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Locally Resonant Periodic Acoustic Media

by

Luke Chalmers

A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the

Faculty of Science
Department of Physics

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Phononic crystals are composite media, with two different elastic materials modulated in a periodic fashion. The two-dimensional system under study is comprised of an array of cylindrical tubes arranged in a square Bravais lattice. The propagation of a time-harmonic acoustic Bloch waves is investigated experimentally and the findings compared with those obtained from the finite element method. The band structure, that is a well-known characteristic of Bloch wave dispersion, is also determined. This demonstrates the existence of band gaps of forbidden transmission at certain frequency ranges. These arise due to the interference when the wavelength of the incident wave is comparable to the periodic spacing of the modulated media.

A second phononic crystal with slotted steel tubes is also studied. The opening in the tube permitting air to flow to the internal cavity, and constituting a Helmholtz resonator. The band structure of such a structure possesses additional band gaps, which arise due to the excitation of the resonators. Furthermore, it has recently been shown that periodically distributed Helmholtz resonator structures, can exhibit negative refraction phenomena. Using the transmission line technique, as outlined by Y.Cheng et al. we demonstrate this behaviour for our experimental system. It is shown that, at certain frequency ranges, the locally resonant phononic crystal exhibits negative effective density and bulk modulus, the two conditions required to obtain a negative index of refraction.
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Chapter 1

Introduction

When a plane wave propagates through an isotropic and homogeneous medium, no obstacles exist to impede its transmission, and it travels undisturbed. If however, the medium through which the wave is travelling contains a discontinuity in terms of impedance, the wavefront will interact with the object, the amount of energy that is reflected and transmitted being proportional to the impedance mismatch. If a medium is comprised of several obstacles arranged in a periodic distribution, ordered scattering occurs from the structure.

In solid state theory the interaction of de Broglie waves with periodic crystalline structures has been extensively studied. The conduction properties of solids are described by considering the ions of the crystal as an infinite periodic potential. The Hamiltonian for an electron supported in a periodic potential of the form $V(x) = V(x + d)$ is given by,

$$H = \frac{p^2}{2m} + V(x), \quad (1.1)$$

which has corresponding Eigenfunctions of the form $\varphi = e^{ik \cdot x} \cdot V(x)$, known as Bloch Eigenfunctions. They consist of a plane wave, $e^{ik \cdot x}$, and a periodic function, or Bloch envelope, $V(x)$, which has the same periodicity as the potential. The Bloch wave vector, $k$, in the exponential, is unique up to a reciprocal lattice vector due to the structures periodic nature. Therefore only the wave vectors inside the first Brillouin zone are considered. For a given wave vector and potential, there are several solutions found by the Schrödinger equation for a Bloch electron. The different solutions are called bands and are separated by a finite spacing at each $k$. If there is a separation which extends over all wave vectors it creates a band gap. The collection of energy Eigenstates within the first Brillouin zone is called the band structure.
Bloch’s theory was the basis for explaining the conduction properties of matter on a quantum mechanical level. However, the theory of Bloch waves applies to any wave like phenomena in a periodic medium. For example, in electromagnetism, a periodic dielectric affects the motion of photons and constitutes a photonic crystal. A periodic elastic medium affects propagating phonons and is called a phononic crystal. It is these phononic, also known as acoustic or sonic crystals, which are the subject of this thesis. They are constructed from periodic repetitions of solid or fluid material, and specifically we concern ourselves with two-dimensional systems. Examples of one, two and three dimensional structures are illustrated in Figure (1.1). Interest into wave propagation through periodic acoustic media was motivated due to the rich and interesting physics the field offers, and initiated by developments in the field of photonics. Furthermore, working at acoustic frequencies has advantages. The longer wavelengths considered mean that the corresponding scale of periodicity is significantly larger. Typically therefore, periodic elastic media can be constructed simply and cheaply, allowing numerous experimental verifications to be provided to accompany the developing theoretical basis. Moreover, if the acoustic wave is contained to propagate within a fluid medium the transverse components of the wave may be disregarded, simplifying band structure calculations to only consider the longitudinal component.

If we consider the one-dimensional scenario, an acoustic wave incident upon such a structure will be partly reflected at every interface, due to the impedance difference between the two media. The reflected components shall then interact with the incident component. Constructive interference causes the energy of the wave to be reflected back upon itself and as such propagation is forbidden. Conversely, if the interference is destructive then all of the energy in the original wave will be transmitted through the crystal. This means that constructive interference of the secondary waves is responsible for the formation of band gaps, while destructive interference leads to the formation of propagation bands. The interference obeys Bragg’s law as given by \( n\lambda = 2d\sin \theta \), where \( d \) is the distance between successive planes. For constructive interference to occur, the
path difference between the interfering waves must be equal to an integer multiple of their wavelength $\lambda$. As the path difference is related to this spacing, constructive interference occurs when the lattice parameter is comparable to the wavelength. Since frequency is inversely proportional to wavelength, the central frequency of the band gap is inversely proportional to the lattice parameter. Therefore it can be written, $\omega_c \sim 1/\lambda \sim 1/a$. The bandwidth of the acoustic spectral gap is directly related to the ratio of the acoustic impedances of the different layers. The position and bandwidth is also dependent upon the direction of a wave’s incidence upon the crystal. This stems directly from the angular dependence in Bragg’s law.

There is an intrinsic limitation with conventional phononic crystals in that the band gaps formed are directly related to the dimensional parameters. To overcome this fundamental restriction, a new class of phononic crystal was developed in which the periodic scattering inclusions are replaced with resonating structures. These are capable of forming separate bands of forbidden transmission. The location of such bands is dependent upon dimensional and material parameters, and can form band gaps at frequencies below a structure’s “Bragg” band. This thesis, details an experimental and theoretical investigation upon a rigid-cylinder phononic crystal (RCPC) and a locally-resonant phononic crystal (LRPC) comprised of Helmholtz resonator inclusions. Steel cylinders are chosen as the periodic scatterers embedded in an air host medium. This simple construction has been used in previous experimental studies of phononic crystals. It is advantageous to work with air as the host medium in the hope of developing practical phononic crystals, that may be utilised in an external environment as acoustic shield barriers. Keeping this potential application in mind, the samples fabricated are designed to be active in the audible frequency range (\sim 20Hz-20kHz).

A theoretical comparison for the experimentally measured data is provided by finite element (FE) computation. This numerical technique is an efficient and flexible tool with which to investigate phononic crystals with different geometrical and material parameters, with both trivial and non-trivial unit cells. The finite element method technique is capable of solving most physical problems governed by partial differential equations. In this report we substitute a time-harmonic pressure dependence into the acoustic wave equation. This reduces the governing mathematics to a Helmholtz equation, where working in the frequency domain, pressure spectra can be obtained. Moreover, by using the Eigenvalue formulation of the wave equation, the acoustic band structure of a single periodic unit cell can be calculated.

The motivation of this thesis is to study locally resonant phononic crystals capable of shielding large ranges of the audible frequency regime. Development of systems which are capable of such performance has obvious commercial applications. In the audible
frequency range with which we are concerned, the wavelength of sound implies very large periodic spacings would be required, which may be impractical. Therefore we use the concept of local resonance to reduce the active frequency of the designed systems. It is also desirable to exhibit some control over the bandwidth of the forbidden gap, and in this thesis the broadening of the acoustic band gap will be addressed. By enlarging the frequency range attenuated by a phononic crystal a larger portion of unwanted noise in the spectrum can be shielded.
Chapter 2

Theory

The first chapter of this thesis serves as an introduction to the concept of band gaps. The term is widely used throughout many different areas of physics to describe a forbidden energy range that arises when a wave type phenomenon encounters a periodic medium. We will begin our discussion from the perspective of a solid state physicist, and consider the simple problem of an electron in a periodic potential. By deriving the energy states of the system, it is possible to understand how forbidden energy bands manifest from diffraction effects at the edge of the first Brillouin zone. Although formulated from a solid state perspective, the behaviour of an electron wave in such a system can be considered analogous to acoustic wave propagation through periodic elastic media which is the subject of this thesis. In addition to providing an understanding of band gap formation, the example also familiarises the reader with the terminology and nomenclature used throughout. The derivation of the electronic band structure presented here is summarised from Kittel [1]. More comprehensive details can be found in Chapters one through seven of this work, or the texts by Anselm [2] and Ziman [3] for a more rigorous introduction with examples.

In the second section of this chapter, the concept of band gaps is extended to introduce periodically modulated dielectric and elastic composites, known as photonic and phononic crystals respectively. Four classes of phononic crystal are introduced, which are categorized according to their material characteristics. The wave equations are presented for elastic and acoustic waves, which are supported in solid and fluid media respectively, and the dispersion relation for each case is discussed.
2.1 Electronic Band Structure

To begin, we consider the behaviour of free valence electrons in the crystal structure of a solid metal with positive ions occupying the lattice points. These are the conduction electrons that experience a periodic potential supported by the ions. The scale of the crystals periodicity is of the order of the de Broglie wavelength of an electron, 1Å, therefore we must study the problem using the principles of quantum mechanics. We will start our discussion with the fundamental axioms of the free electron gas model, and then expand this approximation to look at the simplified problem of an electron in a one-dimensional periodic potential and discuss the special periodic wavefunctions (Bloch waves) that are its solution.

2.1.1 Free Electron Gas

The free electron gas model considers a single electron contained in a potential well. The electron is assumed to be in constant potential \( U = U_0 = 0 \) inside the crystal of length \( L \) in all directions, and \( U = \infty \) outside. Considering the time-independent Schrödinger equation,

\[
-\frac{\hbar^2}{2m} \left( \frac{\partial^2 \Psi(x, y, z)}{\partial x^2} + \frac{\partial^2 \Psi(x, y, z)}{\partial y^2} + \frac{\partial^2 \Psi(x, y, z)}{\partial z^2} \right) + U(r) \Psi(x, y, z) = E \Psi(x, y, z),
\]

with periodic Born-von Karman boundary conditions \( \Psi(r + R) = \Psi(r) \), the three-dimensional solution is defined by,

\[
\Psi = \left( \frac{1}{L} \right)^{3/2} \exp(i \mathbf{k} \cdot \mathbf{r}).
\]

There are infinitely many solutions, each individual solution selected or described by a set of three quantum numbers \( n_x, n_y, n_z \). The solution \( \Psi \) describes a plane wave of amplitude \( (1/L)^{3/2} \) moving in the direction of the wave vector \( \mathbf{k} \). We can extract terms for wavelength of the electron wave, \( \lambda \), momentum, \( \mathbf{p} \), and for an electron of mass \( m \) the velocity \( \mathbf{v} \),

\[
\lambda = \frac{2\pi}{|\mathbf{k}|} = \frac{2\pi}{k}, \quad \mathbf{p} = \hbar \mathbf{k}, \quad \mathbf{v} = \frac{\mathbf{p}}{m} = \frac{\hbar \mathbf{k}}{m}.
\]

The total energy \( E \) is identical to the kinetic energy (as we assume no potential), therefore \( E_{\text{kin}} = 1/2mv^2 = p^2/2m \),

\[
E = \frac{(\hbar k)^2}{2m} = \frac{\hbar^2}{2m} \left( \frac{2\pi}{L} \right)^2 \left( n_x^2 + n_y^2 + n_z^2 \right).
\]
This is an important result of the free electron model. It shows that there are only discrete energy levels for an electron in a box with a constant potential that represents the crystal. Equation (2.4) represents the total energy of the electron as a function of the electron wave vector. Any relation of this kind will from herein be called a dispersion relation.

2.1.2 The Reciprocal Lattice and Bragg Condition

The electron waves defined by the solution to the time-independent Schrödinger equation are like any other wave phenomena in that they experience diffraction effects in a periodic structure such as a crystal. A perfect crystal is comprised of an infinite array of repeating groups of atoms. In crystallography the structure is defined in real space by the groups of atoms and a mathematical construct to which they are attached called the lattice. A three-dimensional lattice can be characterised by three translation vectors \( a_1, a_2, a_3 \) that are chosen such that the arrangement of atoms looks the same when viewed from point \( r \) translated by,

\[
    r' = r + u_1a_1 + u_2a_2 + u_3a_3,
\]

where \( u_1, u_2 \) and \( u_3 \) are arbitrary integers. The translation vectors are known as primitive, \( a_i \), if any two points from which the atomic arrangement is viewed, look the same and always satisfy Equation \( (2.5) \), and there is no unit cell of smaller volume than \( a_1 \cdot a_2 \times a_3 \) that can serve as a building block of the crystal structure. To construct the primitive cell in real space, a first set of construction lines are drawn, connecting a lattice point to all those nearby. It is then possible to draw a second set of lines, located at the mid-point and directed normal to the first. The resulting minimum volume cell is referred to as the Wigner-Seitz cell (Figure 2.1).

**Figure 2.1:** Construction of a Wigner-Seitz cell for a hexagonal Bravais lattice. The first set of construction lines (red) are intersected perpendicular to their midpoint by a second set of construction lines (blue) which define the cell.
To describe diffraction effects we consider the reciprocal lattice, which is the Fourier transform of the space lattice. Its primitive cell is defined by the primitive vectors $b_1, b_2, b_3$ obtained from the space lattice defined by its primitive vectors $a_1, a_2, a_3$ by the equations,

$$b_1 = 2\pi \frac{a_2 \times a_3}{a_1 \cdot a_2 \times a_3}, \quad b_2 = 2\pi \frac{a_3 \times a_1}{a_1 \cdot a_2 \times a_3}, \quad b_3 = 2\pi \frac{a_1 \times a_2}{a_1 \cdot a_2 \times a_3}.$$  \hspace{1cm} (2.6)

The primitive vectors of the reciprocal lattice can be constructed in the space lattice by drawing vectors perpendicular to the three \{100\} planes and taking their length as $2\pi/d_{hkl}$ where $d_{hkl}$ is the spacing between the lattice planes with Miller indices \{hkl\}.

When the electron wave $k$ is incident upon a crystal it will be reflected at a particular set of lattice planes \{hkl\} characterised by its reciprocal lattice vector $g$ only if the Bragg condition is met. If the wave vector of the reflected wave is $k'$, the Bragg condition correlates the three vectors involved $k, k'$ and $g$ through,

$$k - k' = g.$$  \hspace{1cm} (2.7)

**Figure 2.2:** Schematic representation of the Bragg condition. If and only if the three vectors form a closed triangle is the Bragg condition met and the incident wave diffracted. If not, the incoming wave moves through the lattice and emerges on the other side of the crystal, neglecting absorption.

When analysing the structure of a crystal we determine the diffraction pattern in reciprocal space. In X-ray analysis of crystals the incident wave is a well defined plane monochromatic wave, and we wish to know which set of lattice planes will reflect the incoming beam. If we consider the diffraction effects occurring to the free electrons contained in the crystal, however, we are looking at a quasi-continuum of wave vectors. We have all possible directions and many wavelengths. Therefore we ask the question, which particular wave vectors of the infinitely many, meet the Bragg condition for a given crystal lattice plane. This can be explained by considering the Brillouin zone of the crystal, which is the representation of the Wigner-Seitz cell in reciprocal space. It
can be seen from geometrical considerations that all wave vectors ending on the Wigner-Seitz cell walls will meet the Bragg condition for the set of lattice planes represented by the cell wall (Figure 2.3) and are thus diffracted. Of course, the origin must be at the centre of that Brillouin zone. Wave vectors completely in the interior of the 1st Brillouin zone or in between any two Brillouin zones will never get diffracted, they move pretty much as if the potential would be constant, i.e. they behave very close to the solutions of the free electron gas.

![Figure 2.3: The first Brillouin zone depicted in reciprocal space. The Bragg condition is met for wave vectors ending on the zone boundary, provided they originate at the centre of the cell.](image)

It has been shown that all electrons with wave vectors on or near a Brillouin zone are diffracted whereas others are not. This means that electrons with wave vectors near or at a Brillouin zone, $k_{BZ}$, feel the periodic potential while others do not. In other words the $k_{BZ}$ electrons interact with the crystal and this must manifest itself in their energies. It leads to a splitting of the energy value, therefore for a given $k_{BZ}$ electron, instead of $E(k) = (\hbar k)^2 / 2m$ we obtain,

$$E(k_{BZ}) = \frac{(\hbar k)^2}{2m} \pm \Delta E.$$  

(2.8)

The introduction of the term $\Delta E$ means that electrons at the Brillouin zone can have two energies for the same wave vector and thus state. One value is lower than the free electron gas value, the other is higher. Energies between these values are unobtainable for any electrons, causing an energy gap in the $E = E(k)$ relation for all $k$ vectors ending on a Brillouin zone.

To visualise the energy gap we consider a one-dimensional crystal represented by a chain of atoms. In this case the electron wave meeting the Bragg condition will be reflected back on itself. We have,

$$k' = -k,$$  

(2.9)
which leads to the formation of a standing wave. The only solutions of the Schrödinger
equation will be described by the possible superpositions of the two waves of which there
are only two possibilities,

\[ \Psi^+ = \left( \frac{1}{L} \right)^{3/2} \left( \exp (i k \cdot r) + \exp (-i k \cdot r) \right) \] (2.10)

\[ \Psi^- = \left( \frac{1}{L} \right)^{3/2} \left( \exp (i k \cdot r) - \exp (-i k \cdot r) \right). \] (2.11)

Both solutions are now no longer propagating plane waves with a constant probability
density (\(\Psi \Psi^* = \text{constant} \)) throughout the crystal, but standing waves. The probability
density follows a relation given by,

\[ \Psi \Psi^* = \text{constant} \cdot \cos^2 \left( \frac{2 \pi x}{a} \right). \] (2.12)

The maxima of the probability density occurs at the coordinates of the atoms for the
\(\Psi^-\) solution and between the atoms for the \(\Psi^+\) solution. In the first case the potential
energy of the electron is lowered, in the second case it is raised, this is the energy gap.
We have now introduced a potential energy by assuming that the potential is no longer
constant.

It is found that waves with \(k \approx k_{BZ}\) experience diffraction but do not immediately run
back on themselves. After some more reflections however, they do, leading to a splitting
of the energies for all positions on the Brillouin zone. A general relation yields for the
energies of the \(k_{BZ}\) electron waves,

\[ E(k_{BZ}) = \frac{(\hbar k)^2}{2m} \pm |U(g)|, \] (2.13)

with \(U(g)\) the Fourier component of the periodic potential for the reciprocal lattice
vector \(g\) considered. Using this relation we can construct the \(E(k)\) diagram for the free
electron gas model with diffraction effects (Figure 2.4). An energy gap still manifests in
different directions, but at a different position and with different width.

2.1.3 Bloch’s Theorem

The free electron gas model with added diffraction, provides a fundamental understand-
ing of band gap formation. Thus far however the periodicity of the potential has not
been specified. It is defined by the spacing of the ions in a crystal. Therefore at points
\(r\) or \(r + R\) with \(R\) any translation vector,

\[ V(r) = V(r + R). \] (2.14)
Chapter 2. Theory

Figure 2.4: Dispersion relations for a.) the free electron gas model \( E = (\hbar k)^2 / 2m \), and also for b.) the free electron model with diffraction considerations \( E = (\hbar k)^2 / 2m + \Delta E \). The additional term \( \Delta E \) leads to a splitting of the energy and defines the magnitude of the band gap.

It is now possible to obtain wavefunctions \( \Psi_k(r) \) that are solutions of the Schrödinger equation for \( V(r) \). As before, three quantum numbers \( k \) are used as an index to distinguish the solutions. Independent electrons which obey the one electron Schrödinger equation for a periodic potential are called Bloch electrons and obey Bloch’s theorem.

"The Eigenstates of the one electron Schrödinger equation for a periodic potential have the form of a plane wave multiplied by a function which has the periodicity of the Bravais lattice."

Or equivalently,

\[
\Psi_k (r) = u_k (r) \exp (i k \cdot r), \quad \text{with } u_k (r + R) = u_k (r). \quad (2.15)
\]

The periodicity of the lattice also implies,

\[
\Psi_k (r + R) = \Psi_k (r) \exp (i k \cdot R). \quad (2.16)
\]

This means that any function \( \Psi_k(r) \) that is a solution to the Schrödinger equation of the problem, differs only by a phase factor \( \exp (i k \cdot R) \) between equivalent positions in the lattice. Bloch’s theorem has many more forms and does not only apply to electrons in a periodic potential. The periodic function, \( u_k(r) \), can be considered a correction factor, used to generate solutions for periodic potentials from the simple solutions for constant potentials. This implies that \( u_k(r) \) for \( k \) vectors not close to a Brillouin zone will only represent a minor correction, i.e. \( u_k(r) \) should be close to one.
One more difference to the constant potential case is crucial. If we know the wavefunction for one particular \( k \) value, we also know the wavefunctions for infinitely many other \( k \) values too. This follows from yet another formulation of Bloch’s theorem. If \( \Psi_k(r) = u_k(r) \exp(i k \cdot r) \) is a particular Bloch wave solving the Schrödinger of the problem then the following function is also a solution,

\[
\Psi_{k+g}(r) = u_{k+g}(r) \exp(i [k + g] \cdot r) .
\] (2.17)

If \( \Psi_k(r) \) is a solution of the Schrödinger equation for the system, it will always be associated with a specific energy \( E(k) \), which is a constant of the system for the particular set of quantum numbers embodied by \( k \). Since \( \Psi_k(r) \) is identical to \( \Psi_{k+g}(r) \), the specific energy \( E(k + g) \) must be identical to \( E(k) \), or,

\[
E(k + g) = E(k) .
\] (2.18)

This equation does not mean that two electrons with wave vectors \( k \) and \( k + g \) have the same energy, but that any reciprocal lattice point can serve as the origin of the \( E(k) \) function. We can visualise this for the case of an infinitesimally small periodic potential. The \( E(k) \) function is practically the same as in the case of free electrons (Figure 2.5) but starting at every point in reciprocal space.

**Figure 2.5:** Dispersion relations illustrating the influence of Bloch’s theorem. The periodic nature of the crystal means \( E(k + g) = E(k) \), therefore any point in reciprocal space can serve as the origin of the dispersion function.
The diagram shows \( E(k + g) = E(k) \) for dispersion curves with different origins. This gives many energy values for one given \( k \), and due to the periodic nature it can be seen that all possible energy values are contained within the first Brillouin zone (between \(-\frac{1}{2}g_1\) and \(+\frac{1}{2}g_1\) in Figure 2.5). It is therefore sufficient to consider only the 1st Brillouin zone, known as a reduced representation of the band structure. The identical construction, but now for energy functions of a periodic potential as given before is depicted in Figure 2.6(b).

![Figure 2.6: Band structure diagrams for the two dispersion relations of Figure 2.4. Branches outside of the first Brillouin zone have been folded back, i.e. translated by the appropriate lattice vector \( g \). The left symmetric branch is omitted, and in place, the band diagram for a different direction in reciprocal space is shown. The second diagram illustrates an identical construction for a periodic potential. A band gap that exists in both directions (hatched region) is referred to as an absolute or full band gap.](image)

We now have band gaps, regions of unattainable energies, in all directions of the reciprocal lattice. An electron in lowest branch of the diagram with wave vector \( k_1 \) has a definite energy associated with it, however it could also have larger energy values obtained for the same \( k \) but in higher branches of the band diagram. For a translation to the next higher branch an energy \( \Delta E \) is needed.

Periodic potentials such as the Kronig-Penny model can be substituted in and solved for, however, this is as far as we need go with the discussion of electronic band gaps. We have introduced the concept of band gaps manifesting from diffraction effects at the edge of the first Brillouin zone, and furthermore we have shown how the irreducible representation of the band structure is formulated. Both these concepts are essential for understanding the results presented in this thesis. In the next section we look at analogous band gap formation for electromagnetic and vibrational waves propagating in periodic media.
2.2 Photonic and Phononic Crystal

Energy gaps that arise in the band structure of crystalline solids is the basis of all solid state physics. All the properties of electrons in a periodic potential, such as their velocity, effective mass, the density of states and crystal momentum, can be calculated from the band structure and the associated wavefunctions. As aforementioned however, the theory of Bloch waves can be applied to any wave like phenomena in a periodic media. We now move our discussion on to the observations of analogous band gap behaviour for periodically modulated dielectric media, known as photonic crystals, and periodic elastic structures known as phononic media. These support the propagation of electromagnetic and mechanical waves respectively, however be it an ordinary crystal, a dielectric composite or elastic composite, essentially the same Bragg diffractions occur.

2.2.1 Photonic Crystals

Electromagnetic wave propagation through periodic structures was first discussed by Lord Raleigh who studied a multi-layer dielectric stack known as a Bragg mirror in 1887 [4]. A century later the term photonic crystal was first coined by E.Yablonovich [5] and S.John [6] in 1987. Calculations of the photonic band structure were reported in the early 1990’s by several parties for face-centered cubic crystals [7][8][9][10][11]. It was found that such a crystal structure does not possess a complete band gap, in which no Eigenfrequency is permitted for any value of $k$ and for any polarization of the wave [12][13]. However, full photonic band gaps were found to exist for more complex unit cells, such as the diamond structure [7][10][11]. Experimental investigations into macroscopic, periodic constructions of two transparent dielectrics and the corresponding ‘optical’ or ‘photonic’ band structures, were pioneered by E.Yablonovitch. The first reported experimental sample was machined from low-loss dielectric materials containing $\sim 8000$ ‘atoms’, with a lattice parameter of the order $a \sim 13\text{mm}$ [7][14]. This corresponds to wavelengths in the microwave regime, $\omega/2\pi \sim 15\text{GHz}$. Further reports have since scaled down these photonic crystals utilizing the technique of reactive ion etching [15]. There is a motivation to reduce the scale of photonic crystals, for $a \leq 1\mu\text{m}$ would correspond to the optical regime. This offers the potential for several diverse applications associated with semiconductor technology [16]. Periodic arrays of dielectric cylinders in a background medium have also been studied by M.Plihal et al. The two-dimensional band structure, was determined both theoretically [17] and experimentally [18][19][20][21]. For an introduction to the field of photonic crystals, see the review paper by J.D.Joannopoulos et al. [22] or the first book to be published regarding the field by the same authors [23].
2.2.2 Phononic Crystals

Some discussions of elastic wave propagation in periodic structures first appeared in the late 1980’s [24][25], prior to the first band structure calculations of a phononic crystal being reported by M.Kushwaha et al. in 1993 [26]. The Table (2.1) was presented by the authors to compare and contrast the band structure properties of electronic, photonic and phononic crystals.

<table>
<thead>
<tr>
<th>Property</th>
<th>Electronic Crystal</th>
<th>Photonic Crystal</th>
<th>Phononic Crystal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Materials</td>
<td>Crystalline (natural or grown)</td>
<td>Constructed of two dielectric materials</td>
<td>Constructed of two elastic materials</td>
</tr>
<tr>
<td>Parameters</td>
<td>Universal constants, atomic numbers</td>
<td>Dielectric constants of constituents</td>
<td>Mass densities, sound speeds $c_1, c_2$ of constituents</td>
</tr>
<tr>
<td>Lattice constant</td>
<td>1-5Å (microscopic)</td>
<td>0.1µm-1cm (mesoscopic or macroscopic)</td>
<td>Mesoscopic or macroscopic</td>
</tr>
<tr>
<td>Waves</td>
<td>de Broglie (electrons) Ψ</td>
<td>Electromagnetic or light (photons) $E, B$</td>
<td>Vibrational or sound (phonons) $u$</td>
</tr>
<tr>
<td>Polarization</td>
<td>Spin ↑, ↓</td>
<td>Transverse</td>
<td>Coupled trans.-longit. $(\nabla \cdot u \neq 0, \nabla \times u \neq 0)$</td>
</tr>
<tr>
<td>Differential equation</td>
<td>$-\hbar^2 \nabla^2 \Psi + V(r)\Psi = i\hbar \frac{\partial \Psi}{\partial t}$</td>
<td>$\nabla^2 E - \nabla(\nabla \cdot E) = \frac{\epsilon(r)}{c^2} \frac{\partial^2 E}{\partial t^2}$</td>
<td>See Below</td>
</tr>
<tr>
<td>Free particle limit</td>
<td>$W = \frac{1}{2} m k^2$ (electrons)</td>
<td>$\omega = \frac{k}{c} \sqrt{\epsilon}$ (photons) $\omega = \frac{k}{c} \sqrt{\epsilon}$ (phonons)</td>
<td>$\omega = c_1, c_2 k$ (phonons)</td>
</tr>
<tr>
<td>Band gap</td>
<td>Increases with crystal potential, no electron states</td>
<td>Increases with $</td>
<td>\epsilon_a - \epsilon_b</td>
</tr>
<tr>
<td>Spectral region</td>
<td>Radio wave, microwave, optical, x-ray</td>
<td>Microwave, optical</td>
<td></td>
</tr>
</tbody>
</table>

The electronic band structure is found by solving the governing differential equation (Schrödinger equation) for a periodic potential with Bloch form. Therefore, to calculate the phononic band structure we introduce the governing wave equation and apply Bloch’s theorem. Solving for values of $k$ reveals the band structure $|\omega(k)|$.

The differential wave equation has not been defined in Table (2.1) as it can take two forms in a phononic crystal, the elastic or acoustic form. The acoustic wave equation is more important to the scope of this thesis and a derivation is presented in Chapter 3.1.5 to discuss its origins and the significance of its parameters. For now however, it suffices to say that a wave equation governs the propagation, and we seek its Eigenfrequency solutions as a function of wave vector $k$ to plot the dispersion relation (band structure).

The differential wave equation can take two forms in phononic crystals as mechanical waves propagate differently in solid and fluid materials. Therefore it is convenient to consider solid-solid, fluid-fluid and solid-fluid phononic crystals separately. In the case of solid-solid [27][28][29][30][31], mechanical waves propagate throughout the periodic
structure as elastic waves which are vector waves. In fluid-fluid phononic crystals, mechanical waves propagate throughout the structure as acoustic waves which are scalar waves. On the other hand, in the case of solid-fluid phononic crystals, the mechanical waves propagate as elastic waves in the solid material and as acoustic waves in the fluid material. As a side, it is noted that in several investigations, calculations of the solid-fluid band structure is performed, using a fluid-fluid approximation where the solid inclusion is modelled as a fluid with extremely high density and acoustic impedance. This simplifies the calculation, as it is simpler to match fluid-fluid boundaries than couple the longitudinal and transverse components at a solid-fluid boundary.

There are two distinct types of solid-fluid phononic crystal, either solid inclusions embedded in a fluid background or vice versa. A report by P.Langlet et al. [32] detailed a phononic crystal made from vacuum cylinders in a solid background. The waves are confined to propagate only through the background because vacuum does not support the propagation of mechanical waves [33][34]. This means the calculation of the dispersion relation is simplified, as in a solid-vacuum phononic crystal the waves must propagate as elastic waves. The elastic wave equation for an inhomogeneous media is,

\[
\nabla \cdot \left( \rho c_T^2 \nabla u_i + \rho c_L^2 \frac{\partial u}{\partial x_i} \right) + \frac{\partial}{\partial x_i} \left( (\rho c_T^2 - 2\rho c_L^2) \nabla \cdot u \right) = -\rho (\omega(k))^2 u_i, \quad (2.19)
\]

where \( \rho = \rho(r) \), \( c_T = c_T(r) \) and \( c_L = c_L(r) \) are the density and the transverse and longitudinal velocities respectively, describing the mechanical properties of the crystal and, \( u = u_k(r) \) is the spatial part of the displacement vector of the elastic Bloch wave propagating with wave vector \( k \) within the crystal. For a two-dimensional phononic crystal Equation (2.19) reduces to,

\[
\frac{\partial}{\partial x} \left( \rho c_L^2 \frac{\partial u_x}{\partial x} + (\rho c_L^2 - 2\rho c_T^2) \frac{\partial u_y}{\partial y} \right) + ... \\
\frac{\partial}{\partial y} \left( \rho c_L^2 \frac{\partial u_y}{\partial y} + (\rho c_L^2 - 2\rho c_T^2) \frac{\partial u_x}{\partial x} \right) = -\rho (\omega(k))^2 u_x \quad (2.20)
\]

\[
\frac{\partial}{\partial x} \left( \rho c_T^2 \frac{\partial u_x}{\partial x} + \rho c_T^2 \frac{\partial u_y}{\partial y} \right) + \frac{\partial}{\partial y} \left( \rho c_T^2 \frac{\partial u_y}{\partial y} + \rho c_T^2 \frac{\partial u_x}{\partial x} \right) = -\rho (\omega(k))^2 u_y \quad (2.21)
\]

\[
\frac{\partial}{\partial x} \left( \rho c_T^2 \frac{\partial u_z}{\partial x} \right) + \frac{\partial}{\partial y} \left( \rho c_T^2 \frac{\partial u_z}{\partial y} \right) = -\rho (\omega(k))^2 u_z, \quad (2.22)
\]

where \( \rho = \rho(x,y) \), \( c_T = c_T(x,y) \) and \( c_L = c_L(x,y) \) are the mechanical properties of the two-dimensional phononic crystal, and \( u_x = u_x(x,y) \), \( u_y = u_y(x,y) \) and \( u_z = u_z(x,y) \) are the Cartesian coordinates of the spatial displacement vector \( u = u_k(x,y) \). Langlet et al. noted that the wave equations indicate that elastic waves propagating within a two-dimensional phononic crystal can be separated into two distinct forms. Equation
(2.22) describes the propagation of an independent transverse elastic wave, given by the components $u_z$ of the displacement vector (perpendicular to the wave vector, $k$, in the $xy$ plane). Independent transverse waves of this form are called out-of-plane waves. On the other hand, Equations (2.20 and 2.21) describe the propagation of a mixed elastic wave, which is neither transverse nor longitudinal. Because the displacement vector is in the plane of the periodicity $xy$, these mixed elastic waves are called in plane. In order to plot the band structure it was therefore necessary for Langlet et al. to solve Equations (2.22) and (2.20, 2.21) separately to obtain two band structures for out-of-plane and in-plane waves respectively.

The other class of solid-fluid phononic crystal is comprised of solid cylinders in a fluid background. The samples studied in this thesis, are examples of this class. In these phononic crystals, mechanical waves propagate as acoustic (or pressure) waves, that are confined to the continuous fluid medium, due to the rigidity of the cylinders. Acoustic waves travel through these solid-fluid phononic crystals as Bloch waves, of the form,

$$p_k(r, t) = \text{Re} \left[f_k(r)e^{i(k\cdot r - \omega(kt))}\right], \quad (2.23)$$

where $f_k(r)$ is the periodic scalar function with the same periodicity as the underlying phononic crystal, that depends on the particular value of $k$ (equivalent to $u_k(r)$ from Section (2.1.3)). The propagation of these acoustic Bloch waves within the phononic crystal is governed by the acoustic wave equation, which in two dimensions is,

$$\frac{\partial}{\partial x} \left( \frac{1}{\rho} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\rho} \frac{\partial p}{\partial y} \right) = -\frac{1}{\rho c_L^2} (\omega(k))^2 p, \quad (2.24)$$

where $p$ is the spatial part of pressure $p_k(x, y)$. The wave equation can then be solved numerically to obtain the dispersion relation, $[\omega(k)]$. For a phononic crystal comprised of solid cylinders, only one band structure calculation need be performed for the longitudinal wave mode. This is simply because in the fluid medium to which the wave is confined, the acoustic waves do not support a transverse component.

The different classes of phononic crystal have only been briefly outlined here. A full derivation of the acoustic wave equation from fundamental acoustic concepts is presented in the next chapter, with a more detailed discussion of the acoustic dispersion relation being possible after some important concepts from acoustics have been introduced.
2.3 Influence of Parameters

The characteristic material and geometrical parameters of a phononic crystal have a large influence upon whether a sample will exhibit a spectral gap. Before we continue, it is worth outlining these dependencies. Kushwaha reported that the formation and width of band gaps was strongly dependent upon the density contrast as well as differences of the sound velocities and elastic constants of the host and scatter materials [35][36][37][38]. The most favourable situation is to have a large ratio between the velocity in the host material $c_{ot}$ to the velocity in the inclusions $c_{it}$. It was found that there is a critical value, $r_c$, of the ratio $r = c_{ot}/c_{it}$ which is needed in order to open a band gap. The size of the band gap increasing as a function of $r$ until a saturation point as $r \gg r_c$. In addition as the value of $r$ increases the propagating waves are attenuated more severely.

For solid-solid phononic crystals, the ratio between the longitudinal and transverse velocities is also an important quantity. It is defined in both the host material, $r_o = c_{ot}/c_{ot}$, and the scattering inclusions, $r_i = c_{it}/c_{it}$, where $c_{ot}$ and $c_{it}$ are the longitudinal velocities. Smaller values of both $r_o$ and $r_i$ favour the opening of band gaps; however no material is known with a ratio $c_{ot}/c_{it}$ smaller than $\sqrt{2}$ [39]. The density ratio $\rho_o/\rho_i$ was shown by E.N.Economou to be another material parameter that strongly influences band gap formation [40][41]. If both the host and inclusions are fluids then a large density ratio is favourable to form gaps. This is intuitive as a large ratio of $\rho_o/\rho_i$ further increases the impedance mismatch $\rho_o c_{ot}/\rho_i c_{it}$ from the value $c_{ot}/c_{it}$. Conversely, for solid inclusions embedded in a solid host, it was found that reducing the ratio favoured the appearance of gaps.

Several authors have reported on the important role of the packing fraction, upon the bandwidth of the spectral gap. A theoretical investigation of different 3D phononic crystal structures made of steel spheres in a polymer matrix found that the bandwidth of a gap increased with the filling fraction of the corresponding crystal lattice [42]. The authors also discussed the role of the resonant elastic modes of the individual spheres in the formation of large elastic band gaps. They demonstrated in their 3D sample, large complete band gaps open as a result of hybridization between narrow bands due to weak coupling between rigid body resonance modes of single spheres and the continuum bands corresponding to propagation in an effective homogeneous medium.

The symmetry of a phononic crystal is a further parameter that influences the band gap formation. It was reported by D.Caballero et al. [43] that an enlargement of the acoustic band gap can be achieved by reducing the symmetry of the structure, after a similar phenomena was observed for photonic crystals. The full photonic band gap can be enlarged by decreasing the crystal symmetry through the introduction of a two point
basis set. Photonic crystals based on a face-centered-cubic structure do not possess a full band gap, but samples having a diamond structure do because the additional point basis lifts the degeneracy of some bands. Several techniques have been reported to reduce the symmetry of photonic crystals. In two-dimensions such examples include, insetting a different sized rod at the centre of an existing lattice unit cell, lowering the symmetry of the scatterer, using different lattices and deforming the standard lattice. Despite of this it was shown by R.Wang et al. that for a given lattice symmetry, the largest band gaps are obtained when the geometric symmetry of each scatterer is the same as that of the lattice (or its first Brillouin zone). Caballero reported on the band gap behaviour, of phononic crystals with hexagonal lattice symmetry, and investigated the influence of placing a cylinder of increasingly large radius at the centre of a hexagonal cell. The hexagonal lattice is a reduced symmetry of the triangular lattice and it was shown that they always possess a larger complete acoustic band gap than triangular, assuming a constant packing fraction. As the radius of the cylinder increases, the width of the band gap was shown to decrease. This is a counterintuitive result as the packing fraction is being raised. The behaviour arises due to the enhancement of the symmetry as the radius increases and the honeycomb lattice transforms to a triangular one.

C.Goffaux et al. fabricated another form of tunable phononic crystal, realised with an array of rods of square cross-section, which were allowed to rotate about their axes, and this idea was further studied by R.Min et al. The width of the band gap was found to change with the angle with which the rods were rotated. Rotation angles of 35° and 45° increase the packing fraction of the array from 0.4 to 0.5, and the band width was observed to double. The effect of scatterer shape and lattice symmetry on the formation of band gaps was studied independently by both W.Kuang et al. and also by I.Sliwa et al., O.Sigmund et al. and E.A.Rietman et al.

Kuang considered three lattice configurations (triangular, hexagonal and square) and four different scatterer shapes (hexagonal, circular, square and triangular). For a given lattice it was found that the band gap width is maximised when the shape and orientation of the scatterers corresponds to that of the lattice. Also for a given scatterer shape, the width is maximised when the lattice has the largest coordination number, as the crystal symmetries in this case were not reduced by the scatterers.

There is a wealth of published literature detailing the influence of a phononic crystals parameters on band gap formation, with the majority of the early investigations focusing on the material and geometric dependencies. We take into consideration the conclusions reported when designing the phononic crystal for the experimental study in this thesis.
Chapter 3

Acoustics and the Finite Element Technique

This chapter begins with an introduction of important physical and mathematical concepts in acoustics. It presents an approach to finite element techniques for linear time-harmonic acoustics starting from the fundamental axioms of continuum mechanics. Based on these principles, the wave equation is derived. Using a time-harmonic approximation, the Boundary Value Problem of linear time-harmonic acoustics is formulated in the classic (partial differential Helmholtz equation and its boundary conditions) and weak form. Subsequently, the weak form is used as the basis for the discretization process resulting in a Galerkin finite element formulation. The process of the finite element technique is briefly outlined in Section (3.3), but the formulation presented in Section (3.4.3.2) details the specific application of the method to acoustic problems from Ihlenburg’s text [58]. For an introduction to the subject and more particulars of the steps involved several texts are available [59] [60].

The second section of the chapter details the specific finite element software used to generate theoretical comparisons for the experimentally measured results. Two types of analysis are discussed, namely the time-harmonic, which is used to obtain frequency spectra, and the Eigenvalue analysis, used to calculate band structures. In addition subdomain equations and boundary conditions are defined.
3.1 Deriving the Wave Equation

The acoustic wave equation is developed from the fundamental axioms of continuum mechanics. This branch of mechanics concerns the analysis of kinematics and the mechanical behaviour of material modelled as a continuum. To define a continuum, it is assumed that the substance of the object completely fills the space it occupies. This ignores the fact that matter is made from atoms, and is not continuous, therefore models on an inter atomic scale are redundant. At larger length scales however the models are highly accurate. In the derivation, we use the Eulerian representation and the corresponding Eulerian, or spatial, coordinates. This coordinate system is defined for computational convenience as continuum mechanics deals with physical properties of solids and fluids that are independent of any particular coordinate system in which they are observed.

3.1.1 Conservation of Mass

The principle of conservation of mass means that the total mass $M$ of the domain $\Omega$ remains constant during the motion,

$$M(t) = \int_\Omega \rho(x, t)d\Omega,$$  \hspace{1cm} (3.1)

where $x$ and $t$ denote the position vector and time respectively. The principle of conservation of mass implies that the material derivative (or total time derivative) vanishes, i.e.

$$\dot{M} = \frac{dM}{dt} = \int_\Omega \left( \frac{\partial \rho}{\partial t} + \rho \nabla \cdot v \right) d\Omega = 0.$$  \hspace{1cm} (3.2)

The material derivative introduces the flow velocity vector $v$ which results from $\partial x/\partial t$. In addition to the conservation of mass holding globally, it must also be valid for an arbitrary small area of each point of the material which implies the local conservation of mass,

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot v = 0.$$  \hspace{1cm} (3.3)
3.1.2 Balance of Momentum

The principle of balance of momentum means that the time rate of change of momentum is equal to the resultant force $F_R$ acting on the body. With the linear momentum vector $P$, this is defined as,

$$\dot{P} = \frac{dP}{dt} = F_R.$$  \hfill (3.4)

The momentum vector is given by,

$$P = \int_\Omega \rho v d\Omega,$$  \hfill (3.5)

whereas the resultant force combines volume forces and external forces as,

$$F_R = \int_\Omega b \rho d\Omega - \int_\Gamma p n d\Gamma.$$  \hfill (3.6)

The first term on the right hand side is known as the resultant external body force with the external body $b$. This term allows gravity effects to be considered; however, in acoustics the term is irrelevant and is consequently set to zero. The second term represents the resultant contact force which can be transformed into a domain integral using Gauss’ theorem,

$$\int_\Gamma p n d\Gamma = \int_\Omega \nabla p d\Omega.$$  \hfill (3.7)

The material derivative of momentum is given as,

$$\frac{dP}{dt} = \frac{d}{dt} \left( \int_\Omega \rho v d\Omega \right) = \int_\Omega \frac{d(\rho v)}{dt} d\Omega = \int_\Omega \left[ \frac{\partial \rho}{\partial t} v + \rho (\nabla \cdot v) v + \rho \frac{\partial v}{\partial t} + \rho (v \cdot \nabla) v \right] d\Omega.$$  \hfill (3.8)

The first two terms of this integrand disappear with respect to the conservation of mass in Equation (3.2) and (3.3). Therefore Equation (3.8) reduces to,

$$\frac{dP}{dt} = \int_\Omega \left[ \rho \frac{\partial v}{\partial t} + \rho (v \cdot \nabla) v \right] d\Omega.$$  \hfill (3.9)

Incorporating Equations (3.6), (3.7) and (3.9) into Equation (3.4), we obtain the Euler equation,

$$\int_\Omega \left[ \frac{\partial v}{\partial t} + \rho (v \cdot \nabla) v + \nabla p \right] d\Omega = 0.$$  \hfill (3.10)

Or in local form,

$$\rho \frac{\partial v}{\partial t} + \rho (v \cdot \nabla) v + \nabla p = 0.$$  \hfill (3.11)

In continuum mechanics, Euler’s equations of motion comprise the balance of momentum and the balance of balance of momentum, also known as the angular momentum. The latter axiom can be neglected as shear effects are not considered herein, since shear forces are neglected in fluids.
3.1.3 Linearization and Simplification

Problems of linear acoustics commonly refer to small perturbations of ambient quantities, referred to by using the subscript 0. The small fluctuating parts of pressure, density and flow velocity are represented as $\tilde{p}$, $\tilde{\rho}$ and $\tilde{v}$. With this notation we substitute the quantities of pressure, density and flow velocity as,

\begin{align*}
p &= p_0 + \tilde{p} \quad \text{(3.12)} \\
\rho &= \rho_0 + \tilde{\rho} \quad \text{(3.13)} \\
v &= v_0 + \tilde{v}. \quad \text{(3.14)}
\end{align*}

For simplicity it is assumed that there is no ambient flow, i.e. $v_0 = 0$. Substituting for the quantities in Equation (3.3) and considering only the first two terms, we obtain,

\[ \frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \tilde{v} = 0. \quad \text{(3.15)} \]

Similarly Euler’s Equation (3.11) is linearised and simplified as,

\[ \rho_0 \frac{\partial \tilde{v}}{\partial t} + \nabla \tilde{p} = 0, \quad \text{(3.16)} \]

where it is assumed that $\rho_0$ and $p_0$ are independent of time and spatial coordinates.

3.1.4 Constitutive Equation

Sound is a physical phenomenon identified with the propagation of a mechanical disturbance through a medium. The speed of sound is defined as the distance traversed per unit time by a given point on the disturbance, provided the disturbance does not change its shape (as would be the case in an ideal medium). If the disturbance is resolved into spatially harmonic components, an observed point on any one component may be designated by a phase angle, and if directional information is included, then the distance traversed per unit time is called the phase velocity. If the phase velocity is the same at all frequencies, the medium is said to be nondispersive, otherwise it is dispersive. The phase velocity appears in the wave equation, which governs the propagation of sound through the medium. The wave equation is derived shortly in Section (3.1.5) from an equation of motion, a constitutive equation or equation of state and a continuity equation. In an unbounded homogeneous domain the speed of sound depends on the adiabatic bulk modulus and the density of the undisturbed medium. The speed of sound is sensitive to properties such as temperature, pressure, composition and absorption.
The constitutive relations are usually referred to as the equations of state. With respect to thermodynamics, the pressure fluctuation and, thus, sound propagation occurs with negligible heat flow because the changes of state occur so rapidly that there is no time for the temperature to equalise with the surrounding medium. This is the property of an adiabatic process.

The speed of sound is defined as a constant to relate the fluctuating parts of pressure and density to each other as,

\[ \tilde{p} = c^2 \rho. \]  

(3.17)

This is equivalent to,

\[ c = \sqrt{\frac{\partial p}{\partial \rho}}. \]  

(3.18)

The derivation of the speed of sound is different depending on the medium. For gases, we will use relation (3.17) whereas for liquids, we will derive the speed of sound based on Equation (3.18).

### 3.1.4.1 Gases

Consider an ideal gas with the specific heat ratio \( \kappa \). An adiabatic process implies the relation \( p\rho^{-\kappa} \) =constant. Since this relation is valid at any time, it implies,

\[ (p_0 + \tilde{p})(\rho_0 + \tilde{\rho})^{-\kappa} = p_0\rho_0^{-\kappa}, \]  

(3.19)

which can be rewritten as,

\[ 1 + \frac{\tilde{p}}{p_0} = \left(1 + \frac{\tilde{\rho}}{\rho_0}\right)^\kappa. \]  

(3.20)

The right hand side is linearised by,

\[ \left(1 + \frac{\tilde{\rho}}{\rho_0}\right)^\kappa = 1 + \kappa \frac{\tilde{\rho}}{\rho_0}, \]  

(3.21)

which simplifies Equation (3.20), yielding,

\[ \tilde{p} = \left(\kappa \frac{p_0}{\rho_0}\right) \tilde{\rho} = c^2 \tilde{\rho}, \]  

(3.22)

where the speed of sound is denoted by \( c \). The variable \( K \) denoting the adiabatic bulk modulus is introduced as,

\[ c = \sqrt{\frac{K}{\rho_0}} = \sqrt{\frac{\kappa p_0}{\rho_0}}. \]  

(3.23)
3.1.4.2 Liquids

The adiabatic bulk modulus $K$ for liquids is defined as,

$$K = -V \frac{\partial p}{\partial V}. \quad (3.24)$$

Mass conservation can be formulated as $\rho V = m = \text{constant}$. Consequently, we can write,

$$V = \frac{m}{\rho} \quad \text{and} \quad \frac{\partial V}{\partial \rho} = -\frac{m}{\rho^2}. \quad (3.25)$$

Returning the adiabatic bulk modulus, using the chain rule and substituting for volume yields,

$$K = -\frac{m}{\rho} \frac{\partial p}{\partial \rho} \left( \frac{-\rho^2}{m} \right) = \rho \frac{\partial p}{\partial \rho}, \quad (3.26)$$

which allows the speed of sound to be written as,

$$c = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{\frac{K}{\rho_0}}. \quad (3.27)$$

The adiabatic bulk modulus is sometimes replaced by its reciprocal which is called adiabatic compressibility.

3.1.5 Derivation of the Wave Equation

Herein the problem is reduced to one variable, this is the pressure fluctuation, which will subsequently be called the sound pressure. The local conservation of mass (3.3) in its linearised form (3.15), the Euler equation as the balance of momentum (3.11) in its linearised form (3.16) and the constitutive relation of Equation (3.17) are all summarised into one partial differential equation, the wave equation. We start with the constitutive relation (3.17), which is differentiated twice with respect to time,

$$\frac{\partial^2 \tilde{p}}{\partial t^2} = c^2 \frac{\partial^2 \tilde{\rho}}{\partial t^2}. \quad (3.28)$$

Then, derivatives of the density fluctuations are replaced by the local conservation of mass in linearised form (3.15) which gives,

$$\frac{\partial^2 \tilde{p}}{\partial t^2} = -c^2 \rho_0 \frac{\partial (\nabla \cdot \tilde{v})}{\partial t} = -c^2 \rho_0 \nabla \cdot \left( \frac{\partial \tilde{v}}{\partial t} \right). \quad (3.29)$$

Finally, the linearised Euler equation (3.16) is used to substitute for the velocity vector as,

$$\frac{\partial^2 \tilde{p}}{\partial t^2} = c^2 \nabla \cdot \nabla p. \quad (3.30)$$
This is known as the wave equation. In two-dimensions, if we assume that the speed \( c \) is a constant and is not dependent on frequency (dispersionless), then the most general solution is,

\[
p = f(ct - x) + g(ct + x),
\]

(3.31)

where \( f \) and \( g \) are any two twice-differentiable functions. This may be pictured as the superposition of two waveforms of arbitrary profile, one \( f \) travelling up the \( x \)-axis and the other \( g \) down the \( x \)-axis at the speed \( c \). The particular case of a sinusoidal wave travelling in one direction is obtained by choosing either \( f \) or \( g \) to be a sinusoid, and the other to be zero, giving,

\[
p = p_0 \sin (\omega t \pm kx),
\]

(3.32)

where \( \omega \) is the angular frequency of the wave and \( k \) is its wave number. The speed \( c \) included in the acoustic wave equation represents the phase velocity. If \( \omega \) is the number of radians per wave that passes a given location per unit time and \( 1/k \) is the spatial length of the wave per radian, it follows that \( \omega/k = v \) is the speed at which the wave is moving, or the speed at which any fixed phase of the angle is displaced. Therefore, this is called the phase velocity \( v_p \).

### 3.2 Acoustic Dispersion Relation

Before continuing to formulate the boundary value problem and discuss the finite element technique, we shall briefly consider the topic of acoustic dispersion relations. It was discussed in the first chapter that an acoustic dispersion relation is analogous to an electronic band structure, and four classes of phononic crystal were introduced. This thesis is only concerned with rigid cylinder phononic crystal (RCPC’s) embedded in an air medium, and consequentially only with the acoustic wave equation. Now, armed with this equation (3.30), to describe propagation of a sound waves in the fluid, it is possible to find the acoustic dispersion relation and discuss physically what it represents. An overview of the numerical techniques developed to calculate the acoustic band structure is presented shortly, but we now detail what physically happens to the acoustic waves velocity as it passes through the phononic crystal, and how this manifests in the dispersion relation. Until now we have made the assumption that the speed of sound has not been frequency dependent. This is acceptable to describe sound wave propagation through an infinite air expanse, however the finite structure presented by the phononic crystal, is a dispersive medium in which the speed of propagation is frequency dependent. The acoustic dispersion relation represents the separation of a complex sound wave into its frequency components, measured by the rate of change of velocity with frequency.
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Equations (3.23) and (3.27) define the phase velocity of a monochromatic wave. However we require the change of velocity with frequency introducing multiple close frequencies of similar amplitude. Considering the behaviour of such a group leads to a new definition of velocity, known as the group velocity, \( v_g \). If we begin by considering two solutions to the wave equation (3.30) with frequencies \( \omega_1 \) and \( \omega_2 \) that differ by a small amount,

\[
y_1 = a \cos (\omega_1 t - k_1 x) \\
y_2 = a \cos (\omega_2 t - k_2 x).
\]

Superposition of the amplitude and phase gives,

\[
y = y_1 + y_2 = 2a \cos \left[ \frac{(\omega_1 - \omega_2)}{2} t - \frac{(k_1 - k_2)}{2} x \right] \cos \left[ \frac{(\omega_1 + \omega_2)}{2} t - \frac{(k_1 + k_2)}{2} x \right].
\]

This wave system has frequency \( (\omega_1 + \omega_2)/2 \) modulated in time by a very slowly varying envelope of frequency \( (\omega_1 - \omega_2)/2 \) and wave number \( (k_1 - k_2)/2 \). The velocity of this new wave is \( (\omega_1 - \omega_2)/(k_1 - k_2) \) which, if the phase velocities are equal \( \omega_1/k_1 = \omega_2/k_2 \), gives,

\[
\frac{\omega_1 - \omega_2}{k_1 - k_2} = c \frac{(k_1 - k_2)}{(k_1 - k_2)} = c.
\]

In such a case the component frequencies and their superposition, or group, will travel with the same velocity. Suppose now that the two frequency components have different phase velocities so that \( \omega_1/k_1 \neq \omega_2/k_2 \). The velocity of the maximum amplitude of the group is now different from each of the individual velocities and the superposition of

\[
\text{Figure 3.1: Schematic of the phase and group velocity. When } v_p = v_g \text{ the component waves and their envelope move as one. This is an example of a non-dispersive wave. For } v_p = -v_g \text{ the envelope moves in the opposite direction of the component waves. If } v_p > v_g \text{ the component waves move more quickly than the envelope, and conversely, if } v_p < v_g \text{ the component waves move more slowly than the envelope.}
\]
the two waves is no longer constant as the group profile will change with time. This is known as the group velocity, $v_g$, found through,

$$\frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta \omega}{\Delta k}. \quad (3.37)$$

As aforementioned a medium in which the phase velocity is frequency dependent ($\omega/k$ not constant) is known as a dispersive medium and the dispersion relation expresses the variation of $\omega$ as a function of $k$. If a group contains a number of component frequencies which are nearly equal the original expression for the group velocity can be written,

$$\frac{\Delta \omega}{\Delta k} = \frac{d\omega}{dk}. \quad (3.38)$$

Since $\omega = kv_p$ the group velocity,

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk} (kv_p) = v_p + k \frac{dv_p}{dk} = v_p - \lambda \frac{dv_p}{d\lambda}, \quad (3.39)$$

where $k = 2\pi/\lambda$. Usually $dv_p/d\lambda$ is positive, so that $v_g < v_p$. This is known as normal dispersion, but anomalous dispersion can arise when $dv_p/d\lambda$ is negative, so that $v_g > v_p$. Chapter 4.3.2 details the experimental techniques performed to observe points of anomalous dispersion which occur at the band gap frequencies of a phononic crystal.

![Figure 3.2: Dispersion relations for a.) non-dispersive media, straight line with $v_p = v_g$, b.) a normal dispersion curve where the gradient $v = \omega/k > v_g = d\omega/dk$, and, c.) an anomalous dispersion relation where $v_p < v_g$.](image)
3.3 The Finite Element Technique

Most numerical methods in continuum mechanics are based on the principle of deriving a set of equations that describe the behaviour of a small part of the body. By dividing the body into elements and using appropriate compatibility and equilibrium equations to link elements we obtain a reasonably accurate prediction of variables such as stresses and displacements. As the size of elements decreases, accuracy increases but so does computation time.

The finite element method is a numerical analysis technique, widely used since the production of digital computers in the 1950’s. Several variations have been developed to apply the technique to a range of scientific disciplines. Although there may be a large diversity in the formulation of the problem, the finite element method can be categorised by the following process.

Initially, in the analysis of a system, the total region being studied is divided into smaller elements. Each element is assigned a node, positioned on the boundary connecting it to its neighbour. Approximating functions are next defined, in terms of the field variables at the specified nodal points. Typically, polynomial functions are chosen due to the ease with which they can be integrated and differentiated. The matrix equations are next defined, which express the properties in each element. There are four common approaches to define these properties, namely the direct, variational, weighted residual and energy balance approaches. Each has advantages depending on the specific problem being analysed, however all seek to minimise the error between the data so it accurately fits the approximating function.

In order to describe the behaviour of the system as a whole, the individual elements must next be united. The matrix equations are combined from all elements into an overall matrix equation expressing the total solution. The total matrix is of the same form as that of the individual contributing matrix elements, however it contains significantly more terms. It is possible to assemble the solution for each boundary node in such a manner, as the value of the field variable is the same for each element connected to the shared node, providing continuity. Appropriate boundary conditions are next applied to the system equations to represent the physical situation being described, finally allowing solutions to be obtained. To achieve this, the set of simultaneous equations obtained from the combination procedure, are solved for the unknown value of the field variable at each node. It is the manipulation of such large matrix systems that makes the finite element technique such a demanding computational process.
3.4 The Boundary Value Problem

3.4.1 Helmholtz Equation and Boundary Conditions

We consider linear acoustic problems defined in the domain $\Omega$ with the complement $\Omega_c$ and $\Gamma$ representing the closed boundary $\Omega$ and $\Omega_c$. The outward normal is pointing into the complementary domain $\Omega_c$ as shown in Figure 3.3. The wave equation has been formulated to be valid for the sound pressure $\tilde{p}$,

$$\Delta \tilde{p}(x, t) = \frac{1}{c^2} \frac{\partial^2 \tilde{p}(x, t)}{\partial t^2}, \quad x \in \Omega \subset \mathbb{R}^d. \quad (3.40)$$

To complete a solution, the differential equation requires boundary conditions and initial conditions, which will be specified shortly. For time-harmonic problems, the time dependence $\tilde{p}(x, t) = \tilde{p}(x)e^{-i\omega t}$. The time-harmonic sound pressure, or simply pressure $\tilde{p}$ is now represented by $\tilde{p}(x) = p(x)$. Applying the time-harmonic dependence to Equation (3.40) reduces the wave equation to the Helmholtz equation for sound pressure.

Helmholtz Equation;

$$\Delta p(x) + k^2 p(x) = 0, \quad x \in \Omega \subset \mathbb{R}^d. \quad (3.41)$$

We assume admittance boundary conditions being equivalent to Robin boundary conditions which may degenerate to Neumann boundary conditions if the admittance is zero.

Boundary Condition;

$$v_f(x) - v_s(x) = Y(x)p(x), \quad x \in \Omega \subset \mathbb{R}^{d-1}. \quad (3.42)$$
Here the wave-number \( k = \omega/c \) is the quotient of the circular frequency \( \omega = 2\pi f \) (\( f \) denoting frequency) and the speed of sound \( c \), \( Y \) represents the boundary admittance, and the normal fluid particle velocity \( v_f \) is related to the normal derivative of the sound pressure \( p \) by means of the Euler equation in frequency domain,

\[
v_f(x) = \frac{1}{a} \frac{\partial p(x)}{\partial n(x)} = \frac{1}{i\omega\rho_0} \frac{\partial p(x)}{\partial n(x)}.
\]  

(3.43)

In Equation (3.43), \( i \) is the imaginary unit and \( \rho_0 \) the average density of the fluid. The vector \( \mathbf{n}(x) \) represents the outward normal at the surface point \( x \) and \( \partial/\partial n(x) \) is the normal derivative. For time dependence \( e^{i\omega t} \) the constant \( a \) takes the conjugate value \( a = -i\omega\rho_0 \).

In some cases, it is useful to consider the Dirichlet boundary conditions. The Robin condition as formulated in Equation (3.42) is not suited for this case. Instead, we may use the Robin condition as an impedance boundary condition with the impedance \( Z(x) \) as,

\[
Z(x) [v_f(x) - v_s(x)] = p(x), \quad \text{and}, \quad Z(x) = \frac{1}{Y(x)}.
\]  

(3.44)

In case of a homogeneous Dirichlet boundary condition, the value of the impedance is zero and thus leading to \( p(x) = 0 \). Obviously, the inhomogeneous Dirichlet condition results in \( p(x) = p_0(x) \).

The boundary value problem for the time-harmonic case assumes a locally acting boundary admittance relating particle velocities of the fluid and the underlying structure and the sound pressure. For the several problems considered here we simplify the boundary conditions where either \( v_s = 0 \) or \( Y = 0 \).

In addition to fulfilling the Helmholtz equation and the boundary conditions, solutions of external problems require fulfillment of the decay condition at infinity, i.e. the Sommerfeld radiation condition. This is formulated in two steps for the sound pressure,

\[
p = O(r^{-\alpha}) \quad \text{and} \quad \frac{\partial p}{\partial r} - ikp = o(r^{-\alpha}) \quad \text{for} \quad r \to \infty,
\]  

(3.45)

with \( \alpha = (d - 1)/2 \) and \( r \) denoting the distance between an arbitrary point close to a source. Hence, the first expression of Equation (3.45) formulates the decay rate of the solution of the Helmholtz equation \( p \), whereas the second expression requires the left hand side to decay faster than \( r^{-(d-1)/2} \). Clearly, the Sommerfeld condition is a decay condition only for \( d > 1 \). Note that the minus sign on the left hand side of the second part of the Sommerfeld condition changes to a plus sign if the time dependence is chosen to be \( e^{i\omega t} \).
3.4.2 Weak Formulation

A weak formulation is based on introducing the weight function $\chi(x)$ and ‘testing’ it with the Helmholtz operator such that,

$$\int_{\Omega} \chi(x) \left[ \Delta p(x) + k^2 p(x) \right] d\Omega(x) = 0. \quad (3.46)$$

Integrating by parts gives,

$$\int_{\Omega} \chi(x) \left[ \Delta p(x) + k^2 p(x) \right] d\Omega(x) = \int_{\Gamma} \chi(x) a v_f(x) d\Gamma(x)$$

and then,

$$\int_{\Omega} \chi(x) \left[ \Delta p(x) + k^2 p(x) \right] d\Omega(x) = \int_{\Gamma} \chi(x) a v_f(x) d\Gamma(x)$$

$$- \int_{\Omega} \left[ \nabla \chi(x) \cdot \nabla p(x) - k^2 \chi(x) p(x) \right] d\Omega(x) = 0, \quad (3.47)$$

Often, Equation (3.47) represents the starting point for conventional finite element discretization, e.g. Galerkin method. The second part consists of a domain integral and boundary integral. Similarly, the second part of Equation (3.48) consists of one domain integral and two boundary integrals. This domain integral can be transformed into an integral free term using fundamental solutions $G(x, y)$ in the sense of distributions.

Function $G$ represents the solution of the equation,

$$\Delta G(x, y) + k^2 G(x, y) = -\delta(x - y). \quad (3.49)$$

It is known as the free-space Green’s function as well, whereas $\delta(x, y)$ is the Dirac or delta function at the origin $y$. In terms of physics, $G(x, y)$ can be understood as the sound pressure distribution according to a point source in $y$. Together with the harmonic time-dependence of $e^{-i\omega t}$, it represents an outgoing wave. We can write $G$ as,

$$G(x, y) = \frac{1}{2} \sin|kr(x, y)| \quad x \in \Omega \subset \mathbb{R}^1$$

$$G(x, y) = \frac{i}{4} H_0^1(kr(x, y)) \quad x \in \Omega \subset \mathbb{R}^2$$

$$G(x, y) = \frac{1}{4\pi} \frac{e^{ikr(x,y)}}{r(x, y)} \quad x \in \Omega \subset \mathbb{R}^3. \quad (3.50)$$

with $r$ as the Euclidean distance between field point $x$ and source point $y$ as $r(x, y) = |x - y|$. Note that the fundamental solutions are different when the time dependence is chosen to be $e^{i\omega t}$. 
Applying the property of the fundamental solution and the delta function, we find,

\[ \int_{\Omega} \rho(x) \left[ \Delta G(x, y) + k^2 G(x, y) \right] d\Omega(x) = \int_{\Omega} \rho(x) \left[ -\delta(x - y) \right] d\Omega(x) = -\rho(y). \tag{3.51} \]

Hence, Equation (3.48) is rewritten as,

\[ p(y) + \int_{\Gamma} \frac{\partial G(x, y)}{\partial n(x)} \rho(x) d\Gamma(x) = \int_{\Gamma} G(x, y) v_f(x) d\Gamma(x). \tag{3.52} \]

Equation (3.52) is known as representation formula. For \( y \in \Gamma \), it is known as the Kirchhoff-Helmholtz (boundary) integral equation. Note that plus and minus signs of either the first term or the second and the third term may be different if the direction of the normal vector is chosen in opposite direction.

Before entering the discretization process, it will be useful to incorporate the boundary condition (3.42) into the weak formulations (3.47) and (3.52). Furthermore, we substitute for the constant \( a \) as,

\[ a = sk, \quad \text{with} \quad s = i\rho_0 c, \tag{3.53} \]

which explicitly shows wave-number dependency. Recalling the second part of Equation (3.47), introducing the boundary condition and rearranging only unknown terms on the left hand side, i.e. terms including the sound pressure \( p \), we find,

\[ \int_{\Omega} \left[ \nabla \chi(x) \cdot \nabla p(x) - k^2 \chi(x) p(x) \right] d\Omega(x) - sk \int_{\Gamma} \chi(x) Y(x) p(x) d\Gamma(x) = sk \int_{\Gamma} \chi(x) v_s(x) d\Gamma(x). \tag{3.54} \]

We will use Equation (3.55) as the basis for Galerkin discretization using finite elements.
3.4.3 Discretization Process

3.4.3.1 Approximation

Independent of the discretization method, we formulate approximations of our physical quantities. First of all, we approximate the sound pressure $p(x)$ as,

$$p(x) = \sum_{l=1}^{N} \phi_l(x) p_l = \phi^T(x) p,$$

where $p_l$ represents the discrete sound pressure at point $x_l$ and $\phi_l$ is the $l$-th basis function for our approximation. Further, we assume that similar approximations are formulated for the particle velocity of the structure $v_s$ and the boundary admittance $Y$,

$$v_s(x) = \sum_{j=1}^{N} \phi_j(x) v_{sj} = \phi^T(x) v_s,$$

$$Y(x) = \sum_{k=1}^{\tilde{N}} \tilde{\phi}_k(x) Y_k = \phi^T(x) Y.$$

If $v_s$ and $Y$ are explicitly known, these approximations are not necessary for evaluation of the boundary integrals in Equation (3.55). However, there are many practical cases where the structural particle velocity is the result of a finite element simulation and available only as a piecewise defined function. Similarly, the boundary admittance may vary locally or results from other evaluations which motivate the piecewise approximation.

The number of basis function $\phi_l, \phi_j, \tilde{\phi}_k$ is given by $N, N$ and $\tilde{N}$ respectively. If the particle velocity of the structure and the boundary admittance are known functions, $N$ accounts for degree of freedoms. Herein, this coincides with the number of nodes of the finite element mesh. $N$ and $\tilde{N}$ may be equal to each other.
3.4.3.2 Finite Element Method

In many cases, finite element discretization actually refers to a Galerkin discretization of the weak formulation as shown in Equation (3.55). Galerkin discretization means that the basis functions $\phi_l$ which have been used in Equation (3.56) are substituted for the test function $\chi$. Hence we write,

$$
\begin{align*}
sk \int_{\Gamma} \phi_l(x) \left\{ \sum_{j=1}^{N} \bar{\phi}_j(x) v_{s_j} + \sum_{k=1}^{\bar{N}} \bar{\phi}_k(x) Y_k \right\} \left[ \sum_{m=1}^{N} \phi_m(x) p_m \right] \right\} d\Gamma(x) + \\
- \int_{\Omega} \left\{ \nabla \phi_l(x) \cdot \nabla \left[ \sum_{j=1}^{N} \phi_j(x) p_j \right] - k^2 \phi_l(x) \left[ \sum_{j=1}^{N} \phi_j(x) p_j \right] \right\} d\Omega(x) = 0. 
\end{align*}
$$

(3.59)

This can be written in a simpler form by introducing matrices. Herein, we introduce the boundary mass matrix $\Theta$ with entries $\theta_{lj}$ as,

$$
\theta_{lj} = \int_{\Gamma} \phi_l(x) \overline{\phi}_j(x) d\Gamma(x),
$$

(3.60)

the mass matrix $M$ with entries $m_{lj}$ as,

$$
m_{lj} = \int_{\Omega} \phi_l(x) \phi_j(x) d\Omega(x),
$$

(3.61)

the stiffness matrix $K$ with entries $k_{lj}$ as,

$$
k_{lj} = \int_{\Omega} \nabla \phi_l(x) \cdot \nabla \phi_j(x) d\Omega(x),
$$

(3.62)

and the damping matric $C$ with entries $c_{lj}$ as,

$$
c_{lj} = \rho_0 c \int_{\Gamma} \phi_l(x) \left[ \bar{\phi}^T(x) \bar{Y} \right] \overline{\phi}_j(x) d\Gamma(x) = \\
= \int_{\Gamma} \phi_l(x) \left[ \bar{\phi}^T(x) \bar{Y} \right] \overline{\phi}_j(x) d\Gamma(x),
$$

(3.63)

where $\bar{Y}$ assembles the values of the normalised boundary admittances such that $\bar{Y} = \rho_0 c Y$ and $\bar{\bar{Y}} = \rho_0 c \bar{Y}$, respectively. Consequently, we write our system of equations in matrix form as,

$$
(K - ikC - k^2 M) p = sk \Theta v_s = f.
$$

(3.65)

The left hand side shows a matrix polynomial in $k$. In general, $C$ represents a complex matrix because, in general, the boundary admittance is complex. The system matrix consists of three static matrices, $K$, $M$, and, provided the boundary admittance is independent of frequency, $C$. $K$ and $M$ are symmetric and positive definite, $C$ is symmetric but not Hermitian.
3.5 Finite Element Analysis

The finite element software used throughout this thesis as a theoretical comparison is the commercial package, COMSOL Multiphysics (versions 3.2a, 3.4, 3.5 and COMSOL Script v1.0 for interfacing with MATLAB v.7.2 and 9.0). The software is capable of solving all manner of physical phenomena with partial differential equations governing the behaviour. The acoustics module is a specific subpackage used in this thesis. There are several different modules, predefined in the software, covering the common fields of physics, to which the analysis is applicable. Both time-harmonic and Eigenfrequency analysis are performed in the acoustics module, to generate frequency spectra and band structures respectively.

3.5.1 Time-Harmonic Analysis

Two-dimensional representations of phononic crystals with different characteristics are investigated throughout this report. Firstly simple two-dimensional geometries are created in COMSOL using the GUI, which offers similar functionality to most technical drawing and CAD software packages. Periodic arrays of circular dots are defined, arranged in either a square $\Gamma_X$ or $\Gamma_M$ fashion, the array of dots being bounded by an exterior rectangular domain.

The exterior domain typically has subdomain parameters of $\rho = 1.25\text{kg/m}^3$ and $c = 343\text{m/s}$ to represent air, with each periodic scatterer defined by $\rho = 7990\text{kg/m}^3$ and $c = 4915\text{m/s}$ to represent steel. The difference in the acoustic impedance of these two types of domain constitutes the periodic fluid-solid media that categorises a phononic crystal.

Assuming a lossless medium, wave propagation through the domain is governed by the acoustic wave equation. With no damping term present, in COMSOL this takes the form,

$$\frac{1}{\rho_0 c^2} \frac{\partial^2 p}{\partial t^2} + \nabla \cdot \left( -\frac{1}{\rho_0} (\nabla p - q) \right) = Q,$$

where $c = \sqrt{K/\rho_0}$ is the phase velocity of the sound of which $K$ is the adiabatic bulk modulus and $\rho_0$ refers to the density. The additional parameters $q$ and $Q$ which are not present in the derivation of the wave equation (Equation 3.30), represent optional dipole and monopole sources.

The full acoustic wave equation is time dependent, however a harmonic excitation with time dependence of the form $f = f_0 e^{i\omega t}$ causes an equally harmonic response with the
same frequency, thus eliminating the time parameter from the equation \[61\]. Substitution of a time harmonic waveform, assuming the source terms have the same harmonic time dependence, reduces the acoustic wave equation to the Helmholtz equation,

\[ p(x, t) = p(x) e^{i\omega t}. \] (3.67)

Therefore,

\[ \nabla \cdot \left( -\frac{1}{\rho_0} (\nabla p - q) \right) - \frac{\omega^2 p}{\rho_0 c^2} = Q. \] (3.68)

As the time dependence has been removed, the angular frequency enters as a parameter, referred to as working in the frequency domain. Mathematically, the time-harmonic equation is a Fourier transform of the original time-dependent equations, and its solution as a function of \( \omega \) is a Fourier transform of a full transient solution. Therefore a time dependent solution can be retrieved by applying an inverse Fourier transform to a simulation in the frequency domain.

The time-harmonic analysis provides frequency spectra for comparison with experimental measurement. Equation (3.68) is formulated for the acoustic pressure, allowing several characteristic parameters of the acoustic wave to be studied (Table 3.1).  

### 3.5.1.1 Boundary Conditions

Several forms of boundary condition are applied to the domain for the time harmonic analysis. To define the speaker, acoustic sources are applied to boundaries to represent a source of incoming plane waves of the form \( p_0 e^{-ik \cdot r} \). To characterise the wave completely its amplitude \( p_0 \) must be defined as well as the components of its wave direction vector \( n_k \). However, we need only the define direction in our analysis, as COMSOL normalises the components to make \( n_k \) a unit vector. The wave vector is then defined as \( k = \frac{\omega}{c} n_k \), where \( k = \omega/c \) is the wavenumber.

Exterior boundaries of the rectangular domain are defined with radiation conditions, where the surroundings are a continuation of the domain. These absorbing boundaries, approximate an infinite air domain and suppress the existence of interference patterns caused by reflections from the sides of the finite domain. In the time-harmonic analysis the plane wave radiation boundary condition is,

\[ n \cdot \left( \frac{1}{\rho_0} (\nabla p - q) \right) + \frac{ik}{2k} \Delta_T p = \left( \frac{i}{2k} \Delta_T p_0 + (ik - i (k \cdot n)) \frac{p_0}{\rho_0} \right) e^{-i(k \cdot r)}, \] (3.69)

\[ \textsuperscript{†} \text{The variables of local and normal velocity are only available in time-harmonic analysis.} \]
Table 3.1: Acoustics application mode variables of COMSOL Acoustics Module.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Domain</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure</td>
<td>$p$</td>
<td>All</td>
<td>$p$</td>
</tr>
<tr>
<td>Pressure (dB)</td>
<td>$p(dB)$</td>
<td>S</td>
<td>$10 \log \left( \frac{p^2}{p_0^2} \right)$</td>
</tr>
<tr>
<td>Fluid density</td>
<td>$\rho_0$</td>
<td>S</td>
<td>$\rho_0$</td>
</tr>
<tr>
<td>Speed of sound</td>
<td>$c_0$</td>
<td>S</td>
<td>$c_0$</td>
</tr>
<tr>
<td>Dipole source, $x_i$ component</td>
<td>$q_{si}$</td>
<td>S</td>
<td>$q_{si}$</td>
</tr>
<tr>
<td>Dipole source, norm</td>
<td>$\text{norm}(q_s)$</td>
<td>S</td>
<td>$\sqrt{\sum_i q_{si}^2}$</td>
</tr>
<tr>
<td>Local acceleration, $x_i$ component</td>
<td>$a_i$</td>
<td>S</td>
<td>$-\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + q_i$</td>
</tr>
<tr>
<td>Local velocity, $x_i$ component</td>
<td>$v_i$</td>
<td>S</td>
<td>$-ia_i/\omega$</td>
</tr>
<tr>
<td>Local acceleration, norm</td>
<td>$\text{norm}(a)$</td>
<td>S</td>
<td>$\sqrt{\sum_i a_i^2}$</td>
</tr>
<tr>
<td>Normal acceleration</td>
<td>$a_n$</td>
<td>B</td>
<td>$n \cdot \left( -\frac{1}{m} \nabla p + \mathbf{q} \right)$</td>
</tr>
<tr>
<td>Normal velocity</td>
<td>$v_n$</td>
<td>B</td>
<td>$-ia_n/\omega$</td>
</tr>
</tbody>
</table>

where $\Delta_T$ at a given point on the boundary denotes the Laplace operator in the tangent plane at that point. For completeness it should be noted that boundaries can be modelled as rigid perfecting reflecting walls. At such a boundary the normal component of the particle velocity vanishes and the normal acceleration equals zero,

$$ n \cdot \left( \frac{1}{\rho} (\nabla p - \mathbf{q}) \right) = 0. \tag{3.70} $$

For constant fluid density, with no dipole source present, the normal derivative of the pressure is also zero at the boundary, $\partial p / \partial n = 0$. In addition to perfectly absorbing and reflecting boundaries a third class is defined to match the interfaces between the circular dots and rectangular subdomains. Impedance conditions relate the acoustic impedance mismatch between the steel and air domains and evaluate the fractions of the wave which are transmitted and reflected at the boundary. From a physical point of view the acoustic input impedance is the ratio between pressure and normal particle velocity,

$$ n \cdot \left( \frac{1}{\rho_0} (\nabla p - \mathbf{q}) \right) + \frac{i \omega p}{Z} = 0. \tag{3.71} $$
3.5.2 Eigenvalue Analysis

In addition to the time-harmonic analysis a second numerical analysis is performed, to construct the acoustic band structure of the periodic phononic crystal media. The Eigenvalue formulation of the acoustic wave equation solves for the Eigenmodes and Eigenvalues/Eigenfrequencies \[61\],

\[
\nabla \cdot \left( \frac{1}{\rho_0} \nabla p \right) - \frac{\lambda p}{\rho_0 c^2} = 0,
\]

where the Eigenvalue \( \lambda \) is related to the Eigenfrequency \( f \) as \( \lambda = (2\pi f)^2 \).

A single unit cell of the phononic crystal is defined for Eigenmode analysis (Figure 3.4). One Brillouin zone of the periodic structure is sufficient to describe the propagation behaviour entirely, by applying periodic boundary conditions to the extremities of the system. To begin all boundaries are defined with sound hard conditions which are perfectly reflecting. These boundaries are of the Neumann form, required because the problem is a Neumann Eigenvalue analysis. The red boundaries of the figure, represent sources of plane wavefronts \( (p_0 e^{i(k_x \cdot r)}) \), which propagate from left to right, \( e^{ik_x} \), and from the lower to upper, \( e^{ik_y} \), boundaries respectively. The destination boundaries (dashed
green) are defined with periodic conditions, to simulate the situation in which waves exiting the boundaries then re-enter at the source on the other side of the domain. Furthermore, we must define the source and destination vertices (black squares). Vertices 1 – 2 define the left source and 2 – 3 the lower. The corresponding right and upper destination vertices, are defined as points 3 – 4 and 1 – 4 respectively.

Using a loop function in MATLAB it is possible to solve parametrically across values of $k_x$ and $k_y$ between 0 and $\pi$ (the first Brillouin zone is sufficient), and by looping the appropriate variable we can determine the dispersion in three high symmetry directions. When, $k_x = \pi$ and $k_y = 0$, we scan point $\Gamma$ through point $X$ of the Brillouin zone. In the opposing case, $k_x = 0$ and $k_y = \pi$, we obtain the behaviour for transmission along the $\Gamma X'$ direction. This is of an identical form to the $\Gamma X$ band structure as the square lattice is symmetrical in these two scenarios. For the $\Gamma M$ direction both $k_x$ and $k_y$ are looped from 0 to $\pi$ to scan along the diagonal plane. Finally, the third high symmetry direction of the square crystal is the $X M$. We determine the behaviour along this lattice plane using the variables $k_x = \pi$ and $k_y = (0 \rightarrow \pi)$. The different directions are computed separately, then in combining the solutions, we obtain the full acoustic band structure. Any band gaps regions which are identified to extend across all directions are the absolute or complete band gaps of the structure.

### 3.5.2.1 Band Structure of Air

As a preliminary investigation the acoustic band structure of air is calculated. The air domain is an isotropic and homogeneous medium with no periodical scatterer included in the unit cell. For a simple square domain the size of the unit cell in the $k_x$ and $k_y$ directions is chosen to be the same as the lattice constant, $a = 0.022m$, tested experimentally. The resulting band structure for air is included in Figure 3.5. It consists of straight lines passing through the origin with gradient $c_{air}$. The lines are of constant gradient as the air medium is nondispersive, and as such, the speed of sound is not frequency dependent. Theoretically the value of $c_{air}$ at $20^\circ$, and standard atmospheric pressure is taken as 343ms$^{-1}$. Taking the gradient of the calculated result in the first branch of the band structure, the speed of sound is found to be,

$$c_{air} = \frac{d\omega}{dk} = \sqrt{156 \cdot (v/a) / (2\pi)^2} = 343.2m/s. \quad (3.73)$$

This measured result is in excellent agreement with the theoretical value.

---

Footnote:

†COMSOL Script Codes for band structure calculations are included in Appendix A
Figure 3.5: The acoustic band structure of a simple air domain. The reduced $k$ vectors represent the range $\pi/a = 143$ in the $\Gamma X$ direction and $\pi/(a/\sqrt{2}) = 202$ in the $\Gamma M$ and $XM$ directions.
Chapter 4

Experimental Procedure

This chapter details the experimental techniques to measure the acoustic transmission properties of phononic crystals. It begins with a literature review of the first observations of partial and full acoustic band gaps. Following this, specific details are provided about the sample investigated in this report. Two experimental procedures are introduced, the first utilised to measure attenuation data and the second to record phase information. Some further discussion is presented, concerning the accuracy of the experiments and various considerations which have been taken into account.

4.1 Introduction

Details of the first experiments designed to display the existence of an acoustic band gap began to appear during the 1990’s. Typically, in order to identify the phenomenon, the acoustic transmission properties of a periodic structure are analysed to determine any range of frequencies where transmission is forbidden. There was a drive towards experiments of this nature after M.Kushwaha et al. [26] reported the first full band gap calculations for periodic elastic composites in 1993, thus providing a theoretical framework for comparison. It was nearly two years later in 1995 that the first experimental evidence of band gap phenomena was provided by R.James et al. [62]. The band structure of a novel periodic sample was calculated, it being comprised of a one-dimensional array of $N$ Perspex plates immersed in water. It was shown experimentally that the systems frequency spectrum consisted of alternating transmission and gap bands. Each transmission passband was comprised of $(N - 1)$ peaks, corresponding to the total number of water gaps between the perspex sheets. Therefore as $N$ increases, the peaks are more closely spaced and the stop bands became more sharply defined.
Soon after the publication by James et al. [62], which received little attention, more experimental results were reported by F. Meseguer et al. [63]. Their 1995 Nature article received significantly more attention, mainly due to the vast difference of the structure they had conducted their study upon. Instead of fabricating a phononic crystal sample, experiments were performed upon an already existing minimalist sculpture in Madrid by the artist Eusebio Sempere. The sculpture measures 4m in diameter and is considered two-dimensional as opposed to James’ one-dimensional chain array. It is comprised of an array of hollow steel cylinders, which Meseguer believed, but his colleagues debated, should possess a band gap due to its periodicity. The dispute was settled by measuring the acoustic transmission through the sculpture as a function of frequency, and the study revealed that sound propagating perpendicular to the axis of the cylinders was strongly attenuated at 1.67kHz.

Following the first two experimental observations of the acoustic band gap, the first report of an ultrasonic band gap for the longitudinal wave mode was provided by F.R. Montero de Espinosa et al. in 1998 [64]. Small systems were constructed by periodically drilling millimetre holes in an aluminium plate and filling them with air, oil and mercury. The small lattice parameter of the crystal is comparable to ultrasonic wavelengths and a band gap was observed in this regime from 1.00-1.12MHz for the mercury system. It was discovered that the band gap was present irrespective of measurement direction and therefore constituted the first full band gap for a two-dimensional system.

Two further studies [65, 66] were published in 1998; both detailed experiments conducted upon larger scale two-dimensional phononic crystals. The samples were designed to have lattice parameters comparable to wavelengths within the human ears response. Structures that operate in this regime have been of particular interest since the conception of phononic crystals due to the practical soundproofing applications the systems offer. An investigation performed by J.V. Sanchez-Perez et al. [65] studied phononic crystal structures composed of stainless steel and wood cylinders in air. The cylinders were arranged in square and triangular lattice configurations and the transmission was tested along the two high symmetry planes of each crystal. An overlap was observed between the attenuation peaks measured along the two directions and this was considered as the ‘fingerprint’ of the existence of a full band gap. The band structure of the triangular lattice showed band states in the region of the fingerprint that were not excited by the sound transmission. Such a band was coined as a ‘deaf’ band the origins and influences of which will be discussed in more detail in Chapter (5.2).

A second study performed on a larger phononic crystal structure was reported by W.M. Robertson and J.F. Rudy III [66]. The samples were fabricated from electrical conduit cylinders and were arranged to represent square and triangular lattices by inserting
them into wooden bases with the appropriate periodicity. The experimental results showed that both the square and triangular lattices possessed a band gap along the $\Gamma X$ symmetry plane, but no clear band gap could be identified along the $\Gamma M$ and $\Gamma J$ plane of the square and triangular lattice respectively. In an attempt to identify the band gap along these planes, the phase data of the acoustic signal was analysed. Robertson and Rudy found a range where the phase delay shows anomalous dispersion with positive slope, which they interpreted as an indication of a band gap. The existence of the band gap can therefore be identified as a region of anomalous dispersion in the phase. The new technique of phase data analysis pioneered by Robertson and Rudy III provides a more sensitive technique to locate the edges of the band gap.

Learning from the seminal reports published in 1998, the existence of a full band gap in a two-dimensional array of rigid cylinders in air was reported a year later in 1999 by C.Rubio et al. [67], who refined the former experiments in addition to introducing a new variational method to calculate the transmission of sound waves and act as a theoretical comparison. From the Robertson and Rudy III study it was suggested that regions of anomalous phase dispersion coincide with the edges of the band gap. It was in using this conclusion, combined with experimental results and the new variational method that Rubio was able to calculate the band structures of his sample and demonstrate the existence of a full acoustic band gap. A good agreement was found between the theoretical and experimental borders of the band gap along the $\Gamma X$ direction for both lattices. Attenuation bands were also found along the $\Gamma M$ plane of the square lattice and $\Gamma J$ plane of the triangular lattice, therefore proving the existence of a full band gap, although their positions were not consistent with the theoretically obtained dispersion relation. It was concluded that the square lattice configuration, with a filling fraction of $f = 0.41$, possessed a full band gap between 3.1-4.1kHz, in good agreement with the theoretical prediction of 3.3-3.9kHz.

4.2 Sample Preparation

In this work a rigid cylinder phononic crystal was fabricated for experimental study, comprising a periodic array of steel scatterers surrounded by air. The array investigated was constructed from one hundred identical steel tubes with external radius $r_0 = 6.5\text{mm}$, internal radius $r_i = 5.0\text{mm}$ and height 300mm. The constituent units were supported vertically by inserting them into a wooden base with periodically drilled holes. These were arranged appropriately in order to represent the square Bravais lattice.

Figure (4.1) shows the sample under study which was constructed along the $x$-axis, the incident sound direction. Transmission characteristics were determined for the array
along both of the high symmetry planes, namely the $\Gamma X$ and $\Gamma M$. The $10 \times 10$ square Bravais lattice has periodic constant $a = 0.022m$, which corresponds to fraction of volume occupied by the scatterers, i.e. packing fraction of $f_{sq} = \pi r_0^2 / a^2 = 0.274$.

![Figure 4.1: Schematic representation of the phononic crystal constructed from 100 stainless steel tubes, in a square lattice configuration. Also included is a representation of the structures Brillouin zone to clarify the assignment of transmission directions.](image)

Air was selected as the host medium as the scope of this thesis is primarily focused on the attenuation of airborne sound in the audible frequency range. It also has the advantage of simplifying the construction of arrays. If another fluid matrix were to be employed, a containment tank would be required to keep the fluid localised around the scatterers. The use of air as the host medium also ensures that there is a large difference between the acoustic impedance of the host medium ($4.29 \times 10^2 N\cdot s/m^3$) and the steel scattering units ($3.93 \times 10^7 N\cdot s/m^3$). This ensures a very high reflection coefficient at the boundary between the two media. As such, the propagating sound waves are strongly scattered and the majority of the sound energy is concentrated in the host matrix. It is therefore appropriate for us to only consider the longitudinal acoustic wave mode and neglect the influence of any transverse coupling to the solid inclusions. Furthermore, a large impedance mismatch between the host and inclusion materials ensures the ratio $\rho_{\text{air}} / \rho_{\text{steel}}$ is sufficiently large in order to open the spectral gap.

The phononic crystal structure is defined as two-dimensional as the cross-section of the scatterers is invariant in the $z$-axis direction. In addition, for this assumption to be valid, the height of the solid scatterers must be sufficiently larger than the wavelength of the sound used in testing, and also sufficiently larger than the crystals thickness. Therefore the assumption that the system is quasi-two dimensional does not hold at low frequencies and we should expect some diffraction around the crystal at low frequencies.
(a) Rigid cylinder phononic crystal. A square $10 \times 10$ lattice of 13mm diameter steel tubes, with periodicity $a = 0.022$ and filling fraction 0.274.

(b) Zero order transmission experiment. Photograph taken inside the anechoic chamber of the Department of Aeronautical and Automotive Engineering, Loughborough University.

Figure 4.2
4.3 Zero-Order Transmission Experiments

4.3.1 Procedure for Transmission Analysis

Zero order transmission experiments are the original, and still most popular, technique used to determine the attenuation characteristics of a phononic crystal. In order to analyse the sound transmitted across the structure a directional speaker is employed as a source and two calibrated condenser microphones record the signal at different locations.

![Diagram of zero-order transmission experiment](image)

**Figure 4.3:** Zero-order transmission experiment. Microphones positioned at points A and B measure the sound pressure amplitude as a function of frequency, referred to as the sample and control recordings respectively.

A first microphone A is positioned behind the phononic crystal structure and measures the signal after it has propagated through the structure. This provides the sample recording that will have experienced attenuation at some bandwidth of frequencies. In order to identify these frequencies, the signal is compared to a reference signal. This is obtained by the second microphone B that is positioned at the same distance from the source as A and records the sound with no attenuation.

The pressure difference or sound attenuation is found according to the formula,

\[
\text{Sound Pressure Level}(dB) = 20 \log_{10} \left( \frac{A_{\text{micB}}}{A_{\text{micA}}} \right).
\]  (4.1)
4.3.2 Procedure for Phase Data Analysis

A second experimental set up was investigated for the array that is designed to extract and analyse the phase difference of two signals. Both the sample microphone A and the control microphone B are placed collinear with the source that emits a collimated beam. The sample microphone A is located in the same position as for transmission experiments, but the reference microphone B is now alternatively positioned in front of the structure and records the sound before it reaches the array. An analysis of the waveform from both microphones in MATLAB obtains both the sound attenuation and phase delay. The phase delay, $\varphi$, between micA and micB relates the wave vectors and the distances $d_1$, $d_2$ and $L$ in the experimental configuration through,

$$\varphi = \varphi_A - \varphi_B = k_0(d_1 + d_2) + kL,$$  

(4.2)

where $k_0$ and $k$ are the wave vectors in air and inside the phononic crystal respectively. From this relation the sound velocity at each frequency inside the crystal can then be found through,

$$c(f) = c_0 \left[ \frac{c_0 \varphi}{2\pi fL} - \frac{d_1 + d_2}{L} \right]^{-1},$$  

(4.3)

where $c_0$ is the speed of sound in the air matrix, $\varphi$ is the phase delay and $L$ is the thickness of the sample. It is then possible to plot the dispersion relation of the experimental data, by relating the phase velocity to the wave vector $k$ and angular frequency $\omega$ through,

$$c(f) = \omega/k.$$  

(4.4)

\[\text{Figure 4.4: Zero-order phase experiment. Microphones positioned at points A and B measure the sound pressure amplitude as a function of frequency, referred to as the sample and control recordings respectively.}\]
4.3.3 Calibration of Incident Beam

It is essential to the accuracy of both zero-order transmission experiments that the reference beam be analysed. This determines the effects of interference phenomena that occur between the incident wave and the signal reflected from the phononic crystal structure. It is possible to determine the magnitude of this influence by measuring the intensity that reaches microphone B with no structure present in the chamber. For the transmission experiments set up (Figure 4.3), it was found from a direct comparison of both of these signals that there was only a small intensity difference over the entire range of frequencies studied (Figure 4.5(a)). This value is considered as the background noise in the experimental set up. A small enhancement of transmission occurs at Microphone location B with the sample present, from 0.59-6.21kHz arising from reflections.

Due to its position in the phase data experiments, it is anticipated that the reference microphone will be more severely affected by the interference of backscattered waves from the structure, especially for frequencies within the forbidden bandwidth. To investigate this influence, several different microphone positions were explored. Figure 4.5(b) shows a comparison of the average recorded signal to the signal reaching microphone B when no sample is present in the chamber. A direct comparison between the signals shows a maximum intensity difference of ±3dB. Greater levels of interference were observed with the microphone closest to the structure, however even in this worst location, the effect of reflections is not as severe as for the transmission experiments.

The two calibration graphs have been included to give an estimation of the magnitude of the reflections from the sample. The values observed are considered as background noise in the experimental set up, and attenuation of a more significant amplitude can therefore be considered as evidence of Bragg diffraction occurring.

4.4 Generating Signals

The signal generating speaker, Type Realistic Nova 800 (Cat. No. 40-7024), has a linear frequency response in the range from 1-20kHz. This is a sufficiently large range as the maximum audible frequency for humans is approximately 20kHz. In order to generate a continuous rising tone signal across this frequency regime, the speaker was driven by a B&K beat frequency oscillator Type 1022. This was in turn connected to a B&K level recorder Type 2305, allowing an automated scan to be performed. This generated a six second rising tone signal ranging from 0-16kHz, which was transmitted through the phononic crystal arrays.
(a) Amplitude difference between two recordings taken at the control position, with and without the sample present, for the zero-order attenuation procedure. The measurement taken with the sample present is louder in the frequency range from 0.59-6.21kHz, caused by reflections from the side of the sample.

(b) Amplitude difference between two recordings taken at the control position, with and without the sample present, for the zero-order phase procedure.

Figure 4.5: Calibration spectra of (a) the zero order transmission set-up, and (b) the zero order phase experiment.
4.4.1 Plane Wavefronts - Geometrical Considerations

The wavefront of sound incident on the surface of the phononic crystal is assumed to be a full plane wave, i.e. all points of constant phase arrive at the same instance. This condition is essential to ensure that the $k$ vectors are able to fulfill the Bragg condition, therefore we must investigate the validity of the assumption. If we consider the speaker to be a point source radiating omni-directionally (Figure 4.6), the emanating wavefronts of constant phase will be of a circular nature. When such a circular wavefront is incident on the flat crystal surface, there is a time delay between the centre point ($p_{centre}$) arriving at the sample and the rest of the wavefront. This time delay causes a phase difference the magnitude of which can be approximated from geometrical considerations,

$$d = \sqrt{x^2 + \left(\frac{L_w}{2}\right)^2} - x. \quad (4.5)$$

The maximum phase difference with which we are concerned will occur at the edge corner of the phononic crystal represented by the component $p_{edge}$. If the wavelength of the incident sound is equal to the path difference $d$, then the centre point of the wavefront will be $2\pi$ out of phase with the delayed component incident at the edge. In effect the centre point of one wavefront would arrive at the phononic crystals surface at the same time as the wavefront ahead of it reached the edge of the sample. Incident waves with such a phase delay cannot be considered plane. For the assumption of plane waves to be valid the phase difference between the components should be less than $\lambda/8$ out of
phase. This is an approximate limit used in optics, although some texts state that a quarter wave phase difference is sufficiently accurate for acoustic systems [61]. A plot of the phase difference as a function of frequency for the experimental geometry (Section 4.2) is shown in Figure 4.7.

Frequencies below 4.3kHz arrive with less than $\lambda/8$ difference, providing a very good plane wave approximation at frequencies below this limit. Frequencies in the range of 4.3-8.6kHz arrive with up to a quarter wave phase difference, and can still be considered as plane, however with a lower certainty. For frequencies above the 8.6kHz limit, the wavefronts have more than a quarter wave phase difference which means that the plane wave approximation is not valid in this regime. It should be noted that for this estimated calculation the assumption has been made that the speaker is a point source of circular wavefronts. In reality however, the low frequency cone of the speaker has diameter 18cm and is a more planar source than we have assumed.

![Figure 4.7](image_url)

**Figure 4.7:** Percentage phase difference between $p_{centre}$ and $p_{edge}$ as a function of frequency. Phase delay becomes greater as the wavelength of the incident sound becomes smaller. The horizontal lines intersect the points at which phase difference is equal to one eighth and a quarter.
4.5 Recording Signals

A matched pair of Behringer C2 small diaphragm condenser microphones were used to record the sound signals at locations A and B in the experiments. The microphones directivity is classed as unidirectional as they are most sensitive to sound incident from the front. The directional response of the Behringer C2 microphones is illustrated in Figure (4.8), where the sound intensity at a particular frequency is plotted for angles radially from $0^\circ$ to $360^\circ$. This type of sensitivity pattern is known as the cardioid due to its heart shape. A pattern of this form is particularly suited for our experiments as it rejects sound from other directions.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4.8.png}
\caption{Polar sensitivity pattern of the Behringer C2 condenser microphone. The illustration is taken from the Behringer manual.}
\end{figure}

The frequency response of the microphones is stated as 20-20kHz by the manufacturer, however in itself this information is meaningless without a decibel measure of tolerance across the range. The frequency response diagram is shown in Figure (4.9) that plots the microphone sensitivity in decibels over the active range of frequencies for sound incident perfectly on axis, i.e. at $0^\circ$ to the capsule. The response is not linear across the entire range of frequencies, a factor that should not be neglected when interpreting the experimental results. Both microphones are powered by a DC phantom power P48 supply (+48V) via a balanced XLR microphone cable. The BNC output of the phantom power supply is connected to a Creative USB Sound Blaster external sound card (model no. SB0490) via a 3.5mm jack connection. The output of the external sound card is then connected in turn to the analysing computer. Sound files were captured using the Creative Wave Studio software that supports up to a 24bit/96kHz audio recording, which is
more than sufficient for our frequency range of interest. The .WAV sound files are then imported into MATLAB where the Fast Fourier Transform operator decomposes the file into its frequency components and plots the frequency spectrum. Further manipulation in MATLAB allows plots of the phase difference from the two microphones to be obtained, which can in turn be used to find the sound velocity as a function of frequency and the dispersion relation.

### 4.6 Free Field Measurements

It is important to the accuracy of the measurements that free field conditions are simulated and as such the sound behaves as if it were propagating in free space. To achieve such a situation, all acoustic experiments were performed inside an anechoic chamber with dimensions of length, width and height, 11.5 × 7.0 × 6.0m respectively. The chamber is built as a double shell system. It is acoustically isolated from the surrounding building and the reverberation chamber by an air gap, and is supported on vibration isolation mounts. The inner wall is constructed from 230mm thick concrete and the inner room is mounted on VIBREX OMEGA springs. The room has a wire mesh floor that permits the propagation of sound waves and removes errors caused by ground reflections. To absorb sound at the boundaries of the room, the walls of the chamber are lined with glass-fibre wedges (G+H ASONAD) and provide absorption at frequencies above 100Hz. The effective dimensions between wedges are 6.4 × 5.2 × 5.3m of length, width and height respectively, corresponding to an acoustically usable free field range of 4.7 × 3.5 × 3.6m. Free field conditions inside the chamber eliminate the two main sources of error in any acoustic experiment, namely those of background noise and signal interference from reflected wavefronts.
(a) Foam wedges absorb incident sound through a combination of reflection and absorption mechanisms. This eliminates the interference from reflections of smooth surfaces.

(b) Entrance to the anechoic chamber. The thickness of the chambers sound insulation, ensures the environment is isolated from unwanted external noise.

**Figure 4.10:** The anechoic chamber facility at the Department of Aeronautical Engineering, Loughborough University.
Chapter 5

Rigid Cylinder Phononic Crystal

The following chapter details the band gap characteristics of the two-dimensional phononic crystal sample defined in Chapter 4.2. It begins with a discussion of the frequency spectra obtained via zero-order transmission experiments. The data is then used in conjunction with Equation 4.1 to calculate the attenuation level (dB) as a function of frequency, providing a measure of the samples attenuation characteristics to a recognised scale. Following this, Section 5.3 presents the phase data extracted from the zero-order measurements, which is used to construct the acoustic dispersion relation for the sample.

The second half of the chapter details the finite element results and discusses the agreement with the observed experimental behaviour. Wave propagation is simulated as a function of frequency through a scale model of the experimental situation. Plots of the attenuation level as a function of frequency are extracted to serve as a comparison for the experimental data. Finally, theoretical band structure calculations are performed using the finite element for comparison with the experimentally measured behaviour. The dependence of gap width upon the structures packing fraction is also addressed.

5.1 Zero-Order Transmission Experiments

The square lattice phononic crystal defined in Section 4.2 was subjected to the zero-order transmission experiments of Sections 4.3.1 and 4.3.2. The frequency spectra obtained by a Fourier transform of the control (microphone B) and sample (microphone A) recordings are plotted in Figure 5.1. A direct comparison of the two traces in this manner, reveals a frequency range where the amplitude of the sample recording is suppressed relative to the control. This region corresponds to the band gap and is observed.
from 5.45-10.12kHz with the sound incident upon the $\Gamma X$ plane. The gap is centered around 7.79kHz and has a bandwidth of 4.67kHz, however the level of suppression is arbitrary.

There is a simple calculation that can be performed to confirm that the band gap is located at the correct frequency, consistent with Bragg’s law. We know the periodicity of the square lattice along the $\Gamma X$ plane is $a = 0.022m$, and it is the coherent scattering between these equally spaced layers of the crystal that opens a band gap at the Brillouin zone edge $k_{BZ}$. Therefore the centre frequency of the band gap can be estimated through,

$$\omega = v k_{BZ} = \frac{v \pi}{a},$$

which can be rearranged to leave us with,

$$f_c = \frac{v}{2a}.$$  \hspace{1cm} (5.2)

Using the known value of $a$ in conjunction with $v_{air} = 343m/s$ gives a centre frequency of 7.80kHz, in excellent agreement with experimental observation.
Further experiments were performed upon the square lattice sample with sound transmitted along the second high symmetry plane, namely the $\Gamma M$. The frequency spectra measured by the control and sample microphones are plotted for comparison in Figure 5.2. In this orientation it is difficult to identify the band gap region from the data. A suppression in the amplitude of the sample trace can be observed at frequencies between 7.72-12.05kHz, which is considered as evidence of a band gap, although it is difficult to define its boundaries with any certainty. The region of suppressed amplitude is centered around 9.89kHz, the accuracy of which can be verified using a similar calculation to Equation (5.2). The periodicity of the square lattice along the $\Gamma M$ plane is now $a/\sqrt{2}$, which when substituted into $\nu/2a$ gives an estimated central frequency of 11.02kHz. This frequency is contained within the envelope of suppression from 7.95-11.83kHz, however does not match the experimentally observed central frequency of 9.89kHz. The discrepancy of 1.13kHz between the experimental and theoretical results is evidence that the band gap has not formed completely. The cause of this will be discussed further with respect to the attenuation data (Section 5.2).

The frequency spectrum in the $\Gamma M$ plane displays a second region of attenuation, observed above 12.35kHz and continuing through to the highest measured frequency (Figure 5.2). There are two noteworthy points of discussion to be observed about the data.
in this region. Firstly that both the control and sample recordings display a significant drop in amplitude, and secondly that the sample trace is suppressed in this region. The drop in amplitude of both recordings can be attributed to the frequency response of the microphones. Manufacturer guidelines from Section (4.5) suggest a response loss of approximately 4dB upon exceeding 11kHz, and this is observed. This artefact could be compensated for by performing an inverse FT of the microphone response spectrum and subtracting the resulting waveform from the experimental data. One final manipulation of the data by FT would produce frequency spectra of more consistent amplitude response. This process however is a lengthy and unnecessary one, as it merely serves to scale the data, providing no further insight for drawing conclusions.

The real point of interest about the data in this high frequency region above 12.35kHz is the suppression of the sample trace relative to the control. This appears to be further evidence of a spectral gap, as the attenuation occurs close to the predicted location. This experimental result is unexpected but not unique. In fact it is remarkably similar in nature to the frequency spectra obtained by Robertson and Rudy III with sound incident upon the $\Gamma M$ plane of their square lattice sample [66]. They reported a suppression of amplitude starting from 5kHz (the structure has a predicted centre of 6.5kHz) and continuing through to the highest frequency. This behaviour is therefore very similar to the frequency spectrum of Figure (5.2). The consistency in the observed attenuation characteristics of the two separate square arrays, is a consequence of the low filling fraction in both examples. Theoretical considerations from Section (2.3) indicate that the filling fraction determines whether or not a periodic array will exhibit an absolute band gap for all directions [26]. Robertson’s sample was designed with a packing fraction of 0.31, just above the critical value of 0.30 suggested theoretically by Kushwaha. As this value is close to the limit, and the phononic crystal studied in this thesis has a filling fraction below the limit, neither structure possesses a full band gap. Both examples exhibit attenuation in the frequency range around the predicted centre frequency location but are of too low a filling fraction to open a gap for $\Gamma M$ transmission.

5.2 Attenuation Plots

The data obtained in the zero-order transmission experiment can be represented in an alternative manner. The attenuation level can be found through Equation (4.1), to determine the relative amplitude of the two measurements. By plotting the attenuation spectrum as opposed to the two separate traces of sample and control data, it is a simpler task to identify the band gap location. Furthermore, the attenuation levels are calibrated to a recognised decibel scale as opposed to arbitrary standard.
A plot of the attenuation level measured from the transmission experiment, with sound incident upon the ΓX plane, is depicted in Figure 5.3. In the low frequency regime, before the band gap opens at 5.76kHz, a negligible amount of attenuation is experienced, considered to be background noise, attributed to interference from reflections. As the wavelength of the incident sound becomes comparable to the periodicity of the crystal, a sharp increase in the attenuation is observed. For our purposes the band gap is defined as any region of attenuation that exceeds 10dB thus defining the start and finish of the band gap at 5.77kHz and 9.10kHz respectively. A second region of attenuation is observed for ΓX transmission, starting at 12.44kHz. This is attributed to the second spectral band gap. It is an integer harmonic of the fundamental, with an estimated centre frequency of 15.59kHz. In the frequency range tested, we are therefore only able to observe a part of the band gap. It was estimated that above 9kHz the incident wavefront upon the crystal would not be plane. However, the results appear reliable in the high frequency regime.

The relative amplitude of the control and sample recordings for sound incident along the ΓM plane, is presented in Figure 5.4. Similarly to the frequency spectra shown for this direction (Figure 5.2), a significant region of attenuation is observed from 7.70-12.07kHz and also in the high frequency region above 12.18kHz. The attenuation level exceeds the 10dB threshold used to define the band gap at frequencies in the range between
Figure 5.4: Experimental attenuation level for sound incident along the ΓM plane of the square lattice. Attenuation is observed between 7.70-12.07kHz reaching a maxima at 10kHz. A further attenuation peak exists at 3.85kHz alongside two enhanced transmission peaks at 0.86kHz and 12.18kHz.

7.70-12.07kHz. The experimental band gap is therefore centered at 9.89kHz, which does not agree with the theoretical estimate of $f_c = \nu/(\sqrt{2}a) = 11.024\text{kHz}$. Despite this disagreement, the theoretical central frequency is contained within the band gap envelope (as defined by our stipulations) and experiences 9.45dB of attenuation. This is too large to be attributed to background noise and therefore suggests it must be a consequence of Bragg diffraction phenomena.

There are extra attenuation peaks that can be identified in the data of Figure (5.3), that are located at frequencies away from the band gap. Specifically a negative attenuation (transmission) spike can be identified at 1.02kHz that exceeds 10dB of attenuation, also a positive attenuation spike is observed at 3.77kHz. The existence of these peaks can also be observed in Figure (5.4) for sound attenuation along the ΓM plane, with a spike of transmission and attenuation at 0.86kHz and 3.85kHz respectively. An additional negative peak is also present with sound incident upon the ΓM plane at 12.18kHz. These frequencies do not correspond to Bragg scattered wavelengths and therefore, cannot be attributed to Bragg diffraction. Alternatively, the consistency of the peaks to appear at the same location, irrespective of measurement direction, suggests they are instead an artefact of the experimental technique.
There are three alternative mechanisms that can lead to the formation of attenuation peaks. The first of these is a gap in the band that occurs due to Bragg reflections and as already discussed this leads to a broad attenuation peak from 5.44-10.16kHz along the $\Gamma X$ plane. Secondly, there is an energy transfer to Bragg waves of higher order that occurs when the traveling waves leave the phononic crystal structure. The influence of this effect can be studied by placing the sample microphone at several angles, tilted with respect to the original direction, where an enhancement of transmission is observed at certain directions.

The final source that can lead to the appearance of transmission and attenuation peaks is the existence of bands that are not excited by the incident sound wave due to symmetry reasons. This phenomena was first detailed by J.V.Sánchez-Pérez et al. who predicted the existence of a band gap in the dispersion relation that was not detected experimentally [65]. To investigate the behaviour, the pressure distribution of the Eigenmodes was analysed. The phenomena can then be understood with respect to the symmetry of the mode states.

Consider the pressure Eigenmodes solutions of the Helmholtz equation depicted in Figure 5.5. These represent the two lowest modes of the square array of 6.5mm diameter cylinders in air investigated experimentally. The modes shown are near the $M$ point belonging to bands one and two of the structures dispersion relation. Mode 1 has the required symmetry to be excited by a plane wave incident along the $\Gamma M$ direction, however Mode 2 has planes of equal phase along the perpendicular direction and consequently

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5_5.png}
\caption{Eigenmodes of the pressure field for an array of 13mm diameter cylinders in air. The pressure maps show the distribution of the pressure field for the lowest two modes along the $\Gamma M$ plane.}
\end{figure}
cannot be excited. This type of mode is known as ‘deaf’ and it is their existence that leads to the observed transmission peak at 12.18kHz along the crystals $\Gamma M$ plane.

The appearance of this spike 12.18kHz has a significant influence on our interpretation of the results. It occurs at a frequency that interrupts the band gap formation along the $\Gamma M$ plane. If it were not present a region of attenuation would exist from 7.70kHz through to the highest measured frequency. This behavior would then be entirely consistent with that observed by Robertson and Rudy III along the same plane. The transmission peak does exist however, and is located in the suppressed region. Therefore it appears to divide the region into two separate band gaps, whereas in reality they both constitute the first Bragg band.

![Figure 5.6](image)

**Figure 5.6:** Attenuation spectrum for $\Gamma M$ transmission. A band of attenuation averaging approximately 8.7dB is observed, beginning at 7.70kHz through to the highest measured frequency. The presence of a transmission spike at 12.18kHz corresponds to the first deaf mode of the array, which cannot be excited due to symmetry reasons and interrupts the attenuation band.
5.3 Phase Difference

The discussion so far has been concerned with the amplitude data obtained by the zero-order transmission experiment, but we will now turn our attention to the phase data collected. It provides a more sensitive marker to indicate the presence of a band gap, particularly when it is of a small bandwidth or low attenuation level. The phase data from the sample and control microphones is combined to obtain a phase delay diagram along the two high symmetry planes. This relative phase delay was found at each frequency from the attenuation functions, obtained by dividing the complex Fourier transform of the sample time domain waveform by that of the control. Phase delay has a negative slope outside of the band gap, whereas for frequencies where transmission is forbidden, the phase exhibits anomalous dispersion with positive slope. It is this anomalous dispersion that serves as a clearer indicator of the band gap than the suppression of amplitude data alone.

For sound incident along the ΓX crystal plane, a point of anomalous dispersion is observed in the relative phase data (Figure 5.7) at 7.43kHz, compared with the central frequency defined from the attenuation data of 7.79kHz (Figure 5.1) and the theoretical estimate of $f/2a = 7.80\text{kHz}$.

![Figure 5.7: Phase difference, $\phi$ between control and sample measurements, for transmission along the ΓX lattice plane. Anomalous dispersion indicates the presence of the band gap, at 7.43kHz.](image)
5.3.1 Dispersion Relation

The acoustic dispersion relation extracted from the experimental data is now discussed. As detailed in Section 4.3.2 the sound velocity at each frequency was found through,

\[ c(f) = c_0 \left[ \frac{c_0 \varphi}{2\pi f L} - \frac{d_1 + d_2}{L} \right]^{-1}. \]  

The dispersion relation is then found by relating the phase velocity to the wave vector \( k \) and angular frequency \( \omega \) through,

\[ c(f) = \frac{\omega}{k}. \]  

The acoustic dispersion relation is analogous to the electronic band structure discussed in Chapter 2. It represents the separation of a complex sound wave into its frequency components, measured by the rate of change of velocity with frequency.

A plane wave travelling in an infinite expanse of air in the \( \pm z \) direction obeys the linear dispersion relationship \( \omega = \pm c_{\text{air}} k \), as shown in Section 3.5.2.1. The dispersion curve \( [\omega(k)] \) for the allowed propagation of plane waves through air therefore consists of straight lines in the positive frequency half plane passing through the origin and of slope \( \pm c_{\text{air}} \) (the blue line with constant gradient in Figure 5.8).

However, the unit cell contains not just air, but inhomogeneities in the form of the array of metal cylinders. This causes discontinuities to appear in the curves at \( k = \pm m\pi/L_z \) (\( m \) an integer), yielding ranges of frequencies for which the governing Helmholtz equation, describing the propagation of acoustic waves, has no real solution for real \( k \). Indeed \( k \) is entirely imaginary in these band gaps, corresponding to a decaying state, which cannot propagate through the infinite system considered, or an exponentially growing state, which has no physical realisation in a perfectly periodic system. Thus, the band gaps represent ranges of frequency at which no propagation through the periodic structure can occur.

Physically we can envisage this acoustic band gap by considering the velocity of the wave. As the phase velocity is reduced inside the crystal there is a splitting of the phase velocity values. This splitting means that \( k \) vectors ending on the boundary of the first Brillouin zone can attain two energies. One value is lower than the \( \omega = ck \) curve and the other is higher. Such regions of anomalous dispersion exist between 6.82-8.26kHz for \( \Gamma X \) transmission and 8.57-10.87kHz for \( \Gamma M \) transmission.
Figure 5.8: Dispersion $\omega = ck$ has a gradient of phase velocity 343 m/s represents dispersion of acoustic wave through infinite expanse of air. Extended scheme, where the edge of the 1st is 142.8 rads·m$^{-1}$ and 2nd at 285.6 rads·m$^{-1}$. Anomalous dispersion occurs in the region of the band gap, centered at 7.54 kHz for the $\Gamma X$ direction.
5.4 FE Simulation

5.4.1 Attenuation $\Gamma X$

![Figure 5.9: Schematic diagram illustrating the geometry of the FE model used as a comparison for zero-order experimental measurements. The colour of the boundaries distinguishes between the different type of boundary condition applied. The plane wave source is represented by dashed green, absorbing radiation conditions by red, sound hard boundaries by the black lines and impedance conditions by the dashed blue arrows. In both x and y axes, the scale of the units is meters.](image)

A two-dimensional scale representation of the experimental set up was modelled using the finite element method. The geometry is depicted in Figure 5.9, with the appropriate dimensions labelled. The far left boundary is a radiating source of plane wavefronts $(p_o \exp(i\mathbf{k} \cdot \mathbf{r}))$ that travel across the domain from left to right. Absorbing radiation conditions are applied to all other boundaries of the exterior domain to suppress reflections. Each scattering unit has material parameters of sound velocity $\nu = 4915\text{m/s}$ and density $\rho = 7790\text{kg/m}^3$ applied to represent steel, whereas the exterior fluid domain is defined by $\nu = 343\text{m/s}$ and $\rho = 1.25\text{kg/m}^3$ for air. Impedance boundary conditions are applied to the scatterers, so a small amount of transmission may be observed at the boundary, however the large impedance mismatch ensures they are strongly reflecting. The frequency is defined as a parametric variable, allowing several solutions of the model to be obtained across a defined range. By solving the acoustic wave equation for the pressure amplitude it is then possible to synthesize a frequency spectrum at a single point in the
domain. We require two such spectra at points A and B of the domain to represent the locations of the control and sample microphones in the experimental procedure. Using Equation (4.1) the relative amplitudes of the two traces reveals the attenuation characteristics. In order to ensure the reliability of the simulations, at least ten elements per wavelength are required to accurately resolve the wave. The typical range of investigation is 0-16kHz corresponding to a maximum element size of $\lambda_{\text{max}}/10 = 2.14 \times 10^{-3}$m. The large domain has area, $1.65m^2$, therefore approximately $10^5$-elements are required for a sufficiently accurate mesh.

![Figure 5.10: Schematic of the reduced domain geometry. The domain area is 0.12m$^2$, requiring approximately $10^3$-elements for a sufficient mesh density at the highest simulated frequency of 16kHz. The geometry greatly reduces computation time. Diffraction cannot occur around the sides of the array as the absorbing edges of the domain are too close to the edges of the sample (a single lattice period $a$).](image)

A comparison of the data obtained via the finite element method with that measured experimentally is presented in Figure (5.11(b)) (red line with black square markers). A band gap is observed between 5.79-9.18kHz, which coincides with the experimental result, 5.45-10.10kHz. Also observed in the attenuation plot is the existence of a band of enhanced transmission, spanning 1.20-5.45kHz. This is a consequence of reflections from the side of the sample. The frequency spectrum obtained from the geometry of Figure (5.9) is of a low resolution, with only 40 solutions obtainable for the parametric value of $f$ ranging from 0.1-10kHz, with step increments of 250Hz. The spectrum is restricted to such few solutions, due to the large size of the domain (1.65m$^2$) and the inherently...
(a) Frequency spectra obtained by the FE method for the ΓX plane. The control traces (dashed) are taken at point B in the large domain, and with no array present for the reduced domain simulations. It can be observed that the control taken at the side of the crystal is influenced by reflections. It is of a less consistent nature relative to the other controls with propagation through free space.

(b) Comparison of the experimentally measured attenuation spectrum with those calculated for the large and reduced domain geometries. The reduced domain with a 10 x 10 array (red-crosses) shows the best agreement.

**Figure 5.11:** Finite element (a) frequency spectra and (b) attenuation spectra.
long processing times involved. The simulation was run up to 10kHz, which required a maximum element size of \(2.86 \times 10^{-3} \text{m}\) in order to ensure that there were at least 10 elements per wavelength to accurately model the wave. Typically the mesh consisted of approximately \(\sim 10^5\) elements for these large domain simulations.

To increase the accuracy of the frequency spectrum result, a second geometry is defined (Figure 5.10). Several reduced domains were tested, which were significantly less memory intensive, due to the decreased number of elements (\(\sim 10^3\)). Therefore, it is possible to obtain more solutions for the parametric value \(f\), increasing the resolution of the resulting spectrum. It was expected that the reduced geometries would provide less accurate results, as the situation defined is a less faithful representation of the experimental configuration, neglecting the effects of diffraction. However, a comparison of the spectrum with that obtained experimentally (Figure 5.11(b)) shows a much improved agreement. Attenuation for the \(10 \times 10\) reduced domain observed from 5.56-9.42kHz.

Furthermore, the central frequency, \(f_c = 7.48\text{kHz}\), of the band gap predicted using finite element method, agrees with the theoretical estimation, \(f_c = c/2a = 7.80\text{kHz}\), verifying the accuracy of the model. The domain was reduced further to contain 64 cylinders in an \(8 \times 8\) array, 36 cylinders in a \(6 \times 6\) array and 16 cylinders in a \(4 \times 4\) array, corresponding to numbers of elements of \(64 \times 10^3\), \(39 \times 10^3\), and \(19 \times 10^3\) respectively. It is shown that the smaller arrays were perfectly adequate at describing the behaviour of the experimental results, however the agreement deteriorates as the number of scatterers decreases.

A second spike of attenuation positioned around 13.3kHz is observed in the reduced domain spectra. This corresponds to the second Bragg band gap, the first harmonic, at twice the fundamental band gap frequency. The wavelength of the sound is doubled, however the minima and maxima of the wave still coincide with the periodicity of the lattice and exist either on the scatterers or at the equidistance between. The harmonic band gap is of a reduced bandwidth, with higher Q-value than the fundamental, in all of the reduced domains investigated.

The attenuation levels measured experimentally were larger than those predicted by the \(10 \times 10\) reduced domain, peaking at 33dB and 9dB at the most severely attenuated point of the band gap respectively. Multiplication by a scale factor of 3.66 normalises the data, to account for the apparent inaccuracies in the model. The disagreement likely arising from the reference pressure level defined in the model.
5.4.2 Attenuation $\Gamma M$

Two further geometries were defined to simulate sound transmission along the $\Gamma M$ plane of the square lattice. A large domain to accurately represent the experiential set up, which considered the effects of diffraction and reflections, and also a reduced domain neglecting these influences, which resulted in a higher resolution spectrum. A comparison of the control and sample traces reveals that the sound pressure level is suppressed for all frequencies above 8kHz, with a good agreement between both the large and reduced domain geometries (Figure 5.12). The corresponding attenuation level is very similar in nature to that of the experimental data (Figure 5.13(a)), which as aforementioned is also consistent with the results of Robertson and Rudy, for $\Gamma M$ transmission through an array of 23.4mm diameter conduits.

The behaviour of the spectra is unusual with no defined band gap formation, in the theoretically estimated location. It was hypothesized that the drop off above 8kHz could be a consequence of the phase difference across the surface of the sample exceeding $\lambda/8$, i.e. failure of the plane wave approximation. However, the reduced size geometry, ensures full plane wavefronts are incident, and despite this, the spectrum exhibits the same behaviour. Furthermore, the experimental and finite element data for the $\Gamma X$ direction are in good agreement, eliminating this possibility. The explanation for the unusual profile of the $\Gamma M$ spectra is that we are only displaying a portion of the band gap. The region of attenuation observed in the experimental and finite element data, does indeed correspond to the first Bragg band gap, however, the frequency scale is insufficient to observe the end frequency. To investigate this possibility, a simulation was performed across an extended frequency range spanning 1-24kHz in the reduced domain. The resulting spectrum illustrates that an increasing level of attenuation is achieved, beginning at 6.2kHz and reaching a maximum at 16.1kHz (Figure 5.13(b)). It is only above this frequency that the amplitude increases and the band gap finishes. Therefore it can be concluded that the experimentally obtained frequency spectra do not extend a sufficient frequency range to identify the end of the attenuation band.

The extended frequency spectra shows a band gap for $\Gamma M$ transmission reaching a maximum at 16.1kHz and the experimental attenuation data agrees with this trend up to the highest measured frequency, just before this at 16kHz. Neither set of data however, predicts a band gap located at the theoretically estimated frequency of 11.02kHz. Attenuation is observed around this frequency, however there is no evidence of a defined Bragg diffraction peak. As aforementioned this behaviour was observed by other authors. The nature of the band gap and its lack of definition, is a consequence of the phononic crystals packing fraction. The packing fraction chosen of 0.274 is below the
critical value suggested by Kushwaha of 0.3 and as such a band gap does not open. In spite of this, the extended frequency finite element data, displays a second peak of attenuation located at 22.2kHz. This is twice the theoretically estimated fundamental Bragg frequency and is therefore attributed to the first harmonic spectral gap. The location of this second band, although with a large degree of error, ±500Hz, appears as predicted by theory whereas the first band gap does not. The finite element result suggests that the phononic crystal structure of 6.5mm diameter cylinders in air with periodicity 0.022m, does not possess sufficient packing fraction to open the first Bragg gap, however it does display attenuation as expected for the second Bragg frequency.

![Figure 5.12](https://example.com/figure5.12.png)

**Figure 5.12:** Frequency spectra obtained via the FE method for Γ\(M\) transmission. The presence of the array gives rise to increasing levels of attenuation beginning at 8kHz continuing through the highest frequency at 16kHz. This is observed as the sample spectra deviate from the control. The frequency scale represents the experimentally measured range.

\[\text{†This assumption is verified by a numerical analysis investigation of the influence of packing fraction in the following Section 5.4.6.}\]
(a) Comparison of the experimentally measured attenuation levels and the FE model for $\Gamma M$ transmission. The attenuation in the experimental data around 10kHz, coincides with a small rise observed in the reduced domain spectra.

(b) Extended frequency spectrum for $\Gamma M$ transmission, from the finite element method. Due to the high frequency ($f_{\text{max}} = 24\text{kHz}$), the maximum element size is now reduced to $\lambda_{\text{max}} = 0.00143$. For the large domain this corresponds to $10^7$-elements. As such only 12 frequency solutions can be obtained, restricting the spectrum to a 2kHz resolution.

**Figure 5.13:** $\Gamma M$ transmission. (a) Comparison of the frequency spectra from FE, for the large (red) and reduced domain (blue) geometries, with experimental data, and, (b) an extended frequency range simulation.
5.4.2.1 Pressure Maps

The finite element method solves for the pressure magnitude at all points in the meshed domain. This allows the production of surface plots to visualise the solution in two-dimensions. The minima and maxima of the solution are represented by opposing colours with an appropriate transition between. As a standard, COMSOL Multiphysics produces jet colour maps, which are a variation of the HSV colour map that is associated with astrophysical fluid jet simulations from the National Centre for Supercomputer Applications. The lower end of the scale begins with dark blue, then progresses through shades of blue, cyan, green, yellow, red and ending in dark red, representing maximum values. The technicolour diagrams of Figures (5.14) and (5.17) show two-dimensional surface plots of the decibel intensity distribution at various frequencies, for the large and reduced domain geometries respectively, with $\Gamma X$ transmission. The distribution of the instantaneous pressure field is dependent upon the acoustic waves interaction with the phononic crystal structure.

For the large domain simulations, sound is radiated from the linear plane wave at the left boundary of the domain. The wave is confined to propagate down a short waveguide and diffraction effects, when the wavefront exits, cause circular wavefronts to emerge. The phononic crystal, however, is of a sufficient distance from this source, to ensure a full plane wavefront reaches the crystal, i.e. the components of the wavefront incident upon the edge and centre of the samples surface will arrive less than $1/8\lambda$. This is consistent with the experimental procedure.

At low frequencies (Figure 5.14(a)), the inhomogenities represented by the metal cylinders are of subwavelength dimensions. As such, the acoustic wavelength is too large to resolve the periodicity and no Bragg diffraction occurs. The propagating wave experiences the phononic crystal as a homogeneous medium of an increased acoustic impedance. The impedance mismatch between outside and inside the array, $Z_{out}/Z_{in} = \rho_{out}c_{out}/\rho_{in}c_{in}$, governing the proportion of acoustic intensity which is transmitted and reflected. In this first low frequency example, the majority of the energy is transmitted through the crystal, however there is a small reflected component. Interference phenomena occurring between the incident wave and the reflected component, leads to the formation of a standing wave pattern in front of the crystal. These are the equally spaced dark red fringes of an increased acoustic intensity.

The pressure field distribution is not uniform throughout the domain at this low frequency (Figure 5.14(a)). Components of the wavefront incident upon the edge of the sample experience diffraction through some angle. The influence of such diffraction effects around small sized phononic crystals (too small to be defined by an effective
(a) Propagation of an acoustic wave with frequency 4.35kHz. This corresponds to a $k$ vector in the first allowed branch of the band structure.

(b) Propagation at the Bragg frequency of 8.60kHz. Transmission is forbidden and a shadow zone is formed.

(c) Transmission at 9.85kHz, in the second allowed branch of the band structure.

**Figure 5.14:** Two-dimensional surface plots of the acoustic pressure (dB) distribution through the large domain.
medium theory), was first reported by N.García et al. [68]. It was shown that in the low frequency regime in the first allowed band, symmetrical diffraction effects around the sample can lead to the formation of a focused region of high intensity behind the crystal. The components of the wavefront which travel around the edges of the sample experience diffraction through some angle, $\phi_{\text{diff}}$, at the back surface of the array. A line plot is taken through the section $AA'$ of the domain, through the focused point, to reveal the magnitude of the focusing (Figure 5.15).

![Figure 5.15: Cross-sectional line plot, through the section $AA'$. At 4.35kHz the incident acoustic wavelength is larger than the periodicity, and the wave experiences it as a homogeneous medium. Diffraction effects around the sample give rise to a focussed region.](image)

A second pressure map is included to illustrate the field distribution for a frequency within the spectral gap (Figure 5.14(b)). The snapshot represents 8.6kHz, corresponding to the frequency at which the attenuation function reaches its maximum. The wavelength is comparable to the periodic spacing of the phononic crystal, fulfilling the Bragg diffraction condition. Transmission of the incoming acoustic wave is forbidden and the majority of the intensity is reflected. This leads to the formation of a shadow zone behind the phononic crystal. Furthermore, intense reflections from the surface of the sample lead to a standing wave of even greater amplitude for band gap frequencies. At such frequencies, the reflections are comprised of a component from the impedance reflection, which are reinforced by a second component from Bragg reflections. The reflectance properties of phononic crystals were first studied by L.Sanchis et al. [69].
The pressure field distribution of the standing wave field in front of the sample was measured experimentally, utilising a robotic probe attached to motorised grid of tracks. The total pressure in front of the crystal is the superposition of the incident and reflected wave,

\[ \cos(kx - \omega t) + R \cos(-kx - \omega t) = A(x) \cos(-\omega t - \varphi(x)), \]  

(5.5)

where \( R \) is the reflection coefficient, \( \varphi(x) \) is the phase angle, (which is irrelevant for \( 0^\circ \) incidence), and \( A(x) = \sqrt{1 + 2R \cos 2kx + R^2} \). When \( R = 1 \) a full standing wave is formed, generally \( R \neq 1 \) and a partial standing wave is formed. The standing wave ratio is defined by,

\[ SWR^2 = \frac{P_{\text{max}}^2}{P_{\text{min}}^2} = \frac{(1 + R)^2}{(1 - R)^2}, \]  

(5.6)

where \( P^2 \) is the mean square pressure. To determine the standing wave ratio from the COMSOL simulations, we work in terms of the root mean square pressures,

\[ SWR = L_{P_{\text{max}}} - L_{P_{\text{min}}} \]
\[ = 10 \log_{10} \left( \frac{P_{\text{max}}}{P_{\text{ref}}} \right)^2 - 10 \log_{10} \left( \frac{P_{\text{min}}}{P_{\text{ref}}} \right)^2 \]
\[ = 20 \log_{10} \left( \frac{P_{\text{max}}}{P_{\text{min}}} \right), \]  

(5.7)
where $L_{P_{\text{min}}}$ and $L_{P_{\text{max}}}$ are the root mean square pressure levels in dB, $P_{\text{ref}} = 96.4$ dB is an arbitrary value, and $P_{\text{max}}$ and $P_{\text{min}}$ are the maximum and minimum values of the root time averaged pressures respectively. The band gap region is then identified by analysing the standing wave ratio as a function of frequency, at frequencies where reflections are enhanced, observed between 5.6-10 kHz (Figure 5.16).

A final surface plot of the pressure field distribution is taken at 9.85 kHz (Figure 5.14(c)). This is above the first band gap, where $k$ has a real positive value in the second allowed band of the band structure. Transmission is permitted and the sound propagates through the array, aside from a small reflected component due to the impedance mismatch. The amplitude of the pressure is therefore of an approximately equal amplitude in front and behind the crystal.

Three further surface plots are included in Figure (5.17) for the reduced domain. They represent frequencies within the first allowed branch (Figure 5.17(a)), inside the first band gap (Figure 5.17(b)) and in the second allowed branch (Figure 5.17), at 4.6, 7.8 and 11.1 kHz respectively. The diagrams illustrate similar behaviour to the large domain simulations. As a point of note, the pressure distribution throughout the domain is not uniform for the reduced domain surface plots. This is due to the radiation boundary conditions applied, which are defined to absorb plane waves, as opposed to waves propagating perpendicularly to the surface.
(a) Pressure field at 4.6kHz, before the first band gap. The incoming acoustic plane wave is permitted to propagate through the array. The pressure field distribution is not even throughout the domain, with regions of low pressure (blue) at the upper and lower edges of the domain. This is a consequence of the radiation conditions applied to these boundaries, which is discussed further with respect to the phase surface plots (Figure 5.20).

(b) Pressure (dB) at 7.8kHz, the central frequency of the first Bragg band gap. The incoming wave is reflected, giving rise to the formation of a standing wave in front of the crystal and a shadow zone behind the crystal.

(c) Transmission of sound in the second allowed branch of the band structure at 11.1kHz. The pressure amplitude (dB) is approximately equal, pre and post-crystal.

**Figure 5.17**: Surface plots of the pressure field (dB) [a] before, [b] during, and [c] after, band gap formation.
5.4.3 Phase Data

In a similar manner to the experimental technique, the finite element calculated phase data can be used to determine the sound velocity inside the phononic crystal. Consequently the acoustic dispersion relation can be constructed. In themselves points of anomalous dispersion in the phase data also provide an accurate indication of band gap location. The instantaneous phase at a single point in the domain is taken relative to an arbitrary zero. It is found by dividing the complex part of the waveform in the frequency domain by the real component,

\[ \varphi = \tan^{-1} \left( \frac{\Im(P)}{\Re(P)} \right) \].

The instantaneous phase, obtained using the finite element method, for transmission along the \( \Gamma X \) plane of a 10 \( \times \) 10 phononic crystal array of metal cylinders is shown in Figure 5.18. Each cylinder has radius 6.5mm arranged in a square lattice with constant \( a = 0.022m \). The edges of the exterior domain being a single lattice period, \( a \), from the edge of the sample, in order to reduce processing time.

The blue trace represents the sample location, positioned behind the phononic crystal array. Phase for a normal dispersive medium has a linear dependence upon frequency with a negative slope. This behaviour is observed from 0-6.23kHz, the \( k \) vectors for frequencies in this region corresponding to those in the first allowed band state. At 6.23kHz an anomalous point is observed in the phase data, where the gradient becomes positive and the linearity is broken. This occurs when the wavelength of the incident sound is comparable with the periodicity of the crystal, and the corresponding \( k \) vector ends upon the edge of the first Brillouin zone. As such the Bragg condition is fulfilled and the incident wave is diffracted (reflected) back. In the band gap \( k \) is undetermined, causing the phase to briefly exhibit a positive slope. Anomalous dispersion continues until 6.60kHz where the phase regains its constant linear profile, observed as regaining its constant period. The central frequency of this non-linear behaviour at, 6.42kHz, represents the central frequency of the phononic crystal structures first band gap. It is positioned lower than the theoretical estimate of 7.80kHz and the experimental result of 7.79kHz. This is a consequence of the finite size of the array used in the time-harmonic finite element investigation.

Above this first anomaly the phase regains its linear profile in the region from 6.60-13.86kHz, representing the allowed \( k \) values of the second band state, before a further point of anomalous positive gradient occurs at 13.86kHz. This second spike is much sharper than the first extended anomalous region. It denotes the centre frequency of the second band gap, an integer harmonic of the first. As observed in the attenuation
data the finite element predicts the harmonic to be of a smaller bandwidth, and this is consistent with the phase result.

For $\Gamma M$ transmission, a point of anomalous dispersion is identified at 9.84kHz in the post-crystal spectrum, lower than the theoretical estimate of 11.02kHz (Figure 5.19). This peak does however coincide with the maxima observed in our experimental and finite element attenuation spectra. We shall see in the next section how, this is a consequence of the low filling fraction of the sample. Flat regions of anomalous dispersion exist in the pre-crystal phase spectra between 8-10kHz and again from 13kHz through to the highest frequency.

The central frequencies predicted by the FE phase data are lower than the theoretical estimates in both high symmetry planes of the crystal. This is likely a consequence of the finite size of the array. The theoretical estimate assumes an infinite repeating crystal, the periodicity allowing the behaviour of waves in the crystal to be characterised completely by one Brillouin zone. Finite element investigations upon smaller arrays, revealed a weak

![Figure 5.18: Phase profile obtained via the finite element method for the $\Gamma X$ plane, $\tan^{-1}\{\Im(p)/\Re(p)\}$. The sample spectrum (blue) extracted after the array (location B of Figure 4.4) has two points of anomalous dispersion centered at 6.43kHz and 13.86kHz marking the 1st and 2nd Bragg diffraction frequencies respectively. The profile of the pre-crystal control spectrum (red) is significantly more complex, due to interference from reflected components, anomalous dispersion being identified as flat regions between 5.84-8.90kHz and 13.78-14.90kHz.](image-url)
Figure 5.19: Phase profile, $\tan \{\Im(p)/\Re(p)\}$, obtained via the finite element method for the $\Gamma M$ plane. A point of anomalous dispersion is identified in the sample spectra at 9.84kHz corresponding to the 1st band gap. Anomalous dispersion is identified in the pre-crystal control trace (red) between 8.05-10.01kHz and also from 13.04kHz through to the highest measured frequency.

dependency upon the position of the anomalous dispersion and the number of layers in the array. The weakness of the dependency arising from the fact that only two layers were removed in each case. A simulation performed with $20 \times 20$ scatterers configured in a $\Gamma X$ array, illustrates a point of anomalous dispersion in the post-crystal phase data at 7.13kHz. This is closer to the theoretical estimate than for the $10 \times 10$ array. Therefore, it can be concluded that as the number of layers increases the location of the anomalous phase points converge upon the theoretical estimates.
5.4.4 Surface Plots

Surface plots of the instantaneous phase are included for three frequencies, before, during and after the band gap formation (Figure 5.20). Wavefronts of constant phase propagate across the domain from left to right. The wavefronts are slightly curved in nature due to their interaction with the radiation condition applied to the upper and lower boundaries. Radiation conditions are formulated to absorb waves plane onto the surface. As our waves propagate perpendicular to these boundaries, the boundary conditions are not ideal for the situation, however the effect of this curvature is not sufficient to change the conclusions drawn from the results. Furthermore the magnitude of the curvature decreases with frequency, increasing the accuracy of the plane wave assumption proportionally with frequency. It is the curvature of the wavefronts that is responsible for the regions of low pressure amplitude at the lower and upper boundaries of the domain observed on the pressure (dB) field surface plots (Figure 5.17).

The low frequency example (Figure 5.20(a)), corresponds to a $k$ value in the first allowed band of the dispersion relation. Wavefronts experience some delay as they pass through the sample, due to the reduction of the velocity caused by the increased impedance of the medium. No Bragg diffraction occurs inside the structure, and the emerging wavefront is undisturbed being close to plane in nature.

The surface plot at 7.80kHz (Figure 5.20(b)) corresponds to a $k$ vector value that fulfills the Bragg condition ($k_{BZ}$), indeed this is the maximum point of the attenuation spectrum (Figure 5.3). Intense scatterings occur inside the structure, as the wavelength of the sound is comparable to the period of the scatterers, leading to complex interferences. Wavefronts emerging from the crystal are no longer of constant phase and plane, giving rise to the point of anomalous dispersion identified in the phase spectrum, at this frequency. The distribution of phase in front of the sample, reveals that in this area the phase has large positive values (orange and red). This is a consequence of the standing wave formed by interference between the incident and reflected wavefronts. If we superimpose two waves, then similarly to the addition of amplitudes, complex notation tells us that the phase values can also be summed.

The final frame (Figure 5.20(c)), depicts the phase distribution for a $k$ value in the second allowed band. Similarly to the example in the first allowed band, wavefronts incident upon the phononic crystal are free to propagate, experiencing a small delay, before emerging undisturbed and as close to plane in nature as when they entered.
(a) Phase surface plot at 4.6kHz, in the first transmission band. The alternating regions of minima and maxima, define the start of each new wavefront where the phase value shifts from $\pi$ to $-\pi$.

(b) Phase distribution at the central frequency of the band gap, 7.8kHz, where transmission is forbidden. Incident plane wavefronts are reflected back on themselves.

(c) Plane wavefronts propagating through the phononic crystal undisturbed at 11.1kHz, in the second branch of the band structure. It can be observed that the curving of the wavefronts at the top and bottom of the domain is less significant due to the reduced wavelength.

**Figure 5.20:** Surface plots of the instantaneous phase, $\tan^{-1}\{\Im(P)/\Re(P)\}$, for a $10 \times 10$ array of steel cylinders in air. Frequency snapshots are taken (a) before, (b) during, and (c) after, band gap formation.
5.4.5 Phase Difference and Dispersion Relation

The phase delay, pre and post crystal, can be found at each frequency through,

\[ \varphi = \varphi_B - \varphi_A = \tan^{-1} \left( \frac{\Im(P_B)}{\Re(P_B)} \right) - \tan^{-1} \left( \frac{\Im(P_A)}{\Re(P_A)} \right). \]  

(5.11)

Using this measurement, the phase velocity can be determined from the finite element data using the same process as detailed for the experimental data (Equation 4.3). The dependence of \( \omega \) with \( k \) can then be found to obtain the dispersion relation. Two such plots are shown in Figure (5.21) for transmission along the \( \Gamma X \) plane (Figure 5.21(a)) and \( \Gamma M \) plane (Figure 5.21(b)). Dispersion relation has a linear dependence with \( f \) with two point of anomalous dispersion at 5 and 10kHz, in the \( \Gamma X \) direction and 7 and 13kHz in the \( \Gamma M \) direction.
(a) Dispersion relation from the FE phase data. Anomalous dispersion at $\sim 5$kHz and $\sim 10$kHz define the start and finish frequencies of the band gap respectively.

(b) Two points of anomalous dispersion are identified at $\sim 7$kHz and $\sim 13$kHz. This indicates the presence of a band gap centered around 10kHz.

**Figure 5.21**: Dispersion relation for (a) $\Gamma X$, and (b) $\Gamma M$ transmission.
5.4.6 Band Structure of the RCPC

Figure 5.22: Band structure calculations for the experimentally tested unit cell with a packing fraction 0.274. A fundamental band gap opens in the ΓX direction spanning from 6.58-8.47kHz, however the structure does not possess an absolute band gap.

As a final investigative tool we have obtained numerical verification of the band structure using the finite element method. As detailed in Section 3.5.2 the Eigenfrequency analysis is used to calculate the pressure Eigenmodes and Eigenfrequencies of a single unit cell. We imagine the fluid to be comprised of a set of such periodic units of length $L_z$, and insist that at the extremities of the system, periodic boundary conditions are applied, so that plane waves, and not standing waves are basis states. The propagation may then be described entirely in terms of one ‘unit cell’ from 0 to $L_z$. Periodicity in real space leads to periodicity in reciprocal space, so the entire dispersion relation for all frequencies is plotted in the reciprocal space unit cell (the Brillouin zone) for which $|kL_z| \leq \pi$.

The band structure obtained via the finite element for the experimental geometry (Figure 4.1) is shown in Figure 5.22. The reduced frequency is plotted on the vertical axis which is found through $\{(c/a)/2\pi\}^{1/2}$. This is conventional notation, that still represents the same total frequency range investigated experimentally 0-15kHz. Included within this range are the first ten allowed bands for both ΓX and ΓM transmission. All
branches of the structure display some nonlinear behaviour due to the presence of the inhomogenities causing the structure to be a dispersive medium. However, band gaps only open for sound transmitted in the $\Gamma X$ direction. A fundamental spectral gap is identified between the first and second branches ranging from 6.58-8.47kHz. This is centered at 7.52kHz, further verifying the experimental result and finite element wave propagation simulations.

In the $\Gamma M$ and $XM$ directions no band gaps are present. This is because the packing fraction of the sample investigated in this thesis is 0.274, below the theoretical critical limit of 0.31 (Kushwaha [26]) required to open a complete acoustic band gap. By increasing the radius of the cylindrical scatterers in the unit cell from 6.5mm to 7.34mm, with the same lattice constant of $a = 0.022m$ the packing fraction of the array increases from 0.274 to 0.35 (above the critical limit). It can be observed that this structure possesses a complete acoustic band gap in all directions and for all frequencies between 8.37-8.57kHz (Figure 5.23). The absolute band gap is centered at 8.47kHz, in agreement with the theoretical estimate of 7.795kHz.

![Band structure calculation for a phononic crystal with lattice constant, $a = 0.022m$ and packing fraction 0.35. The structure has sufficient packing density to open a complete acoustic band gap in all directions, namely across all three of high symmetry directions the $\Gamma M$, $\Gamma X$ and $XM$.](image-url)
A further band structure calculation is performed for the square lattice of rigid cylinders. The lattice parameter of the elementary unit cell remains a constant size with $a = 0.022\text{m}$, however the radius of the scatterer is increased yet further to 9.61mm. This unit cell corresponds to a packing fraction of 0.6. The complete acoustic band gap is much wider than the sample with packing fraction 0.35, spanning from 7.32-9.17kHz. The increase in bandwidth stemming from the rise in packing fraction. Figure 5.25 shows the dependence of band gap width upon packing fraction for our chosen lattice parameter. The dependence follows relation similar to that of published literature \[35, 36, 37, 38, 65\]. The maximum packing fraction occurs at 0.74, which represents the scenario in which the cylinders are of sufficient radius to touch their neighbour. Illustrated in Figure 5.26 are the Eigenmode solutions of the band structure, for the periodic boundary condition simulation in COMSOL. The frames represent the first nine Eigenmode solutions for $\Gamma X$ transmission, through the same crystal with packing fraction 0.274, used in experiments. Each mode displays a distinct pressure field distribution that occurs between the scatterers in the periodic array. The Eigenfrequencies are plotted on the $\Gamma X$ axis of Figure 5.23.
Figure 5.25: Dependence of band gap width upon the crystal packing fraction. The plot represents the evolution of the absolute band gap, which is present for all directions of transmission.

Figure 5.26: Eigenmode solutions of the single unit cell. The pictures included show acoustic transmission along the ΓX plane, through the experimental geometry of a 6.5mm rigid cylinder with periodicity $a = 0.022m$. 
Chapter 6

Locally Resonant Phononic Crystal

So far the discussion in this thesis has been solely concerned with phononic crystals comprised from solid scattering inclusions, be they two-dimensional cylindrical rods or three-dimensional spheres. The periodical arrangement of the scatterers in the structure causes strong modulations in the density and sound velocity, creating spectral gaps that forbid wave propagation, as observed in Chapter 5. A band gap of this type, that is a consequence of the periodic nature, will from herein be called a ‘Bragg’ type band gap. It has been coined so, due to the Bragg diffraction mechanism that causes it to arise and governs its behaviour.

This chapter details another class of phononic crystal fabricated from inclusions which are capable of exhibiting local resonances. It begins with an introduction to literature in the field, before detailing the locally resonant phononic crystal with Helmholtz resonator inclusions, which are the subject of study in this thesis. Transmission characteristics are measured experimentally and verified via the finite element method.

6.1 Introduction

The location of the Bragg band gap of a phononic crystal can be estimated using $f = v/2a$ (Chapter 5.1). Assuming a constant fluid medium, the wavelength of the attenuated sound is therefore approximately proportional to twice the lattice spacing, and this established relationship has been shown to be reliable in the experimental results detailed so far. As the wavelength of attenuation and the periodicity of the scatterers are two intrinsically linked parameters, the dimensions of a phononic crystal are dictated by its
operating frequency. This means there is very little flexibility in the overall size of a sample when designing a crystal for potential application. It is therefore desirable to be able to exhibit some control over the band gap location, independent of the periodicity. This offers the ability to use phononic crystals for a wider range of applications where designs may be subject to dimensional restrictions and stipulations.

To overcome this inherent restriction imposed, we could consider changing the host matrix material in which the scattering units are embedded. This would directly change the sound velocity in the medium and shift the location of the band gap. This is a flexible method applicable to solid-solid systems. However, the technique is flawed for practical solid-fluid crystals designed for the attenuation of environmental noise. Such phononic crystals require air to be the host medium, to reduce costs. For such low frequency acoustic noise, air is a particularly suitable choice of host medium with a low sound velocity. Therefore, by replacing this with almost any other material, the sound velocity would increase and hence so would the operating frequency of the crystal. This does not fulfill the objective of forming spectral gaps at lattice constants smaller than the relevant phononic crystals wavelength, in fact quite the opposite. The introduction of another host medium also introduces extra costs not incurred when using ‘free’ air.

For completeness, it should be mentioned that the speed of sound in the matrix can also be influenced by environmental parameters such as the temperature and pressure. As a point of interest the reader is directed to a paper by L.Wu et al. [70], reporting a technique to create a tunable phononic crystal by varying the temperature of the surrounding medium.

Aside from adjusting the periodicity, or the matrix sound velocity, there is no other variable that can influence the central frequency around which a Bragg band gap is situated. Therefore, the only way to tailor a phononic crystal to operate at wavelengths below the lattice parameter, is to introduce a second band gap mechanism. The first example of such a structure was reported in a seminal paper by Z.Liu et al. entitled “Locally Resonant Sonic Materials” (LRSM) [71]. A phononic crystal was detailed in which each constituent scatterer exhibited a vibrational resonance. This phenomena was observed to lead to the formation of an additional band of forbidden transmission. As the location of the second ‘resonance’ band gap is determined by the geometry of the scatterers, it is independent of the lattice spacing.

The locally resonant sonic material studied by Z.Liu et al. [71] was comprised of simple microstructure units consisting of a solid core material with a relatively high density and a coating of elastically soft material. In the experiments, centimetre sized lead balls were used as the core material with a 2.5mm layer of silicone rubber. The coated spheres were arranged in a $8 \times 8 \times 8$ simple cubic crystal with lattice constant $a = 1.55$cm. As
the core is free to vibrate in its shell, each unit has an associated resonant frequency. These locally resonant structural units can exhibit effective negative elastic constants at certain frequency ranges. Therefore, if a travelling wave with angular frequency $\omega$ interacts with a medium carrying the local excitation with frequency $\omega_0$, the linear response functions will be proportional to $1/(\omega_0^2 - \omega^2)$. Such an effect is manifest in the electromagnetic frequency response of materials with optical resonances, where a negative dielectric constant $\varepsilon$ implies a purely imaginary wave vector $k = n\omega/c$ (where $n$ is the index of refraction and $c$ the speed of light) and hence exponential attenuation of the electromagnetic wave.

At frequencies away from resonance, the dispersion relation ($\omega$ vs. $k$) of the LRSM structure is linear, as the composite medium behaves as an effective medium in which the elastic waves scatter weakly from the subwavelength scatterers. The localised resonances, however give rise to flat dispersions that are nearly $k$-independent. Coupling with the otherwise linearly dispersed elastic waves opens spectral gaps in the band structure due to the level repulsion effect. The sample was shown to possess two such spectral gaps. At the first gap frequency, the lead core moves as a whole along the direction of wave propagation, with a large strain at the lead-silicone interface. This low frequency resonance was therefore attributed to an oscillation, in which the inner core provides a heavy mass and the silicone rubber provides the soft spring. At the second resonance gap the maximum displacement was observed inside the silicone rubber. The displacement of the core is small but non zero. This is analogous to the ‘optical mode’ in molecular crystals with two atoms per unit cell, where one of the atoms is significantly heavier than the other. Around the resonant frequencies, the response function has large dispersion, leading to a dip in the region of negative elastic constants and hence exponential wave attenuation, followed by resonant transmission when the effective elastic constants satisfy the required condition. The dip is seen to define the lower edge of the band gap, whereas the peak corresponds to the upper boundary.

Further research into the field of locally resonant phononic crystals focussed on similar structures with heavy scattering cores embedded in a soft coating [72][73][74][75]. The resonance mechanism has been extensively studied by C.Goffaux et al. [76], where it was shown that the resonant characteristics could be accurately described by a simple one-dimensional spring and mass model [77]. It was observed that the locally resonant phononic crystals with heavy cores in a soft coating, give rise to a distinctive resonance [78]. The profile has an asymmetric line shape, typical of a Fano resonance, as first reported for the $2s2p^1P$ resonance of He, observed from the inelastic scattering of electrons [79].
6.2 Helmholtz Resonator Inclusions

The first examples of locally resonant phononic crystals all made use of the same constituent units, namely a heavy core coated in a soft layer. Alternatively though there is an entirely different class of resonator that can be substituted for these units. Cavity resonators such as standing wave pipes, quarter-wave tubes and Helmholtz oscillators all exhibit resonance behaviour, the excitation frequency being entirely dependent upon the dimensions of the unit. Such devices can therefore form band gaps independent of the structures lattice spacing. When configured into an array, the local excitation of each resonator causes attenuation of an incident travelling wave, similarly to the previously discussed locally resonant phononic crystals.

It was shown by Z.Hou et al. [80] that a phononic crystal comprised of Helmholtz resonators is a good material for constructing an acoustic lens due to the array’s low acoustic impedance. This will be discussed further in Chapter (7) with respect to refraction phenomena in phononic crystals. For now however, we simply consider the resonance spectral gap formed by Helmholtz resonators. It has a profile of the classical Breit-Wigner (Lorentzian) resonance, which typically have a bandwidth significantly larger than the Fano like resonance gaps reported for the spring-mass LRPC samples. Therefore Helmholtz resonator devices are more suitable for the construction of phononic crystal acoustic barriers, as they are capable of shielding a broader range of noise.

6.2.1 Theory of the Helmholtz Oscillator

A Helmholtz resonator is a container of gas, to which air flow from the external environment is permitted through an open hole, or neck. If an acoustic wave from an external source interacts with the air in the neck it applies a force, increasing the pressure inside the cavity. As the pressure of the acoustic wave oscillates the force exerted decreases and is removed. The air in the cavity is now at a higher pressure than the external medium and this difference serves to push the mass of air in the neck outwards. Due to the inertia of the air, it overshoots its original stationary position, now causing the pressure to fall in the cavity. The mass of air in the neck is therefore pulled back in and it continues to oscillate in such a manner with a frequency, $\omega_0$.

This behaviour can be described quantitatively to obtain an expression for the resonant frequency. Consider a container of volume, $V$, open to the atmosphere through a neck of length, $L$, and of cross-section, $S$. We assume that the pressure, $p_{\text{in}}(t)$, and density, $\rho_{\text{in}}(t)$, inside the container are uniform and depend only on time, and we denote the

\[ \omega_0 \]
pressure in the duct by, \( p(x, t) \). The total mass of air in the neck, \( m \), can be found through,

\[
m = \rho_0 SL', \quad \text{where} \quad L' = L + 1.7a.
\tag{6.1}
\]

If we assume an airtight piston is pushed into the neck, the compression of the fluid in the cavity provides a stiffness [81]. If the piston is moved by a distance, \( x \), the volume of the cavity is changed by \(-\Delta V/V\), resulting in a condensation \( \Delta P/P = -\Delta V/V = Sx/V \). The pressure increase is,

\[
p = \rho_0 c^2 \frac{\Delta \rho}{\rho} = \rho_0 c^2 \frac{Sx}{V}.
\tag{6.2}
\]

The force required to maintain the displacement is \( f = Sp = sx \), where the effective stiffness, \( s \), is found through,

\[
s = \rho_0 c^2 S^2 / V.
\tag{6.3}
\]

The open neck of the resonator radiates sound, providing radiation resistance and a radiation mass. Furthermore, the motion of the fluid in the neck, moving as a unit, provides another mass element. The thermoviscous losses at the neck walls provide additional resistance. The total resistance \( R_m \) is the sum of the radiation resistance \( R_r \) and the resistance \( R_w \) arising from the wall losses. The instantaneous complex driving force produced by a pressure wave of amplitude \( P \) impinging on the resonator opening is described as,

\[
f = S Pe^{i\omega t}.
\tag{6.4}
\]

Therefore, the differential equation for the inward displacement \( x \) of the fluid in the neck is given by,

\[
m \frac{d^2 x}{dt^2} + (R_r + R_w) \frac{dx}{dt} + sx = S Pe^{i\omega t}.
\tag{6.5}
\]

The input mechanical impedance of the resonator is,

\[
Z_m = (R_r + R_w) + i (\omega m - s/\omega),
\tag{6.6}
\]

and resonance occurs when the reactance goes to zero,

\[
\omega_0 = c \sqrt{S/L/V}.
\tag{6.7}
\]

The preceeding hypotheses are only valid provided the section \( S \) of the neck is very small compared to the surface of the walls of the cavity (of volume \( V \)) and the dimensions of the resonator (among which is the length of the tube \( L \)) are very small compared to the wavelength \( \lambda \) considered (that corresponds to the resonant frequency of the whole system). This can be written as,

\[
\sqrt{S} \ll \sqrt[3]{V} \ll \lambda, \quad \text{and} \quad L \ll \lambda.
\tag{6.8}
\]
6.2.2 Individual Two-Dimensional Resonator

A simple two-dimensional Helmholtz resonator was studied theoretically using the finite element technique. The excitation frequency was determined for comparison with the calculated value from Equation (6.7). A circular cavity is modelled with an opening in one side to allow air to enter. This split ring Helmholtz resonator is itself contained within a rectangular domain, through which a plane wave propagates from the left hand side to the right. The pressure field distribution inside the domain is then solved parametrically as a function of frequency. It is possible to then identify the resonant frequency, where the pressure inside the cavity is amplified to its maximum level, as the air exerts a force.

Figure 6.1: Resonance curve of the single Helmholtz inclusion.

Figure 6.1 shows the frequency spectrum synthesized at the central point of the cavity, the inset illustrating the relevant dimensions. The fundamental resonance peak is situated around 4.8kHz, lower than the theoretical result obtained from Equation (6.7) of 5.56kHz. The resonance curve has a bandwidth $\Delta \omega$ of 1kHz at $I(\text{max})/\sqrt{2}$, therefore the quality factor $Q$ can be found from the resonance curves shape using the standard formula relating curve width to $Q$

$$Q = \frac{\omega_0}{\Delta \omega}$$  \hspace{1cm} (6.9)
Using this relation, the quality factor of the finite element resonance curve is measured as $Q = 4.8$. We can formulate a theoretical expression for the quality factor of the driven Helmholtz resonator, by considering the losses from resistance. If we consider the fluid moving in the neck to radiate into the surrounding medium, in the same manner as an open ended pipe, the radiation resistance, $R_r$, can be found through,

$$R_r = \frac{\rho_0 c k^2 S^2}{4\pi} \quad \text{(unflanged)},$$

and the thermoviscous resistance, $R_w$, is found by calculating the absorption coefficients for wall losses, $\alpha_w$, through,

$$\alpha_w = \frac{1}{ac} \left( \frac{\mu \omega}{2 \rho_0} \right)^{1/2} \left( 1 + \frac{\gamma - 1}{\sqrt{Pr}} \right),$$

where $\mu$ is the shear viscosity, $\gamma$, the ratio of specific heats and, $Pr$ the Prandtl number. Using $Q_w = k/2\alpha_w = \omega_0 m/R_w$ the theoretical quality factor of a driven Helmholtz resonator is found through,

$$Q = \frac{\omega_0 m}{R_m}, \quad \text{where } R_m = R_r + R_w.$$ 

Substitution of the relevant dimensions and material parameters used in the finite element model, gives a theoretical estimate of the quality factor $Q = 4.5$.

Two dimensional surface plots of the pressure distribution are included in Figure (6.2). The first plot is taken below resonance where the pressure inside and outside the cavity is approximately equal. The second plot at the excitation frequency of the Helmholtz resonator, shows that the pressure is amplified inside the cavity, which absorbs energy and leads to a shadow zone behind the resonator. The final plot is taken above resonance where again the pressure inside and outside the resonator is equal.

\[\text{†Values of } \mu = 1.85 \times 10^{-5}\text{Pa·s}, \gamma = 1.402 \text{ and } Pr = 0.710, \text{ are obtained from the Table of Physical Properties of Matter, pp528.} \]
Figure 6.2: Surface plots, illustrating the excitation of the Helmholtz resonator. The top frame illustrates the distribution at 2.0kHz, before resonance. The middle frame is taken at the resonance frequency of 5.16kHz, where the pressure inside the cavity is amplified and a shadow region of attenuation is formed. The third frame represents 8.0kHz above resonance.
6.2.2.1 Harmonics

If we expand the range of the frequency spectrum (Figure 6.3), harmonic resonances, also known as overtones, can be observed. These higher order resonance correspond to distinct pressure field distributions within the cavity. The resonant modes that form within the cavity resemble Chladni patterns of a circular plate (Figure 6.4). Each pattern corresponds to an Eigenfrequency solution of the two-dimensional wave equation for a circular membrane with a fixed rim. Modes with no nodal diameters (i.e. the \((n, 0)\) modes) have infinite symmetry belonging to the character group \(c_{\infty V}\). Modes with any integer amount of nodal diameters, or any combination of nodal circles and diameters, have finite symmetry enforced by these lines. They therefore belong to finite symmetry groups \(C_{mV}\). Theoretically, the modes should occur at the same frequency, however Rayleigh’s principle states that in all real systems any small perturbation can enforce the position of nodal diameters. This leads to a splitting of the degenerate pair, and partners A and B of the mode pattern form separately at slightly different values of frequency. As the pair is split one partner corresponds to a pressure minima and the other to a maxima. The perturbation introduced by the machining of the longitudinal slotted opening, is responsible for the ‘fixing’ of the nodal locations in our system.

Figure 6.3: Extended frequency spectrum for the single 6.5mm diameter Helmholtz resonator with a 4mm neck opening. In addition to the fundamental excitation harmonics exist. It should be noted that the fundamental resonant frequency is contained within the audible range, however all higher order harmonics exceed this range and are too high to be perceived.
Figure 6.4: Resonance modes of the air cavity inside the Helmholtz resonator. Modes are classified by the conventional notation for a circular plate, \((n, m)\), where \(m\) is the number of nodal diameters and \(n\) the number of nodal circles.
6.3 Sample Preparation

A two-dimensional locally resonant phononic crystal was fabricated for experimental investigation, comprised of individual Helmholtz resonator inclusions. In total, one hundred units were arranged in a $10 \times 10$ square lattice configuration with a periodicity in both the $\Gamma X$ and $\Gamma X'$ directions of 0.022m. Each resonator has characteristic dimensions of length, internal and external radius of $l = 300\text{mm}$, $r_{\text{int}} = 5\text{mm}$ and $r_{\text{ext}} = 6.5\text{mm}$ respectively. A longitudinal slot, of width $w = 4\text{mm}$, is machined along the substantial length of each tube, allowing air flow to the cavity (Figure 6.5). A small 1cm length remains at the top and bottom of each tube to retain a substantial amount of the tubes rigidity, permitting the tube to be clamped whilst machining. The cross-section of the experimental resonators is the same as that studied in the individual 2D Helmholtz resonator finite element investigation, with an excitation frequency of 4.8kHz, and a theoretically estimated excitation of 5.56kHz.

![Figure 6.5: The $10 \times 10$ locally resonant phononic crystal with Helmholtz resonator inclusions. Each resonator is a slotted tube with a C-shaped cross section.](image)
6.4 Zero-Order Transmission Results

The locally resonant phononic crystal with C-shaped Helmholtz resonator inclusions was subjected to zero-order transmission experiments, with sound incident upon the \( \Gamma X \) crystal plane. A direct comparison of the frequency spectra measured at the control and sample locations is shown in Figure 6.6. Two separate regions of attenuation are observed. The amplitude of the sample recording is suppressed, relative to the control, in the region from 3.98-6.48kHz and then again from 6.78-10.03kHz. These two regions of attenuation can be attributed to two different band gap mechanisms.

![Graph showing frequency spectrum with regions of attenuation labeled as resonance and Bragg.](image)

**Figure 6.6:** Experimentally measure frequency spectrum for the Helmholtz resonator LRPC. Two separate bands of attenuation exist as a consequence of the resonance and Bragg scattering mechanisms. The traces on the graph correspond to four separate measurements, starting at two different frequencies. It is worth noting that the data serves to demonstrate the reproducibility of the results in that the data in all four measurements is almost identical up to 14kHz.

The location of the Bragg band is estimated to be \( c/2a = 7.95\text{kHz} \). This is identical to the rigid cylinder phononic crystal sample due to the same wooden support frame being utilised, therefore with the same periodicity as before of \( a = 0.022\text{m} \). The second band of observed attenuation is centered at 8.46kHz, close to this prediction and it is therefore attributed to the Bragg scattering mechanism. From the experimentally measured results it is therefore observed that the Bragg band is located slightly higher in frequency than the theoretical estimate.
The experimental attenuation spectra, exhibits another band gap, positioned below this first Bragg band. This is the resonance band of attenuation, which arises from the local excitation of the Helmholtz inclusions. We can state this with some certainty as the finite element simulation of a single 6.5mm Helmholtz resonator predicts the excitation frequency of each unit as 4.8kHz (Section 6.2.2). This is very close to the observed central frequency of the band gap at 5.104kHz. Furthermore the theoretical excitation frequency of the slotted tube resonator is 5.56kHz, close to the central frequency of the band. The excitation of the resonators within the array forbids the transmission of waves at these frequencies. We shall see when the band structure is calculated that the resonance modes give rise to flat bands of dispersion in the spectrum. Hybridization of these bands with the otherwise linear bands of the air medium, forces band gaps to open, due to the level repulsion effect. As the first resonance band of attenuation forms below the phononic crystals associated Bragg gap, the structure is therefore an example of a locally resonant phononic crystal which is capable of forming band gaps, lower than the associated Bragg gap.

6.5 Finite Element Simulation

Finite element simulations are performed in the reduced domain, upon a $10 \times 10$ array of Helmholtz resonators. The resulting frequency spectra displays the appearance of the two separate bands of attenuation (Figure 6.7). A first band of attenuation occurs between 3.6-5.8kHz that is attributed to resonance phenomena. This can be defined with some certainty due to the aforementioned theoretical estimates and, as we shall later see in the surface plots, the resonator can be visually identified to resonate in this band. The second band of attenuation is ascribed to the Bragg mechanism, spanning from 6.8-9.6kHz, confirming the experimentally measured data. The central frequency of this attenuation band is 8.2kHz, very close to the observed experimental behaviour, however is again positioned higher than estimated from the simple calculation $f_c = c/2a$.

The agreement between the finite element and experimental spectra, further validates the accuracy of the reduced domain geometries used for theoretical comparison.

In regards to the shifting of both the theoretical and experimental Bragg band to higher frequencies, this is a consequence of the presence of the resonance band. The Bragg gap for the rigid cylinder (RCPC) and locally resonant phononic crystal (LRPC) both end at the same frequency $\sim 9.6$kHz. However, the LRPC Bragg gap starts at a higher frequency than the RCPC Bragg band, due to it being pushed upwards by the presence of the resonance gap. This increase in the start frequency, therefore causes the band to be centered around a higher frequency than the RCPC Bragg gap.
Figure 6.7: Experimentally measure frequency spectrum for the Helmholtz resonator LRPC. Two separate bands of attenuation exist as a consequence of the resonance and Bragg scattering mechanisms. The traces on the graph correspond to four separate measurements, starting at two different frequencies.

The experimentally measured data demonstrates that attenuation levels in both the resonance and Bragg bands are of a similar magnitude. However, this result is not shown by the finite element model. The numerical calculation suggests that the resonance mechanism is much stronger, achieving a maximum attenuation of +150dB, approximately six times larger than the 25dB in the Bragg gap. This extraordinary level of attenuation should not be considered realistic. This is because the real system has diffraction around the crystal. Further simulations of locally resonant phononic crystals with Helmholtz resonators included in this thesis should therefore only be used to determine the frequency location of resonance gaps.

Two-dimensional surface plots are extracted from the simulation to illustrate the pressure and phase field distribution at various frequencies, included in Figures 6.8, 6.9 and 6.10 respectively. In both the pressure and phase diagrams, five surface plots are included to illustrate wave propagation in the first allowed transmission band, at resonance, in the second allowed band, at the Bragg frequency and in the third allowed band above the first Bragg gap.
Chapter 6. *Locally Resonant Phononic Crystal*

(a) Surface plot of the pressure distribution (Pa) at 2.1kHz. Alternating regions of maxima and minima are equally spaced in proportion to the wavelength. The $k$-vector of the wave corresponds to the first allowed band of the dispersion relation, in which the wave propagates through the structure as a homogeneous medium. The majority of the wave energy is transmitted.

(b) Pressure distribution at the fundamental excitation frequency of the Helmholtz resonator inclusions, 5.1kHz. Regions of high pressure exist inside the first row of resonators, each unit acting as an amplifier of gain $Q$. The incident wave is reflected by the oscillating pressure field created by the resonators leading to standing wave formation between the incident and reflected wave in front of the crystal.

**Figure 6.8:** Surface plots of the Pascal pressure intensity throughout the domain at frequencies of (a) 2.10kHz, (b) 5.10kHz.
(a) Pressure at 6.6kHz representing a $k$ value in the second allowed band of the dispersion relation. The acoustic wave is able to propagate through the structure, and appears to converge to a focused beam. This is assumed to be an artefact of the geometry. The wavefronts traveling perpendicularly to the absorbing radiation conditions experience a delay leading to a curving of the plane wavefront. The symmetric nature of the domain causes the interference to give rise to focusing.

(b) The first Bragg band gap, 9.0kHz. Low pressure is observed in the shadow zone behind the crystal. The Bragg diffraction condition is fulfilled, the reflection of the wave forming a standing wave between incident and reflected wavefronts.

(c) Pressure distribution corresponding to a $k$-value in the third allowed band, 11.1kHz. Transmission of the wave is once again permitted.

Figure 6.9: Further surface plots taken at (a) 6.60kHz, (b) 9.0kHz, (c) 11.1kHz.
(a) First allowed band, 3.1kHz. Plane wavefronts of constant phase propagate through the array.

(b) Resonance band gap, 5.1kHz. Anomalous phase is observed in front of and behind the crystal.

(c) Second allowed band, 6.6kHz. Plane wavefronts propagate with spacing $\lambda$.

(d) Bragg band gap, 8.6kHz. Anomalous dispersion arises due to intense reflections.

(e) Third allowed band, 11.6kHz. Plane wavefronts propagate.

Figure 6.10: Surface plots of the instantaneous phase at frequencies of (a) 3.10kHz, (b) 5.10kHz, (c) 6.60kHz, (d) 8.60kHz, (e) 11.6kHz.
6.6 Influence of Layers

Finite element calculations were performed to determine the influence of array thickness upon the resonance and Bragg mechanisms (Figure 6.11). Simulations were performed upon a single row of resonators then subsequently with an increasing number of layers to a maximum of eight. The attenuation achieved in the resonance band is observed to increase with the number of layers. The width of the resonance attenuation band also is increased corresponding to the number of layers. At the 50dB level the two layer sample has a bandwidth of only 0.2kHz as opposed to the four layer sample with a much improved bandwidth of $\sim 1$kHz. As layers are increased to the maximum of eight the bandwidth at the 50dB level remains quite consistent.

In regards to the evolution of the Bragg band gap, the spectrum for a single row of resonators possesses a resonance peak of attenuation but no Bragg gap. This is due to the fact that at least two rows of scatterers are required for the mechanism to exist. As the number of layers increases from one to four, the Bragg band gap becomes more pronounced. In further increasing the array thickness from five to eight layers, the spectra are observed to be very similar. The Bragg gap does not strengthened in terms
of attenuation achieved or bandwidth. An interesting point to note is that from two to four layers, before the Bragg gap converges on stability, the central frequency appears to shift upwards. This is not observed for the resonance gap, which is located around the same excitation frequency regardless of the number of layers in the sample. The result suggests that with only five layers of scatterers, the array can be approximated by an infinite system.

Zero order transmission experiments were performed upon the Helmholtz resonator LRPC with 4, 6, 8 and 10 layers. The measured data (Figure 6.12) demonstrates that the attenuation level in the resonance band gap is of a similar order ($\sim 15 - 20\text{dB}$) for all the samples tested. Moreover, the resonance band gap is of a consistent bandwidth ($\sim 1.8\text{kHz}$) for all samples. The Bragg band for the smallest four layer sample has a narrower bandwidth than the thicker samples, showing that the structure is too small to be assumed infinite. This is consistent with the limit suggested by the finite element calculations. The central frequency of the four layer Bragg band is also positioned at a lower frequency than the other samples, as observed in the finite element spectra. The Bragg band for the 6, 8 and 10 layer samples are of a consistent bandwidth, with a close agreement between the 8 and 10 layer traces.

![Figure 6.12](image.png)

**Figure 6.12:** Experimentally measured attenuation levels for phononic crystals of differing thickness. Only the sample traces for 4, 6, 8 and 10 layers are included to allow a clearer interpretation of the result.
6.7 Influence of Orientation

The orientation of the resonators, relative to the incident sound direction has an influence upon both resonance and Bragg band gap formation. The incidence directions are illustrated in Figure [6.13], labeled as $0^\circ$, $90^\circ$, $180^\circ$ and $270^\circ$, to correspond to sound incident upon the open neck of the chamber, the left, back and right face of the resonator respectively. The attenuation characteristics of the Helmholtz resonator LRPC were measured experimentally in these four scenarios and compared to finite element calculations. The numerically obtained frequency spectra demonstrate that more attenuation is achieved in the Bragg band gap, with the resonators rotated at $90^\circ$ and $270^\circ$ degrees to the sound (red lines Figure 6.14).

![Figure 6.13:](image)

**Figure 6.13:** Classification of the orientation directions.

![Figure 6.14:](image)

**Figure 6.14:** Frequency spectra obtained from FE calculation for the four orientations of sound incidence.
The enhanced attenuation in the Bragg band gap, for $90^\circ$ and $270^\circ$ is a consequence of the scattering surface presented to the incident wave in this situation. To understand this result, consider the illustration of the Bragg scattering mechanism of Figure 6.15. Figure (6.16) shows the $0^\circ$ case, in which sound is incident upon the open face of the resonator. Due to this, the wave is scattered from the inside curved surface, as opposed to the solid surface of a rigid scatterer. These imperfections in the Bragg system lead to the reduced ‘strength’ of the phenomena and weaker attenuation. In the $90^\circ$ and $270^\circ$ situations, incident sound strikes a solid surface and ordered scattering occurs. This situation is more similar to the diffractions that occur from rigid scatterers and as such, more attenuation is achieved.

In the $180^\circ$ case, sound is incident upon the back surface of the resonators. This solid interface should be expected to give rise to strong Bragg interference, leading to similar attenuation levels as in the $90^\circ$ and $270^\circ$ results. However, it is observed that the attenuation achieved is closer to the $0^\circ$ case for sound incident on the open cavity. As discussed
for electronic wavefunctions in the crystal X-ray diffractions of Chapter 2.1.2, waves with $k \approx k_{BZ}$, experience diffraction but do not immediately run back on themselves. However, after some more reflections they do, leading to a splitting of energy for all positions on the Brillouin zone. Analogous to this, frequencies around $f_{Bragg}$ experience reflections from the second row which are then incident upon the open neck of the first row of resonators (Figure 6.17). The scattering is therefore of a less coherent nature, similar to the $0^\circ$ case.

The resonance gap is also dependent upon the orientation of the resonators, in terms of position and bandwidth. In the $0^\circ$ direction, plane wavefronts of sound are incident upon the open neck of the resonator and they efficiently excite an oscillation of the mass of air. It is this motion that causes the incident wavefront to be reflected and form the spectral gap, therefore a broader resonance forms in $0^\circ$ orientation compared to the $90^\circ$ and $270^\circ$. The dependence of power absorption on orientation has been discussed by T.C.Rao [82], where it was shown that the resonant frequency is dependent upon the angle of incidence of the sound. It was reported that the excitation frequency of a Helmholtz cavity is shifted upwards for sound passing perpendicular to the open slot (i.e. the $0^\circ$ and $270^\circ$ scenarios) and this is observed in our calculations, the excitation frequency shifting from 4.8-5.2kHz.

The behaviour predicted by the finite element method is verified by experimental measurement (Figures 6.18 and 6.19). More attenuation is achieved with the resonators in the $0^\circ$ and $180^\circ$ directions. In these orientations it is clear to distinguish between the two different band gap mechanisms. For the $90^\circ$ and $270^\circ$ orientations, however, the mechanisms are indistinguishable and a single band of attenuation is observed extending from 4.0-11.0kHz. The shifting of the resonance gap to a higher frequency and the increased strength of the Bragg diffraction mechanism, leading to the two bands of attenuation overlapping into a single combined envelope, for these orientations.
Figure 6.18: Experimentally measured amplitude data for the four orientation scenarios. Two separate bands of attenuation exist for the 0° and 180° directions, but for 90° and 270° these bands merge into a combined envelope of attenuation.

Figure 6.19: The amplitude data of Figure 6.18 to a recognisable decibel scale.
6.8 Random Distributions of Inclusions

To verify the resonance band gap, zero-order transmission experiments were performed on random distributions of Helmholtz resonators. Three lattice configurations were investigated and comprised of 41, 53 and 44 inclusions. By inserting the tubes into the supporting frame in a random arrangement, the periodic nature of the phononic crystal structure is removed. As such the scattering occurring between the inclusions is incoherent, and Bragg reflections cannot occur. Therefore, we should not expect the Bragg band gap to manifest in such structures. On the other hand, each resonator will still be excited by incident waves. The local excitation of the units still forming linear bands of dispersion which couple with other bands and open a resonance band gap due to the level repulsion effect.

Figure 6.20: Frequency spectra for the three random distributions of resonators. The Bragg mechanism is suppressed due to the loss of periodicity.

Figure 6.21: Random distributions of the Helmholtz resonator cavities.
The experimental frequency spectra for the three array distributions of Figure (6.21) are presented in Figure (6.20). The resonance gap is identified in the spectra of all three samples investigated, confirming the hypothesis that it forms independently of the arrangement of the inclusions. However, the attenuation bands are less well defined than those obtained from the ordered array, and the attenuation levels achieved are approximately half of those obtained experimentally. The lower attenuation levels are a direct consequence of the lower number resonators used in the random arrays.

Chaotic spikes of attenuation are observed at all frequencies above the resonance gap. Due to the nature of the data it is unclear as to which mechanism these may be attributed. A consistent region of attenuation can be identified spanning approximately 7.0-8.4kHz. This band is centered close to the Bragg frequency of the crystal. The breaking of the periodicity has therefore not suppressed the mechanism completely. This arises as the distribution of the resonators is not truly random in nature. The supporting frames in which the inclusions are inserted were the same as those fabricated for the RCPC and LRPC investigations. The square periodicity with which the holes were drilled, meant that the minimum spacing between each unit was never less than 9.0mm. Furthermore, each inclusion has a spacing equal to some integer, $n$, multiple of the lattice constant, $a$, $(n \times a = n \times 0.022m)$. The structures are therefore, still somewhat periodic in their nature. Some coherent scattering is therefore permitted and reflections will lead to attenuation (Figure 6.22).

**Figure 6.22:** Coherent scattering occurring from the quasi-periodic random array of Helmholtz resonators. Due to the consistent periodicity of the structure, some Bragg diffraction phenomena will occur, and as such the Bragg diffraction mechanism is not suppressed entirely.
6.9 Band Structure of the LRPC

Figure 6.23: The acoustic band structure of the Helmholtz resonator LRPC tested experimentally. A flat resonance band gap opens in all directions between 5.98-6.53kHz. This is situated below the structures first characteristic Bragg band gap, which spans 7.42-8.52kHz and is only present in the ΓX direction. The calculation is for sound transmission in the 0° scenario, which is identical to the 180° result.

The full acoustic band structure of the Helmholtz resonator LRPC is included in Figures (6.23) and (6.24). The band structure of Figure (6.23) representing sound transmission in the 0° and 180° cases, where it was found that the solutions are identical. This was confirmed in our experimental measurements and finite element simulations of Section (6.7). Similarly, Figure (6.24) represents the band structure of the LRPC with the resonators orientated at 90° which provides the same result as the 270° scenario. It was shown in the discussion of orientation that these directions can be assumed to be the same due to symmetry considerations. In the 0° and 180° result, two band gaps exist. The first absolute band gap of the structure spans 5.98-6.53kHz, ascribed to resonance. The second band gap is only observed in the ΓX direction and represents the first Bragg band. This fundamental gap now exists between the 2nd and 3rd branches of the dispersion relation, spanning 7.42-8.51kHz. The central frequency of this band, \( f_c = 7.97 \text{kHz} \), being in excellent agreement with the theoretical estimate and experimental observation.
In the 90° and 270° orientations the absolute resonance band gap is again located between 5.98-6.53kHz, identical to the 0° and 180° result. The resonance band gap in the ΓX direction has however been reduced as the sound is passing perpendicular to the opening. It can be seen that the Bragg band is now significantly enlarged, spanning 6.67-8.92kHz. This is due to the increased scattering surface presented to the incident wave in these scenarios. The Bragg gap is so wide that it almost overlaps the resonance band. The behaviour is in good agreement with the experimentally measured data in these orientations where it was observed that the two separate band gaps had combined. Although in the band structure result the band gaps do not overlap, the behaviour is supported here by the calculations.

The Eigenmode solutions are shown in Figure 6.25 These are of a similar nature to those of the rigid cylinder phononic crystal samples. One difference is however striking, the first mode does not represent a pressure field distribution between the scatterers. Instead the first frame illustrates the Eigenmode of the fundamental excitation of the Helmholtz resonator cavity itself, i.e. not of the periodic structure.
Figure 6.25: Pressure maps (Pa), illustrating the first six Eigenmodes of the LRPC band structure (Figure 6.23). The fundamental mode (top left) shows the excitation of the Helmholtz resonator, as the maximum amplitude (red) is contained within the cavity. The second Eigenmode (top right) illustrates a symmetric pressure distribution of the pressure field outside the cavity. This is the first mode of the fluid between the periodic scatterers.
6.10 Multiple Resonances

Two distinct forms of locally resonant phononic crystals exist, composed either of cavity resonators, or scattering units with a heavy core coated in a soft shell which exhibit vibrational resonances. Both mechanisms give rise to drastic peaks in the attenuation spectrum. As aforementioned, the local resonances associated with the soft coating material display an asymmetric line shape typical of a Fano resonance \[78\]. The bandwidth of such attenuation peaks is in general rather small and in order to obtain a broadband attenuation, it is necessary to combine different locally resonant materials to overlap the dips resulting from the several resonant frequencies \[83\]. It is attractive to be able to exhibit some control over the size of the band gap a phononic crystal structure possesses, as the ability to operate over broad frequency ranges, severely expands their potential for commercial applications. Since their conception, several novel applications have been suggested ranging from motorway noise barriers \[84\][85] to earthquake and tsunami shielding \[86\].

It was demonstrated by H.Larabi et al. \[87\] that the transmission properties and band structure of a locally resonant phononic crystal can be significantly influenced by increasing the number and density of the resonating units. Furthermore however, he reported the transmission properties of a more novel design of sample, comprised of multicoaxial inclusions that was investigated theoretically and experimentally. Each solid steel core is coated in multiple layers of alternating soft polymer and steel. The composite units have many characteristic excitation frequencies and display several dips in the transmission spectrum. It was shown that scatterers with three layers, gave rise to two spectral gaps and for a five layer sample, three band gaps formed. Therefore it was concluded that the number of these gaps evolves in relation with the number of coated shells. Larabi also reported that by combining two or more phononic crystals with different parameters, it is possible to overlap some resonance dips and obtain a widening of the frequency gaps.

Using these concepts, we now investigate systems composed of multiple sized Helmholtz resonators to achieve band gap broadening. Two techniques have been studied, namely using mixed arrays of alternating different sized resonators, and secondly a nested system in which the resonators are situated concentrically inside each other.
6.10.1 Mixed Array

6.10.1.1 Two Resonators

A mixed array of two resonator sizes is depicted in Figure (6.26), with the resonators arranged in alternating layers. The dimensions of the two different cavities are included in the diagram. The cavities will be referred to as the 9mm resonator with 15% slot and the 11mm resonator with 5% slot. The percentage measure of slot width, represents the percentage of the circumference which is the open slot. These are chosen such that the resonance bands are separated by a sufficient distance to be identified individually.

![Figure 6.26: A schematic of the mixed locally resonant phononic crystal array. Two inclusions of differing size are arranged in alternating layers, although the same principle can be applied to incorporate more than two resonator sizes.](image)

The finite element calculated frequency spectrum of the mixed LRPC is depicted in Figure (6.27). Also included in the plot are two further traces for LRPC’s comprised from array’s of solely 9mm and 11mm resonators. It is observed that due to the multiple sizes of resonator used, the mixed array possesses two resonance bands of forbidden transmission. The first band spanning from 0.64-4.12kHz corresponds to the excitation of the larger 11mm units, and the second band from 4.8-7.8kHz occurs when the 9mm inclusions are at resonance. The inclusion of the resonators in this cascaded layer arrangement proves to be an efficient method for achieving an increased range of attenuation. There is no loss of attenuation for the mixed system relative to the individual arrays. Furthermore, there is little sacrifice in terms of the width of the gaps. The lower resonance band of the mixed array is shifted to higher frequencies, but remains the same bandwidth as for the single array. The second band loses some bandwidth and is narrower than for the single array, however still covers a useful frequency range $\sim 3$kHz.
The efficiency of the system with little sacrifice in terms of performance, makes it a suitable design with which to achieve broad frequency ranges of attenuation for practical soundproofing applications. It should be noted that the resonator dimensions chosen here are such that the two bands can be distinguished separately. By appropriate choice of sizes, the gaps can be tailored to overlap, giving a single broad band of attenuation.

![Frequency spectrum calculated with the finite element method. Layers of two different resonator sizes, namely 9mm diameter with a 15% slot and 11mm diameter with a 5% slot, are arranged in an alternating fashion.](image)

**Figure 6.27:** Frequency spectrum calculated with the finite element method. Layers of two different resonator sizes, namely 9mm diameter with a 15% slot and 11mm diameter with a 5% slot, are arranged in an alternating fashion.

Five surface plots are included in Figure 6.28 to illustrate the transmission of an acoustic wave through the mixed array. The first frame depicts the sound pressure level of an acoustic wave with a frequency of 500Hz. In this low frequency regime, the wave is permitted to propagate with only a small reflected component that arises due to the impedance difference of air to crystal. The second plot is taken at the most severely attenuated point of the first resonance band gap. The maximum pressure intensity (dark red) is contained within the first layer of the crystal, in the row of larger 11mm diameter resonators. The third frame is taken at 4.3kHz, in the 2nd allowed transmission branch of the crystals band structure. In this band, between the two resonance gaps, the wave is again free to propagate. Frame four illustrates the second resonance band gap at 6.9kHz. It is observed that the maximum pressure lies in the smaller 9mm units. As a note, it appears that the first row of 11mm units is also excited in this figure. However, this is a fringe of maximum of the standing wave in front of the array, caused by the strong reflection from the second row of the crystal. The final frame is taken above both resonances at 8.9kHz where propagation of the acoustic wave is again permitted. Some unexpected focusing occurs, which is an artefact of the radiation boundary conditions.
Figure 6.28: Acoustic wave propagation through the mixed array. The surface plots show the magnitude of the sound pressure level. The top frame illustrates 500Hz, in the first allowed transmission band. Moving down, the second frame depicts the excitation of the 11mm resonators at 1.3kHz. The middle frame illustrates wave propagation in the second branch of the band structure, 4.3kHz. The fourth frame, depicts the excitation of the 9mm resonators, 6.9kHz, and finally the lower frame shows the third transmission branch where propagation is permitted, 8.9kHz.
6.10.1.2 Mixed Array 3 Sizes

A second example of a mixed array is investigated, this time with three different sized Helmholtz resonator inclusions. The three sizes are again chosen, such that their excitation frequencies are sufficiently separated to be identified individually. In this second example the resonance bands are also chosen to be separate from the Bragg frequency. This consideration was not made for the mixed array of the previous section and the resonance band ‘masked’ the presence of the Bragg gap. The frequency spectrum of the three resonator structure is included in Figure (6.30) alongside the spectra for the individual resonator arrays (dashed black lines). Three resonance band gaps can be identified in the mixed spectrum, centered at 3.2kHz, 5.1kHz and 9.8kHz. These correspond to the 17mm, 10mm and 7mm Helmholtz resonators respectively.

There is further, fourth resonance band gap, identified at 12.9kHz. This is attributed to the harmonic excitation of the largest 17mm diameter resonators. The pressure field distribution can be observed in the surface plot of Figure (6.32(c)). There is a single nodal diameter located in the centre of each resonator, running vertically in the domain. This pressure distribution, was predicted in Section (6.2.2), where the harmonic excitation of Helmholtz resonators was discussed. The fundamental resonance can be
considered analogous to a circular membrane vibrating as a whole. The first harmonic resonance identified, corresponds to the second solution where a standing wave is formed in the cavity. The harmonic excitation creates a resonance band gap which has a smaller bandwidth than the fundamental resonance gap of the 17mm resonators. Furthermore, the attenuation achieved for the harmonic band is approximately a third of that achieved at first resonance.

![Figure 6.30: Frequency spectrum of the pressure field intensity behind the mixed Helmholtz resonator phononic crystal array. The mixed array possesses three resonance peaks of attenuation, in addition to a Bragg band gap from its periodicity.](image)

This is because the power absorbed by the Helmholtz resonator is dependent upon the acoustic wavelength at resonance \( \text{[Ref]} \). At the fundamental resonance the whole cavity is of a different pressure. This provides the mass of air in the neck more inertia than the first harmonic where half of the cavity is of an increased pressure and half is of a lowered pressure. The increase in the inertia applied to the mass of air means consequently that more power is radiated. the power radiated is directly related to the power absorbed by the resonator, and this leads to the reduced bandwidth of the harmonic resonance.

The mixed array phononic crystal also demonstrates a Bragg band of attenuation, which arises due to the periodicity of the structure. The region of suppressed amplitude spans from 6.5-9.0kHz, therefore centered around 7.75kHz. The lattice spacing of the alternating layers of resonator size is constant, \( a = 0.022 \text{m} \), for the structure corresponding to a theoretical Bragg frequency of \( \nu/2a = 7.8 \text{kHz} \), in good agreement with the finite element simulation.
Figure 6.31: Surface plots of the instantaneous pressure (dB) distribution for the mixed array with three sizes of resonator. Frequency snapshots are taken (a) in the 1st allowed transmission band, (b) at 17mm 25% resonance, and, (c) at 10mm 15% resonance.
Figure 6.32: Surface plots of the instantaneous pressure (dB) distribution for the mixed array with three sizes of resonator. Frequency snapshots are taken (a) in the 1st Bragg band gap, (b) at 7mm 10% resonance, and (c) at the 1st harmonic resonance band of the 17mm resonators.
6.10.2 Matryoshka - Single Resonator

![Diagram of a Matryoshka resonator comprised of individual Helmholtz cavities. All dimensions are in the units of millimeter.](image)

Using the same principle of multiple resonator sizes, there is another possible arrangement in which the inclusions may be distributed. Smaller resonators can be placed inside larger units in a nested fashion. This type of system has been coined the Matryoshka or ‘Russian doll’ resonators†. The concentric configuration disregards the need for separate layers of different resonator sizes, saving space and reducing the overall thickness of the phononic crystal array. An example of a nested Matryoshka unit is illustrated in Figure 6.33, composed of five Helmholtz resonators.

![Graph showing frequency spectrum for the five ring resonator. Two control traces are included representing propagation through an empty domain, and also with a solid cylindrical scatterer, to provide some estimate of the reflection loss.](image)

†The term has been coined by the present author and colleagues at Loughborough University, namely Dr.D.Elford, Dr.G.Swallowe and Prof.F.Kusmartsev.
The frequency spectra for a single resonator, i.e. not arranged in a periodic array, is included in Figure (6.34), where five sharp resonances are observed. A control trace is included for a solid scatterer inside the domain. This cylinder is chosen to be the same size as the largest ring of the Matryoshka resonator and presents an obstacle to the propagating acoustic wave. The control trace gives an estimate of how much of the attenuation is caused by direct reflection from the object, and allows us to identify the extra attenuation associated with the resonance mechanism.

The excitation behaviour of the 5 ring Matryoshka resonator is illustrated in Figure (6.36). The surface plots show the pressure (Pa) distribution inside the domain at the resonant frequency of the different rings. Maximum and minimum pressures are denoted by the colours red and blue respectively.

Figure 6.35: Surface plots of the Matryoshka resonator at different excitation frequencies. Reading from left to right and top to bottom the excitation frequencies increase as 3.20, 3.73, 4.20, 4.98 & 6.20kHz.
A further frequency spectra for the single Matryoshka inclusion is illustrated in Figure (6.36). Five traces are included to show the attenuation behaviour for an increasing number of nested resonators. It can be seen that the number of resonance peaks evolves with the number of surrounding resonators. It is observed that the single Helmholtz resonator has an excitation frequency of \( \sim 7.3 \text{kHz} \); however the introduction of a second resonator around this first unit, shifts the resonance frequency to a lower value, the peak now being centered around \( \sim 6.7 \text{kHz} \). This is because the extra shell serves to increase the effective length of the neck of the central Helmholtz cavity. Simple calculations suggest that an increase in neck length of \( \sim 15\text{mm} \), provided by the second shell being introduced, will shift the excitation frequency by \( \sim 0.6\text{kHz} \), which is approximately the frequency by which the resonance peak is shifted. As further resonators are placed surrounding the initial cavity, the effective neck length increases further to 30, 45, 60mm for the 3, 4 and 5 ring Matryoshka respectively. The highest frequency peak (associated with the central smallest, high pitched inclusion) is lowered to 6.5, 6.4 and 6.3kHz. Theoretical calculations estimate the fundamental resonance at 6.93kHz which is lowered to 6.23, 5.89, 5.69 and 5.55kHz.

![Figure 6.36: Frequency spectrum for an increasing number of Helmholtz resonator shells. The single cavity result represents the attenuation level calculated by the finite element method. Subsequent traces, for the 2, 3, 4 and 5 ring systems have been shifted by 25dB each time to separate the results and allow a clearer interpretation.](image)
6.10.2.1 Three Resonators

A locally resonant phononic crystal is next investigated, comprised of Matryoshka resonator inclusions. The introduction of many resonators arranged in a periodic fashion causes the bandwidth of the resonance peaks to increase. Therefore the dimensions of the units described in the previous section (Figure 6.33) are unsuitable in such an array. The resonances of the five ring system lie too close together, such that they will become merged and be unable to be identified individually. Therefore a three resonator system is investigated with the same dimensions as used in the three layer mixed array, namely a smallest 7mm diameter resonator with a 10% slot width, surrounded by a 10mm diameter 15% slot and a largest 17mm resonator with a 25% slot width. The structures characteristic frequency spectrum shows the existence of three resonance peaks in addition to a Bragg attenuation band, caused by the crystals periodic nature.

It is observed that the position of the lowest resonant peak, attributed to the 17mm resonator, does not coincide with the control spectrum. The centre of the resonance band is shifted to a higher frequency due to the presence of the smaller cavities positioned inside. These occupy a volume in the 17mm resonator and this reduction in the volume of the cavity corresponds to an increase in the pitch of the resonator.

![Figure 6.37: Transmission spectrum illustrating the pressure field behind a phononic crystal comprised of Matryoshka resonator inclusions. The individual constituent inclusions being comprised of three Helmholtz resonators with dimensions 7mm–10%, 10mm–15%, 17mm–25%.](image-url)
6.10.2.2 Band Structure Matryoshka

Figure 6.38: Band structure calculations for the Matryoshka resonator comprised of 17mm−25% slot, 10mm−10% slot and 7mm−5% slot inclusions. Flat bands exist around the excitation frequencies at 6.3kHz (green 2\textsuperscript{nd} branch), 7.6kHz (cyan 4\textsuperscript{th} branch) and 8.6kHz (dark blue 5\textsuperscript{th} branch), for the resonator sizes respectively.

Figure (6.38) presents the band structure of the three ring Matryoshka resonator. Numerous flat bands exist around the excitation frequencies of the multiple sized cavities. The presence of these degenerate states intersects the continuum bands of the structure, and force a band gap to open around the flat band as a consequence of the level repulsion effect. The lowest frequency band corresponds to the excitation of the largest 17mm resonator starting at 4.48kHz. A second flat band at 7.62kHz represents the spectral gap formed by the 10mm resonators and the third flat band at 8.67kHz being attributed to the 7mm resonators. The presence of multiple minibands contained within a relatively small frequency range demonstrates that locally resonant phononic materials comprised of Matryoshka inclusions provide a broad band of acoustic shielding.
Chapter 7

Refraction and Diffraction Phenomena

7.1 Refraction

A phononic crystal’s ability to reduce the phase velocity of a propagating acoustic wave can be exploited to design refractive acoustic devices. Similarly to optical systems, refraction effects can be tailored to give rise to regions of high intensity, demonstrating the ability to focus transmitted acoustic waves. The phenomenon arises in the first allowed band of the dispersion relation, before the Bragg band gap. In this low frequency region, the wavelength of the sound is larger than the periodic spacing of the structure. Therefore the incoming wave experiences the phononic crystal structure as a homogeneous medium and interacts as such. The refraction arising when the sound wave enters the phononic crystal and the reduced velocity of the sound are analogous to the slowing of an electromagnetic wave such as light when it enters a medium with a higher refractive index. In this chapter the analogy between two well known classical optical systems is investigated. Namely a phononic crystal type Fresnel biprism and biconvex lens are studied. In the final section, we consider the propagation of sound over the sample. The array is considered as a rigid screen to investigate the influence of height on the attenuation level achieved.

7.1.1 Classical Refraction in Phononic Crystals.

The first reported case of refraction in a phononic crystal structure was detailed by F.Cervera et al., who designed acoustic refractive devices using the accepted principles of optical refraction. In the field of optics, refraction of light is permitted because
solid dielectric materials have two characteristic properties - they transmit light at a
velocity lower than that of air and they are transparent to electromagnetic waves in a
certain frequency range. Their ability to reduce the velocity of light causes the focusing
observed through lenticular shapes. The second property of transparency arises because
their electromagnetic impedance $Z = \mu c$ is similar to that of air.

To create a refractive device for acoustic waves it is intuitive to search for a material
which exhibits analogous properties to the two properties that give rise to optical refrac-
tion. The acoustic impedance for a material is defined as $Z = \rho c$ where $\rho$ is the density
of the material and $c$ the speed of sound in the material. This value is always much
larger for a solid medium than of air. As the reflectance and transmittance of a wave at
a boundary is determined by the impedance ratio, there is an inefficient transmission of
energy at the interfaces and the substantial majority is reflected. Due to the periodic
spaces between the scattering units of a phononic crystal, the structure presents less
of a reflective cross section than that of a solid interface. This allows the sound wave
to propagate into the medium and experience refraction, without the majority of the
energy being reflected.

Cervera [89] provides a simple explanation to describe the reduction of the sound velocity
inside the phononic crystal medium. If an array of hard scatterers is rigidly fixed in an
alternating pressure field, the incident wave and the scattered waves are superimposed in
such a way that the sound propagation speed is reduced. In other, words, the following
medium effectively has an increased inertia or density. When dealing with rigid cylinders
in an oscillating fluid (the sound field), we can make the problem equivalent to that of
oscillating cylinders in air at rest. The air around the oscillating cylinders impedes
them from vibrating freely, and this impedance effect results in an effective increase of
the density of the medium [90]. In this way, for a single cylinder, the mass per unit
length increase equals the mass per unit length $S\rho_{\text{air}}$ of the air displaced by the cylinder
[91], where $S = \pi r^2$ is the cross section of a cylinder and $\rho_{\text{air}}$ is the air normal density.
If $N$ is the number of cylinders per unit area, then the increase in the effective density
is given by,

$$
\rho_{\text{eff}} = \rho_{\text{air}} + N\rho_{\text{air}}S = \rho_{\text{air}} (1 + f) .
$$

(7.1)

The phase velocity is inversely proportional to the square root of the density $\rho_{\text{eff}}$ and
compressibility $\kappa_{\text{eff}} = \kappa_{\text{air}}$. Therefore, the speed of sound in a phononic crystal, which
is related to that in air through the refractive index $n$ is,

$$
v_{\text{PC}} = \frac{v_{\text{air}}}{n} = \frac{v_{\text{air}}}{\sqrt{1 + f}} .
$$

(7.2)

The results obtained using this equation were found to agree qualitatively with the values
predicted by Cervera’s phase delay method. This alternative method, as detailed in the
experimental procedure (4.3.2), involves constructing the acoustic dispersion relation of wave vector \((k)\) versus frequency \((f)\) from the phase information, thus obtaining the sound transmission velocity inside the phononic crystal. The refractive index is then estimated using Equation (7.2) as the ratio of the sound velocities, inside and outside of the periodic array.

Although Cervera’s work illustrated clear evidence of refraction, not all phononic crystal structures will display this behaviour. The question posed as to whether a phononic crystal can be described as a refractive medium, is one that is difficult to quantify, and whose answer is related to the size of the structure. A report by N. García et al. [68] suggested that the image focusing observed by Cervera was not caused by refraction mechanism, but by diffraction phenomena around the edges of the samples. He argued that the homogeneous effective medium theory used as a theoretical model was not suitable for long wavelengths, and that the Finite Difference Time Domain (FDTD) method was more appropriate. Using this alternative method and also experimentally, García demonstrated clear evidence of diffraction phenomena. It was shown that, as in conventional optics, lenticular shaped arrays caused focusing. The remarkable result however, was that García could also achieve this effect using a solid rectangular aluminium block.

Due to the high impedance, the sound cannot penetrate the structure and be refracted. Therefore this clearly proves that diffraction is the dominant focusing mechanism, for this system.

Although García’s work showed clear evidence of a phononic crystal diffracting sound waves, his generalised claim that “diffraction is the dominant mechanism over refraction” provoked a response from Cervera and his colleagues, pertaining to “clarify some misconceptions and criticisms” made [92]. Cervera agreed with García’s observations that diffraction was the dominant mechanism in focusing for García’s structures; however only because the systems were six times smaller than the work he had referenced. Cervera stated that if the phononic crystal structure is large enough that it can be described by effective medium theory (where a refractive index can be defined) then refraction dominates over diffraction. However, if an array is smaller than this limit, diffraction is the dominant focusing mechanism.

This conclusion therefore poses the question: what is the minimum phononic crystal size at which its properties can be described by effective values of its acoustical parameters? However, there is no quantifiable answer as refraction and diffraction effects are intrinsically linked. Even if a phononic crystal is large enough to be described by effective medium theory, it will still display some diffraction that is attributable to edge effects; however the contribution to focusing is negligible.
7.1.2 Determining $n$ through Interferometric Techniques.

As discussed so far, there are two approaches that can be used to determine the refractive index of a phononic crystal. Cervera’s Equation (7.2) relates the focal length of a refractive device to the speed of sound inside the phononic crystal. Alternatively the phase delay can be used to construct the acoustic band structure. This is a laborious procedure which is unsuitable for low frequencies at which the phase becomes erratic. Furthermore this technique looks at the linear region of phase velocity in the first transmission band, and as such, the solution for $n$ is therefore unique only to this region.

A third alternative method was detailed by V. Romero-García et al. [93] to determine the refractive index of a phononic crystal. Romero-García uses an acoustic analogy of the Fresnel biprism from conventional optics to determine the refractive index. In optics, the Fresnel biprism consists of two thin prisms joined at the bases that produces a characteristic interference pattern. When a light wave is incident upon the Fresnel biprism, part of the wave is refracted downward and part is refracted upward. This creates an interference region, equivalent to that obtained by two coherent sources. The exact relation between the refractive index and the separation of fringes at a given wavelength is found to be,

$$n = \frac{1}{\sin \frac{\alpha}{2}} \left( \sin \left\{ \frac{1}{2} \left[ \tan^{-1} \left( \frac{sc}{\Delta y 2df} \right) + \alpha \right] \right\} \right),$$

(7.3)

where, $\alpha$ is the prism angle, $d$ is the distance from the source to the vertex of the Fresnel biprism, $\Delta y$ is the fringe spacing in the interference pattern created in a linear cross section $AA'$ and $s$ is distance from the source to the cross section.
7.1.3 COMSOL Simulations

Acoustic wave propagation through a Fresnel biprism phononic crystal comprised of 181 solid scatterers is simulated by the finite element method. The scatterers are arranged in a square lattice configuration with a packing fraction of 0.274 and have subdomain parameters of fluid density $\rho = 7990\text{kg/m}^3$ and sound velocity $v = 4915\text{m/s}$. As the incoming wave propagates through the phononic crystal Fresnel biprism the sound velocity is reduced, and part of the wave is refracted upwards, whilst a second part is refracted downwards. This produces an interference pattern of alternating regions of high and low pressure that can be observed in the decibel pressure map. A line plot of the decibel intensity across a section of the domain is generated to visualise the interference pattern (Figure 7.3). It follows the expected form as predicted by conventional optics, and can be used to determine the fringe spacing. Using this measurement in conjunction with the other dimensional parameters given by the geometry of the system, it is possible to estimate the refractive index of the system using Equation (7.3).

\[ n = \frac{1}{\sin \left( \frac{63.4}{2} \right)} \left( \sin \left\{ \frac{1}{2} \left[ \tan^{-1} \left( \frac{0.65 \cdot 343}{0.1 \cdot 2 \cdot 0.5 \cdot 6800} \right) + 63.4^\circ \right] \right\} \right) = 1.24 \]  \hspace{1cm} (7.4)

Substituting this value of the refractive index into Equation (7.3) gives the speed of sound inside the phononic crystal as,

\[ v_{PC} = \frac{343}{1.28} = 276\text{m/s}. \]  \hspace{1cm} (7.5)
7.1.4 Sonic Crystal Lenses and Diffraction Phenomena

It is possible to further observe the refraction of acoustic waves using a phononic crystal lens, and verify the prediction of the refractive index obtained from the Fresnel biprism technique. In optics a biconvex lens is a device with axial symmetry that transmits and refracts light, converging the beam to a focal point. It was shown by Chao-Hsien Kuo and Zhen Ye [94] that if a phononic crystal is constructed such that the overall system is in a lenticular form, the device can converge a propagating acoustic wave. According to refraction theory, it is possible to relate the object distance, $s_o$, to the image distance, $s_i$, and focal length, $f$, using the formula for thin lenses,

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}. \quad (7.6)$$

If the crystal can be regarded as a refractive medium, the focal length can be related to the radii of curvature of the convex surfaces,

$$\frac{1}{f} = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right). \quad (7.7)$$
Leading to the lensmaker’s formula,

\[
\frac{1}{s_o} + \frac{1}{s_i} = (n - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right).
\]  

(7.8)

As in our simulation we use a symmetric biconvex lens, \( R_1 = -R_2 \), and Equation (7.8) reduces to,

\[
f = \left( \frac{R}{2} \right) \left( \frac{1}{n - 1} \right).
\]  

(7.9)

### 7.1.5 COMSOL Simulation

The focusing ability of a phononic crystal biconvex lens is simulated using the finite element method. The structure is comprised of 612 solid cylindrical scatterers with subdomain parameters of fluid density \( \rho = 7990\text{kg/m}^3 \) and sound velocity \( v = 4915\text{m/s} \). The 6.5mm diameter constituent units are arranged in a square lattice configuration with periodic constant \( a = 0.022\text{m} \) corresponding to a filling fraction of 0.274. As the acoustic wave propagates through the medium its velocity is reduced. Due to the lenticular form of the structure, waves that are incident upon the lens further away from the centre have a shallower angle of incidence and hence are refracted more. By appropriately shaping the lens, components of the wavefront incident upon different parts of the surface will be refracted by different amounts, bringing them into coincidence at a focal point.

The acoustic wave is first influenced by the phononic crystal lens at \( \sim 2\text{kHz} \) where the wavelength of the sound is sufficiently larger than the periodic spacing. Refraction phenomena lead to a focused region of high intensity that is observed on the decibel map surface plot (Figure 7.4(a)). Although there is focusing phenomena occurring at this frequency, the focal point covers a broad area with a maximum intensity of 2.124\text{Pa}. As the frequency is increased the focused region decreases in size and becomes more well resolved, eventually converging to a point. The frequency at which the focal point is the smallest and of highest intensity is \( \sim 5\text{kHz} \). At this frequency, the refraction mechanism is clearly observable, as two bent sound ‘rays’ emerge from the top and bottom of the lens. These coincide at a focal point with a maximum intensity of 3.388\text{Pa}. As the frequency is increased further, the wavelength of the sound becomes comparable to the periodic spacing of the scatterers and Bragg reflections occur.

The refractive index of the phononic crystal lens is found through Equation (7.9). Using the value of \( R = 0.6\text{m} \) and the measured value of \( f = 0.86\text{m} \) from the finite element calculation, the refractive index is found to be \( n = 1.35 \), corresponding to an acoustic velocity in the crystal of 254\text{ms}^{-1}. 
(a) Surface plot of the pressure (Pa) distribution, as an acoustic wave propagates through the biconvex lens. At 2kHz refraction of the incident wave leads to the formation of a poorly resolved focal point. The object distance is the length from the source to the centre of the lens, $s_o = 1.2\text{m}$.

(b) Surface plot at 5kHz, where the focal point is reached its maximum intensity. Two converging sound rays are observable emanating from the lens. The image distance is the length from the centre of the lens to the focal point, $s_i = 3\text{m}$.

(c) Pressure distribution at the central frequency of the band gap, 7.8kHz, where transmission is forbidden. Incident plane wavefronts are reflected back on themselves.

Figure 7.4
7.2 Insertion Losses and Diffraction Phenomena

7.2.1 Introduction

Until this point the phononic crystal structures investigated via the finite element method have all been approximated by two-dimensional geometries. It is acceptable to represent the systems in such a way, as the cross section of the scatterers is invariant as it extends in the $z$-direction and can therefore be assumed as infinite. This assumption is extended to the experimental systems, which although are of a finite length, are sufficiently longer along the $z$-axis relative to the $x$-axis to be assumed as quasi-two-dimensional. By considering the two-dimensional $x-y$ plane it is possible to visualise the pressure field distribution of the acoustic wave as it propagates through the phononic crystal. However, by restricting the discussion to a two dimensional approximation it is not possible to investigate how the sound field is distributed as it passes over the top of a phononic crystal screen. If we consider a projection of the array in the $x-z$ plane, it is possible to determine the influence of barrier height upon the level of attenuation and size of the deaf zone created. In addition this representation also reveals the magnitude of the diffraction phenomena occurring at the edge of the screen.

7.2.2 Theory

To understand how the presence of an array acts as an acoustic barrier, we must consider the effects of diffraction. The theory of diffraction was originally developed for optical systems before being extended to acoustics and other wave phenomena. It is one of the most complicated mathematical problems to analyse with a rigorous solution still being strived for. Due to the inherent complexity, diffraction problems are approximated by simpler methods.

Let us first define the geometry of a phononic crystal structure in the $x-z$ plane. The array of scatterers is considered a thin rigid screen, which extends a height $H$ along the $z$-axis. Acoustic waves are generated at a source $S$ and approach the screen with an incident angle $\varphi_i$. As the acoustic wave propagates over the screen, its interaction with the edge causes it to be diffracted through an angle $\varphi_d$, where its intensity is then measured at a receiver position $P$. It should be noted that in practical examples the reflection of waves from the ground will have some influence upon the attenuation levels behind the screen. However, in the simulations performed it is possible to suppress this effect by enforcing a radiation condition upon this boundary. The exact solution for the diffraction of a plane wave from a semi-infinite barrier was originally given by Sommerfeld \[96\]; however a more convenient approximation was provided by Redfearn...
He used Kirchoff’s diffraction theory [99], which embodies the fundamental concept of the Huygen-Fresnel principle and applied it to the semi-infinite barrier. After some approximations, the sound attenuation is found to be,

\[
[\text{Att}] \frac{1}{2} = -10 \log_{10} \frac{1}{2} \left[ \left\{ \frac{1}{2} - C(v) \right\}^2 + \left\{ \frac{1}{2} - S(v) \right\}^2 \right],
\]

(7.10)

where \([\text{Att}] \frac{1}{2}\) is the diffraction of a half-infinite open space and \(C(v)\) and \(S(v)\) are the cosine and sine Fresnel integrals that are functions of the variable \(v\), where,

\[
v = h_e \sqrt{\frac{2}{\lambda} \left( \frac{1}{a} \frac{1}{b} + \frac{1}{a} \right)},
\]

(7.11)

where \(h_e\) is the effective height of the barrier and \(a\) and \(b\) are the distances from the barrier to the source and receiver respectively. The Fresnel integrals are defined as [99],

\[
C(v) = \int_0^v \cos \left( \frac{\pi}{2} t^2 \right) dt
\]

(7.12)

\[
S(v) = \int_0^v \sin \left( \frac{\pi}{2} t^2 \right) dt.
\]

(7.13)

The solution to these integrals can be approximated, for small values of \(v\), by the series [100],

\[
C(v) = v - \left( \frac{\pi}{2} \right)^2 \frac{v^5}{5 \cdot 2!} + \left( \frac{\pi}{2} \right)^4 \frac{v^9}{9 \cdot 4!} - \ldots
\]

(7.14)

\[
S(v) = \left( \frac{\pi}{2} \right)^3 \frac{v^3}{3 \cdot 1!} - \left( \frac{\pi}{2} \right)^3 \frac{v^7}{7 \cdot 3!} + \left( \frac{\pi}{2} \right)^5 \frac{v^{11}}{11 \cdot 5!} - \ldots
\]

(7.15)
This series does not hold for values of \( v > 1.5 \). Therefore we use a further approximation for large \( v \) \[101\],

\[
C(v) \sim \frac{1}{2} + \frac{1}{\pi v} \sin \left( \frac{\pi}{2} v^2 \right) \tag{7.16}
\]

\[
S(v) \sim \frac{1}{2} - \frac{1}{\pi v} \cos \left( \frac{\pi}{2} v^2 \right). \tag{7.17}
\]

Combining Equations (7.10) and (7.11), Kirchoff’s diffraction theory is used to estimate values of the sound attenuation produced by a semi-infinite barrier. Depending on whether \( v > 0 \) or \( v < 0 \), the observation point \( P \) lies in the illuminated or shadow region respectively. The insertion loss \( R_E \) produced by the barrier can be further related to the path difference of the sound waves through the use of the detour law \[102\]. The detour \( \delta \) is defined as the difference between the distance a sound wave would travel from source to receiver by bending over the barrier versus the direct path of the sound wave with no barrier present. The detour law relates the insertion loss to this ratio of detour and the wavelength by,

\[
R_E \approx 10 \log \left( \frac{2\pi^2 \delta}{\lambda} \right) = 20 \log \left( \frac{\sqrt{2\pi N}}{\tanh \left( \sqrt{2\pi N} \right)} \right) + 5\text{dB}, \tag{7.18}
\]

where \( N \) is the Fresnel number given by,

\[
N = \frac{2\delta}{\lambda}. \tag{7.19}
\]
For Fresnel numbers greater than 0.36, the hyperbolic tan term \(-1 < \tanh \left( \sqrt{2\pi N} \right) \leq 1\) and can be approximated to 1 with less than 1dB margin of error [103]. Therefore Equation (7.18) reduces to,

\[
R_E = 10 \log (2\pi N) + 5\text{dB}.
\]  
(7.20)

Using this formulation the insertion loss created by a barrier can be determined from simple geometric considerations alone.

### 7.2.3 COMSOL Simulations

The insertion loss produced by a solid thin screen barrier is measured as a function of the barrier height using the finite element method. Figure (7.7) depicts an example of the geometry with a barrier extending a distance \(h\) along the \(y\)-axis. A source of circular wavefronts is located at a fixed distance of \(x\) from the left side of the barrier. A second point is chosen on the right hand side of the barrier, at which the sound pressure level is measured, its location also remaining constant as the height is varied. The insertion loss at the analyser point is found relative to a simulation through free space with no barrier present. The reference sound pressure level for propagation through free space at the receiver point was measured as 61.9dB. Impedance boundary conditions are applied to the barrier, the large mismatch of elastic constants ensuring the majority of the wave is reflected and transmission effects are negligible.

![Figure 7.7: A schematic of the finite element geometry defined to study the influence of barrier height on attenuation level, and the size of shadow zone formed. Circular wavefronts are generated at point \(S\), and the acoustic intensity is measured at the receiver point \(P\).](image)

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103 [Reference number here]
The spherical wavefronts which emanate radially outwards from the point source will interact with the barrier differently according to where they strike the surface. The point of a wavefront that is incident directly upon the screen is almost completely reflected back due to the large impedance difference between the screen and air media. Wavefronts generated that are not incident upon the screen pass too high to interact with the barrier and are therefore undisturbed. Such points of the wavefront are absorbed at the exterior boundaries of the domain. Diffraction effects are observed at the point of the propagating wavefront that is incident upon the corner edge of the barrier, the interaction with the structure causing diffraction through a small angle.

![Figure 7.8: A surface plot of the pressure (dB) distribution, at 4kHz. As the acoustic wave propagates over the top of the barrier it is diffracted through an angle, dependent upon the barrier height.](image)

Introducing a barrier into the domain causes the formation of a shadow zone behind the structure. It can be seen that the pressure field inside this zone is not uniformly distributed and that the diffraction effect has given rise to alternating fringes of minimum and maximum intensity. Therefore, we must account for this when measuring the sound pressure level. If a receiver point is chosen such that it lies in either of these extremities, the result taken will not be an accurate reflection of the sound pressure level. To compensate for this the value of the sound pressure level is taken as the most minimal point. It is acceptable to make this assumption based on the fact that the pressure map only shows a snapshot in time. If the map were visualised at some other time instance, the position of the minima and maxima may have changed but the values of intensity will remain consistent. Using this measurement of the minimal sound pressure level in conjunction with the reference level, the insertion loss produced by the barrier can be estimated using Equations (7.10 and 7.20).
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Figure 7.9: Calculated values of the insertion loss produced by a semi-infinite barrier by Kirchoff’s diffraction theory, the detour law and finite element simulation. The fifteen data points represent barrier heights, $h_e$ ranging from 30-150cm in 10cm increments.

From the data collected it can be seen that there is a good agreement between the finite element simulations of insertion loss and the values predicted by theory (Figure 7.9). For lower values of detour, the simulation results are closer to Kirchoff’s theory than the detour law, however this characteristic is reversed as the detour is increased and the values become closer to that of the detour law. The curve fitted to the simulated data has a greater gradient than that of the two parallel theory lines, however this is an artefact of the simulation geometry. As the height of the barrier was increased in each simulation the position of the source remained constant. Therefore as the barrier height increased the angle of incidence to the top edge became shallower. Consequently the diffraction angle is increased, which leads to a small level of extra attenuation [103]. The two theory lines are both plotted however for a constant diffraction angle.
Chapter 8

Negative Refraction

In this Chapter the extraordinary property of negative refraction is addressed. It begins with an introduction to the backward wave negative refraction observed in photonic and phononic crystals. A time-harmonic finite element analysis is presented to display the negative refraction of a propagating acoustic wave. In the final section of the Chapter, a second negative refraction mechanism, called forward wave, is discussed. This mechanism exists in locally resonant phononic crystals. Using the transmission line theory presented by Y.Cheng et al. [104] the negative effective bulk modulus and density is demonstrated for the Helmholtz resonator phononic crystal.

8.1 Negative Refraction in Photonic Crystals

The concept of negative refraction was first suggested for optical metamaterials\footnote{A metamaterial gains its properties from its structure rather than directly from its composition. Initial examples were developed in the 1940’s by W.E.Kock, who fabricated metal lens antennas \cite{107} and metallic delay lenses \cite{107}, microscale structures that manipulate electromagnetic waves.} by V.G.Veselago in 1968 \cite{105}. Classical refraction at the boundary between two media is governed by the ratio of the materials’ refractive indices. In optics this is related to the permittivity $\epsilon$ and permeability $\mu$ through $n = \pm \sqrt{\epsilon/\mu}$. As all known materials have positive values for both $\epsilon$ and $\mu$ - the property of negative refraction is not observed in any naturally occurring material. However, Veselago proposed that an engineered metamaterial structure with simultaneously negative $\epsilon$ and $\mu$, would permit the transmission of light and the product $\epsilon\mu$ is positive and real. It was shown that as light propagates through such a medium, the electric field, magnetic field and wave vector form a left handed set, and are hence known as left handed materials. The nature of the set of vectors $E, H, k$ implies, $S \cdot k < 0$ where $S = E \times H$ is the Poynting vector. The wave vector is opposite to the energy flow, called the backward wave effect.
Three decades passed until a metamaterial was conceived which could exhibit these extraordinary phenomena. J.B. Pendry theoretically proposed that an array of parallel metallic wires aligned along the direction of propagation of the electromagnetic waves could provide the material with a negative permittivity. Additionally, it was shown that metallic C-shaped open rings situated along the propagation axes would lead to a negative permeability. By combining the two inclusions Pendry proved theoretically that the periodic array of wires and rings would exhibit negative refraction characteristics. The design was later realised in 2001, with the first experimental demonstration of backward wave negative refraction (BWNR) reported by D.R. Smith. The phenomena was demonstrated for electromagnetic waves at frequencies within the second branch of the band structure, to achieve the backward wave vector.

8.2 BWNR in Phononic Crystals

The band structure calculations performed using the finite element show the normal modes of the system along the high symmetry directions. However, we can also investigate the modes along any direction inside the crystal. The geometrical set belonging to a particular mode, characterised by a certain frequency, is referred to as an equifrequency contour EFC. Equifrequency contours are formed in $k$-space at all points whose wave vectors correspond to the same frequency. Physically they display the magnitude of the wavevector of a plane wave, propagating in a given medium, as a function of the direction of propagation. For an isotropic medium the equifrequency contours are perfect circles as the magnitude of the wavevector is independent of the direction of propagation (Figure 8.1).

![Figure 8.1: The equifrequency contour (EFC) of an isotropic medium. The group velocity is normal to the EFC contour, as is the phase velocity that is dictated by the direction of the wave vector.](image-url)
At every point the direction of the group velocity in the medium at a given frequency coincides with the direction of the normal to the EFC (pointing towards the increase in frequency). In other words the group velocity, $v_g$, is given by the gradient of $\omega$ as a function of $k$ (as discussed in Section 3.2). On the other hand, the direction of the phase velocity is set by the direction of the wave vector, $k$. In an isotropic medium both the phase and group velocities point in the same direction (Figure 8.1). In anisotropic media the wavevector is direction dependent and the EFC’s are not circular. Phononic crystals are examples of such anisotropic media as the magnitude of the wavevector is modulated by the periodicity.

### 8.2.0.1 EFC Refraction at a Surface

We can use the principles of EFC’s to study the refraction of a plane wave at a surface. Figure (8.2) shows a simple schematic of Snell’s well known law [98]. The theorem states that the wavevectors component tangential to the interface must be conserved as it propagates from one medium to another due of translational symmetry. Furthermore, the parallel components of the wavevectors of the incident and refracted waves be equal.

![Diagram of Snell’s law](image)

The diagram also illustrates the EFC’s of the two media. The EFC’s are drawn to appropriate scales to represent the two different media. Snell’s law tells us that the parallel component in the first media must be equal to that in the second. By projecting the first onto the second, it is possible to determine the direction of propagation of the wave in the solid.
Figure 8.3: Equifrequency surface of infinite square lattice array with periodicity \( a = 0.022 \) m, and scatterer radius 6.5 mm for several frequencies in the 2\(^{nd}\) branch of the band structure. The plane wave expansion method is used to obtain the result. Calculations courtesy of D. Elford.

The EFC calculations for the experimental geometry are included in Figure 8.3. Although the phononic crystal is an anisotropic media, there are some frequencies at which the EFC is a perfect circle. These are included for 11, 12 and 13 kHz, which all correspond to the second branch of the band structure at frequencies that are sufficiently far from the Brillouin zone edge. In this frequency range the wavevector and group velocity are antiparallel and they point in opposite directions. This is a consequence of the fact that frequency is increasing with the decreasing magnitude of wave vector. Therefore, \( \nabla \omega(k) \) points along the inward normal of the EFC contour (Figure 8.4).

Figure 8.4: Left: Schematic of negative refraction, where the incident and refracted components are on the same side of the normal to the interface. Right: Negative refraction illustrated using equifrequency contours. The group velocity points opposite to the wavevector \( k \).
If we interpret this result at the refraction of a surface, Snell’s law tells us that the parallel component must be conserved in both media. However, the wavevector inside the crystal and the direction of wave propagation inside the crystal are now opposite. Therefore, both the incident and reflected components remain on the same side of the normal to the interface (Figure 8.4), and negative refraction is achieved.

8.2.0.2 COMSOL Simulation of BWNR in Phononic Crystals.

A time-harmonic analysis is performed to demonstrate backward wave negative refraction phenomena for a propagating acoustic wave. A phononic crystal is defined, comprising $4 \times 30$ Helmholtz resonator inclusions with a radius 6.5mm and periodicity $a = 0.022$m. The crystal barrier is tilted at $30^\circ$ relative to the perpendicularly travelling acoustic wave (Figure 8.5). Wave propagation is simulated in the frequency range 0-20kHz in order to ensure the range is sufficient to observe the negative refraction in the second branch of the phononic crystals band spectrum. Although the array is comprised of Helmholtz resonators the backward wave effect is not related to this mechanism. The same refraction effects exist in a rigid cylinder phononic crystal of the same periodicity.

![Figure 8.5: Schematic of the geometry used to investigate negative refraction phenomena. The incident acoustic wave propagates from left to right through the domain. At the interface a component of the wave is refracted whilst a second is reflected. Dependent upon the direction of the emanating wavefront we can identify both positive and negative refraction.](image)
Figure (8.6) illustrates the surface plots obtained. Positive refraction effects are observed at a frequency of 7.2kHz, corresponding to a wavevector inside the first allowed transmission branch. The negative refraction phenomena occurs at a frequency of 10.7kHz as predicted by the EFC calculation. From our calculations of the band structure we know this frequency corresponds to a $k$ vector in the second branch of the band spectrum. The negative refraction arises when the wavevector and group velocity are opposite and the magnitude of the wavevector decreases with increasing frequency.
Chapter 8. *Negative Refraction*

(a) Negative refraction of an acoustic wave. The refracted component lies on the same side of the normal as the incident sound wave. As such the wave is refracted upwards in the domain.

(b) Positive refraction phenomena arise in the first allowed transmission band. As in conventional optics the incident and refracted components lie on opposite sides of the normal to the interface. The refracted wave emanates downwards in the domain.

![Figure 8.6](image-url)

Figure 8.6: Manipulation of a propagating acoustic wave as it is incident upon a tilted surface. (a) Negative refraction occurs in the second branch of the crystals band structure 10.7kHz, and, (b) Positive refraction effects through the same device 7.2kHz.
Chapter 8. Negative Refraction

8.3 FWNR in Phononic Crystals

The discussion of negative refraction so far has been concerned with the backward wave effect, where the wavelength of sound is comparable with the periodic constant and strongly modified dispersions are induced due to a band folding effect [113][114][115]. However, locally resonant phononic crystals are also capable of exhibiting negative refraction effects by an entirely different mechanism. It was demonstrated by Z.Liu et al. [71] that a LRPC metamaterial, is capable of exhibiting negative elastic constant in certain frequency ranges. The composite material being comprised of rubber coated lead balls, which has since been further investigated by several other independent parties [78][116][117]. This type of negative refraction is known as forward wave, as the direction of the wavevector and group velocity are not opposite and the behaviour is similar to that of conventional wave phenomena.

N.Fang et al. [118] found negative effective dynamic modulus near the resonant frequency in a structured composite comprised of a 1D array of repeated unit cells with shunted Helmholtz resonators. Y.Cheng et al. [104][119] then extended the proposed transmission line method, using coupled Helmholtz resonators and waveguides to produce anomalous response. At certain frequency ranges, the resonant states are strongly coupled leading to strong dispersive characteristics of the effective density and modulus simultaneously.

The effective bulk modulus and effective density of a phononic crystal media are analogous to the values of permittivity and permeability for dielectric optical media. It is the square root of their product which determines the refractive index. Physically, negative effective density can be thought of as the acceleration of the acoustic particles in a direction opposite of the applied force, whilst a negative compressibility means that the acoustic particles expand under applied pressure [120].

We shall first introduce the transmission line technique as outlined by Y.Cheng et al. [104]. A preliminary investigation is performed to reproduce the published results and verify the method. Subsequently the technique is used to demonstrate a negative effective bulk modulus for the Helmholtz resonators studied in this thesis.

8.3.0.3 Transmission Line Technique - Reproduction of Published Results

Consider the geometry of Figure (8.7), a 1D chain of Helmholtz resonators periodically loaded along a side of a waveguide. Making a long wavelength assumption, the acoustic wave travelling in the waveguide can be considered analogous to an electromagnetic wave propagating along a transmission line. The acoustic counterpart of voltage difference
across an electric circuit part corresponds to the pressure difference across an acoustic
element, while the current at points in the circuit is equivalent to the volume velocity of
the fluid in the corresponding acoustic element. In addition, Helmholtz resonators can
be considered as lumped circuit elements in the shunt branch, permitting the well known
transmission line theory used for electromagnetic waves to study acoustic materials.

![Figure 8.7: Schematic illustrating the dimensional parameters of the Helmholtz res-
onator attached to the transmission line waveguide. The resonators are separated by
the period $L$ with a total of $N$ units.]

The dispersion curves can be obtained from the transmission line system for these parallel
coupled type metamaterials. The non-dissipative propagation constant $K_n$ is a complex
valued function of real frequency $\omega$ and can be obtained through,

$$
\cos (K_n L) = \cos \psi + \frac{F_n \psi \sin \psi}{2 \left( \left( \omega/\omega_0 \right)^2 - 1 \right)}, \quad (n = 1, 2, 3...),
$$

(8.1)

where $F_n = l_2 d_2 w_2 n / l^2 L$ is the total geometrical factor for structure $n$, $\psi = \omega L / c$
measures the lattice constant to incident wavelength and $\omega_0 = 2\pi f_R$ is the resonant
angular frequency.

In electromagnetic metamaterials with local resonances give rise to the simultaneous
negative effective permeability $\mu_{eff}$ and permittivity $\epsilon_{eff}$ which leads to negative refraction
in certain frequency ranges. The electromagnetic wave equations assemble the 1D
telegraphers equations [123],

$$
- \frac{\partial H_z}{\partial x} = -i\omega\epsilon_{eff} E_y \\
- \frac{\partial E_y}{\partial x} = -i\omega\mu_{eff} H_z,
$$

(8.2)

(8.3)

where $H_z$ is the magnetic field in the $z$-direction, $E_y$ is the electric field in the $y$-direction,
and the electromagnetic waves travel along the $x$-direction. In acoustic materials, the
1D microscope acoustic wave equations in the lossless case can be expressed as

\[
- \frac{\partial p}{\partial x} = -i\omega \rho_{\text{eff}} u \quad \text{(8.4)}
\]

\[
- \frac{\partial u}{\partial x} = -i\omega \kappa_{\text{eff}} p, \quad \text{(8.5)}
\]

where \( p \) is the sound pressure and \( u \) is the volume velocity. Here we neglect the harmonic dependence \( e^{i\omega t} \). Comparing (8.4) with (8.2), the compressibility \( 1/\kappa_{\text{eff}} \), the density \( \rho_{\text{eff}} \), and \( p \) and \( u \) correspond to the permeability \( \mu_{\text{eff}} \), the permittivity \( \epsilon_{\text{eff}} \), \( H_z \) and \( E_z \), respectively in the electromagnetic metamaterials.

In order to characterise the acoustic metamaterials by constitutive parameters, we derive the effective parameters \( \kappa_{\text{eff}} \) and \( \rho_{\text{eff}} \). The propagation of the acoustic wave in \( m^{th} \) unit can be described by approximate difference equations,

\[
- \frac{\partial p}{\partial x} = \frac{p_{m+1} - p_m}{L} = \frac{Z}{L} u \quad \text{(8.6)}
\]

\[
- \frac{\partial u}{\partial x} = \frac{u_{m+1} - u_m}{L} = \frac{Y}{L} p, \quad \text{(8.7)}
\]
where \( Y = 1/Z_h \) is the admittance of the Helmholtz resonator. Neglecting the absorption due to the viscosity of the fluid, the bulk modulus of the structure can be found in the lossless case by combining Equations (8.4) and (8.2),

\[
\frac{1}{\kappa_{\text{eff}}} = \frac{1}{\kappa_0} \left( \frac{F \omega_0^2}{\omega_0^2 - \omega^2} \right),
\]

(8.8)

which is a function of frequency, and \( \kappa_0 = \rho_{\text{air}} c_{\text{air}}^2 \) is the modulus of air and \( F = S_2 d_2 / S_c L \) is the volume ratio of resonator cavity to waveguide section. In electromagnetic metamaterials, the general form of permeability in the split ring resonator and wire model is expressed as [110],

\[
\mu_{\text{eff}}(\omega) = \mu_0 \left( 1 - \frac{F \omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma \omega} \right).
\]

(8.9)

Thus the corresponding parameter such as the effective bulk modulus in the acoustic system can be expressed as,

\[
\frac{1}{\kappa_{\text{eff}}} = \frac{1}{\kappa_0} \left( 1 - \frac{F \omega_0^2}{\omega^2 - \omega_0^2 + i\Gamma \omega} \right).
\]

(8.10)

where, \( \Gamma \) is the intrinsic loss of the Helmholtz resonator.
A finite element investigation is performed upon a waveguide of the same dimensions as presented by Y. Cheng et al. [104]. A negative effective bulk modulus is observed around the resonant frequency of the shunted Helmholtz units, implying a negative index of refraction (Figure 8.10). The anomalous response indicates the range of frequencies over which a metamaterial refractive device, made of such units, would display negative refraction phenomena of propagating waves.

**Figure 8.10:** The effective bulk modulus, $\kappa_{\text{eff}}$, obtained by the transmission line technique. Again the calculations presented are for dimensions of the waveguide presented by Y. Cheng [104]. The results obtained are in excellent agreement with the published data.
Effective bulk modulus ($\kappa_{\text{eff}}$), $\kappa_{\text{eff}}$

![Graph](image)

**Figure 8.11:** The evolution of the imaginary component of the function $1/\kappa_{\text{eff}}$ with frequency. Anomalous response is identified around the excitation frequency of 32kHz, where the momentum of the air in the neck of the resonators is at maximum. It is observed that the anomalous response increases with nonlinearly with an increase in the number $N$ periodic unit cells.

### 8.3.0.4 Transmission Line for the 6.5mm Helmholtz Resonators

In the preceding section we have verified the transmission line technique by reproducing the published results [104]. We now redefine the geometry of the Helmholtz resonators to represent the circular inclusions studied experimentally, namely, measurements of internal diameter $r_{\text{int}} = 5\text{mm}$, neck length, $L = 1.5\text{mm}$ and slot radius $a = 2\text{mm}$. Figure (8.12) shows the result calculated for the real part of the effective bulk modulus. A negative effective compressibility is identified at 4.95kHz. This is evidence that the index of refraction will become negative in this frequency range.

The response of the Helmholtz resonator to the fluid is $1/(\omega_0^2 - ω^2)$ [71], and is large around $\omega_0$. The response function changes its phase by 180° and a large dispersion appears when $\omega$ exceeds $\omega_0$. Since the resonator is connected to the waveguide, the oscillating field in the neck of the resonators is coupled to the sound field in the waveguide. The multiple resonators are coupled directly to their neighbours and further coupled indirectly to each other by the propagating wave in the waveguide.
Figure 8.12: The effective bulk modulus as calculated from the transmission line model. At resonance the mass of air in the neck of the Helmholtz inclusions oscillates with enough momentum to retard the acoustic field in the waveguide and negative response appears. This occurs at $\sim 5$kHz for the 6.5mm diameter circular resonators with a 4mm slot width.

The particle velocity in the necks becomes very large at resonance, and the inertia of the oscillating mass of air is consequently large. At resonance the mass of fluid contained within each separate neck vibrate together in phase and will reach a steady state after a few periods. If the frequency of the driven field exceeds $\omega_0$, the oscillators in the unit cell cannot change their phase instantly due to the inertia. Therefore in this case, the incident wave propagating down the waveguide is out of phase with respect to the motion of fluid in the neck, and the fluid in the necks has enough momentum to push the acoustic field. The Helmholtz resonator composites begin to repel the driven force and negative response appears.
Chapter 9

Conclusion

This thesis presents a comprehensive investigation into the properties, characteristics and behaviour of RCPC and LRPC structures. Specifically we study the transmission and manipulation of acoustic waves through a number of devices both experimentally and theoretically. The RCPC is a periodic structure which will forbid the transmission of a propagating acoustic waveform. Analogies of this situation can be drawn with periodic crystal structures where the principles of the well known Bloch waves can be applied to describe the behaviour of the wave. It has therefore become conventional to represent the transmission properties of a PC structure by an acoustic dispersion relation $\omega[k]$. The dispersion relation is directionally dependant due to the different symmetries of the periodic structure.

The RCPC fabricated demonstrated experimentally that it was capable of attenuating sound in the audible frequency range, in both the $\Gamma X$ and $\Gamma M$ crystal planes. Attenuation was also verified using the finite element method; however, the theoretical band structure calculations performed did not show the existence of a band gap for $\Gamma M$ transmission. This is a consequence of the low packing fraction of the structure, and the dependence of the absolute band gap upon packing fraction was investigated.

The LRPC was realised using slotted tube Helmholtz resonators. The inclusion of the resonators means that we can expect flat band degenerate states to appear in the band structure. The repulsive interaction between these bands and the otherwise linear bands of the continuum causes spectral gaps to appear in the dispersion relation around the resonant frequency. These represent the forbidden transmission bands where the acoustic energy of a propagating acoustic wave is amplified and contained, localised inside the Helmholtz resonators.
The attenuation characteristics of the LRPC was tested experimentally and also studied with an extensive finite element investigation. In addition to simulating the propagation of a time-harmonic acoustic wave through the experimental geometry other investigative studies were discussed. The dependence of attenuation level on the crystal thickness was studied, where it was observed that a five layer sample is sufficient to behave as an infinite structure capable of possessing a spectral gap. This result was confirmed with experimental measurements.

In addition to the study of the transmission properties of periodic acoustic structures we have also studied their ability to manipulate sound as it propagates through the media. The spacing between the constituent scatterers presents a lower reflective cross-section to an incident wave than a solid medium. Furthermore, the ability of a phononic crystal to reduce the velocity of a propagating plane wave at certain frequencies make them the ideal media with which to design refractive acoustic devices. Two analogies with conventional optics are made and the refractive indices of a phononic crystal biprism and lens were determined.

In the final chapter of this thesis the phenomena of negative refraction was studied. Strongly modified dispersions leading to negative refraction can be ascribed to two mechanisms. The first is induced by a band folding effect around the Bragg frequency. This is known as BWNR as the group velocity, i.e. the direction of energy transport, points in the opposite direction to the propagating wave vector and phase velocity. The phenomenon occurs around the Bragg frequencies as the wavevector decreases in magnitude as the corresponding frequency increases. The second mechanism that can lead to strongly modified dispersions occurs in LRPC structures around the resonant excitation frequencies. At these frequencies it has been shown that PC materials are capable of exhibiting effective negative elastic constants. The existence of negative bulk modulus and density defines a material with a negative index of refraction. Using a transmission line technique with periodically spaced Helmholtz resonators attached to a waveguide, the existence of a negative effective bulk modulus was demonstrated theoretically for the Helmholtz resonators used in the experimental investigation.

The LRPC sample demonstrated that it was capable of attenuating sound at low frequencies of the audible range, in keeping with the original scope of the thesis. The techniques of mixing cavity sizes achieves a method of control over the frequency range the attenuation spans, and enlarging them sufficiently for practical soundproofing applications. Furthermore, the concentric arrangement of the proposed Matryoshka resonators provide significant space saving, an important consideration when assessing their potential for soundproofing applications.
Appendix A

COMSOL Script

% COMSOL Multiphysics Model M-file
% Generated by COMSOL 3.2a (COMSOL 3.2.0.300, $Date: 2005/12/20 19:02:30 $)

flclear fem

% COMSOL version
clear vrsn
vrsn.name = COMSOL 3.2 ;
vrsn.ext = a ;
vrsn.major = 0;
vrsn.build = 300;
vrsn.rcs = $Name: $ ;
vrsn.date = $Date: 2005/12/20 19:02:30 $ ;
fem.version = vrsn;

% Geometry
[108x510]g1=rect2( 0.022 , 0.022 , base , corner , pos ,
[361x510]{ 0 , 0 }
[409x510], rot , 0 );
g2=circ2( 0.0065 , base , center , pos ,
{ 0.011 , 0.011 }, rot , 0 );
[g3]=geomcopy({g2});
[g4]=geomcopy({g3});
g4=move(g4,[0,0]);
g5=circ2( 0.005 , base , center , pos ,
{ 0.011 , 0.011 }, rot , 0 );
g6=rect2(0.012,0.0080, base , center , pos ,{0.0060,0.011});
g7=rect2( 0.005 , 0.004 , base , center , pos ,
{ 0.0060 , 0.011 }, rot , 0 );
g8=geomcomp({g4,g5}, ns ,{ g4 , g5 }, sf , g4-g5 , edge , none );
g9=geomcomp({g8,g7}, ns ,{ g8 , g7 }, sf , g8-g7 , edge , none );
g10=geomcomp({g1,g9}, ns ,{ g1 , g9 }, sf , g1-g9 , edge , none );

% Geometry
clear s
s.objs={g10};
s.name={ CO1 }; s.tags={ g10 }; fem.draw=struct( s ,s); fem.geom=geomcsg(fem);

n=50
for ii=1:n
    ii
    kx(ii)=(ii-1)/(n-1)*pi
    ky(ii)=(ii-1)/(n-1)*0

    fem.const = {
        ky , num2str(ky(ii)), ...
        kx , num2str(kx(ii))}

% Initialize mesh
fem.mesh=meshinit(fem);

% Refine mesh
fem.mesh=meshrefine(fem, ...
    mcase ,0, ...
    rmethod , regular );

% (Default values are not included)

% Application mode 1
clear appl
appl.mode.class = Acoustics ;
appl.assignsuffix = .aco ;
clear prop
prop.analysis= eigen ;
appl.prop = prop;
fem.appl{1} = appl;
fem.frame = { ref }
;
fem.border = 1;
fem.units = SI ;

% Coupling variable elements
clear elemcpl
% Extrusion coupling variables
clear elem
elem.elem = elcplextr ;
elem.g = { 1 };
src = cell(1,1);
clear bnd
bnd.expr = {{ p ,{}},{}},{{}, p ,{}};
bnd.map = {{ 1 , 1 , 1 },{ 1 , 1 , 1 }};
bnd.ind = {{}, {1},[],{2},[],{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, ... 14}};
src(1) = {{}, bnd, {}};
 elem.src = src;
geomdim = cell(1,1);
clear bnd
bnd.map = {{{}, {}}, {}, {3, 3}};
bnd.ind = {{}, {1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14}, {3, 3}, ... {6}};
geomdim(1) = {{}, bnd, {}};
 elem.geomdim = geomdim;
 elem.var = {pconstr1, pconstr2};
map = cell(1,3);
clear subplot
subplot.type = unit;
map(1) = subplot;
clear subplot
subplot.type = linear;
 subplot.sg = 1;
 subplot.sv = {14, 13};
 subplot.dg = 1;
 subplot.dv = {2, 1};
map(2) = subplot;
clear subplot
subplot.type = linear;
 subplot.sg = 1;
 subplot.sv = {2, 14};
 subplot.dg = 1;
 subplot.dv = {1, 13};
map(3) = subplot;
elem.map = map;
elemcpl(1) = elem;
% Point constraint variables (used for periodic conditions)
clear elem
elem.elem = elpconstr;
elem.g = {1};
clear bnd
bnd.constr = {{0, pconstr2-(p*exp(i*ky))}, {pconstr1-(p*exp(i*kx)), 0}};
bnd.cpoints = {{2, 2}, {2, 2}};
bnd.ind = {{3}, {6}};
 elem.geomdim = {[{}, bnd, {}}};
elemcpl(2) = elem;
fem.elemcpl = elemcpl;

% Multiphysics
fem=multiphysics(fem);
%
% Extend mesh
fem.xmesh=meshextend(fem);

% Solve problem
fem.sol=femeig(fem, ...
    conjugate, on, ...
    symmetric, on, ...
    solcomp,{ p }, ...
    outcomp,{ p }, ...
    neigs,10, ...
    linsolver, spooles );

% Save current fem structure for restart purposes
fem0=fem;

freq(1:10,ii)=real(sqrt(fem.sol.lambda(1:10)));

end

plot(kx,freq(:,::), . )
Appendix B

MATLAB FT Code

```
fs = 1000; % Sample frequency (Hz)
t = 0:1/fs:10-1/fs; % 10 sec sample
x = wavread('C:\Documents and Settings\phlc2-admin\My Documents\Thesis\Wave files\bragg\post crystal square [100] 2.wav');

m = length(x); % Window length
n = pow2(nextpow2(m)); % Transform length
y = fft(x,n); % DFT
f = (0:n-1)*(fs/n); % Frequency range
power = y.*conj(y)/n; % Power of the DFT

plot(f,power)
xlabel('Frequency (Hz)')
ylabel('Power')
title('{\bf Periodogram}')

y0 = fftshift(y); % Rearrange y values
f0 = (-n/2:n/2-1)*(fs/n); % 0-centered frequency range
power0 = y0.*conj(y0)/n; % 0-centered power

plot(f0,power0)
xlabel('Frequency (Hz)')
ylabel('Power')
title('{\bf 0-Centered Periodogram}')

phase = unwrap(angle(y0));

plot(f0,phase*180/pi)
xlabel('Frequency (Hz)')
ylabel('Phase (Degrees)')
grid on
```
Bibliography


[57] E.A.Rietman & J.M.Glynn; Band-gap engineering of phononic crystals: a computational survey of two-dimensional systems, PAPER NUMBER,


