

This model would provide a more realistic fluctuation of the flow properties in the duct, possibly leading to better analysis of engine performance as well as radiated noise.

Another useful application would be to simulate various shapes of the engine cavity to find more efficient layouts. This is particularly applicable to rotary engines where considerably less research has been carried out in this particular area.

These refinements would increase the computational requirements considerably, but not beyond available computational technology.
Bibliography


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