Internal tide generation at the continental shelf modelled using a modal decomposition: two-dimensional results.

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Internal tide generation at the continental shelf
modelled using a modal decomposition:
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Abstract

Stratified flow over topography is studied, with oceanic applications in mind. We develop a model for a fluid with arbitrary vertical stratification and a free surface, flowing over three-dimensional topography of arbitrary size and steepness, with background rotation, in the linear hydrostatic regime. The model uses an expansion of the flow fields in terms of a set of basis functions which efficiently capture the vertical dependence of the flow. The horizontal structure may then be found by solving a set of coupled partial differential equations in two horizontal directions and time, subject to simple boundary conditions. In some cases these equations may be solved analytically, but in general simple numerical procedures are required. Using this formulation, we calculate the internal tide generated by a time-periodic barotropic tidal flow over a continental shelf and slope, in various idealised configurations. We take the topography and fluid motion to be independent of one coordinate direction, and the fluid to be either two-layer or uniformly stratified. For the two-layer case, we derive expressions for the shoreward and oceanward energy fluxes associated with the internal tide. For the uniformly stratified case, we study numerically how the accuracy of the solutions depends upon the number of basis functions used, and show that good solutions and energy flux estimates can often be obtained with only a few basis functions. In both cases our results show that the position of the coastline, through its effect on the form of the barotropic tide, significantly affects the strength of the internal tide generation.

1 Introduction

Internal tides are ubiquitous in the oceans. They are internal waves, generated when stratified fluid is displaced vertically as open ocean tides interact with topography (e.g. Wunsch 1975). Such motions, whose period equals that of the tidal forcing and is thus typically diurnal or semi-diurnal, are often observed near continental slopes (e.g. Pingree and New 1989; Holloway et al. 2001; Lien and Gregg 2001), and near mid-ocean ridges (e.g. Ray and Mitchum 1996). The importance of these internal tides as an agent for mixing in the deep ocean is recognised (e.g. Egbert and Ray 2000; St. Laurent and Garrett 2002), along with their probable role in the dynamics of the large scale ocean circulation (e.g. Munk and Wunsch 1998; Wunsch 2000).

Here we are interested in studying the generation of internal tides. Models of this process remain imperfect, largely because of the complex nature of the topography typically involved. One approach is to use an ocean circulation model (e.g. Holloway and Merrifield 1999; Khatiwala 2003), although when simulating fine-scale internal wave beams there are limitations associated with numerical dissipation and resolution. An alternative approach is to construct analytical or idealised numerical models of the generation process, although it is then necessary to make certain simplifying assumptions. Such studies have, broadly speaking, taken one of two distinct approaches, in both of which one attempts to diagnose the internal wave response to a prescribed time-harmonic barotropic tidal flow.

In the first approach it is assumed that the topography is 'weak', i.e. that the topographic height variations are much less than the depth of the ocean, and that the topographic slopes are small relative to the characteristic slope of the internal waves. Then one can study internal wave generation in a continuously stratified fluid flowing over complex topography in three dimensions. Following the early work of Bell (1975), which was restricted to uniformly stratified flow in a vertically unbounded domain, the effects of a free surface and non-uniform stratification have been consid-
In the second approach it is assumed that the flow fields are two-dimensional, i.e. that the bottom topography, and hence the internal tides, are independent of one horizontal direction. Analytical solutions are then possible for two-layer flow over step topography (Rattray 1960, Weigand et al. 1969), and for uniformly stratified flow over weak topography (Cox and Sandstrom 1962) or a knife-edge ridge (Llewellyn Smith and Young 2003). At the next level of complexity the equations of motion can be reduced to a relatively simple form, and then solved numerically. Uniformly stratified flow over step topography has been considered by Rattray et al. (1969), Prinsenberg et al. (1974), Stigebrandt (1980), and St. Laurent et al. (2003), whilst piecewise linear topography and non-uniform stratification were considered by Prinsenberg and Rattray (1975). Smooth topographies have since been studied, once again for uniform stratification, most notably by Balmforth et al. (2002) for a certain class of vertically unbounded flow, and by Petrelis et al. (2006) using a more general Green’s function approach. Finally, there exist formulations to deal with quite general stratifications and topographic slopes, although these typically need to be solved with quite complicated numerical schemes. Such methods, based upon consideration of wave characteristics, include those of Baines (1974, 1982), Sandstrom (1976) and Craig (1987).

Both approaches have limitations. The weak topography approach cannot be used to study wave generation near the continental shelf-break or at slopes of the same order as the characteristic slope of the internal waves, two regions which are expected to give large internal tides. The two-dimensional approach cannot be used to study wave generation over three-dimensional topography. There is a clear need to find simple approaches for more realistic geometries.

Here we introduce a formulation of the internal tide generation problem that permits simple idealised scenarios to be studied analytically, and more complex scenarios to be studied numerically. It is valid for three-dimensional flows with a free surface and arbitrary vertical stratification, and background rotation, in the linear hydrostatic regime. The method is based upon an expansion of the flow in terms of a set of basis functions which efficiently describe the vertical dependence of the barotropic tide and the internal wave field, and which enable the top and bottom boundary conditions to be satisfied with ease. The horizontal structure and time-dependence may then be found by solving a set of partial differential equations in two horizontal directions and time. In regions of varying topography, there is strong coupling between the equations, corresponding to an interaction between the barotropic and baroclinic modes. If one wishes to proceed further analytically at this stage, then it is necessary to make some additional assumptions about the stratification and the topography. Whilst a major aim is to apply this formulation to study three-dimensional processes, such as internal tide generation in the presence of canyons, here we only present solutions for cases where the topography and fluid motion are both independent of one horizontal coordinate direction. For detailed examples we focus on wave generation at the continental slope, where the tidal flows are generally strong and the topography is relatively steep.

Our approach is as follows. In section 2, we introduce the three-dimensional modal decomposition, and derive a system of coupled partial differential equations for the expansion coefficients. We also derive expressions for the interaction coefficients, determining the strength of the coupling between the modes, in terms of the linear dispersion relation for flow over flat topography. In section 3, we introduce the usual Boussinesq approximation, which distinguishes the barotropic and baroclinic modes, and enables us to derive simplified equations for their interaction. In section 4, we formulate the internal tide generation problem for the case where the topography and fluid motion are both independent of one horizontal direction. In particular, we give an equation for the forcing of baroclinic modes by a prescribed time-harmonic barotropic tidal flow, under the Boussinesq approximation. In section 5 we derive an analytical solution for internal tide generation in a two-layer fluid, for a particular family of curved continental slopes. In section 6 we present corresponding results for a uniformly stratified fluid, and examine to what extent the system may be truncated, perhaps to include just a few modes, whilst still capturing the dynamics of interest. We conclude in section 7.

2 A three-dimensional modal decomposition

We study a layer of stratified fluid, with a free surface, flowing over topography. The motion is three-dimensional, with background rotation, and is taken to be hydrostatic (because the frequency of the motion is small compared with the buoyancy frequency, and the horizontal lengthscale of the motion is large compared with the depth of the fluid). We use a set of Cartesian coordinates $x$, $y$, $z$, with the $z$-axis pointing vertically upwards. At rest, the free surface of the fluid is at $z = 0$, there is a stable density stratification $\rho_0(z)$, and a corresponding hydrostatic pressure
\( p_0(z) \). We will use a set of equations linearised about this state of rest:

\[
\frac{\partial u}{\partial t} - f v = - \frac{1}{\rho_0} \frac{\partial \hat{p}}{\partial x}, \quad \frac{\partial v}{\partial t} + f u = - \frac{1}{\rho_0} \frac{\partial \hat{p}}{\partial y}, \quad \frac{\partial \hat{p}}{\partial z} = -g \hat{p}, \quad \hat{p} = \rho_0 \frac{\partial \hat{p}}{\partial t} + \rho_0 \frac{\partial \rho_0}{\partial z} w = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1a, b, c, d, e}
\]

where \( u, v, w \) are the fluid flow speeds in the \( x, y \) and \( z \) directions respectively, \( \hat{p} \) and \( \rho_0 \) are the pressure and density deviations from the background state, \( f \) is the Coriolis parameter, and \( g \) is the acceleration due to gravity.

We solve (1a–e) for free surface flow over topography defined by \( z = -h(x,y) \). The boundary conditions are \( u \cdot n = 0 \) at \( z = -h \), and the (linearised) condition for continuity of pressure at the free surface, which are, respectively

\[
w = -v \frac{\partial h}{\partial x} - u \frac{\partial h}{\partial y} \text{ at } z = -h, \tag{2a}
\]

\[
\hat{p} = g \rho_0(0) \zeta \text{ at } z = 0, \tag{2b}
\]

where we have introduced the (linearised) vertical particle displacement \( \zeta \), defined by

\[
w = \partial \zeta / \partial t. \tag{3}
\]

### a The normal modes

Our method of solution will depend upon the expansion of the fluid velocity in terms of a set of basis functions. A particularly appropriate set arises from consideration of a simple two-dimensional system with \( f = v = 0 \) and constant \( h \). Looking for solutions of the form \( \zeta = A(x-ct)\phi(z) \), (1a–e) and (2a,b) are satisfied provided

\[
\frac{\partial}{\partial z} \left( \rho_0 \frac{\partial \phi}{\partial z} \right) + \rho_0 N^2 \phi = 0, \tag{4a}
\]

\[
\phi = 0 \text{ at } z = -h, \quad c^2 \frac{\partial \phi}{\partial z} = g \phi \text{ at } z = 0, \tag{4b, c}
\]

where \( N^2(z) = -g \theta / \rho_0 / \partial z / \partial z \). This is an eigenvalue problem for the wave speed \( c \). Typically there will be an infinite discrete set of eigenvalues \( \lambda_n, n = 0, 1, 2, \cdots \), with corresponding eigenfunctions \( \phi_n \) which form a complete set for a certain class of functions on \( -h \leq z \leq 0 \). The eigenfunctions also satisfy an orthogonality condition, which can be established using (4a–c):

\[
\int_{-h}^{0} \rho_0 \frac{\partial \phi_m}{\partial z} \frac{\partial \phi_n}{\partial z} dz = 0, \quad n \neq m. \tag{5}
\]

When \( n = m \) the integral does not vanish, and we write

\[
I_m = \int_{-h}^{0} \rho_0 \left( \frac{\partial \phi_n}{\partial z} \right)^2 \, dz. \tag{6}
\]

Since (4a–c) is homogeneous in \( \phi \), one can uniquely specify each eigenfunction by choosing a value for \( I_m \).

### b The three-dimensional expansion

We now proceed with the description of a general three-dimensional fluid motion with background rotation and topography \( h(x,y) \), as governed by (1a–e). We use \( u \) and \( v \) as our main working variables, and write them in the form

\[
u = \sum_{n=0}^{\infty} U_n(x,y,t) \frac{\partial \phi_n}{\partial z}, \quad v = \sum_{n=0}^{\infty} V_n(x,y,t) \frac{\partial \phi_n}{\partial z}, \tag{7}
\]

where the \( \phi_n \) are the eigenfunctions of (4a–c). However, note that the \( \phi_n \) now depend on \( x \) and \( y \) through their dependence on \( h(x,y) \). To emphasise this, we regard the \( \phi_n \) as functions of \( z \) and \( h \), and write \( \phi_n = \phi_n(z,h) \). Then, differentiating (4b) w.r.t. \( h \) it follows that

\[
\frac{\partial \phi_n}{\partial h} = \frac{\partial \phi_n}{\partial z} \text{ at } z = -h. \tag{8}
\]

Our aim is to find evolution equations for the coefficients \( U_n(x,y,t) \) and \( V_n(x,y,t) \) and \( \nabla_H = e_x \partial / \partial x + e_y \partial / \partial y \), from (1e) and (7)

\[
\frac{\partial \nu}{\partial z} = -\sum_{n=0}^{\infty} \nabla_H \cdot \left( U_n \frac{\partial \phi_n}{\partial z} \right) = -\sum_{n=0}^{\infty} \frac{\partial}{\partial z} \nabla_H \cdot (U_n \phi_n). \tag{9}
\]

Integrating from \(-h \) to \( z \), and exchanging the order of summation and integration on the right-hand side, we find

\[
w = -\sum_{n=0}^{\infty} (U_n \cdot \nabla_H \phi_n) \frac{\partial \phi_n}{\partial z} \bigg|_{z=-h}
\]

\[
-\sum_{n=0}^{\infty} \nabla_H \cdot (U_n \phi_n) + \sum_{n=0}^{\infty} (U_n \cdot \nabla_H \phi_n) \bigg|_{z=-h},
\]

where, on the right-hand side, the first term is just \( w(z = -h) \) evaluated using (2a) and (7), and the third term has been simplified using (4b). But \( \nabla_H \phi_n = \partial \phi_n / \partial h \nabla_H h \), so that the first and third terms cancel using (8), yielding

\[
w = -\sum_{n=0}^{\infty} \nabla_H \cdot (U_n \phi_n). \tag{9}
\]
We can now obtain an expression for the pressure deviation \( \tilde{p} \). Integrating (1c) w.r.t. \( z \), substituting from (2b), differentiating w.r.t. \( t \) and substituting from (1d) and (3), we have

\[
\frac{\partial \tilde{p}}{\partial t} = g \rho_0(0) w(z = 0) + \int_0^z g \frac{\partial \rho_0}{\partial z} w \, dz.
\]

Substituting from (9), and exchanging the order of integration and summation, the right-hand side of this expression may be written as

\[
\sum_{n=0}^{\infty} \nabla_H \cdot \left[ \left( \int_0^z \rho_0 N^2 \phi_n \, dz - g \rho_0(0) \phi_n(z = 0) \right) U_n \right],
\]

so that using (4a,c) one obtains

\[
\frac{\partial \tilde{p}}{\partial t} = - \sum_{n=0}^{\infty} \rho_0(z) \nabla_H \cdot \left( c_n^2 \frac{\partial \phi_n}{\partial z} U_n \right).
\]

Thus, given \( \tilde{p}_r(x, y, z) = \tilde{p}(t = t_r) \),

\[
\tilde{p} = \tilde{p}_r - \sum_{n=0}^{\infty} \rho_0 \nabla_H \cdot \left( c_n^2 \frac{\partial \phi_n}{\partial z} \int_{t_r}^{t} U_n \, dt \right).
\]

The only equations remaining unsatisfied are (1a,b). Using (7) and (10) they become

\[
\begin{align*}
\sum_{n=0}^{\infty} \rho_0 \left( \frac{\partial U_n}{\partial t} - f V_n \right) \frac{\partial \phi_n}{\partial z} & = - \frac{\partial \tilde{p}_r}{\partial x} + \sum_{n=0}^{\infty} \rho_0 \frac{\partial \phi_n}{\partial z} \nabla_H \cdot \left( c_n^2 \frac{\partial \phi_n}{\partial z} \int_{t_r}^{t} U_n \, dt \right), \\
\sum_{n=0}^{\infty} \rho_0 \left( \frac{\partial V_n}{\partial t} + f U_n \right) \frac{\partial \phi_n}{\partial z} & = - \frac{\partial \tilde{p}_r}{\partial y} + \sum_{n=0}^{\infty} \rho_0 \frac{\partial \phi_n}{\partial z} \nabla_H \cdot \left( c_n^2 \frac{\partial \phi_n}{\partial z} \int_{t_r}^{t} U_n \, dt \right).
\end{align*}
\]

Expanding the \( \partial / \partial x_i \nabla_H \) terms, writing \( \partial (c_n^2 \phi_n) / \partial x_i = \partial h / \partial x_i \times \partial (c_n^2 \phi_n) / \partial h \) etc., and using (5) and (6), these become

\[
\begin{align*}
\frac{\partial U_m}{\partial t} - f V_m & - c_m^2 \frac{\partial^2}{\partial x^2} \int_{t_r}^{t} U_m \, dt - c_m^2 \frac{\partial^2}{\partial x \partial y} \int_{t_r}^{t} V_m \, dt = - \frac{1}{I_m} \int_{-h}^{0} \frac{\partial \phi_m}{\partial z} \frac{\partial \tilde{p}_r}{\partial x} \, dz \\
& + c_m^2 \sum_{n \geq 0} \left( I_1(m, n) \left( \frac{\partial^2 h}{\partial x^2} + 2 \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} \right) + I_2(m, n) \left( \frac{\partial h}{\partial x} \right)^2 \right) \int_{t_r}^{t} U_m \, dt \\
& + c_m^2 \sum_{n \geq 0} \left( I_1(m, n) \left( \frac{\partial^2 h}{\partial x \partial y} + \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} + \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} \right) + I_2(m, n) \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} \right) \int_{t_r}^{t} V_m \, dt,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial V_m}{\partial t} + f U_m & - c_m^2 \frac{\partial^2}{\partial y^2} \int_{t_r}^{t} V_m \, dt - c_m^2 \frac{\partial^2}{\partial x \partial y} \int_{t_r}^{t} U_m \, dt = - \frac{1}{I_m} \int_{-h}^{0} \frac{\partial \phi_m}{\partial z} \frac{\partial \tilde{p}_r}{\partial y} \, dz \\
& + c_m^2 \sum_{n \geq 0} \left( I_1(m, n) \left( \frac{\partial^2 h}{\partial y^2} + 2 \frac{\partial h}{\partial y} \frac{\partial h}{\partial x} \right) + I_2(m, n) \left( \frac{\partial h}{\partial y} \right)^2 \right) \int_{t_r}^{t} V_m \, dt \\
& + c_m^2 \sum_{n \geq 0} \left( I_1(m, n) \left( \frac{\partial^2 h}{\partial y \partial x} + \frac{\partial h}{\partial y} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial h}{\partial x} \right) + I_2(m, n) \frac{\partial h}{\partial y} \frac{\partial h}{\partial x} \right) \int_{t_r}^{t} U_m \, dt,
\end{align*}
\]
where

\[ \mathcal{I}_j(m, n) = \frac{1}{c_m^2 c_m} \int_{-h}^{0} \rho_0 \frac{\partial \phi_m}{\partial z} \frac{\partial}{\partial z} \left( c_m^2 \frac{\partial \phi_n}{\partial z} \right) \, dz. \]  

(12)

When \( h \) is constant, the summations on the right-hand side of (11a,b) disappear, and the equations may be reduced to a wave equation for \( U_m \) or \( V_m \).

c Summary

Using the expansion (7), the five equations (1a–e) and two boundary conditions (2a,b) are converted to a linear coupled system (11a,b) for \( U_n(x, y, t) \) and \( V_n(x, y, t), \ n = 0, 1, 2, \cdots \). For a given stratification \( \rho_0(z) \), it is possible to calculate the eigenfunctions \( \phi_n(z, h) \) and the wavespeed relation \( c_n = c_n(h) \) from (4a–c), perhaps numerically, and thus to evaluate the interaction coefficients from (12). The major benefit is that the \( z \)-dependence and the complex geometry \( -h(x, y) \leq z \leq -\zeta(x, y, t) \) have been eliminated from the formulation, and the free-surface and bottom boundary conditions are automatically satisfied. Equations (11a,b) are only subject to initial conditions and lateral boundary conditions. Even so, we are still left with an infinite set of coupled equations, and it remains to be shown how useful this formulation is in practice. We address this in sections 5 and 6.

d The interaction coefficients

Rather than directly performing the integrals in (12), one may derive expressions for \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) in terms of \( N, I_m \) as given by (6), and \( c_m \). The details of this calculation are given in Appendix A, where we employ the notation

\[ \lambda_m(h) = c_m^{-2}(h). \]  

(13)

It is immediately apparent from (A6)–(A9) that \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) depend upon \( I_m \) – different normalisations correspond to different dependencies of the \( \phi_n \) on \( h \), and hence on \( x \) and \( y \), and thus different equations for \( U_n \) and \( V_n \). For the remainder of this work, we fix the \( h \)-dependence by taking

\[ I_m = g \rho_0(0) \hat{I}_m c_m^{-2} = g \rho_0(0) \hat{I}_m \lambda_m, \]  

(14)

where the \( \hat{I}_m \) are positive, dimensionless constants, independent of \( h \), to be chosen later. Then, the \( \phi_m \) are dimensionless, and

\[ \mathcal{I}_1(m, m) = 0, \]  

(15a)

which is convenient mathematically. The other interaction coefficients then reduce to

\[ \mathcal{I}_1(m, n) = \frac{\left( \hat{I}_n \lambda_m \lambda_n / \hat{I}_m \right)^{\frac{1}{2}}}{\lambda_m - \lambda_n}, \quad n \neq m, \]  

(15b)

\[ \mathcal{I}_2(m, m) = \frac{1}{12} \left( \frac{2 \lambda_m^{\prime\prime\prime} c_m^2}{\lambda_m^{\prime\prime} c_m^2} - \frac{3 \lambda_m^{\prime\prime} c_m^2}{\lambda_m^{\prime\prime} c_m^2} - 4 N^2 \lambda_m \right) \]  

\[ + \frac{N^4}{g} - \frac{2 \partial N^2}{g \partial z} \right|_{z=-h}, \]  

(15d)

where we have noted that \( \lambda_m^{\prime\prime\prime} < 0 \), from (A4).

Again, rather than directly performing the integral (6) to impose (14), it is generally easier to use an equivalent condition obtainable from (8) and (A4):

\[ \frac{\partial \phi_n}{\partial z} = \left( \frac{g \rho_0(0) \hat{I}_m \lambda_n}{g \rho_0(-h)} \right)^{\frac{1}{2}} \]  

at \( z = -h \).  

(16)

3 The Boussinesq approximation

Although the formulation is complete using (15a–d) in (11a,b), for many geophysical applications, including those involving internal tides, it proves beneficial to make the Boussinesq approximation. This leads to a clear conceptual difference between a barotropic mode and a set of baroclinic modes, a difference not yet reflected in our formulation. Thus, we suppose that the vertical density differences within the fluid layer are small compared with the background density, i.e.

\[ \frac{\rho_0(-h) - \rho_0(0)}{\rho_0(0)} = \epsilon(h) \ll 1. \]  

(17)

To exploit this, we nondimensionalise (4a–c). We write \( \tilde{z} = z/h \) and \( N = N/\mathcal{N} \), where \( \mathcal{N} \) is an average buoyancy frequency defined by

\[ \mathcal{N}^2 = \frac{1}{h} \int_{-h}^{0} N^2 \, dz = \frac{g}{h} \log(1 + \epsilon) \sim \frac{g\epsilon(h)}{h}, \]  

(18)

the second equality following from the definition of \( N^2 \). Then (4a–c) become

\[ \frac{\partial^2 \phi}{\partial \tilde{z}^2} - \mathcal{N}^2 \log(1 + \epsilon) \left( \frac{\partial \phi}{\partial \tilde{z}} - \frac{gh}{c_m^2} \phi \right) = 0, \]  

(19a)

\[ \phi = 0 \text{ at } \tilde{z} = -1, \quad \frac{c_m^2 \partial \phi}{g \partial \tilde{z}} = \phi \text{ at } \tilde{z} = 0. \]  

(19b,c)
For $\epsilon \ll 1$, there are only two possible leading order balances:

(i) $\phi_{,zz}^2 = 0$, with $gh/c^2 = O(1)$. Then (19b,c) imply $\phi \propto z + 1$, and $c^2 = gh$, i.e. there is only one such mode. We label this mode with the subscript $n = 0$, and refer to it as the barotropic mode. It corresponds approximately to the motion of a surface gravity wave upon a homogeneous fluid. Choosing a normalising constant $\bar{I}_n = 1$ in (14) and (16), to leading order in $\epsilon$ we have

$$\phi_0 = 1 + z/h, \quad \bar{I}_0 = 1, \quad \lambda_0 = (gh)^{-1}.$$  

(ii) Alternatively, $c^2$ scales with $\epsilon gh$, or equivalently $(\mathcal{N}h)^2$. Then, to leading order in $\epsilon$, (19a-c) become

$$\left( \frac{\partial^2}{\partial z^2} + \frac{N^2}{c_n^2} \right) \phi_n = 0, \quad \phi_n = 0 \text{ at } z = -h, 0,$$  

whilst (16) becomes

$$\partial^2 \phi_n / \partial z = \left| \frac{\bar{I}_n \lambda_n}{gh^2} \right|^{1/2} \text{ at } z = -h,$$  

reverting to dimensional variables. We shall choose convenient values for the constants $\bar{I}_n$ from case to case, according to the background stratification. We label these modes with subscripts $n = 1, 2, 3, \ldots$, and refer to them as the baroclinic modes:

$$c_n \sim \mathcal{N}h, \quad \lambda_n \sim (\mathcal{N}h)^{-2}, \quad n \geq 1.$$  

They correspond to internal gravity waves for which there is little surface displacement.

Thus the outcome is the usual Boussinesq approximation in the governing equations, with a 'rigid lid' upper boundary condition for the baroclinic modes. Note the resulting scale separation:

$$c_n/c_0 \sim \epsilon^{1/2}, \quad \lambda_0/\lambda_n \sim \epsilon, \quad n \geq 1,$$  

using (18), (21) and (24). The baroclinic wavespeed is much less than the barotropic wavespeed. This can be exploited to derive simplified versions of the interaction coefficients consistent with the Boussinesq approximation:

(i) We calculate (15b–d) for $m = 0$. For $\mathcal{I}_1(0, n)$ and $\mathcal{I}_2(0, n)$, using (25) we replace $\lambda_0 - \lambda_n$ by $-\lambda_n$ for $n \geq 1$. Then using (20) and (21) for $\bar{I}_0$ and $\lambda_0$, we obtain

$$\mathcal{I}_1(0, n) = - \left| \frac{\bar{I}_n \lambda_n}{gh^2 \lambda_n^2} \right|^{1/2}, \quad n \geq 1, \quad (26a)$$  

$$\mathcal{I}_2(0, n) = - \left| \frac{\bar{I}_n \lambda_n}{gh^2 \lambda_n^2} \right|^{1/2} \left( \frac{\lambda''_n}{\lambda'_n} - \frac{2 \lambda'_n}{\lambda_n} \right), \quad n \geq 1. \quad (26b)$$

For $\mathcal{I}_2(0, 0)$, we note that the first two terms of (15d), which scale like $h^{-2}$, are greater by a factor $\epsilon^{-1}$ than the remaining terms, the largest of which scale like $\mathcal{N}^2/gh$. However, $2\lambda''_m/\lambda_0 - 3\lambda''_0/\lambda_0^2 = 0$ to leading order in $\epsilon$, from (21), so that the leading order contribution to $\mathcal{I}_2(0, 0)$ vanishes:

$$\mathcal{I}_1(0, 0) = 0, \quad \mathcal{I}_2(0, 0) = 0. \quad (26c, d)$$

(ii) We calculate (15b–d) for $m \geq 1$. For $\mathcal{I}_1(m, 0)$ and $\mathcal{I}_2(m, 0)$, using (25) we replace $\lambda_m - \lambda_0$ by $\lambda_m$. Further, in $\mathcal{I}_2(m, 0)$, $\lambda''_0/\lambda_0 = -2h^{-1} \gg \epsilon h^{-1} \sim \lambda'_0/\lambda_m$. Then using (20) and (21)

$$\mathcal{I}_1(m, 0) = \frac{1}{h \lambda_m} \left| \frac{\lambda'_m}{g \bar{I}_m} \right|^{1/2},$ 

$$\mathcal{I}_2(m, 0) = - \frac{2}{h^2 \lambda_m} \left| \frac{\lambda'_m}{g \bar{I}_m} \right|^{1/2}, \quad m \geq 1. \quad (27a, b)$$

Otherwise, no approximation is possible for $\mathcal{I}_{1,2}(m, n)$ with $n \neq m$. For $\mathcal{I}_2(m, m)$, the first three terms of (15d) scale like $h^{-2}$, whilst the fourth and fifth scale like $\epsilon^2 h^{-2}$ and $\epsilon h^{-2}$ respectively. Thus, to leading order in $\epsilon$, for $m \geq 1$:

$$\mathcal{I}_1(m, m) = 0, \quad (27c)$$  

$$\mathcal{I}_2(m, m) = \frac{1}{12} \left( \frac{2 \lambda''_m}{\lambda_m} - \frac{3 \lambda''_0}{\lambda_0^2} - 4 \lambda_m N_b^2 \right), \quad (27d)$$

where

$$N_b = \mathcal{N}(z = -h). \quad (28)$$

Using (26a–d) and (27a–d) in (11a,b) gives equations for the interaction of a spatially varying barotropic mode and a set of spatially varying baroclinic modes. Even when $h$ is not constant, we can still identify the $m = 0$ mode as a barotropic motion, and the $m \neq 0$ modes as baroclinic motions. For instance, using (3), (7), (9) and the boundary conditions in (22), one can show there is no surface displacement or vertically integrated horizontal mass flux associated with the $m \neq 0$ modes.

4 The generation of internal tides

We now apply this formulation to the main problem of interest here: the response of the internal wave field to a barotropic ‘tidal’ flow over bottom topography. For the remainder of this study we shall assume that both the bottom topography and the fluid motion are independent of $y$. Then (11b) reduces to $\partial V_m / \partial t = -f U_m$. 
Substituting into (11a), and differentiating w.r.t. \( t \), we obtain a single governing equation:

\[
\frac{\partial^2 U_m}{\partial x^2} - \frac{1}{c_m^2} \left( \frac{\partial^2}{\partial t^2} + f^2 \right) U_m = -\sum_{n=1}^{\infty} \left\{ 2I_1(m, n) \frac{dh}{dx} \frac{\partial U_n}{\partial x} \right\} U_m + \left[ I_1(m, n) \frac{d^2h}{dx^2} + I_2(m, n) \left( \frac{dh}{dx} \right)^2 \right] U_n \}
\]

We also apply the Boussinesq approximation, and thus use the interaction coefficients of section 3.

We develop solutions for an idealised continental shelf and slope geometry, as illustrated in Figure 1. We suppose that \( h = h_L \) for \( x_C < x < x_L \) (the shelf) and that \( h = h_R \) for \( x > x_R \) (the deep ocean), where \( h_L \) and \( h_R \) are constants, and that \( h(x) \) is continuous for \( x_L \leq x \leq x_R \). We denote the width of the continental shelf by \( L_c = x_L - x_C \), and the width of the slope by \( L_s = x_R - x_L \). Typically we might expect

\[
h_L \approx 200 \text{ m}, \quad h_R \approx 2000 \text{ m}, \quad L_c \approx 100 \text{ km}, \quad L_s \approx 100 \text{ km}.
\]

(30) (31)

We obtain an equation for the barotropic motion by setting \( m = 0 \) in (29):

\[
\frac{\partial^2 U_0}{\partial x^2} - \frac{1}{c_0^2} \left( \frac{\partial^2}{\partial t^2} + f^2 \right) U_0 = \sum_{n=1}^{\infty} \left\{ \frac{\dot{I}_n \lambda_n'}{gh^2 \lambda_n} \right\}^{1/2} \left\{ 2 \frac{dh}{dx} \frac{\partial U_n}{\partial x} \right\} + \left[ \frac{d^2h}{dx^2} + \left( \frac{\lambda_n'}{\lambda_n} - \frac{2\dot{\lambda}_n'}{\lambda_n} \right) \left( \frac{dh}{dx} \right)^2 \right] U_n \}
\]

(32)

where we have used (26a–d) for \( I_1 \) and \( I_2 \). When \( h \) is constant, we recover a wave equation obtainable from the linear shallow water equations for a single fluid layer of constant density, but otherwise the right-hand side describes an interaction with baroclinic modes in regions of varying topography.

Here we assume that the barotropic motion is determined by a tidal forcing of frequency \( \omega \), and that the baroclinic modes can be neglected in determining \( U_0(x, t) \). In principle, one could then replace the terms on the right-hand side of (32) by a tidal forcing, and solve to obtain a barotropic tide \( U_0(x, t) \) (and in turn \( V_0(x, t) \)). However, this is a complex procedure dependent upon ocean margins and other global considerations. Since our interest is the generation of internal tides, which only occurs in regions of varying topography, we only need a local representation of \( U_0(x, t) \) over the continental slope. Such a representation is readily available, since the tidal forcing is approximately constant over the slope, and the lengthscale of the free barotropic motions is much greater than typical slope widths: \( \ell_0/(\omega^2 - f^2)^{1/2} \approx 400 \text{ km} \), even for shallow seas. Then, an approximate solution of (32) is

\[
U_0(x, t) = \frac{Q \left( 1 + \frac{(x - x_L)/L_c}{1 + L_s/L_c} \right) \cos(\omega t)}{h(x)} \cos(\omega t),
\]

(33)

where \( Q \) is the volume flux incident upon the slope from the deep ocean (per unit length along the coastline), which is larger than the flux onto the shelf by a factor \( 1 + L_s/L_c \). From (7) and (20), this corresponds to a barotropic tide

\[
u(x,t) = \frac{Q}{h(x)} \frac{1 + (x - x_L)/L_c}{1 + L_s/L_c} \cos(\omega t),
\]

(34)

with an alongshore component determined by \( \partial v/\partial t = -fu \). The effects of a distant coastline are modelled in the manner of Battisti and Clarke (1982), since (34) vanishes at \( x = x_C \). If we remove the coastline, by taking \( L_c \to \infty \), then \( u(x, t) \) has a spatial dependence \( h^{-1}(x) \), a form often used in simple modelling studies.

Although the barotropic tide is composed of several components of different frequencies, one might model their combined effect by taking

\[
Q \approx 40 \text{ m}^2 \text{s}^{-1}, \quad \omega \approx 1.4 \times 10^{-4} \text{s}^{-1}, \quad f \approx 1 \times 10^{-4} \text{s}^{-1},
\]

(35) (36)

corresponding to a mid-latitude semi-diurnal tidal flow of about \( 10 \text{ cm s}^{-1} \) at the shelf-break, and \( 2 \text{ cm s}^{-1} \) in the deep ocean, for (30) and (31). Vertical particle displacements can be calculated from (3), (9), (20) and (33). For instance, the surface displacement \( \zeta(z = 0) = -Q\dot{\omega}^{-1}(L_c + L_s)^{-1} \sin(\omega t) \), corresponding to a tidal amplitude of about 140 cm over the shelf and slope, for (31).
\[ \frac{\partial^2 U_m}{\partial x^2} - \frac{1}{c_m^2} \frac{\partial^2}{\partial t^2} + f^2 + \frac{1}{12} \left( \frac{3 \lambda_m''}{\lambda_m} \frac{\lambda_m''^2}{\lambda_m} - \frac{2 \lambda_m''}{\lambda_m} \lambda_m' + 4N_b^2 \right) \left( \frac{dh}{dx} \right)^2 U_m \]

\[
= - \frac{1}{h \lambda_m} \frac{\lambda_m'}{\lambda_m} \left\{ \frac{2}{h} \frac{d^2 h}{dx^2} \left( \frac{dh}{dx} \right)^2 \right\} U_0
\]

\[
- \sum_{n \geq 1, n \neq m} \frac{\hat{I}_n \lambda_n' \lambda_n'}{\lambda_m - \lambda_n} \left( \frac{\lambda_m'' \lambda_n'}{\lambda_m} \right)^{3/2} \left\{ \frac{2}{h} \frac{d^2 h}{dx^2} \left( \frac{dh}{dx} \right)^2 \right\} U_n \}, \quad (37)
\]

where we have used (27a–d), and (15b,c) for the other baroclinic interaction terms. When \( h \) is constant we obtain a simple wave equation for \( U_m \), but otherwise there is an interaction with the barotropic mode and the other baroclinic modes.

Since the feedback of the baroclinic modes onto \( U_0(x,t) \) has been neglected, (37) describes the forcing of the internal tide by the prescribed barotropic tide. Using (33), the forcing terms on the right-hand side of (37) are

\[
\frac{h}{\lambda_m} \left\{ \frac{\lambda_m'}{g \tilde{I}_m} \right\} \frac{d^2}{dx^2} \left( \frac{1 + (x - x_L)/L_c}{1 + L_s/L_c} h(x) \right) Q \cos(\omega t). \quad (38)
\]

When \( L_c \to \infty \), this is proportional to \( d^2 h^{-1}/dx^2 \), consistent with the formulation of Baines (1973, 1982), who includes the barotropic tidal forcing as a body force

\[
\frac{zN^2(z)}{\omega} \frac{d}{dx} \left( \frac{1}{\tilde{h}} \right) Q \sin(\omega t) \quad (39)
\]

in the (Boussinesq) equations for the baroclinic motion. Indeed, one can show that the inclusion of \( \rho_0 \times (39) \) in the vertical momentum balance \((1c)\) corresponds to the appearance of the term \( (38) \), with \( L_c \to \infty \), in \( (37) \).

The problem is reduced to finding the baroclinic modal coefficients \( U_m(x,t), m \geq 1 \). Noting the form of the prescribed tidal forcing \((38)\), we choose to write

\[
U_m(x,t) = Q \Re \left( \hat{U}_m(x) e^{-i\omega t} \right) \quad (40)
\]

where the \( \hat{U}_m(x) \) are dimensionless, since \( U_m \) and \( Q \) both have dimensions of \( m^2 \text{s}^{-1} \). Then, using (38), (37) reduces to

\[
\frac{d^2 \hat{U}_m}{dx^2} + \frac{1}{c_m^2} \frac{d^2}{dt^2} \left( \frac{\lambda_m'' \lambda_n'}{\lambda_m} \right)^{3/2} \left( \frac{\lambda_m'' \lambda_n'}{\lambda_m} \right)^{3/2} \left\{ \frac{2}{h} \frac{d^2 h}{dx^2} \left( \frac{dh}{dx} \right)^2 \right\} \hat{U}_n \}, \quad (41)
\]

where \( m \geq 1 \), and where

\[
\omega_f = \sqrt{\omega^2 - f^2} \quad (42)
\]

is required to be real-valued. To complete the formulation of the internal wave response, we simply need to specify boundary conditions for \( \hat{U}_m(x) \). These rely upon consideration of the energy flux away from the generation region.

\[
E = \int_{-h}^{0} \frac{\rho_0}{2} (u'^2 + v'^2 + N^2 \rho^2) \, dz + \frac{g \rho_0(0)}{2} \zeta^2(z = 0),
\]

\[
J = \int_{-h}^{0} \bar{p} u_H \, dz. \quad (43)
\]
Here $u_H = u e_x + v e_y$, and $\zeta(z = 0)$ is the free-surface displacement.

We consider the time-average of the vertically integrated energy flux associated with the baroclinic modes, in regions where $d\tilde{h}/dx = 0$. Since $\partial/\partial y = 0$, we simply consider the $x$-component, denoted by $\langle J_x \rangle$, which takes units of power per unit length (along the coastline). For time-periodic flow of the form (40), with $d\tilde{h}/dx = 0$, from (10) we have

$$\tilde{p} = -\frac{Q \rho_0(z)}{\omega} \sum_{n=0}^{\infty} c_n^2 \frac{\partial \phi_n}{\partial z} \operatorname{Re} \left( \frac{i \tilde{U}_m}{dx} e^{-i \omega t} \right),$$

so that using (5)–(7), (14) and (43) we find

$$\langle J_x \rangle = \frac{g \rho_0(0) Q^2}{2 \omega} \sum_{m=1}^{\infty} I_m \operatorname{Re} \left( i \tilde{U}_m \frac{d\tilde{U}_m}{dx} \right). \quad (44)$$

d Boundary conditions

The boundary conditions we apply for (41) are that $\langle J_x \rangle$ is away from the generation region for $x < x_L$ and $x > x_R$. In these regions (41) becomes $(d^2/dx^2 + \omega_m^2/c_m^2) \tilde{U}_m = 0$, so that we require

$$\tilde{U}_m = \begin{cases} \hat{U}_m(x_L) \exp \left( \frac{-i \omega_m (x - x_L)}{c_m(h_L)} \right) & x < x_L, \\ \hat{U}_m(x_R) \exp \left( \frac{+i \omega_m (x - x_R)}{c_m(h_R)} \right) & x > x_R, \end{cases}$$

(corresponding to (44)). If there is a coastline, in some cases one might modify (45) for $x < x_L$ so that there is wave reflection at the coast (e.g. Gibbs and Middleton 1997).

With (45), there is a unique solution of (41). It is generally most efficient to solve (41) over $x_L \leq x \leq x_R$, with boundary conditions at $x_L$ and $x_R$. Allowing for a discontinuity in $d\tilde{h}/dx$ at these points, such boundary conditions are found by integrating (41) across $x_L$ and $x_R$, using (45) for $d\tilde{U}_m/dx$, and demanding that $\tilde{U}_m$ is continuous (to ensure continuity of mass). One finds that if $d\tilde{h}/dx$ is discontinuous then $d\tilde{U}_m/dx$ is too, which ensures continuity of pressure.

5 Internal tide generation for a two-layer flow

We now consider the two-dimensional internal tide generation problem for a two layer fluid with background density

$$\rho_0(z) = \begin{cases} \rho_{00} & -d < z < 0 \\ \rho_{00} + \Delta \rho & -h(x) < z < -d, \end{cases}$$

where $d$ and $\Delta \rho$ are positive constants. When $\Delta \rho/\rho_{00} \ll 1$, from (17) we have the condition $\epsilon < 1$, and to leading order the baroclinic modes are given by the eigenvalue problem (22) with $N^2 = (g \Delta \rho/\rho_{00}) \delta(1 + z/d)$. As is well known, due to the extreme form of $N$ there is just one baroclinic mode (rather than an infinite set), with wave speed

$$c_1 = c_{\infty} \left( 1 - \frac{d}{h} \right)^{1/2}, \quad \text{where } c_{\infty} = \left( \frac{g \Delta \rho}{\rho_{00}} \right)^{1/2}. \quad (47)$$

The corresponding eigenmode is

$$\phi_1(z, h) = \begin{cases} -z/d & -d < z < 0 \\ (h + z)/(h - d) & -h < z < -d, \end{cases} \quad (48)$$

where we have chosen to use a normalisation constant

$$\tilde{I}_1 = \Delta \rho/\rho_{00}.$$

If we use (48) to model wave generation on a shallow thermocline, typically we might have

$$d \approx 100 \text{ m}, \quad \Delta \rho/\rho_{00} \approx 0.001 \Rightarrow c_{\infty} \approx 1 \text{ m s}^{-1}. \quad (49)$$

We will study the generation of this internal mode by the prescribed tidal flow (34), using the tidally forced baroclinic mode equation (41) for $\tilde{U}_1(x)$. Since there is just one baroclinic mode, there are no baroclinic mode interaction terms, i.e. the sum on the right-hand side of (41) disappears. Further, $N_b = 0$ from (28), and from (47) $\lambda_1 = c_1^2 = c_{\infty}^2$, $h/(h - d)$, so that $2\lambda_1 \lambda_m^2 - 3 \lambda_m^2 = 0$. Using (47) we also choose to write the barotropic forcing in terms of $c_1(x)$, rather than $h(x)$. Then (41) simplifies to

$$\left( \frac{d^2}{dx^2} + \frac{\omega^2}{c_1^2} \right) \tilde{U}_1 = -\frac{d^2}{dx^2} \left[ \left( \frac{1 + (x - x_L)/L_c}{1 + L_s/L_c} \right) c_1^2 \right]. \quad (50a)$$

Boundary conditions are found by integrating (50a) across $x_L$ and $x_R$, and using (45):
where $c_L = c_1(h = h_L)$ is the (constant) baroclinic wavespeed on the shelf, and $c_R = c_1(h = h_R)$ is the (constant) baroclinic wavespeed in the deep ocean.

Given $c_1(x)$, it is simple to solve (50a–c) numerically over $x_L \leq x \leq x_R$. From (46) we may then evaluate

$$\langle J_x \rangle = \frac{\rho_0 Q^2}{2} \left( \frac{g \Delta \rho}{d \rho_0} \right)^{\frac{1}{2}} \left( 1 - \frac{d}{h_R} \right)^{\frac{1}{2}} \left( 1 - \frac{f^2}{\omega^2} \right)^{\frac{1}{2}} \times \left\{ \begin{array}{ll} -J_L & \text{for } x < x_L, \\ J_R & \text{for } x > x_R, \end{array} \right. \quad (51)$$

where

$$J_L = \frac{c_1^4}{c_L^2 c_R^2} |\hat{U}_1(x_L)|^2, \quad J_R = \frac{c_1^4}{c_R^2} |\hat{U}_1(x_R)|^2, \quad (52)$$

are non-dimensional time-averaged vertically integrated baroclinic energy fluxes. For a tidal flow with (30), (35), (36) and (49), the dimensional prefactor in (51) is approximately $5100 \text{ W m}^{-1}$.

\subsection*{An exact solution}

We now suppose that

$$c_1 = c_L + (c_R - c_L)(x - x_L)/L_s, \quad (53)$$

for $x_L < x < x_R$. This serves as a simple model for a continental slope, and permits an analytical solution to be found. The corresponding profiles for $h(x)$ may be found from (47); some examples are shown in Figure 2. Provided the depth of the upper layer $d$ is not close to the shelf depth $h_L$, the profiles show little dependence on $d/h_L$. Indeed, for the slopes in Figure 2, $h(x)$ is given for $x_L \leq x \leq x_R$ to within a maximum relative error of 9% by the limiting formula

$$h(x) \approx h_L \left( 1 - \left( 1 - \frac{h_L}{h_R} \right) \frac{x - x_L}{L_s} \right)^{-1}, \quad d \ll h_L. \quad (54)$$

We turn to solving (50a–c) with (53). We shall characterise the solution by two nondimensional parameters:

$$\theta = \omega f L_s/c_R, \quad \delta_c = 1 - c_L/c_R, \quad (55)$$

which measure the width and height of the slope, respectively.

$$\hat{U}_1 = \frac{1}{1 + L_s/L_c} \left( \frac{c_1}{c_\infty} \right)^2 \left\{ a_1 + a_2 \left( \frac{x - x_L}{L_c} \right) \right\} + a_3 \left( \frac{c_L}{c_1} \right)^{3/2} \cos \left( \gamma \log \frac{c_1}{c_R} \right) + \left( \frac{i \theta}{\delta_c} - \frac{1}{2} \right) \gamma^{-1} \sin \left( \gamma \log \frac{c_1}{c_R} \right) + a_4 \left( \frac{c_R}{c_1} \right)^{3/2} \cos \left( \gamma \log \frac{c_1}{c_L} \right) - \left( \frac{i \theta}{\delta_c} + \frac{1}{2} \right) \gamma^{-1} \sin \left( \gamma \log \frac{c_1}{c_L} \right) \right\}, \quad (56)$$

Then, a particular solution of (50a) is

$$\hat{U}_1 = \frac{1}{1 + L_s/L_c} \left( \frac{c_1}{c_\infty} \right)^2 \left( a_1 + a_2 \left( \frac{x - x_L}{L_c} \right) \right), \quad (55a)$$

where

$$a_1 = -\frac{2 \delta_c}{\theta^2 + 2 \delta_c^2} \left( \delta_c + \frac{2(1 - \delta_c)\theta^2}{\theta^2 + 6 \delta_c^2} \frac{L_s}{L_c} \right), \quad (55b)$$

The kernel of the operator on the left-hand side of (50a) is an arbitrary linear combination of $c_1^{1/2} \cos(\gamma \log c_1)$ and $c_1^{1/2} \sin(\gamma \log c_1)$, where the parameter $\gamma$, possibly imaginary, is given by

$$\gamma = \left( \frac{\theta^2}{\theta^2 + \delta_c^2} - \frac{1}{4} \right)^{1/2}. \quad (56)$$

Thus, we may write the general solution of (50a–c) as
Although (57) appears rather complicated, in this form application of the boundary conditions becomes relatively simple, and (50b,c) give
\[
a_3 = \frac{i}{2\theta D} \left( 2\delta_c + (2\delta_c + i\theta) a_1 + (1 - \delta_c) \frac{L_s}{L_c} a_2 \right),
\]
and
\[
a_4 = -\frac{i}{2\theta D} \bigg( 2\delta_c \left( 1 + \frac{L_s}{L_c} \right) + (2\delta_c - i\theta) a_1 + (1 + 2\delta_c - i\theta) \frac{L_s}{L_c} a_2 \bigg),
\]
(59)
where
\[
D = \cos \left[ \gamma \log(1 - \delta_c) \right] + i\theta (\gamma \delta_c)^{-1} \sin \left[ \gamma \log(1 - \delta_c) \right].
\]
Since \( \gamma \) is either real or imaginary, from (56), the first term of \( D \) is always real and the second is always imaginary, and since these terms cannot vanish simultaneously, \( D \) is always non-zero. Thus, the solution is specified for \( x_L \leq x \leq x_R \), and elsewhere via (45).

We may then calculate the energy fluxes \( J_L \) and \( J_R \), as defined by (52). Note from (55a,b), (58) and (59) that \( a_{1,2,3,4} \) depend only on \( \theta, \delta_c, \) and \( L_s/L_c \). Thus, noting from (57) that \( \hat{U}_1 \propto c_\infty^{-2} \), we see that \( J_L \) and \( J_R \) depend only on \( \theta, \delta_c, \) and \( L_s/L_c \). The typical dependence of \( J_L \) and \( J_R \) on \( \theta \) and \( \delta_c \) is shown in Figure 3, at \( L_s/L_c = 1 \). We interpret this behaviour by considering two scenarios.

b A simplified solution

For typical generation scenarios, \( \delta_c \) is small. For instance, \( \delta_c \ll 1 \) if \( d \ll h_L \), whilst the parameters (30) and (49) give \( \delta_c \approx 0.3 \). Thus we examine the regime \( \delta_c \ll 1 \). Then \( \gamma \approx \theta/\delta_c \) and \( D \approx \exp(-i\theta) \), from (56) and (60), and the solution coefficients reduce to
\[
\begin{align*}
(a_1/a_2) &\approx \frac{2\delta_c}{\theta^2} \left( \delta_c + 2L_s/L_c \right), \\
(a_3/a_4) &\approx \frac{i\delta_c e^{i\theta}}{\theta} \left[ 1 - \frac{2\theta^{-1}L_s/L_c}{1 - (1 + 2\theta^{-1})L_s/L_c} \right].
\end{align*}
\]
Using log\((c_1/c_L) \approx \delta_c (x - x_L)/L_s \) and log\((c_1/c_R) \approx \delta_c (x - x_R)/L_s \), (57) then becomes
\[
\hat{U}_1 \approx \frac{\delta_c}{\theta} \left( 1 + \frac{L_s}{L_c} \right) \left( \frac{c_R}{c_\infty} \right)^2 \left\{ \left( \frac{2L_s}{\theta L_c} \right) + i \right\} e^{\theta(x-x_L)/L_s} - \frac{4L_s}{\theta L_c}.
\]
(61)
Thus \( \hat{U}_1(x_L) = -e^{i\theta} \hat{U}_1^*(x_R) \), so that from (52) \( J_L \) and \( J_R \) are equal to leading order in \( \delta_c \). This leading order value is
\[
J_L = J_R = \frac{\delta_c^2}{\theta^2} \left( 1 + \frac{L_s}{L_c} \right)^{-2} \left[ \left( 2 + \frac{L_s}{L_c} \right) \sin^2 \frac{\theta}{2} \right. \\
+ \left. \frac{L_s^2}{L_c^2} \left( 4\theta^{-1} \sin \frac{\theta}{2} - \cos \frac{\theta}{2} \right)^2 \right].
\]
(62)

Energy is transmitted shorewards and oceanwards at the same rate, and as \( \delta_c \to 0 \), the magnitude of this energy flux tends to zero. The values (62), at \( L_s/L_c = 1 \), are also shown in Figure 3. They approximate the exact values to within 0.01 for \( 0 < \delta_c < 0.3 \) and \( 1 < \theta/\pi < 10 \).

When \( L_s/L_c = 0 \), the solution simplifies considerably. Using (3) and (9) one can show the interface displacement to be
\[
\zeta(z = -d) = -\frac{Q}{\omega \text{Re}} \left( \frac{d}{dx} \left( \frac{c_1^2}{c_\infty^2} + \hat{U}_1 \right) \right) e^{-i\omega t},
\]
so that substituting from (61), with \( L_s/L_c = 0 \), gives
\[
\zeta(z = -d) = -\zeta_0 \left\{ \sin(\omega t) + \frac{1}{2} \sin \left( \frac{\theta(x-x_L)}{L_s} - \omega t \right) \right. \\
\left. - \frac{1}{2} \sin \left( \frac{\theta(x-x_R)}{L_s} + \omega t \right) \right\},
\]
(63)
for \( x_L \leq x \leq x_R \), where \( \zeta_0 = 2Q \delta_c(\omega L_s)^{-1}(c_R/c_\infty)^2 \). For the typical parameters (30), (31), (35), (36) and (49), \( \zeta_0 \approx 1.5 \). Thus the solution above the slope has the form of two counterpropagating waves, plus a spatially uniform term originating from the barotropic tide. However, the solution crucially depends upon the parameter \( \theta \), which from (54) measures the width of the slope relative to the wavelength \( 2\pi c_R/\omega_f \approx 2\pi c_L/\omega_f \) of the internal wave. There are two limiting cases:

(i) When \( \theta = 2n\pi, n = 1, 2, \ldots \), the continental slope accommodates an integer number of wavelengths of the internal wave. Then \( \hat{U}_1 = 0 \) for \( x \leq x_L \) and \( x \geq x_R \), and \( J_L = J_R = 0 \), as can be verified from (62). Such a case is illustrated in Figure 4a, with \( \theta = 2\pi \) and \( \delta_c = 0.05 \). Shown are two complete periods of the tidal motion, plotted using an artificially large value of \( Q \). One can see that the internal tide is more or less localised above the continental slope.

(ii) When \( \theta = (2n - 1)\pi, n = 1, 2, \ldots \), the internal wave does not fit within the continental slope. Then \( \zeta(z = -d) \) is maximised at \( x_L \) and \( x_R \), and \( J_L = J_R = 4\delta_c^2/\theta^2 \), from (62). Such a case is illustrated in Figure 4b, with \( \theta = \pi \) and \( \delta_c = 0.05 \). Thus, at \( L_s/L_c = 0 \) and for \( \delta_c \ll 1 \), \( J_L \) and \( J_R \) have
an alternating sequence of maxima and minima as the width of the slope increases. When \( L_s/L_c \neq 0 \), the same kind of considerations apply, although the locations of the maxima and minima change. However, as can be seen from Figure 3, at \( L_s/L_c = 1 \) the minima and maxima are still at \( \theta \approx 2n\pi \) and \( \theta \approx (2n-1)\pi \) respectively, for \( \delta_c \ll 1 \).

c A narrow step solution

A conspicuous feature of Figure 3 is that the largest outgoing energy fluxes occur in a relatively narrow band where \( \theta \approx \pi \). From (54), this corresponds to the continental slope being narrow compared with the wavelength of the internal wave. This regime might not be typical of internal tide generation at continental margins, since the parameters (30), (31), (36) and (49) imply \( \theta \approx 10 \). Nevertheless, we study this regime by taking \( L_s \to 0 \), with all other parameters fixed. Then \( \theta \to 0 \), \( \gamma \to i/2 \), \( D \to (1 - \delta_c/2)(1 - \delta_c)^{-1/2} \), and

\[
\begin{pmatrix} a_1 \\ a_2 \\
\end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ -1 \\
\end{pmatrix},
\begin{pmatrix} a_3 \\ a_4 \\
\end{pmatrix} \rightarrow \frac{1}{2D} \begin{pmatrix} 1 - ic_L/\omega_f/L_c \\ 1 + ic_R/\omega_f/L_c \\
\end{pmatrix},
\]

Substituting in (57), we find

\[
\hat{U}_1 \to \frac{1}{1 + L_s/L_c} \left( \frac{c_R}{c_\infty} \right)^2 \left( -1 + \frac{i c_L/\omega_f L_c}{c_R/\omega_f L_c} \right),
\]

so that

\[
\hat{U}_1 (x_L) = \frac{\delta_c}{1 - \delta_c} \frac{1}{1 + L_s/L_c} \left( \frac{c_R}{c_\infty} \right)^2 \left( 1 + \frac{i c_L/\omega_f L_c}{c_R/\omega_f L_c} \right),
\]

\[
(64a)
\]

\[
\hat{U}_1 (x_R) = \frac{\delta_c}{1 + L_s/L_c} \left( \frac{c_R}{c_\infty} \right)^2 \left( -1 + \frac{i c_L/\omega_f L_c}{c_R/\omega_f L_c} \right). \tag{64b}
\]

Using (51), we then find that

\[
J_L = \frac{\delta_c^2 (1 - \delta_c)}{(1 + L_s/L_c)^2} \left( 1 + \frac{c_R^2}{\omega_f^2 L_c^2} \right),
\]

\[
J_R = \frac{\delta_c^2}{(1 + L_s/L_c)^2} \left( 1 + \frac{c_L^2}{\omega_f^2 L_c^2} \right).
\]

As \( L_c \to \infty \), \( J_R \) is larger than \( J_L \) by a factor \( (1 - \delta_c)^{-1} \); more energy is transmitted oceanwards than shorewards.

Although (64a,b) is the leading order solution at small \( L_s \), it is also the exact solution for flow over a step topography. This can be obtained more rapidly by putting \( c_1 = c_L + (c_R - c_L)H(x) \) in (56), where \( H(x) \) is the unit step function, or even more rapidly by applying continuity of mass and pressure across the step in each layer of fluid, in the manner of Rattray (1960). A more rigorous approach to this problem has been taken by Barthelemy et al. (2000), accounting for non-hydrostatic effects near the step.

d The effects of a coastline

The effects of a coastline can be evaluated by varying \( L_s/L_c \). The typical dependence of \( J_L \) and \( J_R \) on \( L_s/L_c \) and \( \theta \) is shown in Figure 5, at \( \delta_c = 0.3 \). We have removed the dominant \( \theta \) dependence, as predicted by (62), by displaying \( \theta^2 \times J_{L,R} \). At fixed \( \theta \), \( J_L \) and \( J_R \) can increase or decrease as a coastline is moved closer to the slope. If \( \theta \) is such that there is a minimum in outgoing energy at \( L_s/L_c = 0 \), then \( J_L \) and \( J_R \) increase with \( L_s/L_c \); if \( \theta \) is such that there is a maximum in outgoing energy at \( L_s/L_c = 0 \), then \( J_L \) and \( J_R \) decrease.
Internal tide generation modelled using a modal decomposition

Figure 4: Internal tide generation in a two-layer fluid, for the solution (57), with \( \delta_c = 0.05, d/h_L = 0.5, \) and \( L_s/L_c = 0 \). Shown is the interface displacement at (a) \( \theta = 2\pi \); (b) \( \theta = \pi \).

Figure 5: Nondimensional energy fluxes at \( \delta_c = 0.3 \), determined from the exact two-layer solution (57). Left: \( \theta^2 \times J_L \). Right: \( \theta^2 \times J_R \).

with \( L_s/L_c \). For both \( J_L \) and \( J_R \), variations with \( \theta \) decrease as \( L_s/L_c \) increases.

6 Internal tide generation for a uniformly stratified fluid

We now consider the two-dimensional generation problem for a basic state with uniform stratification \( N \). Typically we might expect

\[
N = 0.0015 \, \text{s}^{-1},
\]

so that \( hN^2/g \ll 1 \). Thus the Boussinesq approximation applies, and we obtain the solution of the eigenvalue problem (22):

\[
c_n = \frac{Nh}{n\pi}, \quad \lambda_n = \frac{n^2\pi^2}{N^2h^2},
\]

Choosing normalising constants

\[
\hat{I}_n = \frac{N^2h_R}{2gn^2},
\]

using (23) the corresponding eigenmodes are

\[
\phi_n = \frac{(-1)^n}{n} \left( \frac{h_R}{h} \right)^{1/2} \sin \left( \frac{n\pi z}{h} \right),
\]

so that the maximum of \( |\partial \phi_n/\partial z| \) is independent of \( n \).

We study the generation of these modes by the prescribed tidal flow (34), using the tidally forced baroclinic mode equation (41). Using (28), (67) and (68), this simplifies to
Boundary conditions are obtained by integrating (69a) across \( x = x_L \) and \( x = x_R \) and using (45). For cases where \( dh/dx \) is continuous at \( x_L \) and \( x_R \), we simply obtain

\[
\begin{align*}
\frac{d^2 \hat{U}_m}{dx^2} & + \frac{1}{h^2} \left( \frac{m^2 \pi^2 \omega_f^2}{N^2} - \left( \frac{1}{4} + \frac{m^2 \pi^2}{3} \right) \left( \frac{dh}{dx} \right)^2 \right) \hat{U}_m = \frac{2h}{\pi} \left( \frac{h}{h_R} \right)^{1/2} \frac{d^2}{dx^2} \left( \frac{1 + (x - x_L)/L_c}{1 + L_s/L_c} \right) \frac{1}{h} \\
& - \sum_{n \geq 1, n \neq m} \frac{2m^2}{m^2 - n^2} \left( 2 \frac{d^2}{dx} \frac{d\hat{U}_n}{dx} + \left( \frac{1}{h} \frac{d^2h}{dx^2} \right) \left( \frac{1}{h} \frac{dh}{dx} \right)^2 \right) \hat{U}_n.
\end{align*}
\]

(69a)

Once (69a–c) has been solved, from (46) we may evaluate

\[
\langle J_x \rangle = \frac{\pi \rho_0 (0) N Q^2}{4} \left( 1 - \frac{f^2}{\omega_f^2} \right) \frac{1}{\pi} \times \begin{cases} -J_L & x < x_L, \\ +J_R & x > x_R, \end{cases}
\]

(70)

where

\[
J_L = \frac{h_R}{h_L} \sum_{m=1}^{\infty} \frac{\left| \hat{U}_m(x_L) \right|^2}{m}, \quad J_R = \sum_{m=1}^{\infty} \frac{\left| \hat{U}_m(x_R) \right|^2}{m}
\]

(71)

are non-dimensional time-averaged vertically integrated baroclinic energy fluxes. For (35), (36), and (65), the dimensional prefactor in (70) is approximately 1300 W m\(^{-1}\).

\( a \) A numerical solution

Although (69a–c) involve an infinite number of modes, we look for solutions with a finite set \( m = 1, 2, \ldots, M \), anticipating that the truncation will be unimportant provided \( M \) is sufficiently large. There remains a system of \( M \)-coupled ordinary differential equations, which may be solved by a variety of methods. We solve a set of initial value problems, stepping out from \( x = x_L \) to \( x = x_R \) using a fourth-order Runge–Kutta scheme with automatic step-size adaptation. In particular, we find a single solution of (69a,b), along with \( M \) linearly independent solutions of (69a,b) with the barotropic forcing omitted. Via an \( M \times M \) matrix inversion, we then find the unique linear combination of these which satisfies (69c). A significant advantage of this method over a standard finite-difference treatment with pre-assigned grid-points is the variable resolution thus obtained. Note that, from (66), the wavelength of the internal waves is proportional to \( h \) and can vary considerably over the slope.

For the remainder of section 6, we consider the particular slope geometry

\[
h = h_L + (h_R - h_L) \sin^2(\pi(x - x_L)/2L_s),
\]

(72)

for \( x_L \leq x \leq x_R \). Then using rescaled variables \( X = (x - x_L)/L_s \) and \( H = h/h_R \), the solutions are seen to depend on three nondimensional parameters: \( h_L/h_R, L_s/L_c \), and a slope parameter \( s \), defined by

\[
s = \max \left( \frac{dh/dx}{\omega_f/N} \right).
\]

(73)

We first examine how the solutions vary with \( s \) (in section 6b), and how they converge as \( M \) increases (in section 6c). For these two sections we set \( L_s/L_c = 0 \) (the case with no coastline), and \( h_L/h_R = 0.5 \) (for ease of visualisation). We then examine the energy fluxes associated with the internal tide, generalising to other values of \( h_L/h_R \) in section 6d, and to other values of \( L_s/L_c \) in section 6e.

\( b \) Subcritical and supercritical solutions

Perhaps the most important parameter for determining the nature of the solutions is \( s \). It is the ratio between the maximum gradient of the continental slope and the ‘characteristic slope’ \( \omega_f/N \) of internal wave motion. There are significant differences between the subcritical regime \( s < 1 \) and the supercritical regime \( s > 1 \). These are illustrated in the top row of Figure 6, where the density is shown for solutions at \( s = 0.5, 1 \) and 2. The solutions are shown at \( t = 1.5\pi/\omega \), which from (34) corresponds to high tide, and with \( Q \) a few hundredths of \( \omega h_R L_s \), which are artificially large volume fluxes for typical slope geometries.

When \( s = 0.5 \) the solution has a distinctly wave-like structure throughout the domain. On the shelf there is an approximately sinusoidal progressive wave, whilst in the deep ocean there is a somewhat more angular looking wave. This is typical when \( s \lesssim 0.6 \). However, as \( s \) increases towards unity, the wave-like structure on the shelf is lost, and is gradually replaced by a beam-like structure.
At $s = 1$ the beam on the shelf is fully developed. It emanates as the tangent from the point where the topographic slope is critical, i.e. where $dh/dx = \omega_f/N$. Here there is a resonant forcing of waves by the topography, leading to the large vertical displacements within the beam. The structure of the beam is time-dependent, as illustrated in Figure 7. Shown are density perturbations due to the internal wave field at the shelf-break $x = x_L$, as resolved by the solution with $M = 256$. The beam corresponds to the rapid changes in the internal wave perturbations near to $z = -0.36 h_R$. Note that the density field alternates between a step-like feature, at $t = 0$ (ebb tide) and $t = \pi/\omega$ (flood tide), and a cusp-like feature, at $t = \pi/2\omega$ (low tide) and $t = \pi/2\omega$ (high tide). We take a closer look at the resolution of these features (which reflect a singularity in the linear inviscid solution) in Section 6c.

As $s$ increases beyond unity, $h(x)$ becomes convex at the critical slope so that the beam can also propagate into the deep ocean, as is evident in Figure 6 at $s = 2$. However, the beam in the deep ocean has a step-like profile when that on the shelf has a cusp-like profile (and vice-versa). Beam-like structures propagating away from both sides of the slope are well-known to be typical of internal tides in the supercritical regime (e.g. Rattray et al. 1969), and have been observed near continental slopes (e.g. Lien and Gregg 2001). In the ocean, as the wave-beams propagate away from the slope they may be broken down by friction (e.g. Prinsenberg et al. 1974), and various reflective and non-linear processes (e.g. New and Pingree 1990, Gerkema 2001, St Laurent and Garrett 2002).

c Truncation and convergence

Since we solve a system truncated to $M$ modes, one might ask if $\hat{U}_m(x)$ obtained at finite $M$ is similar to that obtained as $M \to \infty$. Some typical behaviour is shown in Figure 8, at $s = 1$, $h_L/h_R = 0.5$ and $L_s/L_c = 0$. In the top row we compare $\hat{U}_1(x)$ at $M = 1$, 4 and 256. At $M = 1$, $\hat{U}_1(x)$ is already close to the limiting form at $M = 256$, whilst further changes are barely perceptible beyond $M = 4$. In general $\hat{U}_1(x)$, and the other low-order modes, seem to be largely determined by the barotropic forcing rather than by interactions with other baroclinic modes.
Internal tide generation modelled using a modal decomposition

Figure 7: Time evolution of a solution with $M = 256$ at $s = 1$, $h_L/h_R = 0.5$ and $L_s/L_c = 0$. Shown are vertical profiles at $x = x_L$ of the density perturbations associated with internal wave modes $m = 1, 2, \cdots, 128$. A filter $(m\pi/128)^{-1}\sin(m\pi/128)$ has been applied to the modal coefficients to remove oscillations associated with Gibbs Phenomenon. The vertical dashed line intersecting each profile indicates the time at which each occurs. The horizontal dashed line gives the height at $x = x_L$ of the tangent from the critical slope.

Figure 8: Modal structures $\hat{U}_m(x)$ with varying $m$ and $M$, at $s = 1$, $h_L/h_R = 0.5$ and $L_s/L_c = 0$.

However, the higher-order modes can suffer from a truncation error. To illustrate this, we compare $\hat{U}_{32}(x)$ in solutions with $M = 32$, 64 and 256. As shown in the bottom row of Figure 8, the solution at $M = 64$ is close to the limiting form at $M = 256$, but the solution at $M = 32$ is far from correct. There is a truncation error for modes with $m$ close to $M$, presumably because interactions with neighbouring baroclinic modes $\hat{U}_{M+1}, \hat{U}_{M+2}, \cdots$ are not resolved. For $s \lesssim 1$ this truncation error is benign, but for $s \gtrsim 1$ modes with $m$ close to $M$ can become artificially large. However, as noted above, the low-order modes are not affected. Thus, for solutions at $s \geq 1$, analysis and visualisations are performed with only the bottom half of the calculated modes: $\hat{U}_1, \hat{U}_2, \cdots, \hat{U}_{M/2}$.

The remaining question is just how large need $M$ be to resolve the solution. We start by making a purely visual assessment, at $s = 0.8$, $h_L/h_R = 0.5$ and $L_s/L_c = 0$. Shown in Figure 9 are solutions at $M = 1, 2, 4, 8, 16$ and 32. The most accurate solution, at $M = 32$, has the familiar wave-beam structure in $x \leq x_L$. Although this is not apparent in the solutions at $M = 1$ or 2, it is appearing at $M = 4$, most of the details are captured at $M = 8$, and all of the details are captured at $M = 16$.

Clearly, increasing $M$ beyond some value $M_c$ does
not significantly change the solution. One way to
define \( M_c \) is as the lowest value of \( M \) for which the \( \hat{U}_m(x) \) change by less than 5% of the maximum value of \( |\hat{U}_m| \) (over \( x \) and \( m \)) when \( M \) is doubled. At \( s = 0.8 \), this yields \( M_c = 16 \), in agreement with a visual inspection of Figure 9. At \( s = 0.5 \), this yields \( M_c = 8 \), although as shown in the left-most panels of Figure 6, even just one mode is sufficient to give most solution details in this case.

For \( 0.05 \leq h_L/h_R \leq 0.5 \), as \( s \) increases from 0.25 to 0.9, \( M_c \) typically increases from 4 to 16. Only a few modes are needed for convergence. For \( s \lesssim 0.5 \) the \( \hat{U}_m \) scale like \( m^{-2} \) even for relatively small \( m \), and thus decay quickly. This is consistent with a simple treatment of (69a) in which the baroclinic interactions terms are neglected, so that there is a balance between the terms on the left-hand side (which scale like \( m^2 \hat{U}_m \) when \( s \ll 1 \)) and the barotropic forcing terms on the right-hand side (which scale like unity), giving \( \hat{U}_m \sim m^{-2} \).

There is a different regime when \( s \gtrsim 1 \). The appearance of the beam-like structures, with their fine vertical scales, means that many more modes are needed for convergence. This is reflected in a slower decay of the modal coefficients: \( \hat{U}_m \sim m^{-1/2} \). Therefore about 100 modes would be required for \( \hat{U}_m \) to fall by a factor of 10, suggesting \( M_c \) would be several hundred. Even so, solutions with less than several hundred modes still capture many details of the solutions. For instance, in Figure 6 we show favourable comparisons between solutions using 32 baroclinic modes and 128 baroclinic modes, at \( s = 1 \) and \( s = 2 \).

The \( m^{-1/2} \) spectral decay for \( s \gtrsim 1 \) is consistent with a vertical profile of the form \(|z - z_0|^{-1/2}\) for \( u \), for a beam centred at \( z = z_0 \). Such square root singularities occur in many linear inviscid internal wave solutions (e.g. Robinson 1969, Hurley 1997), and have been noted in connection with internal tides when \( s \geq 1 \) (e.g. Balmforth et al. 2002, Llewellyn Smith and Young 2003). Although the solutions presented here at finite \( M \) do not possess a singularity, one is steadily appearing as \( M \) increases, as illustrated in Figure 10.

### Energy fluxes and small \( M \) solutions

We now discuss the form of the shoreward and oceanward energy fluxes \( J_L \) and \( J_R \) associated with the in-
ternal tide. These show a strong transition as the slope changes from subcritical to supercritical, as shown in Figure 11, once again at $h_L/h_R = 0.5$ and $L_s/L_c = 0$. Whilst $J_L$ is maximised around $s = 1$, $J_R$ increases precipitously around $s = 1$ before increasing more slowly for larger $s$. This is consistent with the development of the beam-like structures shown in Figure 6. Note that $J_L + J_R$, the total energy flux away from the slope, monotonically increases with $s$.

The same kind of behaviour occurs for other values of $h_L/h_R$. Shown in Figure 12 are $J_L$ and $J_R$ as functions of $s$ and $h_L/h_R$, at $L_s/L_c = 0$. The contour scale is logarithmic so that the energy fluxes at $s \ll 1$ can be shown simultaneously with the much larger values at $s > 1$. Whilst $J_L$ equals $J_R$ to within 4% for $s \leq 0.6$, $J_R$ is greater than $J_L$ by a factor of five or more for $1.2 \leq s \leq 2$, consistent with the results of Baines (1974). In contrast, at fixed $s$ the dependence on $h_L/h_R$ is relatively weak (although recall from (70) that the dimensional energy fluxes contain a factor $Q^2 \propto h_R^2$).

As noted in previous studies of internal tides (e.g. Holloway and Merrifield 1999, Petrelis et al. 2006), much of the energy flux is accounted for by the first baroclinic mode. For $s \lesssim 0.7$, about 90% of $J_L$ and $J_R$ is in the first baroclinic mode for the results in Figure 12. This is not too surprising, since a decay of $\bar{U}_m \sim m^{-2}$ implies from (71) that $J \sim m^{-5}$. For $s \gtrsim 1$ a smaller proportion of the energy flux is in the first baroclinic mode, since $\bar{U}_m \sim m^{-1/2}$ now implies $J \sim m^{-2}$. Nevertheless, at least 43% of $J_L$ and 45% of $J_R$ is in the first baroclinic mode for the values shown.

Since much of the energy flux is in the first baroclinic mode, and since we have already noted that the $m = 1$ solution is well predicted even when $M$ is relatively small, there is a possibility of obtaining good energy flux estimates from solutions truncated at small $M$ (even though tens or hundreds of modes might needed to resolve the details of $u(x, z, t)$). Shown in Figure 13 are $J_L$ and $J_R$ as obtained from solutions with $M = 1$, which should be compared with the converged values shown in Figure 12. The $M = 1$ energy flux estimates are each accurate to within 35% for $s \leq 0.8$. For $s > 1$, substantial errors appear in the estimate for $J_L$, whilst the estimate for $J_R$ has a maximum relative error of 75% around $s = 1.1$. More accurate estimates can be obtained at relatively little extra computational cost by increasing $M$ to four or eight.

### The effects of a coastline

As described in section 4, we can model the effects of a coastline by taking $L_s/L_c$ to be non-zero. We fix $h_L/h_R = 0.5$ and consider the variations in $J_L$ and $J_R$ with $s$ and $L_s/L_c$. The results, shown in Figure 14, reveal that both $J_L$ and $J_R$ decrease with $L_s/L_c$ in this parameter range. When $s = 0.25$ the reduction is only 25% as $L_s/L_c$ increases from zero to two. However, for $s \gtrsim 1$, the reduction is by a factor of about $(1+L_s/L_c)^{-5/4}$, so that at $L_s/L_c = 2$ the energy fluxes have fallen by about 75%. Therefore, for typical geometries (31), taking account of the coastline leads to about a 60% reduction of the energy fluxes compared to those for a similar topographic profile in the open.
Figure 12: Nondimensional energy fluxes at $L_s/L_c = 0$ for a uniformly stratified basic state. Left: $J_L$. Right: $J_R$.

Figure 13: Nondimensional energy fluxes determined from $M = 1$ solutions at $L_s/L_c = 0$, for a uniformly stratified basic state. Left: $J_L$. Right: $J_R$.

Note that these reductions are solely due to changes in the barotropic forcing, since the boundary conditions for the baroclinic modes remain unchanged. Thus, a decrease in the strength of the internal tide is to be expected as $L_s/L_c$ increases, since from (34) the strength of the barotropic tide decreases over the slope. This is reflected in the denominator of the barotropic forcing term (38). There can also be changes in the strength of the barotropic forcing associated with derivatives of the numerator, but this effect is not simple to diagnose.

7 Conclusions

In this paper we have constructed a model system of equations to describe the generation of the internal tide by the interaction of the barotropic tide with topography. Although presently restricted to the twin assumptions of linearization and hydrostatic balance in the vertical momentum equation, our model takes full account of three-dimensional topography of arbitrary height and slope, and allows for quite general basic density stratification. These latter two features distinguish our present theory from most previous theories, such as those mentioned in the Introduction, most of which are usually restricted to simplifying hypotheses on the density stratification and/or the topography.
The essence of our model is the exploitation of a novel modal decomposition technique which does not require a uniformly flat bottom topography, as described in section 2. The outcome is an infinite set of modal equations, coupled through the topographic terms, but in practice we expect that only a few modes are needed for a good representation of the solution. Although our model can accommodate free-surface and non-Boussinesq effects, we make the usual Boussinesq approximation (as described in section 3) in the illustrative examples we treat here. For these we have focussed on wave generation at the continental shelf, and have taken some simple, but representative, descriptions of the topography and background stratification. However, in this first paper on this topic, we have restricted attention to two-dimensional topography and flow fields. Further, although our model is capable of describing a full interaction between the barotropic and baroclinic tides, we make the common simplifying assumption here that since the barotropic tide has a much larger wavelength than all the baroclinic modes, it can be specified \textit{a priori}, as detailed in section 4.

We have described two simple models in some detail. The first, described in section 5, is a two-layer stratification, for which there is of course just a single baroclinic mode. For a certain topographic profile, our model yields explicit analytical solutions, from which the properties of the internal mode as a function of the available slope and shelf parameters can be obtained. Then, in section 6, we consider the opposite case of uniform stratification, for which there are an infinite set of modes, which can be found analytically. In this case the system of modal equations must be solved numerically, and it is necessary to truncate to a finite number of modes. The number of modes required for effective convergence depends strongly upon a parameter $s$, defined in (73), giving the ratio of the maximum topographic slope to the characteristic slope of internal waves. When $s \leq 0.5$, only one or two modes are needed. For $0.5 \leq s \leq 0.9$, about 10 modes are needed. For $s \geq 1$, the solution contains singular beams, as expected from the theory of Baines (1974), so that finer and finer structure is revealed as more modes are retained. However, an adequate representation of the beams can be obtained with about 20 modes. In such cases, as the truncation number increases from 1 to 20, the scattered internal tide changes in structure from a “two-layer” picture to a “wave-beam” picture. These examples illustrate how our present model can accommodate both wave-scattering and wave-beam concepts.

We have also studied how the time-averaged vertically integrated energy flux away from the slope depends on various parameters, adding to results from several recent studies (e.g. Khatiwala 2003, St. Laurent et al. 2003, Petrelis et al. 2006). We have emphasised how the location of a coastline, through its effect on the form of the barotropic tide, significantly effects both the shoreward and oceanward energy fluxes. For the uniform stratification, the energy fluxes decrease as the length of the continental shelf is decreased, but for the two-layer stratification the energy fluxes can increase or decrease. For the uniform stratification, we have also shown how solutions truncated to just a few modes can give good energy flux estimates at little computational cost. This is possible because the energy flux is dominated by low-order modes, and these rapidly converge even at low truncation number.

In future work, now underway, we plan to extend...
this present model in a number of directions. First, still within the two-dimensional and Boussinesq hypotheses, we plan to extend the model to observed climatological density stratifications and to actual topographic profiles for typical continental shelves. This involves the same basic approach used here, but with the necessity to calculate the coefficients of the modal equations numerically rather than analytically. Second, we plan to examine the effect of longshore variability in the continental shelf topography, noting that we already have a formulation of the modal equations needed for this task. Third, we plan to incorporate the hitherto neglected nonlinear and non-hydrostatic terms, which we anticipate would replace the present modal equations with a system of coupled Boussinesq-type equations.

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APPENDIX A

The interaction coefficients

We wish to find expressions for the interaction coefficients $I_{1}$ and $I_{2}$, defined in (12), in terms of $N$, $I_{m}$ and $c_{n}$. To do this, we first differentiate (4a) and (4c) $\alpha$ times w.r.t. $h$. We obtain

$$\frac{\partial}{\partial z} \left( \rho_{0} \frac{\partial^{\alpha}}{\partial h^{\alpha}} \left( c_{n}^{2} \frac{\partial \phi_{n}}{\partial z} \right) \right) + \rho_{0} N^{2} \frac{\partial^{\alpha} \phi_{n}}{\partial h^{\alpha}} = 0,$$  \hspace{1cm} \text{(A1a)}

$$\left. \frac{\partial^{\alpha}}{\partial h^{\alpha}} \left( c_{n}^{2} \frac{\partial \phi_{n}}{\partial z} \right) \right|_{z=0} = g \left. \frac{\partial^{\alpha} \phi_{n}}{\partial h^{\alpha}} \right|_{z=0},$$  \hspace{1cm} \text{(A1b)}

where $\phi_{n}$ is an eigenfunction satisfying (4a–c) with eigenvalue $c = c_{n}$. Multiplying (A1a) by $\phi_{m}$, integrating from $z = -h$ to $z = 0$, and substituting for $\rho_{0} N^{2} \phi_{n}$ from (4a), gives

$$\int_{-h}^{0} \phi_{m} \frac{\partial}{\partial z} \left( \rho_{0} \frac{\partial^{\alpha}}{\partial h^{\alpha}} \left( c_{n}^{2} \frac{\partial \phi_{n}}{\partial z} \right) \right) \, dz = -c_{m}^{2} \int_{-h}^{0} \frac{\partial}{\partial z} \left( \rho_{0} \frac{\partial \phi_{m}}{\partial z} \right) \frac{\partial^{\alpha} \phi_{n}}{\partial h^{\alpha}} \, dz = 0.$$

Integration by parts, along with (4b), then gives

$$\int_{-h}^{0} \rho_{0} \frac{\partial}{\partial z} \left( c_{m}^{2} \frac{\partial \phi_{m}}{\partial z} \right) \frac{\partial^{\alpha} \phi_{n}}{\partial h^{\alpha}} \, dz = c_{m}^{2} \left[ \rho_{0} \frac{\partial \phi_{m}}{\partial z} \left( c_{n}^{2} \frac{\partial \phi_{n}}{\partial z} \right) \right]_{-h}^{0} - \left( \rho_{0} \phi_{m} \frac{\partial^{\alpha} \phi_{n}}{\partial h^{\alpha}} \left( c_{n}^{2} \frac{\partial \phi_{n}}{\partial z} \right) \right) \bigg|_{z=0}.$$

But, using (4c) and (A1b), the terms on the right-hand side evaluated at $z = 0$ cancel. Dividing by $c_{m}^{2}$, and substituting from (8) on the right-hand side, we obtain

$$\int_{-h}^{0} \rho_{0} \frac{\partial \phi_{m}}{\partial z} \left( \frac{\partial^{\alpha} \phi_{n}}{\partial h^{\alpha}} \frac{\partial \phi_{n}}{\partial z} - \frac{1}{c_{n}^{2}} \frac{\partial \phi_{n}}{\partial h} \frac{\partial^{2} \phi_{n}}{\partial z^{2}} \right) \, dz = -\rho_{0}(-h) \left( \frac{\partial^{\alpha} \phi_{m}}{\partial h^{\alpha}} \frac{\partial \phi_{n}}{\partial h} \right) \bigg|_{z=-h}. \hspace{1cm} \text{(A2)}$$

Writing

$$\frac{\partial^{\alpha}}{\partial h^{\alpha}} \frac{\partial \phi_{n}}{\partial z} = \frac{\partial^{\alpha}}{\partial h^{\alpha}} \left( \frac{1}{c_{n}^{2}} \frac{\partial^{2} \phi_{n}}{\partial z^{2}} \right) = \lambda_{n} \frac{\partial^{\alpha}}{\partial h^{\alpha}} \left( c_{n}^{2} \frac{\partial \phi_{n}}{\partial z} \right) \frac{\partial^{\alpha-r} \lambda_{n}}{\partial h^{\alpha-r}},$$

where $\lambda_{i}(h) = 1/c_{i}^{2}(h)$, (A2) becomes

$$\left( \lambda_{n} - \lambda_{m} \right) \int_{-h}^{0} \rho_{0} \frac{\partial \phi_{m}}{\partial z} \frac{\partial^{\alpha}}{\partial h^{\alpha}} \left( c_{n}^{2} \frac{\partial \phi_{n}}{\partial z} \right) \, dz + \sum_{r=0}^{\alpha-1} \left( \frac{\alpha}{r} \right) \frac{\partial^{\alpha-r} \lambda_{n}}{\partial h^{\alpha-r}} \int_{-h}^{0} \rho_{0} \frac{\partial \phi_{m}}{\partial z} \frac{\partial^{\alpha}}{\partial h^{\alpha}} \left( c_{n}^{2} \frac{\partial \phi_{n}}{\partial z} \right) \, dz = -\rho_{0}(-h) \left( \frac{\partial^{\alpha} \phi_{m}}{\partial h^{\alpha}} \frac{\partial \phi_{n}}{\partial h} \right) \bigg|_{z=-h}. \hspace{1cm} \text{(A3)}$$

When $n = m$ and $\alpha = 1$, using (6) this reduces to

$$I_{m} = -\rho_{0}(-h) \lambda_{m} \left. \left( \frac{\partial \phi_{m}}{\partial h} \right)^{2} \right|_{z=-h}, \hspace{1cm} \text{(A4)}$$

where the prime denotes differentiation w.r.t. $h$, i.e. $' = \partial \lambda_{m}/\partial h$. Then, dividing (A3) by $c_{m}^{2} I_{m}$, and using (A4) and (12) gives

$$\left( \lambda_{m} - \lambda_{n} \right) \mathcal{I}_{m}(m, n) + \sum_{r=0}^{\alpha-1} \left( \frac{\alpha}{r} \right) \frac{\partial^{\alpha-r} \lambda_{m}(m, n)}{\partial h^{\alpha-r}} \mathcal{I}_{r}(m, n) = \lambda_{m} \left. \frac{\partial^{\alpha} \phi_{n}}{\partial h^{\alpha}} \frac{\partial \phi_{m}}{\partial h} \right|_{z=-h}. \hspace{1cm} \text{(A5)}$$

This relation enables us to evaluate $\mathcal{I}_{m}$ recursively.

We start by taking $n \neq m$. Then setting $\alpha = 1$ in (A5), $\mathcal{I}_{0}$ disappears by (5) and (12), giving

$$\mathcal{I}_{1}(m, n) = \frac{\lambda_{m} - \lambda_{n}}{\lambda_{m} - \lambda_{n}} \left. \frac{\partial \phi_{m}}{\partial h} \right|_{z=-h}, \hspace{1cm} n \neq m.$$ 

Substituting from (A4) for $\partial \phi_{m}/\partial h$ at $z = -h$, taken to be positive, gives

$$\mathcal{I}_{1}(m, n) = \frac{\lambda_{m} - \lambda_{n}}{\lambda_{m} - \lambda_{n}} \left. \left( \frac{\partial \phi_{m}}{\partial h} \right)^{2} \right|_{z=-h}, \hspace{1cm} n \neq m,$$

the first of the required relations. Setting $\alpha = 2$ in (A5), and using (5) again, gives

$$\left( \lambda_{m} - \lambda_{n} \right) \mathcal{I}_{2}(m, n) + 2\lambda_{m} \mathcal{I}_{1}(m, n) = \lambda_{m} \left. \frac{\partial^{2} \phi_{n}}{\partial h^{2}} \frac{\partial \phi_{m}}{\partial h} \right|_{z=-h}. \hspace{1cm} \text{(A6)}$$
Expressions for $\partial^a\phi_n / \partial h^n$ at $z = -h$ are derived in Appendix B. Substituting from (B5), (A6), and, again from (A4), we obtain

$$I_2(m, n) = \frac{\lambda_m}{\lambda_n - \lambda_m} \left( I_m - \frac{\lambda'_n}{\lambda_m} \right) \left( I_m - \frac{\lambda_n'}{\lambda_m} \right) \left( I_m - \frac{\lambda_m'}{\lambda_m} \right) \left( I_m - \frac{\lambda'_n}{\lambda_m} \right) \left( I_m - \frac{\lambda_m'}{\lambda_m} \right)$$

the second of the required relations.

We now take $n = m$. Setting $\alpha = 2$ in (A5) gives

$$\lambda''_m + 2\lambda'_m I_1(m, m) = \lambda'_m \frac{\partial^2 \phi_m / \partial h^2}{\partial \phi_m / \partial h} \bigg|_{z=-h},$$

since $I_0(m, m) = 1$, from (6) and (12). Thus, using (B5) we obtain

$$I_1(m, m) = \frac{1}{2} \left( \frac{I'_m}{I_m} - \frac{\lambda'_m}{\lambda_m} \right),$$

the third of the required relations. Setting $\alpha = 3$ in (A5), with $n = m$, gives

$$\lambda'''_m + 3\lambda''_m I_1(m, m) + 3\lambda'_m I_2(m, m) = \lambda'_m \frac{\partial^3 \phi_m / \partial h^3}{\partial \phi_m / \partial h} \bigg|_{z=-h}.$$

Then substituting from (A8) and (B6) we obtain

$$I_2(m, m) = \frac{\lambda''_m}{6\lambda'_m} - \left( \frac{\lambda''_m}{2\lambda'_m} \right)^2 + \frac{1}{4} \left( \frac{I'_m}{I_m} - \frac{\lambda'_m}{\lambda_m} \right)^2$$

and then eliminate $\rho_0(-h)$ using (A4):

$$\frac{\partial^3 \phi_m}{\partial h^3} = \left( \frac{I'_m}{I_m} - \frac{\lambda'_m}{\lambda_m} \right) \frac{\partial \phi_m}{\partial h} \bigg|_{z=-h}.$$ 

Secondly, $\partial(4a)/\partial z$, (4b), (8) and (B1) imply

$$\frac{\partial^3 \phi_m}{\partial z^3} = \left( \frac{N^4}{g^2} - \frac{N^2}{g^2} \right) - \frac{1}{2} \frac{\partial N^2}{\partial h} \bigg|_{z=-h}.$$

Thirdly, differentiating (8) w.r.t. $h$, we obtain

$$\frac{\partial^2 \phi_m}{\partial h \partial z} = \frac{1}{2} \left( \frac{\partial^2 \phi_m}{\partial h^2} + \frac{\partial^2 \phi_m}{\partial z^2} \right) \bigg|_{z=-h},$$

and substituting from (B1) then gives

$$\frac{\partial^3 \phi_m}{\partial h \partial z} = \frac{1}{2} \left( \frac{\partial^2 \phi_m}{\partial h^2} + \frac{N^2}{g^2} \frac{\partial \phi_m}{\partial h} \right) \bigg|_{z=-h}.$$

Finally, adding (B1) to (B3), differentiating w.r.t. $h$, and substituting from (B2) and (B3), we obtain

$$\frac{\partial^3 \phi_m}{\partial h^3} = \left( \frac{I'_m}{I_m} - \frac{\lambda'_m}{\lambda_m} \right) \frac{\partial \phi_m}{\partial h} \bigg|_{z=-h}.$$ 

With these results in mind, we now differentiate (A4) w.r.t. $h$, and then eliminate $\rho_0(-h)$ using (A4):

$$\frac{\partial^3 \phi_m}{\partial h^3} = \left( \frac{I'_m}{I_m} - \frac{\lambda'_m}{\lambda_m} \right) \frac{\partial \phi_m}{\partial h} \bigg|_{z=-h}.$$ 

the first of the required results. Differentiating again w.r.t. $h$, and substituting from (B3), then (B4), and then (B5), gives the second required result:

$$\frac{\partial^3 \phi_m}{\partial h^3} = \left( \frac{3}{2} \left( \frac{I'_m}{I_m} - \frac{\lambda'_m}{\lambda_m} \right) \right) \frac{\partial \phi_m}{\partial h} \bigg|_{z=-h}.$$ 

$$\frac{\partial^3 \phi_m}{\partial z^3} = \left( \frac{3}{4} \left( \frac{I'_m}{I_m} - \frac{\lambda'_m}{\lambda_m} \right) \right) \frac{\partial \phi_m}{\partial h} \bigg|_{z=-h}.$$ 

References


Internal Tide Generation Modelled using a Modal Decomposition


