Water wave diffraction by segmented permeable breakwaters

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Water Wave Diffraction by
Segmented Permeable Breakwaters

by

Niall David McLean

A Doctoral Thesis

Submitted in partial fulfilment of the requirements for the award of
degree of Doctor of Philosophy of Loughborough University

22nd of October, 1999

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For my mother.
Abstract

This thesis presents an original theoretical investigation, based on diffraction theory (extended for permeable structures by Sollitt & Cross [56]), of the performance of segmented rubble mound breakwaters. The amount of protection offered by such a breakwater is a function of the rubble construction (characterized by porosity and permeability), geometry and spacing of segments as well as depending on the the characteristics of the incident wave field. To explore the influence of these factors on the performance of these breakwaters, the diffraction by three related structures have been considered: a periodic array of impermeable blocks, a single continuous rectangular section permeable structure and a periodic array of permeable blocks in shallow water.

Keywords: Rubble Mound Breakwaters, Infinite Arrays, Boundary Element Method, Systems of Integral Equations, Corner Singularities
Acknowledgements

I would like to thank my supervisor Dr. Phil McIver for all of his efforts, patience and enthusiasm during this work, and sharing some of his extensive knowledge on mathematical modeling of wave phenomena. I would also like to express my gratitude to Dr. Chris Linton and Dr. Paul Martin for their diligent reading and corrections of the first draft of this work.

I would like also to thank all my friends and colleagues over the last five years for their support and encouragement. Especially Salvatore, Bernardo & Sue, Saul, Ruben, Steve, Vassili, Tomas, Mark, Juan, Giambattista, Adriana, David Hulse, Victor, Miriam, David Pizer, Paul, Ewan, Aniruhsda, Navin and Pete. I would also like to acknowledge here the support of the academic and support staff at Department of Mathematical Sciences at Loughborough University. I am especially grateful to Dr. Keith Watling for his tireless maintenance of computing facilities and to Professor Ron Smith, my director of research, and Dr. Maureen McIver for their help and encouragement. At Strathclyde University I would like to acknowledge my gratitude to Dr. Philip Sayer for his patience and understanding during the later stages of this work and for the use of facilities for this purpose. Last but by no means least I would like to thank my family.
## Contents

1 Introduction

1.1 Background .......................................................... 4

1.2 Formulation of linearized diffraction problem for impermeable structures ..... 7

1.3 Synopsis ............................................................. 11

2 Background and derivation of Sollitt & Cross Model ............................ 13

2.1 Introduction .......................................................... 13

2.2 Continuum Approach to Porous Media ..................................... 14

2.2.1 Porosity ........................................................... 14

2.2.2 The Seepage and Average Velocities .................................. 17

2.3 Sollitt & Cross Model .................................................. 18

2.3.1 Unsteady fluid flow through a homogeneous isotropic porous medium ..... 18

2.3.2 Wave Theory In Porous Medium ...................................... 21

2.4 An example of the application of Sollitt & Cross model ..................... 24

2.4.1 Formulation .......................................................... 24

2.4.2 Outline of eigenfunction solution and mode swapping phenomenon ..... 26

2.4.3 Iterative determination of friction factor and physically relevant solution 28

2.4.4 Range of Validity Of Solitt & Cross Model ............................ 29

2.5 Conclusion ............................................................. 30

3 Water Wave Diffraction by a Rectangular Impermeable Segmented Breakwater 31

3.1 Introduction .......................................................... 31

3.2 Formulation .......................................................... 34

3.3 Full Solution .......................................................... 37

3.3.1 Eigenfunction Expansions .......................................... 37

3.3.2 Formulation of Integral Equations .................................... 39
B Chapter 3 appendix
   B.1 Behaviour at Impermeable Corners ................. 110

C Chapter 4 appendix
   C.1 Derivations of Free-surface Green's functions ....... 112
   C.2 Explicit Display Of Singularity In Green's Functions Using Bessel Functions ... 115
   C.3 Series Representation Of Green's Functions for Region 1 .............. 115

D Chapter 5 appendix
   D.1 Derivation Of Green's Function For Region 1 .............. 117
   D.2 Series Representation of Green's Function for Region 1 .............. 118
   D.3 Flow Behaviour at Permeable Block Corners ....... 120
Chapter 1

Introduction

1.1 Background

In one of his many expeditions in search of the lost city of Atlantis, Jacques Cousteau discovered a Minoan breakwater believed to have been constructed about 4500 years ago. Although technology has progressed much since the construction of this ancient structure, breakwaters continue to be employed today with much the same aims: the protection of coastlines and shipping within ports and harbours by reduction of the effects of the incident wave action. In fact with increasing concerns about erosion and flooding of coastal regions, greater development of coastlines worldwide and growing requirements for safe economical port and harbour facilities the demand for such structures has never been greater. Yet even with the wealth of experience accrued through history, successful deployment of a breakwater scheme remains amongst the most challenging problems in coastal engineering. One of the reasons for this is the wide range of factors which need to be accounted for in design. These include direction and magnitude of waves and surges, causes of coastal erosion (if this is what necessitates the scheme), the transport of sediments, the effect of the scheme on the coastal regime and of course economic, social and environmental factors. Additionally in respect of the breakwater itself; it must be capable of withstanding the combined impact forces of wave action and water flow and be built with sufficiently durable material to function over a reasonable life-span. Thus, in practice installation of breakwater system involves extensive cost-benefit and environmental impact studies.

Both in such studies and evaluation of existing breakwater schemes, mathematical models can prove an invaluable resource to the coastal engineer. Although laboratory and full-scale
tests are invariably necessary given the complexity of the problem, mathematical models can still provide a wide variety of useful information and consequently reduce the costs of such studies. They can for example, inform structure and layout design, suggest enhancements to existing schemes and prompt and inform laboratory and full-scale tests. There has consequently been much research activity (to which this thesis aims to contribute) in the last fifty or so years in devising mathematical models for breakwater schemes and solving the associated problems.

A widely used tool in the mathematical modeling of coastal and ocean structures, and one which is central to the consideration of breakwater systems in this thesis, is diffraction theory. It involves representation of the diffraction of waves by a structure by a boundary value problem, the so-called diffraction problem, formulated with a combination of an appropriate inviscid wave wave theory and suitable boundary conditions on the structure. Provided certain criteria are satisfied in this formulation, the solution (typically a velocity potential) gives a good representation of all fluid motions. From this, key design factors of a structure of interest such as the wave forces on it and its performance (for example in reduction of wave effect for a breakwater) can be estimated. The foremost criterion in employment of this theory, is that the structure of interest is large relative to the the wavelength of the incident waves. This ensures that the dominant effect of the structure on the surrounding fluid motions is the diffraction of the incident waves and viscous effects, such as drag, vortex shedding and flow separation are confined within thin boundary layers surrounding the structure and can be neglected. Additionally, an appropriate wave theory needs to be employed. Typically to represent normal operating conditions of breakwater systems, the linearized finite depth Airy water wave theory is employed. This has the advantages that the incoming mixed sea may be Fourier decomposed into a summation of individual frequency component problems and these are linear. Consequently throughout the thesis, diffraction problem are solved with the assumption of monochromatic incident waves, this is often called the frequency domain approach. Results from this approach may then be re-assembled, via transfer functions, to give the structure's response in real mixed seas either as a realization, or in terms of statistics of the response.
Figure 1.1: A Segmented Permeable Breakwater at Kakuda-Hama, Japan (Photograph from Shore Protection Manual [58, Figure 5-32].)

This thesis presents an original theoretical investigation, based on diffraction theory (extended for permeable structures by Sollitt & Cross [56]), of the performance of offshore segmented rubble mound breakwaters (such as illustrated in Figure 1.1). Segmented breakwaters have a number of advantages over those breakwaters consisting of a single, continuous section. For example, the gaps in the structure allow a constant proportion of wave energy to be transmitted, thereby helping to retard tombolo formation (a type of sandbar joining an island or equivalently the breakwater to the mainland), allowing continued longshore transport (important in prevention of coastal erosion); and assisting water quality maintenance in the sheltered region. In theoretical investigations of the performance of segmented breakwater to date (e.g. Dalrymple & Martin [12], Williams & Crull [62] and Evans and Fernyhough [20]) the assumption that the segments are impermeable is made, allowing the usual diffraction theory to be employed. However in practice, the segments in offshore segmented breakwaters often have a rubble mound construction. For this, a variety of different materials ranging from quarry rock to specially fabricated armour units such as tetrapods, cobs and dolos (for
illustration of these see for example Novak et al. [47, page 534]) are used - the latter hav-
ing good interlocking and reflection properties. This use of rubble mound segments has the advantage that the energy of waves penetrating into a sheltered region is effectively reduced by damping within the segments in addition to reflection by the structure. Unfortunately, for such rubble mound structures, the usual diffraction theory (described above) is unavailable as the incident wave field can interact with individual pieces of rubble or armour units. Further, typically the wavelengths of waves incident on the structure are typically large in comparison with these constituent elements and thus viscous effects (neglected by the usual diffraction theory) are important. However, Sollitt & Cross [56], in a study of reflection and transmission of water waves by long continuous section permeable breakwater, provide a mathematical model of wave interaction with permeable structures based on classical porous media flow, which can be incorporated into the usual diffraction theory. Thus facilitated by this theory, diffraction by a permeable segmented breakwater can be modelled.

1.2 Formulation of linearized diffraction problem for impermeable structures

For background a basic outline of diffraction theory for impermeable structures is presented here (note the extension for permeable structures with the Sollitt & Cross [56] theory is tackled in the next chapter). More detailed treatment of this material can be found for example in Dean & Dalrymple [15] or Mei [42].

Throughout, unless otherwise specified, a Cartesian coordinate system \((x, y, z)\) is adopted with \(z\) axis directed vertically upwards and with \(z = 0\) in the plane of the undisturbed free surface. The fluid is supposed to be inviscid and incompressible and its motions irrotational. For irrotational motion the fluid velocity \(u\) may be expressed as the gradient of a scalar potential \(\Phi(x, y, z, t)\), that is

\[
\mathbf{u} = \nabla \Phi. \tag{1.1}
\]

Conservation of mass requires that the divergence of the velocity is zero so that \(\Phi\) satisfies Laplace's equation

\[
\nabla^2 \Phi = 0 \tag{1.2}
\]

throughout the fluid.

The free surface of waves on water is governed by two boundary conditions: the kinematic
free surface condition and the dynamic free surface condition. The kinematic free surface boundary condition states that a particle lying on the free surface remains on the free surface. In particular, denoting the free surface elevation by \( z = \eta(x, y, t) \), the equation of the free surface may be written as

\[
F(x, y, z, t) = z - \eta(x, y, t) = 0, \tag{1.3}
\]

and the condition that a particle on the free surface remains on the free surface as

\[
\frac{DF}{Dt} = 0, \quad \text{on } z = \eta, \tag{1.4}
\]

where \( D/Dt \) denotes the total derivative. Employing the velocity potential defined in equation (1.1) this can be re-written as

\[
\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \eta}{\partial y} \quad \text{on } z = \eta. \tag{1.5}
\]

The dynamic free surface condition on the other hand is derived from the Bernoulli equation (see any text on fluid dynamics, e.g. Batchelor [6]) and the assumption that atmospheric pressure outside the fluid is constant, it is

\[
\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\Phi|^2 + g \eta = 0, \quad \text{at } z = \eta. \tag{1.6}
\]

On the boundary of any fixed impermeable surface, \( S_B \), such as the sea bed or the wetted surface of a fixed impermeable obstacle the potential \( \Phi \) satisfies a no-flow condition. Now since the surface is assumed to be impermeable there can be no flow into the surface and hence the component of velocity normal to the surface must be zero, in particular, in terms of the potential,

\[
\frac{\partial \Phi}{\partial n} = 0. \tag{1.7}
\]

where \( n \) is the outward normal vector to the surface. For the case when the fixed impermeable surface is the sea bed, equation (1.7) becomes

\[
\frac{\partial \Phi}{\partial z} = 0 \quad z = -h. \tag{1.8}
\]

Note throughout the thesis for simplicity the depth \( h \) is supposed to be constant however it could equally be described by a function namely \( h = h(x, y) \).

Thus, the Laplace equation (1.2) together with the boundary condition (1.5)-(1.8) form the diffraction problem for the structure \( S_B \) in water of depth \( h \).

Numerical solution of the full diffraction problems for various structures do exist, however as the kinematic and dynamic free surface conditions (1.4)/(1.5) and (1.6) are non-linear,
these solutions are both difficult to obtain and unwieldy. Thus, to obtain something more manageable, consider, following Mei [42, page 6] that the dimensional variables $\Phi, t, x, y, z$ and $\eta$ can be written in terms of non-dimensional variables as

$$\Phi = A\omega k^{-1} \Phi',$$  
(1.9)
$$t = \omega^{-1} t',$$  
(1.10)
$$(x, y, z) = k^{-1}(x', y', z'),$$  
(1.11)
$$\eta = A\eta',$$  
(1.12)

where the primes denote non-dimensionalised variables, $A$ is the wave amplitude $\omega$ the radian wave frequency and $k$ the wavenumber. (The wave frequency $\omega$ is defined by $2\pi/T$ where $T$ is the wave period and the wavenumber $k$ by $2\pi/L$ where $L$ is the wavelength). Substituting these expressions into the kinematic and dynamic boundary conditions, equations (1.5) and (1.6) respectively gives

$$\frac{\partial \Phi'}{\partial z} = \frac{\partial \eta'}{\partial t} + \epsilon \left( \frac{\partial \Phi'}{\partial x} \frac{\partial \eta'}{\partial x} + \frac{\partial \Phi'}{\partial y} \frac{\partial \eta'}{\partial y} \right) \quad \text{on } z' = \epsilon \eta',$$
(1.13)
$$\frac{\partial \Phi'}{\partial t} + \frac{\epsilon}{2} |\Phi'|^2 + \frac{gk}{\omega^2} \eta' = 0, \quad \text{at } z' = \epsilon \eta',$$
(1.14)

where $\epsilon = kA$ is the wave slope. Next, assuming the wave amplitude is small relative to the wavelength, so $\epsilon \ll 1$ and expanding equations (1.13) and (1.14) about the mean free surface position $y' = 0$, retaining the leading order terms, the linearized conditions

$$\frac{\partial \Phi'}{\partial z'} = \frac{\partial \eta'}{\partial t'}, \quad z' = 0,$$
(1.15)
$$\frac{\partial \Phi'}{\partial t'} + \frac{gk}{\omega^2} \eta' = 0, \quad z' = 0,$$
(1.16)

are obtained. Rewriting the above expressions in terms of the dimensional variables, gives the linearized free surface conditions,

$$\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t}, \quad z = 0$$
(1.17)
$$\eta = -\frac{1}{g} \frac{\partial \Phi}{\partial t}, \quad z = 0,$$
(1.18)

Note the equations (1.17) and (1.18) are often referred to as the linearized kinematic and dynamic free surface conditions respectively. By eliminating $\eta$ from (1.17) and (1.18) the combined linearized free surface condition

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0, \quad z = 0,$$
(1.19)

is obtained.
It is assumed throughout the thesis that all wave motion is sinusoidal in time, and hence the time dependence can be factored out. In particular \( \Phi \) can be written as

\[
\Phi(x, y, z, t) = \Re \{ \phi(x, y, z)e^{-i\omega t} \},
\]

where \( \phi \) is a complex valued potential. (Note the use of the term \( e^{-i\omega t} \) in equation (1.20) is the convention employed throughout the thesis although the term \( e^{i\omega t} \) might equally be employed.) And note by substitution of equation (1.20) into the boundary value problem \( \Phi \) can be replaced with \( \phi \) in all but the combined linearized free-surface condition (1.19) where the time-independent condition,

\[
\frac{\partial \phi}{\partial z} = \frac{\omega^2}{g} \phi, \quad \text{on } z = 0,
\]

is obtained.

Consider the interaction of a wave travelling in the positive \( x \) direction, at angle \( \theta_0 \) to the positive \( x \) axis in the \( (x, y) \) plane, with an impermeable structure which extends through the water depth. In particular, note by application of the separation of variables technique with \( \phi(x, y, z) = Z(z) \hat{\phi}(x, y) \) to solve the boundary value problem for this interaction, it is found that \( \phi \) has the form

\[
\phi(x, y, z) = \frac{-iAg \cosh k(z + h)}{\omega \cosh kh} e^{i(\alpha x + \beta y)} \hat{\phi}(x, y),
\]

where \( \hat{\phi}(x, y) \) satisfies

\[
\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} + k^2 \hat{\phi} = 0,
\]

and \( k \) is a root of the dispersion relation,

\[
\omega^2 = gk \tanh kh.
\]

Further note in the absence of the structure, the solution reduces to

\[
\phi = \phi^I = \frac{-iAg \cosh k(z + h)}{\omega \cosh kh} e^{i(\alpha x + \beta y)},
\]

where \( \alpha = k \cos \theta \) and \( \beta = k \sin \theta \). And note according to Mei [42, page 305] the dispersion relation (1.24) has two real roots \( k = \pm k_0 \) and a denumerable number of imaginary roots \( k = \pm ik_n \) where \( k_n \in \mathbb{R} \) for \( n \in \mathbb{N} \cup 0 \). And thus note in order that the incident wave propagates in the positive \( x \) direction and is non-evanescent, \( k \) in equations (1.22) and (1.23) must equal \( k_0 \), the positive real root. (Note employing the \( e^{i\omega t} \) solution convention, \( k = -k_0 \) would be the appropriate root).
Now since the governing differential equation, Laplace's equation is linear and homogeneous, and in addition all the boundary conditions for $\phi$ are linear, the solutions must satisfy the principle of linear superposition. In particular, if $\phi_1$ and $\phi_2$ are both solutions for $\phi$, then $\phi_3 = a_1\phi_1 + a_2\phi_2$ must also be a solution, where $a_1$ and $a_2$ are arbitrary constants. And this raises the issue of uniqueness for solutions of diffraction problems.

To obtain unique solutions in diffraction problems an additional boundary condition is employed, the so-called radiation condition. To demonstrate this condition note if a monochromatic gravity wave train is incident on structure of interest, the potential $\phi$ can be written

$$\phi = \phi^I + \phi^S,$$  \hspace{1cm} (1.26)

according to the linear superposition where $\phi^I$, the incident wave potential, denote the part of solution associated with the incident wave (the incident mode as given above ) and $\phi^S$, the scattering wave potential, the scattered (diffracted modes) waves. Then the radiation condition states that waves on the free-surface, other than those due to the incident wave potential ($\phi^I$) itself, are due to the presence of the structure (see Newman [46, page 288]). And thus the waves associated with the scattering potential $\phi^S$ must be radiating away from the structure.

\subsection*{1.3 Synopsis}

The amount of protection offered by a segmented rubble mound breakwater is a function of the rubble construction (characterized by porosity and permeability), geometry and spacing of segments as well as depending on the characteristics of the incident wave field. To explore the influence of these factors on the performance of segmented rubble mound breakwaters, the diffraction by three related structures have been considered. Formulation and solution of the appropriate diffraction problems, and results and analysis for each of these are presented here. Firstly, in chapter 3, diffraction by a segmented breakwater consisting of a line of identical impermeable surface piercing blocks is considered. In the formulation of a diffraction problem for this structure, it is assumed that the blocks extend in both the positive and negative directions along the line to infinity and are equally spaced, thus forming a periodic array. This assumption that the segmented breakwater is a periodic array is widely employed (e.g. Dalrymple & Martin [12], Williams & Crull [62] and Evans and Fernyhough [20]) and is not unreasonable for long segmented breakwater with equally spaced segments. In addition it simplifies the diffraction by avoiding consideration of effects at the ends of the
complete breakwater and by permitting, via use of certain periodicity conditions, solution by application of boundary conditions on a single element of the array. In chapter 4, with the Sollitt & Cross theory reviewed in Chapter 2, the reflection and transmission of a single continuous rectangular section permeable structure is considered. Again it is assumed that this structure extends along its length to infinity in both directions, again to avoid treatment of behaviour at the ends of the structure. Also it is assumed that construction of this structure is homogeneous and isotropic. Finally in chapter 5, diffraction by the same periodic segmented breakwater discussed in chapter 3 is reconsidered this time with permeable blocks. All blocks, like the continuous section structure considered in Chapter 4 are supposed to be of homogeneous and isotropic permeable material. Additionally the assumption that the structure stands in shallow water is made to simplify solution.

Solutions for the diffraction problem studied in Chapters 3 and 4 have appeared in the literature (Fernyhough & Evans [20] and Dalrymple, Losada & Martin [13] respectively). The solution of the first problem appeared concurrently with the author's own work on this problem. The solution to the second problem differs from the published solution in use of a Green's function integral equation approach. Chapter 2 presents a review of the Sollitt & Cross [56] theory for water waves and permeable structures and its application. And these chapters inform the study of segmented rubble mound breakwaters in Chapter 5.
Chapter 2

Background and derivation of Sollitt & Cross Model

2.1 Introduction

To model the diffraction of water waves by a permeable structure such as a segmented rubble mound breakwater, it is necessary to model the fluid and wave motions within the structure. One approach is to employ the Navier-Stokes equation together with appropriate boundary conditions on the solid pore boundaries within the structure. However, the pore boundaries of permeable structures are typically complicated and difficult to describe, except for structures with the simplest of geometries. Further, even for steady flows, solution of the boundary value problems formulated for these structures can be very difficult. An alternative, commonly adopted approach to flows in porous media is the continuum approach. This involves association of all points in a permeable structure with a representative velocity field (irrespective of whether the point is in a solid, stationary phase of the media or in moving fluid). With this continuum approach a variety of relations have been derived both theoretically and experimentally to describe the representative velocity fields for porous media with various characteristics. Using such a relation, Sollitt & Cross [56] have derived a boundary value problem, similar to the diffraction problems described in the introduction, to describe the wave motion within a permeable structure. This model has been widely employed previously in theoretical studies of permeable structures and experimentally verified by amongst other Sulisz [60].

In this thesis, the Sollitt & Cross model is used in the theoretical investigation of the
permeable segmented and continuous section breakwaters described in the introduction. Thus in this chapter, a review of development of this model (including a development of the flow equation used) and applications of it in the literature is presented.

2.2 Continuum Approach to Porous Media

There are three concepts in the continuum approach to fluid flow through a porous medium pertinent to understanding the development and use of the Sollitt & Cross model. These are porosity, the discharge velocity and the related conceptual fluid velocity. Thus a brief description of each of these is given in this section. For a more detailed treatment, see for example, Bear [7].

A useful starting point in this task is to formally distinguish the two distinct parts of any porous medium. These are the solid phase or solid matrix (for example the rubble in a rubble mound breakwater) and the void space, that part of the media not occupied by the solid matrix.

2.2.1 Porosity

Informally the porosity of an element in a porous medium is defined as the volume of void space in that element divided by the volume of the element itself. In the continuum approach to porous media, this notion of porosity is refined to give three distinct measures of porosity at every point in a porous medium: the volumetric porosity, the areal porosity and the linear porosity. To demonstrate this, let $P$ denote a point of interest in a porous medium and let $U$ denote a volume (of spherical shape say), with centroid at $P$, which is significantly larger than the volume of any individual pore or particle in the vicinity of $P$ whilst remaining entirely in the medium. Then consider the ratio

$$
\epsilon_U = \frac{U_v}{U},
$$

where $U_v$ denotes the volume of void space in $U$, as the volume $U$ is permitted to contract onto $P$. (Note $\epsilon_U$ is just the informal definition of porosity given above for the volume $U$).

Many aspects of the behaviour of ratio $\epsilon_U$ as $U$ tends to zero are independent both of the choice of $P$ or the medium considered. The key aspects, from the point of view of describing the continuum definitions of porosity, are illustrated in Figure 2.1. Specifically, note that for larger values of $U$, the ratio $\epsilon_U$ may undergo gradual changes as $U$ is contracted, especially
when the domain considered is inhomogeneous (e.g. soil with layers of particles of different sizes). Additionally note that below a certain value of $U$ depending on the distance of $P$ from boundaries of inhomogeneity, these changes tend to decay, leaving only small-amplitude fluctuations that are due to the random distribution of pore sizes in the neighbourhood of $P$. However below a certain critical value $U_0$, large oscillations in the value of $\varepsilon_U$ are observed. This occurs as the dimensions of $U$ approach those of a single pore or particle. Thus in the regime $U < U_0$ (the so-called domain of microscopic effects) no single value $\varepsilon_U$ is particularly representative of the porosity in the vicinity of $P$. Finally, as $U$ tends to 0, converging on $P$, $\varepsilon_U$ becomes either 0 or 1, depending on whether $P$ is located in the solid matrix or void space of the medium.

![Diagram](https://via.placeholder.com/150)

**Figure 2.1: Definition of porosity and representative elementary volume (from Bear [7, page 20])**

On the basis of these observations, the medium's *volumetric* porosity at a point $P$ is defined as

$$\varepsilon(P) = \lim_{U \to U_0} \varepsilon_U = \lim_{U \to U_0} \frac{U_0}{U},$$

(2.2)

and on account of its critical nature, the volume $U_0$ is often called the *representative elementary volume* (REV) or the *physical* (or material) point of the porous medium at the point $P$.

If the procedure just described is repeated for similarly defined plane (or line) elements for $P$, the resultant ratio of void space area (length) to element area (length) is found to exhibit the same trends as the volume ratio defined by equation (2.1). Therefore areal and linear porosities at $P$ can be defined in an analogous way to the volumetric porosity. To be specific, let $A_j$ and $L_k$ denote plane and length elements with centroid at $P$ and norm $j$ and
direction \( k \) respectively. Also let \( A_{jw} \) and \( L_{kv} \) denote the area/length in the elements \( A_j \) and \( L_j \) respectively. Then the medium's areal porosity at the point \( P \) for the norm \( j \), is defined by

\[
\epsilon_j^A(P) = \lim_{A_j \to A_{j0}} \frac{A_{jw}}{A_j},
\]

and its linear porosity at the point \( P \) for the direction \( k \) is defined by

\[
\epsilon_k^L(P) = \lim_{L_k \to L_{k0}} \frac{L_{kv}}{L_k},
\]

where \( A_{j0} \) and \( L_{k0} \) denote the critical area and length respectively below which microscopic effects (like those illustrated above) start to be observed. To signify the importance of the area \( A_{j0} \) and the length \( L_{k0} \) they are called respectively the representative elementary area (REA) and representative elementary length (REL) respectively.

On the basis of the above porosity definitions two important results can be established and important definition (in the context of the wave model to follow) given. Firstly on the basis of the porosity definitions, Bear [7, page 21] establishes that at a point \( P \) in a porous medium, where the volumetric porosity \( \epsilon(P) \) and the areal porosity \( \epsilon_j^A(P) \) with norm \( j \) can be assigned,

\[
\epsilon(P) \approx \epsilon_j^A(P),
\]

irrespective of the norm direction \( j \). Secondly, he states the related result, that at point \( P \) where the areal porosity \( \epsilon_j^A(P) \) with norm direction \( j \) and the linear porosity \( \epsilon_k^L(P) \) with direction \( k \) can be defined

\[
\epsilon_j^A(P) \approx \epsilon_k^L(P),
\]

irrespective of the choice of \( j \) and \( k \). Two important (if not unsurprising) conclusions that can be drawn from equations (2.5) and (2.6). Essentially areal and linear porosities are direction independent. Secondly, essentially only one of the above porosity assignments is needed at a point in a porous medium. Usually, the volumetric porosity is favoured since it is immune to the occasional directional variations experienced by the areal and linear measures. However in some circumstances (for instance on the interface of a rubble mound breakwater with the exterior fluid domain) when volumetric porosity is not defined and it becomes necessary to employ one of the other measures. Thus, henceforth the term “porosity” at a point will be used in the first instance to describe the volumetric porosity and when this may not be assigned, the areal, and as a last resort the linear porosity.

Finally on the basis of this definition of porosity, media for which the porosity is constant at every point are said to be homogeneous. And conversely media which don't satisfy this
criterion are said to be inhomogeneous.

Note, also a medium is said to be isotropic with respect to a property, if that property is independent of direction within the medium. Note also that conversely, a medium is said to be anisotropic with respect to property if it is dependent on direction in the medium.

2.2.2 The Seepage and Average Velocities

To model fluid flow through a rubble mound breakwater, Sollitt & Cross [56] employ another continuum notion, the average microscopic velocity. To define this, let \( v \) denote the actual microscopic fluid velocity field within the medium (defining \( v(P) = 0 \) if the point \( P \) is located in the solid matrix). Then the seepage velocity at a point \( P \) in the medium is defined by

\[
V(P) = \frac{1}{U_{ov}} \int_{U_{ov}} v \, dU_{ov};
\]  

(2.7)

where \( U_{ov} \) denotes the volume of void space in REV for \( P, U_0 \). Thus in an analogous way to which the porosity gives a macroscopic description of the geometry in a porous medium, the average velocity provides a macroscopic description of the fluid flow within it.

The use by Sollitt & Cross [56] of the microscopic velocity is quite unusual. In the majority of literature on fluid flow through porous media which employs the continuum approach, another representation of the fluid flow in media, the seepage velocity is employed. Although not explicitly used by Sollitt & Cross, it is useful (from the point of view of relating their flow equation to existing flowing equations) to present the definition of the discharge velocity and describe the relation between the average velocity and it. Specifically, at a point \( P \) in a porous medium, the discharge velocity is defined by

\[
q(P) = \frac{1}{U_0} \int_{U_0} v \, dU_{ov};
\]  

(2.8)

where recall from the previous section, \( U_0 \) represents the REV of the point \( P \). Further from the definitions (2.7) and (2.8) it is clear that the average velocity and the seepage velocity satisfy the Dupuit-Forchheimer relation,

\[
q = \epsilon V,
\]  

(2.9)

throughout a porous medium, where \( \epsilon \) from the previous subsection is the porosity.

(1) Note Bear called the seepage velocity, the average specific discharge.
2.3 Sollitt & Cross Model

2.3.1 Unsteady fluid flow through a homogeneous isotropic porous medium

Full Nonlinear Expression

In the context of the continuum approach, Sollitt & Cross [56] start the development of their model of wave motion through a porous structure by devising an equation to describe unsteady fluid flow through a homogeneous isotropic porous medium. To obtain this expression, Sollitt & Cross start by arguing that the seepage velocity field in such a medium $V$ must satisfy

$$\frac{\partial V}{\partial t} = -\frac{1}{\rho} \nabla (p + \rho g z) + \text{resistance forces}, \quad (2.10)$$

where $\rho$ denotes the density of the fluid and the resistance forces are those forces which impede fluid motion through the medium. (This is justified in Appendix A.1.1).

Then as a first step in establishing the resistance forces in equation (2.10) they note that for steady flow through a homogeneous isotropic medium, Ward [61] has found (employing an empirical/experimental approach) the one dimensional steady flow equation

$$\frac{1}{\rho} \frac{\partial}{\partial r} (p + \rho g z) = -\frac{\nu}{K_p} q_r - \frac{C_f}{\sqrt{K_p}} |q_r| q_r, \quad (2.11)$$

where $q_r$ denotes the component of the specific discharge (defined by equation (2.8)) in a specified direction $r$ and $\nu$ is the kinematic viscosity of the fluid. In addition, $K_p$ is a medium specific parameter called the \textit{intrinsic permeability} which depends on shape and size of the particles (or pores) in the medium in addition to the porosity. (Detailed consideration of the nature of the intrinsic permeability of a medium together with various expressions derived theoretically and experimentally for it, are presented in Bear [7, Chapter 5]). And finally, $C_f$ is what Ward calls the \textit{turbulent damping coefficient}, which he argues, on the basis of experiment, is equal to 0.550 irrespective of the medium or flow regime considered. It is also worth noting here that equation (2.11) resembles the classic Forchheimer relation, $\partial/\partial r(p + \rho g z) = A q_r + B q_r^2$ and for low Reynolds numbers reduces to something resembling Darcy's law $\partial/\partial r(p + \rho g z) = -\nu p/K_p q_r$; both described in detail by Bear [7, Chapter 5].

Next Sollitt & Cross hypothesize that equation (2.11) with reference to equation (2.10) may be extended for unsteady flow by the addition of a term which evaluates the resistance of the virtual mass of the discrete grains within the medium. This resistance force is given
where $C_m$ is a coefficient known as the virtual mass coefficient of the media grains. Then assuming that $(K_p,C_m,C_f)$ are isotropic and employing the Dupuit-Forchheimer relation (2.9), Sollitt & Cross claim unsteady flow through a porous medium is governed by

$$s \frac{\partial V}{\partial t} = \frac{1}{\rho} \nabla (p + \rho g z) - \frac{\nu}{K_p} \epsilon \nabla \epsilon - \frac{C_f}{\sqrt{K_p}} \epsilon^2 \nabla |V|,$$

(2.13)

where $s$, which Sollitt & Cross call the inertia coefficient, is defined by

$$s = 1 + \frac{1-\epsilon}{\epsilon} C_m.$$  

(Note equation (2.13) resembles the Polubarinova-Kochina equation $\partial/\partial t (p + pgz) = Aq_r + Bq_r^2 + C\partial q_r/\partial t$, see Bear [7, Page 182]).

There are two inconsistencies in Sollitt & Cross’s development of this equation. Note firstly Forchheimer’s equation 2.11 assumes low Reynolds number flow. And note Ward verifies this expression only for Reynolds numbers less than 100. However as observed by Burcharth & Anderson [8, page 255] in the surface level of a breakwater the Reynolds number can be of the order of $10^6$. Secondly Sollitt & Cross’s derivation relies on the hypothesis that an unsteady flow equation for permeable medium can be obtained by a somewhat arbitrary combination of equations (2.10), (2.11 and 2.12).

However on the basis of the resistance to flow models of Burcharth & Anderson [8] and Rumer [54] a flow equation similar to equation (2.13) can be derived. (This is presented in Appendix A). In particular this is of the form,

$$s \frac{\partial V}{\partial t} = \frac{1}{\rho} \nabla (p + \rho g z) - AV - BV |V|.$$  

(2.15)

And its derivation neither relies on the steady flow equation of Ward [61] or the hypothesis that an unsteady flow equation can be obtained just by adding to it, a term relating to inertial forces exerted by the medium on the flow. In addition it takes account of the features of the nature of flow pertinent to wave interaction with a rubble mound breakwater (that is high Reynolds number flow). In particular note that both the flow equation derivation and Burcharth & Anderson’s experiments indicate that $A$ and $B$ are Reynolds number dependent. And thus $A$ and $B$ vary for oscillatory flow through a porous medium, a point overlooked by Sollitt & Cross’s equation who advocate single constant values for $A$ and $B$.

For consistency with Sollitt & Cross, the remainder of the development is discussed with respect to (2.13) (however note this development proceeds identically using equation (2.15).
And note although equation (2.15 (with Reynolds dependent A and B) would seem to have advantageous properties over equation (2.13), it is observed that results of Sollitt & Cross and Sulisz [59, 60] which employ it, compare well with experiment.

Linearized Equation

In the next stage of the development of their model, Sollitt & Cross argue that wave motion through a porous medium can be represented by a linearized version of equation (2.13). Specifically they argue that the dissipative term can be replaced an equivalent linear stress term, namely,

\[
\frac{\nu}{K_p} \epsilon \nabla + \frac{C_f}{\sqrt{K_p}} \epsilon^2 \nabla |\nabla| \sim f \omega \nabla \tag{2.16}
\]

where \(\omega\) is the angular frequency of the periodic motion and \(f\) is a constant parameter which Sollitt & Cross call the dimensionless friction or damping coefficient. (Note \(\omega\) is introduced to make \(f\) dimensionless and for subsequent algebraic expediency.)

Sollitt & Cross both compute \(f\) and justify equation (2.16) on the basis of Lorentz condition of equivalent work. In the context of the present application, this is a requirement that during one wave cycle, the linearized expression in equation (2.16) dissipates the same amount of energy as the nonlinear expression. Specifically note that the work done by a force \(\mathbf{F}\) on the fluid in a volume of medium \(U\) is given by

\[
E(U) = \int_U \rho \mathbf{F} \cdot \nabla dV = \int_U \rho \mathbf{F} \cdot \nabla dV, \tag{2.17}
\]

and thus the power dissipation over one wave period is given

\[
P = \int_t^{t+T} E dt = \int_t^{t+T} \int_U \rho \mathbf{F} \cdot \nabla dV dt. \tag{2.18}
\]

And thus according to the Lorentz condition it is required that

\[
\int_U \int_t^{t+T} \epsilon \rho f \omega \nabla \cdot \mathbf{V} dtdV = \int_U \int_t^{t+T} \epsilon \rho \left( \frac{\nu}{K_p} \epsilon \nabla + \frac{C_f}{\sqrt{K_p}} \epsilon^2 \nabla |\nabla| \right) \cdot \mathbf{V} dtdV, \tag{2.19}
\]

Thus Sollitt & Cross obtains from (2.13) Sollitt & Cross obtain the linearized equation,

\[
s \frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho} \nabla (p + \rho g z) - f \omega \mathbf{u} \tag{2.20}
\]

where according to equation the Lorentz condition given by equation (2.19), the friction factor \(f\) is given by

\[
f = \frac{1}{\omega} \frac{\int_U \int_t^{t+T} \frac{\nu}{K_p} \epsilon |\nabla|^2 + \frac{C_f}{\sqrt{K_p}} \epsilon^2 |\nabla|^3 dtdV}{\int_U \int_t^{t+T} |\nabla|^2 dtdV}. \tag{2.21}
\]
2.3.2 Wave Theory In Porous Medium

Sollitt & Cross's development of a wave theory in a porous medium closely resembles the development of the open sea wave theory described in Chapter 1. Recall that there were three key ingredients in establishing this: the assumptions of incompressible and irrotational flow and the Bernoulli equation.

Potential Flow Field

Sollitt & Cross assume that flow fields associated with wave motion through a porous structure is incompressible and that the wave frequency term $e^{-i\omega t}$ can be factored out in it. On the basis of the linearized flow equation (2.20) and the second assumption they first demonstrate that the flow associated with the wave motion in the medium is irrotational. Specifically, writing the seepage velocity field $V$ as $V(x, y, z, t) = e^{-i\omega t}\tilde{V}(x, y, z)$ and substituting this in equation (2.20), it reduces to

$$\omega(f - is)V = \frac{-1}{\rho}\nabla(p + \rho gz),$$

and by taking the curl of this equation, it becomes

$$\omega(f - is)\nabla \times V = \frac{-1}{\rho}\nabla \times \nabla(p + \rho gz) = 0,$$

since $\nabla \times (\cdot) = 0$. Thus assuming $(f - is) \neq 0$, the seepage velocity field is irrotational and there exists a potential $\Phi(x, y, z, t)$ such that

$$V = \nabla \Phi.$$  \hspace{1cm} (2.24)

Further substituting equation (2.24) into equation (2.20), it becomes

$$\nabla \left( s\frac{\partial \Phi}{\partial t} + \frac{1}{\rho}(p + \rho gz) - f\omega \Phi \right) = 0,$$

and consequently,

$$s\frac{\partial \Phi}{\partial t} + \frac{1}{\rho}(p + \rho gz) - f\omega \Phi = C(t),$$

where $C(t)$ is a time dependent constant. Note equation (2.26) resembles Bernoulli's equation and is employed by Sollitt & Cross in analogous way. Note also by defining $\Phi'$ by

$$\Phi' = \Phi - \int^t C(t) \, dt,$$

which is clearly also a velocity potential for the seepage velocity $V$, that $\Phi'$ satisfies

$$s\frac{\partial \Phi'}{\partial t} + \frac{1}{\rho}(p + \rho gz) - f\omega \Phi' = 0,$$

according to equation (2.26). Thus without loss of generality $C(t)$ can be and is set equal to zero.
Linearized Free-surface Conditions

Sollitt & Cross next develop linearized free-surface conditions for waves in homogeneous isotropic structure which penetrates the free-surface in essentially the same way as such conditions were developed for the open ocean in Chapter 1. The only deviation from this is that instead of employing the Bernoulli equation, equation (2.26) is used. Specifically letting \( \eta(x, z, t) \) denote the wave profile in such a structure, the linearized kinematic free-surface condition is obtained identically as

\[
\frac{\partial \Phi}{\partial z} = \frac{\partial \eta}{\partial t}. \tag{2.29}
\]

Again the dynamic free surface condition consists of the assumption that the pressure at free surface constant. In fact without loss of generality it can be assumed that \( p = 0 \) at the free-surface and consequently by equation (2.26), the dynamic free surface condition in the structure can be stated as

\[
\left( s \frac{\partial \Phi}{\partial t} + g z - f \omega \Phi \right)_{z=\eta} = 0. \tag{2.30}
\]

Then the Taylor series expansion of equation (2.30) equation about \( z = 0 \) (as was done for dynamic free surface condition for the ocean ocean in Chapter 1) is

\[
\left( s \frac{\partial \Phi}{\partial t} + g z - f \omega \Phi \right)_{z=\eta} = \left( s \frac{\partial \Phi}{\partial t} + g z - f \omega \Phi \right)_{z=0} + \eta \left( s \frac{\partial^2 \Phi}{\partial z \partial t} + g - f \omega \frac{\partial \Phi}{\partial z} \right)_{z=0} + \ldots = 0. \tag{2.31}
\]

Thus discarding all but first order terms, the linearized dynamic free-surface condition

\[
\eta = -\frac{1}{g} \left( s \frac{\partial \Phi}{\partial t} - f \omega \Phi \right)_{z=0} \tag{2.32}
\]

is obtained.

Note by substituting equation (2.32) in equation (2.29) a combined linear kinematic and dynamic boundary condition,

\[
\frac{\partial \Phi}{\partial z} = -\frac{1}{g} \left( s \frac{\partial^2 \Phi}{\partial t^2} - f \omega \frac{\partial \Phi}{\partial t} \right) \text{ on } z = 0 \tag{2.33}
\]

is obtained. Further writing \( \Phi(x, y, z, t) = e^{-i\omega t} \phi(x, y, z) \) and substituting in equation (2.33), it reduces to

\[
\frac{\partial \phi_2}{\partial z} - \frac{\omega^2(s + i f)}{g} \phi_2 = 0 \quad \text{on } z = 0. \tag{2.34}
\]
Matching and Bed Boundary Conditions

Modelling the diffraction by a porous structure using Sollitt & Cross's approach involves employing the wave theories developed here and in Chapter 1 to model the wave motion within and external to the structure respectively. To determine a physically relevant description of the flow in both parts, it is necessary additionally to apply two matching conditions on the boundary of the structure. Note also that Sulisz \[60\] and Yu & Chwang \[64\] consider, using Sollitt & Cross's approach, porous structures consisting of regions of homogeneous isotropic porous media with different dissipative properties. And similar matching conditions need also to be applied on the interface between such regions. Sollitt & Cross and the others establish all these matching conditions on the basis of continuity of pressure and mass flux.

To demonstrate the formulation of these matching conditions, let \(p_i, \phi_i, s_i, f_i, \epsilon_i (i = 1, 2)\) denote the pressure, potential and inertia and friction coefficients respectively for two regions with a common boundary \(S_B\). Note according to equation (2.13) and (2.20), \(s = 1, f = 0\) and \(\epsilon = 1\) leads to the same to the same boundary value problem for the external fluid region as stated in Chapter 1. Then for a point \(P\) on the boundary \(S_B\), the pressure is given by equation (\ref{eq:2.35}) as

\[
p_1 = \rho (s_1 \frac{\partial \Phi_1}{\partial t} - gz - f_1 \omega \Phi_1),
\]

\[
p_2 = \rho (s_2 \frac{\partial \Phi_2}{\partial t} - gz - f_2 \omega \Phi_2),
\]

and thus by continuity of pressure

\[
(s_1 \frac{\partial \Phi_1}{\partial t} - f_1 \omega \Phi_1) = (s_2 \frac{\partial \Phi_2}{\partial t} - f_2 \omega \Phi_2).
\]

Further writing \(\Phi_i(x, y, z, t) = e^{-i \omega t} \phi_i(x, y, z)\) for \(i = 1, 2\), this reduces to

\[
(s_1 + i f_1)\phi_1 = (s_2 + i f_2)\phi_2.
\]

Also for \(P \in S_B\) consider the mass flux \(M_f\) through a small element \(A\) of the surface \(S_B\) with centroid at \(P\). In terms of the actual microscopic velocity field \(v\) (for region 1 and 2) the mass flux through \(A\) is given by

\[
M_f = \int_A \rho v \cdot n \, dA,
\]

where \(n\) is the unit normal to \(A\) pointing (without loss of generality) from region 1 into region 2. Then from the perspective of region \(i\) note that

\[
M_f = \rho \int_{A_i} v \cdot n \, dA_i \cong \rho A_{vi} V_1(P) \cdot n,
\]

23
where $A_{ei}$ denote the void space in region $i$ in $A$ and $V_i$ denote the seepage velocity in region $i$ ($i = 1, 2$). Thus continuity of mass flux implies
\[
\epsilon_1 V_1(P) \cdot n = \epsilon_2 V_2(P) \cdot n,
\] (2.41)
or in terms of the potentials
\[
\frac{\partial \Phi_1}{\partial n} = \frac{\partial \Phi_2}{\partial n},
\] (2.42)
\[
\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n}.
\] (2.43)

2.4 An example of the application of Sollitt & Cross model

Dalrymple, Losada & Martin [13] have employed the Sollitt & Cross model (just described) to study the reflection and transmission by a porous block under oblique wave attack. The block considered is supposed to be infinitely long, consist of a homogeneous isotropic medium, to stand fixed in a region of constant depth and to pierce the undisturbed free-surface. This work provides a simple yet highly illustrative example of the application of the Sollitt & Cross model and reveals some interesting features encountered in its use (for example the phenomenon of mode swapping). Thus with reference to the preceding review of the model development, the formulation of the boundary value problem and the preliminary steps in its solution for this interaction are presented here. Note that a full solution of the formulated boundary value problem which adopts a different solution approach to that employed by Dalrymple et al. is presented later in Chapter 4.

2.4.1 Formulation

![Side cross-section of porous structure](image)

Figure 2.2: Side cross-section of porous structure
A monochromatic wave train of specified frequency is assumed obliquely incident on the structure, travelling in the positive direction at angle \( \theta \) to the \( x \) axis. Employing the Sollitt and Cross [56] theory developed here, the wave field can be described by the velocity potentials \( \Phi_i (i = 1, 2, 3) \) in each of the respective regions. Dalrymple et al. [13] note that the \( y \) variation of the solution in each region must be same to satisfy the matching conditions between the regions. Thus defining \( \lambda \) as the \( y \) component of the incident wave number \( k \), that is \( \lambda = k \sin \theta \), and assuming time dependent motions of frequency \( \omega \) throughout, each velocity potential may be written

\[
\Phi_i (x, y, z, t) = \Re \left[ e^{i(\lambda y - \omega t)} \phi_i (x, z) \right] \quad (i = 1, 2, 3),
\]

and the full problem can be stated as

\[
(\nabla^2 - \lambda^2) \phi_i = 0, \quad (i = 1, 2, 3),
\]

subject to the following boundary conditions:

the bed condition,

\[
\frac{\partial \phi_i}{\partial z} = 0, \quad \text{when } z = -h \quad (i = 1, 2, 3),
\]

the free surface conditions (according to equation (2.34),

\[
\frac{\partial \phi_i}{\partial z} - \frac{\Gamma_i}{h} \phi_i = 0, \quad \text{at } z = 0 \quad (i = 1, 2, 3),
\]

where

\[
\Gamma_1 = \Gamma_3 = \frac{\omega^2 h}{g}, \quad (2.48)
\]

\[
\Gamma_2 = (s + if) \Gamma_1 = \frac{\omega^2 h (s + if)}{g}, \quad (2.49)
\]

and the matching conditions,

\[
\phi_1 (0, z) = (s + if) \phi_2 (0, z), \quad (2.50)
\]

\[
\frac{\partial \phi_1}{\partial x} (0, z) = \epsilon \frac{\partial \phi_2}{\partial x} (0, z), \quad (2.51)
\]

\[
\phi_3 (b, z) = (s + if) \phi_2 (b, z), \quad (2.52)
\]

\[
\frac{\partial \phi_3}{\partial x} (b, z) = \epsilon \frac{\partial \phi_2}{\partial x} (b, z). \quad (2.53)
\]

In addition to the boundary conditions stated above, the radiation condition, that all waves modes in regions 1 and 3 other than the incident one propagate away from the structure, is imposed.
2.4.2 Outline of eigenfunction solution and mode swapping phenomenon

Dalrymple et al. employ a matched eigenfunction solution approach to solve the boundary value problem given above. This is described here to demonstrate the approach, for later use in the alternative solution of the the problem in Chapter 4 and for description of the mode swapping phenomenon.

To start with, consider the solution of the boundary value problem in the regions 1 and 3. By employing separation of variables to solve (2.45) and applying the boundary conditions (2.46) and (2.47) (for \(i = 1, 3\) respectively) together with the radiation condition, the eigenfunction expansions

\[
\phi_1(x, z) = (e^{i(k_0^2 - \lambda^2)^{1/2}x} + R_0e^{-i(k_0^2 - \lambda^2)^{1/2}x})I_0(z) + \sum_{n=1}^{\infty} R_ne^{-i(k_n^2 - \lambda^2)^{1/2}x}I_n(z),
\]

\[
\phi_3(x, z) = T_0e^{i(k_0^2 - \lambda^2)^{1/2}(x-b)}I_0(z) + \sum_{n=1}^{\infty} T_ne^{i(k_n^2 - \lambda^2)^{1/2}(x-b)}I_n(z),
\]

are obtained, where a family of evanescent modes is included in both expansions to satisfy the matching conditions on the block and where the branch of the square root \((k^2 - \lambda^2)^{1/2}\) which satisfies

\[
\Re\{(k^2 - \lambda^2)^{1/2}\} \geq 0 \quad \text{and} \quad \Im\{(k^2 - \lambda^2)^{1/2}\} \geq 0
\]

is taken. Note that the subscript zero refers to the incident and reflected waves, whereas the subscripts, \(n > 0\), refers to the evanescent modes.

The depth dependency of the problem is provided by the \(I_n(z)\) which is defined by

\[
I_n(z) = \frac{-ig \cosh k_n(z + h)}{\omega \cosh k_nh}, \quad n = 0, 1, 2, \ldots
\]

where \(k_0\) is the real positive eigenvalue and \(k_n (n = 1, 2, \ldots)\) denote the purely imaginary
eigenvalues (located in the \( \Im m k \geq 0 \) part of the complex plane) of the dispersion relation

\[
\Gamma_1 = kh \tanh kh. \tag{2.58}
\]

It is well known that the set of eigenfunctions \( \{ \cosh k_n(z + h), n = 0, 1, 2, \ldots \} \) is a complete orthogonal set with

\[
\int_{-h}^{0} \cosh k_m(z + h) \cosh k_n(z + h) \, dz = \delta_{mn} N^2(k_n), \tag{2.59}
\]

where \( \delta_{mn} \) denotes the Kronecker delta and the normalization factor \( N^2(k_n) \) is defined for \( n = 0, 1, 2, \ldots \) by

\[
N^2(k_n) = \frac{\sinh 2k_n h + 2k_n h}{4k_n}. \tag{2.60}
\]

Similarly by employing separation of variables and applying the boundary conditions (2.46) and (2.47) for region 2, the eigenfunction expansion

\[
\phi_2(z, x) = \sum_{n=1}^{\infty} (A_n e^{i(K_n^2 - \lambda^2)^{1/2}x} + B_n e^{-i(K_n^2 - \lambda^2)^{1/2}(x-b)}) P_n(z), \tag{2.61}
\]

is obtained, where the branch of the square root \( (K_n^2 - \lambda^2)^{-1/2} \) with

\[
\Re (K_n^2 - \lambda^2)^{-1/2} \geq 0, \text{ and } \Im (K_n^2 - \lambda^2)^{-1/2} \geq 0, \tag{2.62}
\]

is taken for each \( n \), \( (n = 1, 2, \ldots) \). Note, writing \( \alpha_n = \Re (K_n^2 - \lambda^2)^{1/2} \) and \( \beta_n = \Im (K_n^2 - \lambda^2)^{1/2} \) for \( n = 1, 2, \ldots \) then its clear that the terms,

\[
A_n e^{i(K_n^2 - \lambda^2)^{1/2}x} P_n(z) = A_n e^{-\beta_n x} e^{i\alpha_n x} P_n(z) \quad (n = 1, 2, 3, \ldots), \tag{2.63}
\]

represent the modes propagating through the medium in the positive \( x \) direction, and the terms,

\[
B_n e^{-i(K_n^2 - \lambda^2)^{1/2}(x-b)} P_n(z) = B_n e^{\beta_n(x-b)} e^{-i\alpha_n(x-b)} P_n(z) \quad (n = 1, 2, 3, \ldots), \tag{2.64}
\]

represent those propagating through the medium in the negative \( x \) direction. Further by the choice of the square root branch in (2.62), it is clear that \( \beta_n \geq 0 \) for each \( n \) and consequently as the modes described by equations (2.63) and (2.64) travel through the medium, their energies are dissipated as expected.

Further in the porous block, the depth dependency becomes

\[
P_n(z) = -\frac{ig \cosh K_n(z + h)}{\omega \cosh K_n h} \quad n = 1, 2, \ldots \tag{2.65}
\]
where the eigenvalues \( K_n (n = 1, 2, \ldots) \) denote the roots of the porous medium dispersion relation,

\[
\Gamma_2 = (s + if) \Gamma_1 = K h \tanh K h, \tag{2.66}
\]

with positive imaginary part. Further note that in the majority of circumstances, the eigenfunctions \( \{ \cos K_n(z + h), n = 1, 2, \ldots \} \) satisfies a "pseudo-orthogonality" condition similar to the condition (2.67),

\[
\int_{-h}^{0} \cosh K_m(z + h) \cosh K_n(z + h) \, dz = \delta_{mn} N^2(K_n). \tag{2.67}
\]

where

\[
N^2(K_n) = \frac{\sinh K_n h + 2K_n h}{4K_n} \tag{2.68}
\]

(Note equation (2.67) is a "pseudo-orthogonality" relation in the sense that true orthogonality would require

\[
\int_{-h}^{0} \cosh K_m(z + h) \cosh K_n(z + h) \, dz = \delta_{mn} N^2(K_n). \]

However as first observed by Dalrymple et al. [13] there are certain incident wave frequencies when the porous dispersion relation (2.66) has double roots and in this case the separation of variables solution (2.61) is incomplete. Note a double root may be detected by the failure of the pseudo-orthogonality condition (2.67), in particular when

\[
N^2(K_n) = 0. \tag{2.69}
\]

Further Dalrymple em et al. describe this phenomenon with the terms "mode swapping" and "coalescence of wave-number", and note it has gone unnoticed in several other works in this area including Sollitt and Cross [56]. Further with reference to an appropriate Green's function for the problem they identify the additional "non-separable" terms needed for completion of the eigenfunction expression in this case. Recently, McIver [39] has provided a detailed explanation for this phenomenon and a rigorous justification that the additional "non-separable" terms employed by Dalrymple et al. [13] are both necessary and sufficient by utilizing the theory of non-self-adjoint operators.

### 2.4.3 Iterative determination of friction factor and physically relevant solution

A physically meaningful solution of boundary value problem arising from application of Sollitt & Cross model depends on appropriate choice of the inertia coefficient \( s \) and friction factor
On the basis of comparison with experiments, Sulisz suggests that a value close to $s = 2$ is the most appropriate for the inertia coefficient. (However for consistency with the much of the rest of the literature based on the model, $s = 1$ is adopted throughout this thesis). An appropriate value for the friction factor for a given wave structure interaction on the other-hand is determined in two parts. Firstly, appropriate values of the coefficients in the nonlinear unsteady flow equation (2.13) (or equation (2.15) if it is used) need to be determined. Note Madsen [41] provides various empirical formulae for these. Next to determine an appropriate value of $f$, Sollitt & Cross suggest the following five steps:

(a) Assume an initial value for $f$, for example $f = 1.0$.

(b) Solve the dispersion relation for $n$ values.

(c) Solve boundary value problem which arises.

(d) On the basis of the solution in the porous structure, calculate $V_2 = \nabla \Phi_2$ and calculate,

$$f = \frac{1}{\omega} \int_U \int_t^{t+T} \frac{\nu}{K_p} \epsilon|V_r|^2 + \frac{C_f}{\sqrt{K_p}} \epsilon^2 |V_r|^3 \, dt dV$$

$\int_U \int_t^{t+T} |V_r|^2 \, dt dU$

where $V_r = \Re \{V_1\}$.

(e) Compare the calculated $f$ with assumed $f$ and iterate if necessary.

Note Madsen [41] suggest employing the bisection method in the iterative process. And for the $s = 1$ case he suggests starting with $f = 1$ and $f = 0$ in this. Note the solution of the boundary value problem for this latter case is known a priori, it's just the incident wave, since this is the case where no block is present. And thus this is a good choice.

2.4.4 Range of Validity Of Solitt & Cross Model

Finally note Madsen [41, page 391] and Sollitt & Cross [56, page 1842] observe by comparison with experiments that the model is most successful when the wave height exceeds the particle diameter of the medium. And they note that when the wave amplitude is less than or approximately equal to the particle diameter, the incident wave can interact with the individual particles in the medium, leading to reflections directly off the particle surfaces. And clearly in this regime continuum approaches to flow through the permeable structures such as Sollicit & Cross's model are not really appropriate - which explains the failure of the model in this regime.
2.5 Conclusion

This chapter has attempted to review the development of the Sollitt & Cross model and discuss its use and limitations. Detailed and critical consideration has been given to the flow equation (2.13) on account of its central role both in the development and use of the model. And on the basis of the work of Burcharth & Anderson [8], it is suggested that a vector form of Polubarinova-Kochina unsteady equation (with flow regime dependent coefficients $A$, $B$ and $C$) is perhaps a more appropriate equation to use in the model than equation (2.13). However it is also that use of Sollitt & Cross equation in the model gives results that compare well with experiment. In application of the model, the phenomenon of mode swapping discovered by Dalrymple, Losada & Martin [13] was discussed. The dissipative nature of the model was pointed out in section 2.4.2 with reference to equations (2.63) and (2.64). Finally the range of validity of the Sollitt & Cross model has been discussed.
Chapter 3

Water Wave Diffraction by a Rectangular Impermeable Segmented Breakwater

3.1 Introduction

One of the main goals of this work, as the reader will recall from the introduction, is to model the diffraction of a monochromatic gravity wave train by a periodic array of permeable blocks (representing a typical segmented rubble mound breakwater). Application of the Sollitt & Cross [56] theory discussed in the previous chapter, to study this diffraction results in a three dimensional boundary value problem. To try and identify potential solution techniques (and possible pitfalls with these), a diffraction problem for the same geometry but with impermeable blocks has been considered. As demonstrated in the previous chapter, this can be formulated equivalently via the classical diffraction theory for impermeable structures (described in the introduction) or by taking the $\epsilon = 0$ (zero porosity) case of the permeable problem. Further as one might expect the solution of this problem is simpler than the general permeable case since no wave motions can take place within the structure. Specifically, in the impermeable case, the depth variation of the solution can be factored out in a straightforward way, making the problem effectively two dimensional. Additionally, the impermeable boundary conditions are easier to apply than those for the permeable case. Thus this is a good choice of trial problem for investigation of solution techniques that may be appropriate to permeable problem. Formulation and solution of the impermeable problem is the subject
Numerous studies have been made of diffraction by periodic gratings and arrays which inform the current study. For instance, in acoustics, Achenbach et al. [2, 3] solved diffraction problems involving screens and cylinders respectively employing a Green's function/integral equation approach. In the area of water waves, Dalrymple & Martin [12] solved a diffraction problem for an array of thin breakwaters under normal wave attack using a matched eigenfunction approach with a least squares and alternatively a variational approach to tackle the matching problem which arises. More general cases of this problem, including oblique incidence and arrays of non-collinear breakwater have been tackled by Porter & Evans [50] (correcting an error in Williams et al. [62]) and Williams et al. [63] and respectively. These both employ the Green's function/integral equation approach employed by Achenbach et al. [2, 3]. Linton & Evans [31] and Linton [32] have considered diffraction problems for arrays of cylinders (in the contexts of both acoustics and water waves) and parallel plates (in the context of acoustics) using a combination of eigenfunction and multipole solution methods.

For the solution of the current problem the matched eigenfunction approach employed by Dalrymple & Martin [12] together with an integral equation approach to the matching problems which arise has been employed. Briefly, formulation and solution of this problem proceeds as follows. Firstly, using the Bloch theory employed by Linton & Evans [31] and Linton [32] together with a symmetry argument described by McIver [37], the problem is formulated in symmetric and anti-symmetric parts on a single repeating element of the array. Eigenfunction expansions for these symmetric and anti-symmetric sub-problems with undetermined coefficients are then found for the gap region and the outer region (comprising of the regions upwave and downwave from the structure). It is then noted that to ensure analytic continuation of the full solution, the outer and gap solutions must satisfy certain matching conditions on their common interfaces. And these are employed together with certain orthogonality properties of the eigenfunction expansions to derive a set of integral equations. Finally via numerical solution of these integral equations using a collocation method with specially designed boundary elements which attempt to account for the anticipated velocity singularities at the block corners, the undetermined coefficients in the eigenfunction expansions are fully specified.

During the completion of this work, Fernyhough & Evans [20] published an alternative solution of the boundary value problem. Like the present solution, this employs a similar matched eigenfunction and integral equation approach to solution but with some additional
sophistication which gives it two key advantages over the present approach. Firstly on the basis of shrewd manipulation of the matched eigenfunction expressions for the gap and outer regions and application of orthogonality, they formulate integral equations with positive definite integral operators. And on the basis of this property, they establish bounds on components of their solution (see Fernyhough & Evans [20, equations (2.66) and (2.67)]) which guarantee the convergence of their method. Secondly, they employ a Galerkin method in their solution (typically more efficient than the simple collocation employed above), with basis functions which have the same singular behaviour as the anticipated velocity singularities at the block corners (see Fernyhough & Evans[20, equations (2.70) and (2.72)]). And this approach takes much more precise account of this singular behaviour than is done in the present approach. And thus it is expected (and demonstrated below) that the present method performs less efficiently than that of Fernyhough & Evans[20]. Nevertheless as demonstrated below it also performs well.
3.2 Formulation

Geometry

The interaction of a gravity wave train with a periodic array of identical impermeable blocks which extend through water of constant depth $h$ is considered. The blocks which lie in a line as illustrated in Figure 3.1 are each of length $2c$ and width $2d$ and are separated from each other by a gap of width $2l$. Throughout the chapter, a right-handed Cartesian coordinate system $(x, y, z)$ illustrated in Figure 3.1 is employed. In this system, the $x$ and $y$ axes lie parallel with the width and length respectively of the blocks with the $x$ axis pointing in the direction of wave propagation. The $z$ axis is normal to the undisturbed free surface and points out of the fluid and the origin is located in the undisturbed free surface at the centre of a gap.

![Figure 3.1: The periodic array.](image-url)
A non-evanescent wave of small amplitude $A$ and radian frequency $\omega$ is supposed to attack the array at an angle $\theta_0$ to the $x$ axis (illustrated in Figure 3.1). Employing the linearized water wave theory for water of finite depth as described in Chapter 1, the flow about the structure can be described by the velocity potential $\Phi$,

$$\Phi(x, y, z, t) = \Re \left\{ \frac{iAg \cosh k(h + z)}{\omega \cosh kh} e^{-i\omega t} \phi(x, y) \right\},$$

(3.1)

where $\phi$ is a solution of

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0,$$

(3.2)

and the wavenumber $k$ is the positive real root of

$$\omega^2 = gk \tanh kh.$$

(3.3)

And note since the waves are propagating at angle $\theta_0$ to the $x$ axis, the incident mode is represented by

$$\phi^I(x, y) = e^{i(\alpha_0 x + \beta_0 y)}, \quad (\alpha_0 = k \cos \theta_0, \ \beta_0 = k \sin \theta_0)$$

(3.4)

In addition, $\phi$ satisfies the following boundary conditions as a consequence of the geometry of the structure. Firstly, since the blocks are impermeable, $\phi$ satisfies the no flow condition,

$$\frac{\partial \phi}{\partial n} = 0,$$

(3.5)

on the boundary of the blocks, where $n$ is a normal to the block boundary. And secondly to obtain a unique solution to this problem it is necessary to apply the radiation condition as described in chapter 1.

**Geometrical Simplifications**

Solution of the boundary value problem (3.2)-(3.5) is considerably simplified by exploitation of the periodicity and symmetry in the problem.

In particular adopting the arguments described by Linton & Evans [31] the periodicity of the array can be exploited. Assuming the array extends to infinity in both the positive and negative $y$ directions then according to Bloch's theorem (see Ashcroft & Mermin [4, page 133]) the potential $\phi$ is of the form,

$$\phi(x, y) = e^{i\beta_0 y} \psi(x, y),$$

(3.6)
where \( \psi(x, y) \) is periodic in \( y \) with period \( 2b \) where \( b = c + l \). Further it follows from equation (3.6) that it's only necessary to consider the problem in the strip \( |y| < b \) subject to the two independent conditions,

\[
\phi(x, b) = e^{i2\beta_0 b} \phi(x, -b), \quad (3.7)
\]
\[
\frac{\partial \phi}{\partial y}(x, b) = e^{i2\beta_0 b} \frac{\partial \phi}{\partial y}(x, -b). \quad (3.8)
\]

Figure 3.2: Periodic element of the array

And the symmetry of the array about the \( y \) axis can be exploited to simplify the solution by employing an argument described by McIver [37, page 68]. Note in particular that the solution \( \phi \) (as a continuous function on the domain) can be decomposed as

\[
\phi(x, y) = \phi^s(x, y) + \phi^a(x, y), \quad (3.9)
\]

where the symmetric potential \( \phi^s \) is an even function of \( x \) and the anti-symmetric potential \( \phi^a \) is an odd function of \( x \). And as consequence of the array symmetry about \( x = 0 \), the boundary value problem can be decomposed using equation (3.9) into two simpler boundary value problems for the symmetric and anti-symmetric potentials on the region \( x \leq 0 \) (as taken here, or equivalently \( x \geq 0 \)). In particular in the respective symmetric and anti-symmetric boundary value problems, \( \phi^s \) and \( \phi^a \) are required to satisfy (3.2)-(3.5) and the radiation condition, and in addition,

\[
\frac{\partial \phi^s}{\partial x} = 0, \quad x = 0, \quad (3.10)
\]
\[
\phi^a = 0, \quad x = 0. \quad (3.11)
\]

to ensure continuity of velocity and pressure on \( x = 0 \).

Finally it facilitates the use of the method of separation of variables in the solution of this problem to consider the symmetric and anti-symmetric problem on two adjoining regions of the fluid domain: the regions \( |x| \geq d \) (region 1) and \( |x| \leq d \) (region 2). In this scheme, if \( \phi^s_1 \),
\( \phi_1^a \) and \( \phi_2^a \) and \( \phi_2^b \) denote the solutions to the respective symmetric and and anti-symmetric problems on regions 1 and 2, analytic continuation of the solution requires the matching conditions,

\[
\begin{align*}
\phi_1^a(-d, y) &= \phi_2^a(-d, y), \\
\phi_1^b(-d, y) &= \phi_2^b(-d, y), \\
\frac{\partial \phi_1^a}{\partial x}(-d, y) &= \frac{\partial \phi_2^a}{\partial x}(-d, y), \\
\frac{\partial \phi_1^b}{\partial x}(-d, y) &= \frac{\partial \phi_2^b}{\partial x}(-d, y),
\end{align*}
\]

are satisfied for \(|y| < l|.

### 3.3 Full Solution

#### 3.3.1 Eigenfunction Expansions

In this subsection, symmetric and anti-symmetric eigenfunction for regions 1 and 2 are sought. Firstly, in region 1, employing separation of variables, the Bloch conditions (3.7) and (3.8) and the radiation condition for the regions \( x < -d \) and \( x > d \), the full eigenfunction expansions,

\[
\phi_1(x, y) = e^{i(\alpha x + \beta y)} + \sum_{n=-\infty}^{\infty} R_n e^{-i(\alpha x - \beta y)}, \quad (x \leq -d)
\]

\[
\phi_1(x, y) = \sum_{n=-\infty}^{\infty} T_n e^{i(\alpha x + \beta y)}, \quad (x \geq d)
\]

are obtained, where

\[
\alpha_n = \begin{cases} 
  i(\beta_n^2 - k^2)^{1/2} & (n \leq -r, \ n \geq s) \\
  (k^2 - \beta_n^2)^{1/2} & (-r \leq n \leq s)
\end{cases}
\]

\[
\beta_n = \beta_0 - \frac{n\pi}{b},
\]

and

\[
\begin{align*}
r &= \left[ \frac{(1 + \sin \theta_0) \frac{kb}{\pi}}{\pi} \right], \\
s &= \left[ \frac{(1 - \sin \theta_0) \frac{kb}{\pi}}{\pi} \right].
\end{align*}
\]

(Notes the modes \( n \leq -r, \ n \geq s \) are evanescent and the modes \(-r \leq n \leq s \) are non-evanescent or travelling modes.)
Then by manipulating equation (3.16) with sgn functions and absolute value functions, equations (3.16) and (3.17) can be decomposed into the symmetric and anti-symmetric eigenfunction expansions (satisfying the symmetric and anti-symmetric problems),

\[ \phi_1^+(x, y) = \frac{1}{2} e^{-i(\alpha_0|x| - \beta_0 y)} + \sum_{n=-\infty}^{\infty} R_n^e e^{i(\alpha_n|x| + \beta_n y)}, \]  

\[ \phi_1^-(x, y) = -\text{sgn}(x) \left( \frac{1}{2} e^{-i(\alpha_0|x| - \beta_0 y)} + \sum_{n=-\infty}^{\infty} R_n^e e^{i(\alpha_n|x| + \beta_n y)} \right), \]  

where

\[ \text{sgn}(x) = \begin{cases} 
-1, & \text{if } x < 0 \\
1, & \text{if } x \geq 0 
\end{cases} \]  

and the coefficients \( R_n \) and \( T_n \) satisfy according to equation (3.9),

\[ R_n = R_n^e + R_n^t, \]  

\[ T_n = R_n^e - R_n^t, \]

for all integer \( n \). Note the undetermined coefficients \( R_n \) and \( T_n \), the so-called reflection and transmission coefficients, denote the amplitudes of the wave modes reflected and transmitted by the array respectively. It's also useful to note here (for use in the next subsection) that the lateral eigenvectors \( \{ e^{i\beta_n y}, n \in \mathbb{N} \} \) satisfy the orthogonality condition,

\[ \frac{1}{2b} \int_{-b}^{b} e^{i\beta_n y} e^{-i\beta_m y} dy = \delta_{mn}. \]  

Eigenfunction solutions for the symmetric and anti-symmetric boundary value problems in region 2 are obtained in a similar way. In particular, by employing separation of variables to solve Helmholtz equation (3.2) in region 2 and applying the zero flow condition (3.5) on the side walls of the gap, the eigenfunction expansion,

\[ \phi_2(x, y) = \sum_{n=0}^{\infty} \{ M_n \cos r_n x + N_n \sin r_n x \} \mu_n(y) + \{ P_n \cos s_n x + Q_n \sin s_n x \} \nu_n(y), \]  

is obtained where \( r_n \) and \( s_n \) are defined by,

\[ r_n = \left( k^2 - \frac{\pi^2 n^2}{l^2} \right)^{1/2}, \]  

\[ s_n = \left( k^2 - \frac{(2n + 1)^2 \pi^2}{4l^2} \right)^{1/2}, \]

and the lateral eigenvectors are defined by,

\[ \mu_0(y) = \frac{1}{(2l)^{1/2}}, \]  

\[ \mu_n(y) = \frac{1}{l^{1/2}} \cos \frac{\pi ny}{l} \quad (n \neq 0), \]  

\[ \nu_n(y) = \frac{1}{l^{1/2}} \sin \frac{\pi(2n + 1)y}{2l}. \]
(It is noted here that equation (3.28) can probably be simplified. In particular in the solution of Fernyhough & Evans [20] they obtain a much simpler expression directly using the same approach but under a different coordinate system). And as cosine and sine are even and odd functions respectively, it is straightforward to decompose \( \phi_2 \) into the symmetric and anti-symmetric parts,

\[
\phi^s_2(x, y) = \sum_{n=0}^{\infty} M_n \cos r_n x \cdot \mu_n(y) + P_n \cos s_n x \cdot \nu_n(y),
\]

\[
\phi^a_2(x, y) = \sum_{n=0}^{\infty} N_n \sin r_n x \cdot \mu_n(y) + Q_n \sin s_n x \cdot \nu_n(y).
\]

Note also, that like the lateral eigenfunctions for region 1, the lateral eigenfunctions for region 2 satisfy the orthogonality (in fact orthonormality) relations for \( m, n \in \mathbb{N} \),

\[
\int_{-l}^{l} \mu_m(y) \mu_n(y) \, dy = \delta_{mn},
\]

\[
\int_{-l}^{l} \nu_m(y) \nu_n(y) \, dy = \delta_{mn},
\]

\[
\int_{-l}^{l} \mu_m(y) \nu_n(y) \, dy = 0.
\]

### 3.3.2 Formulation of Integral Equations

Next, to determine the (as yet) undetermined coefficients in the symmetric and anti-symmetric eigenfunction solutions found above, integral equations are formulated using the remaining boundary conditions (the no flow condition on the lateral sides of the blocks and the matching conditions (3.12)-(3.15)) and the orthogonality relations given above. To illustrate this integral equation formulation (learned from Evans and McIver [18]), attention is focused on the symmetric problem. Let \( U^s(y) \) denote the symmetric part of the velocity on the interface \( z = -d \), then clearly according to equation (3.22), the symmetric eigenfunction expression for region 1,

\[
U^s(y) = \sum_{n=-\infty}^{\infty} \alpha_n R^s_n e^{i(\alpha_n d + \beta_n y)} \quad (|y| \leq b).
\]

However with reference to the matching condition (3.14), \( U^s(y) \) may be written equivalently on \([-l, l]\) in terms of equation (3.34), the symmetric eigenfunction expansion for region 2, as

\[
U^s(y) = \sum_{n=0}^{\infty} r_n M_n \sin r_n d \cdot \mu_n(y) + P_n \sin s_n d \cdot \nu_n(y). \quad (|y| < l).
\]

Next employing the orthogonality relation (3.27) and applying the no flow condition (3.5) on the lateral sides of the blocks to (3.39), the expression for the coefficients \( R^s_n \),

\[
R^s_n = \frac{1}{2} \left\{ \delta_{n0} e^{-i\alpha_n d} - \frac{1}{i\alpha_n b} \int_{-l}^{l} U^s(y) e^{-i\beta_n y} \, dy \right\} e^{-i\alpha_n d}, \quad (n \in \mathbb{Z})
\]
is obtained. And similarly employing the orthonormality conditions (3.36)-(3.38) to (3.40) the expressions for the coefficients $M_n$ and $P_n$, 

$$M_n = r_n^{-1} \csc r_n d \int_{-l}^{l} U^s(y) \mu_n(y) \, dy, \quad (n \in \mathbb{N}) \quad (3.42)$$

$$P_n = s_n^{-1} \csc s_n d \int_{-l}^{l} U^s(y) \nu_n(y) \, dy, \quad (n \in \mathbb{N}) \quad (3.43)$$

are obtained. Finally applying the matching condition (3.12) to the symmetric potentials (3.22) and (3.34) and substituting the integral expressions (3.41)-(3.43) for the respective coefficients, the integral equation, 

$$\int_{-l}^{l} K^s(y, y') U^s(y') \, dy' = e^{-i(\alpha_0 d - \beta_0 v)}, \quad (3.44)$$

is obtained, where

$$K^s(y, y') = \sum_{n=0}^{\infty} \left( r_n^{-1} \cot r_n d \mu_n(y) \mu_n(y') + s_n^{-1} \cot s_n d \nu_n(y) \nu_n(y') \right) - \frac{i}{2b} \sum_{n=-\infty}^{\infty} \alpha_n^{-1} e^{i\beta_n(v-v')} \quad (3.45)$$

Thus on the basis of the solution of the integral equation (3.44) for $U^s(y)$, all the unknown coefficients of the symmetric eigenfunction solutions (3.22) and (3.34) can be determined via equations (3.41)-(3.43), completely solving the symmetric problem.

Similarly applying the same procedure to the anti-symmetric problem, the anti-symmetric coefficient expressions,

$$R_n = \frac{1}{2} \left\{ \delta_{00} e^{-i\alpha_0 d} - \frac{1}{i\alpha_0 b} \int_{-l}^{l} U^a(y) e^{-i\beta_n v} \, dy \right\} e^{-i\alpha_0 d}, \quad (n \in \mathbb{Z}) \quad (3.46)$$

$$N_n = r_n^{-1} \sec r_n d \int_{-l}^{l} U^a(y) \mu_n(y) \, dy, \quad (n \in \mathbb{N}) \quad (3.47)$$

$$Q_n = s_n^{-1} \sec s_n d \int_{-l}^{l} U^a(y) \nu_n(y) \, dy. \quad (n \in \mathbb{N}). \quad (3.48)$$

and the integral equation, 

$$\int_{-l}^{l} K^a(y, y') U^a(y') \, dy' = -e^{-i(\alpha_0 d - \beta_0 v)}, \quad (3.49)$$

are obtained, where

$$K^a(y, y') = \sum_{n=0}^{\infty} \left( r_n^{-1} \tan r_n d \mu_n(y) \mu_n(y') + s_n^{-1} \tan s_n d \nu_n(y) \nu_n(y') \right) + \frac{i}{2b} \sum_{n=-\infty}^{\infty} \alpha_n^{-1} e^{i\beta_n(v-v')} \quad (3.50)$$

And on the basis of the solution of the integral equation (3.49) for $U^a(y)$, all the unknown coefficients of the anti-symmetric eigenfunction solutions (3.23) and (3.35) can be determined via equations (3.46)-(3.48), completely solving the anti-symmetric problem.
3.3.3 Solution of Integral Equations

In a program written by the author the coefficients of the symmetric and anti-symmetric eigenfunction expansions are determined for various values of the non-dimensionalised parameters $k_1, b/l, d/l, \theta_0$ via numerical solution of the integral equations (3.44) and (3.49).

And there are three issues to be dealt with in solution of each of these integral equations. Firstly, there are the difficulties involved in evaluating the kernels $K^s(y, y')$ and $K^a(y, y')$, each of which involves an infinite oscillatory series. Secondly, both equations are first kind Fredholm integral equations, a class for which ill-conditioning problems in numerical solution are well documented (see Jones [27, pages 258-259], Delves & Mohamed [16, Chapter 12] and Porter & Chamberlain [49, page 28]). And thirdly, as noted in the chapter introduction, it is anticipated that the velocity field has singularities at the block corners. In particular, it is demonstrated in Appendix B.1 at the corners

$$|\nabla \phi(\rho, \varphi)| \cong A \rho^{-1/3},$$  \hspace{1cm} (3.51)

where $(\rho, \varphi)$ represent polar coordinates attached to the block corner. And the presence of these singularities can disrupt the convergence of all numerical solution strategies.

In the program described above, numerical solution of the integral equations has been implemented using an elementary collocation method. To illustrate this approach, successfully employed by Macaskill [35] in numerical solution of first and second kind Fredholm integral equations, consider the archetypal first kind Fredholm integral equation,

$$\int_a^b K(y, y') U(y') \, dy' = V(y), \quad (b > a)$$  \hspace{1cm} (3.52)

Then dividing the interval $a \leq y \leq b$ into $N$ segments $(y_j, y_{j+1})$ with $y_j < y < y_{j+1}$, $j = 1, \ldots, N$, taking $y_1 = a$ and $y_{N+1} = b$, and approximating $U(y) = U_j = \text{constant}$ on each segment $(y_j, y_{j+1})$, equation (3.52) can be approximated by

$$\sum_{j=1}^N U_j \int_{y_j}^{y_{j+1}} K(y, y') \, dy' = V(y).$$  \hspace{1cm} (3.53)

Further considering this approximate equation at $y = \bar{y}_i$ ($i = 1, \ldots, N$), the midpoints of each segment, the linear system

$$AU = V,$$  \hspace{1cm} (3.54)

is obtained where the matrix $A$ is defined by

$$A = \left[ \int_{y_j}^{y_{j+1}} K(\bar{y}_i, y') \, dy' \right]_{ij},$$  \hspace{1cm} (3.55)
and the vectors $U$ and $V$ by

\[
U = [U_1, U_2, \ldots, U_N]^T, \quad (3.56)
\]
\[
V = [V(\bar{y}_1), \ldots, V(\bar{y}_N)]^T. \quad (3.57)
\]

solution of this linear system then results in the solution on $[a, b],

\[
U(y) = \sum_{i=1}^{N} U_i N_i(y) \quad (3.58)
\]

where

\[
N_i(y) = \begin{cases} 
0 & \text{if } y \notin (y_i, y_{i+1}) \\
1 & \text{if } y \in (y_i, y_{i+1})
\end{cases} \quad (3.59)
\]

**Kernel Evaluation**

The kernels $K^a(y, y')$ and $K^s(y, y')$ are computed by truncation of the series and by employment of the asymptotics of the discarded part. To illustrate this, note it is clear that $K^a(y, y')$ and $K^s(y, y')$ can be re-written as

\[
K^a(y, y') = \sum_{n=0}^{\infty} K^a_n(y, y'), \quad (3.60)
\]
\[
K^s(y, y') = \sum_{n=0}^{\infty} K^s_n(y, y'), \quad (3.61)
\]

where

\[
K^a_n(y, y') = r_n^{-1} \tan r_n d\mu_n(y') \mu_n(y) + s_n^{-1} \tan s_n d\nu_n(y') \nu_n(y) + \frac{i}{2b} ((1 - \delta_{0n}) \alpha_n^{-1} e^{i\beta_n (y - y')} + \alpha_n^{-1} e^{i\beta_n (y - y')}), \quad (3.62)
\]
\[
K^s_n(y, y') = r_n^{-1} \cot r_n d\mu_n(y') \mu_n(y) + s_n^{-1} \cot s_n d\nu_n(y') \nu_n(y) - \frac{i}{2b} ((1 - \delta_{0n}) \alpha_n^{-1} e^{i\beta_n (y - y')} + \alpha_n^{-1} e^{i\beta_n (y - y')}), \quad (3.63)
\]

And note for large $n$ that

\[
K^a_n(y, y') \sim \tilde{K}_n(y, y') = \frac{1}{n\pi} \{ \cos \left( \frac{n\pi}{b} (y - y') \right) + \cos \left( \frac{n\pi}{b} (y - y') \right) \}, \quad (3.64)
\]
\[
K^s_n(y, y') \sim \tilde{K}_n(y, y'). \quad (3.65)
\]

and further according to Abramowitz and Stegun [1, Equation 27.8.6 (i)] that

\[
\sum_{n=1}^{\infty} \tilde{K}_n(y, y') = -\frac{1}{\pi} \ln \left( 4 \sin \left( \frac{\pi |y - y'|}{2l} \right) \sin \left( \frac{\pi |y - y'|}{2b} \right) \right). \quad (3.66)
\]
And consequently for sufficiently large $N$, $K^s(y, y')$ and $K^a(y, y')$ can be approximated by

$$K^a(y, y') \approx K_0^a(y, y') + \sum_{n=1}^{N} (K_n^a(y, y') - \tilde{K}_n(y, y'))$$

$$-\frac{1}{\pi} \ln(4 \sin(\frac{\pi |y - y'|}{2l}) \sin(\frac{\pi |y - y'|}{2b})), \quad (3.67)$$

$$K^s(y, y') \approx K_0^s(y, y') + \sum_{n=1}^{N} (K_n^s(y, y') + \tilde{K}_n(y, y'))$$

$$+\frac{1}{\pi} \ln(4 \sin(\frac{\pi |y - y'|}{2l}) \sin(\frac{\pi |y - y'|}{2b})). \quad (3.68)$$

Note additionally that the expressions (3.67) and (3.68) indicate (if not prove) that the symmetric and anti-symmetric kernels are singular at $y = y'$. And Jones [27, Page 259] notes that “from the standpoint of numerical work, the more singular a kernel the better for an integral equation of the first kind”.

**Account for velocity singularities in collocation method**

In implementation of the collocation method (described above) to the solution of equations (3.44) and (3.49) in the program (also described above), account for the anticipated singular behaviour of the velocity field is made by choice of the boundary elements. Informed by Macaskill [35], who has employed a similar scheme to account for endpoint square root velocity singularities and the technique described by Davis & Rabinowitz [14, page 143] for evaluation of certain singular integrals using advantageous properties of Gaussian quadrature, the following assignment of boundary elements is made. For odd $N$ (the number of elements), the midpoint of each element is assigned by

$$\bar{y}_n = l x_n \quad (n = 1, \ldots, N) \quad (3.69)$$

where $\{x_n\}$ denote the zeros of the $N^{th}$ order Jacobi polynomial $P_N^{-1/3,-1/3}(x)$ (see Abramowitz & Stegun [1, Expression 22.2.1]) arranged in ascending order, and the element end-points $y_j$ are assigned by $y_1 = -l$ and the recurrence relation,

$$y_{n+1} = 2\bar{y}_n - y_n \quad (n = 1, \ldots, N). \quad (3.70)$$

Note that the present scheme is inappropriate for even $N$ where the recurrence relation overshoots the interval $[-l, l]$. Also compare the present scheme with that of Macaskill [35] who has employed elements centered on Chebychev points (zeros of first kind Chebychev polynomials to be precise. And note the present scheme is inappropriate for even $N$ when the recurrence relation overshoots the interval $[-l, l]$. 

43
There are two good justifications for allocating the boundary element using such an approach. Firstly equations (3.69) and (3.70) (and equivalently the assignment of Macaskill [35]) assign shorter elements near the interval endpoints where the singular behaviour, described in the present problem (3.51), is most pronounced and progressively larger elements towards the centre of the gap. Secondly it is noted to deal with singular integrals of the form
\[ \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta f(x) \, dx \quad (-1 < \alpha < 0 - 1 < \beta < 0) \]  
where \( f(x) \) is a suitably differentiable function (see Davis & Rabinowitz [14, page 75]), Davis & Rabinowitz [14, page 143] advocate the use of Gaussian Quadrature Rule,
\[ \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta f(x) \, dx \approx \sum_{m=1}^{M} w_m^{(\alpha,\beta)} f(x_m^{(\alpha,\beta)}). \] 
where the abscissae \( \{x_m^{(\alpha,\beta)}\} \) are the zeros of the Jacobi polynomials \( P_M^{(\alpha,\beta)}(x) \) and the weights \( w_m^{(\alpha,\beta)} \) also depend on the form of these polynomials. Note that Davis & Rabinowitz [14, page 75]) demonstrate this quadrature method is uniformly convergent and as described there, in some sense optimal. Clearly according to (3.51) the integral equations (3.44) and (3.49) are of the form (3.72) with \( \alpha = -1/3 \) and \( \beta = -1/3 \). And it is thus speculated that use of the boundary elements centered on the abscissae \( x_m^{(-1/3,-1/3)} \), gives the collocation method some of the advantages of the quadrature rule (3.72).

3.4 Results

Fernyhough & Evans [20] have plotted various graphs of reflection and transmission coefficients against angle of incidence and wavenumber. And some of these graphs have been re-plotted here using the present solution to investigate its performance and also illustrate some of the features of the performance of arrays of impermeable blocks.

To verify the computation of the reflection and transmission coefficients in the program described above, and calibrate the number of boundary element used, two checks have been performed. Firstly note according to Linton & Evans [31, Equation (2.32)] the travelling reflection and transmission coefficients satisfy the energy relation,
\[ \sum_{m=-r}^{s} \alpha_m (|R_m|^2 + |T_m|^2) = \alpha_0 \] 
where recall \( r \) and \( s \) are defined by (3.20) and (3.21) respectively. And this relation has been checked in all runs of the program. Secondly, in trial runs for given parameters, the program
has been run for $2M + 1$ and $4M + 3$ boundary element respectively and truncation of the series for $K^*(y,y')$ and $K^a(y,y')$ for $N$ and $2N$ elements in equations (3.67) and (3.68). And on the basis of this experiments, it has been found that about 21 boundary elements and truncation of the series with $N = 900$ elements gives about 3 decimal places of accuracy in the reflection and transmission coefficients. (Not for contrast Fernyhough & Evans use a minimum of 5 basis functions in their application of Galerkin’s method).

Figure 3.3: (a) $|R|$ and (b) $|T|$ against $\theta_0$ for $kb = 0.5$, $d/b = 0.8$: (i) $l/b = 0.1$ (———), (ii) $l/b = 0.4$ (—— ———), (iii) $l/b = 0.6$ (—— ———), (iv) $l/b = 0.8$ (—— ———)

In figure 3.3, the moduli of the reflection and transmission coefficients $R_0$ and $T_0$ (with $R = R_0$ and $T = T_0$ respectively) are plotted against the angle of incidence $\theta_0$ for $kb = 0.5$ and various values of $l/b$. (Note for the range of parameters considered there is only one travelling) mode according to (3.20) and (3.21) Further note from (a) that for each value of $l/b$, $|R|$ strictly decreases as $\theta_0$ increases until an angle with zero reflection is reached. And beyond this angle, $|R|$ increases rapidly, tending to 1 as $\theta_0$ tends to 90 degrees. And note from (b) that $|T|$ increases as $\theta_0$ increases until reaching 1 (complete transmission) at the angle of zero reflection before decreasing rapidly, tending to 0 as $\theta_0$ reaches 90 degrees.

In respect of gap size figure 3.3 demonstrates that for larger gap lengths the plots of $|R|$ and $|T|$ flatten towards 0 and 1 respectively and the angle of zero reflection is lowered. In particular considering arrays with two gap sizes: $|R|$ is lower and $|T|$ higher for the larger gap size provided the angle of incidence is not in the vicinity of the angle of zero reflection for that gap size. Note in the vicinity of the angle of zero reflection of the array with larger gap size, this trend reverses: $|R|$ is higher and $|T|$ lower for the array with the larger gap size.

Finally note figure 3.3(a) was also plotted by Fernyhough & Evans [20, Figure 5] and the
two graphs compare favourably.

In Figure 3.4 the moduli of the reflection and transmission coefficients $R_n$ and $T_n$ of the travelling modes are plotted against $k_1$ for $d/l = 0.5$, $b/l = 2.55$ and with an angle of incidence, $\theta_0$, of 30 degrees. Note this graph demonstrates that as $kd$ increases additional travelling modes appear in the solution. In particular note from the graph the modes $R_1$ and $T_1$ appear at $k_1 = 0.7$, the modes $R_2$ and $T_2$ appear at $k_1 = 1.6$, and finally the modes $R_{-1}$ and $T_{-1}$ and $R_3$ and $T_3$ appear about $k_1 = 2.4$. And note these cut-off frequencies are consistent with with equations (3.20) and (3.21).

This graph illustrates when there is more than one travelling mode in the solution, the performance of the array is difficult to interpret. And for this reason Fernyhough & Evans advocate the use of the total reflected wave energy by, and the total transmitted energy through, the structure as a proportion of the incident wave energy,

$$R_{total} = \sum_{n=-r}^{s} \frac{\alpha_n}{\alpha_0} |R_n|^2,$$

$$T_{total} = \sum_{n=-r}^{s} \frac{\alpha_n}{\alpha_0} |T_n|^2.$$ 

in such situations.
In Figure 3.5 \( R_{\text{total}} \) and \( T_{\text{total}} \) are plotted against \( kl \) for \( d/l = 0.5, b/l = 2.55 \) and three angles of incidence, \( \theta_0 \). Note figure 3.5(b) was also plotted by Fernyhough & Evans [20, Figure 5] and the two compare favourably (taking account of the two non-dimensionalisations used). The general trend (ignoring the spikes at the cut-off frequencies which again appear according to (3.20) and (3.21)) seems to be that \( T_{\text{total}} \) drops and \( R_{\text{total}} \) increases with increasing \( kl \). Note this is what might expect: the shorter the wavelength relative to the width of the structure the greater the diffraction. However as pointed out by Fernyhough & Evans, in certain ranges of parameters interference effects conspire to act against the general trend. For example note for \( \theta_0 = 60^\circ \) degrees in the range \( 1.5 \leq kl \leq 2.5 \) \( R_{\text{total}} \) is decreasing and \( T_{\text{total}} \) increases.

These interference effects are further illustrated by figure 3.6 (again plotted by Fernyhough & Evans [20, Figure 7]). In particular Fernyhough & Evans [20] note that after the cut-off at
\[ \theta_0 = \sin^{-1}(2\pi/5 - 1) \] the larger \( l/b \), the larger \( T_{\text{total}} \) as expected, but as \( \theta_0 \) increases beyond \( \theta_0 \approx 77.3 \) this behaviour is reversed.

### 3.5 Conclusion

In this chapter, diffraction of water waves by a periodic array of impermeable blocks has been considered by employment of linear diffraction theory. On the basis of the diffraction theory an appropriate boundary value problem has been formulated and solved using a matched eigenfunction approach. Specifically, employing symmetric and anti-symmetric eigenfunction expansions for the gap and outer regions, it was shown that solution of the formulated boundary value problem can be reduced to solution of two integral equations. And these integral equations are solved using a boundary element method with boundary elements designed to account for expected singular behaviour at the block corners.

Taking account of the original motivation for tackling this problem (to inform solution for the permeable case) and of Fernyhough & Evan's published solution, the present study has concentrated more on comparison of their solution with the present one rather than on the features of interaction. In particular it has been noted that Fernyhough & Evan's solution methodology has a stronger theoretical justification (particularly in respect of proof of convergence of their method and more accurate treatment of the velocity singularities at the block corners). (Recall Fernyhough & Evans reformulate their integral equations in terms of positive definite integral operators and employ Galerkin's method with a special basis which model the velocity singularities at the block corners. And it has been confirmed, via replotting the graphs in Fernyhough & Evans [20] with the current solution, that their solution methodology is more efficient. However it is noted that the present solution methodology also performs well, in spite of Fernyhough and Evans concerns (see Fernyhough & Evans [20, Page 264]) about the convergence/efficiency of such approaches.

And this result gives some motivation as to a suitable approach to tackle the the boundary integral equations, which arise in an analogous way in solution for the permeable array problem considered in Chapter 5. In particular note that the boundary of these integral equations consists of a closed rectangular curve around one of the array blocks, and that the collocation method described above together with the choice of elements used, can be trivially generalized for solution of these equations. And the above result gives encouragement (if not proof) as to the convergence of such an approach. Application of Fernyhough & Evans
method on the otherhand is much less straightforward. Note in particular that the Galerkin method, which is central to their approach is only applicable when the boundary of integral equation for which the solution is sought, can be parameterized explicitly by a single variable. However the determination of an explicit parameterization of the permeable array integral equations boundary, which involve corners, together with a suitable basis which models the velocity at these corners is not a particularly trivial tasks. And consequently, even if it offers greater prospect of demonstrating the convergence, no attempt has been made to generalize Fernyhough & Evans method to tackle the integral equations in the permeable case.
Chapter 4

Reflection and Transmission from a Rectangular Single Continuous Section Permeable Breakwater

4.1 Introduction

In Chapter 3 diffraction of water waves by an impermeable array was considered by a matched eigenfunction and integral equation approach. However application of such an approach in the permeable case is not straightforward on account of the matching conditions that need to be applied. To illustrate this, recall that determination of the eigenfunction series for the gap region in the impermeable case relied on the application of a zero normal velocity condition on the block walls. However the matching condition that apply on these walls in the permeable case do not facilitate finding a similar eigenfunction expression for the gap region (or within the blocks themselves) so straightforwardly.

However note Ijima et al. [26] and Sulisz [59, 60] employ a different approach, for solution of boundary value problems formulated for wave permeable array interactions, which avoids such detailed determination of eigenfunction expressions. In particular note that Ijima et al. [26] has, using two dimensional Green’s functions coupled with eigenfunction series and a boundary integral equation approach, considered the full three dimensional problem of wave diffraction by vertically sided porous structures of arbitrary shape. And similarly note that Sulisz has employed two dimensional Green’s functions and a boundary integral equation approach to model reflection and transmission by infinitely long surface-piercing permeable
structures of any cross-section in regions of varying depth.

Thus to learn this approach and inform solution of the permeable array problem, a Green’s function and integral approach has been employed to re-solve the boundary value problem formulated and solved by Dalrymple et al. [13] for wave reflection and transmission by a permeable block. And this work is presented here. Recall formulation of this problem is also presented in Chapter 3 with the thesis $e^{-i\omega t}$ solution convention. And in addition recall that Dalrymple et al. employ a matched eigenfunction approach to solution of this problem.

Although the purpose of this work was largely scholarly it has been noted that the approach has certain advantages over the eigenfunction solution approach. In particular, note the non-self-adjoint nature of the problem (discussed in Chapter 2) in the porous region is completely transparent and the consequent double root phenomenon (also discussed in chapter 2) requires no special consideration or treatment. Secondly in this problem, no computation of the roots of the porous dispersion relation is necessary utilizing this method. In fact, if only the reflection and transmission coefficients of non evanescent modes are required, it’s only necessary to determine the incident wave number. By contrast for the eigenfunction solution, Dalrymple et al. (who employed numerical methods) and recently McIver [39] (who has developed an analytical theory) devote considerable effort to the location of these roots.

Solution proceeds as follows. Firstly, appropriate Green’s functions are chosen for the problem (those used by Dalrymple et al. are demonstrated to be appropriate). Then employing Green’s theorem and the matching conditions prescribed by Sollitt & Cross a system of simultaneous integral equations is obtained. And finally by numerical solution of this system using an adapted Nystrom method a complete solution of the problem is obtained.

### 4.2 Boundary Value Problem

Recall formulation of boundary value problem for the work of Dalrymple et al. was presented in Chapter 2. In particular, recall that for each of the three regions illustrated in Figure 4.1, the velocity potential $\Phi_i$ has the form,

$$\Phi_i(x, y, z, t) = \Re[e^{i(\lambda y - \omega t)}\phi_i(x, z)] \quad (i = 1, 2, 3), \quad (4.1)$$

and $\phi_i(x, z) (i=1,2,3)$ must satisfy

$$(\nabla^2 - \lambda^2)\phi_i = 0, \quad (i = 1, 2, 3), \quad (4.2)$$
Figure 4.1: porous structure from above

and the following boundary conditions:

the bed condition,

\[ \frac{\partial \phi_i}{\partial z} = 0, \text{ when } z = -h \quad (i = 1, 2, 3), \quad (4.3) \]

the free surface conditions,

\[ \frac{\partial \phi_i}{\partial z} - \frac{\Gamma_i}{h} \phi_i = 0, \quad \text{at } z = 0 \quad (i = 1, 2, 3), \quad (4.4) \]

where

\[ \Gamma_1 = \Gamma_3 = \frac{\omega^2 h}{g}, \quad (4.5) \]
\[ \Gamma_2 = (s + if) \Gamma_1 = \frac{\omega^2 h(s + if)}{g}, \quad (4.6) \]

and the matching conditions,

\[ \phi_1(0, z) = (s + if) \phi_2(0, z), \quad (4.7) \]
\[ \frac{\partial \phi_1}{\partial x}(0, z) = \epsilon \frac{\partial \phi_2}{\partial x}(0, z), \quad (4.8) \]
\[ \phi_3(b, z) = (s + if) \phi_2(b, z), \quad (4.9) \]
\[ \frac{\partial \phi_3}{\partial x}(b, z) = \epsilon \frac{\partial \phi_2}{\partial x}(b, z), \quad (4.10) \]

where recall from Chapter 2, \( \epsilon \) is the porosity of the block, and \( s \) and \( f \) are Sollitt & Cross's inertia coefficient and friction factor respectively.

In addition to the boundary conditions stated above, the radiation condition, that all waves modes in regions 1 and 3 other than the incident one propagate away from the structure, is imposed.
4.3 Full Solution

4.3.1 Green's Functions

To employ the integral equation approach in the solution of the above boundary value problem it's necessary to determine suitable two dimensional Green's functions for each region. Adapting the argument of Mei[42, page 308] for the current problem, appropriate functions satisfy the following boundary value problem,

\[ \nabla^2 G_i - \lambda^2 G_i = -\delta(x-\xi)\delta(z-\eta), \quad (i = 1, 2) \tag{4.11} \]

\[ \frac{\partial G_i}{\partial z} = 0 \quad \text{at } z = -h, \tag{4.12} \]

\[ \frac{\partial G_i}{\partial z} - \frac{\Gamma_i}{h} G_i = 0, \quad \text{at } z = 0 \quad (i = 1, 2), \tag{4.13} \]

where \((\xi, \eta)\) denotes a source point and \(G_1(x, z, \xi, \eta)\) and \(G_2(x, z, \xi, \eta)\) are functions for regions 1 and 2 respectively. In addition it is assumed that \(G_i \quad (i = 1, 2, 3)\) satisfy the radiation condition. Note also a region 1 Green's function is also appropriate for region 3.

Using Fourier cosine and inverse Fourier cosine transforms (as Mei [42, page 370] does for his problem with Fourier and inverse Fourier transforms) it can be demonstrated that (see Appendix C.1),

\[ G_1(x, z, \xi, \eta) = -\frac{1}{\pi} \int_0^\infty \frac{\cosh \gamma(z_\leq + h)\{\Gamma_1 \sinh \gamma z_\geq + \gamma h \cosh \gamma z_\geq\}}{\gamma\{\Gamma_1 \cosh \gamma h - \gamma h \sinh \gamma h\}} \cos \mu(x - \xi) \, d\mu, \tag{4.14} \]

where the smile (\(\smile\)) denotes that the path of integration passes underneath a pole located at \(\mu = (k^2 - \lambda^2)^{1/2}\), is a solution of the above problem for region 1. Similarly, as described by Dalrymple et al. [13, pages 642-643],

\[ G_2(x, z, \xi, \eta) = -\frac{1}{\pi} \int_0^\infty \frac{\cosh \gamma(z_\leq + h)\{\Gamma_2 \sinh \gamma z_\geq + \gamma h \cosh \gamma z_\geq\}}{\gamma\{\Gamma_2 \cosh \gamma h - \gamma h \sinh \gamma h\}} \cos \mu(x - \xi) \, d\mu, \tag{4.15} \]

is a solution for region 2. In both cases \(\gamma\) is defined by

\[ \gamma = \sqrt{\mu^2 + \lambda^2}, \tag{4.16} \]

and \(z_\leq = \min\{z, \eta\}\) and \(z_\geq = \max\{z, \eta\}\).

It's constructive here to present a few properties and alternative forms of the Green's functions (4.14) and (4.15), for use in the following sections. Firstly note if the field point \((x, z)\) is close to the source point, then,

\[ G_i(x, z, \xi, \eta) \sim \frac{1}{2\pi} \ln r, \quad (i = 1, 2), \tag{4.17} \]
where,
\[
r = \sqrt{(x - \xi)^2 + (z - \eta)^2}.
\]

And this property of the Green's functions is employed in the derivation of the boundary integral equations. Secondly note as demonstrated in Appendix C.2, the Green's functions (4.14) and (4.15) can be rewritten in the forms,
\[
G_1(x, z, \xi, \eta) = \frac{1}{2\pi} \left\{ K_0(\lambda r) + K_0(\lambda r') \right\}
- \frac{1}{\pi} \int_0^\infty \frac{\cosh \gamma (\eta + h) \cosh \gamma (z + h) e^{-\gamma h} (\Gamma_1 + \gamma h) \cos \mu (x - \zeta)}{\gamma \{ \Gamma_1 \cosh \gamma h - \gamma h \sinh \gamma h \}} d\mu, \quad (4.18)
\]
\[
G_2(x, z, \xi, \eta) = \frac{1}{2\pi} \left\{ K_0(\lambda r) + K_0(\lambda r') \right\}
- \frac{1}{\pi} \int_0^\infty \frac{\cosh \gamma (\eta + h) \cosh \gamma (z + h) e^{-\gamma h} (\Gamma_2 + \gamma h) \cos \mu (x - \zeta)}{\gamma \{ \Gamma_2 \cosh \gamma h - \gamma h \sinh \gamma h \}} d\mu, \quad (4.19)
\]
where \(K_0(z)\) represents the modified Bessel's function of the second kind of order zero and \(r' = ((x - \xi)^2 + (z + \eta + 2h)^2)^{1/2}\).

These forms have the advantage that the singular behaviour of both Green's functions near their sources is contained explicitly in the term \(\frac{1}{2\pi} \{ K_0(\lambda r) + K_0(\lambda r') \}\) and are also useful in solution of the boundary integral equations. Finally, series forms of (4.14) and (4.15) can be found by employing some complex analysis (demonstrated by Mei [42] for another Green's function). Note in the appendix of Dalrymple et al. [13] they derive a series form for \(G_2(x, z, \xi, \eta)\) in connection with determining the eigenfunction terms for a double eigenvalue. However, of more significance in the current problem, being employed in the next section, is the series form for \(G_1(x, z, \xi, \eta)\). It is demonstrated in Appendix C.3 that,
\[
G_1(x, z, \xi, \eta) = i \sum_{n=0}^\infty \frac{\chi_n(\eta) \chi_n(z) e^{-i(k_n^2 - \lambda^2)^{1/2} |z - \zeta|}}{2(k_n^2 - \lambda^2)^{1/2}}, \quad (4.21)
\]
where \(k_0 = k\) and \(k_n\) \((n = 1, 2, \ldots)\) are the positive real root and the roots on the positive part of the imaginary axis respectively of the linear dispersion relation,
\[
\Gamma_1 = kh \tanh kh, \quad (4.22)
\]
and \(\chi_n(z)\) is defined by
\[
\chi_n(z) = \frac{1}{N_n} \cosh k_n(z + h), \quad (4.23)
\]
where
\[
N_n = \frac{\sqrt{\sinh 2k_n h + 2kh}}{2\sqrt{k_n}}. \quad (4.24)
\]
Note that the set \(\{\chi_n(z)\}\) satisfy the orthonormality relation
\[
\int_{-h}^0 \chi_m(z) \overline{\chi_n(z)} \, dz = \delta_{mn}. \quad (4.25)
\]
4.3.2 Integral Equation Formulation

Figure 4.2: Contours used in application of Green’s function

In order to obtain a system of integral equations appropriate for the current solution of the boundary value problem, consider the closed contours $S_i$ ($i = 1, 2, 3$) illustrated in Figure 4.2. Note that $S_i$ ($i = 1, 2, 3$) are defined in terms of their constituent parts by

$$S_1 = S_{B_1} \cup S_{F_1} \cup S_{F_2} \cup S_{-\infty}, \quad (4.26)$$
$$S_2 = S_{B_2} \cup S_{P_1} \cup S_{F_2} \cup S_P, \quad (4.27)$$
$$S_3 = S_{B_3} \cup S_{P_2} \cup S_{F_3} \cup S_{\infty}, \quad (4.28)$$

where $S_{-\infty}$ and $S_{\infty}$ are defined by

$$S_{\pm\infty} = \lim_{x \to \pm\infty} \{(x, z) : -h < z < 0\}. \quad (4.29)$$

Suppose $(\xi, \eta)$ lies in the closed domain $D_i$ (the region enclosed by the contour $S_i$) ($i = 1, 2$ or 3) and let $D_i^0(\xi, \eta)$ denote a disc (or segment, if $(\xi, \eta)$ is located on boundary $S_i$) with centre $(\xi, \eta)$, radius $\epsilon$ ($\epsilon > 0$) and boundary $B_i^0(\xi, \eta)$ which is contained in $D_i$. Then note that for any $(\xi, \eta) \in D_i$ ($i = 1, 2, 3$), the solution $\phi_i(x, z)$ and more significantly the Green’s function $G_i(x, z, \xi, \eta)$ satisfies the Helmholtz equation in $D_i - D_i^0(\xi, \eta)$ and applying Green’s theorem to $\phi_i(x, z)$ and $G_i(x, z, \xi, \eta)$ on $D_i - D_i^0(\xi, \eta)$ the equation

$$\int_{S_i + B_i^0(\xi, \eta)} \left( \phi_i \frac{\partial G_i}{\partial n} - G_i \frac{\partial \phi_i}{\partial n} \right) ds = 0, \quad (4.30)$$

is obtained, where the normal $n$ is supposed to point of out of region enclosed by $S_i + B_i^0(\xi, \eta)$. Further, employing the logarithmic singularity of the Green’s function in the region of their source given by (4.17), note that

$$\lim_{\epsilon \to 0} \int_{B_i^0(\xi, \eta)} \left( \phi_i \frac{\partial G_i}{\partial n} - G_i \frac{\partial \phi_i}{\partial n} \right) ds = C(\xi, \eta) \phi_i(\xi, \eta), \quad (i = 1, 2, 3), \quad (4.31)$$

55
where

\[
C(\xi, \eta) = \begin{cases} 
1 & \text{if } (\xi, \eta) \notin S_i \\
\frac{1}{2} & \text{if } (\xi, \eta) \in S_i \text{ (not at a corner)} \\
\frac{1}{4} & \text{if } (\xi, \eta) \in S_i \text{ (at a corner)}
\end{cases}
\]

and thus

\[
C(\xi, \eta)\phi_i(\xi, \eta) + \int_{S_i} \left( \phi_i \frac{\partial G_i}{\partial n} - G_i \frac{\partial \phi_i}{\partial n} \right) dS_i = 0, \quad (i = 1, 2, 3). \tag{4.32}
\]

Next considering the application of the boundary conditions for \((\xi, \eta)\) in each element \(D_i\), first note that \(\phi_i(x, z)\) and \(G_i(x, z, \xi, \eta)\) satisfy the same boundary conditions at the seabed (equation (4.3)) and at the mean sea-level (equation (4.4)) in each region and thus,

\[
\int_{S_{p_i}} \left( \phi_i \frac{\partial G_i}{\partial n} - G_i \frac{\partial \phi_i}{\partial n} \right) dS_i = \int_{S_{p_i}} \left( \phi_i \frac{\partial G_i}{\partial n} - G_i \frac{\partial \phi_i}{\partial n} \right) dS_i = 0, \quad (i = 1, 2, 3) \tag{4.33}
\]

Further for \((\xi, \eta) \in D_i \ (i = 1, 3)\) note by considering (4.21) with \(k_n = \mathcal{K}_n\) (clearly \(\mathcal{K}_n > 0, \in \mathbb{R}\)), its clear that in limit as \(|z| \to \infty\) that,

\[
G_1(x, z, \xi, \eta) \sim \frac{i}{2(k^2 - \lambda^2)^{1/2} \Xi(x)} \Xi(z) \Xi(0) \Xi(\eta) \tag{4.34}
\]

Similarly according to the radiation condition, the solutions in regions 1 and 3 satisfy

\[
\phi_1(x, z) \sim (e^{i(k^2-\lambda^2)^{1/2}z} + Re^{-i(k^2-\lambda^2)^{1/2}z}) \chi_0(z), \quad \text{as } z \to -\infty, \tag{4.35}
\]

\[
\phi_3(x, z) \sim Te^{i(k^2-\lambda^2)^{1/2}(z-b)} \chi_0(z), \quad \text{as } z \to \infty, \tag{4.36}
\]

where \(R\) and \(T\) denote the principal reflection and transmission coefficients respectively and the expressions on the right hand sides of (4.35) and (4.36) are obtained from the eigenfunction solutions (adapted for consistency with the frequency term change) found by Dalrymple et al. [13] for regions 1 and 3. Consequently employing the orthogonality relation (4.25),

\[
\int_{S_{-\infty}} \left( \phi_1 \frac{\partial G_1}{\partial n} - G_1 \frac{\partial \phi_1}{\partial n} \right) dS_1 = -\frac{1}{h} e^{i\xi} \chi_0(\eta) \tag{4.37}
\]

\[
\int_{S_{\infty}} \left( \phi_3 \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \phi_3}{\partial n} \right) dS_3 = 0. \tag{4.38}
\]

Finally employing the matching conditions (4.7) and (4.8) on \(P_1\) and (4.9) and (4.10) on \(P_2\) obtain for \((\xi_1, \eta_1) \in D_i \ (i = 1, 2 \text{ and } 3),

\[
C(\xi_1, \eta_1)\phi_1(\xi_1, \eta_1) = \frac{1}{h} e^{i\xi_1} \chi_0(\eta_1) - \int_{-\infty}^{0} \left( p_1(z) \frac{\partial G_1}{\partial z}(0, z, \xi_1, \eta_1) - G_1(0, z, \xi_1, \eta_1) u_1(z) \right) dz, \tag{4.39}
\]

\[
C(\xi_2, \eta_2)\phi_2(\xi_2, \eta_2) = \int_{-\infty}^{0} \left( p_1(z) \frac{\partial G_2}{\partial z}(0, z, \xi_2, \eta_2) - G_2(0, z, \xi_2, \eta_2) u_1(z) \right) dz, \tag{4.40}
\]
\[- \int_{-h}^{0} \left( \frac{p_2(z)}{\delta} \frac{\partial G_2(b, z, \xi, \eta)}{\partial x} - G_2(b, z, \xi, \eta) \frac{u_2(z)}{\epsilon} \right) \, dz, \quad (4.40)\]

\[C(\xi, \eta) \partial_3(\xi, \eta) = \int_{-h}^{0} \left( \frac{p_2(z)}{\delta} \frac{\partial G_1(0, z, \xi, \eta)}{\partial x} - G_1(0, z, \xi, \eta) u_2(z) \right) \, dz, \quad (4.41)\]

where \(\delta = s + i\) and the pressures \(p_1(z)\) and \(p_2(z)\) and the velocities \(u_1(z)\) and \(u_2(z)\) on the interfaces \(P_1\) and \(P_2\) (as seen from regions 1 and 3) are defined by

\[
p_1(z) = \phi_1(0, z), \quad p_2(z) = \phi_1(b, z),
\]

\[
u_1(z) = \frac{\partial \phi_1}{\partial x}(0, z), \quad u_2(z) = \frac{\partial \phi_1}{\partial x}(b, z),
\]

Thus note by determining \(\{p_1(z), p_2(z), u_1(z), u_2(z)\}\) on the interfaces \(P_1\) and \(P_2\), equations (4.39)-(4.41) give a complete solution of the problem.

Considering the expressions (4.39)-(4.41) on the interfaces \(x = 0\) and \(x = b\) and employing the matching conditions (4.7) and (4.9), the following system of integral equations,

\[
\frac{1}{2} p_1(\eta_1) = \frac{1}{h} x_0(\eta_1) + \int_{-h}^{0} G_1(0, z, 0, \eta_1) u_1(z) \, dz, \quad (4.42)
\]

\[-\frac{1}{2\delta} p_1(\eta_1) = \int_{-h}^{0} \left( \frac{p_2(z)}{\delta} \frac{\partial G_2(b, z, 0, \eta_1)}{\partial x} - G_2(b, z, 0, \eta_1) \frac{u_2(z)}{\epsilon} + G_2(0, z, 0, \eta_1) \frac{u_1(z)}{\epsilon} \right) \, dz, \quad (4.43)\]

\[-\frac{1}{2\delta} p_2(\eta_2) = \int_{-h}^{0} \left( \frac{p_1(z)}{\delta} \frac{\partial G_2(0, z, b, \eta_2)}{\partial x} - G_2(0, z, b, \eta_2) \frac{u_1(z)}{\epsilon} + G_2(b, z, b, \eta_2) \frac{u_2(z)}{\epsilon} \right) \, dz, \quad (4.44)\]

\[-\frac{1}{2} p_2(\eta_2) = - \int_{-h}^{0} G_1(b, z, b, \eta_2) u_2(z) \, dz, \quad (4.45)\]

is obtained, where the symmetry property of the Green's functions,

\[
\frac{\partial G_i}{\partial x}(\xi, z, \xi, \eta) = 0 \quad (i = 1, 2)
\]

about the \(z\) axis has been employed. Equations (4.42) to (4.45) form a system of four simultaneous equations in four unknowns \(\{p_1(z), p_2(z), u_1(z), u_2(z)\}\) which would seem to have a reasonable prospect of solution.

### 4.3.3 Solution Of System of Integral Equations

**Nystrom Method**

Numerical solution of the system of integral equations has been implemented in a program by the author using a Nystrom method (see for example Delves & Mohamed [16]) adapted to tackle the singular behaviour of Green's functions near their sources. To illustrate this
approach consider the integral equation (4.45). Then employing the form of $G_1(x, z, \xi, \eta)$ given by (4.18), equation (4.45) can be rewritten as

$$\frac{1}{2} p_2(\eta) = -\int_{-h}^{0} \left( u_2(z) G_1(b, z, b, \eta) - u_2(\eta) S(z, \eta) \right) dz - u_2(\eta) \int_{-h}^{0} S(z, \eta) dz,$$

where $S(z, \eta)$ is defined by

$$S(z, \eta) = \frac{1}{2\pi} \{ K_0(\lambda|z - \eta|) + K_0(\lambda|z + \eta + 2h|) \}. \quad (4.48)$$

Note that in the first integral, the singular behaviour has been "removed" and is thus amenable without too much concern to Gaussian integration methods. In particular, employing the Gauss-Legendre abscissae $\eta_j$ and weights $w_j$, $(j = 1, 2, \ldots, n)$ for the interval $[-h, 0]$, the system $(i = 1, \ldots, n)$,

$$\frac{1}{2} p_2(\eta_i) = -w_i G_1^*(b, \eta_i, b, \eta_i) u_2(\eta_i) - u_2(\eta_i) \int_{-h}^{0} S(z, \eta_i) dz$$

$$- \sum_{j=1, j \neq i}^{n} w_j [G_1(b, \eta_j, b, \eta_j) u_2(\eta_j) - S(\eta_j, \eta_i) u_2(\eta_i)], \quad (4.49)$$

where from equation (4.18),

$$G_1^*(x, z, \xi, \eta) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{\cosh \gamma(x + h) \cosh \gamma(z + h) e^{-\gamma h} (\Gamma_1 + \gamma h) \cos \mu(x - \xi)}{\gamma \{ \Gamma_1 \cosh \gamma h - \gamma h \sinh \gamma h \}}. \quad (4.50)$$

Applying the same separation out of singular behaviour and application of the Gauss-Legendre rules for the other equations, then the system,

$$\frac{1}{2} P_1 = \frac{1}{h} \chi_0 + A u_1, \quad (4.51)$$

$$-\frac{1}{2\delta} P_1 = \frac{1}{\delta} D P_2 - \frac{1}{\epsilon} C u_2 + \frac{1}{\epsilon} B u_1, \quad (4.52)$$

$$-\frac{1}{2\delta} P_2 = \frac{1}{\delta} D P_1 + \frac{1}{\epsilon} C u_1 - \frac{1}{\epsilon} B u_2, \quad (4.53)$$

$$\frac{1}{2} P_2 = -A u_2, \quad (4.54)$$

is obtained where the vectors $\{ \chi_0, P_1, P_2, \nu_1, \nu_2 \}$ are defined by

$$\chi_0 = (\chi_0(\eta_1), \chi_0(\eta_2), \ldots, \chi_0(\eta_n))^T \quad (4.55)$$

$$p_i = (p_i(\eta_1), p_i(\eta_2), \ldots, p_i(\eta_n))^T \quad (i = 1, 2) \quad (4.56)$$

$$u_i = (u_i(\eta_1), u_i(\eta_2), \ldots, u_i(\eta_n))^T \quad (i = 1, 2) \quad (4.57)$$

and the matrices are defined $(1 \leq i, j \leq n)$ by,

$$A_{ij} = \begin{cases} w_j G_1(0, \eta_j, 0, \eta_i) & (i \neq j), \\ S_i + w_i G_1^*(0, \eta_i, 0, \eta_i) & (i = j) \end{cases} \quad (4.58)$$

58
\[ B_{ij} = \begin{cases} w_j G_2(0, \eta_j, 0, \eta_i) & (i \neq j), \\ S_i + w_i G_2^*(0, \eta_j, 0, \eta_i) & (i = j) \end{cases} \quad (4.59) \]

\[ C_{ij} = w_j G_1(b, \eta_j, 0, \eta_i) = w_j G_1(0, \eta_i, b, \eta_i) \quad , \]

\[ D_{ij} = w_j \frac{\partial G_2}{\partial x}(b, \eta_j, 0, \eta_i), \quad (4.61) \]

where

\[ S_i = - \sum_{j=1}^{n} w_j S(\eta_j, \eta_i) + \int_{-h}^{0} S(z, \eta_i) \, dz \quad (4.62) \]

and

\[ G_2^*(x, z, \xi, \eta) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\cosh \gamma(\eta + h) \cosh \gamma(z + h) e^{-\gamma h}(\Gamma_2 + \gamma h) \cos \mu (z - \xi)}{\gamma (\Gamma_2 \cosh \gamma h - \gamma h \sinh \gamma h)} \, d\mu. \quad (4.63) \]

Note also that the Green's function properties,

\[ G_i(b, z, b, \eta) = G_i(0, z, 0, \eta), \quad (i = 1, 2) \quad (4.64) \]

\[ G_i(0, z, b, \eta) = G_i(b, z, 0, \eta), \quad (i = 1, 2) \quad (4.65) \]

\[ \frac{\partial G_i}{\partial x}(0, z, b, \eta) = \frac{\partial G_i}{\partial x}(b, z, 0, \eta) \quad (i = 1, 2) \quad (4.66) \]

have been employed.

**Computation of the Green's and associated functions**

The employment of the solution strategy described above requires accurate computations of the Green's functions \( G_1 \) and \( G_2 \), together with the associated functions \( G_1^* \), \( G_2^* \) and \( G_2^\times \) (the \( x \) derivative of \( G_2 \)). To demonstrate the approach adopted in the program, consider the evaluation of \( G_1^*, G_2^* \) and \( G_2^\times \). (Note in the program, computations of \( G_1, G_2 \) and \( G_2^\times \) are made on the basis of the expressions in (4.18) and (4.19) respectively which involve \( G_1^* \) and \( G_2^* \) explicitly.) There are two key issues in computation of \( G_1^*, G_2^* \) and \( G_2^\times \). The first is the treatment of the deformed path of integration around the pole at \( \mu = (k^2 - \lambda^2)^{1/2} \) in the definition of \( G_1^* \) (and \( G_1 \)). The second issue, which affects computation of \( G_1^*, G_2^* \) and \( G_2^\times \), is that they all involve integrals of oscillatory functions on infinite intervals.

To demonstrate the approach employed, note that \( G_1^* \) and \( G_2^* \) can be written in form

\[ G_1^*(x, z, \xi, \eta) = -\frac{1}{\pi} \left( \int_{0}^{\infty} \frac{F_1(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi) \, d\mu \right), \quad (4.67) \]

\[ G_2^*(x, z, \xi, \eta) = -\frac{1}{\pi} \left( \int_{0}^{\infty} \frac{F_2(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi) \, d\mu \right), \quad (4.68) \]
where

\[ F_i(\mu, z, \eta) = \frac{\cosh \gamma (\eta + h) \cosh \gamma (z + h) e^{-\gamma h} (\Gamma_i + \gamma h)}{\Gamma_i \cosh \gamma h - \gamma h \sinh \gamma h} \quad (i = 1, 2). \] (4.69)

Then note in the program, that the deformed path of integration in the definition of \( G_1^* \) is dealt with, with reference to an application of some residue theory to the integral. In particular note by application of Cauchy’s Residue Theorem (see for example Stewart & Tall [57, page 213]), \( G_1^* \) can be expanded as

\[
G_1^*(x, z, \xi, \eta) = -\frac{1}{\pi} \left( \int_0^{\infty} \frac{F_1(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi) d\mu + i\pi \text{Res} \left( \frac{F_1(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi), (k^2 - \lambda^2)^{1/2} \right) \right),
\] (4.70)

where

\[
\int_0^{\infty} \frac{F_1(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi) d\mu = \lim_{\alpha \to 0} \left( \int_0^{\mu_0 - \alpha} \frac{F_1(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi) d\mu + \int_{\mu_0 + \alpha}^{\infty} \frac{F_1(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi) d\mu \right),
\] (4.71)

is the Cauchy principal value integral (see Stewart & Tall [57, page 224]) of \( F_1(\mu, z, \eta) \cos \mu (x - \xi) / \gamma \) on the interval \([0, \infty)\). Further note according to another standard result of residue theory (lemma 12.2 in Stewart & Tall [57, page 216]),

\[
\text{Res} \left( \frac{F_1(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi), (k^2 - \lambda^2)^{1/2} \right)
= \lim_{\mu \to (k^2 - \lambda^2)^{1/2}} (\mu - (k^2 - \lambda^2)^{1/2}) \frac{F_1(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi)
= -\frac{e^{-kh \cosh kh (\Gamma_1 + k) \cos (k^2 - \lambda^2)^{1/2}(x - \xi)) \chi_0(\eta) \chi_0(z)}{2kh(k^2 - \lambda^2)^{1/2}}.
\] (4.72)

And thus \( G_1^* \) is evaluated from equation (4.70).

To deal with the infinite integrals for \( G_1^*(x, z, \xi, \eta) \) (in equation (4.70)), \( G_2^*(x, z, \xi, \eta) \) and \( G_{2z}^*(x, z, \xi, \eta) \) the integrals are truncated and evaluated numerically and summed with asymptotic approximations estimating the discarded parts. To demonstrate the origin of the asymptotic approximations, consider the behaviour of

\[
\frac{F_i(\mu, z, \eta)}{\gamma} \cos \mu (x - \xi) \quad (i = 1, 2),
\] (4.73)

for large \( \mu \). In particular note by re-writing the trigonometric functions in \( F_i(\mu) \) in exponential form, it is clear that

\[
F_i(\mu, z, \eta) = \frac{(e^{\gamma (\eta + h)} + e^{-\gamma (\eta - h + 2h)}) (1 + e^{-2\gamma (x + h)})}{2(e^{-2\gamma h} + \frac{(e^{-\gamma h})}{(e^{\gamma h})})} \sim \frac{1}{2} e^{\gamma (\eta + z)} \sim \frac{1}{2} e^{\mu (\eta + z)} \quad (4.74)
\]
for large $\mu$. And thus note according to Abramowitz and Stegun [1] equation (5.1.4) that for sufficiently large $a$,

$$\int_a^\infty \cos \mu (x - \xi) \frac{F_1(\mu)}{\gamma} d\mu \sim -\frac{1}{2} \int_a^\infty \cos \mu (x - \xi) e^{-\mu|\eta + z|} d\mu$$

$$= -\frac{1}{2} \text{Re}\{E_1[(|\eta| + z) + i|z - \xi|]a]\}$$

where $E_1[z]$ represents the first order exponential integral of $z$. The integral over an infinite domain for $G_{zy}(x, z, \xi, \eta)$ is tackled in a similar way. Note

$$G_{zy}(x, z, \xi, \eta) = \frac{1}{\pi} \int_0^\infty \frac{\mu F(\mu)}{\gamma} \sin \mu (x - \xi) d\mu$$

and for sufficiently large $a$,

$$\int_a^\infty \mu \sin \mu (x - \xi) \frac{F(\mu)}{\gamma} d\mu \sim -\frac{1}{2} \int_a^\infty \sin \mu (x - \xi) e^{-\mu|\eta + z|} d\mu$$

$$= -\frac{1}{2} a \text{Im} E_0[a(|z + \eta| - i(x - \eta))]$$

where $E_1[z]$ represents the first order exponential integral of $z$.

### 4.4 Results

The efficacy of the Green's function/Integral equation approach has been verified by running the program discussed in the previous section for a selection of the wave train - porous block interactions considered by Dalrymple et al. [13]. This is demonstrated here by the graphs shown in Figure 4.3 (the $f = 1$ case) and in Figure 4.4 which compare very favourably with Dalrymple et al. [13, Figures 9 and 10].

In each of these figures (and additionally in Figure 4.5 the absolute value of the reflection coefficient, $R$ is plotted for block with various dimensions and properties and various incident wave trains. (Recall from Chapter 3, that the reflection coefficient is the ratio of the reflected wave amplitude over the incident wave amplitude). Employing the orthogonality condition (2.67) and the eigenfunction expansion solutions in regions 1 and 3 given in equation (2.54) and (2.55), it is clear that the reflection coefficient is given by

$$R = R_0 = -1 + \int_{-h}^0 \phi_1(0, z) \chi_0(z) dz,$$

(4.80)

or alternatively employing the horizontal velocity component,

$$R = R_0 = 1 + \frac{i}{(k^2 - \lambda^2)^{1/2}} \int_{-h}^0 \frac{\partial \phi_1}{\partial x}(0, z) \chi_0(z) dz.$$
Similarly the transmission coefficient (the ratio of the amplitude of the transmitted non-
evanescent mode to the amplitude of the incident mode) is given by

\[ T = T_0 = \int_{-h}^{0} \phi_3(x, b) \chi_0(z) \, dz \] (4.82)

or employing the horizontal velocity component,

\[ T = T_0 = \frac{-i}{(k^2 - \chi^2)^{1/2}} \int_{-h}^{0} \frac{\partial \phi_3}{\partial x} (x, b) \chi_0(z) \, dz. \] (4.83)

In addition to comparison with results from Dalrymple et al. [13] two tests have been
made on the validity of the program. Firstly the reflection/transmission coefficients have been
computed using both the potential and velocity forms given above. Secondly the program
has been run for blocks with the trivial parameters \( \epsilon = 1 \) and \( \delta = 1 \) corresponding to no block
being present. And the program gives correctly values of zero and one for the reflection and
transmission coefficients respectively. Two additional checks could have been made. Firstly
verification that in the case that \( \epsilon = 0 \) corresponding to a totally impermeable blocks the
reflection and transmission coefficient should be one and zero respectively could be checked.
(For the program it wasn’t possible to employ this check as \( \epsilon \neq 0 \) was implicitly assumed
to simplify the linear system from four vector equations in four unknowns to two vector
equations in two unknowns). And secondly the following energy relations,

\[ \Im \int_{-h}^{0} \phi_1(0, z) \frac{\partial \phi_1^*}{\partial x} (0, z) \, dz = \frac{1}{(k^2 - \chi^2)^{1/2}} (|R|^2 - 1) \] (4.84)

\[ \Im \int_{-h}^{0} \phi_1(b, z) \frac{\partial \phi_1^*}{\partial x} (b, z) \, dz = \frac{1}{(k^2 - \chi^2)^{1/2}} |T|^2, \] (4.85)

\[ \Im \left\{ \frac{1}{\delta} \int_{-h}^{0} \left( \phi_1(0, z) \frac{\partial \phi_1^*}{\partial x} (0, z) - \phi_1(b, z) \frac{\partial \phi_1^*}{\partial x} (b, z) \right) \, dz \right\} = \frac{f}{h} \int_{0}^{b} |\phi_2(z, 0)|^2 \, dz. \] (4.86)

found by applying Green’s theorem to the solution \( \phi_i \) and its complex conjugate \( \phi_i^* \) \((i = 1, 2, 3)\)
in each of the three regions. And note the first two of these have been checked.

On the basis of experimentation with the number of boundary elements. It has been
found that 21 boundary elements give approximately three decimal places of accuracy for the
parameters in each of the plots shown.

Detailed consideration of the reflection and transmission properties of infinitely long
porous block under oblique wave attack has been made by Dalrymple, Losada & Martin
[13]. Nevertheless the graphs present a useful to opportunity to point some of the features of
reflection and transmission from a porous structure under oblique wave attack.
Figure 4.3: $|R|$ and $|T|$ against angle of incidence, $\Gamma_1 = 0.2012$, $\delta = 1 + i$, $\epsilon = 0.4 : f = 1 (- - - )$, $f = 3 (---)$, $f = 5 (-----)$

In figure 4.3 the magnitudes of the reflection and transmission coefficients $R = R_0$ and $T = T_0$ are plotted against the angle of incidence for three blocks with the same dimensions and porosity but three different friction factors. For each structure, it is observed that from 0 degrees, $|R|$ decreases with increasing angle until a minimum is reached. And after this angle it increases rapidly tending to 1 at 90 degrees. It is also observed that $|T|$ increases with increasing angle until a maximum is reached at the angle of minimum reflection and then decreases rapidly towards 0 at 90 degrees.

And observe from this figure that there are two clear effects of increasing the friction factor on the performance of a permeable block against angle. Firstly for higher values of the friction factor the angle of the minimum reflection increases. Secondly for higher friction there is greater reflection and less transmission (as would be expected as the magnitude of the friction factor describes the amount of energy dissipated in a permeable structure). Note the minimum and maximums of $|R|$ and $|T|$ also increase and decrease respectively with increasing friction factor.
In figure 4.4 the magnitudes of the reflection and transmission coefficients $R = R_0$ and $T = T_0$ are plotted against $kh$ for three blocks with the same dimensions and porosity but three different friction factors. For each structure, it is observed that from $kh = 0$, $|R|$ increases with increasing $kh$ reaching a maximum for a certain value of $kh$. And after this it decreases slowly. It is also observed that $|T|$ decreases with increasing $kh$ eventually hitting zero. (Note that $|R|$ and $|T|$ are annotated the wrong way round in the figure.)

And observe from this figure that there are two clear effects of increasing the friction factor on the performance of a permeable block against $kh$. Observe firstly, that for higher values of the friction factor $|R|$ increases and $|T|$ decreases. Secondly observe that for higher friction factor, that $|T|$ diminishes more rapidly against $kh$ as one might expect in view that the friction factor quantifies the dissipative properties of a porous structure as described in Chapter 2. Thirdly observe that the maximum of $|R|$ occurs at lower values of $kh$ for higher values of the friction factor.
Finally in figure 4.5 observe the magnitudes of the reflection and transmission coefficients $R = R_0$ and $T = T_0$ are plotted against $kh$ for three blocks each of a different thickness but with the same friction factor and porosity. For each structure, it is observed that from $kh = 0$, $|R|$ increases with increasing $kh$ reaching a maximum for a certain value of $kh$. And after this it decreases slowly. It is also observed that $|T|$ decreases with increasing $kh$ eventually hitting zero.

And observe from this figure that there are two clear effects of increasing the block thickness on the performance of a permeable block against $kh$. Observe firstly that $|T|$ diminishes much more rapidly against $kh$ by increasing the block thickness. Secondly observe that the maximum of $|R|$ is greater and occurs at smaller value of $kh$ for greater thicknesses.

4.5 Conclusion

In this chapter the problem of determining the reflection and transmission from an infinitely long porous structure by water waves, originally formulated and solved by Dalrymple et al. [13], has been re-considered. An integral equation / Green’s function solution has been presented here as an alternative to the matched eigenfunction solution employed by Dalrymple et al. [13].

This solution has been successfully validated by comparison of generated results with those of Dalrymple et al.. In addition it has been noted that this approach has certain advantages over the matched eigenfunction solution of Dalrymple et al. [13] in the solution of this problem. Firstly, the solution employs the widely regarded (and as demonstrated here, highly efficient) boundary integral equation approach. Secondly, this solution doesn’t require the such detailed consideration of the dispersion relations (beyond the determination of the incident wavenumber) as the matched eigenfunction solution and thirdly treatment of the “mode swapping” phenomena is transparent in the solution requiring no additional terms or detailed consideration. In addition, it is also noted that the Green’s functions employed here, perhaps offer some advantage over the simple log terms utilized by Sulisz [59] and [60], by already satisfying the bed and free surface conditions.

Future work based on this solution described here might include consideration of the types of problems solved by Sulisz [59] and [60] for oblique incidence. Additionally it would be interesting to have quantitative measures of the relative efficiencies of the Dalrymple et al. solution, the current solution, and Sulisz solution employing his Green’s function.
Chapter 5

Diffraction by permeable rectangular segmented breakwater

5.1 Introduction

This chapter presents a shallow wave solution for the interaction of a gravity wave train with a periodic array of rubble mound breakwaters. There are two main motivations for studying this problem. The first is to understand the effect of porosity on diffraction through a gap in a breakwater. This extends the work of Fernyhough & Evans [20] on diffraction through a impermeable array discussed in Chapter 3. The second is to understand the effect that the addition of gaps has on the performance of a rubble mound breakwater with prescribed resistance factors (porosity, friction factor). This extends the work of Dalrymple, Losada & Martin [13] discussed in the previous chapter. Additionally, the solution extends the work of Ijima et al. [26] who considered flow about a porous structure in isolation.

As before, the problem is formulated using the Sollitt & Cross [56] approach to wave interaction with a rubble mound structure. During the formulation of the boundary value problem it is assumed that the structure is set in a regime of shallow water. This assumption (previously employed in porous media wave interaction problems by amongst others Madsen [40] and Dalrymple et al. [13]) simplifies the problem by reducing it from a three-dimensional problem to a two dimensional one - the vertical variation in the solution being factored out. The use of this approximation seems reasonable from an engineering point of view as generally offshore breakwaters are located in relatively shallow water. And also note that Dalrymple et al. [13] have shown for the structure considered in the last chapter that shallow water
approximation gives good comparison with the full solution of the three dimensional problem.

Again, as in previous chapter solution of the boundary value problem is achieved by the employment of Green's functions and numerical solution of the resultant integral equation. Solution of array problems with this approach has a long history (see for example Achenbach et al. [2, 3] and Williams et al. [62, 63]). A large literature exists on the derivation and computation of appropriate Green's solutions for this problem. A review of various methods of computation of periodic Green's functions for Helmholtz equation together with a novel new approach has recently been presented by Linton [33]. Numerical solution of the system of integral equations which arises has been achieved with a boundary element method adapted from that used in Chapter 3.
5.2 Formulation

Geometry

The interaction of a gravity wave train with a periodic array of porous blocks is considered. The blocks are supposed to be constructed of the same homogeneous isotropic porous material and to extend through water of constant depth $h$. Introducing Cartesian coordinates, the undisturbed free-surface is located at $z = 0$ and the sea-bed at $z = -h$. The blocks, each of length $2l$, width $b$, lie along the $y$-axis as illustrated in figure 5.1. In addition, it is assumed that the distance between the centres of adjacent gaps is $2d$, and thus the gap size is $2c$, where $c = (d - l)$.

\[
\begin{array}{c}
\text{Figure 5.1 : element of array}
\end{array}
\]

Full Linearized Water Wave Formulation
The diffraction problem for this structure is developed in much the same way as those structures tackled in chapters 3 and 4. In the following region 1 denotes the fluid domain external to the array and region 2 denotes the domain bounded by the blocks. The usual Airy linear theory is employed in description of wave motion in region 1. The Sollitt & Cross [56] theory described in chapter 3 is used to describe wave motions in region 2 and on the interface of the two regions. Thus assuming time-dependent motion of frequency $\omega$ throughout, the wave field potentials may be written as

$$\hat{\phi}_i(x, y, z, t) = \text{Re}[e^{-i\omega t} \Phi_i(x, y, z)], \quad (i = 1, 2),$$

and the full problem can then be stated as

$$\nabla^2 \Phi_i = 0, \quad (i = 1, 2),$$

subject to the following boundary conditions:

the bed condition,

$$\frac{\partial \Phi_i}{\partial z} = 0, \quad \text{when } z = -h, \quad (i = 1, 2)$$

the free surface conditions,

$$\frac{\partial \Phi_1}{\partial z} - \frac{\Gamma_i}{h} \Phi_i = 0, \quad \text{at } z = 0 \quad (i = 1, 2),$$

where

$$\Gamma_1 = \frac{\omega^2 h}{g},$$

$$\Gamma_2 = (s + if) \Gamma_1 = \frac{\omega^2 h(s + if)}{g}.$$ 

and the matching conditions,

$$\Phi_1 = (s + if) \Phi_2,$$

$$\frac{\partial \Phi_1}{\partial n} = \epsilon \frac{\partial \Phi_2}{\partial n},$$

on the boundaries of each block where the normal $n$ points out of region 1.

It is assumed that a wave of the form

$$\Phi_1(x, y, z) = e^{i k (x \cos \theta_0 + y \sin \theta_0)} \cosh (z + h),$$

(5.9)
having wavelength $\lambda = 2\pi/k$ and making an angle $\theta_0$ with the positive $x$-direction, $-\pi/2 \leq \theta_0 \leq \pi/2$ is incident on the array. Note in order that $\Phi^I$ satisfies the bed and free surface conditions, $k$ needs to satisfy the dispersion relation,

$$\Gamma_1 = kh \tanh kh$$

Similarly, waves in the rigid porous region have wave mode with wave number $K$ satisfying

$$\Gamma_2 = Kh \tanh Kh$$

Equation (5.9) is more conveniently written,

$$\Phi^I(x, y, z) = e^{(\alpha x + \beta y)} \cosh k(z + h)$$

where $\alpha = \alpha_0 = k \cos \theta_0$ and $\beta = \beta_0 = k \sin \theta_0$.

In addition $\Phi_1$ must satisfy a radiation condition as described in Chapter 1. In particular with the exception of incident mode, all other modes propagate away from the structure.

_Shallow Water Approximation_

A considerable simplification of the problem is made, by making the shallow water assumption $0 < kh \ll 1$. Expanding $\Phi_i$ (for $i = 1, 2$) about the free surface $z = 0$,

$$\Phi_i(x, y, z) = \Phi_i(x, y, 0) + z \frac{\partial \Phi_i(x, y, 0)}{\partial z} + \frac{1}{2} z^2 \frac{\partial^2 \Phi_i(x, y, 0)}{\partial z^2} + \ldots, \quad (5.13)$$

which by the free surface conditions (5.4) and Laplace's equation maybe written as,

$$\Phi_i(x, y, 0) + z \frac{\Gamma_i}{h} \Phi_i + \frac{1}{2} z^2 \left( \frac{\partial^2 \Phi_i}{\partial x^2} - \frac{\partial^2 \Phi_i}{\partial y^2} \right)(x, y, 0) + \ldots \quad (5.14)$$

Then defining $\phi_i(x, y) = \Phi_i(x, y, 0)$ for $i = 1, 2$, $\Phi_i(x, y, z)$ maybe approximated by

$$\Phi_i(x, y, z) = \phi_i(x, y) + z \frac{\Gamma_i}{h} \phi_i(x, y) - \frac{1}{2} z^2 \nabla^2 \phi_i(x, y) \quad (5.15)$$

Further applying the bed condition (5.3) to (5.15) in regions 1 and 2, the expression

$$\frac{\Gamma_i}{h} \phi_i + h \nabla^2 \phi_i = 0, \quad (5.16)$$

is obtained. And note by considering (5.10) and (5.11) with $0 < kh, |Kh| \ll 1$, it's clear that

$$\Gamma_1 = (kh)^2, \quad (5.17)$$

$$\Gamma_2 = (Kh)^2, \quad (5.18)$$
and that equation (5.16) may be written as

\[(\nabla^2 + k^2)\phi_1 = 0,\]  
(5.19)

in region 1, and

\[(\nabla^2 + K^2)\phi_2 = 0,\]  
(5.20)

in region 2 where the root \(K\) of equation (5.18) is chosen such that,

\[\Im mK \geq 0,\]  
(5.21)

to reflect that the energy of the waves decays as they propagate through the structure.

**Periodicity Conditions**

As in Chapter 3, it is only necessary to consider the problem on a single element of the array (illustrated in figure 5.2) provided \(\phi_1(x, y)\) satisfies the two Bloch conditions

\[\phi_1(x, d) = e^{2i\beta d}\phi_1(x, -d),\]  
(5.22)

\[\frac{\partial \phi_1}{\partial y}(x, d) = e^{2i\beta d}\frac{\partial \phi_1}{\partial y}(x, -d),\]  
(5.23)

as a consequence of the periodicity of the array.

![Figure 5.2: Periodic Element](image)

**5.3 Solution**

**5.3.1 Green’s Functions**

A Green’s solution approach is employed to solve the boundary value problem (c.f. Achenbach et al. [2, 3] and Williams et al. [62, 63]). For this purpose, it is necessary to find suitable
Green's functions for regions 1 and 2. In region 1, a periodic Green's function $G_1(x, y, \xi, \eta)$ is sought. This is subject to the boundary value problem,

\[
(\nabla + k^2)G_1 = -\delta(x - \xi)\delta(y - \eta) \tag{5.24}
\]

\[
G_1(x, d, \xi, \eta) = e^{-2i\beta d}G_1(x, -d, \xi, \eta) \tag{5.25}
\]

\[
\frac{\partial G_1}{\partial y}(x, d, \xi, \eta) = e^{-2i\beta d} \frac{\partial G_1}{\partial y}(x, -d, \xi, \eta), \tag{5.26}
\]

\[
0G_1(-d, \xi, \eta) = e^{-2i\beta d}G_1(x, -d, \xi, \eta), \tag{5.27}
\]

and $G_1$ is assumed to satisfy the radiation condition described in Chapter 1. Note the difference of sign in the exponentials of the periodicity conditions for the solution and the Green's function ensures that

\[
\phi_1(x, d) \frac{\partial G_1}{\partial y}(x, d, \xi, \eta) - G_1(x, d, \xi, \eta) \frac{\partial \phi_1}{\partial y}(x, d) = \phi_1(x, -d) \frac{\partial G_1}{\partial y}(x, -d, \xi, \eta) - G_1(x, -d, \xi, \eta) \frac{\partial \phi_1}{\partial y}(x, -d). \tag{5.28}
\]

In region 2, it is sufficient that $G_2(x, y, \xi, \eta)$ satisfies

\[
(\nabla + K^2)G_2 = -\delta(x - \xi)\delta(y - \eta), \tag{5.29}
\]

Green's functions for regions 1 and 2, satisfying the boundary value problem (5.24)-(5.28) and the radiation condition, are

\[
G_1(x, y, \xi, \eta) = \frac{1}{4i}H_0(kr)
- \frac{1}{2\pi} \int_0^\infty \frac{(\cos 2\beta d - e^{-2k\gamma d})\cosh k\gamma(y - \eta) - i\sin 2\beta d\sinh k\gamma(y - \eta)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)} \cos k(x - \xi)t \, dt, \tag{5.30}
\]

\[
G_2(x, y, \xi, \eta) = \frac{1}{4i}H_0(Kr), \tag{5.31}
\]

where

\[
\gamma = \begin{cases} 
-i(1-t^2)^{1/2} & (t \leq 1) \\
(t^2-1)^{1/2} & (t > 1)
\end{cases} \tag{5.32}
\]

\[
r = ((x - \xi)^2 + (y - \eta)^2)^{1/2} \tag{5.33}
\]

and $H_0(= H_0^0)$ is the Hankel function of first kind of zeroth order. The Green's function for region 2 is the usual wave source for Helmholtz equation. The Green's function expression (5.30) for region 1 was obtained by Linton [31]. It is obtained (as expanded on in Appendix D.1) by combining the equivalent wave source for region 1 ($H_0(kr)$) with a Fourier integral.
solution and applying the above periodicity conditions. Note that this Green’s function can be written in the form,

\[ G_1(x, y, \xi, \eta) = -\frac{i}{4d} \sum_{m=-\infty}^{\infty} \frac{1}{\alpha_m} e^{i(\alpha_m|x-\xi|-\beta_m(y-\eta))} \]  \hspace{1cm} (5.34)

where

\[ \alpha_m = \begin{cases} 
    i{(\beta_m^2 - k^2)^{1/2}} & (m \leq -r, \ m \geq s) \\
    (k^2 - \beta_m^2)^{1/2} & (-r \leq m \leq s) 
\end{cases} \]  \hspace{1cm} (5.35)

\[ \beta_m = \beta - \frac{m\pi}{d} \]  \hspace{1cm} (5.36)

This shows that the integral form is identical to the periodic Green function employed by Achenbach et al. [2, 3] and Williams et al. [62, 63]. However as demonstrated by Linton [33] evaluation of the integral form given by (5.30) is computationally more efficient than these forms.

It is useful here to note using equations (5.30), (5.31) and Abramowitz & Stegun equation (9.1.8) that as \((x, y) \rightarrow (\xi, \eta)\),

\[ G_1(x, y, \xi, \eta) \sim \frac{1}{2\pi} \log k r, \]  \hspace{1cm} (5.37)

\[ G_2(x, y, \xi, \eta) \sim \frac{1}{2\pi} \log K r. \]  \hspace{1cm} (5.38)

5.3.2 Integral Equation Formulation

Figure 5.3 : Contours for Green’s Theorem
Green's solutions for region 1 and 2 are found by applying Green's Theorem to the solutions and Green's functions, in their respective regions. Note by applying Green's Theorem to the region between $S$ and $L$ excluding $B_c$ (illustrated in figure 5.3) to the solution $\phi_1(x, y)$ and the Green's function $G_1(x, y, \xi, \eta)$ with source point located there, the expression,

$$\int_{S+L+B_c} \left( \phi_1(x, y) \frac{\partial G_1(x, y, \xi, \eta)}{\partial n} - G_1(x, y, \xi, \eta) \frac{\partial \phi_1(x, y)}{\partial n} \right) ds = 0,$$

is obtained where $n$ is the normal to the surfaces $L, S$ and $B_c$ pointing out of region. And note if $(\xi, \eta)$ is located on the contour $S$ then $B_c$ is taken as a half-circle so as to exclude in the region bounded by the half-circle and its intersection with the contour $S$. Assuming $(\xi, \eta)$ doesn't lie on $L_2$ or $L_4$ then by equation (5.28),

$$\int_{L_2+L_4} \left( \phi_1(x, y) \frac{\partial G_1(x, y, \xi, \eta)}{\partial n} - G_1(x, y, \xi, \eta) \frac{\partial \phi_1(x, y)}{\partial n} \right) ds = 0. \tag{5.40}$$

Employing separation of variables, the radiation and periodicity conditions (5.22) and (5.23) the solution in region 1 is for $x \leq 0$ of the form,

$$\phi_1(x, y) = e^{i(\alpha_0 x + \beta_0 y)} + \sum_{m=-\infty}^{\infty} R_m e^{-i(\alpha_m x - \beta_m y)} , \tag{5.41}$$

and for $x \geq b$ of the form,

$$\phi_1(x, y) = \sum_{m=-\infty}^{\infty} T_m e^{i(\alpha_m(x-b) + \beta_m y)}, \tag{5.42}$$

where $R_m$ and $T_m$ denote the reflection and transmission coefficients respectively. Note that the eigenfunctions $\{e^{i\beta_m y}\}$ satisfy the orthogonality relation,

$$\frac{1}{2d} \int_{-d}^{d} e^{i\beta_m y} e^{-i\beta_n y} dy = \delta_{mn}. \tag{5.43}$$

Further note for $x \ll 0$,

$$\phi_1(x, y) \sim e^{i(\alpha_0 x + \beta_0 y)} + \sum_{m=-r}^{s} R_m e^{-i(\alpha_m x - \beta_m y)}, \tag{5.44}$$

$$G_1(x, y, \xi, \eta) \sim \frac{i}{4d} \sum_{m=-r}^{s} \frac{1}{\alpha_m} e^{-i(\alpha_m (x-\xi) + \beta_m (y-\eta))}, \tag{5.45}$$

where denoting the integer part of $x$ by $[x],$

$$r = \left[ (1 + \sin \theta_0) \frac{kd}{\pi} \right], \tag{5.46}$$

$$s = \left[ (1 - \sin \theta_0) \frac{kd}{\pi} \right], \tag{5.47}$$

and thus,

$$\int_{L_1} \left( \phi_1(x, y) \frac{\partial G_1(x, y, \xi, \eta)}{\partial n} - G_1(x, y, \xi, \eta) \frac{\partial \phi_1(x, y)}{\partial n} \right) ds = e^{i(\alpha_0 \xi + \beta_0 \eta)} = \chi_0(\xi, \eta). \tag{5.48}$$

74
Similarly note when \( x \gg b \),
\[
\phi_1(x, y) \sim \sum_{m=-r}^{r} T_m e^{i(\alpha_m(x-b)+\beta_m y)},
\]
(5.49)
\[
G_1(x, y, \xi, \eta) \sim -\frac{i}{4\pi} \sum_{m=-r}^{r} \frac{1}{\alpha_m} e^{i(\alpha_m(x-\xi)-\beta_m(y-\eta))},
\]
(5.50)

and thus
\[
\int_{L_3} \left( \phi_1(x, y) \frac{\partial G_1}{\partial n}(x, y, \xi, \eta) - G_1(x, y, \xi, \eta) \frac{\partial \phi_1}{\partial n}(x, y) \right) ds = 0
\]
(5.51)

Finally note according to (5.37)
\[
\int_{B_3} \left( \phi_1(x, y) \frac{\partial G_1}{\partial n}(x, y, \xi, \eta) - G_1(x, y, \xi, \eta) \frac{\partial \phi_1}{\partial n}(x, y) \right) ds = C(\xi, \eta) \phi_1(\xi, \eta).
\]
(5.52)

where
\[
C(\xi, \eta) = \begin{cases} 
-3/4 & \text{if } (\xi, \eta) \text{ is located at any one of the block corners} \\
-1/2 & \text{if } (\xi, \eta) \text{ is elsewhere on the block boundary} \\
-1 & \text{elsewhere}
\end{cases}
\]
(5.53)

And thus combining equations (5.40), (5.48) and (5.51), equation (5.39) reduces to
\[
C(\xi, \eta) \phi_1(\xi, \eta) + \int_{S} \left( \phi_1(x, y) \frac{\partial G_1}{\partial n}(x, y, \xi, \eta) - G_1(x, y, \xi, \eta) \frac{\partial \phi_1}{\partial n}(x, y) \right) ds = -\chi_0(\xi, \eta)
\]
(5.54)

And note similarly by applying Green's Theorem to the region enclosed by the block, the expression,
\[
C(\xi, \eta) \phi_2(\xi, \eta) - \int_{S} \left( \phi_2(x, y) \frac{\partial G_2}{\partial n}(x, y, \xi, \eta) - G_2(x, y, \xi, \eta) \frac{\partial \phi_2}{\partial n}(x, y) \right) ds = 0
\]
(5.55)
is obtained. Further letting \( P(s) = \phi_1(z(s), y(s)) \) and \( V(s) = \frac{\partial \phi_1}{\partial n}(z(s), y(s)) \) and applying the matching conditions (5.7) and (5.8) to equations (5.54) and (5.55), the expressions
\[
-C(\xi, \eta) \phi_1(\xi, \eta) = \chi_0(\xi, \eta) + \int_{S} \left( P(s) \frac{\partial G_1}{\partial n}(x, y, \xi, \eta) - G_1(x, y, \xi, \eta) V(s) \right) ds,
\]
(5.56)
\[
C(\xi, \eta) \phi_2(\xi, \eta) = \int_{S} \left( \frac{P(s)}{\delta} \frac{\partial G_2}{\partial n}(x, y, \xi, \eta) - G_2(x, y, \xi, \eta) \frac{V(s)}{\epsilon} \right) ds.
\]
(5.57)

are obtained.

Finally choosing the source point \((\xi, \eta)\) to lie on the boundary of the structure in equations (5.56) and (5.57), and applying the matching condition (5.7), the boundary-integral equations,
\[
C(\xi, \eta) P(s_1) = \chi_0(\xi, \eta) + \int_{S} \left( P(s) \frac{\partial G_1}{\partial n}(x, y, \xi, \eta) - G_1(x, y, \xi, \eta) V(s) \right) ds,
\]
(5.58)
\[
-\frac{1}{\delta} C(\xi, \eta) P(s_1) = \int_{S} \left( \frac{P(s)}{\delta} \frac{\partial G_2}{\partial n}(x, y, \xi, \eta) - G_2(x, y, \xi, \eta) \frac{V(s)}{\epsilon} \right) ds,
\]
(5.59)
are obtained where \( x(s_1) = \xi \) and \( y(s_1) = \eta \). And note the solutions of the coupled integral equations (5.58) and (5.59), \( P(s_1) \) and \( V(s_1) \), together with expressions (5.56) and (5.57) provide a complete description of flow field both inside and outside the structure.

### 5.3.3 Solution of System of Integral Equations

Numerical solution of the system of integral equations has been achieved using the Boundary Element Method (BEM) and is outlined in this section.

**Boundary Element Method**

A key factor in the choice of boundary elements in this problem is the anticipated singular nature of velocity field near the corners of the blocks. Note it is demonstrated in Appendix D.4 (on the basis of an argument of Liggett & Liu [28, page 53]) that there is the same kind of velocity singularity in the vicinity near a permeable corner as there is for an impermeable corner. In particular letting \( \phi(\rho, \varphi) \) denote the velocity potential for such a flow where \( (\rho, \varphi) \) are radial and angular coordinates fixed to the corner, then in the vicinity of the corner, Appendix D.4 shows

\[
|\nabla \phi(\rho, \varphi)| \approx \frac{A(\epsilon)}{\rho^{1/3}}, \tag{5.60}
\]

where \( A(\epsilon) \) denotes the strength of the singularity. Notice also when \( \epsilon = 1, \delta = 1 \) (the case where no blocks are present) the solution of the system is just the incident mode \( \chi_0(x, y) \).

This suggests that as the porosity \( \epsilon \) increases to 1 the singularity strength \( A(\epsilon) \) decreases to 0. It's hence computationally advantageous to choose a partition of the block which has a greater concentration of elements in the vicinity of the corner than elsewhere where the normal velocity is changing more rapidly.

To define such a partition, recall that the Gauss-Jacobi weights and abscissae used in the numerical evaluation of the integral,

\[
I = \int_{a_1}^{a_2} f(x)(x - a_1)^{c_1}(a_2 - x)^{c_2} \, dx \tag{5.61}
\]

are determined by the weight function,

\[
w(x) = (x - a_1)^{c_1}(a_2 - x)^{c_2} \tag{5.62}
\]

and in particular the choice of parameters \( c_1 \) and \( c_2 \). Now suppose that \( \{y_j, w_j\}_{j=1,...,m} \) and \( \{x_j, w_j\}_{j=1,...,n} \) denote the Gauss-Jacobi abscissae and weights \( (c_1 = -1/3, \ c_2 = -1/3) \) on
the intervals \([-l, l]\) and \([0, b]\) respectively. Then for odd \(m\) and \(n\) define the boundary element

centre-points \(\{Q_j\}_{j=1,\ldots,2(m+n)}\) by

\[
Q_j = \begin{cases}
(0, y_j) & \text{if } 1 \leq j \leq m \\
(x_{j-m}, l) & \text{if } m + 1 \leq j \leq m + n \\
(b, y_{2m+n-j+1}) & \text{if } m + n + 1 \leq j \leq 2m + n \\
(z_{2(m+n)-j+1}, -l) & \text{if } 2m + n + 1 \leq j \leq 2(m + n),
\end{cases}
\]  

(5.63)

and element endpoints by \(Q_1 = (0, -l)\) and the recursion relation,

\[
Q_j = 2Q_{j-1} - Q_{j-1} \quad j = 2, \ldots, 2(m+n).
\]  

(5.64)

Note this definition places endpoints at each corner of the block: \(Q_{m+1} = (0, l), Q_{m+n+1} = (b, l)\) and \(Q_{2m+n+1} = (b, -l)\) (it is also convenient to define \(Q_{2(m+n)+1} = Q_1\)). In addition to having a greater concentration of elements near the block corners note that application of the BEM produces solution (pressure and velocity) at the Gauss-Jacobi abscissae on each side. So it is possible to use the Gauss-Jacobi quadrature rules (which account for the corner velocity singularities) to evaluate additional integrals involving the velocity e.g. the reflection and transmission coefficients.

Applying the BEM with this choice of elements, results in the following linear system,

\[
AP - BV = -\chi_0, \tag{5.65}
\]

\[
\frac{1}{\delta} CP - \frac{1}{\epsilon} DV = 0, \tag{5.66}
\]

where the pressure vector \(P\), velocity vector, \(V\) and incident mode vector, \(\chi_0\), are defined by

\[
P = (P(Q_1), \ldots, P(Q_{2(m+n)}))^T, \tag{5.67}
\]

\[
V = (V(Q_1), \ldots, V(Q_{2(m+n)}))^T, \tag{5.68}
\]

\[
\chi_0 = (\chi(Q_1), \ldots, \chi(Q_{2(m+n)}))^T \tag{5.69}
\]

and the \(2(m + n) \times 2(m + n)\) matrices \(A, B, C\) and \(D\) are defined by

\[
A_{ij} = -\frac{1}{2} \delta_{ij} + \int_{Q_j}^{Q_{j+1}} \frac{\partial G_1}{\partial n}(Q(s), \overline{Q}_i) \, ds, \tag{5.70}
\]

\[
B_{ij} = \int_{Q_j}^{Q_{j+1}} G_1(Q(s), \overline{Q}_i) \, ds, \tag{5.71}
\]

\[
C_{ij} = \frac{1}{2} \delta_{ij} + \int_{Q_j}^{Q_{j+1}} \frac{\partial G_2}{\partial n}(Q(s), \overline{Q}_i) \, ds, \tag{5.72}
\]

\[
D_{ij} = \int_{Q_j}^{Q_{j+1}} G_2(Q(s), \overline{Q}_i) \, ds. \tag{5.73}
\]

77
Note in a program written by the author to implement the above solution, the main computational difficulty is to evaluate the integral in the Green's function for region 1 (equation (5.30)). In particular note from Appendix D.3 that the integrand has the poles,

\[ t_n = (1 + \gamma_n^2)^{1/2} = \frac{1}{k} (k^2 - \beta_n^2)^{1/2} = \frac{1}{k} \alpha_n, \]

and according to equation (5.35) for \(-r \leq n \leq s\) the poles are on the real axis and otherwise are on the positive imaginary axis. And as for the Green's functions employed chapter 4, it is necessary to account for the poles on the real axis in evaluating the integral. And this could be achieved by application of residue theory as in Chapter 4. However when there are several poles on the real axis this becomes quite an arduous task.

Fortunately however, by employing a very elegant transform developed by McIver & Bennett [38] to the integral in the definition of \(G_1\) this problem can be simply overcome. This transform is (reformulating it in terms of present problem),

\[ t = e^{-i\pi/4} \left\{ \frac{s}{kd} \left( 2 + \frac{is}{kd} \right) \right\}^{1/2}, \gamma = \frac{s}{kd} - i. \]

And note in the \(s\) plane all singularities of the integrand in \(G_1\) lie on the imaginary axis and, in particular the poles on the \(Re\) axis are transformed onto \(0 \leq Re s \leq k\). Further The path of integration in the \(s\) plane runs from the origin to \(ik\) along the imaginary axis, passing to the right of any poles, and then moves off the axis along \(s = ik\) in the positive \(Re s\) direction. And as all of the poles are on the imaginary axis, the path of integration may (and is) now be deformed to run along the positive \(Re s\) axis which is free of singularities. And thus the transformed integral can be simply truncated for a large value and evaluated using numerical integration as is done in the program described above.

5.4 Results And Analysis

On the basis of the above solution, the performance of rectangular porous arrays in shallow water has been investigated. Of principal interest in this, has been to ascertain the relative importance of porosity, permeability and gap size on the performance of segmented breakwaters. And this has been achieved by consideration of the reflected and transmitted energies computed from the solution (as done with the solutions for the other structures presented in the previous chapters) for a range of arrays and incident waves.
Reflection and Transmission Coefficient Calculation and Validation

As done for the structures considered in the previous chapters, reflection and transmission from the porous array are studied by consideration of the reflection and transmission coefficients. Specifically recall the reflection coefficients $R_m$ and transmission coefficients $T_m$ represent the coefficients of the eigenfunction solution expressions (5.41) and (5.42) respectively. Further note by application of the orthogonality condition (5.43) to equations (5.41) and (5.42), the expressions,

$$R_m = -\delta_{om} + \frac{1}{2d} \int_{-d}^{d} \phi_1(0, y)e^{-i\beta_m y} dy = \delta_{om} - \frac{1}{i2\alpha_m d} \int_{-d}^{d} \frac{\partial \phi_1}{\partial x}(0, y)e^{-i\beta_m y} dy$$  \hspace{1cm} (5.76)

$$T_m = \frac{1}{2d} \int_{-d}^{d} \phi_1(b, y)e^{-i\beta_m y} dy = \frac{1}{i2\alpha_m d} \int_{-d}^{d} \frac{\partial \phi_1}{\partial x}(b, y)e^{-i\beta_m y} dy,$$ \hspace{1cm} (5.77)

can be obtained. Recall such expressions were employed in calculation of the reflection and transmission coefficients in Chapters 3 and 4.

The expressions (5.76) and (5.77) could also be employed here to calculate the coefficients. However recall in the integral equation solution described above, the pressure on the block, $P(s)$, and the fluid velocity normal to the block, $V(s)$ are determined. And consequently to evaluate the expressions (5.76) and (5.77) from $P(s)$ and $V(s)$, it is necessary to use equation (5.56) to compute $\phi_1(0, y)$ (or $\partial \phi_1/\partial x(0, y)$) and $\phi_1(b, y)$ (or $\partial \phi_1/\partial x(b, y)$) on the intervals $y \in (-d, -l)$ and $y \in (l, d)$. However in view that the coefficients of the non-evanescent modes of (5.41) and (5.42) are of most interest, a different more efficient approach is employed. Specifically note by applying Green’s theorem to $\phi_1$ and the functions $e^{i(\alpha_m z - \beta_m y)}$ and $e^{-i(\alpha_m z + \beta_m y)}$ respectively on the region between $S$ and $L$, employing the eigenfunction expressions (5.44) and (5.49) for the solution on $L_1$ and $L_3$ respectively, the expressions

$$R_m = \frac{1}{4i\alpha_m d} \int_S \left( P(s) \frac{\partial}{\partial n} (e^{i(\alpha_m z - \beta_m y)}) - e^{i(\alpha_m z - \beta_m y)} V(s) \right) ds,$$ \hspace{1cm} (5.78)

and

$$T_m = \frac{e^{i\alpha_m b}}{4i\alpha_m d} \left\{ 4i\alpha_0 d \delta_{om} + \int_S \left( P(s) \frac{\partial}{\partial n} (e^{-i(\alpha_m z + \beta_m y)}) - e^{-i(\alpha_m z + \beta_m y)} V(s) \right) ds \right\},$$ \hspace{1cm} (5.79)

are obtained for the coefficients of non-evanescent modes ($-r \leq n \leq s$). And clearly computation of the reflection and transmission coefficients for the non-evanescent modes with (5.78) and (5.79) is more efficient than with (5.76) and (5.77) since no additional determination of $\phi_1$ away from the block is necessary.

A program has been written to implement both the numerical solution of the integral equations (5.58) and (5.59) given above and compute the reflection and transmission coefficients (for non-evanescent modes) using equations (5.78) and (5.79) respectively.
To verify the program and calibrate the number of boundary elements required in the integral equation solution and truncations in Green’s function computations, three tests have been employed. Firstly the program has been verified for the impermeable case ($\epsilon = 0$) against the impermeable array specific code described in Chapter 3 and the results of Fernyhough & Evans [20]. Specifically note for this special case,

$$\Phi(x, y, z) = \frac{iA g \cosh k(z + h)}{\omega \cosh kh} \phi_1(x, y)$$  \hspace{1cm} (5.80)

is a solution of the full boundary value problem (5.2)-(5.11) and consequently the program should reproduce identically the reflection and transmission coefficients generated by Fernyhough and Evans [20] or the impermeable array code. (Note the program is demonstrated to satisfy this condition in figure 5.4 which there is no discernable difference between the impermeable and permeable codes). Secondly, note by application of Green’s theorem to $\phi_1$ and its complex conjugate $\bar{\phi}_1$ on the region between $S$ and $L$ (again employing the eigenfunction solutions (5.44) and (5.49) on $L_1$ and $L_2$ respectively), the energy relation

$$\Re m \int_S \phi_1 \frac{\partial \bar{\phi}_1}{\partial n} \, ds = 2d\{-\alpha_0 + \sum_{m=-r}^{s} \alpha_m (|R_m|^2 + |T_m|^2)\},$$  \hspace{1cm} (5.81)

is obtained. (Compare this with the relation (3.73) found in Chapter 3.) And this relation has been verified for all reflection and transmission coefficients computed.

Thirdly and finally the program has been verified by comparison of reflection and coefficients computed with $m$ and $2m + 1$ elements per side. And this has been used as criterion to determine the number elements to use. Specifically when the reflection/transmission coefficients coincide for $m$ and $2m + 1$ elements respectively, the larger number of elements has been taken.

And on the basis of this analysis it is found that between 17 and 21 boundary elements per side and truncation of the transformed green’s function integral at $s = 40$ gives 2-3 decimal places of accuracy in the coefficients for $0 \leq kd \leq 5$. 

![Graph showing reflection and transmission coefficients](image)
Figure 5.4: $|R|$ and $|T|$ plotted against angle of incidence $\theta_0$, for $\delta = 1 + i$, $\epsilon = 0$, $kd = 0.5$, (a) “$b/d = 0.7072$, $l/d = 0.3536$” and (b) “$b/d = 1$, $l/d = 0.5$”. The results generated by the impermeable array code used in Chapter 3 and the porous array code of the current chapter are represented by the impermeable and broken lines respectively.

With the goal of understanding the influences of gap size, porosity and the friction factor on the performance of the porous arrays, reflection and transmission by the nine array/incident wave configurations listed in Table 5.1 have been considered. Note that typical permeable breakwaters have porosities $\epsilon \approx 0.4$ and that $f = 1, 3, 5$ represents a range of friction factors considered by Dalrymple et al. [13].

<table>
<thead>
<tr>
<th>$b/d$</th>
<th>$l/d$</th>
<th>$\delta$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.25</td>
<td>0.5, 0.75, 0.875, 1</td>
<td>$1 + i$</td>
</tr>
<tr>
<td>2</td>
<td>0.25</td>
<td>0.5, 0.75, 0.875, 1</td>
<td>$1 + i$, $1 + 3i$, $1 + 5i$</td>
</tr>
<tr>
<td>3</td>
<td>0.25</td>
<td>0.5, 0.75, 0.875, 1</td>
<td>$1 + i$, $1 + 3i$, $1 + 5i$</td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
<td>0.5, 0.75, 0.875, 1</td>
<td>$1 + 10i$</td>
</tr>
</tbody>
</table>

Table 5.1: Arrays and Blocks Considered

For the blocks without gaps ($l/d = 1$) note the reflection and transmission coefficients are computed using the plane wave approximation expressions,

$$R_0 = \frac{i(m^2 - 1) \sin[(K^2 - \lambda^2)^{1/2} b]}{2m \cos[(K^2 - \lambda^2)^{1/2} b] - i(1 + m^2) \sin[(K^2 - \lambda^2)^{1/2} b]},$$

$$T_0 = \frac{2m}{2m \cos[(K^2 - \lambda^2)^{1/2} b] - i(1 + m^2) \sin[(K^2 - \lambda^2)^{1/2} b]},$$

based on the expressions derived by Dalrymple et al. [13, Equations (4.2)], where $m$, known as the dimensionless admittance of the block is defined by

$$m = \frac{\epsilon}{\delta} \left(\frac{K^2 - \lambda^2}{k^2 - \lambda^2}\right)^{1/2}.$$ 

Note for consistency with the present solution, expressions (5.82)-(5.84) have been re-cast from those derived by Dalrymple et al. [13, Equations (4.2)] in terms of the $e^{-i\omega t}$ solution convention. And note also according to this convention, the branch of the square roots in (5.82)-(5.83) with negative imaginary part is taken. (Recall from equations (2.63) and (2.64) that this ensures energy is dissipated in a porous block). Finally note that in shallow water limit, the plane wave and shallow water approximation solutions for the block are identical. And consequently it’s consistent to compare reflection and transmission computed
using (5.82) and (5.83) and the present array solution for similar blocks and arrays in shallow water.

In figures 5.5 to 5.22 the reflection and transmission coefficients (and where appropriate the reflection and transmission energies) are plotted for each of the array/incident wave configurations listed in Table 5.1. To illustrate clearly the relative influences of the porosity, friction factor and gap size on the performance of arrays, two figures are plotted for each row of the table. Specifically, in the first figure, the graphs are collated according to gap length (in each graph, it is constant) permitting straightforward inspection of the influence of porosity (friction factor) for arrays of each gap size. And in the second figure, the graphs are collated according to porosity (friction factor), permitting straightforward inspection of influence of gap length for arrays of each porosity. And further this system of collation seems to provide the best basis of comparison of all the arrays considered.

In figures 5.5 and 5.6 the magnitudes of the reflection and transmission coefficients \( R(= R_0) \) and \( T(= T_0) \) are plotted against angle of incidence \( \theta_0 \) for \( kd = 0.5 \) and the parameters in the first row of the table 5.1. Note for \( kd = 0.5 \) there is only one travelling mode according to equations (5.46) and (5.47).

To start with, observe in figure 5.5 (5.6) that the individual plots of \( |R| \) and \( |T| \) for the permeable arrays have much in common with those for the impermeable arrays and permeable blocks of similar dimensions also plotted. In particular, note \( |R| \) strictly decreases as \( \theta_0 \) increases until an angle of zero reflection is reached. Beyond this angle, note that \( |R| \) increases rapidly, tending to 1 as \( \theta_0 \) tends to 90 degrees. Also note that \( |T| \) increases as \( \theta_0 \) increases until a maximum is reached at the angle of zero reflection. Note for the impermeable arrays this maximum is 1 corresponding to complete transmission. However for the permeable block and arrays this maximum is less than 1. And note after this angle, \( |T| \) decreases rapidly, tending to 0 as \( \theta_0 \) reaches 90 degrees.

Next observe in figure 5.5 the effect of varying porosity on the performance of the arrays of each gap size. Firstly note that the angle of zero reflection decreases with increasing porosity. Secondly note considering arrays with the same dimensions but two porosities (for example in 5.5(a)) that: \( |R| \) is lower and \( |T| \) higher for the array with the larger porosity provided the angle of incidence is not in the vicinity of the angle of zero reflection for that porosity. And note in the vicinity of the angle of zero reflection of the array with larger porosity, this trend reverses: \( |R| \) is higher and \( |T| \) lower for the array with the larger porosity. And observe in figure 5.5 that the effects of porosity become more pronounced the smaller the gap size is.
And observe in figure 5.6 the effect of varying gap size on the performance of the arrays of each porosity. Firstly note the angle of zero reflection also decreases with increasing gap length (and hence this depends both on porosity and gap length). Secondly note considering arrays with two gap sizes: |\( R \)| is lower and |\( T \)| higher for the larger gap size provided the angle of incidence is not in the vicinity of the angle of zero reflection for that gap size. Note in the vicinity of the angle of zero reflection of the array with larger gap size, this trend reverses: |\( R \)| is higher and |\( T \)| lower for the array with the larger gap size. And observe in figure 5.6 that the effects of gap size become more pronounced the smaller the porosity is.

Finally compare the respective values of |\( T \)| for the permeable and impermeable arrays for each gap length in figures 5.5 (5.6). Note clearly the transmission is the key indicator of performance of these structures as breakwaters. And observe that other than reducing the peak transmission, the dissipative property of the permeable arrays considered here offer little or no advantage in performance over the impermeable arrays of the same dimensions. However such comparison indicates that the influence of the dissipative effects of the medium in the permeable arrays is gap size dependent. Specifically note in figure 5.5(a), all the permeable arrays perform slightly better than the impermeable array. In figure 5.5(b) both the \( \epsilon = 0.4 \) and \( \epsilon = 0.6 \) arrays perform worse than the impermeable array and the \( \epsilon = 0.2 \) has equivalent performance. And for cases 5.5(c) the impermeable array completely outperforms the porous arrays. And hence it can be concluded that dissipative effects of permeable arrays are most (least) influential the larger (smaller) the gap size is.

In figures 5.7 and 5.8 the magnitudes of the reflection and transmission coefficients \( R(= R_0) \) and \( T(= T_0) \) are plotted against angle of incidence \( \theta_0 \) for \( kd = 1.0 \) and the parameters in the first row of the table 5.1. Note for \( kd = 1.0 \) there is again only one travelling mode according to equations (5.46) and (5.47).

Comparison of the figures 5.7 (5.8) with figures 5.5 (5.6) reveals some of the features of the effect of wave frequency on the performance of permeable arrays. To start with, note that the general features of the individual plot of |\( R \)| and |\( T \)| for \( kd = 1.0 \) and \( kd = 0.5 \) are much the same. Next observe that the angle of zero reflection appears to be independent of \( kd \). And observe that for \( kd = 1.0 \) the values of |\( R \)| are generally higher and of |\( T \)| are generally lower than those obtained for \( kd = 0.5 \). There are two good physical justifications for this phenomenon. Firstly, the wavelength of the incident wave train is smaller relative to the array width for \( kd = 1.0 \) and consequently the wave is more prone to diffraction than in the previous case. Secondly, since the wavelength is smaller relative to the array (block)
width, more energy will be dissipated by the porous medium of the structure (see equation (2.64)). Finally note for \( kd = 1.0 \), the impermeable arrays perform much better relative to the permeable arrays for each gap size than for \( kd = 0.5 \).

In figures 5.9 and 5.10, the total reflected and transmitted energies \( R_{\text{total}} \) and \( T_{\text{total}} \) (defined by equations (3.74) and (3.75) respectively) are plotted against \( kd \) for \( \theta_0 = 30 \) degrees and the parameters in the first row of table 5.1. Note for the range of \( kd \) considered additional travelling modes appear in the array solutions at the cut-off frequencies \( kd = 2\pi/3 \) and \( kd = 4\pi/3 \) according to equations (5.46) and (5.47). And hence for the reasons discussed in Chapter 3, plots of the total reflected and transmitted energies are more informative than those for the individual coefficients. Note for the permeable block that \( R_{\text{total}} \) and \( T_{\text{total}} \) are computed using

\[
R_{\text{total}} = |R_0|^2, \tag{5.85}
\]
\[
T_{\text{total}} = |T_0|^2 \tag{5.86}
\]

with \( R_0 \) and \( T_0 \) are computed using equations (5.82) and (5.83) since according to the eigenfunction solution of Dalrymple et al. [13] there is always only one non-evanescent mode for the permeable block.

To start with, observe in figure 5.9, the individual plots of \( R_{\text{total}} \) and \( T_{\text{total}} \) against \( kd \) for the permeable arrays, have much in common (like those in figures 5.5-5.8) with those for the impermeable arrays and permeable blocks of similar dimensions. Observe that the general trend is that \( R_{\text{total}} \) increases and \( T_{\text{total}} \) decreases with increasing \( kd \) with \( R_{\text{total}} \) starting at zero and \( T_{\text{total}} \) at one respectively. Note in addition that there are similar spikes for permeable arrays at the cut-off frequencies (as one might reasonably expect) as there for the impermeable arrays, although these are much less pronounced.

Next observe in figure 5.9 the effect of varying porosity on the performance of the permeable arrays of each gap size with respect to \( kd \). Note for the permeable arrays the general trend is the same as with the porous block (and also as in figures 5.5 and 5.7). In particular increasing the porosity reduces \( R_{\text{total}} \) and increases \( T_{\text{total}} \) respectively. And note as in figures 5.5 and 5.7 the effects of variation of porosity become more pronounced the smaller the gap size is.

And observe in figure 5.10 the effect of varying gap size on the performance of the arrays of each porosity with respect to \( kd \). Note for the permeable arrays the general trend is the same as with the impermeable (and also as in figures 5.5 and 5.7). In particular increasing
the gap size reduces $R_{total}$ and increases $T_{total}$ respectively. And note as in figures 5.6 and 5.8 the effects of variation of gap size become more pronounced the smaller the porosity.

Next compare the respective values of $T_{total}$ for the permeable and impermeable arrays for each gap length in figure 5.9β. Recall the transmission is the key indicator of performance of these structures as breakwaters. And observe in the figures there are certain regimes of $kd$ where the permeable arrays perform better than the impermeable arrays and vice-versa and associated cross-over values of $kd$ where the behaviour swaps over.

Specifically observe in figure 5.9β (a) that for $kd \leq 1.2$ each of permeable arrays considered performs better than the impermeable array of the same dimensions. And note at $kd \approx 2$ the plot of $T_{total}$ for impermeable array crosses over the permeable array plot for porosity $\epsilon = 0.6$. Then for $2 \leq kd \leq 4$ the impermeable array performs better than the all the permeable arrays considered until $kd = 4$ beyond which the $\epsilon = 0.2$ array and the impermeable array perform equivalently for remainder of the range of $kd$ considered.

The various regimes of $kd$ where the performance of the permeable array is better than that of the impermeable arrays and vice-versa can be interpreted by consideration of porous medium dissipation and diffraction effects. In particular note when $kd$ is small, diffraction is limited and the main reduction of the transmitted energy is via dissipation in the structure. However note as $kd$ increase, diffraction starts to play a larger role in the performance, resulting here in lower $T_{total}$ for the impermeable arrays than the permeable arrays. And note for second regime after the second cross-over (seen in figures 5.9β (a) and (b) $\epsilon = 0.2$) the dissipative effects become as influential as the diffraction. Finally observe that as the gap size decreases the value of $kd$ for the first cross-over for each porosity decreases.

In figures 5.11 and 5.12 the magnitudes of the reflection and transmission coefficients $R(=R_0)$ and $T(=T_0)$ are plotted against angle of incidence $\theta_0$ for $kd = 0.5$ and the parameters in the second row of the table 5.1. Note for $kd = 0.5$ there is only one travelling mode according to equations (5.46) and (5.47). And these graphs explore the influence of friction factor on the performance of permeable arrays. (Note for comparison the impermeable array of each gap size are also plotted).

To start with observe that the individual plots of $|R|$ and $|T|$ in figures 5.11 and 5.12 are much as those described in figures 5.7. However note for various values of friction factor the minimum of $|R|$ is greater than zero.
Next, consider the effect of varying the friction factor on the performance of the arrays for each gap size against angle (as illustrated in figure 5.11). Observe firstly that the effect of increasing the friction factor is to increase $|R|$ and decrease $|T|$. (This is to be expected as the friction factor is the principal dissipative parameter in the Sollitt & Cross model). And note this is not subject to reversal near the minimum of reflections as for porosity graphs. Also observe that as the friction increases both the minimum of $|R|$ and the angle at which it occurs increases. And observe in figure 5.11 that the effects of friction become more pronounced the smaller the gap size is.

Next, consider the effect of varying the gap length on the performance of the arrays for each value of the friction factor against angle (as illustrated in figure 5.12). Observe firstly that the effect of decreasing the gap length is also to increase $|R|$ and decrease $|T|$. And note this is not subject to reversal near the minimum of reflections as for porosity graphs. Also observe that as the gap length decreases both the minimum of $|R|$ and the angle at which it occurs increases. And observe in figure 5.12 that the effects of gap length become more pronounced the higher the friction factor is.

Finally compare the respective values of $|T|$ for the permeable and impermeable arrays for each gap length in figures 5.11. It appears there seems to be a critical friction for each gap size in order that the permeable arrays perform better than the impermeable arrays. And note that these graphs confirm again that the dissipative effects of the medium are most pronounced when the gap size is largest.

In figures 5.13 and 5.14 the magnitudes of the reflection and transmission coefficients $R(= R_0)$ and $T(= T_0)$ are plotted against angle of incidence $\theta_0$ for $kd = 1.0$ and the parameters in the second row of the table 5.1. Note in comparison with figures 5.11 and 5.12 these graphs consider the influence of higher frequency on the performance of arrays with parameter in row 2 of 5.1. As observed in comparison of figure 5.7 with figure 5.5 the effect of higher frequency is principally to increase $|R|$ and decrease $|T|$ for each of the arrays considered (for the reasons described in Figure 5.7). And note as before the impermeable arrays considered perform better against the permeable arrays at this frequency.

In figures 5.15 and 5.16 the magnitudes of $R_{\text{total}}$ and $T_{\text{total}}$ are plotted against $kd$ for $\theta = 30$ degrees and the parameters in the third row of table 5.1. Note as in figures 5.9 and 5.10 for the range of $kd$ considered additional travelling modes appear in the array solutions at the cut-off frequencies $kd = 2\pi/3$ and $kd = 4\pi/3$ according to equations (5.46) and (5.47).
Observe again in figure 5.15 that as the friction is increased the reflection increases and the transmission decreases and that this phenomenon is most pronounced the smaller the gap size. And in figure 5.16 observe again that as the gap size decreases that the reflection increases and the transmission decreases and that this most pronounced the higher the friction factor. Also observe again the various regimes where the permeable arrays perform better than the impermeable arrays (and refer to the discussion of figure 5.9 and 5.10 for discussion of this phenomenon).

In figures 5.17 and 5.18 the magnitudes of the reflection and transmission coefficients \( R(= R_0) \) and \( T(= T_0) \) are plotted against angle of incidence \( \theta_0 \) for \( kd = 0.5 \) and the parameters in the fourth row of the table 5.1. Note again for \( kd = 0.5 \) there is only one travelling mode according to equations (5.46) and (5.47). And Figure 5.17 illustrate some interesting features of arrays and blocks with higher friction factors.

Observe firstly that the reflection coefficients for the permeable arrays for each gap size follows the same trends as the permeable block (and as described in figure 5.5) with one difference. In particular note that as the porosity increases the minimum of the reflection increases and the angle at which it occurs decreases. (Compare this with the permeable block where the angle of the minimum decreases but there is no change in the minimum itself).

Observe next that the plots of transmission for the permeable arrays are quite different. Observe that the the relative plots for each porosity of transmission are very close to one another for each gap size. And additionally observe that the higher porosity the lower the transmission. And both of these trends are in marked contrast to figures 5.5 and 5.7 and 5.17(d). (Observe also in contrast that against gap size for each porosity the graphs are arranged as usual).

These phenomena might be explained as follows. Note when the friction factor is high enough for the wave permeable array interaction, the array performs more like a solid array. And in particular the wave energy prefers to transmit through the gaps rather than through the medium. Secondly in these circumstances the higher the porosity the more likely for the wave to enter into the array permeable blocks and be dissipated. Note for figure 5.17

Observe that in in figure 5.18 the same trends with gap size are observed as in figure 5.6 and 5.8 are repeated And note these phenomena are observed again for higher frequencies in figures 5.19-5.22.
Figure 5.5: $|R|$ and $|T|$ against $\theta_0$ for $kd = 0.5$, $\delta = 1 + i$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and (i) $\epsilon = 0$ (---), (ii) $\epsilon = 0.2$ (---), (iii) $\epsilon = 0.4$ (---), (iv) $\epsilon = 0.6$ (---).

Figure 5.6: $|R|$ and $|T|$ against $\theta_0$, for $kd = 0.5$, $\delta = 1 + i$, $b/d = 0.25$, for (a) $\epsilon = 0$, (b) $\epsilon = 0.2$, (c) $\epsilon = 0.4$, (d) $\epsilon = 0.6$ and (i) $l/d = 0.5$ (---), (ii) $l/d = 0.75$ (---), (iii) $l/d = 0.875$ (---), (iv) $l/d = 1$ (---).
Figure 5.7: $|R|$ and $|T|$ against $\theta_0$ for $kd = 1.0$, $\delta = 1 + i$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and (i) $\epsilon = 0$ (-----), (ii) $\epsilon = 0.2$ (-----), (iii) $\epsilon = 0.4$ (-----), (iv) $\epsilon = 0.6$ (-----).

Figure 5.8: $|R|$ and $|T|$ against $\theta_0$, for $kd = 1.0$, $\delta = 1 + i$, $b/d = 0.25$, for (a) $\epsilon = 0$, (b) $\epsilon = 0.2$, (c) $\epsilon = 0.4$, (d) $\epsilon = 0.6$ and (i) $l/d = 0.5$ (-----), (ii) $l/d = 0.75$ (-----), (iii) $l/d = 0.875$ (-----), (iv) $l/d = 1$ (-----).
Figure 5.9a: $R_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $\delta = 1 + i$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and (i) $\epsilon = 0$ (-----), (ii) $\epsilon = 0.2$ (---), (iii) $\epsilon = 0.4$ (----), (iv) $\epsilon = 0.6$ (-----).

Figure 5.9b: $T_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $\delta = 1 + i$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and (i) $\epsilon = 0$ (-----), (ii) $\epsilon = 0.2$ (---), (iii) $\epsilon = 0.4$ (----), (iv) $\epsilon = 0.6$ (-----).
Figure 5.10a: $R_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $\delta = 1 + i$, $b/d = 0.25$, for (a) $\epsilon = 0$, (b) $\epsilon = 0.2$, (c) $\epsilon = 0.4$, (d) $\epsilon = 0.6$ and (i) $l/d = 0.5$ (-----), (ii) $l/d = 0.75$ (-----), (iii) $l/d = 0.875$ (-----), (iv) $l/d = 1$ (-----).

Figure 5.10b: $T_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $\delta = 1 + i$, $b/d = 0.25$, for (a) $\epsilon = 0$, (b) $\epsilon = 0.2$, (c) $\epsilon = 0.4$, (d) $\epsilon = 0.6$ and (i) $l/d = 0.5$ (-----), (ii) $l/d = 0.75$ (-----), (iii) $l/d = 0.875$ (-----), (iv) $l/d = 1$ (-----).
Figure 5.11: $|R|$ and $|T|$ against $\theta_0$ for $kd = 0.5$, $s = 1$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ with for $\epsilon = 0.5$: (i) $f = 5$ (---), (ii) $f = 3$ (-- --), (iii) $f = 1$ (-----) and (iv) $\epsilon = 0$ (------).

Figure 5.12: $|R|$ and $|T|$ against $\theta_0$, for $kd = 0.5$, $s = 1$, $\epsilon = 0.4$, $b/d = 0.25$, for (a) $f = 1$, (b) $f = 3$, (c) $f = 5$, (d) and (i) $l/d = 0.5$ (-----), (ii) $l/d = 0.75$ (-----), (iii) $l/d = 0.875$ (------), (iv) $l/d = 1$ (-----).
Figure 5.13: $|R|$ and $|T|$ against $\theta_0$ for $kd = 1.0$, $s = 1$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and with for $\epsilon = 0.5$: (i) $f = 5$ (-----), (ii) $f = 3$ (-----), (iii) $f = 1$ (-----) and (iv) $\epsilon = 0$ (-----).

Figure 5.14: $|R|$ and $|T|$ against $\theta_0$, for $kd = 1.0$, $s = 1$, $\epsilon = 0.5$, $b/d = 0.25$, for (a) $f = 1$, (b) $f = 3$, (c) $f = 5$, (d) and (i) $l/d = 0.5$ (-----), (ii) $l/d = 0.75$ (-----), (iii) $l/d = 0.875$ (-----), (iv) $l/d = 1$ (-----).

93
Figure 5.15α : $R_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $s = 1$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and for $\epsilon = 0.4$: (i) $f = 5$ (---), (ii) $f = 3$ (----), (iii) $f = 1$ (-----) and (iv) $\epsilon = 0$ (———).

Figure 5.15β : $T_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $s = 1$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and for $\epsilon = 0.4$: (i) $f = 5$ (---), (ii) $f = 3$ (----), (iii) $f = 1$ (-----) and (iv) $\epsilon = 0$ (———).
Figure 5.16a: $R_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $s = 1$, $\epsilon = 0.4$, $b/d = 0.25$, for (a) $f = 1$, (b) $f = 3$, (c) $f = 5$ and (i) $l/d = 0.5$ (- - - - -), (ii) $l/d = 0.75$ (---), (iii) $l/d = 0.875$ (-----), (iv) $l/d = 1$ (-----).

Figure 5.16b: $T_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $s = 1$, $\epsilon = 0.4$, $b/d = 0.25$ for (a) $f = 1$, (b) $f = 3$, (c) $f = 5$ and (i) $l/d = 0.5$ (- - - - -), (ii) $l/d = 0.75$ (---), (iii) $l/d = 0.875$ (-----), (iv) $l/d = 1$ (-----).
Figure 5.17: $|R|$ and $|T|$ against $\theta_0$ for $kd = 0.5$, $\delta = 1 + 10i$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and (i) $\epsilon = 0$ (——), (ii) $\epsilon = 0.2$ (— — —), (iii) $\epsilon = 0.4$ (— — — —), (iv) $\epsilon = 0.6$ (— — — — —).  

Figure 5.18: $|R|$ and $|T|$ against $\theta_0$, for $kd = 0.5$, $\delta = 1 + 10i$, $b/d = 0.25$, for (a) $\epsilon = 0$, (b) $\epsilon = 0.2$, (c) $\epsilon = 0.4$, (d) $\epsilon = 0.6$ and (i) $l/d = 0.5$ (— — — — —), (ii) $l/d = 0.75$ (— — —), (iii) $l/d = 0.875$ (— — —), (iv) $l/d = 1$ (——).
Figure 5.19: $|R|$ and $|T|$ against $\theta_0$ for $kd = 1.0, \delta = 1 + 10i, b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and (i) $\epsilon = 0$ (---), (ii) $\epsilon = 0.2$ (----), (iii) $\epsilon = 0.4$ (-----), (iv) $\epsilon = 0.6$ (--------).

Figure 5.20: $|R|$ and $|T|$ against $\theta_0$, for $kd = 1.0, \delta = 1 + 10i, b/d = 0.25$, for (a) $\epsilon = 0$, (b) $\epsilon = 0.2$, (c) $\epsilon = 0.4$, (d) $\epsilon = 0.6$ and (i) $l/d = 0.5$ (-----), (ii) $l/d = 0.75$ (-----), (iii) $l/d = 0.875$ (-----), (iv) $l/d = 1$ (-----).
Figure 5.21a: $R_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $\delta = 1 + 10i$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and (i) $\epsilon = 0$ (-----), (ii) $\epsilon = 0.2$ (----), (iii) $\epsilon = 0.4$ (-----), (iv) $\epsilon = 0.6$ (-----).

Figure 5.21b: $T_{total}$ against $kd$ for $\theta_0 = 30^\circ$, $\delta = 1 + i$, $b/d = 0.25$, for (a) $l/d = 0.5$, (b) $l/d = 0.75$, (c) $l/d = 0.875$, (d) $l/d = 1$ and (i) $\epsilon = 0$ (-----), (ii) $\epsilon = 0.2$ (----), (iii) $\epsilon = 0.4$ (-----), (iv) $\epsilon = 0.6$ (-----).
Figure 5.22α: $R_{\text{total}}$ against $kd$ for $\theta_0 = 30^\circ$, $\delta = 1 + 10i$, $b/d = 0.25$, for (a) $\epsilon = 0$, (b) $\epsilon = 0.2$, (c) $\epsilon = 0.4$, (d) $\epsilon = 0.6$ and (i) $l/d = 0.5$ (--- ---), (ii) $l/d = 0.75$ (--- ---), (iii) $l/d = 0.875$ (--- ---), (iv) $l/d = 1$ (-----).

Figure 5.22β: $T_{\text{total}}$ against $kd$ for $\theta_0 = 30^\circ$, $\delta = 1 + 10i$, $b/d = 0.25$, for (a) $\epsilon = 0$, (b) $\epsilon = 0.2$, (c) $\epsilon = 0.4$, (d) $\epsilon = 0.6$ and (i) $l/d = 0.5$ (--- ---), (ii) $l/d = 0.75$ (--- ---), (iii) $l/d = 0.875$ (--- ---), (iv) $l/d = 1$ (-----).
5.5 Conclusion

In this chapter the interaction of water waves with periodic arrays of permeable blocks in shallow water has been considered with the Sollitt & Cross model. Using the model a boundary value problem for the interaction was formulated and by the employment of a shallow water approximation, the number of spacial coordinates in this, was reduced from three to two. Further the reduced boundary value problem has been solved using a Green's function and integral equation approach. (Note the boundary integral equations obtained are solved using a collocation scheme informed by the one employed in Chapter 3.) And finally from this solution, the non-evanescent reflection and transmission coefficients have been computed.

On the basis of consideration of the reflection and transmission coefficients for various wave frequencies and arrays of varying dimensions, the general performance of arrays of permeable blocks have been investigated. This investigation has been based principally on three comparisons. Firstly the performance of the permeable arrays and impermeable arrays of the same dimensions have been compared. Secondly the performance of permeable arrays and permeable blocks of the same thickness and dissipative properties (that is, equal friction factor $f$ and porosity $\epsilon$) have been compared. And finally the performance of permeable arrays of various dimensions and dissipative properties have been compared.

And on the basis of the graphs plotted several observations have been made. Firstly it is observed that in contrast to impermeable arrays, permeable arrays prevent the phenomenon of complete transmission. For the permeable arrays it is observed for moderate friction factor that increasing the porosity reduces reflection and increases transmission (as with the permeable block). It is also observed that the higher the friction factor the lower the transmission and the higher the reflection (again as with the permeable block). Next it has been observed that the smaller the gap size the more pronounced the differences between plots of various porosities and friction factors. However it is noted by contrast that the dissipative effect on the performance of permeable arrays are most pronounced when the the gap size is large (or more importantly when the blocks are small and diffraction effects diminished).

And comparing the performance of impermeable and permeable arrays of the same dimension against frequency, it is observed that there regimes where the impermeable arrays perform better than the permeable arrays and vice-versa. Most obvious of these regimes is for small frequency, where the permeable arrays perform better than the permeable arrays as diffraction effects are minimal. Finally it is observed in the later figures that for higher frictions the permeable arrays behave in a different way. In particular it was observed that the plots of
transmission cluster close together and the higher the porosity the lower the transmission.

From a mathematical point of view there seems to be two obvious enhancements in the present work that can be made. Note firstly note in the plots against angle of incidence the program (or more particularly the Green’s function computation) fails for angles greater than 86 degrees. And it observed that the Green’s function computations could be improved directly by some asymptotic consideration of the integral (note at present the infinite integral are just truncated at an appropriate point). Alternatively the more efficient Green’s function computation algorithms of Linton [33] using Ewald’s method might be implemented. Secondly more efficient integral equation solution might be pursued (using Galerkin’s method or more sophisticated boundary element methods such as described in the next chapter).

From a physical point of view there are three obvious enhancements that can be made. Firstly note on the basis of energy relation (5.81) it is observed that the quantity

$$E_D = \Im \int_S \phi_1 \frac{\partial \phi_1}{\partial n} \, ds = 2d\{-\alpha_0 + \sum_{m=-r}^{s} \alpha_m(|R_m|^2 + |T_m|^2)}$$

represents the energy dissipation by the array. And hence represents an interesting quantity that might be plotted.

Secondly note as observed by Sulisz [60] the friction factor has a strong dependence (see equation (2.21)) on frequency. Therefore Sulisz recommends the use of the dimensionless friction factor,

$$\hat{f} = \omega \sqrt{h/gf},$$

instead. And this parameter, as Sulisz describes, has a much weaker dependence on frequency (although it still varies slowly with it) than the friction factor. Hence the dimensionless friction factor is a much more accurate indicator of the dissipative property of a given permeable structure in respect of incident waves of any frequency. And consequently, plots such as figures 5.9-5.10 would become more representative of individual permeable structures if constant values of the dimensionless friction factor $\hat{f}$ rather than the friction factor $f$ were used.

Thirdly note the relationship between the friction factor and porosity has been for simplicity overlooked in this study. However observe that Madsen [40, page 179] has obtained, on the basis of an energy argument and a shallow water approximation, a closed form expression for the friction factor $f$. This is

$$f = \frac{\epsilon}{kb} \left[ - \left( 1 - \frac{kbA}{2\omega} \right) + \sqrt{\left( 1 + \frac{kb\alpha}{2\omega} \right)^2 + \frac{16B}{3\pi} \frac{a_1 b}{h}} \right].$$
where $a_i$ denote the amplitude of the incident wave and $A$ and $B$ are the coefficients in the porous medium flow equation (2.15). (Note in the context of the Sollitt & Cross equation (2.13) $A = \nu \epsilon / K_p$ and $B = C_f \epsilon^2 / \sqrt{K_p}$). And this expression can be used to study this relationship. And also the influences of permeability (adopting Sollitt & Cross's flow equation) and porosity on the performances of periodic arrays could be investigated.
Chapter 6

Conclusion

6.1 Summary

In this thesis, four aspects of mathematical modelling of offshore breakwaters have been considered. In chapter 2 the formulation of the Sollitt & Cross [56] model has been reviewed and its limitations and possible improvements (via use of more accurate flow equations) outlined. In chapter 3 diffraction by a periodic array of impermeable blocks was investigated by formulation and solution of an appropriate diffraction problem. Results presented compare favourably with those of Fernyhough & Evans [20] which appeared concurrently. In chapter 4 a new solution of the diffraction problem for an infinitely long rectangular porous block originally formulated and solved by Dalrymple et al. [13] is presented. The solution here employed a Green's function / integral equation approach similar to one used by Sulisz [60] instead of the matched eigenfunction approach employed by Dalrymple et al. [13]. This approach has the advantage over the eigenfunction approach that it is not necessary to solve the porous dispersion relationship. In addition the Green’s function employed here perhaps have an advantage over that employed by Sulisz [60], in satisfying the free surface and bed conditions (for constant depth) and additionally incorporating obliquely incident waves. Finally diffraction by a periodic array of permeable blocks in shallow water, representative of a segmented offshore rubble mound breakwater, has been considered. Results may be broadly summarized as follows. Reflection by a permeable segmented breakwater is almost always less than for a solid breakwater, or other array with lower porosity. Transmission is typically higher for arrays with higher porosity when the wavelength is large relative to the structure and can become smaller for wavelengths that are small relative to the structure as the
damping effect of the media starts to have a significant influence. As expected, increasing the gap size allows more transmission and less reflection. Finally, data obtained for all three structures considered shows many of the same trends except near the cut-off frequencies and angles, which are both features of the periodic geometry.

6.2 Improvements in Integral Equation Solution

In the solution of the systems of integral equations (3.44)-(3.49) in Chapter 2 and (5.58)-(5.59) in Chapter 5, a boundary element method based on Gauss-Jacobi abscissae was employed. Two simple variations on this are now apparent.

To justify the first of these, recall Macaskill [35] employs the points

\[ x_i = a + (b - a) \sin \frac{\pi}{N} \left( \frac{k}{N} \right) \quad k = 2i - 1 - N \quad \text{with} \quad i = 1, \ldots, N \]  

(6.1)

to account for square root singularities at the endpoints of the interval \([a, b]\). In Chapter 2, this choice of points was justified by the observation that these points are the abscissae for the Gauss-Chebyshev integration rule and the weight function for this rule (namely \((x - a)^{1/2}(b - x)^{1/2}\)) has the form of the singular behaviour of the velocity at the block corners. Another good reason for this choice is illustrated by considering the change of variables \(\theta(x) = \arcsin x\) and integrals of the form

\[ I = \int_{-1}^{1} \frac{f(x)}{(1 - x^2)^{1/2}} dx. \]  

(6.2)

where \(f\) is a continuous function on \([-1, 1]\). Under this change of variables, equation (6.2) becomes

\[ I = \int_{-\pi/2}^{\pi/2} f(\sin \theta) d\theta, \]  

(6.3)

which is no longer singular. In addition, note that under this change of variables, the Chebyshev points \(x_i\) \((a = -1, b = 1)\) become equally spaced on \([-\pi/2, \pi/2]\). For integrals with the cube root singularities typified by blocks considered in Chapters 2 and 5, a similar change of variables which removes the singular behaviour is \(\theta(x) = x^2 F(1/2, 1/3; 3/2; x^2)\) (see Abramowitz and Stegun [1, page 556] equation 15.1.1 for a definition of the hypergeometric function \(2F_1(x_1, x_2, x_3, x)\)). This transforms

\[ I = \int_{-1}^{1} \frac{f(x)}{(1 - x^2)^{1/3}} dx \]  

(6.4)

to

\[ I = \int_{0}^{\theta(1)} f(\theta^{-1}(z)) d\theta. \]  

(6.5)

104
However under this transform the Gauss-Jacobi abscissae employed are not equally spaced. This suggests the following alternative choice of points on which to center the boundary elements. Let $\theta_n$ be equally spaced along the interval $[\theta(-1), \theta(1)]$ then take $x_n = \theta^{-1}(\theta_n)$. The justification of the choice of point would seem to be more rigorous than that given for the use of the Gauss-Jacobi abscissae. However the use of these points overlooks the significance of the Gauss-Jacobi abscissae in Gaussian integration and establishing the collocation points is more difficult.

The second variation involves employing functions which represent the anticipated behaviour of the solution on the boundary elements near the corners instead of assuming constant values. Such schemes are suggested by Liggett [28, page 37]. Specifically recall from Chapter 2, the solution to each integral equation is approximated by expressions of the form

$$U(y) = \sum_{i=1}^{N} U_i N_i(y), \quad (6.6)$$

where

$$N_i(y) = \begin{cases} 
0 & \text{if } y \not\in (y_i, y_{i+1}) \\
1 & \text{if } y \in (y_i, y_{i+1}). 
\end{cases} \quad (6.7)$$

An alternative scheme using a function approximation of the solution might be to assign

$$N_1(y) = \begin{cases} 
0 & \text{if } y \not\in (y_1, y_{i+1}) \\
(y - y_1)^{-1/3} & \text{if } y \in (y_1, y_2), 
\end{cases} \quad (6.8)$$

and similarly

$$N_N(y) = \begin{cases} 
0 & \text{if } y \not\in (y_N, y_{N+1}) \\
(y_{N+1} - y)^{1/3} & \text{if } y \in (y_N, y_{N+1}). 
\end{cases} \quad (6.9)$$

This approach clearly uses a more analytically correct description of the solution near the endpoints (it also bears a resemblance to the expansion functions chosen by Porter & Evans [50] and Fernyhough & Evans [20] in their Galerkin approach to solution of integral equations). However, use of these expressions leads to more complicated integrals to evaluate both in solution and calculation of coefficients. Thus any advantage in fewer elements required might be offset by longer times required in evaluation of integrals.

### 6.2.1 Further Work

So far, understanding of the results obtained from the permeable periodic array is limited. From the results obtained so far, it is clear much more data needs be obtained and graphs plotted. For example, little data has been obtained so far for permeable structure with larger
friction factors or various inertia factors. Also data has only be obtained for very low \( kd \) values.

In the longer term, there are a number of interesting extensions of this work. Firstly, an obvious goal is the solution of the full three-dimensional diffraction problem for the periodic array of permeable blocks. Secondly, consideration of periodic arrays of permeable structures other than blocks would seem a valuable goal since breakwaters do not have such an idealized structure. A simple extension of the work of Chapter 5 would be consideration of non-collinear structures. Thirdly, the more difficult problem of diffraction by breakwater and segmented breakwater of finite length is also of practical interest. Fourthly, segmented breakwater consisting of elements with varying porosity such as crown breakwaters also merits consideration. Another related issue is improvement of, or replacement of the Sollitt & Cross model of wave interaction with rubble mound structures which models wave interaction structure on the interfaces of the structure and by consideration of the free surface only. Fifthly, nothing has been said in the current work about wave breaking on such structures. Finally consideration of the effects on an actual shoreline or harbour protected by such structures is of interest.
Appendix A

Chapter 2 appendix

A.1 Derivation of Flow Equation

A.1.1 Resistance Force Balance

To establish equation (2.10) (equation (1a) of Sollitt & Cross [56]), let \( P \) denote an arbitrary point in the medium, the unit vector \( \mathbf{r} \) denote an arbitrarily chosen direction and \( \Lambda_r(P) \) denote the smallest cylindrical element enclosing the REV of \( P \), with centroid at \( P \) and major axis in the direction \( \mathbf{r} \). Then consider the forces (illustrated in Figure A.1) acting on the fluid in \( \Lambda_r(P) \) in the \( \mathbf{r} \) direction.

There are four forces acting on the fluid in the cylindrical element \( \Lambda_r(P) \) in the positive \( \mathbf{r} \) direction. There are the pressure forces, \( p\varepsilon dA \) and \( -(p + \frac{\partial p}{\partial r} dr)\varepsilon dA \) acting on the left hand and right hand faces of \( \Lambda_r(P) \) respectively where \( p \) denotes pressure of the fluid in the medium and \( dA \) the cross-sectional area of \( \Lambda_r(P) \). (Note that \( \varepsilon dA \) denotes the area of fluid exposed to the pressure on the faces since the ratio of void space to ). Also there is the component of the gravitational force in the \( \mathbf{r} \) direction, \( -(\rho g\varepsilon dAdr)\cos \theta \), where \( dr \) denotes the length of the element \( \Lambda_r(P) \) and \( \theta \) denotes the smallest angle between the positive \( z \) axis and the directional vector \( \mathbf{r} \). Finally as consequence of the unknown resistance the medium exerts on the fluid passing through \( \Lambda_r(P) \), there is the resistance force \( -F_R \).

Thus on the basis, of these forces, an average flow description of the flow through the medium can be written. Specifically noting the definition of the seepage velocity from section and employing Newton’s law, the component of seepage velocity in the \( \mathbf{r} \) must satisfy

\[
\rho \varepsilon dAdr \frac{DV_r}{Dt} = p\varepsilon dA - (p + \frac{\partial p}{\partial r} dr)\varepsilon dA - (\rho g\varepsilon dAdr)\cos \theta - F_R, \tag{A.1}
\]
where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (V \cdot \nabla)$ is the substantial derivative operator with respect to the seepage velocity. Further noting that $\cos \theta = \partial z/\partial r$ and that the local acceleration $\frac{\partial V_r}{\partial t}$ typically dominates over the convective term $(V \cdot \nabla)V_r$, equation (A.1) simplifies to

$$
\frac{\partial V_r}{\partial t} = -\frac{1}{\rho} \frac{\partial}{\partial r} (p + pgz) - \frac{F_R}{\rho cdAdr}.
$$

which is essentially a directional component of equation (1a) of Sollitt & Cross [56].

### A.1.2 Resistance Force Estimation

Both Rumer [54] and Burcharth & Andersen [54, page 251] estimate the resistance force $F_R$ on the basis of two assumptions (Rumer for low Reynolds number steady flows and Burcharth & Andersen for unsteady and more realistic flows). They first assume that the medium comprises of an arrangement of almost identical particles with representative diameter, $a$, cross-sectional area $\alpha a^2$ and volume $\beta a^3$. Thus in the cylindrical element $\Lambda_r(P)$ there are approximately

$$
N = \frac{(1 - \epsilon)dAds}{\beta a^3}.
$$

particles. And secondly they argue that the resistance force

$$
F_R = N\lambda F_p,
$$

where $F_p$ denotes the resistance force exerted by an individual particle in isolation and $\lambda$ is a factor which accounts for inter-particle forces. Since the particles (rubble or armour units)
in a breakwater are small relative to the wave motion, the force $F_p$ can be estimated by Morison's equation (see Dean & Dalrymple [15, page 221]) as

$$F_p = \frac{1}{2} C_D \rho a^2 |V_r| V_r + C_M \rho \beta a^3 \frac{D V_r}{Dt}$$  \hspace{1cm} (A.5)

where $C_M$ is a shape dependent parameter known as the inertia coefficient (known to be 1.5 for spherical particles) and $C_D$ is a shape and flow dependent parameter known as the drag coefficient. Thus employing equation (A.3) and (A.4) the resistance force $F_R$ may be estimated as

$$F_R = \frac{(1-\epsilon) \alpha d A d s}{2 \beta d} \lambda C_D |V_r| V_r + \rho \lambda C_M (1-\epsilon) d A d s \frac{\partial V_r}{\partial t}$$  \hspace{1cm} (A.6)

where it is again assumed that the local acceleration $\partial V_r / \partial t$ dominates over the convective term $(V \cdot \nabla)V_r$. And finally, substituting this in equation (A.2) the approximate flow equation

$$\frac{\partial V_r}{\partial t} = -\frac{1}{\rho} \frac{\partial}{\partial r} (p + \rho g z) - \frac{(1-\epsilon) \alpha}{2 \beta \epsilon d} \lambda C_D |V_r| V_r + \lambda C_M \frac{1-\epsilon}{\epsilon} \frac{\partial V_r}{\partial t}$$  \hspace{1cm} (A.7)

is obtained.

The steady low Reynolds number case has been studied by Rumer [54]. In particular, according to Batchelor [6, page 244, equation (4.10.8)] the drag coefficient for a sphere in isolation is

$$C_{D, s} = \frac{24}{R} (1 + \frac{3}{16} R) + \ldots$$  \hspace{1cm} (A.8)

(Note Chester & Breach [9] have found that the drag experienced by the fluid as a consequence of a sphere is of the form,

$$D = 6 \pi a U (1 + \frac{3}{8} R + \frac{9}{40} R^2 (\log R + \gamma + \frac{5}{3} \log \frac{323}{360}) + \frac{27}{80} R^3 \log R + O(R^3)),$$  \hspace{1cm} (A.9)

where $\gamma$ is Euler's constant,) for moderate Reynolds numbers, $R = \frac{\rho a V_r}{\nu}$. Thus introducing the parameters $\lambda_1$ and $\lambda_2$ to account for inter-particle viscous drag effects, $C_D$ has the form

$$C_D \approx \frac{\lambda_1 \nu}{\rho V_r} + \lambda_2 + \ldots,$$  \hspace{1cm} (A.10)

for a permeable structure. And therefore for low and moderate Reynolds numbers, equation (A.7) becomes,

$$\frac{\partial V_r}{\partial t} = -\frac{1}{\rho} \frac{\partial}{\partial r} (p + \rho g z) - \frac{(1-\epsilon) \alpha}{2 \beta \epsilon d} \lambda \lambda_1 \nu V_r - \frac{(1-\epsilon) \alpha}{2 \beta \epsilon d} \lambda \lambda_2 |V_r| V_r + \lambda C_M \frac{1-\epsilon}{\epsilon} \frac{\partial V_r}{\partial t}$$  \hspace{1cm} (A.11)

Thus combining equations (A.7) and (A.11) it is clear that flow through permeable medium can be described by

$$s \frac{\partial V}{\partial t} = -\frac{1}{\rho} \nabla (p + \rho g z) - AV - BV |V|.$$  \hspace{1cm} (A.12)

where $A$ and $B$ and $s$ depend on porosity, particle shape and Reynolds number.
Appendix B

Chapter 3 appendix

B.1 Behaviour at Impermeable Corners

Consider the solution $\phi$ of the boundary value problem (3.2)-(3.5), around the block corner illustrated above, and note in particular, that in the close vicinity of the block that $\phi$ satisfies effectively the 2-D Laplace Equation. To demonstrate this, consider that $\phi$ satisfies Helmholtz equation, which in terms of the polar coordinates system $(r, \theta)$ illustrated above is

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + k^2 \phi = 0.
$$

(B.1)

And note that by rescaling $r$ as,

$$
R = \frac{r}{L}
$$

(B.2)

where $L$ is some as yet unspecified length scale associated with the corner, that equation (B.1) can be re-written as

$$
\frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \phi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \phi}{\partial \theta^2} + L^2 k^2 \phi = 0.
$$

(B.3)
And hence choosing $L$ such that $|L^2 k| \ll 1$, equation (D.28) reduces effectively to the 2-D Laplace equation,

$$\nabla^2 \Phi = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0.$$ \hspace{1cm} (B.4)

Next note by application of separation of variables (see Mei [43, page 83]) to equation (B.4), $\phi$ has the general solution,

$$\phi(R, \theta) = R^m (A_m \cos m\theta + B_m \sin m\theta) \quad (m \geq 0),$$ \hspace{1cm} (B.5)

in the vicinity of the block (assuming that $\phi$ is bounded) and by application of the body boundary condition (3.5) on each side of the block to (B.5), the linear system,

$$B_m m R^{m-1} \sin \frac{3m\pi}{2} + B_m m R^{m-1} \cos \frac{3m\pi}{2} = 0,$$ \hspace{1cm} (B.7)

is obtained. And consequently for non-homogeneous solutions of the boundary value problem,

$$m = \frac{2}{3} m_1$$ \hspace{1cm} (B.8)

for some $m_1 \in \mathbb{N}$. And thus in close vicinity of the corner ($R \ll 1$),

$$\phi(R, \theta) \sim A_{2/3} R^{2/3} \sin \frac{2}{3} \theta,$$ \hspace{1cm} (B.9)

and in terms of the fluid velocity component $\nabla \phi(R, \theta)$, it's clear that

$$|\nabla \phi(R, \theta)| \sim A_{2/3} R^{-1/3}.$$ \hspace{1cm} (B.10)
Appendix C

Chapter 4 appendix

C.1 Derivations of Free-surface Green’s functions

The free-surface Green’s functions in (4.14) and (4.15) used in solution of the boundary value problem (4.2)-(4.10) are obtained by solution of the boundary value problem (4.11)-(4.13) using the Fourier cosine transform in $X$, where $X = z - \xi$. In particular, applying this transform to the boundary value problem (4.11)-(4.13), it becomes

$$\frac{d^2 G_i}{dz^2} - \gamma^2 G_i = \frac{1}{2} \delta(z - \eta), \quad (C.1)$$

$$\frac{dG_i}{dz} = \frac{\Gamma_i}{h} G_i, \quad \text{when } z = 0, \quad (C.2)$$

$$\frac{dG_i}{dz} = 0 \quad \text{when } z = -h, \quad (C.3)$$

where

$$\bar{G}_i(\mu, z, \eta) = \int_0^\infty G_i(X, z, \eta) \cos \mu X \, dX, \quad (C.4)$$

denotes the Fourier cosine transform in $X$ of the function $G_i$ ($i = 1, 2$) and $\gamma = (\mu^2 + \lambda^2)^{1/2}$ as in (4.16). Note also by applying the inverse transform to $\bar{G}_i$,

$$G_i(x, z, \xi, \eta) = G_i(X, z, \eta) = \frac{2}{\pi} \int_0^\infty \bar{G}_i(\mu, z, \eta) \cos \mu X \, d\mu. \quad (C.5)$$

Consequently from solution of the boundary value problem (C.1)-(C.3), a simpler problem, solution for $G_i$ in the original boundary value problem can be obtained.

On account of the singularity of the right side of (C.1) at $z = \eta$, the transformed boundary value problem (C.1)-(C.3) is solved initially in two parts: (i) for $z > \eta$ and (ii) for $z < \eta$. In particular, for $z > \eta$, the general solution of (C.1) when $z > \eta$ is of the form

$$\bar{G}_i = A \cosh \gamma(z + h) + B \sinh \gamma(z + h). \quad (C.6)$$
and by application of the boundary condition (C.2) to (C.6) it's clear that

$$A(\Gamma_i \cosh \gamma h - \gamma h \sinh \gamma h) = B(\gamma h \cosh \gamma h - \Gamma_i \sinh \gamma h).$$  \hspace{1cm} (C.7)

And similarly by application of the boundary condition (C.3) to the general solution of (C.1) on the interval $0 \leq z \leq \eta$ (which evidently has the same form as (C.6)) it is clear that for $0 \leq z \leq \eta$,

$$\mathcal{G}_i = C \cosh \gamma (z + h).$$  \hspace{1cm} (C.8)

Next to determine $A$, $B$ and $C$, it is assumed that $\mathcal{G}_i$ is continuous at $z = \eta$ and this has two consequences. Firstly, by continuity of $\mathcal{G}_i$ at $z = \eta$ gives

$$A \cosh \gamma (h + \eta) + B \sinh \gamma (h + \eta) = C \cosh \gamma (h + \eta).$$  \hspace{1cm} (C.9)

Secondly, integrating (C.1) across the source, it is clear that

$$\lim_{\epsilon \to 0} \left\{ \frac{d\mathcal{G}_i}{dz} \right\}_{\eta - \epsilon}^{\eta + \epsilon} = -\frac{1}{2},$$  \hspace{1cm} (C.10)

and then substituting (C.6) for $z < \eta$ and (C.8) for $z > \eta$ in (C.10), that

$$A \sinh \gamma (h + \eta) + B \cosh \gamma (h + \eta) - C \sinh \gamma (h + \eta) = -\frac{1}{2\gamma},$$  \hspace{1cm} (C.11)

noting by continuity of $\mathcal{G}_i$ at $z = \eta$ that

$$\lim_{\epsilon \to 0} \int_{\eta - \epsilon}^{\eta + \epsilon} \mathcal{G}_i dz = 0.$$  \hspace{1cm} (C.12)

And thus equations (C.7), (C.9) and (C.11) constitutes a system of three simultaneous linear in three unknowns, $A$, $B$ and $C$, respectively. And thus solving this system by elimination, it is clear that for $z < \eta$,

$$\mathcal{G}_i = -\frac{\cosh \gamma (h + z)(\gamma h \cosh \gamma h + \Gamma_i \sinh \gamma h)}{2\gamma(\Gamma_i \cosh \gamma h - \gamma h \sinh \gamma h)},$$  \hspace{1cm} (C.13)

and for $z > \eta$ that,

$$\mathcal{G}_i = -\frac{\cosh \gamma (h + \eta)(\gamma h \cosh \gamma z + \Gamma_i \sinh \gamma z)}{2\gamma(\Gamma_i \cosh \gamma h - \gamma h \sinh \gamma h)}.$$  \hspace{1cm} (C.14)

Noting that the forms for $G_i$ for $z < \eta$ and $z > \eta$ are the same with only $z$ and $\eta$ interchanged, $\mathcal{G}_i$ can be written

$$\mathcal{G}_i = -\frac{\cosh \gamma (z_\geq + h)(\Gamma_2 \sinh \gamma z_\geq + \gamma h \cosh \gamma z_\geq)}{2\gamma(\Gamma_2 \cosh \gamma h - \gamma h \sinh \gamma h)},$$  \hspace{1cm} (C.15)

where $z_\geq = \max\{z, \eta\}$ and $z_\leq = \min\{z, \eta\}$. 

113
Finally solutions of the boundary value problem (4.11)-(4.13) are obtained by application of the inverse Fourier cosine transform (C.4). However note for the function for region 1 the inverse Fourier transform needs a slight modification. In particular note that the functions $\mathcal{G}_i \ (i = 1, 2)$ have poles in the complex $\mu$ plane when

$$\Gamma_i \cosh \gamma h - \gamma h \sinh \gamma h = 0. \quad (i = 1, 2) \tag{C.16}$$

respectively. And note since equation (C.16) is a rearrangement of the dispersion relations for region 1 and 2, respectively, its clear that $\mathcal{G}_1$ has poles at

$$\mu = \pm(k_n^2 - \lambda^2)^{1/2} \quad (n \in \mathbb{N} \cup \{0\}), \tag{C.17}$$

and $\mathcal{G}_2$ has poles at

$$\mu = \pm(R_n^2 - \lambda^2)^{1/2} \quad (n \in \mathbb{N} \cup \{0\}), \tag{C.18}$$

where expressions (2.56) and (2.62) designate the respective root branch taken. Note all the poles of $G_1$ and of $G_2$ (given $f \neq 0$) are complex except for $\mu = \pm(k_0^2 - \lambda^2)^{1/2}$. And thus in employing the inverse cosine Fourier transform to $\mathcal{G}_1$, it is necessary to deform the contour of integration in the complex plane around the pole at $\mu = (k_0^2 - \lambda^2)^{1/2}$ in order that the integral exists. Further doing so underneath (rather than above) the pole ensures that $G_1$ satisfies the radiation condition. And thus taking this into account, application of the inverse Fourier transform to $\mathcal{G}_1$ and $\mathcal{G}_2$ gives

$$G_1(x, z, \xi, \eta) = -\frac{1}{\pi} \int_0^\infty \frac{\cosh \gamma(z_0 + h)\{\Gamma_1 \sinh \gamma z_0 + \gamma h \cosh \gamma z_0\}}{\gamma\{\Gamma_1 \cosh \gamma h - \gamma h \sinh \gamma h\}} \cos \mu(x - \xi) \, d\mu, \tag{C.19}$$

and

$$G_2(x, z, \xi, \eta) = -\frac{1}{\pi} \int_0^\infty \frac{\cosh \gamma(z_0 + h)\{\Gamma_2 \sinh \gamma z_0 + \gamma h \cosh \gamma z_0\}}{\gamma\{\Gamma_2 \cosh \gamma h - \gamma h \sinh \gamma h\}} \cos \mu(x - \xi) \, d\mu. \tag{C.20}$$

respectively, where the smile ($\smile$) on the integral in equation (C.19) denotes that the path of integration is deformed in the complex $\mu$ plane underneath the pole on the positive real axis at $\mu = (k_0^2 - \lambda^2)^{1/2}$. 

114
C.2 Explicit Display Of Singularity In Green's Functions Using Bessel Functions

To verify that the Green's function $G_1$ and $G_2$ can be written equivalently in the forms (4.18) and (4.19) respectively note that for $z > \eta$ that $G_2$ can be expanded as

$$G_2(x, z, \xi, \eta) = -\frac{1}{\pi} \int_0^\infty \frac{\cosh \gamma (\eta + h) \cos \mu (x - \xi)}{\gamma} \left\{ \frac{\cosh \gamma (z + h) e^{-\gamma h} (\Gamma_1 + \gamma h)}{\Gamma_2 \cosh \gamma h - \gamma h \sinh \gamma h} - e^{-\gamma (z + h)} \right\} d\mu. \quad (C.21)$$

where $\Gamma_2 \sinh \gamma z + \gamma h \cosh \gamma z$ has been expanded as

$$\Gamma_2 \sinh \gamma z + \gamma h \cosh \gamma z = \cosh \gamma (z + h) (\cos \gamma h - \sin \gamma h) (\Gamma_2 + \gamma h) - e^{-\gamma (z + h)} \{ \Gamma_2 \cosh \gamma h - \gamma h \sinh \gamma h \}. \quad (C.22)$$

Further note using Gradshtein & Ryzhik [24, Equation 3.961(2)] it can be demonstrated that

$$\frac{1}{2\pi} \{ K_0(\lambda r) + K_0(\lambda r') \} = \frac{1}{2\pi} \int_0^\infty \frac{e^{-\gamma \tau} \cos \mu (x - \xi)}{\gamma} (e^{\gamma \tau} + e^{-\gamma (\eta + 2h)}) d\mu$$

$$= \frac{1}{\pi} \int_0^\infty \frac{e^{-\gamma \tau} \cos \mu (x - \xi)}{\gamma} e^{-\gamma h} \cosh \gamma (\eta + h) d\mu \quad (C.23)$$

where

$$r = \{(x - \xi)^2 + (y - \eta)^2\}^{1/2}, \quad (C.24)$$

$$r' = \{(x - \xi)^2 + (y + \eta + 2h)^2\}^{1/2}, \quad (C.25)$$

represent the distance of field point from the source, and from the image of the source in the bed respectively. And thus note by combining (C.23) with (C.21), the expression,

$$G_2(x, z, \xi, \eta) = \frac{1}{2\pi} \{ K_0(\lambda r) + K_0(\lambda r') \}$$

$$- \frac{1}{\pi} \int_0^\infty \frac{\cosh \gamma (\eta + h) \cosh \gamma (z + h) e^{-\gamma h} (\Gamma_2 + \gamma h) \cos \mu (x - \xi)}{\gamma \{ \Gamma_2 \cosh \gamma h - \gamma h \sinh \gamma h \}} d\mu, \quad (C.26)$$

is obtained. And note following a similar procedure for $z < \eta$ equation (C.26) is again obtained and that the same course can be followed to verify the expression (4.19) for $G_1$.

C.3 Series Representation Of Green's Functions for Region 1

The Green's function for regions 1 maybe represented by series by applying residue theory to the integral representations. To obtain the series representation of $G_1$, its integral representation as given by (4.14) is first rearranged. In particular note by expanding the cosine
term in terms of exponentials in (4.14), \( G_1 \) becomes
\[
G_1(x, z, \xi, \eta) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F_1(\mu, z, \eta)}{\gamma} e^{i\mu(x-\xi)} d\mu, \tag{C.27}
\]
where \( F_1(\mu, z, \eta) \) is given by equation (4.69) and the path of integration is deformed to pass above the pole at \( \mu = -(k_n^2 - \lambda^2)^{1/2} \) and below the pole at \( \mu = (k_n^2 - \lambda^2)^{1/2} \) in order that \( G_1 \) satisfies the radiation condition. Note from equation (C.17) that \( F_1(\mu, z, \eta)e^{i\mu(x-\xi)}/\gamma \) has poles at \( \pm(k_n^2 - \lambda^2)^{1/2} \). And note the residues at these poles are given by
\[
\text{Res}\left( \frac{F(\mu, z, \eta)}{\gamma} e^{i\mu(x-\xi)}, \pm(k_n^2 - \lambda^2)^{1/2} \right) = \mp \frac{\chi_n(\eta)\chi_n(z)e^{\pm(k_n^2 - \lambda^2)^{1/2}(x-\xi)}}{2(k_n^2 - \lambda^2)^{1/2}}, \tag{C.28}
\]

Next consider the two cases: (i) \( (x - \xi) > 0 \) and (ii) \( (x - \xi) < 0 \). And for the first case, let \( S_R = Re^{it} \) \( (0 \leq t \leq \pi) \) and note that according to the residue theorem applied on the region bounded by \( S_R \) and the real axis (deformed as described above), that
\[
\lim_{R \to \infty} \left( \int_{-R}^{R} \frac{F_1(\mu, z, \eta)}{\gamma} e^{i\mu(x-\xi)} d\mu + \int_{S_R} \frac{F_1(\mu, z, \eta)}{\gamma} e^{i\mu(x-\xi)} d\mu \right)
\]
\[
= 2\pi i \sum_{n=0}^{\infty} \text{Res}\left( \frac{F_1(\mu, z, \eta)}{\gamma} e^{i\mu(x-\xi)}, (k_n^2 - \lambda^2)^{1/2} \right). \tag{C.29}
\]
In addition note since \( (x - \xi) > 0 \) that
\[
\lim_{R \to 0} \int_{S_R} F(\mu)e^{i\mu(x-\xi)} d\mu = 0 \tag{C.30}
\]
and thus,
\[
\int_{-\infty}^{\infty} F(\mu, z, \eta)e^{i\mu(x-\xi)} d\mu = -2\pi i \sum_{n=0}^{\infty} \frac{\chi_n(\eta)\chi_n(z)e^{-i(k_n^2 - \lambda^2)^{1/2}(x-\xi)}}{2(k_n^2 - \lambda^2)^{1/2}}, \tag{C.31}
\]
Similarly for \( (x - \xi) < 0 \) let \( S_{R'} = Re^{-it} \) \( (0 \leq t \leq \pi) \) and note that according to the residue theorem applied on the region bounded by \( S_{R'} \) and the real axis (deformed as described above), that
\[
\lim_{R' \to \infty} \left( \int_{-R'}^{R'} \frac{F_1(\mu, z, \eta)}{\gamma} e^{i\mu(x-\xi)} d\mu + \int_{S_{R'}} \frac{F_1(\mu, z, \eta)}{\gamma} e^{i\mu(x-\xi)} d\mu \right)
\]
\[
= -2\pi i \sum_{n=0}^{\infty} \text{Res}\left( \frac{F_1(\mu, z, \eta)}{\gamma} e^{i\mu(x-\xi)}, -(k_n^2 - \lambda^2)^{1/2} \right). \tag{C.32}
\]
In addition note since \( (x - \xi) < 0 \) that
\[
\int_{-\infty}^{\infty} F(\mu, z, \eta)e^{i\mu(x-\xi)} d\mu = -2\pi i \sum_{n=0}^{\infty} \frac{\chi_n(\eta)\chi_n(z)e^{-i(k_n^2 - \lambda^2)^{1/2}(x-\xi)}}{2(k_n^2 - \lambda^2)^{1/2}}, \tag{C.33}
\]
Thus combining equations (C.31) and (C.33), it is clear that
\[
G_1(x, z, \xi, \eta) = \frac{i}{2\pi} \sum_{n=0}^{\infty} \frac{\chi_n(\eta)\chi_n(z)e^{i(k_n^2 - \lambda^2)^{1/2}(x-\xi)}}{2(k_n^2 - \lambda^2)^{1/2}}. \tag{C.34}
\]
Appendix D

Chapter 5 appendix

D.1 Derivation Of Green's Function For Region 1

The Green's function for region 1 is assumed to be the superposition of a wave source (Hankel function) and a Fourier transform solution. In particular it is assumed that $G_1$ is of the form,

$$4iG_1(x, y, \xi, \eta) = H_0(kr) - \frac{2i}{\pi} \int_0^\infty (Ae^{k\gamma y} + Be^{-k\gamma y}) \frac{\cos k(x - \xi) t}{\gamma} dt,$$  \hspace{1cm} \text{(D.1)}

where

$$\gamma = \begin{cases} -i(1 - t^2)^{1/2} & (t \leq 1) \\ (t^2 - 1)^{1/2} & (t > 1) \end{cases}$$  \hspace{1cm} \text{(D.2)}

and

$$r = ((x - \xi)^2 + (y - \eta)^2)^{1/2}.$$  \hspace{1cm} \text{(D.3)}

Further note according to Gradshteyn & Ryzhik [24, Equation 3.961(2)] and Abramowitz & Stegun [1, Equation 9.6.4] it can be demonstrated that

$$H_0(kr) = -\frac{2i}{\pi} \int_0^\infty \frac{e^{-k(\gamma|y-\eta|)}}{\gamma} \cos k(x - \xi) t dt$$  \hspace{1cm} \text{(D.4)}

and consequently $G_1$ maybe written,

$$G_1(x, y, \xi, \eta) = \begin{cases} -\frac{2i}{\pi} \int_0^\infty [e^{-k\gamma(y-\eta)} + Ae^{k\gamma y} + Be^{-k\gamma y}] \frac{\cos k(x - \xi) t}{\gamma} dt & (d > y > \eta) \\ \frac{2i}{\pi} \int_0^\infty [e^{k\gamma(y-\eta)} + Ae^{k\gamma y} + Be^{-k\gamma y}] \frac{\cos k(x - \xi) t}{\gamma} dt & (-d < y < \eta). \end{cases}$$  \hspace{1cm} \text{(D.5)}

Further note by applying periodicity conditions (5.25) and (5.26) to equation (D.5), the expressions,

$$e^{-k\gamma(d-\eta)} + Ae^{k\gamma d} + Be^{-k\gamma d} = e^{-2\gamma d}[e^{-k\gamma(d+\eta)} + Ae^{-k\gamma d} + Be^{k\gamma d}],$$  \hspace{1cm} \text{(D.6)}
\[-e^{-k\gamma(d-\eta)} + Ae^{k\gamma d} - Be^{-k\gamma d} = e^{-2\beta d}[e^{-k\gamma(d+\eta)} + Ae^{-k\gamma d} - Be^{k\gamma d}],\]  
(D.7)

are obtained. And solving (D.6) and (D.7) for \(A\) and \(B\), it is found that

\[A = -\frac{e^{-k\gamma\eta}(e^{-2i\beta d} - e^{-2k\gamma d})}{2(\cos 2\beta d - \cosh 2k\gamma d)},\]  
(D.8)

\[B = \frac{e^{k\gamma\eta}(e^{-2k\gamma d} - e^{2i\beta d})}{2(\cos 2\beta d - \cosh 2k\gamma d)}.\]  
(D.9)

Thus note by substituting equations (D.8) and (D.9) into equation (D.1), the integral expression for \(G_1\),

\[G_1(x, y, \xi, \eta) = \frac{1}{4i} H_0(kr) - \frac{1}{2\pi} \int_0^\infty \frac{(\cos 2\beta d - e^{-2k\gamma d}) \cosh k\gamma(y - \eta) - i \sin 2\beta d \sinh k\gamma(y - \eta)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)} \cos k(x - \xi)t \, dt,\]  
(D.10)

is obtained, where the smile (\(-\)) denotes that the contour of the integration is deformed to pass beneath all the poles on the positive real axis. This ensures that the Green's function satisfies the radiation condition. In particular note that the poles of the Green's function for region 1 are found by solving

\[\cosh 2k\gamma d - \cos 2\beta d = 0.\]  
(D.11)

Solutions of this equation are

\[\gamma_n = \frac{-i}{k} \left(\beta - \frac{n\pi}{d}\right) = \frac{-i}{k} \beta_n \quad (n \in \mathbb{Z}),\]  
(D.12)

and consequently the poles of the Green's function are \(t = \pm t_n\) where

\[t_n = (1 + \gamma_n^2)^{1/2} = \frac{1}{k} \left(k^2 - \beta_n^2\right)^{1/2} = \frac{1}{k} \alpha_n, \quad (n \in \mathbb{Z})\]  
(D.13)

where \(\alpha_n\) and \(\beta_n\) for \(n \in \mathbb{Z}\) are defined by equations (5.35) and (5.36) respectively.

### D.2 Series Representation of Green's Function for Region 1

A series representation of \(G_1\) can be obtained by applying residue theory to equation (D.5). In particular note by expanding the cosine term in terms of exponentials in (D.5), \(G_1\) becomes

\[G_1(x, y, \xi, \eta) = -\frac{1}{4\pi} \int_{-\infty}^\infty \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t} \, dt\]  
(D.14)

where

\[F(y, \eta, t) = (\cosh 2k\gamma d - \cos 2\beta d)e^{-k\gamma(y-\eta)} + (\cos 2\beta d - e^{-2k\gamma d}) \cosh k\gamma(y - \eta) - i \sin 2\beta d \sinh k\gamma(y - \eta))/(\cosh 2k\gamma d - \cos 2\beta d),\]  
(D.15)
and the path of integration is deformed in the complex \( t \) plane to pass beneath the poles \( \{t_n\} \) and above the poles \( \{-t_n\} \) \((-r \leq n \leq s\) on the real axis. (Recall \( t_n \) \((n \in \mathbb{N})\) are defined in (D.13).) And note the residues at all the poles are given by

\[
\text{Res}\left( \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t}, \pm t_n \right) = \lim_{t \to \pm t_n} \left( \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t} \right) = \pm \frac{1}{2\alpha_n} e^{-i\beta_n(y-\eta)} e^{\pm i\alpha_n(x-\xi)} \quad (D.16)
\]

Next consider the two cases: (i) \((x - \xi) > 0\) and (ii) \((x - \xi) < 0\). And for the first case, let \( S_R = Re^{it} \) \((0 \leq t \leq \pi)\) and note that according to the residue theorem applied on the region bounded by \( S_R \) and the real axis (deformed as described above), that

\[
\lim_{R \to \infty} \left( \int_{-R}^{R} \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t} dt + \int_{S_R} \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t} dt \right)
\]

\[
= 2\pi i \sum_{n=-\infty}^{\infty} \text{Res}\left( \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t}, -t_n \right).
\]

(D.17)

In addition note since \((x - \xi) > 0\) that

\[
\lim_{R \to \infty} \int_{S_R} \frac{F(x, y, \xi, \eta, t)}{\gamma} e^{ik(x-\xi)t} dt = 0,
\]

and thus for \((x - \xi) > 0\),

\[
\int_{-\infty}^{\infty} \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t} dt = -\frac{i\pi}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} e^{i(\alpha_n(x-\xi) - \beta_n(y-\eta))}. \quad (D.18)
\]

Similarly for \((x - \xi) < 0\) let \( S_{R'} = Re^{-it} \) \((0 \leq t \leq \pi)\) and note that according to the residue theorem applied on the region bounded by \( S_{R'} \) and the real axis (deformed as described above), that

\[
\lim_{R \to \infty} \left( \int_{-R}^{R} \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t} dt + \int_{S_{R'}} \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t} dt \right)
\]

\[
= -2\pi i \sum_{n=-\infty}^{\infty} \text{Res}\left( \frac{F(y, \eta, t)}{\gamma} e^{-ik(x-\xi)t}, -t_n \right).
\]

(D.20)

In addition note since \((x - \xi) < 0\) that

\[
\lim_{R \to \infty} \int_{S_{R'}} \frac{F(x, y, \xi, \eta, t)}{\gamma} e^{ik(x-\xi)t} dt = 0,
\]

and thus for \((x - \xi) < 0\),

\[
\int_{-\infty}^{\infty} \frac{F(y, \eta, t)}{\gamma} e^{ik(x-\xi)t} dt = -\frac{i\pi}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} e^{-i(\alpha_n(x-\xi) + \beta_n(y-\eta))}. \quad (D.21)
\]

(D.22)

Thus combining equations (D.19) and (D.22) it is clear that,

\[
G_1(x, y, \xi, \eta) = -\frac{i}{4d} \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} e^{i(\alpha_n|x-\xi| - \beta_n(y-\eta))}. \quad (D.23)
\]
D.3 Flow Behaviour at Permeable Block Corners

Consider the permeable corner illustrated above. Introducing polar coordinates \( (r, \theta) \) with origin at the corner, the velocity potential in the region of the corner must satisfy the boundary value problem,

\[
\begin{align*}
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi_i}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi_i}{\partial \theta^2} + k_i^2 \phi_i &= 0 \quad (i = 1, 2), \\
\phi_1 &= \delta \phi_2, \\
\frac{\partial \phi_1}{\partial n} &= \epsilon \frac{\partial \phi_2}{\partial n},
\end{align*}
\]

where \( n \) denotes the normal to the porous media (pointing into the media) and \( k_1 = k \), \( k_2 = K \). Rescaling \( r \) as,

\[
R = \frac{r}{L}
\]

where \( L \) is some as yet unspecified length scale associated with the corner, then equation (D.24) becomes

\[
\frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \phi_i}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \phi_i}{\partial \theta^2} + L^2 k_i^2 \phi_i = 0. 
\]

Now if \( |L^2 k_i| \ll 1 \) for both \( i = 1, 2 \) equation (D.28) is equivalent to the 2-D Laplace equation,

\[
\nabla^2_R \phi_i = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \phi_i}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \phi_i}{\partial \theta^2} = 0. 
\]

Liggett & Liu [28, page 53] present an argument for 2-D Laplace type flow around a permeable corner (with slightly different boundary conditions). In particular assuming that \( \phi_i \quad (i = 1, 2) \) are bounded, the general solution of equation (D.29) in regions 1 and 2 is

\[
\phi_i(R, \theta) = R^n (A_{i,m} \cos m\theta + B_{i,m} \sin m\theta) \quad (i = 1, 2),
\]
is obtained. Further note from (D.30), that the $x$ and $y$ derivatives of $\phi_i(R, \theta)$ \quad (i = 1, 2) are given by

\[
\frac{\partial \phi_i}{\partial x}(R, \theta) = mR^{m-1}\{A_{i,m} \cos[(m-1)\theta] - B_{i,m} \sin[(m-1)\theta]\}, \quad (D.31)
\]

\[
\frac{\partial \phi_i}{\partial y}(R, \theta) = mR^{m-1}\{A_{i,m} \sin[(m-1)\theta] + B_{i,m} \cos[(m-1)\theta]\} \quad (D.32)
\]

respectively. Hence applying the boundary conditions (D.25) and (D.26) obtain the following linear system

\[
\begin{bmatrix}
R^m & 0 & -\delta R^m & 0 \\
0 & mR^{m-1} & 0 & -\epsilon mR^{m-1} \\
R^m \cos \frac{3m\pi}{2} & -R^m \sin \frac{3m\pi}{2} & -\delta R^m \cos \frac{3m\pi}{2} & \delta R^m \sin \frac{3m\pi}{2} \\
R^{m-1} \sin \frac{3m\pi}{2} & mR^{m-1} \cos \frac{3m\pi}{2} & \epsilon mR^{m-1} \sin \frac{3m\pi}{2} & \epsilon mR^{m-1} \cos \frac{3m\pi}{2}
\end{bmatrix}
\begin{bmatrix}
A_{1,m} \\
B_{1,m} \\
A_{2,m} \\
B_{2,m}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\quad (D.33)
\]

Further note the determinant of the system (D.33) is,

\[
\begin{vmatrix}
R^m & 0 & -\delta R^m & 0 \\
0 & mR^{m-1} & 0 & -\epsilon mR^{m-1} \\
R^m \cos \frac{3m\pi}{2} & -R^m \sin \frac{3m\pi}{2} & -\delta R^m \cos \frac{3m\pi}{2} & \delta R^m \sin \frac{3m\pi}{2} \\
R^{m-1} \sin \frac{3m\pi}{2} & mR^{m-1} \cos \frac{3m\pi}{2} & \epsilon mR^{m-1} \sin \frac{3m\pi}{2} & \epsilon mR^{m-1} \cos \frac{3m\pi}{2}
\end{vmatrix}
= R^{4m-2}(\epsilon^2 - \delta^2) \sin^2 \frac{3m\pi}{2}. \quad (D.34)
\]

And consequently for non-homogeneous solutions of the boundary value problem,

\[
m = \frac{2}{3} m_1 \quad (D.35)
\]

for some $m_1 \in \mathbb{N}$. And thus in close vicinity of the corner ($R \ll 1$),

\[
\phi_1(R, \theta) \sim A_{1,2/3} R^{2/3} \sin \frac{2}{3} \theta, \quad (D.36)
\]

and in terms of the fluid velocity component $\nabla \phi(R, \theta)$, it’s clear that

\[
|\nabla \phi_1(R, \theta)| \sim A_{1,2/3} R^{-1/3}. \quad (D.37)
\]
Bibliography


[34] C.M. Linton & P. McIver, "Green’s functions fo water waves in porous structures", Pre-print (1999)


