Effective optimal control of a fighter aircraft engine

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EFFECTIVE OPTIMAL CONTROL OF A FIGHTER AIRCRAFT ENGINE

by

Sa'ad Mustafa Mahmoud, B.Sc, M.Sc.

A doctoral thesis
submitted in partial fulfilment of the requirements
for the award of
the degree of Doctor of Philosophy of the Loughborough University of Technology
February, 1988

Formerly of the Department of Transport Technology
Loughborough University of Technology

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To

My Dear Parents
DECLARATION

This thesis is the outcome of the research carried out by the author in the Department of Transport Technology at Loughborough University of Technology and represents the independent work of the author; the work of others been referenced where appropriate.

The author also certifies that neither the thesis nor the original work contained herein has been submitted to any other institution for a degree.

Sa'ad Mustafa Mahmoud
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SUMMARY

Typical modern fighter aircraft use two-spool, low by-pass ratio, turbojet engines to provide the thrust needed to carry out the combat manoeuvres required by present-day air warfare tactics. The dynamic characteristics of such aircraft engines are complex and non-linear. The need for fast, accurate control of the engine throughout the flight envelope is of paramount importance and this research was concerned with the study of such problems and subsequent design of an optimal linear control which would improve the engine's dynamic response and provide the required correspondence between the output from the engine and the values commanded by a pilot.

A detailed mathematical model was derived which, in accuracy and complexity of representation, was a large improvement upon existing analytical models, which assume linear operation over a very small region of the state space, and which was simpler than the large non-analytic representations, which are based on matching operational data. The non-linear model used in this work was based upon information obtained from DYNGEN, a computer program which is used to calculate the steady-state and transient responses of turbojet and turbofan engines. It is a model of fifth order which, it is shown, correctly models the qualitative behaviour of a representative jet engine. A number of operating points were selected to define the boundaries used for the flight envelope. For each point a performance investigation was carried out and a related linear model was established. By posing the problem of engine control as a linear quadratic problem, in which the constraint was the state equation of the linear model, control laws appropriate for each operating point were obtained. A single control was effective with the linear model at every point. The same control laws were then applied to the non-linear mathematical model adjusted for each operating point, and the resulting responses were carefully studied to determine if one single control law could be used with all operating points. Such a law was established. This led, naturally, to the determination of an optimal linear tracking control law, and a further investigation to determine whether there existed an optimal non-linear control law for the non-linear model. In the work presented in this dissertation these points are fully discussed and the reasons for choosing to find an optimal linear control law for the non-linear model by solving the related two-point, boundary value problem using the method of quasilinearisation are presented. A comparison of the effectiveness of the respective optimal control laws, based upon digital simulation, is made before suggestions and recommendations for further work are presented.
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LIST OF SYMBOLS AND ABBREVIATIONS

- **a**: Constant, coefficient used in S.O.C. method, and used as a parameter in a filter
- **a₀, a₁, a₂**: Constants, used as coefficients in S.O.C. method
- **b**: Constant, used as parameter in a filter
- **c**: Constant, used as parameter in a filter, and a constant in the gradient method
- **d_ij**: Boundary values in Q.L. method
- **m**: Dimension of control vector
- **n**: Dimension of state vector
- **p**: Dimension of output vector
- **s**: Laplace variable
- **t**: Time
- **t**: t-test parameter
- **u_m, u_m**: Elements of control vector
- **w**: Command input, and part of integral controller
- **x**: State variable
- **x_i, x_i**: Elements of state vector
- **y_p**: Elements of output vector
- **z_p**: Elements of desired response vector

- **H**: Hamiltonian function
- **J, J₁, J₂, Jₑ**: Performance indices
- **L**: Cost functional
- **Z**: Desired response variable
- **τ**: Reverse time
- **ψ₁, ψ₁**: Costate variable

\[ A, A₁, A₁, B, B₁ \]  
\[ C, C₁, D₁, D₁, E \]  
\[ \hat{P}, G, \hat{G}, \hat{H}, k, M \]  
\[ \hat{M}, P, \hat{P}, P, P, Q \]  
\[ Q, S, T, φ \]  

\[ \text{Matrices} \]
e  Error vector
f  Vector function
g  Forcing function vector
v  Filter state variable
u  Control vector
x  State vector
y  Output vector
z  Desired response vector
R  Augmented state vector
κ  Augmented state vector function
ψ  Adjoint vector

Abbreviations

A.R.E.  Algebric Riccati equation
CL     Feedback matrix based on OUTREG
GM     Gradient method
LIN    Linear mathematical model of the F100 engine
LQP    Linear quadratic problem
M.R.E.  Matrix Riccati equation
O.T.P.  Optimal tracking problem
O.T.R.  Optimal tracking regulator problem
OUTREG Output regulator problem
PLA    Power lever angle
P.W.L.T. Piecewise linearization technique
Q.L.   Quasilinearization method
S.O.C.  Specific optimal control
STATREG State regulator problem
SL     Feedback matrix based on STATREG
Chapter 1

Introduction

1.1 Introduction  
1.2 Multivariable Control of a Jet Engine  
1.3 Scope of the Thesis
1.1 Introduction

Early simple gas turbine engines had only a single input variable, namely fuel flow, to effect control [Zucrow (1958) and Saravanamuttoo and Fawke (1969)]. An ordinary speed governor was used, the only additional complication being the need to limit the degree of over- or under-fuelling during transients. One of the earliest moves toward improving the performance of aircraft engines was the development of the two-spool turbojet, whereby the increase in allowable pressure ratio resulted in a decrease in specific fuel consumption. Straight turbojets have been superseded, to a large extent, by the by-pass, or fan, engine for subsonic applications, where developments of the by-pass engine have led to three-spool configurations for large engines, and two-spool geared-fan configurations for small engines.

The need to achieve versatility and improved performance has led to a requirement for modern engines to operate over wider ranges, often close, at times, to component fatigue and stability limits. This increased operating range has resulted in the development of more sophisticated gas turbines with multiple controls (e.g., several fuel flows, variable nozzle, variable compressor and intake geometry), and will lead undoubtedly to increasingly complex engines. For example, future variable-cycle engines may incorporate variable fan and turbine geometry to optimize overall engine performance.

The current control philosophy for present engines is to adopt separate control systems for each of the major components with the minimum of communication between the control systems. For instance, the variable geometry intake, the main engine and reheat systems would all have their own local controllers. Interactions between them may be large enough to cause serious degradation in the transient performance of the overall system. Therefore, a control strategy using the multivariable approach is a better method of handling such systems more effectively. Further, by introducing interaction into the controller, multivariable control can be used to optimise the performance in some chosen manner, and, can be used particularly, to minimise the transient excursion of critical parameters near their limits.

1.2 Multivariable Control of a Jet Engines

The application of multivariable control of gas turbine engines has two advantages. First it can compensate for any possible loss in performance due to interaction, and next, it can optimise the performance in some way by exploiting this same interaction. Various methods for achieving these advantages have been developed using various performance criteria such as minimizing a performance index, positioning the closed-loop poles, reducing interaction,
manipulating the characteristic loci etc. Engine data can also come in a variety of forms such as state-space models, frequency response curves, transfer functions and simulations. The methods of effective control system design can be regarded as essentially forming two classes: time domain and frequency domain methods.

In the time domain methods, a feedback control is calculated according to some predefined criterion such as minimizing a performance index i.e., optimal control [Athans and Falb (1966)], or positioning the closed-loop poles (pole-assignment) [Munro (1969) and Ali and Mahmoud (1987)]. In this fashion the design requirement is transferred from a choice of feedback control to a choice of appropriate performance criterion to be extremized. The most common of these methods is that employing a state-space model of the plant, to which analytical methods may be applied. A typical procedure is to derive a state-space model of the process to be controlled, to choose a performance index to be minimized or maximized (optimal control) and to calculate the corresponding controller. If the resulting design is unsatisfactory, usually the index is then altered and the sequence repeated until an acceptable design is found.

An example of such an approach was that adopted by the National Aeronautics and Space Administration (NASA) for multivariable control of a Pratt and Whitney F100 turbofan engine. Using the Linear Quadratic Regulator (LQR) synthesis technique, the aim was to 'optimise' the transient response of the gas turbine throughout its flight envelope.

The NASA procedure, starting from a digital thermodynamic engine simulator [Sellers and Daniele (1975)], was to linearise about a number of different operating conditions [Geyser (1978)]. Having chosen a suitable performance index, NASA workers applied LQR theory to derive a feedback controller. Trim integral terms were added to the controller to remove steady-state offsets. This was carried out at each of the linearised operating points. The NASA program was carried out in three phases:

1. digital simulation
2. a real-time hybrid computation
3. tests on the gas turbine engine in a high altitude test facility

NASA work has been reported in a number of references, such as Miller and Hackney (1976), DeHoff and Hall (1976), Adams, Dettoff and Hall (1977), Szuch, Soeder, Seldner and Cwynar (1977), Szuch, Skira and Soeder (1977), DeHoff and Hall (1978), Lehtinen, DeHoff and Hackney (1979), Lehtinen, Bruce, Costakis, William, Soeder, James and Seldner (1983), and Soeder (1983).
The second class of methods, the frequency domain approach, uses frequency response data [MacFarlane (1979)] and attempts to reduce the problem to a series of non-interacting single-loop designs. These frequency domain methods first aroused the interest of gas turbine engineers in the 1960's when attempts were made to regulate, using engine fuel and nozzle area respectively, the HP and LP shaft speeds of an Olympus two-shaft turbojet. Significant interaction was experienced and a means was sought for extending the controller design procedure to cope.

At that time, a suitable control theory for such an approach was being developed at the University of Manchester Institute of Science and Technology (UMIST) [Rosenbrock (1966) and MacFarlane (1969)]. From this work resulted the formulation of the Inverse Nyquist Array (INA) design method [Rosenbrock (1969)]. In this technique compensators are sought to reduce interaction to an extent sufficient to allow single-loop theory to be applied.

Starting from measured data for the Olympus engine, which was supplied by the National Gas Turbine Establishment (NGTE) in the UK, transfer function and state-space models were derived. Mueller (1967) and McMorran (1969) then showed that, employing only frequency response data, the INA method was effective for designing satisfactory regulators for such models.

Hodge (1970) applied this method to the reheat system of the real engine, integrating reheat fuel and nozzle area control. Thrust was regulated by maintaining predefined levels of jet-pipe pressure and temperature. A similar study on this engine was carried out by Dixon (1971), including engine fuel and nozzle area control.

Although other examples of multivariable control theory applied to gas turbine models exist in the literature, for example, Tiwari (1977), Cejj, Schafer, Sain and Hoppner (1977) and National Engineering Consortium (1977), the studies carried out by NASA and UMIST are, the only published applications, to date, which are supported by experimental evidence.

On the other hand, non-linear aspects of gas turbine multivariable control have received little attention. For instance, in the NASA experiment, engine limit protection appears to have been achieved without any regard to the transient behaviour as the limit is approached [Skira, DeHoff and Hall (1980)]. What attention there has been, has been devoted largely to scheduling regulator gains with operating condition, typically by straightforward linear interpolation.
1.3 Scope of the Thesis

The main requirement of any jet engine control system is the control of thrust; however, this is subject to several constraints including:

1. prevention of 'surge', a flow instability caused by large-scale stalling of flow over the compressor blades as a result of adverse pressure gradient. This is an extremely undesirable condition and can result in the sudden loss of thrust, combustion being extinguished, or, in very severe cases, can result in the entire engine being wrecked. It can also be caused by large fluctuations of intercompressor pressure as a result of disturbances in the afterburner section being transmitted back along the bypass duct.

2. maximum temperatures in the turbines must not be exceeded to prevent the blades from melting or from being seriously weakened.
   (See Sobey and Suggs (1963).)

In addition, the required level of thrust ought to be maintained with minimum fuel consumption.

Modern fighter aircraft use two-spool, low by-pass ratio, turbojet engines to provide the thrust needed to carry out the combat manoeuvres required by present-day air warfare tactics. The dynamic characteristics of such aircraft engines are complex and non-linear. The need for fast control of the engine throughout the flight envelope which ensures the above requirement is of great importance and this research was concerned with the study of such problems and the subsequent design of an optimal control which would improve the engine's dynamic response and provide the required correspondence between the output from the engine and the values commanded by a pilot.

It was the special intention of this research work to investigate the possibility of the existence of a single, linear, feedback control law, which would provide acceptable dynamic performance over the entire flight regime of the engine when represented by its non-linear model. If such a control law was shown to exist by the research study, then its form would be determined and its effectiveness confirmed.

In chapter 2, a detailed mathematical model of the two-spool, low by-pass ratio, F100 turbojet engine is derived which, in accuracy and complexity of representation, is a considerable improvement upon existing analytical models, which all assume linear operation over a very small region of the state space. The model was also simpler than the large
non-analytic representations, which are based on matching operational data. The non-linear model used in this work was based upon information obtained from DYNGEN, a computer program which is used to calculate the steady-state and transient responses of turbojet and turbofan engines. It is a model of fifth order which, it is shown, correctly models the qualitative behaviour of a representative jet engine. A number of operating points were selected to define the boundaries used for the flight envelope. For each point a performance investigation was carried out and a related linear model was established.

In chapter 3, a brief description of the state and output regulator problems of the linear quadratic problem, LQP, is given, with more attention being paid to their numerical solution and in particular, the numerical solution of the matrix Riccati equation, M.R.E. By posing the problem of engine control as a linear quadratic problem, in which the constraint was the state equation of the linear model, control laws appropriate for each operating point were obtained and dealt with in chapter 4. A single control was found to be effective with the linear model at every point. The same control laws were then applied to the non-linear mathematical model, adjusted for each operating point, and the resulting responses were carefully studied to determine if one single control law could be used with all operating point. Such a law was established.

In chapter 5, the linear optimal tracking problem, (O.T.P.), is presented: a successful solution based on this problem guarantees that the output vector of a linear observable system will be close to a desired response vector without excessive expenditure of control energy. An optimal tracking control law which provided the desired output response was obtained at each operating point of the flight envelope. The application of the optimal tracking control law (derived from one operating point )to some other operating point, produced no successful results due to the effect of the nozzle area control variable. The optimal tracking regulator problem, (O.T.R.), was a version of the optimal tracking problem, in which the output vector of the system was brought back to zero, in some specified manner. Successful results were obtained when the O.T.R. was applied at each operating point, and a single control law derived from the O.T.R. was capable of reasonably regulating the engine throughout the flight envelope.

The application of optimal control theory to a non-linear system leads to the formulation of the problem as a non-linear, two-point, boundary-value problem. In chapter 6 the formulation of this problem is presented, together with an account of the numerical techniques which are capable of solving such problem. The method of quasilinearization and the invariant imbedding method are discussed, and an algorithm for solving such a problem was developed and tested using several cases studies.

In chapter 7, a report of the attempts made to solve the non-linear, two-point, boundary value
problem, (T.P.B.V.P), which results from the application of the optimal control theory to the non-linear engine model to optimize its dynamic response is presented. The analytical form of T.P.B.V.P was highly complicated, and its solution proved to be very difficult. Several methods were used to solve this problem, but only one method (a combined approach from piecewise linearization and the gradient method) provided a useful approximate solution to the T.P.B.V.P. The case in which the above method was applied, was the optimal non-linear tracking problem, in which it was required that the thrust response of the engine be close to some desired response. A comparison of the effectiveness of the respective control laws, based upon digital simulation was made and is presented before suggestions and recommendations for further work are presented in chapter 8.
Chapter 2
Mathematical Models of the Engine

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2.8 Concluding Remarks 38
2.1 Introduction

In this chapter, a brief description of the F100 afterburning turbofan jet engine is given, for which OMAR, a fifth order non-linear mathematical model, has been developed in the course of this research work. OMAR is based on DYNGEN, the non-analytical jet engine simulation. This mathematical model OMAR correctly describes most of the dynamic behaviour of the jet engine throughout its operating range, and it has good agreement with the results obtained from the use of DYNGEN. Several operating points were chosen to define several flight mission boundaries. Performance studies were made about each of these points and a linear analytical model, LIN, was derived.

2.2 Engine Description

The engine selected for this research was the Pratt & Whitney F100 turbofan jet engine with afterburner, which is fitted on the General Dynamics F-16 light-weight fighter and the McDonnell Douglas F-15, Eagle [Jane's All The World's Aircraft (1984)]. A schematic drawing of this engine is shown in Figure (2.1). The F100 is a twin-spool, axial-flow turbofan with a bypass ratio of 0.7 at sea-level, static, standard-day conditions and is a representative of current high technology jet engines. A single inlet is used for fan airflow and engine core airflow. Airflow leaving the fan is separated into two flow streams; one stream passing through the core, and the other stream passing through the annular fan duct. A three-stage fan is driven by a two-stage, low pressure turbine. A two-stage, high pressure, air-cooled turbine drives the 10-stage compressor. The fan has variable trailing-edge compressor inlet guide vanes. These inlet vanes are positioned to improve inlet aerodynamics and fan efficiency. The compressor has rear rows of variable stator vanes; these vanes are positioned to improve starting and to provide good compressor operating characteristics. Airflow bleed is extracted at the compressor exit and is discharged through the fan duct for starting. Bleed is also extracted to provide turbine cooling and to satisfy installation requirements. The main combustor consists of an annular diffuser and a chamber with 16 fuel nozzles. The engine core and fan duct streams combine in an afterburner that consists of a diffuser and five concentric fuel manifolds. The afterburner discharges through a variable convergent-divergent nozzle. The variable nozzle area geometry provides nearly optimum nozzle area and expansion ratio through the operating range. Figures (2.2) and (2.3) show the maps of the low and high pressure compressors respectively*. Figures (2.4) and (2.5) show the maps of the low and high pressure turbines respectively*.

* The variables involved in these figures are defined later.
Figure 2.1: Schematic Drawing of the F-100 Jet Engine Which Powers the General Dynamics F-15 Eagle.

<table>
<thead>
<tr>
<th>MAIN NOZZLE THROAT AREA</th>
<th>AV</th>
</tr>
</thead>
<tbody>
<tr>
<td>COMBUSTOR FUEL FLOW RATE</td>
<td>WP</td>
</tr>
<tr>
<td>AFT BURNER PRESSURE</td>
<td>P1</td>
</tr>
<tr>
<td>COMBUSTOR INTERNAL ENERGY</td>
<td>U4</td>
</tr>
<tr>
<td>COMBUSTOR EXIT PRESSURE</td>
<td>P4</td>
</tr>
<tr>
<td>LP COMPRESSOR/TURBINE RPM</td>
<td>NF</td>
</tr>
<tr>
<td>HP COMPRESSOR/TURBINE RPM</td>
<td>NC</td>
</tr>
</tbody>
</table>

Legend:
- LP: Low Pressure
- HP: High Pressure
- TURBINE
- COMP: Compressor
- AFTER BURNER
- PROPELLING NOZZLE
- INLET
Low Pressure Compressor Map

Figure 2.2 Low Pressure Compressor Map
Figure 2.3 High Pressure Compressor Map
High Pressure Turbine Map

Figure 2.4 High Pressure Turbine Map
Figure 2.5 Low Pressure Turbine Map
DYNGEN - The Simulation Program

DYNGEN, a Fortran program for analyzing steady-state and transient performance of jet engines, is based on three earlier NASA-computer programs SMOTE, McKinney (1967), GENGEN, Koenig Fishbach (1972), and GENGENII, Fishbach and Koening (1972), which are capable of calculating the steady-state performance of turbojet and turbofan engines at design and off-design operating conditions. However, as the need to predict the transient performance of these engines becomes more important in preliminary design and control studies, DYNGEN was developed to provide this added capability. The result is a digital program which can perform both steady-state and transient calculations at design and off-design conditions on a variety of jet engine configurations. Thus DYNGEN can solve 16 non-linear differential equations, uses data maps and thermodynamic tables, and is coded with data taken from experimental measurements on a real prototype of the F100 engine. This simulation program was originally written in FORTRAN IV for the NASA Lewis Research Centre machine, an IBM 7904 model 2 which has a 64-bit word length. DYNGEN was input manually into a Honeywell Multics at Loughborough University of Technology Computer Centre, (a computer which has a 36-bit word length) but certain modifications had to be made to the program to make it as 'machine independent' as possible [Appendix A], while direct contact was made with its author for information and the latest program amendments.

On the whole, DYNGEN performed quite well. Comparisons with published simulation runs from NASA Lewis Research Centre, Sellers (1975), verified that the program worked as expected. However, DYNGEN failed in some operating conditions which required large variations in fuel flow, because such conditions cause a number of engine parameters to operate in a region not defined by the component maps. Experiments with DYNGEN can provide knowledge of how to avoid such extreme conditions. This matter was also reported by Shearer (1977).

OMAR - The F100 Analytical Non-linear Model

OMAR was developed because it was fundamental that an analytical model for the F100 engine be available for the application of optimal control techniques. OMAR was intended to be an approximation of DYNGEN based on the specified engine configuration. There is no general agreement as to what the order of the mathematical model of any jet engine system should be. It is a physical, not a mathematical, entity and, consequently, every mathematical model must be an approximation to reality. In general, the higher is the order of the model, the more accurate is the approximation. The order that was selected was fifth and its selection was a function of two parameters: first, how good was the agreement with DYNGEN, and second, the computational limitations imposed by the algorithms required...
by the control study which followed. OMAR includes five differential equations, which
govern the dynamic behaviour of the system, along with 36 algebraic equations which
express the relationship between the various engine variables. Since the F100 is a two-spool
turbofan engine with an afterburner, then the most appropriate candidates as state variables
were chosen as listed hereafter:

1. The high pressure compressor speed, \( N_C \).
2. The low pressure compressor (fan) speed, \( N_F \).
3. The combustor exit pressure, \( P_4 \), a variable which is strongly affected by changes in
   the control variable, \( W_{FB} \), the combustor fuel flow.
4. The combustor internal energy, \( U_4 \), a variable which is related by a constant to the
   combustor exit temperature, \( T_4 \).
5. The afterburner exit pressure, \( P_7 \), a variable which is strongly affected by changes in the
   main nozzle throat area, \( A_8 \).

The dynamics of the above selected state variables and their related outputs can be described
by the general state and output equations,

\[
\dot{x} = f(x, u, t) \quad (2.1)
\]
\[
y = g(x, u, t) \quad (2.2)
\]

where \( x \) is the state vector \( (x_1 \ x_2 \ \ldots \ x_n)^T \), \( x \in \mathbb{R}^n \)
\( u \) is the input vector \( (u_1 \ u_2 \ \ldots \ u_m)^T \), \( u \in \mathbb{R}^m \)
\( y \) is the output vector \( (y_1 \ y_2 \ \ldots \ y_p)^T \), \( y \in \mathbb{R}^p \)
\( t \) is the independent variable, time

The unsteady power balance, continuity and energy equations were used to establish
equation (2.1). For example:

1. The power output from a turbine must equal the power absorbed by the compressor
   plus the power \( (M \ \frac{d\theta}{dt}) \) required to provide rotor acceleration:

   \[
   W_T \Delta h_T = W_c \Delta h_c + M \frac{d\theta}{dt} \quad (2.3)
   \]

   \[
   \text{where: angular velocity} \quad \frac{d\theta}{dt} = \frac{2\pi}{60} N
   \]

   \[
   \text{torque} \quad M = I \alpha
   \]

   \[
   \text{angular acceleration} \quad \alpha = \frac{d}{dt} \left( \frac{d\theta}{dt} \right) = \frac{2\pi}{60} \cdot \frac{dN}{dt}
   \]
Hence: \[ M \frac{d\theta}{dt} = \left( \frac{2\pi}{60} \right) \left( \frac{dN}{dt} \right) \frac{2\pi}{60} N = \left( \frac{2\pi}{60} \right)^2 I N \frac{dN}{dt} \] (2.4)

Then equation (2.3) will become:

\[ \dot{W}_T \Delta h_T = \dot{W}_c \Delta h_c + \left( \frac{2\pi}{60} \right)^2 I N \frac{dN}{dt} \] (2.5)

where:
- \( I \) is the moment of inertia of the rotor. The rotor represents the assembly of a turbine disc, compressor disc and the shaft connecting them.
- \( N \) is the rotor speed, in revs/min.
- \( \Delta h \) is the enthalpy change.
- \( W \) is the mass flow rate.

The suffix \( T \) indicates turbine and the suffix \( c \) denotes compressor.

Equation (2.5) is used to establish the differential equations which govern both NC and NF when the proper values are used.

2. The rate of mass stored in any control volume is proportional to the time derivative of the pressure of that control volume (\( dP/dt \)):

\[ \dot{W}_\text{out} = \dot{W}_\text{in} - \left( \frac{V}{\gamma RT} \right) \frac{dP}{dt} \] (2.6)

where:
- \( V, T, P \) are the volume, temperature and the pressure of that control volume respectively.
- \( R \) is the gas constant.
- \( \gamma \) is the ratio of the specific heats.

Equation (2.6) is used to establish the differential equations which govern both \( P_4 \) and \( P_7 \), when the proper values are used.

3. In unsteady flow, energy storage is accounted for by two terms: one reflecting the rate of change of specific internal energy (\( dU/dt \)), and another reflecting energy storage caused by mass storage:

\[ \dot{W}_\text{out} h_\text{out} = \dot{W}_\text{in} h_\text{in} - (\dot{W}_\text{in} - \dot{W}_\text{out})U - \left( \frac{PV}{RT} \right) \frac{dU}{dt} \] (2.7)
Equation (2.7) is used to establish the differential equation which govern U4.

Algebraic relationships relating the rest of the engine variables were derived as a starting point, using both the theoretical relationships, developed by Brenan and Leake (1975) and Longenbaker and Leake (1978) and data generated from extensive runs of DYNGEN. In some situations where the theoretical form was unavailable; polynomial, linear and exponential forms were used, whatever seemed to best fit the situation. In most cases, the variables used in OMAR corresponded to those of DYNGEN. Table 2.1 is a list of all variables used, and Table 2.2 gives a listing of the inputs, state variables and outputs. Tables 2.3 and 2.4 are listings of the values of the constants and the variables of the model for the specified engine. Equations (2.8) to (2.41) represent a listing of the non-linear relationships existing between the state variables and the intermediate variables. Equations (2.42) to (2.46) represent a listing of the non-linear state equations.

In OMAR, as DYNGEN, there are certain physical engine constraints that must not be exceeded. For this study, those constraints are:

1. For high pressure turbine inlet temperature, T4, a value of 3500°F is the upper limit.

2. Surge margins for both the low and high pressure compressors, i.e. ZF and ZC. The surge margin is a measure of how far from or near to the operating point the surge line is. It is a fraction; the larger the fraction, the closer to surge line or the smaller the surge margin. The surge margins must never reach a value of unity, else surge will take place.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Variable Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALTP</td>
<td>altitude, (ft)</td>
</tr>
<tr>
<td>P1</td>
<td>standard pressure, (atm)</td>
</tr>
<tr>
<td>T1</td>
<td>standard temperature, ( ^\circ R ) { function of ALTP }</td>
</tr>
<tr>
<td>AM</td>
<td>Mach number of aircraft</td>
</tr>
<tr>
<td>( \eta I )</td>
<td>adiabatic inlet efficiency</td>
</tr>
<tr>
<td>P2</td>
<td>fan inlet total pressure, (atm)</td>
</tr>
<tr>
<td>T2</td>
<td>fan inlet total temperature, ( ^\circ R )</td>
</tr>
<tr>
<td>( \frac{T}{T} )I</td>
<td>inlet temperature ratio</td>
</tr>
<tr>
<td>( \frac{P}{p} )I</td>
<td>inlet isentropic pressure ratio</td>
</tr>
<tr>
<td>A8</td>
<td>main nozzle throat area, (ft(^2))</td>
</tr>
<tr>
<td>CNC</td>
<td>corrected compressor rotor speed</td>
</tr>
<tr>
<td>CNF</td>
<td>corrected fan rotor speed</td>
</tr>
<tr>
<td>FG</td>
<td>thrust, (lbf)</td>
</tr>
<tr>
<td>NC</td>
<td>compressor rotor speed, (rpm)</td>
</tr>
<tr>
<td>NF</td>
<td>fan rotor speed, (rpm)</td>
</tr>
<tr>
<td>PCMAX</td>
<td>compressor pressure ratio at surge</td>
</tr>
<tr>
<td>PFMAX</td>
<td>fan pressure ratio at surge</td>
</tr>
<tr>
<td>P21</td>
<td>fan exit (compressor inlet) pressure, (atm)</td>
</tr>
<tr>
<td>P3</td>
<td>compressor exit pressure, (atm)</td>
</tr>
<tr>
<td>P4</td>
<td>combustor exit pressure, (atm)</td>
</tr>
<tr>
<td>P7</td>
<td>afterburner exit pressure, (atm)</td>
</tr>
<tr>
<td>T21</td>
<td>fan exit (compressor inlet) temperature, ( ^\circ R )</td>
</tr>
<tr>
<td>T3</td>
<td>compressor exit temperature, ( ^\circ R )</td>
</tr>
<tr>
<td>T4</td>
<td>combustor exit temperature, ( ^\circ R )</td>
</tr>
<tr>
<td>T50</td>
<td>high pressure turbine exit temperature, ( ^\circ R )</td>
</tr>
<tr>
<td>T55</td>
<td>low pressure turbine exit temperature, ( ^\circ R )</td>
</tr>
<tr>
<td>T7</td>
<td>afterburner exit temperature, ( ^\circ R )</td>
</tr>
<tr>
<td>U4</td>
<td>combustor internal energy, (Btu/lbm)</td>
</tr>
<tr>
<td>WAC</td>
<td>compressor airflow rate, (lbf/sec)</td>
</tr>
<tr>
<td>WAF</td>
<td>fan airflow rate, (lbf/sec)</td>
</tr>
<tr>
<td>WA3</td>
<td>airflow rate into combustor, (lbf/sec)</td>
</tr>
</tbody>
</table>

Table 2.1 Symbols for Variables
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Variable Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>WCMAX</td>
<td>maximum compressor airflow rate, (lbm/sec)</td>
</tr>
<tr>
<td>DWCMAX</td>
<td>correction term for maximum compressor airflow rate</td>
</tr>
<tr>
<td>WFB</td>
<td>fuel flow rate into combustor, (lbm/sec)</td>
</tr>
<tr>
<td>WFMAX</td>
<td>maximum fan airflow rate, (lbm/sec)</td>
</tr>
<tr>
<td>WG4</td>
<td>gaseous flow rate out of combustor, (lbm/sec)</td>
</tr>
<tr>
<td>WG50</td>
<td>gaseous flow rate out of high pressure turbine, (lbm/sec)</td>
</tr>
<tr>
<td>WG55</td>
<td>gaseous flow rate out of low pressure turbine, (lbm/sec)</td>
</tr>
<tr>
<td>WG7</td>
<td>gaseous flow rate out of afterburner, (lbm/sec)</td>
</tr>
<tr>
<td>ZC</td>
<td>compressor surge margin</td>
</tr>
<tr>
<td>ZF</td>
<td>fan surge margin</td>
</tr>
<tr>
<td>PRF</td>
<td>pressure ratio across fan</td>
</tr>
<tr>
<td>PRC</td>
<td>pressure ratio across compressor</td>
</tr>
<tr>
<td>WFA</td>
<td>fuel flow rate to afterburner, (lbm/sec)</td>
</tr>
<tr>
<td>FGM</td>
<td>momentum thrust, (lbf)</td>
</tr>
<tr>
<td>FGP</td>
<td>pressure thrust, (lbf)</td>
</tr>
<tr>
<td>FG</td>
<td>gross thrust, (lbf)</td>
</tr>
<tr>
<td>V8</td>
<td>main nozzle stream velocity, (ft/sec)</td>
</tr>
<tr>
<td>CVMNOZ</td>
<td>main nozzle thrust coefficient</td>
</tr>
<tr>
<td>TFF</td>
<td>turbine flow function, (lbm - (\sqrt{\text{R}}) - (\text{in}^2/\text{sec}) - lbf)</td>
</tr>
<tr>
<td>DH</td>
<td>turbine delta enthalpy (temperature corrected) ((H_{in} - H_{out})/T_{in}) ((\text{Btu/lbm -} ^\text{oR}))</td>
</tr>
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Table 2.1 Symbols for Variables cont.
<table>
<thead>
<tr>
<th>Variable Description</th>
<th>Symbol</th>
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<tr>
<td>fuel flow</td>
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<td>nozzle area</td>
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</tr>
<tr>
<td>compressor rotor speed</td>
<td>NC</td>
</tr>
<tr>
<td>fan rotor speed</td>
<td>NF</td>
</tr>
<tr>
<td>burner exit pressure</td>
<td>P4</td>
</tr>
<tr>
<td>afterburner exit pressure</td>
<td>P7</td>
</tr>
<tr>
<td>high pressure turbine inlet energy</td>
<td>U4</td>
</tr>
<tr>
<td>thrust</td>
<td></td>
</tr>
<tr>
<td>high pressure turbine inlet temperature</td>
<td>T4</td>
</tr>
<tr>
<td>compressor surge margin</td>
<td>ZC</td>
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<tr>
<td>fan surge margin</td>
<td>ZF</td>
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Table 2.2  Input, State and Output Variables
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(J)</td>
<td>mechanical equivalent of heat</td>
<td>778.26</td>
</tr>
<tr>
<td>(R)</td>
<td>gas constant</td>
<td>.0252</td>
</tr>
<tr>
<td>(γ)</td>
<td>ratio of specific heats</td>
<td>1.4</td>
</tr>
<tr>
<td>(T1)</td>
<td>fan inlet temperature</td>
<td>518.668</td>
</tr>
<tr>
<td>(IC)</td>
<td>high pressure rotor polar moment of inertia</td>
<td>3.8</td>
</tr>
<tr>
<td>(IF)</td>
<td>low pressure rotor polar moment of inertia</td>
<td>4.5</td>
</tr>
<tr>
<td>VCOMB</td>
<td>combustor volume</td>
<td>1.65</td>
</tr>
<tr>
<td>VA FBN</td>
<td>afterburner volume</td>
<td>49.77</td>
</tr>
<tr>
<td>CVMNOZ</td>
<td>nozzle thrust coefficient</td>
<td>.9494</td>
</tr>
<tr>
<td>NCRE F</td>
<td>high pressure rotor reference speed</td>
<td>10070</td>
</tr>
<tr>
<td>NFREF</td>
<td>low pressure rotor reference speed</td>
<td>9651</td>
</tr>
<tr>
<td>(η)</td>
<td>combustor efficiency</td>
<td>20.71175</td>
</tr>
<tr>
<td>(CPC)</td>
<td>compressor specific pressure</td>
<td>.24</td>
</tr>
<tr>
<td>(CPF)</td>
<td>fan specific pressure</td>
<td>.24</td>
</tr>
<tr>
<td>(CVB)</td>
<td>combustor specific volume</td>
<td>.20279</td>
</tr>
<tr>
<td>(CPHT)</td>
<td>high pressure turbine specific pressure</td>
<td>.22589</td>
</tr>
<tr>
<td>(CPLT)</td>
<td>low pressure turbine specific pressure</td>
<td>.27938</td>
</tr>
<tr>
<td>(PCBL C)</td>
<td>percent of compressor exit air bled for cooling</td>
<td>.16</td>
</tr>
<tr>
<td>(PCBLDU)</td>
<td>percent of bleed air which leads into fan duct</td>
<td>.208</td>
</tr>
<tr>
<td>(PCBL HP)</td>
<td>percent of bleed air put into high pressure turbine</td>
<td>.726</td>
</tr>
<tr>
<td>(PCBLLP)</td>
<td>percent of bleed air put into low pressure turbine</td>
<td>.066</td>
</tr>
</tbody>
</table>

Table 2.3  Constants at Take-off Point
<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>WFB</td>
<td>2.75</td>
<td>PFMAX</td>
<td>3.3733</td>
</tr>
<tr>
<td>A8</td>
<td>2.94826</td>
<td>WAF</td>
<td>220.89</td>
</tr>
<tr>
<td>NC</td>
<td>11928.5</td>
<td>WCMAX</td>
<td>54.577</td>
</tr>
<tr>
<td>NF</td>
<td>9904.1</td>
<td>DWCMAX</td>
<td>1.5735</td>
</tr>
<tr>
<td>P4</td>
<td>23.974</td>
<td>PCMAX</td>
<td>10.3024</td>
</tr>
<tr>
<td>U4</td>
<td>587.386</td>
<td>WAC</td>
<td>137.758</td>
</tr>
<tr>
<td>P7</td>
<td>2.4514</td>
<td>WA3</td>
<td>115.717</td>
</tr>
<tr>
<td>CNF</td>
<td>1.0262</td>
<td>WG50</td>
<td>134.469</td>
</tr>
<tr>
<td>T21</td>
<td>744.32</td>
<td>WG4</td>
<td>118.467</td>
</tr>
<tr>
<td>CNC</td>
<td>0.9888</td>
<td>WG55</td>
<td>135.923</td>
</tr>
<tr>
<td>T3</td>
<td>1470.82</td>
<td>T55</td>
<td>1790.74</td>
</tr>
<tr>
<td>T4</td>
<td>2896.52</td>
<td>T7</td>
<td>1412.54</td>
</tr>
<tr>
<td>T50</td>
<td>2105.77</td>
<td>WG7</td>
<td>223.64</td>
</tr>
<tr>
<td>P3</td>
<td>25.399</td>
<td>FG</td>
<td>13364</td>
</tr>
<tr>
<td>P21</td>
<td>2.9829</td>
<td>ZC</td>
<td>0.80784</td>
</tr>
<tr>
<td>WFB MAX</td>
<td>203.939</td>
<td>ZF</td>
<td>.83548</td>
</tr>
</tbody>
</table>

Table 2.4 Equilibrium Values for Engine Variables at Take-off Point
The following equations represent the non-linear relationships which exist between the state variables and the intermediate variables:

\[
\frac{T}{T_1} = 1.0 + 0.2 \left( AM \right)^2 \tag{2.8}
\]

\[
\frac{P}{P_1} = \left( \frac{T}{T_1} \right)^{3.5} \tag{2.9}
\]

\[
\eta_i = \begin{cases} 
1.0 & \text{if } AM \leq 1.0 \\
1.0 - 0.075(AM-1.0)^{1.35} & \text{if } AM > 1.0 
\end{cases} \tag{2.10}
\]

Then

\[
T_2 = \left( \frac{T}{T_1} \right) T_1 \tag{2.11}
\]

\[
P_2 = \left( \frac{P}{P_1} \right) P \eta_i \tag{2.12}
\]

\[
CNF = \frac{NF}{NFREF} = \frac{NF}{9651} \tag{2.13}
\]

\[
T_{21} = T_2 + 214.2732 \cdot CNF^2 - 48(A8 - 2.94826) \tag{2.14}
\]

\[
CNC = \frac{NC}{NCREF \sqrt{T_{21}/T_2}} = \frac{NC}{10070 \sqrt{T_{21}/518.68}} \tag{2.15}
\]

\[
T_3 = T_{21} + 743.2722 \cdot CNC^2 - 68(A8 - 2.94826) \tag{2.16}
\]

\[
T_4 = U_4/CVB \tag{2.17}
\]

\[
T_{50} = 0.727T_4 \tag{2.18}
\]

\[
P_3 = 1.05944P_4 \tag{2.19}
\]

\[
P_{21} = -6.20568 - 0.0129774T_{21} - 0.0185376P_3 \tag{2.20}
\]

\[
WF_{\text{MAX}} = 261.01 \cdot CNF - 63.196 \tag{2.21}
\]

\[
PF_{\text{MAX}} = 3.516739 \cdot CNF - 0.23561 \tag{2.22}
\]

\[
K = 1.0 - e^{-2.31326(PF_{\text{MAX}} - (P_{21}/P_2))} \tag{2.22a}
\]

\[
WAF = WF_{\text{MAX}} + 28.502K \tag{2.22b}
\]
\[
\text{WCMAX} = 137.54 - 457.987\,\text{CNC} + 564.325\,\text{CNC}^2 - 188.113\,\text{CNC}^3
\] (2.23)

\[
\text{DWCMAX} = 6.492 - 4.9749\,\text{CNC}
\] (2.24)

\[
\text{PCMAX} = 26.43184 - 89.0484\,\text{CNC} + 109.724\,\text{CNC}^2 - 36.575\,\text{CNC}^3
\] (2.25)

\[
\text{WAC} = \frac{(P21/P2)}{\sqrt{T21/518.668}} \left\{ \text{WCMA}X + \text{DWCMAX} \right\} K1
\] (2.26)

\[
K1 = 1.0 - e^{-0.36(\text{PCMAX} - \text{P3}/P21)}
\] (2.26a)

\[
\text{PRF} = P21/P2
\] (2.27)

\[
\text{PRC} = P3/P21
\] (2.28)

\[
\text{WA3} = (1.0 - \text{PCBL})\,\text{WAC}
\] (2.29)

\[
\text{WG50} = \frac{301.97\,\text{P4}}{\sqrt{T4}} + 3.9699\,\text{P7}
\] (2.30)

\[
\text{WG4} = \text{WG50} - \text{PCBL}\,(\text{PCBL})\,\text{WAC}
\] (2.31)

\[
\text{WG55} = \text{WG50} + \text{PCBL}\,(\text{PCBL})\,\text{WAC}
\] (2.32)

\[
\text{T55} = 106.002 + 0.8615\,T50 - 0.10458\,(\text{CNC})\sqrt{T21(T50)}
\] (2.33)

\[
\text{T7} = 0.49661\,T55 + 205.886\,\text{P7}
\] (2.34)

\[
\text{WG7} = \frac{1121.786\,\text{P7}(A8)}{\sqrt{T7}}
\] (2.35)

\[
\text{FGP} = 2116.217\,(0.53978\,\text{P7} - \text{P2})\,A8
\] (2.36)

\[
\text{V8} = \sqrt{1934.415\,\text{T7} + 68558.36}
\] (2.37)

\[
\text{FGM} = \text{CVMNOZ(V8)(WG7/G)}
\] (2.38)

\[
\text{FG} = \text{FGP} + \text{FGM}
\] (2.39)

\[
\text{ZC} = \frac{\text{PRC} - 1.0}{\text{PCMAX} - 1.0}
\] (2.40)

\[
\text{ZF} = \frac{\text{PRF} - 1.0}{\text{PCMAX} - 1.0}
\] (2.41)
OMAR dynamics described by the general state equation (2.1) are detailed here in the following non-linear differential equations:

\[
\frac{dN_C}{dt} = \left(\frac{30.2}{\pi}\right) \frac{J}{KC(\text{NC})} \left[\text{CPC}(WAC)(T21-T3)+\text{CPHT}(WG50)(T4-T50)\right] \quad (2.42)
\]

\[
\frac{dN_F}{dt} = \left(\frac{30.2}{\pi}\right) \frac{J}{IR(\text{NF})} \left[\text{CPF}(WAF)(T2-T21)+\text{CPLT}(WG55)(T50-T55)\right] \quad (2.43)
\]

\[
\frac{dP4}{dt} = \frac{R(\gamma)(T4)}{V\text{COMB}} \left[W43 + WFB - W4\right] \quad (2.44)
\]

\[
\frac{dP7}{dt} = \frac{R(\gamma)(T7)}{V\text{AFBN}} \left[W4 - W43 + WAF + WFA - W7\right] \quad (2.45)
\]

\[
\frac{dU4}{dt} = \frac{CVB(R)(T4)}{V\text{COMB}(P4)} \left[T4\{W4 - WFB - W43\} + \gamma(T3 W43 - T4 W4 + T4(1 + \eta) WFB\}\right] \quad (2.46)
\]
2.5 Aircraft Operating Range

The main segments of a typical mission profile are take-off, climb, cruise, descent and landing, as shown schematically in Figure (2.6). The details of those segments depend on the specific mission of an aircraft. In this work, certain similarities (apart from afterburning features) between the Rolls-Royce SPEY and the F100, were useful in choosing a demonstration flight envelope. Although several operating points were investigated, only six of them were used in this work. These operating points are:

1. Take-off at sea level, standard day conditions
2. Climb at 4000 ft., Mach number 0.3
3. Subsonic cruise at 35000 ft., Mach number 0.6
4. Approach at 1500 ft., Mach number 0.15
5. Sea level, standard day, take-off, military rating
6. Supersonic flight at 35000 ft., Mach number 1.3

Operating conditions 5 and 6 require the use of afterburning. Afterburning (or reheat) is a method of augmenting the basic thrust of an engine to improve the aircraft take-off, climb and (for military aircraft) combat performance. Figure (2.7) shows the effect of afterburning on the rate of climb for military rating.

While in supersonic flight a considerable increase in exhaust jet velocity is required. The afterburner reheat from after the turbines, permitting it to accelerate to an appropriate level above the flight velocity and boosting the thrust to overcome the increased drag. Figures (2.8) and (2.9) show the effect of afterburner fuel flow on engine thrust, afterburner temperature and on the main nozzle area, specific fuel consumption and afterburner pressure respectively.

It is worth mentioning here that Figures (2.8) and (2.9) demonstrate one of the main principles of afterburning, which is: when afterburning is selected the gas temperature increases and the nozzle opens to give an extra area suitable for the resultant increase in the volume of the gas stream. This prevents any increase in pressure occurring that would affect the functioning of the engine and enables afterburning to be used over a wide range of engine speed.
Figure 2.6: Example of a typical mission profile

Figure 2.7: Afterburning and its effect on the rate of climb
Fig. 2.8 Effect of Afterburner Fuel Flow On Engine Thrust and T7

Fig. 2.9 Effect of Afterburner Fuel Flow on A8, SFC & P7
2.6 OMAR and DYNGEN Comparison

Figures (2.10), (2.11), (2.12) and (2.13) show the dynamic responses for both OMAR and DYNGEN at sea level condition for the engine variables: high pressure compressor corrected speed, turbine inlet temperature, high pressure compressor surge margin and the thrust respectively. From these transients, one can notice that there is a reasonable agreement throughout, but in general the time response of OMAR is faster than that of DYNGEN.

Table 2.5 shows a statistical analysis for OMAR and DYNGEN variables throughout the operating points:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean Me</th>
<th>Standard Deviation STDEV</th>
<th>Standard Error of the Mean SE</th>
<th>Mean</th>
<th>t-Test value 't'</th>
</tr>
</thead>
<tbody>
<tr>
<td>NC</td>
<td>24</td>
<td>252</td>
<td>103</td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>NF</td>
<td>157.4</td>
<td>242.2</td>
<td>98.9</td>
<td>1.59</td>
<td></td>
</tr>
<tr>
<td>P4</td>
<td>0.351</td>
<td>0.42</td>
<td>0.171</td>
<td>2.05</td>
<td></td>
</tr>
<tr>
<td>P7</td>
<td>0.0815</td>
<td>0.1090</td>
<td>0.0445</td>
<td>1.83</td>
<td></td>
</tr>
<tr>
<td>U4</td>
<td>4.67</td>
<td>6.34</td>
<td>2.59</td>
<td>1.81</td>
<td></td>
</tr>
<tr>
<td>FG</td>
<td>221</td>
<td>426</td>
<td>174</td>
<td>1.27</td>
<td></td>
</tr>
<tr>
<td>T4</td>
<td>83.0</td>
<td>102.8</td>
<td>42.0</td>
<td>1.98</td>
<td></td>
</tr>
<tr>
<td>ZC</td>
<td>0.0172</td>
<td>0.0279</td>
<td>0.0114</td>
<td>1.51</td>
<td></td>
</tr>
<tr>
<td>ZF</td>
<td>0.0077</td>
<td>0.0262</td>
<td>0.0107</td>
<td>0.72</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.5 Statistical Analysis for DYNGEN and OMAR Variables Throughout the Operating Points
Fig. 2.10 CNC Comparison for DYNGEN and OMAR

Legend

DYNGEN STEP INPUT
OMAR STEP INPUT

Fig. 2.11 T4 Comparison for DYNGEN and OMAR

Legend

DYNGEN STEP INPUT
OMAR STEP INPUT
Fig. 2.12 ZC Comparison for DYNGEN and OMAR

Fig. 2.13 FG Comparison for DYNGEN and OMAR
where the notations are defined as follows:

\[ X = \text{the raw score} \]

\[ N = \text{the number of cases} \]

\[ M_e = \frac{\sum X}{N} \]

\[ d = \text{the deviation score obtained by subtracting the mean from each row score} = X - M_e \]

\[ \text{STDEV} = \sqrt{\frac{\sum d^2}{N-1}} \]

\[ \text{SE MEAN} = \text{Standard Error of the Mean} = \frac{\text{STDEV}}{\sqrt{N}} \]

The statistical significance can be measured using the t-test, where 't' is given by:

\[ t = \frac{\sum D}{\sqrt{N\sum D^2 - (\sum D)^2}} \]

\[ D = \text{the difference between each subject's two scores.} \]

The above equations were based on the analysis of Cohen and Holliday (1979). For the type of data generated from OMAR and DYNGEN comparison, the two-tailed test was useful and the critical value of that t-test distribution was 4.03. If the 't' value of a test is greater than any one of the above values for the similar case, then the level of error will be significant. From Table 2.5, the 't' values are all within the acceptable range, that make the level of error insignificant.

Figures (2.14), (2.15), (2.16) and (2.17) show the comparison between OMAR and DYNGEN for the engine outputs FG, T4, ZC and ZF throughout the operating points presented as bar charts.

### 2.7 Linear Engine Models

A linearised model of the F100 Pratt & Whitney engine was derived from OMAR around each of the six operating points of Section (2.5). A linear model is represented by the following state and output equations:

\[ \dot{x} = Ax + Bu \quad (2.47) \]

\[ y = Cx + Du \quad (2.48) \]
Fig. 2.14 Steady State Comparison for DYNGEN and OMAR

Fig. 2.15 Steady State Comparison for DYNGEN and OMAR
Fig. 2.16 ZC Steady State Comparison for DYNGEN and OMAR

Fig. 2.17 ZF Steady State Comparison for DYNGEN and OMAR
where $x$ is the state vector $\mathbb{R}^n$, $u$ is a control vector $\mathbb{R}^m$, $y$ is an output vector $\mathbb{R}^p$, $A$, $B$, $C$, $D$ are matrices of appropriate order. The state, control and output vectors are defined as follows:

$$
\begin{align*}
   x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \\
   &\quad \text{High pressure compressor speed} \quad \text{NC} \\
   &\quad \text{Low pressure compressor speed} \quad \text{NF} \\
   &\quad \text{Combustor internal pressure} \quad \text{P4} \\
   &\quad \text{Afterburner pressure} \quad \text{P7} \\
   &\quad \text{Combustion chamber internal energy} \quad \text{U4} \\

   u &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
   &\quad \text{Combustor fuel flow rate} \quad \text{WFB} \\
   &\quad \text{Exhaust nozzle area} \quad \text{A8} \\

   y &= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \\
   &\quad \text{Engine Thrust} \quad \text{FG} \\
   &\quad \text{Turbine inlet temperature} \quad \text{T4} \\
   &\quad \text{HP Compressor surge margin} \quad \text{ZC} \\
   &\quad \text{LP Compressor surge margin} \quad \text{ZF} \\
\end{align*}
$$

In Equations (2.1), (2.2) if a 'datum' state $x_d$ and corresponding input $u_d$ are chosen, the behaviour of a small perturbation in the state $\Delta x$, caused by a small perturbation in the input, $\Delta u$, is modeled using the first terms of a Taylor expansion:

$$
\begin{align*}
   x + \Delta x &= f(x_d, u_d, t) + \left. \frac{\partial f}{\partial x} \right|_{x_d, u_d} \Delta x + \left. \frac{\partial f}{\partial u} \right|_{x_d, u_d} \Delta u \quad (2.49)
\end{align*}
$$

where

$$
\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \ddots \\ \frac{\partial f_n}{\partial x_1} & & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}
$$
From equations (2.47), (2.48) and (2.49):

\[
\frac{\partial f}{\partial u} = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \ldots & \frac{\partial f_1}{\partial u_m} \\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \ldots & \frac{\partial f_2}{\partial u_m} \\
\vdots & \vdots & & \vdots \\
\frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \ldots & \frac{\partial f_n}{\partial u_m}
\end{bmatrix}
\]

Two methods were used to perform the partial derivatives of equation (2.49):

1. A very tedious hand-calculated derivation in which the analytical forms of those derivatives were obtained.


All results from methods 1 and 2 were identical.

The reasons for using these methods were:

1. The first and the second partial derivatives of the state equations were needed in an analytical rather than numerical form, for the control studies to follow.

2. Because of the special nature of the numerical derivative process, a means of checking the quality of its output was required. So both methods were used in conjunction.
The linear models of the engine around the chosen operating points are specified as follows:

1. LIN1 is the linear model resulted from the linearization of OMAR about operating point 1.
2. LIN2 is the linear model resulted from the linearization of OMAR about operating point 2.
3. LIN3 is the linear model resulted from the linearization of OMAR about operating point 3.
4. LIN4 is the linear model resulted from the linearization of OMAR about operating point 4.
5. LIN5 is the linear model resulted from the linearization of OMAR about operating point 5.
6. LIN6 is the linear model resulted from the linearization of OMAR about operating point 6.

In [Appendix B] the A, B, C and D matrices for, the linear models LIN1, LIN2, LIN3, LIN4, LIN5 and LIN6 respectively, are presented. The analytical formulas used to generate the elements of the A matrix are presented in [Appendix C].

2.8 Concluding Remarks

OMAR, a fifth order non-linear mathematical model of the F100 jet engine was developed. It was based on DYNGEN, a Fortran program for analyzing steady state and transient performance of jet engines. OMAR was intended to be an approximation of DYNGEN based on the specified engine configuration, and it was developed because it was essential that an analytical model of F100 engine should be available for the application of optimal control techniques. Several points were chosen to define a number of flight envelope boundaries and a linear analytical model, LIN, was derived about each of these points. At later stages of this work, OMAR was used to produce a piecewise linear model of the engine, i.e. a linear analytical model at each communication interval; this process proved to be useful. It was noticed that the time responses with OMAR were faster than those from DYNGEN and with values of overshoot which were a little higher, but in general, the agreement between the responses was sufficiently close to ensure that OMAR was as good a model as that from DYNGEN. Some minor corrective tuning may still be needed when any of the control laws based on OMAR are applied on a real engine, and, in particular, in those regions near the surge limits.
Chapter 3

Solution of a Linear Quadratic Problem

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3.1 Introduction

In an attempt to control the dynamic response of the multivariable jet engine system presented in Chapter 2, the Linear Quadratic Problem, LQP, technique was adopted.

The solution to this multivariable technique is a feedback control law, which is relatively easy to synthesize, and if the problem is correctly formulated, then this feedback law will guarantee stability around the given operating point.

This solution will also ensure optimality by minimizing a quadratic type performance criterion that contains weighting matrices for the required variables, where the dynamics of those variables are governed by the choice of those weighting matrices.

In LQP the control problem may be formulated in three different forms, namely:

1. a state regulator problem
2. an output regulator problem
3. an optimal tracking problem

In the first two forms, the problem solution will bring the deviations in the state and output vectors rapidly back to zero, and without excessive control expenditure, while in the third form the problem solution will be such that it makes the output vector try to follow closely some specified vector.

In this chapter a brief description of the state and output regulator problems is given, with more attention being paid to their numerical solution, while the optimal tracking problem and its application are discussed in later chapters. A detailed account of these problems can be found in a number of text books such as Athans and Falb (1966) and Schultz and Melsa (1967).

3.2 State Regulator Problem

The solution of the state regulator problem leads to an optimal feedback system with the property that the components of the state vector \( x(t) \) are kept near to zero without excessive expenditure of control energy. The optimal control \( u^*(t) \) is obtained by minimizing a quadratic performance index of the type,
\[ J_1 = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_0^{t_f} (x^T(t) Q x(t) + u^T(t) G u(t)) \, dt \quad (3.1) \]

subject to the constraint,

\[ \dot{x}(t) = A x(t) + B u(t) \quad (3.2) \]

where \( x(t) \) is the state vector \( \in \mathbb{R}^n \), \( u(t) \) is the control vector \( \in \mathbb{R}^m \), \( A \) is the plant matrix, of order \([n \times n]\), \( B \) is the driving matrix, of order \([n \times m]\), \( Q \) is the state weighting matrix, of order \([n \times n]\), \( G \) is the control weighting matrix, of order \([m \times m]\) and \( S \) is a matrix, of order \([n \times n]\), which penalizes the existence of errors in the state variables at the final time.

If the value of the state vector is zero at the final time \( t_f \), then the term \( x^T(t_f) S x(t_f) \) is zero, i.e. the \( S \) matrix is zero.

The Hamiltonian function, \( H \), associated with the performance index of (3.1), is defined as follows:

\[ H = \frac{1}{2} \left( x^T(t) Q x(t) + u^T(t) G u(t) \right) + \psi^T(t) \left( A x(t) + B u(t) \right) \quad (3.3) \]

where \( \psi(t)^* \) is the co-state vector \( \in \mathbb{R}^n \), and it is the solution of the vector differential equation,

\[ \dot{\psi}(t) = -\frac{\partial H}{\partial x(t)} \quad (3.4) \]

which reduces to

\[ \dot{\psi}(t) = -A^T \psi(t) - Q x(t) \quad (3.5) \]

From the transversality condition,

\[ \psi(t_f) = S \]

+ Only time-invariant systems are considered in this development

* Occurences of the symbol \( \psi \), without an index, is to be interpreted as a vector quantity throughout the dissertation.
For the Hamiltonian, $H$, to be minimized with respect to $u(t)$ then:

$$\frac{\partial H}{\partial u(t)} = G \dot{u}(t) + B^T \psi(t) = 0$$  \hspace{1cm} (3.6)$$

$$u^o(t) = - G^{-1} B^T \psi(t)$$  \hspace{1cm} (3.7)$$

The solution to (3.5) can be shown to be (see e.g. Athans and Falb (1966)),

$$\psi(t) = P(t) x(t)$$  \hspace{1cm} (3.8)$$

where the matrix, $P(t)$, is positive definite and the solution of the matrix equation,

$$\dot{P}(t) = - Q + A^T P(t) - P(t) A + P(t) B G^{-1} B^T P(t)$$  \hspace{1cm} (3.9)$$

From equation (3.7) and equation (3.8), the control law is,

$$u^o(t) = - G^{-1} B^T P(t) x(t)$$

$$= k(t) x(t)$$  \hspace{1cm} (3.10)$$

where

$$k(t) = - G^{-1} B^T P(t)$$  \hspace{1cm} (3.11)$$

For the inverse of the matrix, $G$, to exist, it is necessary that $G$ be positive definite.

The feedback matrix, $k(t)$, which is rectangular matrix of order $[m \times n]$, is a function of time: to determine $k(t)$ requires the solution of the matrix Riccati equation, (M.R.E.), i.e. (3.9).

For equation (3.10), and hence (3.11), to be true the following Jacobian matrix must be positive definite, i.e.

$$\begin{bmatrix}
\frac{\partial^2 H}{\partial x^2(t)} & \frac{\partial^2 H}{\partial x(t) \partial u(t)} \\
\frac{\partial^2 H}{\partial u(t) \partial x(t)} & \frac{\partial^2 H}{\partial u^2(t)}
\end{bmatrix} \geq 0$$  \hspace{1cm} (3.12)$$
which reduces to

\[
\begin{bmatrix}
Q & 0 \\
0 & G
\end{bmatrix} > 0
\]  \hspace{1cm} (3.13)

Since \(G\) is positive definite, then \(Q\) must be at least positive semi-definite for equation (3.13) to be true.

Using equation (3.10), equation (3.2) could be rewritten as:

\[
\dot{x}(t) = A x(t) - B G^{-1} B^T \psi(t)
\]  \hspace{1cm} (3.14)

Hence,

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\psi}(t)
\end{bmatrix} = M \begin{bmatrix}
x(t) \\
\psi(t)
\end{bmatrix}
\]  \hspace{1cm} (3.15)

where, \(M\), is the system canonical matrix, given by

\[
M = \begin{bmatrix}
A & -B G^{-1} B^T \\
-Q & -A^T
\end{bmatrix}
\]  \hspace{1cm} (3.16)

### 3.3 Output Regulator Problem

For the case where direct control of the output vector is required, a performance index, such as (3.17), is minimized:

\[
J_2 = \frac{1}{2} y^T(t_f) S y(t_f) + \int_0^{t_f} \left( y^T(t) Q y(t) + u^T(t) G u(t) \right) dt
\]  \hspace{1cm} (3.17)
with
\[ y(t) = C x(t) + D u(t) \]  \hspace{1cm} (3.18)

where \( y(t) \) is the output vector, \( C \) is a matrix of the order \([p \times n]\), \( D \) is a matrix of the order \([n \times m]\), \( Q \) is the output weighting matrix of order \([p \times p]\) and \( G \) is an input weighting matrix of order \([m \times m]\) \( S \) is a matrix weighting terminal values of the output vector and is of order \([p \times p]\).

The associated Hamiltonian, \( H \), is expressed as:
\[ H = \frac{1}{2} \left[ \left( C x(t) + D u(t) \right)^T Q \left( C x(t) + D u(t) \right) + u^T(t) G u(t) \right] + \psi^T(t) \left( A x(t) + B u(t) \right) \]  \hspace{1cm} (3.19)

For \( H \) to be minimized with respect to \( u(t) \), then:
\[ \frac{\partial H}{\partial u(t)} = D^T Q C x(t) + (G + D^T Q D) \dot{u}(t) + B^T \psi(t) = 0 \]  \hspace{1cm} (3.20)
or
\[ u^0(t) = - (G + D^T Q D)^{-1} \left[ D^T Q C x(t) + B^T \psi(t) \right] \]  \hspace{1cm} (3.21)

also
\[ \frac{\partial H}{\partial x(t)} = - \psi(t) = C^T Q C x(t) + C^T Q D u(t) + A^T \psi(t) \]  \hspace{1cm} (3.22)

The solution to equation (3.22) is,
\[ \psi(t) = \hat{P}(t) x(t) \]  \hspace{1cm} (3.23)

where the matrix, \( \hat{P}(t) \), is positive definite and is the solution to a matrix Riccati equation viz equation (3.24),
\[ \dot{\hat{P}}(t) = - \hat{Q} - \hat{A}^T \hat{P}(t) - \hat{P}(t) \hat{A} + \hat{P}(t) B G^{-1} B^T \hat{P}(t) \]  \hspace{1cm} (3.24)
Equation (3.24) is the same as equation (3.9) of the state regulator problem, provided that the appropriate matrices are used, and they are:

\[
\begin{align*}
\hat{A} &= A - B \hat{G}^{-1}D^TQ_C \\
\hat{G} &= G + D^TQD \\
\text{and} \\
\hat{Q} &= C^T[Q - DGD^{-1}D^TQ]C
\end{align*}
\] (3.25, 3.26, 3.27)

Substituting equation (3.23) in equation (3.21) gives,

\[
u^0(t) = -(G + D^TQD)^{-1}[B^T\hat{P}(t) + D^TQC]x(t)
\] (3.28)

or

\[
u^0(t) = \hat{k}(t)x(t)
\] (3.29)

where

\[
\hat{k}(t) = -(G + D^TQD)^{-1}[B^T\hat{P}(t) + D^TQ C]
\] (3.30)

\(\hat{k}(t)\) is a time-varying rectangular \([m \times n]\) matrix, and to determine its value, the solution of the matrix Riccati equation, M.R.E., (3.24) is required.

It is important to point out here that this analysis which leads to the construction of the output regulator matrix Riccati equation (3.24), is essential for the development of the optimal tracking problem, as discussed in Chapter 5.

3.4 Numerical Solution of The Optimal Control Problem

In order to find the optimal control laws (3.10) and (3.28) for both the state and output regulator problems, the matrix Riccati equations (3.9) and (3.24) must be solved.

For reasons of practical implementation, numerical difficulties and performance comparison studies, two approaches for the solution of the Riccati equation were considered:

1. The complete solution of the matrix Riccati differential equation was obtained, where the solution matrix, \(P(t)\), was a time-dependent matrix, which started from a given initial condition and ended up with a steady-state value in a finite time interval. A feedback law based on this approach will be termed as \textit{time-varying feedback}.
2. The steady-state solution of the matrix Riccati equation when, \( \dot{P}(t) = 0 \) was next obtained. In this case the matrix Riccati equation reduced to an algebraic one and was then termed the algebraic Riccati equation, A.R.E. Its solution, the matrix \( P(t) \), will be a constant matrix. A feedback law based on this approach is termed \textit{static feedback}.

3.4.1 Numerical Solution of The Matrix Riccati Equation

In theory the Riccati matrix \( P(t) \) can be found by integrating either equation (3.9) or (5.24) backwards in time from a known final condition. The handling of a problem like this has been reported by several authors. It was mentioned by Lee (1968) for example, that the Riccati equation cannot be solved explicitly by the method of quadratures and it was pointed out that the non-linear Riccati differential equation can be solved by transferring it to a second order linear differential equation with time-varying coefficients\(^+\). This approach, however, effectively proves to be useful only for single-state variable systems, because when it is tried for higher order systems, the complications in the analytical procedure makes it almost impossible to carry on with this approach.

The algorithm proposed by Athans and Falb (1966) was successful for low order and simple models, but it does not work for larger and stiff models, like the one used in this work. Similar results were obtained for the cases reported in Hsu and Meyer (1968) and Noton (1965), while Jacobs (1984) handled his [2x2] model using an analogue simulation.

Advanced variable-step numerical integration procedures have been suggested by Owens (1981), while Soroka and Shaked (1986) have proposed the successive sequential method, SDM, for the solution of the matrix Riccati equation, an approach which is different from that used in numerical integration techniques. In this work, the algorithm presented by Gear (1971) for solving ordinary stiff differential equations was adopted together with the Numerical Algorithms Group – NAG-advanced numerical integration routines. It proved to be successful, in particular for systems like the one used in this research.

In equation (3.9) \( \dot{P}(t) \) is a positive definite and symmetric matrix. If \( P(t) \) is symmetric at a time \( t \), then it remains so. The \( n^2 \) elements of the \( P(t) \) matrix can therefore be evaluated by integrating ordinary (although non-linear) differential equations and since \( P(t) \) is symmetric

\[ + \text{This is, of course, an example of the well known general principle that any non-linear dynamic system can be adequately represented by an equivalent time-varying linear differential equations} \]
all the elements are not required: only \( n(n+1)/2 \) need be integrated. This matter was disputed by Kalman and Englar (1966), who have stated that unless a certain corrective action is made at each integration step, then the symmetry of the matrix \( P(t) \) may be lost. Their statement was "at each step \( P(t) \) is symmetrized before proceeding by replacing it with

\[
\frac{P(t) + P^T(t)}{2}
\]

Symmetrization is absolutely essential because otherwise uncontrollable round-off error may accumulate in the antisymmetric part of \( P(t) \)." A similar remark was made by Kortüm (1979).

Although the approach adopted in this work for solving the M.R.E. did not use the above-mentioned continuous symmetrization, the symmetry of the \( P(t) \) matrix has been retained throughout the entire time domain (results will be shown later). The same results were obtained, when the systems reported in the above-mentioned reports were tested. On this point, one can conclude that Kalman and Englar's statement is not applicable for the developed methods of solution for the class of problems mentioned here, since the available integration routines are more robust than those used in 1966.

As regards the boundary values for this integration process, they are deduced from the requirement that:

\[
J(x, t_f) = S, \text{ which is satisfied if,}
\]

\[
P(t_f) = S, \text{ however in this work } S = 0, \text{ hence } P(t_f) = 0.
\]

Anticipating numerical integration of the differential equations, it is clearly more convenient to start with initial values. Therefore, one would use the reverse time \( \tau \), where:

\[
\tau = t_f - t
\]

and integrate

\[
\frac{dP(\tau)}{d\tau} = Q + A^T P(\tau) + P(\tau) A - P(\tau) B G^{-1} B^T P(\tau)
\]

with the initial conditions \( P(\tau) = 0, \tau = 0 \).

The same approach is valid for equation (3.24). Although for the system under
consideration, it was possible to integrate backward in time from the final conditions with the same results being obtained as with the use of (3.31) approach.

Once the solution of the matrix Riccati equation is completed, then each value of the matrix \( P(t) \) will be stored as an array at each relevant communication interval, for all \( t \in [t_0, t_f] \). Then the differential system of equation (3.15) will be solved forward in time from \( t_0 \) to \( t_f \) while reading the previously stored \( P(t) \) values at each required communication interval. Figure (3.1) show a block diagram representation for the solution of the matrix Riccati equation, M.R.E.

Figure (3.1a) illustrates the behaviour of the solution, \( P(t) \), of the M.R.E., as the final time \( t_f \) varies (for demonstration sake only the element \( P(1,1) \) is presented here, other elements of \( P(t) \) behaving in a similar manner). From figure (3.1a) two points can be noticed. First, for each value of \( t_f \), the matrix \( P(t) \) goes to the terminal value (which is zero in this case, since \( S = 0 \)), that is, if the solution is going forward using the reverse time \( \tau \) principle, or vice versa, \( P(t) \) starts from zero if the solution is using the backward time \( t \) direction. Second, at any particular point \( t \) in time, the Riccati solution \( P(t) \) appears to converge to a specific value \( P_\infty \) that is independent of time \( t \), where \( P_\infty \) can be obtained from the limiting operation case, i.e.

\[
P_\infty = \lim_{t_f \to \infty} P(t)
\]

Using equations like (3.11), one can present the effect of varying final time \( t_f \) on the behaviour of the feedback matrix \( k(t) \). Figures (3.2) and (3.3) demonstrate that effect for the feedback matrix elements \( k(1,1) \) and \( k(1,4) \). For a small value of \( t_f \) such as 0.05 and 0.1, it is quite interesting to note the very rapid response of the feedback elements, so that they can satisfy the final boundary value of the Riccati matrix \( P(t) \). Whether the original system response (engine dynamics) will behave in a similar manner under the same conditions, is a matter to be discussed later. Those figures were based on the linear model LIN1 of Chapter 2, and for the state regulator problem solution, with the following weighting matrices:

\[
\begin{align*}
Q & = \text{diagonal} \ [1,1,1,1] \\
G & = \text{diagonal} \ [100,300]
\end{align*}
\]

Regarding the output regulator problem solution, the first point mentioned above will not be valid, i.e. if \( P(t_f) \) is zero, then \( k(t_f) \) will not be zero. This point could be explained by equation (3.30) and will be dealt with in the next chapter. Figure (3.4a) gives a block diagram representation of the regulator problem with time-varying feedback.
3.4.2 Solution of The Steady State Riccati Equation

When the performance indices of (3.1) and (3.12) have an infinite time interval, i.e. the matrix \( P(t) \) will be in its steady-state condition (\( dP(t)/dt = 0 \)), then equations (3.9) and (3.24) become the algebraic Riccati equations, A.R.E., that is:

\[
-Q - A^T P - PA + PB G^{-1} B^T P = 0 \tag{3.34}
\]

and

\[
-Q - \hat{A}^T \hat{P} - \hat{P} \hat{A} + \hat{P} B G^{-1} B^T \hat{P} = 0 \tag{3.35}
\]

In this case, the differential system is:

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\psi}(t)
\end{bmatrix} = \bar{M}
\begin{bmatrix}
x(t) \\
\psi(t)
\end{bmatrix} \tag{3.36}
\]

where \( \bar{M} \) is the system canonical matrix,

\[
\bar{M} = \begin{bmatrix}
A & -B G^{-1} B^T \\
-Q & -A^T
\end{bmatrix} \tag{3.37}
\]

In this work, the solution of the A.R.E. was based on the method proposed by Marshal and Nicholson (1970). The method is purely algebraic and depends only on having a good algorithm for determining the eigenvalues and eigenvectors of the \( \bar{M} \) matrix. In the case that matrix \( \bar{M} \) has repeated eigenvalues, then the modal matrix (whose columns are the eigenvectors of \( \bar{M} \)), will become singular, hence its inverse is not defined, which is a disadvantage of this method. Figure (3.4b) gives a block diagram representation of the regulator problem with static feedback.

A \( P \) matrix and its feedback matrix based on this algebraic procedure, were compared to those based on the solution of equation (3.33). Results based on the example of Section 3.4.1 were presented in Table 3.1. The same comparison was made for various sets of data, and the results were identical. This exercise was useful in testing the software based on the above two algorithms.

+ This is disadvantage can be overcome by using Schur vectors, as proposed by Laub(1981).
$P(t)$ matrix is generated backwards in time.

Figure 3.1: Block Diagram Representation of the Matrix Riccati Equation (M. R. E.)
Figure 3.1a The Effect Of Final Time $t_f$ On Riccati Matrix Element $P(1,1)$
Figure 3.2 Effect of final time $t_f$ on feedback matrix element $k(1,1)$
Figure 3.3 Effect Of Final Time $t_f$ On Feedback Matrix Element $k(1,4)$
Figure 3.4a: Block diagram Representation of the Regulator Problem with Time-Varying Feedback Matrix.
Figure 3.4b: Block Diagram Representation of The Regulator Problem with Constant Feedback Matrix.
### Riccati Matrix from M.R.E. solution:

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<th>A.R.E. Solution</th>
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### Feedback Matrix from M.R.E. solution:

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### Riccati Matrix from A.R.E. solution:

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### Feedback Matrix from A.R.E. solution:

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### Table 3.1 Comparison Between the Solution of the M.R.E. (the limiting condition) with that of the A.R.E. and Their Feedback Matrices

#### 3.5 The Symmetry Analysis of the Solution of the Riccati Equation

A procedure to provide a measure of anti-symmetry for the solution of the matrix Riccati equation $P(t)$ was developed. In this procedure a matrix $\tilde{P}(t)$, which is a symmetrized version of $P(t)$ is defined as:
\[
\bar{P}(t) = \frac{P(t) + P^T(t)}{2}
\] (3.38)

The matrix \(\bar{P}(t)\) is the best available estimate of a base value for comparison, about which deviation may be measured. Then a comparison is made between each element of the strict lower (or the higher) triangle of the matrix \(P(t)\), with its corresponding element in the symmetrized version \(\bar{P}(t)\). The resulting relative errors are stored in the vector \(\text{ERR}(t)\), a vector of dimension \((n_x(n-1)/2)\), and is defined as:

\[
\text{ERR}_{ij} = \frac{|P_{ij} - \bar{P}_{ij}|}{|P_{ij}|}, \text{ all are time dependent, or}
\]

\[
\text{ERR}_{ij} = \frac{e_{ij}}{|P_{ij}|}
\] (3.39)

Out of those \((n_x(n-1)/2)\) relative error values of \(\text{ERR}(t)\), a single global measure of error \(\text{GRR}(t)\) will be more convenient to present and compare. \(\text{GRR}(t)\) is defined as the ratio of the sum of the absolute values of the relative error (from \(\text{ERR}(t)\)) divided by the sum of the absolute values of the appropriate symmetrized values (i.e. of the chosen triangle of \(\bar{P}(t)\)), that is:

\[
\text{GRR}(t) = \frac{\sum e_{ij}(t)}{\sum |\bar{P}_{ij}(t)|}, \text{ for the chosen triangle}
\] (3.40)

where \(\text{GRR}(t)\) represent a single measure of the deviation in \(P(t)\) matrix (obtained from the numerical solution of the matrix Riccati equation) from its symmetrized version \(\bar{P}(t)\).

Figures (3.5) and (3.6) show the variation of \(\text{GRR}\) as a function of the reverse time \(\tau\), for the following cases, respectively.

1. The M.R.E. solution with \(Q\) diagonal \([1,1,1,1,1]\)
   
   \(G\) diagonal \([100,300]\)

2. The M.R.E. solution with \(Q\) diagonal \([1000,1,1,1,1]\)
   
   \(G\) diagonal \([10,30]\)
Figure (3.7) shows the variation of the elements of the relative error vector $\text{ERR}(t)$ with the reverse time $\tau$. It is worth mentioning here that the data plotted in figures (3.5), (3.6) and (3.7) were intentionally left without curve fitting, because it was thought that presenting them in this way will be more descriptive to the idea behind these data.

The main conclusions deduced from the above figures are:

1. The absolute values of $\text{GRR}(t)$ and $\text{ERR}(t)$, are very small, i.e. of the order of $10^{-11}$ to $10^{-13}$ for all $t \in [t_0, t_f]$, which means that they are insignificant for all practical purposes.

2. Although the deviations are very small in value, nevertheless, they have been suppressed with time, i.e. as the solution proceeds. The same trend was noticed, when other sets of the weighting matrices were tried.

3.6 The Choice of Weighting Matrices

The LQP control law and system response are greatly influenced by the weighting matrices $Q$ and $G$ which are chosen to be diagonal. Selection of these weighting factors is not an easy task, since no direct relationship has ever been demonstrated between the weighting factors and the optimum system response. Attempts have been made by many authors to establish such a relationship but all were used for special cases, such as procedures suggested by Bryson and Ho (1975), & Harvey and Stein (1978). In this work a trial-and-error technique was used in conjunction with the plots which show the relationship between the elements of the control matrix versus the weighting matrices $Q$ and $G$ until a satisfactory response was obtained.

3.7 Concluding Remarks

In this chapter, the solution of the state and output regulator problems of the linear quadratic problem, LQP, were presented. These solutions led to the formulation of the matrix Riccati equation, M.R.E., when the final time, $t_f$, in the chosen performance index was finite, and to the algebraic Riccati equation, A.R.E., when $t_f$ was infinite, i.e. the limiting condition or the steady-state case of the M.R.E. More attention was paid to the numerical solution of the M.R.E., and in particular the numerical integration procedure capable of solving stiff differential systems, like those of the F100 engine. Comparisons were made between the A.R.E. solutions and those from the M.R.E. steady state conditions: identical results were obtained. The numerical solution of the M.R.E. requires more computer time and space than that of the A.R.E., no advantages of the former approach were apparent at this stage of the work. The merits of the numerical solution of the M.R.E. will be shown in the later stages of this research.
Figure 3.5 GRR Variation With Reverse Time For Case No. 1

Figure 3.6 GRR Variation With Reverse Time For Case No. 2
Figure 3.7 Variation of ERR With The Reverse Time
Chapter 4
Solution of the Linear Quadratic Problem Applied to LIN and OMAR

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4.1 Introduction

The aim of the work reported in this chapter was to find out a single effective control law, which was able to regulate the dynamic responses of the linear, and ultimately the non-linear, engine models, LIN and OMAR throughout the engine's entire operating range. The effective linear control law was obtained from solving a Linear Quadratic Problem, LQP, in the manner presented in the previous chapter.

To gain some insight into the problem, and to solve the linear control problem, the linear analysis is performed first. In this analysis, the time-varying and the steady-state solutions of both the LQP state and output regulator problems were applied to LIN. Comparison of the results lead to the conclusion that it was best to implement the steady-state solution of the output regulator problem to regulate LIN throughout its operating range. This steady-state solution was applied to optimize the dynamic responses of LIN at each operating condition, using the relevant choice of the weighting matrices $Q$ and $G$. Results of this procedure were obtained for each condition. Then a search was made amongst all those local optimal control laws to determine a single effective one which was capable of regulating LIN throughout the operating range. The results showed such a control law was feasible.

Regarding the non-linear model, a linear, optimal, feedback law (synthesized from the LQP technique) was used at each operating point to regulate the engine's dynamic response. Results of some operating points were presented. Then a search was made for a single effective control law, which could be applied throughout the flight envelope; again the results showed that it was feasible to do this.

4.2 State Regulation Applied to LIN

In this section the state regulator problem is considered, in particular using the solution of equation (3.11) to regulate the dynamic response of LIN, a model derived in Chapter 2 and presented in detail in [Appendix B]. Equations (3.1), (3.10) and (3.11) are re-written here as:

\[
J_1 = \frac{1}{2} x^T(t_f) S x(t_f) + \int_0^{t_f} (x^T(t) Q x(t) + u^T G u(t)) \, dt
\]

\[u^o(t) = k(t) x(t)\]  \hspace{1cm} (4.1)  \hspace{1cm} (4.2)

and

\[k(t) = - G^{-1} B^T P(t)\]   \hspace{1cm} (4.3)

with the same definition of parameters as being used in chapter 3.

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It is most important to point out here that the flight operational requirements dictate that the regulation process must bring the deviation in engine thrust, FG, back to zero as soon as possible, without crossing the compressor's surge margins, ZC and ZF, or without exceeding the maximum allowable value of the turbine inlet temperature, T4. (Cohen, Rogers and Saravanamuttoo (1979), Kerrebnrock (1977) and Rolls-Royce Limited (1973)).

The control strategy used was that when the state vector, \( x(t) \), has been excited by some initial vector \( x(o) \), then it should be brought back to zero as soon as possible without excessive expenditure of control energy. This was achieved by using the feedback matrix of (4.2), while minimizing the performance index of (4.1) with a proper choice of the weighting matrices, \( Q \) and \( G \).

Since the output vector \( y(t) \) of (4.5) is directly related to the state vector \( x(t) \) of (4.4), then \( y(t) \) also returned to zero when \( x(t) \) did so, thereby fulfilling the flight operational requirements.

It may be helpful to the reader to present here the general equation of LIN with definitions of the parameters, i.e:

\[
\dot{x}(t) = Ax(t) + Bu(t) \quad (4.4)
\]

\[
y(t) = Cx(t) + Du(t) \quad (4.5)
\]

where \( x(t) \) is the state vector \( \mathbb{R}^n \), \( u(t) \) is a control vector \( \mathbb{R}^m \), \( y(t) \) is an output vector \( \mathbb{R}^p \), A, B, C, D are matrices of appropriate order. The state, control and output vectors are defined as follows:

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 
\end{bmatrix}
\]

- High pressure compressor speed \( NC \)
- Low pressure compressor speed \( NF \)
- Combustor internal pressure \( P4 \)
- Afterburner pressure \( P7 \)
- Combustion chamber internal energy \( U4 \)

\[
u = \begin{bmatrix}
u_1 \\
u_2 
\end{bmatrix}
\]

- Combustor fuel flow rate \( WFB \)
- Exhaust nozzle area \( A8 \)
The model LIN1 (take-off operating condition) was used for testing the algorithms, and for determining the trends in the system response.

To judge the quality of the controlled response of the engine, a simulation of the uncontrolled engine is needed first. Figures (4.1a), (4.1b), (4.2) and (4.3) show the open loop response of the state and output vectors respectively of LIN1. From those figures, one notices that in general a settling time of about 0.9 sec. is obtained.

Regarding the controlled response, the choice of the weighting matrices $Q$ and $G$ in the performance index of (4.1) is very important. To find a proper choice of $Q$ and $G$, the procedure outlined in Section (3.6) was used, and various sets were tried. The following set was found to be the most suitable choice for this operating point, viz:

\[
Q = \text{diagonal } [3.00E+2, 5.00E+2, 1.00E+2, 1.0, 1.0] \\
G = \text{diagonal } [2.00E+8, 5.00E+8]
\]

To find the feedback matrix $k(t)$ and also $u(t)$, the Riccati matrix $P(t)$ must be known. Figure (4.4) shows the solution of the matrix Riccati equation $P(t)$ with reverse time $\tau$, where $\tau = t_f - t$. From this figure one can see that $P(\tau)$ starts from zero and reaches its steady-state value at about $t = 0.25$ sec.

The following points should be noticed:

1. Following the analysis of matrix symmetry of Section (3.5), where it was shown that the Riccati matrix $P(\tau)$, and also $k(t)$ have retained their numerical symmetry throughout the domain of the solution. Only $n(n+1)/2$ elements of this matrix were presented in Figure (4.4). Those $n(n+1)/2$ elements (15, since $n=5$) represent the upper triangle of the Riccati matrix.

2. The idea of using the reverse time $\tau$, for solving the differential equation backwards in time, is done merely for numerical convenience. Whether this approach is applied, or the direct backward integration method is used, where the time interval is negative, depends solely on the quality of the software available for solving numerically the differential
system and upon the dynamic system itself. In this work, both approaches were used, and identical results were obtained. Intentionally, the solution of the Riccati equation was presented for the reverse time approach (figures such as 4.4), because it is believed that presenting it in this way demonstrates better the idea that the matrix Riccati equation, M.R.E., was solved backward in time, from the given final value to its steady-state one.

3. The solution of the Riccati equation $P(\tau)$ was stored at each communication interval, for all $\tau \in [0, t_f]$ and then equation (4.3) was used to calculate the feedback matrix $k(t)$ forward in time, such as:

$$
\begin{array}{c|c|c|c|c|c|c}
 t & \tau & P(t) & k(t) \\
\hline
 0 & t_f & P_{ss} & k_{ss} \\
 t_f & 0 & P(t_f) & k(t_f) \\
\end{array}
$$

Table 4.1 Boundary Values for the Riccati and Feedback Matrices

where $P_{ss}$ and $k_{ss}$ are the steady-state values of the Riccati matrix and of the feedback matrix respectively, i.e

$$
k_{ss} = -G^{-1}B^TP_{ss} \quad (4.6)
$$

$$
P(t_f) = 0, \; \text{since} \; S \Delta = 0 \quad (4.7)
$$

These results are based on the fact that the final time $t_f$ is large enough for the system to reach its steady-state.

4. Figure (4.5) demonstrates the solution of the feedback matrix $k(t)$ with forward time, $t$, where its boundary values [stated in table (4.1)] are shown to be satisfied. From this figure, it is clear that for almost 75% of the time range (from $t=0$ to $t=0.75$) the dynamical system of (4.4) has been subjected to the effect of the steady-state value of the feedback matrix $k(t)$, and to the dynamical value of $k(t)$ for the rest of the time, which is very small (about 0.25 sec.). This shows that the dynamic response of the engine model LIN1 is dominated by the steady-state value of the Riccati matrix, $P_{ss}$, rather than by its time-varying value, $P(t)$. In other words, the settling time $t_s$ of the controlled system is determined by the steady-state value of the Riccati matrix $P_{ss}$. Since the value of $P_{ss}$ for a given system depends on the choice of the
weighting matrices $Q$ and $G$, then the settling time $t_s$ of the controlled system is dominated by the choice of $Q$ and $G$. The same thing was noticed for the other operating points of LIN.

Figures (4.6) and (4.7) show the controlled response of LIN1 state vector and the control law (based on the feedback matrix of Figure (4.5)) respectively.

Figures (4.8) and (4.9) show the controlled response of LIN1 output vector, subjected to the control law of Figure (4.7). It should be noted that the controlled system has reached its steady-state value at about 0.6 sec., which is an improvement on the open loop system.

4.2.1 The Effect of the Choice of the Time Interval in the Performance Index

To find out the effect of the final time, $t_f$, on the dynamic response of the controlled system, several runs were made for the system presented in the previous section, where each run was made for a different value of $t_f$. Figure (4.10) shows the response of the output variable, $FG$, for three cases: $t_f = 1.0, 0.6$ and $0.4$ sec. respectively. The uncontrolled response is also shown for comparison. From the figure, it can be seen that the lower is the value of $t_f$ the greater is the likelihood that the output variable, $FG$, will not be fully restored to zero. Moreover, the dynamic response is seen to be more oscillatory and exhibits a greater overshoot. This is quite clear for the case of $t_f=0.4$. The behaviour was identical for the rest of the output vector and the state vector, but not for the Riccati matrix, or the feedback matrix, or the control vector, since they fulfilled the final condition of equations (4.6) and (4.7) for any value of $t_f$ (as was pointed out in Section (3.4.1) and shown in Figures (3.1), (3.2), and (3.3) for that example). All this reasoning accords with remark no. 4 made in the last section. The final conclusion is therefore that, for a given set of $Q$ and $G$, if the specified final time $t_f$ is less than $t_s$, then the system can not fulfill the final boundary condition for that choice of $t_f$.

This conclusion leads to the approach of adopting the application of the steady-state solution of the matrix Riccati equation, M.R.E., for the whole response time: it is the same as using the solution of the algebraic Riccati equation, A.R.E., since it is easier and faster to compute. Figures (4.11) and (4.12) show the response of the state and control vectors respectively, when the system has been subjectd to the steady-state solution of the M.R.E. In support of the above conclusion, it can been seen that Figure (4.11) is identical to that of (4.6) and that Figure (4.12) is identical to that of (4.7).
Figure 4.1a Open Loop Response of LIN1 NC, NF & U4

Figure 4.1b Open Loop Response of LIN1 P4 & P7
Figure 4.2 Open Loop Response of LIN1 Thrust & T4

Figure 4.3 Open Loop Response of LIN1 ZC & ZF
Figure 4.4 Variation of LIN1 Riccati Matrix Elements With Reverse Time
Figure 4.5 Variation Of The Feedback Matrix $k(m \times p)$ With Time
Figure 4.6 Controlled Response of LIN1 State Variable

Figure 4.7 Variation of The Optimal Control Law With Time
Figure 4.8 Controlled Response of LN1 Thrust & T4

Figure 4.9 Controlled Response of LN1 ZC & ZF
Effect of Final Time $t_f$

Figure 4.10 Effect of Final Time $t_f$ on Settling Time

Legend
- ■ FG Response Using $t_f = 1$
- □ FG Response Using $t_f = 0.6$
- ● FG Response Using $t_f = 0.4$
- ○ FG Open Loop Response
Figure 4.11 UN1 State Variable Based on The M.R.E Steady State Solution

Figure 4.12 Steady State M.R.E Optimal Control Law
4.3 Output Regulation Applied to LIN

In this section the application of the output regulator problem, and in particular its solution of equation (3.30), to regulate the dynamic response the output vector of LIN is discussed. Equations (3.17), (3.29) and (3.30) are re-written here as:

\[ J_2 = \frac{1}{2} y^T(t_f) S y(t_f) + \int_0^{t_f} (y^T(t) Q y(t) + u^T(t) R u(t)) \, dt \]  

(4.8)

\[ u(t) = \hat{k}(t) x(t) \]  

(4.9)

\[ \hat{k}(t) = -(G + D^T Q D)^{-1} \left[ B^T \hat{p}(t) + D^T Q C \right] \]  

(4.10)

with the same parameter definition as used in chapter 3.

The control strategy was that when the state vector \( x(t) \) had been excited by an initial vector, \( x(0) \), then the output vector \( y(t) \) was to be brought back to zero as soon as possible, but without excessive expenditure of control energy, or crossing the engine constraints. This control strategy was used as another way of fulfilling the flight operational requirements which were stated in Section (4.2).

Again the model LIN1 was used first, for testing the algorithms and for obtaining some insight to the problem solution. To obtain the optimal feedback law of (4.9), the solution of the output regulator M.R.E. of (4.11) was required, viz:

\[ \dot{\hat{p}}(t) = -Q \hat{A}^T \hat{p}(t) - \hat{p}(t) \hat{A} + \hat{p}(t) \hat{B} G^{-1} B^T \hat{p}(t) \]  

(4.11)

The same definitions of \( \hat{A} \), \( \hat{Q} \) and \( \hat{G} \) which were used in chapter 3 were retained here.

Using the procedure outlined in Section (3.6), the following set of weighting matrices was found to be the most suitable choice for this condition, that is:

\[ Q = \text{diagonal} \left[ 1.00E+5, 2.00E+5, 1.00E+2, 1.00E+2 \right] \]

\[ G = \text{diagonal} \left[ 9.00E+10, 1.00E+9 \right] \]
Figures (4.13a) and (4.13b) show the solution of \( P(t) \) of equation (4.11). The high numerical values of some elements of this matrix, and in particular \( P(4,4) \), should be noted. These high values occur as a result of the high values which appear in the \( C \) and \( D \) matrices of the LIN model, and it is not then surprising that such high-valued elements had a major effect upon the feedback matrix and consequently the control signal, \( u^0(t) \). Figures (4.14) and (4.15) show the response of the state vector \( x(t) \) and the control variables \( u^0(t) \) respectively. Figures (4.16) and (4.17) present the controlled response of the elements of the output vector \( y(t) \). The good response of the thrust, in particular, should be noted.

Figures (4.18) and (4.19) show the variation of the elements of the feedback matrix \( k(t) \) with forward time, \( t \). The way in which \( k(2,4) \) varies, is noteworthy because of the way in which it has been defined, i.e. equation (4.10). In this equation, \( k(t) \) comprises two parts, one related to the Riccati matrix, \( P(t) \), the other to the term \( D^TQC \). Thus, even when \( P(t) \) fulfills its final condition at \( t_f \), i.e.

\[
\hat{P}(t_f) = 0
\]  
(4.12)

\( k(t_f) \) is not null, because the term \(- (G + D^TQD)^{-1}D^TQC\) is finite. In this condition, \( k(t_f) = 0 \) only when the \( D \) matrix of LIN is also zero.

Regarding the effect of final time \( t_f \), the same reasoning used in the remark no. 4 of sections (4.2) and (4.2.1), has once more been demonstrated here. Results similar to those of figure (4.10) were obtained.

Again the conclusion here is to choose the steady-state solution of the M.R.E. (which is the same as the solution of the algebraic Riccati equation A.R.E.). Figures (4.20), (4.21), (4.22) and (4.23) present the system response, based on the steady-state M.R.E. solution. In figure (4.23) it is noted that \( k(2,4) \) has retained its steady-state value which is determined from

\[
-(G + D^TQD)^{-1}[B^TP_{ss} + D^TQC].
\]

4.4 Comparison Between State and Output Regulation

The two approaches presented in section (4.2) and section (4.3) were applied to LIN throughout the operating range. From the results, the output regulator technique was adopted subsequently, since in all of those results the thrust response based on OUTREG was better than that based on STATREG, while the engine was kept within its working limits. Figures (4.24) and (4.25) illustrate the superiority of OUTREG. Figure (4.24) shows the comparison for the thrust variable at the take-off operating condition; figure (4.25) does the same, but for the subsonic cruise operating condition.
Figure 4.13a The Riccati Matrix For The Output Regulator Problem

Legend
- \( P(1,1) \)
- \( P(1,2) \)
- \( P(1,3) \)
- \( P(1,5) \)
- \( P(2,2) \)
- \( P(2,3) \)
- \( P(2,5) \)
- \( P(3,3) \)
- \( P(3,4) \)
- \( P(4,4) \)
- \( P(4,5) \)

Figure 4.13b The Riccati Matrix For The Output Regulator Problem
Figure 4.14 Response of LIN1 Using Output Regulator Control

Figure 4.15 Optimal Control of Output Regulator Problem
Figure 4.16 UN1 FG & T4 Responses Using Output Regulator Control

Figure 4.17 UN1 ZC &ZF Responses Using Output Regulator Control
Figure 4.18 Variation of Feedback Matrix Elements With Time

Figure 4.19 Variation of Feedback Matrix Elements With Time
Figure 4.20 FG & T4 Responses Using Steady State Output Regulator Problem

Figure 4.21 Steady State M.R.E Optimal Control Based on The Output Regulator Problem
Figure 4.22 ZC & ZF Based on The Steady State Output Regulator Problem

Figure 4.23 Static Feedback Matrix From The Output Regulator Problem
Figure 4.24 STATREG & OUTREG Comparison at Take-off Condition
Figure 4.25 STATREG & OUTREG Comparison at Subsonic Cruise Condition
4.5 A Globally Effective Control Law

In an attempt to regulate the dynamic response of LIN throughout the flight envelope the dynamic response of the engine was optimized at each operating point, by using the output regulator problem technique, with an appropriate choice of the weighting matrices, Q and G. The outcome of this process is presented as follow:

#### Operating Point 1, LIN1

The weighting matrices were:

\[Q = \text{diagonal } [2.0E+5, 2.0E+5, 1.0E+2, 1.0E+2]\]
\[G = \text{diagonal } [15.0E+10, 3.0E+10]\]

**RICCATI MATRIX P1**

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**FEEDBACK MATRIX k1**

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#### Operating Point 2, LIN2

The weighting matrices were:

\[Q = \text{diagonal } [7.0E+7, 6.0E+7, 1.0E+2, 1.0E+2]\]
\[G = \text{diagonal } [2.0E+13, 3.0E+13]\]

**RICCATI MATRIX P2**

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The weighting matrices were:

\( Q = \text{diagonal } [2.0\times10^6, 2.0\times10^3, 3.0\times10^4, 3.0\times10^4] \)

\( G = \text{diagonal } [2.2\times10^{11}, 2.8\times10^{12}] \)

**Operating Point 3 , LIN3**

**Riccati Matrix P3**

\[
\begin{bmatrix}
3.77141E+02 & 1.54420E+01 & -6.59677E+04 & -5.30954E+06 & -1.05756E+02 \\
1.54420E+01 & 4.96923E+02 & 7.77690E+03 & -5.11628E+05 & 2.62243E+03 \\
-6.59677E+04 & 7.77690E+03 & 1.13959E+08 & 9.01848E+09 & 2.04462E+06 \\
-5.30954E+06 & -5.11628E+05 & 9.01848E+09 & 8.38462E+11 & 1.48375E+08 \\
-1.05756E+02 & 2.62243E+03 & 2.04462E+06 & 1.48375E+08 & 6.03260E+04 \\
\end{bmatrix}
\]

**Feedback Matrix k3**

\[
\begin{bmatrix}
2.67686E-06 & -4.07497E-05 & -3.34695E-02 & 1.26322E-00 & -9.66768E-04 \\
\end{bmatrix}
\]

**Operating Point 4 , LIN4**

**Riccati Matrix P4**

\[
\begin{bmatrix}
4.21296E+05 & 1.32168E+06 & -9.64939E+06 & -1.12508E+09 & 1.12264E+05 \\
1.32168E+06 & 1.19968E+07 & 1.19503E+08 & 7.67474E+09 & 2.38974E+06 \\
-9.64939E+06 & 1.19503E+08 & 1.03865E+10 & 8.42883E+11 & 4.52921E+07 \\
-1.12508E+09 & 7.67474E+09 & 8.42883E+11 & 8.31139E+13 & 8.68065E+09 \\
1.12264E+05 & 2.38974E+06 & 4.52921E+07 & 8.68065E+09 & 1.14553E+07 \\
\end{bmatrix}
\]
**FEEDBACK MATRIX k4**

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Operating Point 5  ,LIN5

The weighting matrices were:

Q= diagonal [3.0E+7,14.0E+7,1.0E+2,1.0E+2]
G= diagonal [11.0E+13,1.0E+14]

**RICCATI MATRIX P5**

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<td>7.19021E+08</td>
<td>1.02593E+06</td>
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**FEEDBACK MATRIX k5**

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<td>1.20771E-02</td>
<td>-1.65175E+00</td>
<td>2.48144E-04</td>
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</tbody>
</table>

Operating Point 6  ,LIN6

The weighting matrices were:

Q= diagonal [9.0E+7,2.0E+8,3.0E+9,1.0E+9]
G= diagonal [6.0E+13,3.0E+12]

**RICCATI MATRIX P6**

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<td>1.46734E+07</td>
<td>4.87251E+08</td>
<td>2.04174E+05</td>
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<td>1.71465E+08</td>
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<td>1.63683E+07</td>
<td>1.60382E+09</td>
<td>8.59326E+06</td>
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</table>
Next a search was made among the feedback matrices presented above to determine if a single feedback matrix was capable of effectively regulating the engine at every operating point, and not merely at the point for which it was derived. The feedback matrix $k_1$ of OP1, was found to be the most suitable to perform this task, in ways to be decided below.

Figures (4.26), (4.27) and (4.28) show the controlled and the uncontrolled time histories of the output and control vectors, respectively, for the climb point. Two types of controlled responses are demonstrated in those figures: the first is the response of model LIN2 when the feedback matrix $k_2$ was used in close-loop control, and the second is when the feedback matrix calculated from the take-off point was used in the control operating on LIN2. From those figures it can be seen that, in the second case, there is not much deterioration in the quality of the closed-loop response of LIN2, especially the speed at which the thrust variable settles, the limits of the surge margins, and the level of the control signals demands. The same trend in the dynamic responses prevails at the other operating points. Figures (4.29), (4.30) and (4.31) present the same point for the subsonic cruise condition, in which the feedback matrices $k_3$ and $k_1$ were applied with LIN3. Meanwhile, figures (4.32), (4.33) and (4.44) present the responses of the same variables, but for the approach operating point, where $k_4$ and $k_1$ have been applied to LIN4. Regarding the operating conditions with afterburning, again the same kind of response occurs. Figures (4.35), (4.36) and (4.37) present the controlled response of the output and control variables, when both the feedback matrices, $k_3$ and $k_1$, were applied with LIN5. Figures (4.38), (4.39) and (4.40) present the controlled response of the supersonic cruise condition, where $k_6$ and $k_1$ have been applied to LIN6. Figure (4.41) show the model LIN1 with both feedback matrices $k_1$ and $k_1$; the clear difference in thrust variable response due to those different feedback matrices, at this very critical operating point is easily identified.

The same trend has been noticed when the procedure presented above was applied to the different jet engine linear models reported in Michael and Farrar (1973). Although those linear models represent the engine at static, sea level, standard day, operating conditions. Nevertheless, they were representing different operating conditions, because they were for different power lever angles PLA, and that included the maximum PLA, i.e the take-off rating. Again it was noticed that take-off feedback matrix is the most suitable one for running the other operating conditions effectively.
Climb Condition

Figure 4.26 FG & T4 Responses For Control Laws CL1 & CL2
Climb Condition

Figure 4.27 ZC & ZF Responses For Control Laws CL1 & CL2
Figure 4.28  WFB & A8 Responses For Control Laws CL1 & CL2
Figure 4.29  FG & T4 Responses For Control Laws CL1 & CL3
Figure 4.30 ZC & ZF Responses For Control Laws CL1 & CL3
Subsonic Cruise Condition

Figure 4.31 - WFB & A8 Responses For Control Laws CL1 & CL3

Legend
- WFB Response Using CL3
- A8 Response Using CL3
- WFB Response Using CL1
- A8 Response Using CL1

Figure 4.31 WFB & A8 Responses For Control Laws CL1 & CL3
Approach Condition

Figure 4.32  FG & T4 Responses For Control Laws CL1 & CL4
Figure 4.33 ZC & ZF Responses for Control Laws CL1 & CL4
Figure 4.34: WFB & A8 Responses For Control Laws CL1 & CL4
Take-off With Afterburning Condition

![Graph showing FG and T4 responses for control laws CL1 and CL5.](image)

Legend
- FG Open Loop Response
- T4 Open Loop Response
- FG Response Using CL5
- T4 Response Using CL5
- Y1 Response Using CL1
- T4 Response Using CL1

Figure 4.35 FG & T4 Responses For Control Laws CL1 & CL5
Take-off With Afterburning Condition

Figure 4.36 ZC & ZF Responses For Control Laws CL1 & CL5
Take-off With Afterburning Condition

![Graph showing WFB & A8 responses for control laws CL1 & CL5. Legend includes symbols for WFB Response Using CL5, A8 Response Using CL5, WFB Response Using CL1, and A8 Response Using CL1.]

Figure 4.37  WFB & A8 Responses For Control Laws CL1 & CL5
Supersonic Cruise Condition

Figure 4.38 FG & T4 Responses For Control Laws CL1 & CL6
Supersonic Cruise Condition

Figure 4.39 ZC & ZF Responses For Control Laws CL1 & CL6
Supersonic Cruise Condition

![Graph showing WFB & A8 Responses for Control Laws CL1 & CL6]

**Legend**
- ■ WFB Response Using CL6
- □ A8 Response Using CL6
- ● WFB Response Using CL1
- ○ A8 Response Using CL1

**Figure 4.40** WFB & A8 Responses For Control Laws CL1 & CL6
Take-off Operating Condition With CL1 & CL3

Legend
- FG Response Using CL1
- T4 Response Using CL1
- FG Response Using CL3
- T4 Response Using CL3

Figure 4.41 FG & T4 Responses For Control Laws CL1 & CL3 Based On LN1
4.6 Regulation of the Non-linear Model

In this section it is reported the work carried out to investigate the problem of regulating the dynamic response of the non-linear model engine OMAR, throughout the defined flight envelope, but using only linear feedback control laws, which were synthesized by solving the LQP in the fashion outlined in preceding section. Each linear feedback law was applied at its equivalent operating condition. Then the law corresponding to the take-off condition was applied to the engine at other points. From the results obtained this control law was found to be the best candidate for effectively regulating the engine throughout the defined operating range, a similar result to that obtained in the preceding section. An heuristic explanation for this results based on thermodynamic cycle principles of the two spool jet engine can be proposed as follows:

For pure jet and low bypass-ratio aircraft engine, the nearest operating point to the surge line is the take-off operating point. As the flight proceeds from take-off through climb to high altitude cruise, the surge margin increases, i.e the operating line gets further from the surge line. [Harmon (1981).] This is opposite to the results for engine of higher bypass-ratios (4 to 6) where surge is more likely to happen at cruise rather than at take-off. Hence, for this engine, since the take-off point is the nearest to the surge line throughout the flight range, then it will be safer and more practical to use a feedback based on this point to control the other operating points, rather than the other way round. Figure (4.41b) show a block diagram representation for the controlled non-linear engine model using the LQP feedback law.

Three cases were presented here to demonstrate the above conclusion, viz:

1 - OMAR at take-off, with control laws CL1 and SL1 (of STATREG).

2 - OMAR at climb, with control laws CL1 and CL2.

3 - OMAR at subsonic cruise, with control laws CL1 and CL3.

The responses of OMAR's thrust, turbine inlet temperature, surge margins, and the fuel flow and the nozzle area are shown in figures (4.42a), (4.42b), (4.43) and (4.44) respectively, at the take-off operating condition, when OMAR was subjected to the control laws CL1 and SL1. It is clear from figure (4.42a), that the thrust response with CL1 is better than that with SL1.

---

+ This F100 jet engine is of low bypass-ratio, 0.7
The responses of OMAR's thrust, turbine inlet temperature, surge margins, and control variables are represented in figures (4.45a), (4.45b), (4.46) and (4.47) respectively, at the climb operating condition when it was subjected to control laws CL1 and CL2. Figures (4.48a), (4.48b) and (4.49) show the response of OMAR thrust, turbine inlet temperature and surge margins, at the subsonic cruise condition. Similar results were obtained at other operating conditions.

From these results it was clear that it was possible to regulate the dynamic response of the non-linear engine model OMAR at any operating point, using a control law derived from the LQP solution based on a model defined at another operating point.

4.7 Concluding Remarks

In regulating the dynamic response of the linear model of the engine, LIN, the application of the control laws based on the solution of the output regulator problem proved to be more desirable than those based on the state regulator problem. From studying the responses of LIN at different operating points, the feasibility of using a single control law to control the dynamic response of LIN effectively throughout the operating range was established. The same result was noted when different control laws based on OUTREG were applied to OMAR.
Figure 4.41b: Closed-Loop control of the Non-Linear Engine

\[
x = f(x, u, t)
\]
Figure 4.42a OMAR FG Responses For Control Laws CL1 & SL1
OMAR at Take-off

Figure 4.42b OMAR T4 Responses For Control Laws CL1 & SL1
OMAR at Take-off

Legend
- ZC Open Loop Response
- ZC Response Using CL1
- ZC Response Using SL1
- ZF Open Loop Response
- ZF Response Using CL1
- ZF Response Using SL1

Figure 4.43 OMAR ZC & ZF Responses For Control Laws CL1 & SL1
OMAR at Take-off

Figure 4.44 OMAR WFB & A8 Responses For Control Laws CL1 & SL1
OMAR at Climb

![OMAR at Climb Graph](image)

**Legend**
- ▪ FG Open Loop Response
- ◊ FG Response Using CL1
- ○ FG Response Using CL2

*Figure 4.45a OMAR FG Responses For Control Laws CL1 & CL2*
OMAR at Climb

Figure 4.45b OMAR T4 Responses For Control Laws CL1 & CL2
Figure 4.46 OMAR ZC & ZF Responses For Control Laws CL1 & CL2
Figure 4.47 OMAR WFB & A8 Responses For Control Laws CL1 & CL2
OMAR at Subsonic Cruise

Figure 4.48a  OMAR FG Responses For Control Laws CL1 & CL3
OMAR at Subsonic Cruise

Figure 4.48b OMAR T4 Responses For Control Laws CL1 & CL3

Legend
- T4 Open Loop Response
- T4 Response Using CL1
- T4 Response Using CL3
OMAR at Subsonic Cruise

Figure 4.49 OMAR ZC & ZF Responses For Control Laws CL1 & CL3
Chapter 5

Linear Optimal Tracking Problem

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5.8 Concluding Remarks 153
5.1 Introduction

In this chapter a brief account of the optimal tracking problem is given, to show that the response of the output vector of a linear observable system can be made to be close to some desired response.

The objective was to determine a single control law, based on the solution of this optimal tracking problem, which could be used with the linear model of the engine LIN throughout its flight envelope.

The tracking regulator problem is a version of the optimal tracking problem, in which the output vector of the system can be brought back to zero, in some specified manner. It was found out that a single control law, based on this tracking regulator problem, was capable of regulating LIN throughout the operating range.

5.2 The Optimal Tracking Problem

Given a linear, observable system described by the equations

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

(5.1)

where \( x(t) \) is the state vector in \( \mathbb{R}^n \), \( u(t) \) is a control vector in \( \mathbb{R}^m \) and \( y(t) \) is the output vector in \( \mathbb{R}^p \). \( A, B, C \) and \( D \) are matrices of appropriate order. The observability matrix,

\[
T = [C : CA : CA^2 \ldots CA^{n-1}]
\]

has full rank, \( n \).

Suppose that a vector \( z(t) \) is the desired output, of dimension \( p \), the same dimension as \( y(t) \). The objective is to control the system in such a manner as to make the output vector \( y(t) \) close to the vector \( z(t) \) without excessive expenditure of control energy. The error vector \( e(t) \) is defined as

\[
e(t) = z(t) - y(t)
\]

(5.3)
The cost function $J_e$ to be minimized, is chosen to be:

$$J_e = \frac{1}{2} e^T(t_f) S e(t_f) + \frac{1}{2} \int_0^{t_f} \left( e^T(t) Q e(t) + u^T(t) G u(t) \right) dt$$  \hspace{1cm} (5.4)$$

with the assumption that $t_f$ is specified, $G$ is positive definite, and $Q$ and $S$ are positive semi-definite. Since the error vector is a function of $z(t)$, $x(t)$, and $u(t)$, i.e.

$$e(t) = z(t) - \left( C x(t) + D u(t) \right)$$  \hspace{1cm} (5.5)$$

and since it is required that the terminal error shall be zero, i.e. $S = 0$, then the cost function, $J_e$, of equation (5.4) can be written as:

$$J_e = \frac{1}{2} \int_0^{t_f} \left[ (z(t) - (C x(t) + D u(t)))^T Q (z(t) - (C x(t) + D u(t)) + u^T(t) G u(t) \right] dt$$  \hspace{1cm} (5.6)$$

The Hamiltonian for this tracking problem is given by

$$H = \frac{1}{2} \left[ z^T(t) z(t) - z^T(t) Q (C x(t) + D u(t)) - (C x(t) + D u(t))^T Q z(t) 
+ \left( C x(t) + D u(t) \right)^T Q (C x(t) + D u(t) + u^T(t) G u(t) \right] 
\psi(t) \left( A x(t) + B u(t) \right)$$  \hspace{1cm} (5.7)$$

To minimize the Hamiltonian,

The condition $\frac{\partial H}{\partial u(t)} = 0$ yields the equation,

$$\frac{\partial H}{\partial u(t)} = D^T Q C x(t) + \left( G + D^T Q D \right) u^0(t) + B^T \psi(t) - D^T Q z(t) = 0$$  \hspace{1cm} (5.8)$$
then

\[ \mathbf{u}^*(t) = -\left( \mathbf{G} + \mathbf{D}^T \mathbf{Q} \mathbf{D} \right)^{-1} \left[ \mathbf{D}^T \mathbf{Q} \mathbf{C} \mathbf{x}(t) + \mathbf{B}^T \psi(t) - \mathbf{D}^T \mathbf{Q} \mathbf{z}(t) \right] \]  

(5.9)

where \( \psi(t) \) is the system adjoint vector,

\[
\dot{\psi}(t) = \mathbf{C}^T \mathbf{Q} \mathbf{C} \mathbf{x}(t) + \mathbf{C}^T \mathbf{Q} \mathbf{D} \mathbf{u}(t) + \mathbf{A}^T \psi(t) - \mathbf{C}^T \mathbf{Q} \mathbf{z}(t)
\]

(5.10)

Substituting equation (5.9) in equation (5.1) we get

\[
\dot{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{A} - \mathbf{B} \left( \mathbf{G} + \mathbf{D}^T \mathbf{Q} \mathbf{D} \right)^{-1} \mathbf{D}^T \mathbf{Q} \mathbf{C} \\
+ \mathbf{B} \left( \mathbf{G} + \mathbf{D}^T \mathbf{Q} \mathbf{D} \right)^{-1} \mathbf{D}^T \mathbf{Q} \mathbf{z}(t) \end{bmatrix} \mathbf{x}(t) - \mathbf{B} \left( \mathbf{G} + \mathbf{D}^T \mathbf{Q} \mathbf{D} \right)^{-1} \mathbf{B}^T \psi(t)
\]

(5.11)

Substituting equation (5.9) in equation (5.10) we get

\[
\dot{\psi}(t) = -\mathbf{C}^T \mathbf{Q} \mathbf{C} \mathbf{x}(t) + \mathbf{C}^T \mathbf{Q} \mathbf{D} \left( \mathbf{G} + \mathbf{D}^T \mathbf{Q} \mathbf{D} \right)^{-1} \mathbf{D}^T \mathbf{Q} \mathbf{x}(t)
\]

\[
+ \mathbf{C}^T \mathbf{Q} \mathbf{D} \left( \mathbf{G} + \mathbf{D}^T \mathbf{Q} \mathbf{D} \right)^{-1} \mathbf{B} \dot{\psi}(t) + \mathbf{A}^T \psi(t) + \mathbf{C}^T \mathbf{Q} \mathbf{z}(t)
\]

(5.12)

Combining equations (5.11) and (5.12) to obtain the canonical equation of the optimal tracking system:

\[
\begin{bmatrix} \dot{\mathbf{x}}(t) \\
\dot{\psi}(t) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{A}}^T \\
\hat{\mathbf{Q}} & -\hat{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\
\psi(t) \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{E}} \\
\hat{\mathbf{H}} \end{bmatrix} \mathbf{z}(t)
\]

(5.13)

where
This is a system of $2n$ linear differential equations, with the desired output vector $z(t)$ acting as the forcing function. It can be shown [Athans and Falb (1966) and Anderson and Moore (1971)] that the state vector $x(t)$ and the co-state vector $\psi(t)$ are related by the equation

$$\psi(t) = P(t)x(t) - g(t) \quad (5.20)$$

where $P(t)$ is an $[n \times n]$ matrix and $g(t)$ is an $n$-column vector. If equation (5.20) is differentiated with respect to time, then

$$\dot{\psi}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t) - \dot{g}(t) \quad (5.21)$$

From equations (5.13) and (5.20) we obtain

$$\dot{\psi}(t) = -Qx(t) - \hat{A}^T(P(t)x(t) - g(t)) + \hat{H}z(t) \quad (5.22)$$

Using $\dot{x}(t)$ from equation (5.11) and $\dot{\psi}(t)$ from equation (5.22), together with $\psi(t)$ from equation (5.20), equation (5.21) will become

$$-Qx(t) - \hat{A}^T P(t)x(t) + \hat{A}^T g(t) + \hat{H}z(t) = P(t)\hat{A}x(t) - P(t)\hat{F}P(t)x(t) + P(t)\hat{F}g(t) + P(t)\hat{E}z(t) + x(t)\hat{F}P(t) - \dot{g}(t) \quad (5.23)$$

From equation (5.23) it can be concluded that
1. The \([n \times n]\) matrix \(P(t)\) must satisfy the Riccati matrix differential equation

\[
\dot{P}(t) = P(t) \hat{A} + \hat{A}^T P(t) - P(t) \hat{F} P(t) + \hat{Q} \tag{5.24}
\]

2. The column vector \(g(t)\) must satisfy the vector differential equation

\[
\dot{g}(t) = \left[ P(t) \hat{F} - \hat{A}^T \right] g(t) + \left[ P(t) \hat{E} - \hat{H}^T \right] z(t) \tag{5.25}
\]

Since the terminal error should be zero, i.e. the \(S\) matrix in equation (5.4) is zero, then the boundary conditions for the differential equations (5.24) and (5.25) are

\[
P(t_f) = 0 \quad \tag{5.26}
\]

\[
g(t_f) = 0 \quad \tag{5.27}
\]

From examining equation (5.24) and its boundary condition (5.26), it can be seen that neither depends upon the independent desired output \(z(t)\). This means that once the system, the cost function, and the terminal time \(t_f\) are specified, then the matrix \(P(t)\) is completely specified. From comparison of equation (5.24) with that of the output regulator problem (3.24) one finds that they are completely identical, which means that the feedback structure of the optimal tracking system is the same as the feedback structure of the optimal output regulator system.

The essential difference between the optimal tracking problem and the output regulator problem is however the vector, \(g(t)\). One can think of the vector \(g(t)\) as a forcing function to the system of equation (5.11), and one can regard the vector \(z(t)\) as the forcing function to the dynamical system which generates the signal \(g(t)\), i.e. equation (5.25). To carry on with the above analysis, \textit{a priori} knowledge of the desired output \(z(t)\) must be available (which is the case in this work), i.e. \(z(t)\) must be a known function of time.

5.3 Mathematical Model of the Desired Output

The desired output vector \(z(t)\) for the optimal tracking problem of section (5.2) was obtained by applying a step function to an appropriate filter. A second order filter was used\(^+\) and may be represented as a transfer function, viz:

\[^+\] A second order filter was chosen since this is an appropriate representation of the dynamics of the actuator used on the F100 engine.
\[
\frac{z_1(s)}{w(s)} = \frac{(1 + cs)}{(1 + as)(1 + bs)} \tag{5.28}
\]

where \(a, b\) and \(c\) are parameters of the system.

Since a state space formulation has been used in the optimal tracking problem of section (5.2), it will be convenient to transform equation (5.28) into a state space one. Re-arranging equation (5.28) gives,

\[
z_1(s) \left[ (1 + as)(1 + bs) \right] = w(s) \left[ 1 + cs \right] \tag{5.29}
\]

or

\[
(z_1(s) \left[ 1 + (a+b)s + (ab)s^2 \right] = w(s) \left[ 1 + cs \right] \tag{5.30}
\]

or

\[
(ab) \ddot{z}_1(t) + (a+b) \dot{z}_1(t) + z_1(t) = w(t) + c \dddot{w}(t) \tag{5.31}
\]

\[
\ddot{z}_1(t) = -\left\{ \frac{a+b}{ab} \right\} \dot{z}_1(t) - \left\{ \frac{1}{ab} \right\} z_1(t) + \left\{ \frac{1}{ab} \right\} w(t) + \left\{ \frac{c}{ab} \right\} \dddot{w}(t) \tag{5.32}
\]

\[
\ddot{z}_1(t) - \left\{ \frac{c}{ab} \right\} \dddot{w}(t) = \left\{ \frac{a+b}{ab} \right\} \dot{z}_1(t) - \left\{ \frac{1}{ab} \right\} z_1(t) + \left\{ \frac{1}{ab} \right\} w(t) \tag{5.33}
\]

Let

\[
v_2(t) = z_1(t) \tag{5.34}
\]

\[
v_1(t) = \dot{z}_1(t) - \left\{ \frac{c}{ab} \right\} w(t) \tag{5.35}
\]

Then

\[
\dot{v}_2(t) = \ddot{z}_1(t) \tag{5.36}
\]

\[
\dot{v}_1(t) = \dddot{z}_1(t) - \left\{ \frac{c}{ab} \right\} \dddot{w}(t) \tag{5.37}
\]

Substitution of equation (5.37) in equation (5.33) yields,

\[
\dot{v}_1(t) = -\left\{ \frac{a+b}{ab} \right\} \dot{z}_1(t) - \left\{ \frac{1}{ab} \right\} z_1(t) + \left\{ \frac{1}{ab} \right\} w(t) \tag{5.38}
\]

\[+ z_1(t), \text{ defined in equation (5.44), is the filter output and represents the first element of the desired response vector, } z(t)^T = [z_1(t), z_2(t), ..., z_p(t)], \text{ the remaining elements of } z(t) \text{ are zero since it is only the thrust, } FG, \text{ that is required to be close to } z_1(t). \text{ } w(s) \text{ is a command input to the filter.}\]
Substitution of equation (5.35) and (5.34) in (5.38) yields,

\[
\dot{v}_1(t) = - \left[ \frac{a+b}{ab} \right] v_1(t) + \left[ \frac{c}{ab} \right] w(t) - \left[ \frac{1}{ab} \right] v_2(t) + \left[ \frac{1}{ab} \right] w(t) \tag{5.39}
\]

or

\[
\dot{v}_1(t) = - \left[ \frac{a+b}{ab} \right] v_1(t) - \left[ \frac{1}{ab} \right] v_2(t) - \left[ \frac{a+b}{ab} \right] \frac{c}{ab} - \left[ \frac{1}{ab} \right] \right] w(t) \tag{5.40}
\]

Substituting equation (5.34) in (5.35) yields,

\[
\dot{v}_2(t) = v_1(t) + \left[ \frac{c}{ab} \right] w(t) \tag{5.41}
\]

Equations (5.41) and (5.42) can be expressed in matrix form as:

\[
\begin{align*}
\dot{v}(t) &= A_1 v(t) + B_1 w(t) \tag{5.43} \\
z_1(t) &= C_1 v(t) + D_1 w(t) \tag{5.44}
\end{align*}
\]

where \(v(t)\) is the filter state vector in \(\mathbb{R}^2\), \(w(t)\) is the filter command input, and \(z_1(t)\) is the filter output, \(A_1, B_1, C_1\) and \(D_1\) are matrices of the appropriate order and are defined as:

\[
A_1 = \begin{bmatrix}
-\left(\frac{a+b}{ab}\right) & -\left(\frac{1}{ab}\right) \\
1 & 0
\end{bmatrix}
\]

\[
B_1 = \begin{bmatrix}
-\left(\frac{ac+bc-ab}{ab}\right) \\
\left(\frac{c}{ab}\right)
\end{bmatrix}
\]

\[
C_1 = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad D_1 = \begin{bmatrix}
0
\end{bmatrix}
\]
Figure (5.1) shows a block diagram representation of this filter dynamics in its Laplace form (equation 5.28).

Figure (5.2) shows a block diagram representation of the same filter dynamics in its state space form, equations (5.43) and (5.44).

5.4 Numerical Solution of the Tracking Problem

Since the aim of the optimal tracking problem is to make the output of the system y(t) close to the desired vector z(t), the first step in a numerical solution of this problem is to construct the z(t) vector. Using the analysis of section (5.3), it is possible to specify the required response of z(t) by a proper choice of the filter parameters a, b and c, where z(t) will be solved forward in time from t=0 to t=tf; it is stored at each required communication interval.

It was pointed out in section (3.3), and shown in section (5.2), that the formulation of the matrix Riccati equation, based on the output regulator problem, is essential for the solution of the optimal tracking problem. Therefore, the second step in the solution of the optimal tracking problem is to construct the output regulator matrix Riccati equation (like that of equation (3.19)), where the final time tf has been specified from the first step, i.e. the z(t) solution. This matrix Riccati equation is solved backwards in time, from t= tf to t=0, using the same numerical procedure developed in section (3.4); values of the Riccati matrix, P(t), are stored as an array for each communication interval.

The third step of this solution is to solve for the forcing function g(t), where the knowledge of P(t) and z(t) makes it possible to compute g(t), by solving equation (5.25) backwards in time for all t ∈ [t₀, tf]. Starting from g(tf), the g(t) solution can be found and stored at each relevant communication interval. Again the numerical procedure of section (3.5) can be used to solve equation (5.25). Figure (5.3) shows a block diagram representation for the g(t) solution, where block 2 in this figure shows the calculation of z₁(t) and block 1 shows the construction of the Riccati equation and the solution of the Riccati matrix P(t).

After z(t), P(t) and g(t) have been computed for all t ∈ [t₀, tf], and each equivalent value has been stored for the relevant communication interval, then the optimal control vector u₀(t) can be computed. To construct the vector u₀(t) for the optimal tracking problem, substitute equation (5.20) in equation (5.9), viz:
\[ u^0(t) = \left( G + D^T Q D \right)^{-1} \left[ D^T Q C x(t) + B^T P(t) x(t) - B^T g(t) - D^T Q z(t) \right] \] (5.45)

or
\[ u^0(t) = \hat{G}^{-1} \left[ D^T Q C + B^T P(t) \right] x(t) + \hat{G}^{-1} B^T \hat{g} + \hat{G}^{-1} D^T Q z(t) \] (5.46)

The vector \( u^0(t) \) is calculated forwards in time \( t \), from \( t=0 \) to \( t = t_f \) after reading the stored (pre-computed) values of \( z(t) \), \( P(t) \) and \( g(t) \) at each relevant communication interval.

Figure (5.4) shows a block diagram representation of \( u^0(t) \) calculation, where the vector \( x(t) \) has been generated from the solution of the system dynamics of equation (5.13).

5.5 Solution of the Optimal Tracking Problem Applied to LIN

When a specific thrust command is required for any operating point of LIN, then using the optimal tracking problem (O.T.P.) to provide specific thrust command appeared to be an effective technique. The optimal tracking analysis detailed in section (5.2) was applied to LIN at each operating point, where equation (5.9 or 5.46) was used to generate the required control law at that operating point, while equations (5.21), (5.24) and (5.25) were used to produce \( \psi(t) \), \( P(t) \) and the forcing function vector, \( g(t) \), respectively. The use of analysis of section (5.3) enabled the profile of the desired thrust, namely, \( z_1(t) \), to be generated; in particular equations (5.43) and (5.44) with an input command of \( w = 500 \).

Starting with the application of the optimal tracking problem, O.T.P., at the take-off operating point, LIN1, the optimal control law is CTR1. This control law has been derived using the following weighting matrices:

\[
Q = \text{diagonal} \left[ 1.00\text{E}+4, 10.0, 1.0, 1.0 \right]
\]

\[
G = \text{diagonal} \left[ 1.00\text{E}+8, 1.00\text{E}+8 \right]
\]

Figure (5.5) shows the time response of the engine thrust \( FG \) compared with the desired thrust signal \( z_1(t) \), together with the turbine inlet temp. \( T_4 \). It should be noticed from this figure that \( FG \) and \( z_1(t) \) are almost identical\( ^{+} \), i.e. the engine thrust has followed the prescribed desired response, while the amount of overshoot in the turbine inlet temp. \( T_4 \) was kept within acceptable working limits: in this operating point the maximum allowable overshoot in \( T_4 \) is 703.48.

\( ^{+} \) This very small difference between \( FG \) and \( z_1(t) \) was left intentionally for demonstrative purposes; it could be cancelled by increasing the numerical value of the weighting matrix element \( Q(1,1) \).
Figure 5.1: Block Diagram Representation of a Second Order Filter

\[ \frac{(1+cs)}{(1+as)(1+bs)} \]

Figure 5.2: Block Diagram of State Space Representation of a Second Order Filter
Figure 5.3: Block Diagram Representation of the Generation of the Forcing Function $g(t)$.

$P(t)$ matrix has been generated backwards in time.
Figure 5.4: Block Diagram Representation of the Optimal tracking Problem.
Figure (5.6) shows the responses of the engine surge margins $Z_C$ and $Z_F$; the deviations in the numerical values of both variables were kept within the working surge margins of the low and high pressure compressors, i.e. the operating point surge margins were still below the surge lines of both the high and low pressure compressors. In this operating condition the maximum allowable deviation in the numerical value of $Z_C$ is 0.1921, while that of $Z_F$ is 0.1645.

Figure (5.7) shows the optimal responses of the engine control variables, the fuel flow rate $WFB$ and the exit nozzle area $A_8$, while figure (5.8) presents the responses of the state variables when subjected to the control of figure (5.7).

Figures (5.9) and (5.9a) show the response of the forcing function vector $g(t)$, which is the numerical solution of equation (5.25). From this figure it should be observed that at time $t=t_f$ ($t_f=1.0$), the value of $g(t_f)$ is equal to zero, which is the required final condition value of $g(t)$. The split of the $g(t)$ vector into two figures (5.9 and 5.9a) was done merely for convenience of demonstration, because of the high numerical value of the $g_4(t)$ element. Figures (5.10) and (5.10a) show the response of the Riccati matrix, $P(t)$, with the reverse time $\tau$, which is the numerical solution of the matrix Riccati equation, M.R.E., equation (5.24). In these figures note that at $\tau=0$, $P(0)=0$, the required final value of $P(t)$, when the M.R.E. is solved with time, $t$. Again the split of the matrix $P(t)$ into two figures was done because of the high numerical value of the element $P(4,4)$ and those like it in figure (5.10a).

Next, the O.T.P., was applied to the climb operating condition LIN2. Figure (5.11) shows the response of the engine thrust $F_G$ compared with the desired profile $z_1(t)$, together with the response of the turbine inlet temperature $T_4$. From this figure it is clear that $F_G$ response is very much the same as $z_1(t)$, i.e. the engine thrust response follows the desired response, while the overshoot in the turbine inlet temperature $T_4$ in this operating condition is lower than that of the take-off point, because LIN2 operates at a lower energy level than LIN1.

Figure (5.12) shows the engine surge margins $Z_C$ and $Z_F$. Normally their maximum allowable deviations are higher than those of LIN1, so it can be seen that this operating condition has remained within the working surge margins.

The optimal control law of this operating condition is denoted OTR2, and has been derived using the following weighting matrices:

\[
Q = \begin{bmatrix} 1.00E+5, & 10.0, & 1.0, & 1.0 \end{bmatrix}
\]

\[
G = \begin{bmatrix} 1.00E+9, & 1.00E+9 \end{bmatrix}
\]
Figure (5.13) shows the optimal response of the engine control variables WFB and A8, which cause the variables of LIN2 model to respond in the manner presented in figures (5.11) and (5.12).

For the application of the O.T.P. to the subsonic cruise operating condition LIN3, the resulting optimal control law is OTR3, which has been derived using the following weighting matrices.

\[
Q = \text{diagonal} \begin{bmatrix} 1.00E+10, 10.0, 1.0, 1.0 \end{bmatrix}
\]

\[
G = \text{diagonal} \begin{bmatrix} 9.00E+15, 5.00E+13 \end{bmatrix}
\]

Figure (5.14) shows the response of the engine thrust FG compared with the desired one \( z_I(t) \), together with the turbine inlet temperature T4. From this figure it should be noticed once more that FG and \( z_I(t) \) are practically identical. The rest of the engine variables at this operating point were not presented here, since, they each kept within the allowable working limits.

To avoid repetition, the thrust responses of the engine at the other operating points LIN4, LIN5 and LIN6, have not been presented because they were similar to those presented in figures (5.5), (5.11) and (5.14), while other engine variables at those operating points were all maintained within the working limits of the engine. As a matter of record, for the O.T.P. with LIN4, the optimal control law was denoted OTR4; it had been derived using the following weighting matrices:

\[
Q = \text{diagonal} \begin{bmatrix} 1.00E+11, 10.0, 1.0, 1.0 \end{bmatrix}
\]

\[
G = \text{diagonal} \begin{bmatrix} 6.00E+13, 1.00E+14 \end{bmatrix}
\]

For the O.T.P. with LIN5, the optimal control law was denoted OTR5; it had been derived using the following weighting matrices:

\[
Q = \text{diagonal} \begin{bmatrix} 3.00E+11, 10.0, 1.0, 1.0 \end{bmatrix}
\]

\[
G = \text{diagonal} \begin{bmatrix} 9.00E+13, 2.00E+14 \end{bmatrix}
\]

For the O.T.P. with LIN6, the optimal control law was denoted OTR6; it had been derived using the following weighting matrices:
LIN1 Optimal Tracking Problem

Figure 5.5 Time Responses of FG & T4 For LIN1 Using O.T.P
Figure 5.6 Response of LNI ZC & ZF Using The Optimal Tracking Problem

Figure 5.7 Response of LNI WFB & A8 Using The Optimal Tracking Problem
Figure 5.8 LJN1 State Variable Response Using Optimal Tracking Problem
Figure 5.9a Response of the Variable \( g_4 \) Using Optimal Tracking Problem

Figure 5.9 Response of the \( g \) Vector Using Optimal Tracking Problem
The Optimal Tracking Problem Riccati Matrix

Figure 5.10 Variation of LN1 Riccati Matrix Elements With Reverse Time

Figure 5.10a Variation of LN1 Riccati Matrix Elements With Reverse Time
LIN2 Optimal Tracking Problem

Figure 5.11 Time Responses of FG & T4 For LIN2 Using O.T.P
Figure 5.12 Response of LIN2 ZC & ZF Using The Optimal Tracking Problem

Figure 5.13 Response of LIN2 WFB & A8 Using The Optimal Tracking Problem
LIN3 Optimal Tracking Problem

Figure 5.14 Time Responses of FG & T4 For LIN3 Using O.T.P
Q = diagonal\[1.00E+10, 10.0, 1.0, 1.0]\]
G = diagonal\[3.00E+13, 9.00E+13]\]

5.6 Application of a Single Control Law Over the Engine Flight Envelope

In the preceding section, it was shown that the individual solutions to the optimal tracking problem worked successfully at each operating point. What is intended in this section is to present an account of an attempt to explore the possibility that the solution to the optimal tracking problem, based on one particular operating point, could be applied to all other operating points of LIN. This meant that both the Riccati matrix $P(t)$ and the forcing function vector $g(t)$ had to be generated from one operating point and then used at another operating point. The rest of the solution at the new operating point was then proceeded with on that basis. Figure (5.15) shows a block diagram representation of this principle, where block B in the figure shows the generation of both $P(t)$ and $g(t)$ at some other different operating point and then being fed to block A, which represents the dynamics of another different operating point. For example, the control law obtained as a solution for LIN3, say, was used with LIN1, LIN2, LIN4 etc.

Several case studies were made, but for brevity only two are presented here. The first study is at operating point LIN5.

Fig(5.16) shows the response of thrust FG, corresponding to LIN5 with the desired profile $z_1(t)$. There are two conditions:

The first was the O.T.P. when solved based on the model LIN5, i.e. both the matrix $P(t)$, and the vector, $g(t)$, were calculated on the basis of LIN5. The O.T.P. control law CTR5 was then applied to the model dynamics representing LIN5.

The second was when the control law obtained from the O.T.P. based upon LIN1 was applied to LIN5. That is, both the matrix $P(t)$, and vector $g(t)$, were calculated on the basis of LIN1, but the control law CTR1, was then applied to the model dynamics representing LIN5.

Note from figure (5.16) that in the first case the response of FG was identical to $z_1(t)$, a successful condition, but in the second condition the difference between $z_1(t)$ and FG was large.
Figure (5.17) shows the response of LIN5 control variables WFB and A8 for the two conditions discussed above. The difference between the control vectors in both conditions is quite clear, in particular the nozzle area control variable, A8.

The second case study is operating point LIN1:

Figure (5.18) shows the response of LIN1 thrust FG compared with the desired signal $z_1(t)$ for two conditions. The first condition was when both the $P(t)$ matrix and the $g(t)$ vector were calculated locally on the basis of LIN1 and applied to the LIN1 model, i.e. the O.T.P. control law CTR1 was applied to LIN1. Note that the LIN1 FG response was nearly the same as $z_1(t)$. The second condition was when the $P(t)$ matrix and the $g(t)$ vector were calculated on the basis of the climb operating point, LIN2, and was then fed to the operating point model, LIN1. From figure (5.18) it can be seen that LIN1 thrust for the second condition is not very near to the desired signal $z_1(t)$. From the operational viewpoint this is not a favourable condition. Figure (5.19) shows the response of LIN1 control variables WFB and A8 for the same two conditions. From this figure the difference in the fuel flow responses for both conditions can be seen to be marked, while the responses of the nozzle area A8 were not very different. This small difference in nozzle area responses for these two conditions lead to the small difference in the thrust responses of the same two conditions, which was noticed in figure (5.18). This was similar to the first case study of LIN5.

This trend prevailed throughout the several case studies which were conducted over the flight envelope: that is, the larger the difference in nozzle area responses, the larger is the difference in thrust responses for any two conditions under consideration, and vice versa.

It was concluded that this observed relationship between the thrust and the nozzle area, makes it impossible for the solution to the optimal tracking problem, based upon one operating point, to make the thrust at another operating point be close to the desired response, i.e. there is no single control law which can be derived from the optimal tracking problem for one operating point, which is capable of giving the desired commanded thrust response throughout the engine's operating range.

From the viewpoint of the thermodynamic performance of the jet engine, there is a sound justification for this conclusion. That is, when an engine has been subjected to two different control laws, each of them will drive the engine to a different value of exit nozzle area A8, hence the engine has been driven onto two different working lines, since changing A8 changes the level of the engine working line [Harman (1981) and Hill and Peterson (1965)]. Figure (5.20) illustrates this principle.
Figure 5.15: Block Diagram Representation of the Optimal tracking Problem calculated from one operating point and applied to another.
Figure 5.16 LIN5 Thrust Response Based On LIN5 & LIN1 O.T.P

Legend
- The Desired Thrust
- LIN5 FG Based On LIN1 O.T.P
- LIN5 FG Based On LIN5 O.T.P

Figure 5.17 LIN5 WFB & A8 Using Optimal Tracking Problem Of LIN5 & LIN1
Figure 5.18 LIN1 Thrust Response Based On LIN1 & LIN2 O.T.P

Legend
- The Desired Thrust
- LIN1 FG Based On LIN1 O.T.P
- LIN1 FG Based On LIN2 O.T.P

Figure 5.19 LIN1 WFB & A8 Using Optimal Tracking Problem Of LIN1 & LIN2
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Figure 5.20 Lines of constant nozzle area showing working lines and surge margin.
This research work has indicated that those two different control laws drive the engine to two
different operating points (although those operating points correspond to the same altitude),
and hence the engine runs at two different thrust levels.

5.7 The Tracking Problem Used As Regulator Problem

The optimal tracking analysis of section (5.2) was used to obtain a control strategy similar to
that of the output regulator problem, that is to say "when the state vector \( x(t) \) has been
excited by an initial vector, \( x(o) \), then the output vector, \( y(t) \), is to be brought back to zero
as soon as possible, but without excessive expenditure of control energy, or infringe the
engine constraints". The control strategy of the optimal tracking regulator problem, O.T.R.,
was "when the state vector, \( x(t) \), has been excited by an initial vector, \( x(o) \), then the output
vector \( y(t) \) is to be brought back to zero, following a specific desired response, but without
excessive expenditure of control energy, or infringement of the engine constraints."

The main difference between these two control strategies is that in the O.T.R., the output
vector \( y(t) \) must decay to zero following a specific desired profile, and that can be achieved
through using the control law obtained from the optimal tracking analysis.++

This desired profile signal \( z_1(t) \) which is the first element of the vector \( z(t) \), was generated
from a first order filter, whose general form was chosen to be:

\[
\frac{z_1(t)}{w(s)} = \frac{1}{1 + as}
\]  

(5.47)

where \( a \) is the system parameter.

For the autonomous case, i.e. where \( w(s) = 0 \), then

\[
z_1(s) (1+as) = 0
\]  

(5.48)

++ It is important to point out here that, although in theory it is possible to make the whole of
the output vector \( y(t) \) follow a desired function, in this section, as in section (5.5), it was
only the first variable FG of the output vector, which was forced to follow the desired
response, but other output variables T4, ZC and ZF, were kept within the
working limits. That was done for two reasons: first, the operational importance of the
engine thrust, FG, and, secondly, it was more convenient for computation.
Equation (5.47) may be written as a state equation, that is:

\[ \dot{z}_1(t) = A_1 z_1(t) \]  

(5.49)

where in the special case of this work \( A_1 \) is a \([1x1]\) matrix, with value \(-1/a\). The solution to (5.49) is:

\[ z_1(t) = z_1(o) e^{-t/a} \]  

(5.50)

As previously \( z_1(t) \) is the first element of the desired response vector \( z(t) \). Figure (5.21) shows a block diagram representation of equation (5.49), it is shown as part of figure (5.22).

To regulate LIN according to the O.T.R. control strategy, the optimal tracking analysis of section (5.2) was applied to LIN at each of the operating points. Equation (5.9) was used to generate the required control function at that operating point, and equations (5.21), (5.24) and (5.25) were used to generate \( \psi(t) \), \( P(t) \) and the forcing function vector \( g(t) \). Equation (5.49) was used to generate the profile of the desired \( z_1(t) \). Figure (5.22) shows a block diagram representation of the tracking regulator problem, O.T.R.

Starting with the application of the O.T.R. to the take-off point LIN1, the control law at this point is OTR1, and was derived using the following weighting matrices:

\[ Q = \text{diagonal} [1.00E+7, 2.00E+4, 100.0, 100.0] \]

\[ G = \text{diagonal} [3.00E+11, 5.00E+10] \]

These \( Q \) and \( G \) matrices were obtained using the procedure outlined in section (3.6).

Figure (5.23) shows the time response of the engine thrust, \( FG \), with control law OTR1, the thrust response using the output regulator problem, OUTREG, control law CL1, is also shown. The inlet temperature \( T4 \) using OTR1 was compared with that using control law CL1. Although the overshoot in \( T4 \) (as a result of using OTR1) was higher than that which occurred when using CL1, nevertheless, \( T4 \) remained within the acceptable working limits. In figure (5.24) is shown the engine surge margin (\( ZC \) and \( ZF \)) responses, using both OTR1 and CL1; from this figure it can be seen that both were kept within the working limits in both conditions. It is quite clear from inspecting the thrust responses of figure (5.23) that the advantage of using control law OTR1 over that of CL1 is a more rapid response, especially when the other variables were kept within the limits, as mentioned above.
The second case presented here is the subsonic cruise condition, LIN3, where the tracking regulator problem control law used in this condition was OTR3, which was derived using the following weighting matrices:

\[
Q = \text{diagonal} \begin{bmatrix} 1.00E+7, 2.00E+3, 100.0, 100.0 \end{bmatrix}
\]
\[
G = \text{diagonal} \begin{bmatrix} 5.00E+12, 8.00E+13 \end{bmatrix}
\]

Figure (5.25) shows the thrust FG and T4 responses for the two conditions: first, using OTR3 control law, and, next, using the OUTREG CL3 control law. Figure (5.26) shows the engine surge margins ZC and ZF. Here the conclusion is similar to the case of figure (5.23) and figure (5.24), where thrust response using OTR3 was better than that using CL3 control law, while T4, ZC and ZF were kept within the working limits.

Figure (5.27) shows the engine thrust response corresponding to LIN2 operating point but for different test conditions, these conditions were:

a) open loop
b) LIN2 using the OUTREG control law, CL2
c) LIN2 using the O.T.R. control law, OTR2

For comparison the desired thrust signal, \( z_1(t) \), has also been shown. It is quite clear from figure (5.27) that the response corresponding to condition c is better than those of conditions a and b, being almost everywhere identical with \( z_1(t) \).

The O.T.R. control law OTR2 was calculated using the following weighting matrices:

\[
Q = \text{diagonal} \begin{bmatrix} 6.00E+8, 3.00E+6, 100.0, 100.0 \end{bmatrix}
\]
\[
G = \text{diagonal} \begin{bmatrix} 1.00E+14, 1.00E+14 \end{bmatrix}
\]

Figures (5.28), (5.29), (5.30) and (5.31) present the same responses as Figure (5.27) but for the operating conditions LIN3, LIN4, LIN5 and LIN6 respectively. From those figures, a general trend can be observed, namely: that the thrust responses, based on the O.T.R. control laws, are better in general than those based on the OUTREG control laws. It is important here to point out that all other engine variables for the operating conditions described in the above figures, were kept within the working limits for each operating point.

For the purpose of record the O.T.R. control law OTR4 was calculated using the following weighting matrices:
The O.T.R. control law OTR5 was calculated using the following weighting matrices:

\[
Q = \text{diagonal}[9.00E+7, 1.00E+6, 300.0, 100.0]
\]
\[
G = \text{diagonal}[1.00E+9, 2.00E+7, 300.0, 100.0]
\]

The O.T.R. control law OTR6 was calculated using the following weighting matrices:

\[
Q = \text{diagonal}[1.00E+10, 1.00E+2, 100.0, 100.0]
\]
\[
G = \text{diagonal}[5.00E+15, 9.00E+13]
\]

Regarding the matter of finding a single control law to regulate LIN throughout the operating range, it was shown in Chapter 4, that the output regulator problem OUTREG, control law CL1 was capable of reasonably regulating LIN throughout the flight envelope. What was intended in the work being reported in this section was to explore the possibility of finding a single control law, based on the tracking regulator problem, which would be able to regulate LIN effectively throughout the operating range, and to establish whether the dynamic response of LIN using this O.T.R. control law was better or worse than that which resulted from using CL1.

Several studies were made at points throughout the flight envelope: once more it was found that use of the control law, OTR1, based on the take-off operating point, for the other operating points of LIN, was better.

Figure (5.32) shows a block diagram representation of using the OTR control law calculated at one operating point and applied to another operating point of LIN. Of the studies mentioned, three are now presented here to support the finding. Figure (5.33) shows the thrust response of LIN for the following conditions:

a) open loop
b) using the O.T.R. control law OTR1
c) using the OUTREG control law CL1

and were compared with the desired response $z_1(t)$. 

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It is clear from this figure that both conditions b and c are better than condition a, but none is near to the desired signal $z_j(t)$. Although the thrust response in condition b is somewhat better than that of c, in terms of settling time, it was the thrust value during the transient period of condition b, which was higher than that of condition c. This might be of some practical significance.

Figure (5.34) shows LIN2 control variables WFB and A8 for both of the above conditions b and c. It is clear from this figure that condition b requires a slightly higher value of WFB, but the most important thing is that A8 came back to zero with OTR1, i.e. to the same operating point and not to some other, as happened in the analysis of section (5.6).

Figure (5.35) shows thrust response of LIN5 for conditions similar to those of figure (5.33), but based on the afterburning take-off operating point. Figure (5.36) shows LIN5 surge margins ZC and ZF responses for the conditions b and c of figure (5.35). All were kept within the working limits.

Figure (5.37) shows thrust response of LIN6 for conditions similar to those of Figure (5.33), but based on the supersonic cruise operating point, while Figure (5.38) shows control variable response of LIN6 for the conditions b and c of figure (5.37).

For the several studies made throughout the flight envelope, two points should be mentioned here: first, for the application of OTR for LIN, all the engine variables were kept within the working limits; second, the conclusions drawn from a comparison of the thrust responses from figure(5.33) and (5.34) were obtained for every operating point throughout the operating range of the engine.

5.8 Concluding Remarks

In this chapter, a summary of the optimal tracking problem applied to a linear observable system was presented, and showed that it is possible to make the output vector of a dynamical system be close to some desired response. A numerical procedure for solving the problem resulted from the application of the optimal tracking problem, O.T.P., to the linear model of the jet engine was developed. From the solution of this problem it was concluded that:

1 - The O.T.P was successful in providing the desired response locally, but no single control law derived on the basis of this O.T.P. was able to provide a global solution.
2 - The optimal tracking regulator problem, O.T.R. was successfully used at each operating point of the engine flight envelope, and it was possible for a single control law, based on this O.T.R., to provide a useful and practical solution for the global problem.

3 - The nozzle area, A8, control variable, was the most likely reason for the difference in responses between the O.T.R. and the O.T.P., when they were derived from one operating point and applied to another.
Figure 5.21: Block diagram of the first order filter

Figure 5.22: Block Diagram Representation of the Optimal tracking Regulator Problem (O.T.R.).
Figure 5.23  LIN1 Thrust & T4 Based on Output Regulator Problem and O.T.R

Figure 5.24  LIN1 ZC & ZF Based on Output Regulator Problem and O.T.R
Figure 5.25 LIN3 FG & T4 Based On OUTREG and O.T.R

Figure 5.26 LIN3 ZC & ZF Based on Output Regulator Problem and O.T.R
Climb Condition

Legend
- The Desired Thrust
- LIN2 Open Loop Thrust
- LIN2 Thrust Using OUTREG CL2
- LIN2 Thrust Using OTR2

Figure 5.27 LIN2 Thrust Using OUTREG and O.T.R
Subsonic Cruise Condition

![Graph showing engine thrust vs time with legend](image)

Legend:
- ■ The Desired Thrust
- □ LIN3 Open Loop Thrust
- ● LIN3 Thrust Using OUTREG CL3
- ○ LIN3 Thrust Using OTR3

Figure 5.28 LIN3 Thrust Using OUTREG and O.T.R
Approach Condition

Figure 5.29 LIN4 Thrust Using OUTREG and O.T.R
Take-off With Afterburning Condition

Figure 5.30 LIN5 Thrust Using OUTREG and O.T.R
Supersonic Cruise Condition

![Graph showing Supersonic Cruise Condition](image)

Legend
- ■ The Desired Thrust
- □ LIN6 Open Loop Thrust
- ● LIN6 Thrust Using OUTREG CL6
- ○ LIN6 Thrust Using OTR6

Figure 5.31 LIN6 Thrust Using OUTREG and O.T.R
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Figure 5.34 LIN2 WFB & A8 With Control Laws OTR1 Of O.T.R & CL1 Of OUTREG
Figure 5.35 LIN5 Thrust Using The O.T.R OTR1 & The OUTREG CL1

Figure 5.36 LIN5 ZC & ZF Using Control Laws OTR1 & CL1 Of O.T.R & OUTFREG
Figure 5.37 LIN6 Thrust Using The O.T.R OTR1 & The OUTREG CL1

Legend
- The Desired Thrust
- LIN6 Open Loop Thrust
- LIN6 Thrust Using CL1
- LIN6 Thrust Using OTR1

Figure 5.38 LIN6 WFB & A8 Using Control Laws OTR1 Of O.T.R And CL1

Legend
- LIN6 WFB Using CL1
- LIN6 A8 Using CL1
- LIN6 WFB Using OTR1
- LIN6 A8 Using OTR1
Chapter 6

The Non-linear Problem

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6.1 Introduction

The determination of the optimal control of any non-linear system is, in general, a very difficult task. Only in the most restricted cases can an analytical solution be obtained. Therefore, the determination of optimal trajectories and their associated optimal controls must often be carried out numerically by means of some iterative procedure on a computer.

The application of optimal control theory to a non-linear system leads to the formulation of the problem as a non-linear, two-(or multi-)point, boundary-value problem.

In this chapter, the formulation of the problem is presented, together with an account of the numerical techniques which are capable (in theory) of solving such problems. Two of those techniques, the method of quasilinearization, and the invariant imbedding method, will be discussed in detail, together with how they are linked to form a useful technique for the solution of the problem.

6.2 Problem Formulation

A formal statement of the problem is to determine the optimal control vector, \( u^0(t) \), which will minimize a performance index:

\[
J = \int_0^\infty \mathcal{L}(x(t), u(t), t) \, dt
\] (6.1)

subject to the constraint of the system dynamics;

\[
x'(t) = f(x(t), u(t), t)
\] (6.2)

with \( x(t_0) = x_0 \)

where \( x(t) \) is the state vector of the system \( x(t) \in \mathbb{R}^n \) and \( u(t) \) is the control vector \( u(t) \in \mathbb{R}^m \).

The Hamiltonian of the system is given by:

\[
H = \mathcal{L}(x(t), u(t), t) + \psi^T(t) f(x(t), u(t), t)
\] (6.3)

where \( \psi(t) \) is the system adjoint vector \( \psi(t) \in \mathbb{R}^n \), obtained from:
Since $x(t_f)$ is unknown, then $\psi(t_f) = 0$. [This is the transversality condition, Burghes and Graham (1980)]

The optimal control vector $u^0(t)$ is determined from:

$$\frac{\partial H}{\partial u^0(t)} = 0$$ (6.5)

Thus the optimal control vector is influenced considerably by the choice of the cost function $L$ and the constraint $f$.

If this optimal control vector $u^0(t)$, is substituted for $u(t)$ in equation (6.2) and in the associated equation for the adjoint vector, $\psi(t)$, equation (6.4), then those equations become

$$\dot{x}(t) = f\left(x(t), \psi(t), t\right), \quad x(0) = x_0$$ (6.6)

$$\dot{\psi}(t) = g\left(x(t), \psi(t), t\right), \quad \psi(t_f) = 0$$ (6.7)

Combining equations (6.6) and (6.7) together will form a single, new vector equation given by:

$$\dot{R}(t) = \kappa(R(t)) = \begin{bmatrix} f\left(x(t), y(t), t\right) \\ g\left(x(t), y(t), t\right) \end{bmatrix}$$ (6.8)

where $R(t)$ is the new vector $R(t) \in \mathbb{R}^{2n}$, and defined as:

$$R(t) = \begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix}$$ (6.9)

---

* Early accounts of this condition relate to the calculus of variations [Bliss (1946), Bolza (1960) and Schulz and Melsa (1967)].
where the boundary conditions of this problem are:

\[ R_j(t_j) = d_j \]  \hspace{1cm} (6.10)  

where \( j = 1, 2, \ldots, 2n \); \( t_i = t_0 \) or \( t_f \).

or

\[ R_{j=1, \ldots, n}(t_0) = x(0), \quad \text{and} \quad R_{j=n+1, \ldots, 2n}(t_f) = \psi(t_f) = 0 \]

Equations (6.8) and (6.10) represent an example of what is known as non-linear two-point boundary value problems, T.P.B.V.P. In general the analytical solution of such problems is not available (apart from some restricted cases), and their numerical solutions prove to be difficult. In order to find \( x(t) \), equation (6.8) must be integrated forwards in time, from \( t=t_0 \) to \( t=t_f \), provided \( \psi(t) \) is known (which cannot be the case, since \( \psi(t_0) \) is unknown). The same condition applied for \( \psi(t) \), i.e. in order to find \( \psi(t) \), equation (6.8) must be integrated backwards in time from \( t=t_f \) to \( t=t_0 \), provided \( x(t) \) is known (which is not the case, since \( x(t_f) \) is unknown). Thus, there is a problem of initializing the solution process. In addition to this, the non-linear two-point boundary value problems associated with the application of optimal control theory, prove to be even more difficult to solve because of the instability of the adjoint equations generated by this sort of application.

Several techniques for solving the T.P.B.V.P. have been reported, and include: Finite Difference Method, Discrete Dynamic Programming, Gradient Method, Second Variation Method, Shooting-and-Matching, Quasilinearization and Invariant Imbedding [Sage (1966), Lee (1968), McCue (1967), Kelley (1962), Speed, Brown and Goodwin (1970), Keller (1975), Noton (1965), Dixon (1972)].

Most of the above-mentioned methods have the following points in common:

1. They are a problem-dependent techniques, i.e. a technique which proves to be useful in solving a specific problem may not prove to be useful when applied to another problem.

2. They depend heavily on estimates of the unspecified boundary values and numerical difficulties are likely to occur when they are applied to optimal control problems.

3. They require a fairly large amount of either computer memory, or computer running time, or both.
Procedures for solving the T.P.B.V.P. which combine two or more of the above-mentioned techniques have been reported. Roddy (1985) has developed a composite procedure for solving the non-linear T.P.B.V.P. where this procedure combines the Gradient Method, the Finite Difference Method and the Shooting-and-Matching technique. This composite procedure proved to be successful in certain cases but not for others, so the method of quasilinearization had to be used later in his work.

Sage (1968) and Lee (1970) had proposed a scheme which combines the invariant imbedding procedure with the quasilinearization technique, to form what is known as a predictor-corrector method. For the purpose of this research, this latest scheme is the one to be adopted.

6.3 Quasilinearization

Quasilinearization is a technique whereby a non-linear, two-point (or multi-point), boundary value problem is transformed into a more readily solvable linear, non-stationary boundary value problem. To apply the quasilinearization method, (Q.L.), to the system of equation (6.8), it is necessary to start by finding some nominal solution which satisfies the boundary conditions (6.10). Then the T.P.B.V.P. of (6.8) has to be linearized about this nominal trajectory, which may be termed (for the purpose of this work) $R^k(t)$. Next the resulting linear T.P.B.V.P. is solved to obtain a new trajectory, $R^{k+1}(t)$, which, under suitable conditions, will be closer to the true solution, $R(t)$, than was $R^k(t)$. Repeating this process, by linearizing the system of (6.8) about the new trajectory $R^{k+1}(t)$, will produce a new linear T.P.B.V.P. which is a better approximation to the non-linear system of (6.8). Solving this new linear T.P.B.V.P. will give a new trajectory $R^{k+2}(t)$, which is even closer to the true solution of the (6.8) system. If this process were to continue, then the sequences of the vectors or trajectories:

$$\{R^k(t), R^{k+1}(t), R^{k+2}(t), \ldots, R^{k+l}\}$$

converge to $R(t)$, which is the true solution of the non-linear T.P.V.B.P.

According to the Q.L. method, this sequence of vectors obeys:

$$\dot{R}^{k+1}(t) = \kappa(R^k,t) + \left[ J_{R^k} \kappa(R^k,t) \right] [R^{k+1}(t) - R^k(t)]$$  \hspace{1cm} (6.11)
where
\[
J_k \kappa (R^k, t)
\]
is the Jacobian matrix, of order \([2n \times 2n]\), and the \(ij\)th component of which is given by
\[
\frac{\partial \kappa_i}{\partial R_j}
\]
or:
\[
J_k \kappa (R^k, t) = \begin{bmatrix}
\frac{\partial \kappa_1}{\partial R_1} & \frac{\partial \kappa_1}{\partial R_{2n}} \\
\frac{\partial \kappa_{2n}}{\partial R_1} & \frac{\partial \kappa_{2n}}{\partial R_{2n}}
\end{bmatrix}
\] (6.12)

It can be observed that equation (6.11) represents just the first two terms of a Taylor series expansion of equation (6.8) on the \((k+1)\)th iteration:
\[
\dot{R}^{k+1}(t) = \kappa R^{k+1}(t)
\] (6.13)
expanded about the \(k\)th iteration
\[
\kappa R^{k+1}(t) = \kappa (R^k, t) + \left[ \frac{\partial \kappa(R,t)}{\partial R} \right]_{R=R^k(t)} \left[ R^{k+1}(t) - R^k(t) \right]
\] (6.14)

The general solution of (6.11) has been shown (by references given in the previous section) to be of the form:
\[
R^{k+1}(t) = \phi \left( t, t_0 \right) R^{k+1}(t_0) + PP^{k+1}(t)
\] (6.15)

where \(\phi(t,t_0)\) is the transition matrix of the linearized system, of order \([2n \times 2n]\), and \(PP(t)\) is the particular integral solution of equation (6.11) and it is a vector \(\mathbb{R}^{2n}\).
Following the work of Spingarn (1970), the transition matrix $\phi(t,t_0)$ was evaluated by integrating the following matrix equation:

$$\frac{\partial \phi^{k+1}(t,t_0)}{\partial t} = \left[ \begin{array}{c} J_k \\ R_k \end{array} \right] \phi^k(t,t_0)$$ (6.16)

where

$$\phi^{k+1}(t_0,t_0) \Delta = I$$ (6.17)

The particular integral vector $PP(t)$ obeys the differential equation

$$PP^{k+1}(t) = \kappa (R^k, t) - \left[ \begin{array}{c} J_k \\ R_k \end{array} \right] R^k(t) + \left[ \begin{array}{c} J_k \\ R_k \end{array} \right] PP^{k+1}(t)$$ (6.18)

with the boundary conditions $PP^{k+1}(t_0) = 0$ (6.19)

In order to determine the initial condition vector, $R^{k+1}(t_0)$, the right hand side of equation (6.15) is to be equated to the given boundary conditions, the result will be a $2n$ linear, simultaneous, algebraic equation, the solution to which provides the unknown vector $R^{k+1}(t_0)$.

Thus:

$$\phi_{js} \left( \begin{array}{c} t_i \\ t_0 \end{array} \right) R^{k+1}_s(t_0) + PP_j(t) = d_{ij}$$ (6.20)

where

$$j = 1, 2, ... 2n$$

$$s = 1, 2, ... 2n$$

$d_{ij}$ is constructed in the manner of equation (6.10)

Once $R^{k+1}(t_0)$ is determined, then $R^{k+1}(t)$ is immediately available from the forwards integration of (6.11) from $t=t_0$ to $t=t_f$. 

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Finally, in the quasilinearization technique, instead of being solved directly, the non-linear differential equation is solved recursively as a series of linear differential equations. The linear equation is obtained by using the first and second terms in the Taylor's series expansion of the original non-linear equation. This technique is a generalized Newton-Raphson formula for functional equations. The main advantage of this technique is that if the procedure converges, it converges quadratically to the solution of the original equation. Quadratic convergence means that the error in the (k+1)th iteration tends to be proportional to the square of the error in the kth iteration. The advantage of quadratic convergence, of course, lies in the rapidity of convergence.

The main difficulty with the method arises from the fact that, in using the superposition principle, a set of algebraic equations must be solved. Thus, the ill-conditioning phenomenon in solving a set of linear algebraic equations can render the superposition principle useless. Furthermore, this technique requires a fairly good initial estimate of the solution, and the resulting linearised adjoint equations can be highly unstable.

6.4 Algorithm Test

A computer code was developed, based on the quasilinearization technique, in order to test and assess both the algorithm and the code before applying them to jet engine control. Several cases were studied and some of them are presented here:


Consider the non-linear process described by the following state equation:

\[ \dot{x}(t) = -x^3(t) + u(t), \quad x(0) = 1 \quad (6.21) \]

It is required to find the optimal control \( u(t) \) which minimises the following performance index:

\[ J = \frac{1}{2} \int_0^1 \left( x^2(t) + 0.01u^2(t) \right) dt \quad (6.22) \]
The Hamiltonian for this problem is

\[ H = \frac{1}{2} \left( x^2(t) + 0.01 u^2(t) \right) + \psi(t) \left( -x^3(t) + u(t) \right) \]  

(6.23)

The adjoint equation is

\[ \dot{\psi}(t) = 3x^2(t)\psi(t) - x(t), \quad \psi(1) = 0 \]  

(6.24)

The gradient of the Hamiltonian with respect to the control should be zero to satisfy equation (6.5)

\[ \therefore \frac{\partial H}{\partial u(t)} = 0.01u(t) + \psi(t) = 0 \]  

(6.25)

\[ \therefore u(t) = -100\psi(t) \]  

(6.26)

Then the system canonical equation could be written as:

\[ x(t) = -x^3(t) - 100\psi(t), \quad x(0) = 1 \]  

(6.27)

\[ \dot{\psi}(t) = 3x^2(t)\psi(t) - x(t), \quad \psi(1) = 0 \]  

(6.28)

Let

\[ R(t) = \begin{bmatrix} x(t) & \psi(t) \end{bmatrix}^T \]  

(6.29)

\[ \dot{R}(t) = \begin{bmatrix} x(t) & \dot{\psi}(t) \end{bmatrix}^T \]  

(6.30)

Hence the system Jacobian matrix, \( J_R \), is given as:

\[ J_R = \begin{bmatrix} -3x^2(t) & -100 \\ 6x(t)\psi(t) - 1 & -3x^2(t) \end{bmatrix} \]  

(6.31)

and

\[ \kappa(R(t) - J_R R(t)) = \begin{bmatrix} 3x^3(t) \\ -6x^2(t)\psi(t) \end{bmatrix} \]  

(6.32)
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Substituting equations (6.29), (6.30), (6.31) and (6.32) in equation (6.16) and equation (6.18), and integrating the resulting differential equations from their known initial values, will give the required complementary function solutions \( \phi(t) \) and the particular integral solutions \( PP(t) \). In order to find \( R^{k+1}(t_0) \) we substitute \( \phi(t_f), PP(t_f) \) and the given boundary values in equation (6.20), we get the following simultaneous equations:

\[
\begin{bmatrix}
1 & 0 \\
\phi_{21}(1) & \phi_{22}(1)
\end{bmatrix}
\begin{bmatrix}
\chi^{k+1}(0) \\
\psi^{k+1}(0)
\end{bmatrix}
+
\begin{bmatrix}
0 \\
PP_2(1)
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

(6.33)

where (1) represents at \( t=t_f=1 \)
(0) represents at \( t=t_0=0 \)

Substituting \( R^{k+1}(0) \) from (6.33), (6.30) and (6.31) into (6.11) will give the trajectory \( R^{k+1}(t) \). Repeating the same process by using \( R^{k+1}(t) \) as the nominal trajectory to find \( R^{k+2}(t) \). The optimal control is finally given by equation (6.26).

Figures (6.1) and (6.2) show the Q.L. solution of the above problem, and these results were identical to those presented by Speedy, Brown, Goodwin (1970).

In this problem (like others reported in the references of Section 6.2), one notices that the optimal control \( u^o(t) \) defined by equation (6.5) is specified in open loop form, i.e. as a function of time. Theoretically, it is possible that the optimal controller can be formed in closed loop form. However, the corresponding T.P.B.V.P. is then even more difficult to solve since it involves non-linear partial differential equations. Furthermore, the optimal control is highly dependent, in a non-linear manner, on the initial state vector \( x(0) \). This means that for most non-linear control systems, only open loop control laws are available, even though closed loop control laws are more desirable. This desirability of closed loop control laws led to the development of specific optimal control (S.O.C.).
2. Problem two: Specific Optimal Control

The specific optimal control problem, S.O.C., is defined in the following manner:

\[ \dot{x} = f(x, u, t), \quad x(t_0) = x_0 \quad (6.34) \]

it is intended to determine the unknown parameters in the control law of the form:

\[ u = h(y, b) \quad (6.35) \]

where \( y \in \mathbb{R}^p \) is a known function of the state, and \( b \in \mathbb{R}^q \) is a constant vector to be determined such that a performance index of the form

\[ J_e = \int_{t_0}^{t_f} \eta(x, u, t) \, dt \quad (6.36) \]

is minimized.

The above problem can be reformulated as follows:

\[ \dot{x} = f[x, h(y, b), t] \quad (6.37) \]

\[ J_e(b) = \int_{t_0}^{t_f} \eta[x, h(y, b), t] \, dt \quad (6.38) \]

and since \( y \) is a known function of \( x \) then:

\[ \dot{x} = f(x, b, t) \quad (6.39) \]

\[ J_e(b) = \int_{t_0}^{t_f} \eta(x, b, t) \, dt \quad (6.40) \]

with

\[ b = 0 \quad (6.41) \]

In order to demonstrate this technique, and how it is possible to solve the above problem using the Q.L. method, then the following problem is to be solved using the computer code developed in problem one. Given the system described by:
\[ x' = -x^2 + u, \quad x(0) = 10.0 \quad (6.42) \]

It is desired to employ an S.O.C. of the form \( u(t) = ax(t) \), where \( a \) is to be chosen so as to minimize

\[ J(a) = \frac{1}{2} \int_0^{1.0} \left( x^2 + u^2 \right) dt \quad (6.43) \]

The problem is reformulated by adjoining the equation \( a = 0 \) to the system dynamics. Thus,

\[ x' = -x^2 + ax, \quad x(0) = 10.0 \quad (6.44) \]

\[ \dot{a} = 0 \quad (6.45) \]

\[ J(a) = \frac{1}{2} \int_0^{1.0} \left( x^2 + a^2 x^2 \right) dt \quad (6.46) \]

The S.O.C. problem now becomes a problem of obtaining the value of \( a \) which minimizes the cost function (6.46), subject to the associated differential constraints. The Hamiltonian of the system can be written as:

\[ H = \frac{1}{2} \left( x^2 + a^2 x^2 \right) + \sum_{i=1}^{i=2} \psi_i f_i \]

\[ = \frac{1}{2} \left( x^2 + a^2 x^2 \right) + \psi_1 \left( -x^2 - ax \right) \quad (6.47) \]

\[ \dot{\psi}_1 = -\frac{\partial H}{\partial x} = -x - a^2 x + 2x \psi_1 - a \psi_1 \quad (6.48) \]

\[ \dot{\psi}_2 = -\frac{\partial H}{\partial a} = -\psi_1 x - a^2 \quad (6.49) \]

+This problem was reported by Sage (1966) and McLean (1974)
Equations (6.44), (6.45), (6.48) and (6.49) represent the system canonic equations subject to boundary conditions, i.e.

\[
\begin{align*}
    \dot{x} &= -x^2 + ax, \quad x(0) = 10.0 \\
    \dot{a} &= 0, \quad \psi_2(0) = 0 \\
    \dot{\psi}_1 &= -x - a^2 x + 2x \psi_1 - a \psi_1, \quad \psi_1(1.0) = 0 \\
    \dot{\psi}_2 &= -\psi_1 x - a^2 x, \quad \psi_2(1.0) = 0
\end{align*}
\]

where \( R = \begin{bmatrix} x & a & \psi_1 & \psi_2 \end{bmatrix}^T \)

Hence the canonical system Jacobian matrix, \( J_R \), is given as:

\[
J_R = \begin{bmatrix} (-2x a) & (x) & 0 & 0 \\
0 & 0 & 0 & 0 \\
(2\psi_1 - 1.0 - a^2) & (-\psi_1 - 2ax) & (-a + 2x) & 0 \\
(-\psi_1 - 2ax) & (-x^2) & (-x) & 0 \end{bmatrix}
\]

and

\[
\kappa(R) - J_R R = \begin{bmatrix} x^2 - ax \\
0 \\
2a^2 \psi_1 x + a\psi_1 \\
2ax^2 + \psi_1 x \end{bmatrix}
\]

From equation (6.50), the Q.L. equation is given as:

\[
R^{k+1} = \kappa(R^k) + J_R^k \left[ R^{k+1} - R^k \right]
\]
Substituting equations (6.50), (6.51) and (6.52) in equation (6.16) and (6.18), and integrating the resulting differential equations from their known initial values will give the required complementary function solution \( \phi(t) \), and the particular integral solution \( PP(t) \). \( R_k^{k+1}(x) \) is to be found using equation (6.10) with the known boundary values of (6.50), then equation (6.53) is to be solved forwards in time to find the trajectory \( R_k^{k+1}(t) \). Repeating the same process again, a in Problem One, until convergence takes place. Results obtained here were identical to those presented by Sage (1966) and McLean (1974).

Figures (6.3) and (6.4) show the Q.L. solution of the state variable \( x(t) \), and the constant \( a \) of the control variable \( u \). Figures (6.5) and (6.6) show the Q.L. solution of the adjoint variables \( \psi_1(t) \) and \( \psi_2(t) \) respectively.

3. Problem Three: A conventional three term feedback controller

In this problem it is intended to add a derivative and integral controllers to the proportional control function of problem two, and then to use the Q.L. method to solve the new problem. Given the system described by:

\[
x' = -x^2 + u, \quad x(0) = x_0
\]

it is desired that the control variable, \( u \), should take the form:

\[
u = a_0 x + a_1 x^2 + a_2 \int_{t_0}^{t} x \, dt
\]

(6.55)

where \( a_0, a_1, a_2 \) are to be chosen such as to minimize:

\[
J(a) = \int_{t_0}^{t} \left( x^2 + u^2 \right) \, dt
\]

(6.56)

Now let \( w = \int_{t_0}^{t} x \, dt \), then \( \dot{w} = x \)

(6.57)
Substitute (6.58) in (6.54) yield:

\[
\dot{x} = \frac{a_o x - x^2}{(1-a_1)} + \frac{a_2 w}{(1-a_1)} \quad (6.59)
\]

Substitute (6.59) in (6.58) yield:

\[
u = a_o x + a_1 \left[ \frac{x (a_o - x)}{(1-a_1)} + \frac{a_2 w}{(1-a_1)} \right] + a_2 w
\]

or

\[
u = \frac{1}{(1-a_1)} \left[ a_o x - a_1 x^2 + a_2 w \right] \quad (6.60)
\]

\[
\therefore \quad \nu^2 = \frac{1}{(1-a_1)^2} \left[ a_o^2 x^2 + a_1^2 x^4 - 2a_o a_1 x^3 + a_2^2 w^2 + 2a_2 a_0 w x - 2a_1 a_2 w x^2 \right]
\]

The Hamiltonian of the system is:

\[
H = \frac{1}{2} \left( \dot{x}^2 + \nu^2 \right) + \sum_{i=1}^{i=5} \psi_i f_i \quad (6.61)
\]

or

\[
H = \frac{1}{2} x^2 + \frac{1}{2(1-a_1)^2} \left[ a_o^2 x^2 + a_1^2 x^4 - 2a_o a_1 x^3 + a_2^2 w^2 + 2a_2 a_0 w x - 2a_1 a_2 w x^2 \right]
\]

\[
+ \psi_1 \left[ \frac{x (a_o - x)}{(1-a_1)} + \frac{a_2 w}{(1-a_1)} \right] \quad (6.63)
\]

\[
\psi_i = -\frac{\partial H}{\partial x_i}
\]
Then the system canonic equation subject to boundary conditions could be written as:

\[
\begin{align*}
\dot{x} &= \frac{a_0 x - x^2}{(1 - a_1)} + \frac{a_2 w}{(1 - a_1)}, \quad x(0) = x_0 \quad (6.64) \\
\dot{a}_0 &= 0, \quad \psi_2(0) = 0 \quad (6.65) \\
\dot{a}_1 &= 0, \quad \psi_3(0) = 0 \quad (6.66) \\
\dot{a}_2 &= 0, \quad \psi_4(0) = 0 \quad (6.67) \\
\dot{w} &= x, \quad \psi_5(0) = 0 \quad (6.68) \\
\dot{\psi}_1 &= -2x - \frac{1}{(1-a_1)^2} \left[ 2a_0^2 x + 4a_1^2 x^3 - 6a_0 a_1 x^3 + 2a_0 a_2 w - 4a_1 a_2 x w \right] \\
&\quad + \psi_1 \left[ \frac{2x - a_0}{1-a_1} \right] - \psi_5, \quad \psi_1(t_f) = 0 \quad (6.69) \\
\dot{\psi}_2 &= -\frac{1}{(1-a_1)^2} \left[ 2a_0^2 x^2 - 2a_1 x^3 + 2a_0 x w \right] - \left[ \frac{\psi_1 x}{1-a_1} \right], \quad \psi_2(t_f) = 0 \quad (6.70) \\
\dot{\psi}_3 &= -\frac{1}{(1-a_1)^2} \left[ 2a_1 x^4 - 2a_0 x^3 - 2a_2 w x^3 \right] + \left[ \frac{2}{(1-a_1)^3} \right] \\
&\quad + \left[ a_0^2 x^2 + a_1 x^4 - 2a_0 a_1 x^3 + a_2^2 w^2 + 2a_2 a_0 x w - 2a_1 a_2 w x^2 \right] \\
&\quad - \psi_1 \left[ \frac{x(x_0 - x)}{(1-a_1)^2} + \frac{a_2 w}{(1-a_1)^2} \right], \quad \psi_3(t_f) = 0 \quad (6.71)
\end{align*}
\]
\[ \dot{\psi}_4 = - \frac{1}{(1-a_1)^2} \left[ 2a_2w^2 + 2a_0xw - 2a_1wx^2 \right] - \left[ \frac{\psi_1w}{1-a_1} \right] \], \quad \psi_4(t) = 0 \quad (6.72) \\

\[ \dot{\psi}_5 = - \frac{1}{(1-a_1)^2} \left[ 2a_2w + 2a_0ax - 2a_1a_2x^2 \right] - \left[ \frac{\psi_1a_2}{1-a_1} \right] \], \quad \psi_5(t) = 0 \quad (6.73) \\

Equations (6.64) to (6.73) can be combined into a single vector equation:

\[ \dot{R}(t) = \kappa(R(t)) \quad (6.74) \]

where

\[ R = \begin{bmatrix} x & a_o & a_1 & a_2 & w & \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 \end{bmatrix}^T \quad (6.75) \]

The elements of the canonical system Jacobian matrix, \( J_R \), are given as:

\[ J_R(1,1) = \begin{bmatrix} a_0 - 2x \end{bmatrix}, \quad J_R(1,2) = \begin{bmatrix} x \end{bmatrix} \]

\[ J_R(1,3) = \begin{bmatrix} a_0x - x^2 + a_2y \end{bmatrix} \]

\[ J_R(1,4) = \begin{bmatrix} w \end{bmatrix}, \quad J_R(1,5) = \begin{bmatrix} a_2 \end{bmatrix} \]

\[ J_R(5,1) = 1.0 \]

\[ J_R(6,1) = -2 - \frac{1}{(1-a_1)^2} \left[ 2a_2^2 + 12a_1^2x^2 - 12a_0a_1x - 4a_1a_2w \right] + \begin{bmatrix} 2\psi_1 \end{bmatrix} \]

\[ J_R(6,2) = - \frac{1}{(1-a_1)^2} \left[ 4a_0x - 6a_1x^2 + 2a_2y \right] - \begin{bmatrix} \psi_1 \end{bmatrix} \]

\[ J_R(6,3) = - \frac{1}{(1-a_1)^2} \left[ 8a_1x^3 - 6a_0x^2 - 4a_2wx - 2\psi_1x + a_0\psi_1 \right] - \begin{bmatrix} \frac{2}{(1-a_1)^3} \end{bmatrix} \]

\[ \begin{bmatrix} 2x & a_0^2 + 4a_1^2x^3 - 6a_0a_1x^2 + 2a_0a_2w - 4a_1a_2wx \end{bmatrix} \]
\[ J_R(6,4) = - \frac{1}{(1-a_1)^2} \left[ 2a_0w - 4a_1wx \right] \]

\[ J_R(6,5) = - \frac{1}{(1-a_1)^2} \left[ 2a_0a_2 - 4a_1a_2x \right] \]

\[ J_R(6,6) = - \left[ \frac{a_0 - 2x}{1-a_1} \right] \]

\[ J_R(6,10) = -1.0 \]

\[ J_R(7,1) = - \frac{1}{(1-a_1)^2} \left[ 4a_0x - 6a_1x^2 + 2a_2w \right] - \left[ \frac{\Psi_1}{1-a_1} \right] \]

\[ J_R(7,2) = \left[ \frac{-2x^2}{(1-a_1)^2} \right] \]

\[ J_R(7,3) = \left[ \frac{2x^3 - \Psi_1x}{(1-a_1)^2} \right] - \left[ \frac{2}{(1-a_1)^3} \right] \left[ 2a_0x^2 - 2a_1x^3 + 2a_2wx \right] \]

\[ J_R(7,4) = - \left[ \frac{2wx}{(1-a_1)^2} \right], \quad J_R(7,5) = - \left[ \frac{2a_2x}{(1-a_1)^2} \right] \]

\[ J_R(7,6) = - \left[ \frac{x}{1-a_1} \right] \]

\[ J_R(8,1) = - \frac{1}{(1-a_1)^2} \left[ 8a_1x^3 - 6a_0x^2 - 4a_2wx + \Psi_1a_0 - 2x\Psi_1 \right] \]

\[ - \left[ \frac{2}{(1-a_1)^3} \right] \left[ 2a_0^2x + 4a_1x^3 - 6a_1a_0x^2 + 2a_1a_2w - 4a_1a_2wx \right] \]
\[ J_R(8,2) = -\frac{1}{(1 - a_1)^3} \left[ -2x^3 + \psi_1 x \right] - \left[ \frac{2}{(1 - a_1)^3} \right] \left[ 2a_o x^2 - 2a_1 x^3 + 2a_2 w x \right] \]

\[ J_R(8,3) = -\left[ \frac{2x^4}{(1 - a_1)^2} \right] - \left[ \frac{2}{(1 - a_1)^3} \right] \left[ 2a_1 x^4 - 2a_o x^3 - 2a_2 w x^2 \right] \]

\[ + \psi_1 x a_o - \psi_1 x^2 + a_2 w \psi_1 \] \[ - \left[ \frac{2}{(1 - a_1)^3} \right] x^4 - 2a_o x^3 - 2a_2 w x^2 \]

\[ - \left[ \frac{6}{(1 - a_1)^4} \right] \left[ a_o x^2 + a_1 x^4 - 2a_2 a_1 x^3 + a_2 w + 2a_2 a_o x w - 2a_1 a_2 w x^2 \right] \]

\[ J_R(8,4) = \left[ \frac{2 w x^2 - w \psi_1}{(1 - a_1)^2} \right] - \left[ \frac{2}{(1 - a_1)^3} \right] \left[ 2a_2 w^2 + 2a_o x w - 2a_1 w x^2 \right] \]

\[ J_R(8,5) = \left[ \frac{2a_o x^2 - a_1 \psi_1}{(1 - a_1)^2} \right] - \left[ \frac{2}{(1 - a_1)^3} \right] \left[ 2a_2 a_o x - 2a_1 a_2 x^2 \right] \]

\[ J_R(8,6) = \left[ \frac{- a_o x + x^2 - a_2 w}{(1 - a_1)^2} \right] \]

\[ J_R(9,1) = \left[ \frac{-2a_o w + 4a_1 w x}{(1 - a_1)^2} \right] \]

\[ J_R(9,2) = -\left[ \frac{x w}{(1 - a_1)^2} \right] \]
\[ J_R(9,3) = \left[ \frac{2w^2 - \psi_1w}{(1 - a_1)^2} \right] - \left[ \frac{2}{(1 - a_1)^2} \right] \left[ 2a_2w^2 + 2a_0w - 2a_1w^2 \right] \]

\[ J_R(9,4) = -\left[ \frac{2w^2}{(1 - a_1)^2} \right] \]

\[ J_R(9,5) = -\left[ \frac{1}{(1 - a_1)^2} \right] \left[ 4a_2w + 2a_0x^2 \right] - \left[ \frac{\psi_1}{1 - a_1} \right] \]

\[ J_R(9,6) = -\left[ \frac{y}{1 - a_1} \right] \]

\[ J_R(10,1) = -\left[ \frac{2a_0a_2 + 4a_1a_2x}{(1 - a_1)^2} \right], \quad J_R(10,2) = -\left[ \frac{2a_2x}{(1 - a_1)^2} \right] \]

\[ J_R(10,3) = -\left[ \frac{1}{(1 - a_1)^2} \right] \left[ 2a_2x^2 + \psi_1a_2 \right] - \left[ \frac{2}{(1 - a_1)^2} \right] \left[ 2a_2w + 2a_0a_2x + 2a_1a_2x^2 \right] \]

\[ J_R(10,4) = -\left[ \frac{1}{(1 - a_1)^2} \right] \left[ 4a_2w + 2a_0w - 2a_1x^2 \right] - \left[ \frac{\psi_1}{1 - a_1} \right] \]

\[ J_R(10,5) = -\left[ \frac{2a_2}{(1 - a_1)^2} \right], \quad J_R(10,6) = -\left[ \frac{a_2}{1 - a_1} \right] \]

The rest of the elements of \( J_R \) are zero.

The analytical complexity of this problem has made it somewhat tedious to analytically evaluate the vector \( \kappa(R) - J_R \mathbf{R} \). Since it is possible to evaluate the vectors \( \kappa, \mathbf{R} \) and the matrix \( J_R \) at each communication interval, then the NAG routines for matrix multiplication and matrix subtraction, have been used to evaluate \( \kappa(R - J_R)\mathbf{R} \) at each communication interval.

The rest of this problem is similar to that of Problems One and Two. Figures (6.7), (6.8), (6.9), (6.10) and (6.11) show the Q.L. solution of Problem Three for the state vector \([x \ a_0 \ a_1 \ a_2 \ w]^T\).
6.5 Invariant Imbedding

As mentioned earlier in this chapter, the techniques of modern control theory often involve the solution of non-linear, two-point, boundary value problems. These problems are generally difficult to solve analytically and numerically. The split in boundary conditions, makes a simple integration impossible, and iterative techniques are time-consuming and often complex. The invariant imbedding method is a technique whereby the original two-point boundary values problem is effectively reduced to an initial value problem, through a process of direct evaluation of the missing initial (or terminal) boundary conditions. This procedure has been reported in many of the references stated in Section (6.2).

The derivation of the invariant imbedding equation, can be outlined as follows:

given a two-point boundary value problem described by:

\[ x'(t) = f(x(t), \psi(t), t) \]  \hspace{1cm} (6.76)
\[ \psi(t) = g(x(t), \psi(t), t) \]  \hspace{1cm} (6.77)

where \( x(t) \) and \( \psi(t) \) are \( n \)-dimensional vectors, with the boundary conditions as:

\[ x(t_o) = a, \quad \psi(t_f) = b \]  \hspace{1cm} (6.78)

where \( t_o \) is the starting value of the independent variable \( t \), and \( t_f \) is its final value.

It is desirable to find the initial value of the vector \( \psi(t) \), i.e. \( \psi(t_o) \), so that it is possible to solve the above problem as an initial condition one, with initial conditions \( x(t_o) \), and the starting point of the process, \( t_o \). Thus let:

\[ x(t_o) = C \]  \hspace{1cm} (6.79)

and

\[ \psi(t_o) = r(C, t_o) \]  \hspace{1cm} (6.80)

where \( r \) will be considered as the dependent variable, while \( C \) and \( t_o \) are independent variables.

With a perturbation of \( \Delta t_o \), equation (6.77) can be expanded as:
Substituting equations (6.80) and (6.77) in (6.81) yields:

\[ r(C + \Delta C, t_o + \Delta t_o) = r(C, t_o) + g(C, r, t_o)\Delta t_o + O(\Delta^2) \]  

where \( \lim_{\Delta \to 0} \frac{O(\Delta^2)}{\Delta} = 0 \)

The left hand side of equation (6.82) could be expanded in a Taylor series, and using equation (6.77) yields:

\[ r(C + \Delta C, t_o + \Delta t_o) = r(C, t_o) + \frac{\partial r}{\partial C} \Delta C + \frac{\partial r}{\Delta t_o} \Delta t_o + O(\Delta^2) \]  

From equations (6.77) and (6.78), \( \Delta C \) can be written as:

\[ \Delta C = g(C, r, t_o) + O(\Delta^2) \]

Then, by equating the right-hand sides of equations (6.82) and (6.83) and substituting equation (6.84) for \( \Delta C \), the invariant imbedding equation

\[ \frac{\partial r}{\partial t_o} + \left[ \frac{\partial r}{\partial C} \right] f(C, r, t_o) = g(C, r, t_o) \]  

results when we take the limit as \( \Delta t_o \) approaches zero. This partial differential equation governs the dependence of the missing initial conditions on \( \psi \) as a function of the initial value

\[ \]  

+ The invariant imbedding technique can be used to find other missing conditions; for example, if \( x(0)=a, x(t_f)=b \), and it is required to find the final condition on \( \psi \), then the invariant imbedding equation will take the form

\[ \frac{\partial r}{\partial t_f} + \left[ \frac{\partial r}{\partial C} \right] f(C, r, t_f) = g(C, r, t_f) \]
of the process duration, $t_o$, and the initial conditions on $x, C$. Equation (6.85) can be solved analytically by the method of characteristics. However, it is possible to make an assumption of approximate linearity and thereby derive a set of related equations which can be solved numerically as an initial condition problem. The latter approach will be adopted here.

To demonstrate this technique, the following problem is presented: given the system

$$\dot{x}(t) = -x^2(t) + u(t), \quad x(0) = x_0$$  \hspace{1cm} (6.86)

$$J = \frac{1}{2} S^2 x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left( ax(t)^2 + u^2(t) \right) dt$$  \hspace{1cm} (6.87)

The canonical equations are:

$$\dot{x}(t) = -x^2(t) - \psi(t) = f$$ \hspace{1cm} (6.88)

$$\dot{\psi}(t) = -ax(t) + 2\psi(t)x(t) = g$$ \hspace{1cm} (6.89)

$$\psi(t_f) = Sx(t_f)$$

If we imbed so as to compute an initial condition for the adjoint, then equation (6.85) for this problem, could be written as:

$$\frac{\partial x}{\partial t_o} + \left[ \frac{\partial r}{\partial C} \right] (-C^2 - r) = -aC + 2rC$$ \hspace{1cm} (6.90)

with

$$x(t_o) = C, \quad \psi(t_o) = r(C, t_o)$$ \hspace{1cm} (6.91)

We guess a solution of the form

$$\psi(t_o) = r(C, t_o) = m(t_o) C + n(t_o)$$ \hspace{1cm} (6.92)

Substituting equation (6.92) in equation (6.90) yields:
\[ \dot{m}C + \dot{n} + m(\dot{C}^2 - mC - n) = -\dot{a}C + 2(mC + n)C \]

Equating the coefficients of powers \(C^0\) and \(C^1\) yields:

\[ \dot{n} = nm, \quad \dot{m} = m^2 + n - a \]  \hspace{1cm} (6.93)

At the start of the computation, where \(t = t_f\):

\[ \psi(t_f) = \psi(t_f)C + n(t_f) = S x(t_f) \]  \hspace{1cm} (6.94)

for the case \(S = 0\), then \(m(t_f) = 0, n(t_f) = 0\)  \hspace{1cm} (6.95)

Equation (6.93) is then integrated backwards in time with the prescribed end conditions (6.95) from \(t_f\) to \(t_0\) \((t_f=1.0, a=1)\) yielding:

\[ \psi(t_o) = \psi(C, t_o) = m(t_o)C + n(t_o) \]

\[ = m(t_o) x(t_o) + n(t_o) \]  \hspace{1cm} (6.96)

Since \(x(t_o)\) is known, and \(m(t_o)\) and \(n(t_o)\) have just been computed, then an (approximate) solution for \(\psi(t_o)\) is obtained.

Figure (6.12) shows the solution of the variable \(m\) in equation (6.93) with reverse time, while figure (6.13) shows the invariant imbedding solution of the state variable \(x\), compared with that of the Q.L. method, in this figure it is clear that the two solutions are very close. Similar result was obtained when both the Q.L. and invariant inbedding were applied to problem one.
6.6 Concluding Remarks

The application of the optimal control theory to a non-linear system leads to the formulation of the problem as a non-linear, two-point, boundary value problem. Several techniques for solving the T.P.B.V.P. have been reported, but most of them are problem-dependent techniques, i.e. what proves to be useful in solving a certain problem may not prove the same when applied to another problem.

Two methods for solving the two-point, boundary value problem were presented: First, the quasilinearization method and second, the invariant imbedding method. The quasilinearization method was shown to be successful in solving a number of problems and that included those of a highly complicated analytical nature, such as that of Problem Three. The main features of this technique is that if the procedure converges, it converges quadratically; the technique requires a fairly good initial estimate of the solution; and the resulting linearized adjoint equations can be highly unstable. The invariant imbedding solution was shown to be very close to that of the Q.L. method, which makes it useful as the source of an initial guess at a starting trajectory for the Q.L. solution.
Figure 6.1 State Variable Response In Case Study No. 1
Figure 6.2 Co-state Variable Response in Case Study No. 1
Figure 6.3 State Variable Response In Case Study No.2
Figure 6.4 State Variable No. 2 Response In Case Study No. 2
Figure 6.5 Co-State Variable No.1 Response In Case Study No. 2
Figure 6.6 Co-State Variable No. 2 Response In Case Study No. 2
Figure 6.7 State Variable No.1 Response In Case Study No.3

Figure 6.8 State Variable No.2 Response In Case Study No.3
Figure 6.9 State Variable No. 3 Response In Case Study No. 3

Figure 6.10 State Variable No. 4 Response In Case Study No. 3
Figure 6.11 State Variable No.5 Response In Case Study No.3
Figure 6.12 Solution Of Equation 6.93 With Reverse Time

Figure 6.13 State Variable Responses Using Q.L & Inv. Imb. Methods

Legend
- Q.L. Solution
- Inv. Imb. Solution
Chapter 7
Non-linear Control of the Non-linear Engine Model

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7.1 Introduction

In this chapter a report of the attempts made to solve the non-linear, two-point, boundary value problem, which results from the application of the optimal control theory to the non-linear engine model, OMAR, to optimize its dynamic response is presented. Owing to the analytical nature of OMAR, this T.P.B.V.P is of highly complicated analytical form, and its solution proved to be very difficult. Several methods were used to solve this T.P.B.V.P, and they were:

1 - invariant imbedding

2 - a combined approach from piecewise linearization and invariant imbedding methods

3 - quasilinearization, which consisted of:
   a - optimal control, where \( u^0 \) was obtained from \( \partial H/\partial u^0 = 0 \)
   and b - specific optimal control, (S.O.C.)

4 - a combined approach from piecewise linearization and the gradient methods

Method No. 4 was used to obtain an approximate solution for the optimal non-linear tracking problem, for in this problem it is required that the thrust response (the case of slam acceleration) be close to some desired response.

7.2 Problem Formulation

Given the non-linear engine model, OMAR, developed in chapter 2 and presented here,(all the variables in this model have retained the same definitions stated in chapter 2), as:

\[
\frac{dNC}{dt} = \left(\frac{30}{\pi}\right)^2 \frac{J}{IC(NC)} \left[CPC(WAC)(T21-T3)+CPHT(WG50)(T4-T50)\right] \tag{7.1}
\]

\[
\frac{dNF}{dt} = \left(\frac{30}{\pi}\right)^2 \frac{J}{IF(NF)} \left[CPF(WAF)(T2-T21)+CPLT(WG55)(T50-T55)\right] \tag{7.2}
\]

\[
\frac{dP4}{dt} = \frac{R(T4)}{VCOMB} \left[WA3 + WFB - WG4\right] \tag{7.3}
\]
\[
\frac{dP7}{dt} = \frac{R(y)(T7)}{VAFBN} \left[ WG4 - WA3 + WAF + WFA - WG7 \right] 
\]

(7.4)

\[
\frac{dU4}{dt} = \frac{CVB(R)(T4)}{VCOMB(P4)} \left[ T4\{WG4 - WFB - WA3\} + \gamma(T3 WA3 - T4 WG4 \\
+ T4(1 + \eta)WFB\} \right] 
\]

(7.5)

Where the following equations represent the non-linear algebraic relationships existing between the state variables and the intermediate variables:

\[
(T/T)_1 = 1.0 + 0.2 \ AM^2 \\
(P/P)_1 = (T/T)_1^{3.5} \\
\eta_I = \begin{cases} 
1.0 & \text{if } AM \leq 1.0 \\
1.0 - 0.075(AM - 1.0)^{1.35} & \text{if } AM > 1.0 
\end{cases}
\]

\[
T2 = (T/T)_1T1 \\
P2 = (P/P)_1P1\eta_I \\
CNF = \frac{NF}{NFREF} = \frac{NF}{9651} \\
T21 = T2 + 214.2732 \ CNF^2 - 48(A8 - 2.94826) \\
CNC = \frac{NC}{NCREF\sqrt{T21/T2}} = \frac{NC}{10070\sqrt{T21/518.68}} \\
T3 = T21 + 743.2722 \ CNC^2 - 68(A8 - 2.94826) \\
T4 = U4/CVB \\
T50 = 0.727T4 \\
P3 = 1.05944 P4
\]
\[ P_{21} = -6.20568 - 0.0129774 P_3 \]
\[ W_{F\text{MAX}} = 261.01 \ \text{CNF} - 63.196 \]
\[ P_{F\text{MAX}} = 3.516739 \ \text{CNF} - 0.23561 \]
\[ K = 1.0 - e^{2.31326(P_{F\text{MAX}} - (P_{21}/P_2))} \]
\[ W_{AF} = W_{F\text{MAX}} + 28.502 K \]
\[ W_{C\text{MAX}} = 137.54 - 457.987 \ \text{CNC} + 564.325 \ \text{CNC}^2 - 188.113 \ \text{CNC}^3 \]
\[ D_{W\text{CMAX}} = 6.492 - 4.9749 \ \text{CNC} \]
\[ P_{C\text{MAX}} = 26.43184 - 89.0484 \ \text{CNC} + 109.724 \ \text{CNC}^2 - 36.575 \ \text{CNC}^3 \]
\[ W_{C\text{MAX}} = 137.54 - 457.987 \ \text{CNC} + 564.325 \ \text{CNC}^2 - 188.113 \ \text{CNC}^3 \]
\[ D_{W\text{CMAX}} = 6.492 - 4.9749 \ \text{CNC} \]
\[ P_{C\text{MAX}} = 26.43184 - 89.0484 \ \text{CNC} + 109.724 \ \text{CNC}^2 - 36.575 \ \text{CNC}^3 \]
\[ W_{C\text{MAX}} = \frac{(P_{21}/P_2)}{\sqrt{T_{21}/518.668}} \left\{ W_{C\text{MAX}} + D_{W\text{CMAX}} \right\} K_1 \]
\[ K_1 = 1.0 - e^{-0.36(P_{C\text{MAX}} - P_3/P_{21})} \]
\[ P_{RF} = P_{21}/P_2 \]
\[ P_{RC} = P_3/P_{21} \]
\[ W_{A3} = (1.0 - P_{CBL}) W_{AC} \]
\[ W_{G50} = \frac{301.97 \ P_4}{\sqrt{T_4}} + 3.9699 \ P_7 \]
\[ W_{G4} = W_{G50} - P_{CBLH} P_{CBLC} W_{AC} \]
\[ W_{G55} = W_{G50} + P_{CBLH} P_{CBLC} W_{AC} \]
\[ T_{55} = 106.002 + 0.8615T_{50} - 0.10458(C\text{NC}) \sqrt{T_{21}T_{50}} \]
\[ T_7 = 0.49661 \ T_{55} + 205.886 \ P_7 \]
\[ W_{G7} = \frac{1121.786 \ P_7(A_8)}{\sqrt{T_7}} \]
\[ \text{FGP} = 2116.217 \ (0.53978 \ P7 - P2) \ A8 \]
\[ \text{V8} = \sqrt{1934.415 \ T7 + 68558.36} \]
\[ \text{FGM} = \text{CVMNOZ}(V8)(WG7/G) \]
\[ \text{FG} = \text{FGP} + \text{FGM} \]
\[ \text{ZC} = \frac{\text{PRC} - 1.0}{\text{PCMAX} - 1.0} \]
\[ \text{ZF} = \frac{\text{PRF} - 1.0}{\text{PCMAX} - 1.0} \]

The following definitions were used:

state vector: \( X(t) = [NC \ NF \ P4 \ P7 \ U4]^T \) \hfill (7.6)\(^+\)

steady state value of state vector \( XSS = [NCSS \ NFSS \ P4SS \ P7SS \ U4SS]^T \) \hfill (7.7)

control vector \( U(t) = [WFB \ A8]^T \) \hfill (7.8)

steady state value of control vector \( USS = [WFBSS \ A8SS]^T \) \hfill (7.9)

The objective of the work reported in this chapter was to apply optimal control theory to the above non-linear engine model, to optimize its dynamic performance. The control strategy chosen was: when the engine is perturbed from its steady-state operating condition, by some initial state vector \( X(o) \), it is to be brought back to its steady-state condition, as quickly as possible and without excessive expenditure of control energy or without violating the operational limitations of the engine. The optimal control vector \( u(t) \) capable of achieving this control strategy, can be obtained by minimizing some quadratic performance index of the type:

\[ + \text{The reader should see equations (7.11) to (7.13) to learn why these vectors [equations (7.6) to (7.9)] have been denoted with uppercase letters.} \]
\[ J = \frac{1}{2} \mathbf{x}^T(t_f) \mathbf{Sx}(t_f) + \frac{1}{2} \int_0^{t_f} \left( \mathbf{x}^T(t) \mathbf{Qx}(t) + \mathbf{u}^T(t) \mathbf{Gu}(t) \right) dt \]  \hspace{1cm} (7.10)

where:

\[ \mathbf{x}(t) = \mathbf{X}(t) - \mathbf{XSS} \]  \hspace{1cm} (7.11)

\[ \dot{\mathbf{x}}(t) = \dot{\mathbf{X}}(t) \]  \hspace{1cm} (7.12)

\[ \mathbf{u}(t) = \mathbf{U}(t) - \mathbf{USS} \]  \hspace{1cm} (7.13)

Because it is intended that at \( t = t_f \), \( \mathbf{x}(t_f) = 0 \), i.e.:

\[ \mathbf{X}(t_f) = \mathbf{XSS} \]  \hspace{1cm} (7.14)

then \( S = 0 \)  \hspace{1cm} (7.14a)

The Hamiltonian function, \( H \), associated with the above performance index, is defined as:

\[ H = \frac{1}{2} \left( \mathbf{x}^T(t) \mathbf{Qx}(t) + \mathbf{u}^T(t) \mathbf{Gu}(t) \right) + \psi_1 \dot{x}_1 + \psi_2 \dot{x}_2 + \psi_3 \dot{x}_3 + \psi_4 \dot{x}_4 + \psi_5 \dot{x}_5 \]  \hspace{1cm} (7.15)

where

\[ \psi = - \frac{\partial H}{\partial \mathbf{x}} \]  \hspace{1cm} (7.16)

equation (7.16) could be re-written as:

\[ \dot{\psi}_1 = - \left[ \mathbf{Q}(1,1) x_1 + \psi_1 \frac{\partial \dot{x}_1}{\partial x_1} + \psi_2 \frac{\partial \dot{x}_2}{\partial x_1} + \psi_3 \frac{\partial \dot{x}_3}{\partial x_1} + \psi_4 \frac{\partial \dot{x}_4}{\partial x_1} + \psi_5 \frac{\partial \dot{x}_5}{\partial x_1} \right] \]  \hspace{1cm} (7.17)

\[ \dot{\psi}_2 = - \left[ \mathbf{Q}(2,2) x_2 + \psi_1 \frac{\partial \dot{x}_1}{\partial x_2} + \psi_2 \frac{\partial \dot{x}_2}{\partial x_2} + \psi_3 \frac{\partial \dot{x}_3}{\partial x_2} + \psi_4 \frac{\partial \dot{x}_4}{\partial x_2} + \psi_5 \frac{\partial \dot{x}_5}{\partial x_2} \right] \]  \hspace{1cm} (7.18)

\[ \dot{\psi}_3 = - \left[ \mathbf{Q}(3,3) x_3 + \psi_1 \frac{\partial \dot{x}_1}{\partial x_3} + \psi_2 \frac{\partial \dot{x}_2}{\partial x_3} + \psi_3 \frac{\partial \dot{x}_3}{\partial x_3} + \psi_4 \frac{\partial \dot{x}_4}{\partial x_3} + \psi_5 \frac{\partial \dot{x}_5}{\partial x_3} \right] \]  \hspace{1cm} (7.19)

\[ \dot{\psi}_4 = - \left[ \mathbf{Q}(4,4) x_4 + \psi_1 \frac{\partial \dot{x}_1}{\partial x_4} + \psi_2 \frac{\partial \dot{x}_2}{\partial x_4} + \psi_3 \frac{\partial \dot{x}_3}{\partial x_4} + \psi_4 \frac{\partial \dot{x}_4}{\partial x_4} + \psi_5 \frac{\partial \dot{x}_5}{\partial x_4} \right] \]  \hspace{1cm} (7.20)

\[ \dot{\psi}_5 = - \left[ \mathbf{Q}(5,5) x_5 + \psi_1 \frac{\partial \dot{x}_1}{\partial x_5} + \psi_2 \frac{\partial \dot{x}_2}{\partial x_5} + \psi_3 \frac{\partial \dot{x}_3}{\partial x_5} + \psi_4 \frac{\partial \dot{x}_4}{\partial x_5} + \psi_5 \frac{\partial \dot{x}_5}{\partial x_5} \right] \]  \hspace{1cm} (7.21)
\( u(t) \) can be obtained by \( \frac{\partial H}{\partial u} = 0 \), which yields

\[
\begin{align*}
\dot{u}_1 &= -G^{-1}(1,1) \left[ \psi_1 \frac{\partial \dot{x}_1}{\partial u} + \psi_2 \frac{\partial \dot{x}_2}{\partial u} + \psi_3 \frac{\partial \dot{x}_3}{\partial u} + \psi_4 \frac{\partial \dot{x}_4}{\partial u} + \psi_5 \frac{\partial \dot{x}_5}{\partial u} \right] \quad (7.22) \\
\dot{u}_2 &= -G^{-1}(2,2) \left[ \psi_1 \frac{\partial \dot{x}_1}{\partial u} + \psi_2 \frac{\partial \dot{x}_2}{\partial u} + \psi_3 \frac{\partial \dot{x}_3}{\partial u} + \psi_4 \frac{\partial \dot{x}_4}{\partial u} + \psi_5 \frac{\partial \dot{x}_5}{\partial u} \right] \quad (7.23)
\end{align*}
\]

In order to construct the system canonical equations, all the partial differential terms in equations (7.17) to (7.23) must be evaluated analytically, i.e. the differential terms:

\[
\frac{\partial \dot{x}_i}{\partial x_j} \quad i = 1, 2, \ldots, 5 \\
\frac{\partial \dot{x}_i}{\partial u_m} \quad i = 1, 2, \ldots, 5 \\
\text{and} \\
\frac{\partial \dot{x}_i}{\partial \psi_j} \quad i = 1, 2, \ldots, 5 \\
\]

Substituting equations (7.22) and (7.23) in equations (7.1) to (7.5), and using equations (7.24) and (7.25) in equations (7.17) to (7.23), then the system canonical equation may be written as:

\[
\begin{align*}
\text{WFB} &= \text{WFBSS} - ((\psi_1 \text{DXU}(1,1) + \psi_2 \text{DXU}(2,1) + \psi_3 \text{DXU}(3,1)) \\
&\quad + \psi_4 \text{DXU}(4,1) + \psi_5 \text{DXU}(5,1))/G(1,1)) \\
\text{A8} &= \text{A8SS} - ((\psi_1 \text{DXU}(1,2) + \psi_2 \text{DXU}(2,2) + \psi_3 \text{DXU}(3,2)) \\
&\quad + \psi_4 \text{DXU}(4,2) + \psi_5 \text{DXU}(5,2))/G(2,2)) \\
\text{CNF} &= \text{NF/XNLPDS} \\
\text{T21} &= (\text{T2+TX*(CNF**2)-48.0*(A8-2.9482558))} \\
\text{CNC} &= \text{NC/(XNHPDS*SQRTMI/72))} \\
\text{T3} &= (\text{T21+TX3*(CNC**2)-68.0*(A8-2.9482558))}
\end{align*}
\]

+ Throughout this chapter, for ease of representation, \( x_i \) is used to represent \( x_i \), \( u_m \) is used to represent \( u_m \), and \( \psi_i \) is used to represent \( \psi_i \).
T4 = (U4/CVB)  
T50 ST4*T4  
P3 = (1.05944*P4)  
P21 = ((-XX+0.0129774*T21)-0.0185376*P3)  
WFMAX = (261.01*CNF-63.916)  
PFMAX = (3.516739*CNF-0.23561)  
WAF = WFMAX+28.502*(1.0-EXP(-2.313268*(PFMAX-(P21/P2))))  
WCMAK = (137.54-457.987*CNC+564.325*(CNC**2)-188.113*(CNC**3))  
DWCMAX = (6.492-4.974*CNC)  
PCMAX = (YY-89.0484*CNC+109.72*(CNC**2)-36.57*(CNC**3))  
WAC = (P21/P2)/SQRTMI/T21)*(WCMAX+DWCMAX*  
(1.0-EXP(-0.36*(PCMAX-(P3/P21))))  
RAFAN = P21/P2  
RACOMP = P3/P21  
EFFIC = 20.71175  
WA3 = (1.0-PCBLUC)*WAC  
WG50 = SP4*P4/SQRT(T4)+3.96992*P7  
WG4 = WG50-PCBLHP*PCBLUC*WAC  
WG55 = WG50+PCBLLP*PCBLUC*WAC  
T55 = 106.002+ST5*T50-0.10458*CNC*SQRT(T21*T50)  
T7 = (T55+ST7/P7)/SP7  
WG7 = (112.784*A8*P7)/SQRTM  
PCBLDU = 1.0-PCBLHP-PCBLLP  
PS8 = 0.539782*P7  
FGP = CAPSF*(PS8-P2)*A8  
V8 = SQRT(ST8*T7+68558.365)  
FGM = CVMNOZ*V8*WG7/G  
FG = FGP+FGM  
ZC = ((P3/P21)-1.0)/(PCMAX-1.0)  
ZF = ((P21/P2)-1.0)/(PFMAX-1.0)  

DO 20 I = 1,5  
DP3(3) = 1.05944  
DCNF(2) = 1/XNLPS  
DT21(I) = -48*DA8(I)  
DT21(2) = 214.2733*2.0*CNF*DCNF(2)-48*DA8(2)  
DCNC(I) = NC*DT21(I)))/2*T2*XNPDS*(T21/T2)**1.5  
DCNC(I) = (SQRT(T21/T2)-0.5*NC*(DT21(1)/T2)*(T21/T2)**(-0.5))/  
(XNPDS*(T21/T2))  

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DT3(I) = DT21(I)+743.2722*2.0*CNC*DCNC(I)-68.0*DA8(I)
DP21(I) = 0.0129774*DT21(I)
DT4(5) = 1/CVB
DT50(5) = 0.727*DT4(5)
IF (I.LEQ.3) DP21(3) = 0.0129774*DT21(3)-0.0185376*DP3(3)
DWFMX(2) = 261.01*DCNF(2)
DPFMX(2) = 3.51639*DCNF(2)
DWAF(I) = (-28.502*2.313268*DP21(I)/P2)*EXP(-2.31368*(PFMAX-(P21/P2)))
DWAF(2) = DWFX(2)+28.502*2.31326*(DPFMX(2)-(DP21(2)/P2))
*EXP(-2.31326*(PFMAX-(P21/P2)))
DWCMX(I) = -457.987*DCNC(I)+2.0*564.325*CNC*DCNC(I)-3*188.113
*(CNC**2)*DCNC(I)
DDWMX(I) = 4.974*DCNC(I)
DPCMX(I) = -89.0484*DCNC(I)+2*109.72*CNC*DCNC(I)-3*36.57
*(CNC**2)*DCNC(I)
WAC1 = (P21/P2)/SQRT(T21/T2)
WAC2 = WCMAX+DWCMAX*(1.0-EXP(-0.36*(PFMAX-(P3/P21))))
WAC1(I) = (-P21/P2)*0.5*((T21/T2)**(-0.5))*(DT21(I)/T2)
+SQRT(T21/T2)*(DP21(I)/P2)/(T21/T2)
WAC2(I) = DWMX(I)+DWCMAX*(-EXP(-0.36*(PFMAX-(P3/P21))))
*EXP(-0.36*(PCMX(I)+P3*(P21**(-2.0))*DP21(I))))+
(1.0-EXP(-0.36*(PFMAX-(P3/P21))))*DDWMX(I)
IF (I.LEQ.3) DWAC2(3) = DWCMX*(EXP(-0.36*(PFMAX-(P3/P21))))*
(-0.36*(DP21(3)-P3*DP21(3)/P21**2.0)))+
(1.0-EXP(-0.36*(PFMAX-(P3/P21))))*DDWMX(3)+DWCMX(3)
WAC(I) = WAC1*DWAC2(I)+WAC2*DWAC1(I)
WAC3(I) = (1.0-PCBLC)*WAC(I)
DWG50(3) = 279.218/SQRT(T4)
DWG50(4) = 3.96992
DWG50(5) = (-0.5*279.21*P4/SQRT(T4**3))*DT4(5)
DWG4(I) = DWG50(I)-PCBLC*PCBLC*DWAC(I)
DWG55(I) = DWG50(I)+PCBLLP*PCBLC*DWAC(I)
DT55(I) = -0.10458*(((CNC*(0.5)*DT21(I)*T50)/SQRT(T21*T50))
+SQR(T21*T50)*DCNC(I))
IF (I.LEQ.5) DT55(5) = 0.8615*DT50(5)-0.5*0.10458*CNC*DT21(5)
+T50*DT21(5))/SQRT(T21*T50))-0.10458
*SQRT(T21*T50)*DCNC(5)
DT7(I) = DT55(I)/SP77
DT7(4) = (DT55(4)+414.582)/SP77
SM \* P7*A8

DSKI(I) = P7*DA8(I)

IF (LEQ.4) DSKI(4) = A8+P7*DA8(4)

DWG7(I) = -0.5*1121.784*P7*DT7(I)/SQR(T7)**3+(1121.78*P7/SQR(T7))*DA8(I)

IF (LEQ.4) DWG7(4) = -0.5*1121.784*P7*DT7(4)/SQR(T7)**3+(1121.78*P7/SQR(T7))*DA8(4)+A8*(1121.78/SQR(T7))

PP1 = EXP(-2.31368*(PFMAX-(P21/P2))

DPPI(I) = 2.31368*(DP21(I)/P2)*PP1

IF (LEQ.2) DPPI(2) = -2.31368*(DPFMX(2)-(DP21(2)/P2))*PP1

PP2 = EXP(-0.36*(PCMAX-(P3/P2))

DPP2(I) = -0.36*(DPCMX(I)+(P3*DP21(I)/P2**2))*PP2

IF (LEQ.3) DPP2(3) = -0.36*(DPCMX(3)-(P21*DP3(3)-P3*DP21(3))

PPI = EXP(-2.31368*(PFMAX-(P21/P2))

DPPI(I) = 2.31368*(DP21(I)/P2)*PPI

IF (I. EQ. 2) DPPI(2) = -2.31368*(DPFMX(2)-(DP21(2)/P2))*PPI

PP2 = EXP(-0.36*(PCMAX-(P3/P2))

DPP2(I) = -0.36*(DPCMX(I)+(P3*DP21(I)/P2**2))*PP2

IF (I. EQ. 3) DPP2(3) = -0.36*(DPCMX(3)-(P21*DP3(3)-P3*DP21(3))

FIP = 0.5*(P21/P2)*((T21/T2)**(-0.5))

DFIP(I) = 0.5*(((T21/T2)**(-0.5))*(DP21(I)/P2)+0.5*(P21/P2)

*(-0.5)*((T21/T2)**(-1.5))*(DT21(I)/T2)

PIP(I) = -FIP*(DT21(I)/T2)+SQR(T21(T2))*(DP21(I)/P2)

PK(I) = -0.36*(DPCMX(I)+P3*(P21**(2.0)))*DP21(I))

IF (LEQ.3) PK(3) = -0.36*(DPCMX(3)-(P21*DP3(3)-P3*DP21(3)))

SPP7 = SSP7*(WG4-WA3+WAF-WG7)

SSU4 = (CVB*RA*T4)/VCOMB*P4

20 CONTINUE

SSNC(1) = 91.189*AJ/(PMH*NC)

DSSNC(1) = -SSNC/NC

SSU4(3) = -SSU4/P4

SSU4(5) = (SSU4/T4)*DT4(5)

PP10 = (CPC*WAC*(DT21(1)-DT3(1))+CPC*(T21-T3)*DWAC(1))

20 CONTINUE

SSNC = 91.189*AJ/(PMIHP*NC)

SSNF(91.189*AJ/(PMILP*NF))

SSP4 = (RA/VCOMB)*GAMSTR*T4

SSP7 = (RA*GAMSTR/VAFBN)

SSU4 = SSP7*(WG4-WA3+WAF-WG7)

SSU4 = (CVB*RA*T4)/(VCOMB*P4)

DNCDT(1) = SSNC*(CPC*WAC*(DT21(1)-DT3(1))+CPC*(T21-T3)*DWAC(1))

+ (CPC*WAC*(T21-T3)+CPHT*WG50*(T4-TSO))*(-SSNC/NC)

DNCDT(2) = SSNC*(CPC*WAC*(DT21(2)-DT3(2))+CPC*(T21-T3)*DWAC(2))
DNCDT(3) = SSNC*(CPC*(T21-T3)*DWAC(3)+CPHT*(T4-T50)*DWG50(3)+CPC*WAC*(DT21-DT3(3)))+CPC*WAC*(DT21-DT3(3))

DNCDT(4) = SSNC*(CPHT*(T4-T50)*DWG50(4)+CPC*WAC*(DT21(4)-DT3(4)))+CPC*(T21-T3)*DWAC(4)

DNCDT(5) = SSNC*(CPHT*(T4-T50)*DWG50(5)+CPHT*WG50*(DT4(5)-DT50(5)))+CPC*WAC*(DT21(5)-DT3(5))+CPC*(T21-T3)*DWAC(5)

DO 40 I = 1,5

DNFDT(l) = SSNF*(CPF*(T2-T21)*DWAF(I)-CPF*WAF*DT2(I)+CPLT*(T5O-T55)*DWG55(I)-CPLT*WG55*DT55(I))

DP4DT(I) = SSP4*(DWA3(I)+DWFB(I)-DWG4(I))

DSP7(I) = SSP7*(DWG4(I)-DWA3(I)+DWAF(I)-DWG7(I))

DP7DT(I) = SSP7*T7*(DWG4(I)-DWA3(I)+DWAF(I)-DWG7(I))=SP7*DT7(I)

DU4DT(I) = SSU4*(T4*(DWG4(I)-DWFB(I)-DWA3(I))+GAMSTR*(T3*DWA3(I)+WA3*DT3(I)-T4*DWG4(I)+(1.0+EFFIC)*DWFB(I)*T4))

40 CONTINUE

DNFDT(2) = SSNF*(-CPF*WAF*DT2(2)+CPF*(T2-T21)*DWAF(2)-CPLT*WG55*DT55(2)+CPLT*(T5O-T55)*DWG55(2))+(CPF*WAF*(T2-T21)+CPLT*WG55*(T50-T55))*(-SSNF/NF)

DNFDT(5) = SSNF*(CPF*(T2-T21)*DWAF(5)-CPF*WAF*DT2(5)+CPLT*(T5O-T55)*DWG55(5)+CPLT*WG55*(DT50(5)-DT55(5)))

DP4DT(5) = SSP4*(DWA3(5)+DWFB(5)-DWG4(5))+(WA3*WFB-WG4)*GA0/VCOMB)*GAMSTR*DT4(5)

DU4DT(3) = SSU4*(T4*(DWG4(3)-DWFB(3)-DWA3(3))=GAMSTR*(T3*DWA3(3)+WA3*DT3(3)-T4*DWG4(3)+(1.0+EFFIC)*DWFB(3)*T4))-(T4*(WG4-WFB-WA3)+GAMSTR*(T3*WA3-T4*WG4+T4*(1.0+EFFIC)*WFB))*TU44/P4)

DU4DT(5) = SSU4*(T4*(DWG4(5)-DWFB(5)-DWA3(5))+(WG4-WFB-WA3)*DT4(5)+GAMSTR*(T3*DWA3(5)+WA3*DT3(5)-T4*DWG4(5)-WG4*DT4(5)-T4*(1.0+EFFIC)*DWFB(5)+WFB*(1.0+EFFIC)*DT4(5))+(T4*(WG4-WFB-WA3)+GAMSTR*(T3*WA3-T4*WG4+T4*(1.0+EFFIC)*WFB))*(SSU4*DT4(5)/T4)

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The system canonical equations

\[ \dot{NC} = ((91.189 \cdot AJ)/(PMI \cdot NC)) \cdot (CPC \cdot WAC \cdot (T21 - T3) + CPHT \cdot WG50 \cdot (T4 - T50)) \]  
(7.26)

\[ \dot{NF} = ((91.189 \cdot AJ)/(PMILP \cdot NF)) \cdot (CPF \cdot (WAF) \cdot (T22 - T21) + CPLT \cdot WG55 \cdot (T50 - T55)) \]  
(7.27)

\[ \dot{P4} = (RA/VCOMB) \cdot GAMSTR \cdot T4 \cdot (WA3 + WFB - WG4) \]  
(7.28)

\[ \dot{P7} = RA \cdot GAMSTR \cdot (T7/VAFBN) \cdot (WG4 - WA3 + WAF - WG7) \]  
(7.29)

\[ \dot{U4} = ((CVB \cdot RA \cdot T4) / (VCOMB \cdot P4)) \cdot (T4 \cdot (WG4 - WFB - WA3) + GAMSTR \cdot (T3 \cdot WA3 - T4 \cdot WG4 + T4 \cdot (1.0 + EFFIC) \cdot WFB)) \]  
(7.30)

\[ \psi_1 = - (Q(1,1) \cdot x_1 + \psi_1 \cdot DNCDT(1) + \psi_2 \cdot DNFDT(1) + \psi_3 \cdot DP4DT(1) + \psi_4 \cdot DP7DT(1) + \psi_5 \cdot DU4DT(1)) \]  
(7.31)

\[ \psi_2 = - (Q(2,2) \cdot x_2 + \psi_1 \cdot DNCDT(2) + \psi_2 \cdot DNFDT(2) + \psi_3 \cdot DP4DT(2) + \psi_4 \cdot DP7DT(2) + \psi_5 \cdot DU4DT(2)) \]  
(7.32)

\[ \psi_3 = - (Q(3,3) \cdot x_3 + \psi_1 \cdot DNCDT(3) + \psi_2 \cdot DNFDT(3) + \psi_3 \cdot DP4DT(3) + \psi_4 \cdot DP7DT(3) + \psi_5 \cdot DU4DT(3)) \]  
(7.33)

\[ \psi_4 = - (Q(4,4) \cdot x_4 + \psi_1 \cdot DNCDT(4) + \psi_2 \cdot DNFDT(4) + \psi_3 \cdot DP4DT(4) + \psi_4 \cdot DP7DT(4) + \psi_5 \cdot DU4DT(4)) \]  
(7.34)

\[ \psi_5 = - (Q(5,5) \cdot x_5 + \psi_1 \cdot DNCDT(5) + \psi_2 \cdot DNFDT(5) + \psi_3 \cdot DP4DT(5) + \psi_4 \cdot DP7DT(5) + \psi_5 \cdot DU4DT(5)) \]  
(7.35)

where the boundary values are given as:

\[
\begin{align*}
X(t_o) &= X(o) \\
\psi(t_f) &= 0
\end{align*}
\]

With this type of boundary conditions, equations (7.26) to (7.35) represent an example of what is known as the non-linear two-point boundary value problem.

Some remarks about the way in which equations (7.26) to (7.35) were presented and used can be usefully made here:
There are 5 independent state variables, where the partial derivatives are to be taken with respect to them, and they are indexed as follows:

- NC, the high pressure compressor speed, has an index of 1
- NF, the low pressure compressor speed, has an index of 2
- P4, the combustor internal pressure, has an index of 3
- P7, the afterburner exit pressure, has an index of 4
- U4, the combustor internal energy, has an index of 5

The partial derivatives of any variable, VAR, with respect to the independent variable vector, can be expressed as:

\[ \frac{\partial \text{VAR}}{\partial x_i} \quad \text{or} \quad D \text{VAR}(i) \quad i = 1, 2, \ldots, 5 \]

where the character D denotes \( \partial / \partial x_i \). If \( i \) has a fixed index value, then the term will be the partial derivative of VAR with respect to that independent variable which has the same index value. For example:

- \( D \text{CNC}(1) = \frac{\partial \text{CNC}}{\partial x_1} = \frac{\partial \text{CNC}}{\partial x_1} \)
- \( DT4(5) = \frac{\partial T4}{\partial x_5} = \frac{\partial T4}{\partial x_5} \)
- \( D \text{WAF}(2) = \frac{\partial \text{WAF}}{\partial x_2} = \frac{\partial \text{WAF}}{\partial x_2} \)
- \( D \text{WG50}(4) = \frac{\partial \text{WG50}}{\partial x_4} = \frac{\partial \text{WG50}}{\partial x_4} \)
- \( D \text{NC}(3) = \frac{\partial \text{NC}}{\partial x_3} = \frac{\partial \text{NC}}{\partial x_3} \)
- \( D \text{NF}(2) = \frac{\partial \text{NF}}{\partial x_2} = \frac{\partial \text{NF}}{\partial x_2} \)
- \( D \text{U4}(5) = \frac{\partial \text{U4}}{\partial x_4} = \frac{\partial \text{U4}}{\partial x_4} \)

etc.
2. There are two independent control variables, indexed as follows:

- $WFB$, the combustor fuel flow rate, has an index of $1$
- $A8$, the exhaust nozzle area has an index of $2$.

In a manner similar to step 1,

$$DXU(1,1) = \frac{\partial X1}{\partial U1} = \frac{\partial NC}{\partial WFB}$$
$$DXU(1,2) = \frac{\partial X1}{\partial U2} = \frac{\partial NC}{\partial A8}$$
$$DXU(2,1) = \frac{\partial X2}{\partial U1} = \frac{\partial NF}{\partial WFB}$$
$$DXU(2,2) = \frac{\partial X2}{\partial U2} = \frac{\partial NF}{\partial A8}$$

$$DXU(5,1) = \frac{\partial X5}{\partial U1} = \frac{\partial U4}{\partial WFB}$$
$$DXU(5,2) = \frac{\partial X5}{\partial U2} = \frac{\partial U4}{\partial A8}$$

In [Appendix D] the analytical work leading to the partial derivatives of the variables obtained with respect to $U1$ and $U2$ is shown.

3. The use of computer code featured in the presentation of the partial derivatives was adopted because

a) of the way in which each variable is related to the others; and
b) some of the partial derivatives have the same analytical form for different indexed independent variables

c) the need (later on) for having the second partial derivatives for the same above-mentioned independent variables
4. A thorough procedure of checking those analytical forms of partial derivatives was made, using both the NAG Library routines and the Advanced Continuous Simulation Language - ACSL. Identical results were obtained.

In the next two sections, are reported the attempts made to solve numerically the above-mentioned non-linear two-point boundary value problem, i.e. equations (7.26) to (7.35). The first approach was an implementation of the invariant imbedding technique, so that a very approximate solution to the problem could be found. Then this approximate solution was used as an initial guess for the second approach to be used, namely, the quasilinearization method, in which a very good initial guess of the solution is a crucial factor for the success of this method reaching a final solution.

7.3 Invariant Imbedding Solution

It was shown in Chapter 6 that the invariant imbedding method is a technique whereby the original two-point boundary values problem can be effectively reduced to an initial value problem, by means of a process of direct evaluation of any missing initial (or final) boundary conditions. The intention here was to try to use this invariant imbedding method to solve the two-point boundary values problem of the previous section, i.e. equations (7.26) to (7.35). Those equations can be expressed in their general form as:

\[
\dot{x}(t) = f(x(t), \psi(t), t) \tag{7.36}^+
\]

\[
\dot{\psi}(t) = g(x(t), \psi(t), t) \tag{7.37}
\]

where \(x(t)\) and \(\psi(t)\) are n-dimensional vectors (where n=5 in this problem), with the boundary conditions as:

\[
x(t_0) = a, \psi(t_f) = 0 \tag{7.38}
\]

\(t_0\) is the starting value of the independent variable, \(t\), and \(t_f\) is its final value.

---

\(x(t)\) has retained the same definition of the previous section, i.e. \(x(t) = X(t) - XSS\)
\(dx = dX\) or for example \(dX_1 = dNC, dX_2 = dNF\) and so on.
It is desirable to find the initial value of the vector \( \psi(t) \), i.e. \( \psi(t_0) \), so that it is possible to solve the above problem as an initial condition one, i.e. with initial conditions \( x(t_0) \) and \( \psi(t_0) \) at the starting point of the process, \( t_0 \). Let:

\[
x(t_0) = C
\]  

(7.39)

and

\[
\psi(t_0) = r(C, t_0)
\]  

(7.40)

\( r \) is considered as the dependent variable, while \( C \) and \( t_0 \) are taken as independent variables.

Following the same process of analysis of section (6.5), then the invariant imbedding equation may be written as:

\[
\frac{\partial r}{\partial t_0} + \left[ \frac{\partial r}{\partial C} \right] f(C, r, t_0) = g(C, r, t_0)
\]  

(7.40a)

Equation (7.40a) could be expanded as:

\[
\frac{\partial \psi_1}{\partial t_0} + \frac{\partial \psi_1}{\partial C} \dot{C} + \frac{\partial \psi_1}{\partial \dot{C}} \ddot{C} + \frac{\partial \psi_1}{\partial \dot{P}_4} \dot{P}_4 + \frac{\partial \psi_1}{\partial \dot{P}_7} \dot{P}_7 + \frac{\partial \psi_1}{\partial \dot{U}_4} \dot{U}_4 = \psi_1
\]  

(7.41)

\[
\frac{\partial \psi_2}{\partial t_0} + \frac{\partial \psi_2}{\partial C} \dot{C} + \frac{\partial \psi_2}{\partial \dot{C}} \ddot{C} + \frac{\partial \psi_2}{\partial \dot{P}_4} \dot{P}_4 + \frac{\partial \psi_2}{\partial \dot{P}_7} \dot{P}_7 + \frac{\partial \psi_2}{\partial \dot{U}_4} \dot{U}_4 = \psi_2
\]  

(7.42)

\[
\frac{\partial \psi_3}{\partial t_0} + \frac{\partial \psi_3}{\partial C} \dot{C} + \frac{\partial \psi_3}{\partial \dot{C}} \ddot{C} + \frac{\partial \psi_3}{\partial \dot{P}_4} \dot{P}_4 + \frac{\partial \psi_3}{\partial \dot{P}_7} \dot{P}_7 + \frac{\partial \psi_3}{\partial \dot{U}_4} \dot{U}_4 = \psi_3
\]  

(7.43)

\[
\frac{\partial \psi_4}{\partial t_0} + \frac{\partial \psi_4}{\partial C} \dot{C} + \frac{\partial \psi_4}{\partial \dot{C}} \ddot{C} + \frac{\partial \psi_4}{\partial \dot{P}_4} \dot{P}_4 + \frac{\partial \psi_4}{\partial \dot{P}_7} \dot{P}_7 + \frac{\partial \psi_4}{\partial \dot{U}_4} \dot{U}_4 = \psi_4
\]  

(7.44)

\[
\frac{\partial \psi_5}{\partial t_0} + \frac{\partial \psi_5}{\partial C} \dot{C} + \frac{\partial \psi_5}{\partial \dot{C}} \ddot{C} + \frac{\partial \psi_5}{\partial \dot{P}_4} \dot{P}_4 + \frac{\partial \psi_5}{\partial \dot{P}_7} \dot{P}_7 + \frac{\partial \psi_5}{\partial \dot{U}_4} \dot{U}_4 = \psi_5
\]  

(7.45)

where \( \dot{C}, \dot{C}, \dot{P}_4, \dot{P}_7, \dot{U}_4, \psi_1, \psi_2, \psi_3, \psi_4 \) and \( \psi_5 \) are to be substituted from equations (7.26) to (7.35).
Equations (7.41) to (7.45) are a set of non-linear partial differential equations with six independent variables, namely $t_o$, NC, NF, P4, P7 and U4. Numerical integration of such equations proved to be very difficult; even the NAG library routines which handle partial differential equations were not able to accommodate a problem of this nature. An attempt was made to use the method of lines for solving this problem, but owing to the complex nature of this problem, too much time and resources were needed to proceed with this approach, a matter which is in any case beyond the immediate concern of this research, so this approach was abandoned. A finite element approach was considered for the solution of this problem, but again, because of the same limitations faced in the first approach, this proposed technique was not started with.

Finally, the numerical integration approach to the above problem, was replaced by a second approach in which linearity was assumed between the non-linear adjoint and the state vectors, i.e.

if

$$x(t_o) = C, \quad \psi(t_o) = r(C, t_o)$$

then a solution of the form:

$$\psi(t_o) = r((C, t_o)) = P(t_o)C + N(t_o)$$  \hspace{1cm} (7.46)

where $P$ is an $[n \times n]$ matrix, and $N$ is a vector of $\mathbb{R}^n$ was guessed. From equation (7.46):

$$\frac{\partial r}{\partial t_o} = PC + N = Px + N$$  \hspace{1cm} (7.47)

$$\frac{\partial r}{\partial C} = P$$  \hspace{1cm} (7.48)

Substituting equations (7.47) and (7.48) in equation (7.40a) yields:

$$\dot{PC} + \dot{P} + P f(C, r, t_o) = g(C, r, t_o)$$  \hspace{1cm} (7.49)

or

$$\dot{Px} + \dot{N} + P \dot{x} = \dot{\psi}$$  \hspace{1cm} (7.50)

where $\dot{x}$ and $\dot{\psi}$ are to be obtained from equation (7.26) to (7.35).
To construct the ordinary differential equations in P and N (like the procedure of section (6.5), equation (6.93)), one has to find the coefficients of the state vector elements on each side of equation (7.50), and then equate the coefficients of the state vector elements of the same powers, i.e. equating the coefficients of $x^0, x^1, \ldots$ at both sides of equation (7.50). Because of the analytical complexity of this system, this process of state variable separation and equating of coefficients is very difficult, indeed for moderately complex problems it is nearly impossible.

To proceed with this second approach, certain approximations had to be made, and the only reasonable approximation was to express the original non-linear engine dynamics, by a piecewise linear approximation viz:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$  \hspace{1cm} (7.51)

where $A(t), B(t)$ are time-varying matrices, of appropriate order, and which are generated at each communication interval++. The trend of using non-linear system piecewise linearization for gas turbine engine control was reported recently by Munro (1986), Yates (1986), Chrysanthou (1986), and work by other researchers*.

In this work tests were carried out to determine the difference in dynamic responses between that of the non-linear system and that of the piecewise-linear version, i.e. the form of equation (7.51)

A relative error vector, $REV(t)$, is defined as:

$$REV(t) = \frac{|e(t) - x(t)|}{x(t)}$$  \hspace{1cm} (7.52)

where $x(t)$ is the state vector of the piecewise-linear system, defined by the following equation:

++ See footnote of section 3.4.1

*Private communication with personnel from 'Dowty Electronics' where it was pointed out that they have used this approach.

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\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \]  \hspace{1cm} (7.53)

where \( e(t) \) is defined as:

\[ e(t) = X(t) - XSS \]  \hspace{1cm} (7.54)\(^+\)

where \( X(t) \) is the state vector of the original non-linear system defined in equation (7.6), and XSS is the steady-state value of the state vector defined in equation (7.7).

Figure (7.1) shows for the open loop case at the take-off operating condition, the time responses of some elements of the matrix \( A(t) \) of equation (7.53), and figure (7.2) shows the variation with time of the relative error vector, \( REV(t) \), for the same conditions as figure (7.1). Figure (7.3) shows the variation of \( REV(t) \) with time for the case where a feedback law of the form:

\[ u = kx \]  \hspace{1cm} (7.54)

was applied to both systems of equations (7.53) and (7.54) at the same previous operating conditions. \( k \) was a \([2\times5]\) feedback matrix, based on the solution of the A.R.E., for the linear model LIN1, and using the following weighting matrices:

\[
\begin{align*}
Q & = \text{diagonal } [1.0\times10^2, 5.0\times10^4, 1.0\times10^2, 1.0, 1.0] \\
G & = \text{diagonal } [3.0\times10^9, 9.0\times10^9]
\end{align*}
\]

It was not surprising that the relative error vector, \( REV(t) \), in figure (7.2) had a different pattern of time variation from that of figure (7.3), owing to the change in the closed loop dynamic nature for both conditions under consideration at this operating point. A number of studies were made, including different initial state vectors, which did not violate the engine's operational limitations, and different feedback matrices obtained by use of different weighting matrices, \( Q \) and \( G \). The following remarks apply:

\[ + e(t) \text{ is the same as } x(t) \text{ of equation (7.11) and is used for convenience of presentation.} \]
1. In all cases, the relative error vector, \( \text{REV}(t) \), goes to zero by the end of the simulation period \( t_f \).

2. The closed-loop patterns presented higher values in \( \text{REV}(t) \), than those from the open loop.

3. In most cases, the maximum value in \( \text{REV}(t) \) was associated with the state variable No.5, i.e. \( U_4 \), the combustor internal energy. The lowest values were associated with the states variables, \( NC \) and \( NF \), the high and low pressure compressor speeds. The \( \text{REV}(t) \) value associated with the other state variables, \( P_4 \) and \( P_7 \), were higher than those of \( NC \) and \( NF \), but lower than that of \( U_4 \). Nevertheless, from the point of view of the engine's thermodynamic performance, it is the high and low pressure compressor speeds which in general have the dominating effect on the engine performance. There have been even some engine models reported with only two states variables, \( NC \) and \( NF \),(Longenbaker and Leake (1977)).

4. Throughout the range of cases studied, the maximum \( \text{REV}(t) \) value experienced was of the order of 17 per cent, associated with \( U_4 \), and a maximum of the order of 8 per cent was associated with \( NC \) or \( NF \): these values were associated with a closed-loop condition.

Since the aim here was to proceed with the approach of combined invariant imbedding method and piecewise linearization technique, so that an approximate solution to the two boundary values problem of equations (7.26) to (7.35) could be found, the maximum value of \( \text{REV}(t) \) noted above did not appear to pose a major obstacle to the achievement of such a preliminary solution.

Using both the piecewise linearization and the invariant imbedding methods, the problem was formulated as:

given the system
\[
\dot{x} = A(t) x(t) + B(t) u(t), \quad x(t_0) = x_0
\] (7.55)

and a performance index to be minimized viz:

\[
J = \frac{1}{2} x(t_f) S x(t_f) + \frac{1}{2} \int_0^{t_f} \left[ x^T(t) Q x(t) + u^T(t) G u(t) \right] dt
\] (7.56)

then the associated Hamiltonian is:

\[
H = \frac{1}{2} \left( x^T(t) Q x(t) + u^T(t) G u(t) \right) + \psi^T(t) \left( A(t) x(t) + B(t) u(t) \right)
\] (7.57)
The linear two-point boundary value problem was be expressed as:
\[
\begin{align*}
\dot{x} &= A(t)x(t) - B(t)G^{-1}B^T(y(t)) \quad x(t_0) = x_0 \\
\dot{\psi} &= -Qx(t) - A^T(t)\psi(t) \quad \psi(t_f) = Sx(t_f) = 0
\end{align*}
\]
where \( S = 0 \)

For this problem one can use the invariant imbedding equation (7.40a), with \( x = C \) and \( \psi = r \) to obtain
\[
\frac{\partial r}{\partial t_0} + \frac{\partial r}{\partial C} \left[ A(t)C - B(t)G^{-1}B^T(t)r \right] = -QC - A^T(t)r
\]  
(7.61)

Since the system equations are linear, then one is justified in assuming a solution, valid for arbitrary \( C \), and arbitrary initial time \( t_0 = t_a \), of the form:
\[
\psi(t_a) = r(C, t_a) = P(t_a)C + N(t_a)
\]  
(7.62)

Substituting this assumed solution into the invariant imbedding equation, yields:
\[
\dot{P}C + \dot{N} + P \left[ AC - BG^{-1}B^TPC - BG^{-1}B^TN \right] = -QC - A^TPC - A^TNP
\]  
...
(7.63)

This can be a solution for arbitrary \( C \) only if:
\[
\dot{P} = -P(t)A(t)A^T(t)P(t) + P(t)B(t)G^{-1}B^T(t)P(t) - Q
\]  
(7.64)
\[
\dot{N} = \left[ P(t)B(t)G^{-1}B^T(t) - A^T(t) \right]N(t)
\]  
(7.65)
\[ \dot{P} = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)G^{-1}B^T(t)P(t) - Q \] (7.64)

\[ \dot{N} = \begin{bmatrix} P(t)B(t)G^{-1}B^T(t) - A^T(t) \end{bmatrix} N(t) \] (7.65)

which must satisfy, at the start of the computation

\[ \psi(t_f) = P(t_f)C + N(t_f) = Sx(t_f) = SC \] (7.66)

where \( t = t_f \)

Thus, the initial conditions for the \( P \) equation and the \( N \) equation are, since \( C \) is arbitrary:

\[ N(t_f) = 0, \quad P(t_f) = S \] (7.67)

Clearly, the solution to \( N(t) \) is \( 0 \).

So, if the matrix Riccati equation (7.64), is to be solved backwards in time for \( P(t) \), starting at \( t_f \), with initial condition \( S (S=0, \text{for this case}) \), then we obtain at any \( t \leq t_f \) the value of \( \psi(t) \) which, at time \( t \), would be the initial condition vector required to produce the optimal solution.

Figure (7.4) shows the solution of equation (7.64), for the operating condition around LIN1, and the weighting matrices:

\[ Q = \text{diagonal} \begin{bmatrix} 1.00E+2, 1.00E+2, 1.00E+1, 1.00, 1.00 \end{bmatrix} \]

\[ G = \text{diagonal} \begin{bmatrix} 1.00E+9, 1.00E+8 \end{bmatrix} \]
In this figure, the elements in the upper triangle of the Riccati Matrix, \( P(t) \), are presented for reverse time; from this figure one notices that \( P(t) \) has no steady-state value like the case where the matrices, \( A \) and \( B \), are constants. Hence it can be concluded that this pattern of variation in \( P(t) \) is due to the fact that the matrices \( A(t) \), \( B(t) \) are themselves functions of time, and that the A.R.E. is no longer valid in this situation. Figures (7.5) and (7.6) represent the variation with time of the states and control vectors for the above operating condition.

Symmetry analysis for the Riccati matrix \( P(t) \) of equation (7.64), was carried out in the same manner of that reported in section (3.5), and similar results were obtained, i.e. the \( P(t) \) matrix retained its symmetry throughout the calculation.

Figure (7.7) shows the state vector response of the non-linear engine model, OMAR, at the take-off operating point, when subjected to the optimal control of figure (7.6). Figure (7.8) represents the variation of the fuel flow, WFB, and the nozzle area, \( A_8 \), for OMAR at the same condition.

Figure (7.9) shows the solution of equation (7.64) for the operating condition around LIN2, and figures (7.10) and (7.11) represents the states and control vectors variations with time for this condition.

Figure (7.12) shows the states vector response of OMAR, at climb operating point, when subjected to the optimal control of figure (7.11), while figure (7.13) shows the variations of WFB and \( A_8 \) at this condition.
Figure 7.1 Responses of Some Elements of The A(t) Matrix

Figure 7.2 Relative Error Vector Variation With Time at Lift-off Point
Figure 7.3 Relative Error Vector Variation For a Controlled Case
The Riccati Matrix $P(n \times n)$

Figure 7.4 Variation Of Riccati Matrix With Reverse Time Around UN1
Figure 7.5 Controlled Response of Around LIN1 State Vector

Figure 7.6 Variation of The Optimal Control Law With Time
Figure 7.7 Controlled Response of OMAR State Vector at Take-off

Figure 7.8 Variation of WFB & A8 For OMAR at Take-off
The Riccati Matrix $P(n \times n)$

Figure 7.9 Variation Of Riccati Matrix With Reverse Time Around LN2
Figure 7.10 Controlled Response of Around LIN2 State Vector

Figure 7.11 Variation of The optimal Control Law With Time
Figure 7.12 Controlled Response of OMAR State Vector at Climb

Figure 7.13 Variation of WFB & A7 For OMAR at Climb
7.4 Quasilinearization

In the preceding chapter it was pointed out that the Q.L. method is a technique whereby a non-linear two-point, boundary value problem is transformed into a more readily solvable linear, non-stationary value problem. The intention here in this section is to attempt to explain the use of the Q.L. method (outlined in section (6.3) ) to solve the non-linear two-point boundary value problem of section (7.2). The relevant equations, i.e. (7.26) to (7.35), can be expressed as:

\[ \dot{R}(t) = \kappa(R(t)) \triangleq \begin{bmatrix} f(x(t), \psi(t), t) \\ g(x(t), \psi(t), t) \end{bmatrix} \]  

where \( R(t) \) is a new vector \( \in \mathbb{R}^{2n} \), defined as:

\[ R(t) \triangleq \begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix} \]

The boundary conditions of this problem are:

\[ R_j(t_i) = d_{ij} \]  

where \( j = 1, 2, \ldots, 2n; \ t_i = t_o \) or \( t_f \) or

\[ R_{j=1,n}(t_o) = x(o), \quad \text{and} \quad R_{j=n+1,2n}(t_f) = \psi(t_f) = 0 \]

According to the Q.L. method, the problem solution obeys:

\[ \dot{R}^{k+1}(t) = \kappa(R^k, t) + \left[ J_{R, k} \kappa(R^k, t) \right] \left[ R^{k+1}(t) - R^k(t) \right] \]

where

\[ J_{R, k} \kappa(R^k, t) \]

is the canonical system Jacobian matrix, of order \([2n \times 2n]\), and the ijth component of which is given by
\[
\frac{\partial \kappa_i}{\partial R_j}
\]

or:

\[
J_{Rk} \kappa(R^k, t) = \begin{bmatrix}
\frac{\partial \kappa_1}{\partial R_1} & \frac{\partial \kappa_1}{\partial R_{2n}} \\
\frac{\partial \kappa_2}{\partial R_1} & \frac{\partial \kappa_2}{\partial R_{2n}} \\
\vdots & \vdots \\
\frac{\partial \kappa_{2n}}{\partial R_1} & \frac{\partial \kappa_{2n}}{\partial R_{2n}}
\end{bmatrix}
\]  (7.92)

Applying equation (7.92) to equations (7.26) to (7.35), we obtain the following [10*10] Jacobian matrix, \(J_R\), i.e:

\[
J_R(1,j) = DNCDT(j) \quad (7.93)
\]

\[
J_R(2,j) = DNFDT(j) \quad (7.94)
\]

\[
J_R(3,j) = DP4DT(j) \quad (7.95)
\]

\[
J_R(4,j) = DP7DT(j) \quad (7.96)
\]

\[
J_R(5,j) = DU4DT(j) \quad (7.97)
\]

\[
J_R(6,j) = DS1DT(j) = \frac{\partial \dot{\psi}_1}{\partial R_j} \quad (7.98)
\]

\[
J_R(7,j) = DS2DT(j) = \frac{\partial \dot{\psi}_1}{\partial R_j} \quad (7.99)
\]

\[
J_R(8,j) = DS3DT(j) = \frac{\partial \dot{\psi}_3}{\partial R_j} \quad (7.100)
\]

\[
J_R(9,j) = DS4DT(j) = \frac{\partial \dot{\psi}_4}{\partial R_j} \quad (7.101)
\]

\[
J_R(10,j) = DS5DT(j) = \frac{\partial \dot{\psi}_5}{\partial R_j} \quad (7.102)
\]

In all the above equations, \(j\) takes a value from 1 to 10, and represents the index of the independent variables that partial differentiations are to be taken with respect to, i.e. the \(R\) vector, where:
It is worth noting that for the values of \( j = 1 \) to \( j = 5 \) in equations (7.93) to (7.97), all the elements have been derived before, when the system canonical equations were constructed. Concerning the other elements, i.e. for the values of \( j = 6 \) to \( j = 10 \), it is required to establish the partial derivatives with respect to the co-state vector, \( \psi \). Because of the analytical complexity of the problem, the following indirect approach was used, i.e.:

\[
\begin{align*}
J_{R}(1,6) &= \frac{\partial \dot{\mathbf{N}}}{\partial \psi_1} = \frac{\partial \dot{\mathbf{N}}}{\partial \psi_2} \left( \frac{\partial W_{FB}}{\partial \psi_1} \right) + \frac{\partial \dot{\mathbf{N}}}{\partial \psi_3} \left( \frac{\partial W_{FB}}{\partial \psi_2} \right) \\
&= -DXU(1,1)\left(\left(\frac{DXU(1,1)}{G(1,1)}\right) - DXU(1,2)\right)\left(\frac{(DXU(1,2))}{G(2,2)}\right) \quad (7.104) \\
J_{R}(1,7) &= \frac{\partial \dot{\mathbf{N}}}{\partial \psi_2} = \frac{\partial \dot{\mathbf{N}}}{\partial \psi_3} \left( \frac{\partial W_{FB}}{\partial \psi_2} \right) + \frac{\partial \dot{\mathbf{N}}}{\partial \psi_4} \left( \frac{\partial W_{FB}}{\partial \psi_3} \right) \\
&= -DXU(1,1)\left(\left(\frac{DXU(2,1)}{G(1,1)}\right) - DXU(1,2)\right)\left(\frac{(DXU(2,2))}{G(2,2)}\right) \quad (7.105) \\
J_{R}(1,8) &= \frac{\partial \dot{\mathbf{N}}}{\partial \psi_3} \\
&= -DXU(1,1)(DXU(3,1)/G(1,1)) - DXU(1,2)(DXU(3,2)/G(2,2)) \quad (7.106) \\
J_{R}(1,9) &= \frac{\partial \dot{\mathbf{N}}}{\partial \psi_4} \\
&= -DXU(1,1)(DXU(4,1)/G(1,1)) - DXU(1,2)(DXU(4,2)/G(2,2)) \quad (7.107) \\
J_{R}(1,10) &= \frac{\partial \dot{\mathbf{N}}}{\partial \psi_5} \\
&= -DXU(1,1)(DXU(5,1)/G(1,1)) - DXU(1,2)(DXU(5,2)/G(2,2)) \quad (7.108) \\
J_{R}(2,6) &= \frac{\partial \dot{\mathbf{N}}}{\partial \psi_1} \\
&= -DXU(2,1)\left(\left(\frac{DXU(1,1)}{G(1,1)}\right) - DXU(2,2)\right)\left(\frac{(DXU(1,2))}{G(2,2)}\right) \quad (7.109) \\
J_{R}(2,7) &= \frac{\partial \dot{\mathbf{N}}}{\partial \psi_2} \\
&= -DXU(2,1)\left(\left(\frac{DXU(2,1)}{G(1,1)}\right) - DXU(2,2)\right)\left(\frac{(DXU(2,2))}{G(2,2)}\right) \quad (7.110)
\end{align*}
\]
\[ J_{R(2,8)} = \frac{\partial N F}{\partial \psi_3} \]

\[ = -DXU(2,1) (DXU(3,1)/G(1,1)) - DXU(2,2) (DXU(3,2)/G(2,2)) \quad (7.111) \]

\[ J_{R(2,9)} = \frac{\partial N F}{\partial \psi_4} \]

\[ = -DXU(2,1) (DXU(4,1)/G(1,1)) - DXU(2,2) (DXU(4,2)/G(2,2)) \quad (7.112) \]

\[ J_{R(2,10)} = \frac{\partial \dot{p}_4}{\partial \psi_5} \]

\[ = -DXU(2,1) (DXU(5,1)/G(1,1)) - DXU(2,2) (DXU(5,2)/G(2,2)) \quad (7.113) \]

\[ J_{R(2,6)} = \frac{\partial \dot{p}_4}{\partial \psi_1} \]

\[ = -DXU(3,1) (DXU(1,1)/G(1,1)) - DXU(3,2) (DXU(1,2)/G(2,2)) \quad (7.114) \]

\[ J_{R(3,7)} = \frac{\partial \dot{p}_4}{\partial \psi_2} \]

\[ = -DXU(3,1) (DXU(2,1)/G(1,1)) - DXU(3,2) (DXU(2,2)/G(2,2)) \quad (7.115) \]

\[ J_{R(3,8)} = \frac{\partial \dot{p}_4}{\partial \psi_3} \]

\[ = -DXU(3,1) (DXU(3,1)/G(1,1)) - DXU(3,2) (DXU(3,2)/G(2,2)) \quad (7.116) \]

\[ J_{R(3,9)} = \frac{\partial \dot{p}_4}{\partial \psi_4} \]

\[ = -DXU(3,1) (DXU(4,1)/G(1,1)) - DXU(3,2) (DXU(4,2)/G(2,2)) \quad (7.117) \]

\[ J_{R(3,10)} = \frac{\partial \dot{p}_4}{\partial \psi_5} \]

\[ = -DXU(3,1) (DXU(5,1)/G(1,1)) - DXU(3,2) (DXU(5,2)/G(2,2)) \quad (7.118) \]
\[
J_R(4,6) = \frac{\partial P_7}{\partial \psi_1} = \frac{\partial P_7}{\partial \psi_1} \cdot \frac{\partial WFB}{\partial \psi_1} + \frac{\partial P_7}{\partial A_8} \cdot \frac{\partial A_8}{\partial \psi_1}
\]
\[
= -DXU(4,1)(DXU(1,1)/G(1,1)) - DXU(4,2)(DXU(1,2)/G(2,2)) \quad (7.119)
\]
\[
J_R(4,7) = \frac{\partial P_7}{\partial \psi_2}
\]
\[
= -DXU(4,1)(DXU(2,1)/G(1,1)) - DXU(4,2)(DXU(2,2)/G(2,2)) \quad (7.120)
\]
\[
J_R(4,8) = \frac{\partial P_7}{\partial \psi_3}
\]
\[
= -DXU(4,1)(DXU(3,1)/G(1,1)) - DXU(4,2)(DXU(3,2)/G(2,2)) \quad (7.121)
\]
\[
J_R(4,9) = \frac{\partial P_7}{\partial \psi_4}
\]
\[
= -DXU(4,1)(DXU(4,1)/G(1,1)) - DXU(4,2)(DXU(4,2)/G(2,2)) \quad (7.122)
\]
\[
J_R(4,10) = \frac{\partial P_7}{\partial \psi_5}
\]
\[
= -DXU(4,1)(DXU(5,1)/G(1,1)) - DXU(4,2)(DXU(5,2)/G(2,2)) \quad (7.123)
\]
\[
J_R(5,6) = \frac{\partial U_4}{\partial \psi_1}
\]
\[
= -DXU(5,1)(DXU(1,1)/G(1,1)) - DXU(5,2)(DXU(1,2)/G(2,2)) \quad (7.124)
\]
\[
J_R(5,7) = \frac{\partial U_4}{\partial \psi_2}
\]
\[
= -DXU(5,1)(DXU(2,1)/G(1,1)) - DXU(5,2)(DXU(2,2)/G(2,2)) \quad (7.125)
\]
\[
J_R(5,8) = \frac{\partial U_4}{\partial \psi_3}
\]
\[
= -DXU(5,1)(DXU(3,1)/G(1,1)) - DXU(5,2)(DXU(3,2)/G(2,2)) \quad (7.126)
\]
The approximate invariant imbedding solution derived in the previous section (the solution resulted from combined piecewise linearization and the invariant imbedding methods) was used as initial guess vector to evaluate the above constructed Jacobian matrix, so that equations (7.129) and (7.130) could be integrated forward in time, to evaluate \( \phi(t_f), PP(t_f) \). Using \( \phi(t_f), PP(t_f) \) and the known initial and final boundary values, \( d_{ij} \), equation (7.131) was solved for \( R^{(k+1)}(t_f) \). The new solution vector \( R^{(k+1)}(t) \) was obtained by integrating equation (7.128) forward in time starting from the initial boundary value vector \( R^{(k+1)}(t_o) \). In examining the vector \( R^{(k+1)}(t) \), it was noticed that the co-state variables solution was numerically unstable, and when the vector \( R^{(k+1)}(t) \) was used as the initial guess for the next
iteration to generate $R^{(k+2)}(t)$, the solution process was terminated because of a singularity (an attempt to invert an ill-conditioned matrix) that took place when an attempt was made to solve equation (7.131), again for $R^{(k+2)}(t)$. Although it is known that the co-state variables (associated with optimal control problems) are unstable in the forward direction of time, it was the initial guess vector which had the important effect in this context. In order to solve the above T.P.B.V.P another approach was used, that is, specific optimal control (S.O.C.). In this approach (as shown in chapter 6, Problem Three) the control variables may take the forms:

$$u_1 = a \times$$
$$u_2 = b \times$$

where
$$a = [a_1 a_2 a_3 a_4 a_5]$$
$$b = [b_1 b_2 b_3 b_4 b_5]$$

and
$$\dot{a} = 0$$
$$\dot{b} = 0$$

This S.O.C. technique is useful in two ways:

1. If it is to succeed, then it will provide a closed-loop control for the engine which is more convenient than the open-loop control that could be achieved from the first solution (equations (7.26) to (7.35)).

2. It reduces the analytical complexity of the problem, by setting a specific form of the control variables (equations (7.132) and (7.133)), and not obtaining the control vector from setting $\partial H/\partial u=0$

The disadvantage of this technique is the big increase in problem size, from that of the first (the problem presented by equations (7.26) to (7.35)), where another 10 state variables must be added to the first state vector, and another 10 co-state variables must be added to the co-state vector of the first problem.

The $R(t)$ vector was modified to become:

$$R(t) = [x_1 x_2 x_3 x_4 x_5 a_1 a_2 a_3 a_4 a_5 b_1 b_2 b_3 b_4 b_5 \psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_6 \psi_7$$
$$\psi_8 \psi_9 \psi_{10} \psi_{11} \psi_{12} \psi_{13} \psi_{14} \psi_{15}]^T$$

(7.138)
Another extra 20 differential equations were then added to the system canonical equations (equations (7.26) to (7.35)). The first 10 state equations are

\[
\begin{align*}
\dot{a} &= 0 \\
\dot{b} &= 0
\end{align*}
\]

and the other 10 co-state equations are

\[
\dot{\psi}(6\rightarrow10) = -\frac{\partial H}{\partial a}
\]

\[
\dot{\psi}(11\rightarrow15) = -\frac{\partial H}{\partial b}
\]

The new system differential equations will comprise:

\[
\begin{bmatrix}
\dot{x} \\
\dot{a} \\
\dot{b} \\
\dot{\psi}
\end{bmatrix}
\]

The new system Jacobian matrix was a \([30\times30]\) matrix, the elements of which are the partial derivatives of the above differential equations, equation (7.141), with the respect to the state vector, \(R(t)\), equation (7.138). It was a lengthy and tedious process, but when its results where compared with those obtained using the same checking procedure of section (7.2) (remark 4), identical results where obtained, which endorsed the quality of those analytical derivations.

In order to solve this S.O.C. problem numerically, the same sequence of steps used in the first problem was used once more, but because of the size and nature of this problem (it is a stiff differential system), neither of the main frame computers available at Loughborough University of Technology computer centre\(^+\), were able to handle the problem. The alternative was to use the Cyber 205 supercomputer\(^*\) at the University of Manchester regional computer centre, UMRCC.

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\(^+\) Honeywell Multics DPS8/70M computer and A Prime 750 computer

\(^*\) The Cyber 205 at UMRCC and Cyber 205 diagnostic Handbook
When this problem was run using the Cyber 205, the process was automatically terminated because the transition matrix of equation (7.131) was singular. It is known from theoretical considerations [Desoer (1970)] that the transition matrix corresponding to the matrix $J_R$ is non-singular; however, numerical evaluation of the elements of the system transition matrix, $(S.T.M)$, depends on the condition of the elements of the $J_R$ matrix. In this work these elements were determined numerically and represent the slopes of non-linear functions at particular operating points. Hence there was no method of conditioning the $J_R$ matrix to obtain an accurate $S.T.M$ which retained the property of non-singularity. To get an approximate solution, so that the process could be started, the singular value decomposition subroutine from the NAG library was used to obtain a generalized inverse of the matrix. This NAG subroutine was useful in obtaining a solution for equation (7.131), $R^{k+1}(t_0)$, but when this $R^{k+1}(t_0)$ was used in integrating equation (7.128) forward in time, the same thing happened as in the first problem, i.e the co-state vector solution went unstable, and the process "blew up". This instability of the co-state vector in forward time, made the quasilinearization technique not very useful for solving the T.P.V.B.P which resulted from the application of the optimal control theory to the non-linear engine model, OMAR. What was needed was to try to find some other approach, where it was possible to avoid the numerical integration of the co-state variables in forward time. The gradient method, (G.M), was thought to be useful in this situation.

7.5 Gradient Method

It is required to determine an optimal control vector, $u^0(t)$, which will minimize the performance index:

$$ J = \int_{0}^{T_f} L(x(t), u(t), t) \, dt $$  \hspace{1cm} (7.142)

subject to the constraint of the system dynamics;

$$ \dot{x}(t) = f(x(t), u(t), t) \quad x(t_0) = x_0 $$ \hspace{1cm} (7.143)

The Hamiltonian of the system is given by:

$$ H = L \left( x(t), u(t), t \right) + \psi^T(t) f \left( x(t), u(t), t \right) $$ \hspace{1cm} (7.144)
\[ \dot{\psi}(t) = -\frac{\partial H}{\partial x(t)} \quad \text{with} \quad \psi(t_f) = 0 \quad (7.145) \]

The optimal control vector \( u^0(t) \) is determined from:

\[ \frac{\partial H}{\partial u^0(t)} = 0 \quad (7.146) \]

The complete gradient method is described by the following algorithm:

1. Assume a control vector \( u^1 \), which is an estimate of the optimal control \( u^0 \).

2. Integrate state equations, (7.143), in forward time using \( u^1 \), and calculating the value of the performance index \( J \).

3. Integrate the adjoint equations in reverse time, using \( u^1 \) and \( x^1 \).

4. Calculate the function \( \frac{\partial H}{\partial u} \) at \( (u^1, x^1, \psi^1) \).

5. Update the control vector \( u^1 \) such as:

\[ u^{i+1} = u^i + c \frac{\partial H}{\partial u}(u^i, x^i, \psi^i) \quad (7.147) \]

The constant 'c' is usually found by linear search so that the performance index is minimized. The new control vector, \( u^{i+1} \), is then used in the state equations (step 2) and the process is repeated as before.

The main advantage of this algorithm is that the adjoint equations are solved in the stable direction, i.e., in the reverse time direction.
7.6 Non-linear Optimal Tracking Problem

Given non-linear engine model, OMAR, described by equations (7.1) to (7.5) and the associated algebraic relationships, which relate the intermediate variables. Suppose that the variable $Z$ is the desired thrust output, then the objective is to control this non-linear engine model, in such a manner as to make the thrust variable, $FG$, close to $Z$ without excessive expenditure of control energy, and ensuring that other output variables $T4$, $ZC$ and $ZF$ are kept within their operational limits.

The optimal control vector $u(t)$ which is capable of achieving this objective can be obtained by minimizing a performance index of the type:

$$J_e = \frac{1}{2} \int_{0}^{T} \left[ (Q(1,1)(Z - FG))^2 + u^T(t) G u(t) \right] dt \quad (7.148)$$

The Hamiltonian function, $H$, associated with this performance index, is defined as

$$H = \frac{1}{2} \left( Q(1,1)(Z - FG)^2 + u^T(t) G u(t) \right) + \psi_1 x_1 + \psi_2 x_2 + \psi_3 x_3 + \psi_4 x_4 + \psi_5 x_5$$

.......... \quad (7.149)

where $x(t), \dot{x}(t), u(t)$ are defined in equations (7.11), (7.12) and (7.13)

where

$$\dot{\psi} = - \frac{\partial H}{\partial x} \quad (7.150)$$

where (7.150) could be written as:

$$\dot{\psi}_1 = \left[ Q(1,1)F1 + \psi_1 \frac{\partial x_1}{\partial x_1} + \psi_2 \frac{\partial x_2}{\partial x_1} + \psi_3 \frac{\partial x_3}{\partial x_1} + \psi_4 \frac{\partial x_4}{\partial x_1} + \psi_5 \frac{\partial x_5}{\partial x_1} \right] \quad (7.151)$$

$$\dot{\psi}_2 = \left[ Q(2,2)F2 + \psi_1 \frac{\partial x_1}{\partial x_2} + \psi_2 \frac{\partial x_2}{\partial x_2} + \psi_3 \frac{\partial x_3}{\partial x_2} + \psi_4 \frac{\partial x_4}{\partial x_2} + \psi_5 \frac{\partial x_5}{\partial x_2} \right] \quad (7.152)$$

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\[ \psi_3 = \left[ Q(3,3)F_3 + \psi_1 \frac{\partial x_1}{\partial x_3} + \psi_2 \frac{\partial x_2}{\partial x_3} + \psi_3 \frac{\partial x_3}{\partial x_3} + \psi_4 \frac{\partial x_4}{\partial x_3} + \psi_5 \frac{\partial x_5}{\partial x_3} \right] \quad (7.153) \]

\[ \psi_4 = \left[ Q(4,4)F_4 + \psi_1 \frac{\partial x_1}{\partial x_4} + \psi_2 \frac{\partial x_2}{\partial x_4} + \psi_3 \frac{\partial x_3}{\partial x_4} + \psi_4 \frac{\partial x_4}{\partial x_4} + \psi_5 \frac{\partial x_5}{\partial x_4} \right] \quad (7.154) \]

\[ \psi_5 = \left[ Q(5,5)F_5 + \psi_1 \frac{\partial x_1}{\partial x_5} + \psi_2 \frac{\partial x_2}{\partial x_5} + \psi_3 \frac{\partial x_3}{\partial x_5} + \psi_4 \frac{\partial x_4}{\partial x_5} + \psi_5 \frac{\partial x_5}{\partial x_5} \right] \quad (7.155) \]

where

\[ F_1 = \frac{\partial F_1}{\partial x_1} \]

\[ F_2 = \frac{\partial F_2}{\partial x_2} \]

\[ F_3 = \frac{\partial F_3}{\partial x_3} \]

\[ F_4 = \frac{\partial F_4}{\partial x_4} \]

\[ F_5 = \frac{\partial F_5}{\partial x_5} \]

\[ u^0(t) \] can be obtained by \( \frac{\partial H}{\partial u^0} = 0 \), which yields

\[ \frac{\partial H}{\partial u_1} = -G^{-1}(1,1) \left[ D_1 \psi_1 \frac{\partial x_1}{\partial u_1} + \psi_2 \frac{\partial x_2}{\partial u_1} + \psi_3 \frac{\partial x_3}{\partial u_1} + \psi_4 \frac{\partial x_4}{\partial u_1} + \psi_5 \frac{\partial x_5}{\partial u_1} \right] \quad (7.156) \]

\[ \frac{\partial H}{\partial u_2} = -G^{-1}(2,2) \left[ D_2 \psi_1 \frac{\partial x_1}{\partial u_2} + \psi_2 \frac{\partial x_2}{\partial u_2} + \psi_3 \frac{\partial x_3}{\partial u_2} + \psi_4 \frac{\partial x_4}{\partial u_2} + \psi_5 \frac{\partial x_5}{\partial u_2} \right] \quad (7.157) \]

where

\[ D_1 = \frac{\partial F_1}{\partial u_1} \]

\[ D_2 = \frac{\partial F_2}{\partial u_2} \]

where the analytical expressions of \( F_1, F_2, F_3, F_4, F_5, D_1 \) and \( D_2 \) are derived in a similar manner to that presented in \( \text{[Appendices A, C and D]} \).
Equations (7.26) to (7.30) and (7.151) to (7.155) represent the canonical equations of the non-linear tracking problem, which is a non-linear, two-point, boundary value problem. Considering the numerical difficulties in solving the T.P.B.V.P experienced in sections (7.3) and (7.4), and in particular the instability of the co-state variables in forward time, the gradient method of section (7.5) was found to be a useful algorithm for attempting to solve the T.P.B.V.P of this tracking problem.

Although the gradient method is not highly dependent on the initial guess, like the Quasilinearization method, it may not converge to a final (optimal) solution, even after a large number of iterations. Recent work using the gradient method for solving T.P.B.V.P were reported by Elsayed (1985) and Roddy (1985). Elsayed presented the final trajectory, i.e. the one where no significant progress in the solution was noticed, as the optimal solution for his problem, while Roddy in his work has noticed that the process of convergence has stopped before reaching the optimal solution of his problem, since he had the knowledge of what should be the optimal solution for that problem from another algorithm.

Before presenting the solution which resulted from applying the G.M. to the problem of this section, two remarks are made:

1 - Since it is required that the thrust FG is to be close to Z, then the performance index should have its minimum value when those two variables are very close, i.e. if the G.M. method is to converge to the optimal solution, then the error between FG and Z must be near zero. This means that a knowledge of the anticipated solution should be available.

2 - In order to improve the efficiency of the convergence process, a solution which is approximate to the optimal one is used as an initial guess for starting the gradient process. This approximate solution was obtained by using a combination of the piecewise linearization and the linear optimal tracking problem of chapter 5. Figure (7.14) show a block diagram representation of this approach and it will be termed the piecewise-linear optimal tracking problem, (P.W.L.T). Then the G.M. method was used to improve this approximate solution and make it closer to the final solution. Although the solutions obtained by this method were not the exact solutions in a mathematical sense, nevertheless they were successful from the practical point of view, i.e. practically they were close enough to the desired solution.
The steps of the solution process could be summarized as:

1- The initial solution guess was established by application of the control law obtained from piecewise-linear optimal tracking problem (P.W.L.T) to the non-linear engine model, OMAR.

2- Using the controls of OMAR which resulted from step one, the state equations (7.26) to (7.30) were integrated forwards in time and the value of $J_e$ was calculated simultaneously using the state and control trajectories of steps one and two. Then the co-state equations (7.151) to (7.155) were integrated backwards in time.

4 - Update the controls by determining

$$u_{1i+1} = u_{1i} + c \frac{\partial H}{\partial u_1}$$

$$u_{2i+1} = u_{2i} + c \frac{\partial H}{\partial u_2}$$

and repeat from step two.

A process of trial and error was used to find the value of $c$ which gave an approximate minimum value of $J_e$.

For their practical importance, two cases are presented in this section. The first case was a slam acceleration from 70 percent to full throttle lift-off power rating, at a standard day condition. The second case was for the approach operating condition, where it was required to abandon the approach and to accelerate with about a 50 percent increase in power rating.

Figures (7.15a), (7.15b), (7.15c) and (7.15d) show the time responses of some elements of the $A(t)$, $B(t)$ matrices, while figures (7.16a) and (7.16b) show the solution of the matrix Riccati equation with reverse time using those $A(t)$ and $B(t)$ matrices. Figures (7.17) and (7.18) present the control and the thrust responses for this P.W.L.T. Figures (7.19), (7.19a) and (7.20) represents the responses of OMAR fuel flow, nozzle area, state vector, turbine temperature and the thrust, when subjected to control of the P.W.L.T, i.e that of figure (7.17). Figures (7.21), (7.22), (7.23) and (7.24) show the responses of OMAR control, state and output variables resulted for the gradient method solution for the lift-off case. Figure (7.25) show various responses of OMAR thrust at the same operating condition, these are:
1 - The desired thrust response $Z$.

2 - The thrust response resulted from the application of the P.W.L.T controls to OMAR

3 - OMAR thrust response from the gradient method solution.

4 - Open-loop thrust response

Although both of the thrust response of points 2 and 3 were better than that of the open-loop, the gradient method response was closer to the desired response $Z$.

Figures (7.26a), (7.26b), (7.26c) and (7.26d) show the time responses of some elements of $A(t)$ and $B(t)$ matrices while figure (7.27) show the solution of the M.R.E. with reverse time using those $A(t)$, $B(t)$ matrices for the approach condition. Figures (7.28) and (7.29) show the control and the thrust responses of the P.W.L.T. Figures (7.30), (7.31) and (7.32) show the responses of OMAR fuel flow, nozzle area, surge margin, turbine temperature and the thrust when it was subjected to the controls of the P.W.L.T method i.e figure (7.28). Figures (7.33) and (7.34) present OMAR gradient method responses of the fuel flow, nozzle area and surge margins. Figure (7.35) show a similar comparison to that of figure (7.25) but for the approach condition, similar remarks to those of figure (7.25) apply.

From the previous figures of the two cases studied, it was noticed that the Gradient method algorithm was capable for providing an approximate solution to the optimal non-linear tracking problem while keeping the rest of the engine variables within their working limits.

7.7 Concluding Remarks

The non-linear two point boundary value problem which resulted from the application of optimal control theory to the non-linear engine model, OMAR, was very complicated analytically and very difficult to solve numerically.

Methods like the Invariant Imbedding and the Quasilinearization were used in an attempt to solve this problem, but they did not prove successful. Although these methods are theoretically applicable to the solution of T.P.B.V.P (as shown in chapter 6) they are not general, i.e. they do not guarantee a successful numerical solution to every problem. Thus the only way of determining their effectiveness for a particular problem is to apply them to the problem. The Invariant Imbedding technique failed to provide a successful solution to the above problem because of its analytical complexity. In the case of the Q.L method it is necessary to question whether the deviation from the anticipated solution is due to the way in
which the equation is formulated or to the nature of the problem. A thorough check was made to ensure that the equations were formulated correctly (as shown in section (7.2)). The difficulty in solving the problem was due to the ill-conditioned transition matrix caused by the method in which the elements of the Jacobian matrix were numerically determined. In this method these elements were determined numerically from the slopes of non-linear functions (it is noted that these non-linear functions were both complicated and stiff non-linear differential equations) at different time stations, and interpolation was used to determine the values of the variables of the non-linear functions in between time stations. Since there was no control of the accumulated error in this procedure, the quality of the elements of the Jacobian matrix were not guaranteed.

The gradient method showed a positive capability for providing an approximate solution to this problem which proved to be useful from the practical point of view. The use of the piecewise-linear optimal tracking problem also proved to be helpful in this context.
Note: In this figure matrices A, B, C, D are time-varying

Figure 7.14: Block Diagram Representation of the Optimal tracking Problem using piecewise linearization.
Figure 7.15a Time Variation of $A(1,1)$

Figure 7.15b Time Variation of $A(3,3)$
Figure 7.15c Time Variation of $B(1,2)$

Figure 7.15d Time Variation of $B(3,1)$
Figure 7.16a Riccati Matrix for The Piecewise Linearized Tracking Problem
Figure 7.16b Riccati Matrix for The Piecewise Linearized Tracking Problem

Legend
- P(1, 4)
- P(2, 3)
- P(2, 4)
- P(3, 3)
- P(3, 4)
- P(3, 5)
- P(4, 4)
- P(4, 5)
Figure 7.17 Control Vector Resulted From Fig 7.16 Riccati Matrix

Figure 7.18 Thrust Response Resulted From Control Vector of Fig.7.17
Figure 7.19 OMAR WFB & A7 When Subject to P.W.L.T Control of Fig. 7.17

Figure 7.19a OMAR State Vector When Subject to Control of Fig. 7.17
Figure 7.20 OMAR Thrust & T4 When Subject to Control of Fig.7.17
Figure 7.21 OMAR WFB & A7 Resulted From Gradient Method

Figure 7.22 State Vector Response When Subject to Control of Fig. 7.21
Figure 7.23 OMAR Thrust & T4 When Subject to G.M Control of Fig. 7.21

Figure 7.24 OMAR Surge Margins When Subject to G.M Control of Fig. 7.21
Legend
- Desired Thrust Response
- Thrust Response Using P.W.L.T
- Thrust Response Using G.M
- FG Open Loop Response

Figure 7.25 OMAR Thrust Responses With Different Control Laws
Figure 7.26a Time Variation of $A(1,5)$

Figure 7.26b Time Variation of $A(2,4)$
Figure 7.26c Time Variation of $B(1,2)$

Figure 7.26d Time Variation of $B(3,1)$
Figure 7.27 Riccati Matrix for The Piecewise Linearized Tracking Problem
Figure 7.29 Thrust Response Resulted From Control Vector of Fig. 7.28

Figure 7.28 Control Vector Resulted From Fig. 7.27 Riccati Matrix
Figure 7.30 OMAR WFB & A8 When Subject to Control of Fig 7.28

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The work undertaken in the conduct of this research aimed at determining a single feedback control law which could effectively control the dynamic response of a modern fighter aircraft engine throughout its flight envelope. The engine chosen for this study was the Pratt & Whitney F100 turbofan jet engine, with afterburning. Several reasons dictated this choice of the engine; they were:

1. The F100 represents an example of the new generation of jet engines used to power modern fighter aircraft, like the F-15 and F-16.

2. The engine has many of the advanced technology aspects of jet engines, such as:
   - the variable geometry of low and high pressure compressor inlet and stator vanes, used to improve the aerodynamic efficiency;
   - airflow bleed, extracted from the compressor to provide for turbine cooling;
   - the use of an afterburner which discharges through a variable convergent-divergent nozzle, where the variable nozzle area geometry provides nearly optimum nozzle area and expansion ratio throughout the operating range of the engine.

3. A reasonable quantity of engine data was available from the published literature, which proved helpful in the process of constructing the engine mathematical models. An example of such data has been shown in figures (2.2) to (2.5).

4. The work carried out by NASA using this engine (reported earlier in this dissertation) was helpful in gaining insight to the problem in general and into the engine dynamics and control, in particular.

A starting point, therefore, was to have a good analytical non-linear model of the engine on which to perform the control studies. DYNGEN was available, a Fortran program which can be used to analyze the steady-state and transient performance of turbojet and turbofan engines at design and off-design operating conditions. When DYNGEN was coded with proper data of the F100 (data maps and thermodynamics tables) a good digital simulation of the F100 engine was achieved. Based on this digital simulation, OMAR, a fifth order non-linear mathematical model of the engine was developed and was reported in chapter 2. OMAR was developed because it was fundamental that an analytical model of the F100 should be available for the application of optimal control techniques. Several points were chosen to define a number of flight envelope boundaries and a linear analytical model, LIN, was derived about each of these points. At later stages of this work, OMAR was used to produce a piecewise linear model of the engine, i.e. a linear analytical model at each
communication interval; this process proved to be useful. It was noticed that the time responses with OMAR were faster than those from DYNGEN and with values of overshoot which were a little higher, but in general, the agreement between the responses was sufficiently close to ensure that OMAR was as good a model as that from DYNGEN. Some minor corrective tuning may still be needed when any of the control laws based on OMAR are applied on a real engine, and, in particular, in those regions near the surge limits.

The first phase of the control study in this work was to try to optimize the dynamic responses of the engine about each of the specified operating points which formed the boundaries of the flight envelope, i.e. the dynamic responses of the linearized model about each of its operating conditions. This led to the formulation of the linear quadratic problem, LQP, a brief account of which was given in chapter 3. The state and output regulator problems are part of the LQP, and their solution leads either to the formation of a matrix Riccati equation, M.R.E., or to its special case, the algebraic Riccati equation, A.R.E., when only the steady-state condition is considered. The feedback gains obtained as a solution of the Riccati equation, correspond only to a particular performance index which was minimized. If the resulting response was unsatisfactory, the performance index was altered (by altering the weighting matrices, for example) and the process was repeated until an acceptable design was reached. A limitation of this technique is that there is no direct procedure which enables a designer to choose a performance index (or its weighting matrices) so that the desired response can be achieved directly. Another reported disadvantage is the need for large amounts of computing time and memory for the solution of the M.R.E., and, in particular, for systems of large dimension. As was shown in chapter 3, for a system of dimension n, the number of differential equations requiring to be integrated is n(n+1)/2; for example if n=30, then this number is 465, a large number of differential equations which have to be integrated. Nevertheless, it is the view of the author that with the computers now available this problem of size does not cause any significant problem in solving such equations.

In chapter 4, the linear quadratic problem was applied at each operating point, and appropriate control laws were obtained. The following were remarked:

1. While the engine was kept within its working limits control laws derived from the output regulator problem gave better thrust responses from the engine than those produced by control laws derived from the state regulator problem. This led to the adoption of the output regulator technique throughout the flight envelope.
2. Since the system matrices were time-invariant, control laws based on the solution of the M.R.E. offered no advantages in terms of the controlled system settling time on over those based on the solution of the A.R.E. It was the weighting matrices of the chosen performance index which had the dominant effect on the settling time, rather than the final time, $t_f$, of the performance index at that stage of the work. This led to the adoption of the A.R.E. solution.

3. An appropriate feedback matrix was obtained at each operating point, and a search was made among these feedback matrices to determine if a single feedback matrix was capable of effectively regulating the engine at every operating point, not merely at the point from which it was derived. The feedback matrix of the take-off operating point was found to be the most suitable to perform this overall task.

4. The above appropriate feedback matrices were applied to OMAR, each feedback matrix being capable of regulating the engine at the operating point from which it was derived. The feedback matrix of the take-off operating point was capable of practically regulating the engine throughout the operating range.

In chapter 5, a summary of the linear optimal tracking problem, O.T.P., applied to a linear observable system was presented, and it was shown that it is possible to make the output vector of a dynamical system reasonably close to some desired response. To obtain a solution of this problem, the solution of the matrix Riccati equation, M.R.E., was essential. At this stage of the work, the merits of the control law based upon the M.R.E. outweighed those based upon the A.R.E. Control laws based on the O.T.P. were successful in providing locally the desired responses of the output vectors, but it is emphasized here that no single control law was able to provide a global solution. Control laws derived from the optimal tracking regulator problem, O.T.R., were successful in providing locally the desired responses, and it was possible for a single control law (take-off condition) to provide a useful solution for the global problem. The difference in responses between the O.T.P. and the O.T.R. in providing a global solution of the problem was caused by the manner in which the control variable, the nozzle area, $A_8$, affected the thermodynamic performance of the engine.

The second phase of the control study in this work was to investigate the possibility of applying optimal control theory to optimize the dynamic response of the engine's non-linear mathematical model. The application of optimal control theory to a non-linear system leads to the formulation of the problem as a non-linear, two-point, boundary value problem.
A number of methods can be used to solve the T.P.B.V.P., but most of them are problem-dependent techniques and the determination of the optimal control must often be carried out numerically by means of an iterative procedure using a digital computer. At the start of this phase of the control study, two methods were thought to be useful for providing a successful solution to the T.P.B.V.P. relating to the optimization of the dynamic response of OMAR. An account of these two methods, the method of quasilinearization, (Q.L.), and the invariant imbedding method, was given in chapter 6. The quasilinearization is a technique whereby a non-linear, two-point, boundary value problem is transformed into a more readily solvable linear, non-stationary, boundary problem. The main advantage of this technique is that if the procedure converges, its convergence to the solution of the original equation is quadratic. Quadratic convergence means that the error in the (k+1)th iteration tends to be proportional to the square of the error in the kth iteration. The main difficulty with the method arises from the fact that, in using the superposition principle, a set of algebraic equations must be solved. Thus, the ill-conditioning phenomenon which can sometimes arise in solving a set of linear algebraic equations can render the superposition principle useless. Furthermore, this technique requires a fairly good initial estimate of the solution, and the resulting linearized adjoint equations can then be highly unstable. The Q.L. method was shown to be successful in solving a number of problems including those of a highly complicated analytical nature.

The second technique was the invariant imbedding method; it is a technique whereby the original two-point, boundary value problem was effectively reduced to an initial value problem, through a process of direct evaluation of the missing initial (or terminal) boundary conditions. The problem solution from the invariant imbedding method can be used as an initial estimate of the solution to start the numerical solution process of the Q.L. method. Results in chapter 6 show that the two solutions can be close.

In chapter 7, the invariant imbedding method was first tried to solve the T.P.B.V.P. which resulted from the application of the optimal control theory to OMAR. The problem was reduced to a set of non-linear partial differential equations with six independent variables. Numerical integration of such equations proved to be very difficult; even the NAG library routines which handle partial differential equations were unable to accommodate a problem of this nature. An attempt was made to use the method of lines for solving this problem, but owing to the complex nature of this problem, too much time and resources were needed to proceed with this approach; the numerical procedure was a matter which was in any case beyond the immediate concern of this research. Then, the numerical integration approach to the above problem, was replaced by a second approach in which linearity was assumed between the non-linear adjoint and state vectors, but owing to the analytical complexity of the resulting set of equations, the process of variable separation was not feasible. Therefore
this approach was unsuccessfull for the problem as posed. To proceed with this second approach, certain approximations had to be made, the only reasonable approximation being to express the original non-linear system dynamics, OMAR, by a series of piecewise linear approximations, i.e. a linear system with time-varying matrices. This solution led to the formulation of the matrix Riccati equation with time varying system matrices. Once again the numerical solution of the M.R.E. proved to be essential for reaching a problem solution. The approximate solution obtained from the above approach was used as the initial estimate solution for the Q.L. method, when it was applied to solve the above non-linear T.P.B.V.P. The Q.L. solution was unsuccessful due to the numerical instability of the adjoint variables, a result which made the Q.L. method ineffective in this context. The gradient method was used to provide an approximate (although good) solution to the T.P.B.V.P. resulting from the application of the non-linear optimal tracking problem applied to OMAR.

The solution obtained from the linear optimal tracking problem with time-varying system matrices, was used as the starting solution to initiate the gradient method process. The same process was carried out at each operating point, where the gradient method produced an approximate solution to the problem, in the sense that the engine thrust response was near to the desired response.

In summary, the chief objective of this research was to establish if a single linear feedback control law could be found to provide acceptable dynamic response over the flight regime of an engine when represented by its non-linear mathematical model. Such a control law was found and its effectiveness has been shown in this dissertation by means of the results of simulation studies. First, a number of linear models representing the engine's performance at six operating points were constructed and validated (chapter 2). Next, by means of the theory of the LQP, a number of optimal linear feedback control laws were found (chapter 4) corresponding to these linear models. Then the performance of these linear feedback control laws were assessed when used with the linear models for the other operating points, and from this study it was found that all the laws worked at every operating point but that the control law derived on the basis of the take-off condition provided the best performance over the flight envelope (chapter 4). These linear feedback controls were subsequently used to control the non-linear model at the six operating points. Here it was established that only the linear control law corresponding to take-off would provide acceptable control over the whole flight regime (chapter 4). To determine the true effectiveness of this single control law an attempt to establish a specific optimal control law, i.e. a linear control whose structure was defined a priori and whose parameters were to be optimised by means of solving a T.P.B.V.P. using the method of quasilinearization was considered. The numerical and computational problems associated with this approach precluded it providing a useful solution and details of this attempt were presented in chapter 7. The approach adopted for
investigating the engine's behaviour in response to commands (tracking) was nearly identical to that summarized above. The fact that no candidate tracking control law could provide global tracking performance, although adequate regulation performance could be obtained, led to the consideration of finding an optimal non-linear control law to provide global tracking performance. From studying the work associated with the Q.L. method the possibility emerged of finding an optimal non-linear control function by using piecewise linearization and the gradient method. This approach provided a successful solution locally.

8.2 Suggestions for Further Work

This section includes suggestions and recommendations about further work relating to engine simulation and effective mathematical models, and control studies.

8.2.1 Engine Simulation and Mathematical Models

It is obvious that any mathematical technique which can provide a simulation of the transient behaviour of the engine would be of great value, and could be expected to result in an improved understanding of engine dynamics and control problems without the need to endanger a valuable engine. The judicious use of such techniques should yield savings in both development time and cost. The following points are made for consideration:

1. The computer program DYNGEN, which was the starting point of this research, is still a useful and valuable tool for jet engine simulation, and it could be used for further work in this area of research. To make the use of DYNGEN more effective, it would need to be made completely 'machine independent'. Although the modifications made to DYNGEN in the course of this work had almost made it of that kind, one non-standard Fortran 77 feature had to be left, namely the 'recursive call procedure'. In this procedure the subroutine 'ENGBAL', is the main subroutine. It controls all the engine balancing loops, check tolerances and the number of loops; it calls itself indirectly through other subroutines in the program. This recursive call procedure is not allowed in a number of computers, and it is recommended that this process is avoided in any improved revision of DYNGEN. It is also recommended that some modifications be made to improve the way in which a number of the thermodynamic process were calculated in DYNGEN.
2. Subroutines should be added to DYNGEN, to make it possible for the feedback control law calculated on the basis of optimal control theory to operate directly on DYNGEN.

3. Although OMAR was a good approximation of DYNGEN, nevertheless, a better agreement between them could be achieved by applying a number of measures. For example, by an increase in the number of state variables in OMAR to make it as near as possible to those of DYNGEN; by improving some of the algebraic relationships between the intermediate variables by using higher powers in the equations of interpolation; by using specific heats as a function of temperature rather than as constant values and by better formulation of components maps.

8.2.2 Control Studies

The following points are made for consideration:

1. A better and more direct method for determining the appropriate weighting matrices of the performance indices chosen to be minimized in the LQP technique should be developed, such that the desired response is directly obtained.

2. Some improved computational method which reduces the computational time needed to solve the M.R.E., particularly for systems of large dimension should be found.

3. The use of piecewise linearization or time-varying systems rather than time invariant system for the LQP, O.T.R. and the O.T.P should be studied with due regard to the special stability problems involved in such time-varying systems.

4. The effect of the presence of noise on the quality of the controlled responses, should be studied and the techniques of using state estimation to implement the feedback control should also be investigated.

5. More attention should be paid to the computational techniques used in the numerical solution of the non-linear, two-point, boundary value problems, such as the method of lines and the finite element method which can be used in solving the non-linear partial differential systems which result from the invariant imbedding method, and numerical techniques which can avoid the ill-conditioning phenomenon in the Q.L. method.
6. The use of composite techniques for solving T.P.B.V.P., in which a solution from one technique will be used as the initial guess in the next method has been found to be effective in this research and should be investigated further.

7. Analogue simulation of dynamical systems, and the use of microprocessors to implement the control laws derived from the optimal control theory on a real time system, and what benefits could be achieved from using the present technology in microprocessors.

8. It was established during the course of this work that it is possible to use a single control law which is capable of practically controlling the engine throughout its operating range: this being the aim of the work. It is recommended then, that an investigation should be made to determine whether the techniques used (and their above proposed modifications) can be applied more widely to other non-linear dynamical systems.
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Appendix A

Fortran Standardization of DYNGEN

Program DYNGEN was originally written in Fortran iv and developed for use, without modification, on IBM 7094 Model 2 computers. It is therefore not surprising to learn that the program required a number of modifications before it could be usefully employed on the computer system used by the author (a Honeywell Multics DPS8/70M). The purpose of this appendix is to document the principal changes made to the program, as published, so that it would comply with the requirements of the American National Standard Fortran 77, to result in a more portable (machine-independent) program.

The motivation for the standardization arose because the available program was readable, well organized and documented, and largely conformed to the standard apart from a number of localized deviations. These were sufficient to cause compilation errors on the Multics system, while others caused run-time errors whose reasons were far from obvious when the program was first used.

The standardization exercise has largely been successful apart from the fact that the program has a subroutine (ENGBAL) which, indirectly calls itself. This is a situation that is prohibited by the Fortran standard. However, the author has taken advantage of the so-called 'dynamic linking' facility on the Multics system to get round this problem, and still preserve the intent of DYNGEN.

The main modifications (including changes for ANSI Fortran standardization) to DYNGEN can be summarized as follows:

1. Common blocks: Each named common block has been changed to have the same length in each subroutine in which it is declared. For example, the longest declared length found for each of the named common blocks, ALL1, ALL2, ALL3, ALL4 and ALL5 was 80 storage units (i.e. any entity that is of type integer or real). All occurrences of these blocks were changed to be 80 storage units long throughout DYNGEN.

+ Dynamic linking allows a program to link a subroutine or function at the point in the program's execution when the subprogram is required.
2. Block data program units: All of DYNGEN's block data program units were unnamed, ANSI Fortran 77 allows at most one block data subprogram to be unnamed, therefore all block data program units were named.

3. Namelist statement: Namelist is a non-standard Fortran feature that allows annotated input/output lists to be made. This namelist was replaced by a list-directed read or write.

4. Hollerith data: Entities defined as hollerith data type were changed to the character data type, for example, the following hollerith data item 6HCONFAN would be changed to the character item 'CONFAN', i.e. the sequence of statements:

   INTEGER TEXT
   TEXT = 6HCONFAN

   would be changed to

   CHARACTER*6 TEXT
   TEXT = 'CONFAN'

5. Unavailable system routine: Subroutine OVERFL was an IBM 7094 system routine which was used to detect whether an arithmetic overflow has occurred. This routine was not available and was substituted with a dummy routine of the same name. Should an arithmetic overflow arise the Honeywell Multics operating system automatically detects it and stops the program with an appropriate error message.

6. Initializing entities in common blocks: The IBM 7094 sets all uninitialised values to zero, an operation which is required at the start of DYNGEN's numerical process. In this modification all numeric values in common blocks were initially set to zero using assignment statements, where each of these common blocks were equivalenced to a corresponding one-dimensional array with the same number of storage units. This array was the set to zero using a DO loop and thus setting all the elements of the corresponding common block to zero.
7. Double precision: A double precision version of DYGEN has been generated. This involved: using the statement IMPLICIT DOUBLE PRECISION (A-H, O-Z) at the head of each program unit; ensuring that all real constants in call to subroutines were changed to double precision constants; and specific intrinsic functions such as FLOAT, AMIN1, AMX1 and ALOG respectively were changed to their generic equivalents REAL, MIN, MAX and LOG respectively.

8. Interactive execution: A simple facility has been added to DYGEN to allow it to be executed interactively. Write and read statements have been added to prompt the user to supply the input file name and the name of the results file.
Appendix B

A, B, C and D Matrices of LIN

This appendix show the matrices A, B, C and D of the six operating points used to define the boundaries of the flight envelope (section 2.5).

THE A, B, C and D MATRICES OF LIN1

A Matrix

<table>
<thead>
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B Matrix

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C Matrix

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D Matrix

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THE A, B, C and D MATRICES OF LIN2

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C Matrix

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291
### The A, B, C and D Matrices of LIN3

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### The A, B, C and D Matrices of LIN4

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<td>0.3077240 0.2771490 -262.66300 -132.96700 2.4479200</td>
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### A Matrix

```
-12.507300 -2.0847400 0.6899340 -5.6338900 1.3458600 0.9517810
```

### B Matrix

```
B Matrix
0.  5703.0100
0.  2591.4700
61.928900 -999.30300
0.  -40.832800
31789.700  -591.51900
```

### C Matrix

```
C Matrix
3.110E-04  0.  0.  9606.2300  -0.0840982
0.  0.  0.  0.  4.9311900
-1.492E-04  -1.276E-04  0.0442247  0.  0.
0.  1.210E-04  -0.0082744  0.  0.
```

### D Matrix

```
D Matrix
0.  4293.4500
0.  0.
0.  0.1348420
0.  -0.2623740
```

### THE A, B, C and D MATRICES OF LINE 6

### A Matrix

```
-8.2373700 -4.2182500 1782.6000  1115.7000  23.421700
0.4719160 -4.2383300  730.33100  595.83500  16.529000
0.8431420  1.2348800 -384.51000 -241.58200  4.2392400
-0.0183596 -0.0059500  9.1477600 -82.167300  0.0040189
1.7028900 -8.8395300 -2966.9400  -2592.2200  -94.894700
```

### B Matrix

```
B Matrix
0.  7421.4300
0.  1954.8100
80.884800 -1481.8500
0.  6.8246400
47981.200  6433.0900
```

### C Matrix

```
C Matrix
2.596E-04  0.  0.  9026.0200  -0.0774988
0.  0.  0.  0.  4.9311800
-9.588E-05  -2.235E-04  0.0614118  0.  0.
0.  3.240E-04  -0.0614118  0.  0.
0.  3.240E-04  -0.0184226  0.  0.
```

### D Matrix

```
D Matrix
0.  3494.7700
0.  0.
0.  0.2668020
0.  -0.4831450
```

---

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Appendix C

The A Matrix Elements

This appendix show a computer listing of the formulas used to generate the elements of the A matrix.

\[
\begin{align*}
CNF &= \text{NF}/\text{XNLPS}D \\
T_{21} &= (T_2 + TX \cdot (CNF^2) - 48.0 \cdot (A8 - 2.9482558)) \\
CN &= NC/(XNHPDS \cdot DSQRT(T_{21}/T_2)) \\
T_3 &= (T_{21} + TX3 \cdot (CNC^2) - 68.0 \cdot (A8 - 2.9482558)) \\
T_4 &= (U_4/CVB) \\
T_{50} &= ST_4 \cdot T_4 \\
P_3 &= (1.05944 \cdot P_4) \\
P_{21} &= ((-XX + 0.0129774 \cdot T_{21}) - 0.0185376 \cdot P_3) \\
WF_{MAX} &= (261.01 \cdot CNF - 63.916) \\
PF_{MAX} &= (3.516739 \cdot CNF - 0.23561) \\
WAF &= (WF_{MAX} + 28.502 \cdot (1.0 - \exp(-2.313268 \cdot (PF_{MAX} - (P_{21}/P_2)))) \\
WC_{MAX} &= (137.54 - 457.987 \cdot CNC + 564.325 \cdot (CNC^2) - 188.113 \cdot (CNC^3)) \\
DW_{CMA}X &= (6.492 - 4.974 \cdot CNC) \\
PC_{N4AX} &= YY - 89.0484 \cdot CNC + 109.72 \cdot (CNC^2) - 36.57 \cdot (CNC^3) \\
W_{C} &= ((P_{21}/P_2)/DSQRT(T_{21}/T_2)) \cdot (WC_{MAX} + DW_{CMA}X \cdot (1.0 - \exp(-0.36 \cdot \text{PC}_{N4AX} - (P_3/P_{21})))) \\
RAFAN &= P_{21}/P_2 \\
RACOMP &= P_3/P_{21} \\
EFFIC &= 20.71175 \\
WA_3 &= (1.0 - \text{PCBL}_{HC}) \cdot WAC \\
WG_{50} &= SP_4 \cdot P_4/DSQRT(T_4) + 3.96992 \cdot P_7 \\
WG_4 &= WG_50 - PC_{BLHP} \cdot PC_{BLC} \cdot WAC \\
WG_{55} &= WG_50 + PC_{BLLP} \cdot PC_{BLC} \cdot WAC \\
T_{77} &= (T_{55} + ST_7 \cdot P_7)/SP_{77} \\
WG_7 &= (1121.784 \cdot A8 \cdot P_7)/DSQRT(T_7) \\
PCBLDU &= 1.0 - PC_{BLHP} - PC_{BLLP} \\
PS_8 &= 0.539782 \cdot P_7 \\
FGP &= CAPSF \cdot (PS_8 - P_2) \cdot A8 \\
V_8 &= DSQRT(ST_8 \cdot T_7 + 68558.365) \\
FGM &= CVMNOZ \cdot V_8 \cdot WG_7/G \\
FG &= FGP + FGM \\
ZC &= ((P_3/P_{21}) - 1.0) \cdot (PC_{MAX} - 1.0) \\
ZF &= ((P_{21}/P_2) - 1.0) \cdot (PF_{MAX} - 1.0) \\
DP_3(3) &= 1.05944 \\
DO 10 I = 1, 5 \\
DCNF(2) &= 1/XNLPSD \\
DT_{21}(I) &= 48 \cdot DA_8(I) \\
DT_{21}(2) &= 214.2733 \cdot 2.0 \cdot CNF \cdot DCNF(2) - 48 \cdot DA_8(2) \\
DCNC(I) &= (NC \cdot DT_{21}(I))/2 \cdot T_2 \cdot XNHPDS \cdot (T_{21}/T_2)^{**1.5} \\
DCNC(1) &= (DSQRT(T_{21}/T_2) - 0.5 \cdot NC \cdot (DT_{21}(1)/T_2) \cdot (T_{21}/T_2)^{**(-0.5)})/ (XNHPDS \cdot (T_{21}/T_2)) \\
DT_3(I) &= DT_{21}(I) + 743.2722 \cdot 2.0 \cdot CNC \cdot DCNC(I) - 68.0 \cdot DA_8(I) \\
DP_2(1) &= 0.0129774 \cdot DT_{21}(I) \\
DT_4(5) &= 1/CVB \\
DT_5(5) &= 0.727 \cdot DT_4(5) \\
IF (I.EQ.3) &= 0.0129774 \cdot DT_{21}(3) - 0.0185376 \cdot DP_3(3) \\
DF_{WFMX}(2) &= 261.01 \cdot DCNF(2) \\
DP_{FMX}(2) &= 3.516739 \cdot DCNF(2) \\
DWAF(I) &= (-28.502 \cdot 2.313268 \cdot DP_2(1)/P_2) \cdot \exp(-2.31368 \cdot (PF_{MAX} - (P_{21}/P_2))) \\
DWAF(2) &= DW_{FMX}(2) + 28.502 \cdot 2.313268 \cdot (DP_{FMX}(2) - (DP_{21}(2)/P_2)) \cdot \exp(-2.313268 \cdot (PF_{MAX} - (P_{21}/P_2))) \\
\end{align*}
\]
DWCMX(I)=-457.987*DCNC(I)+2.0*564.325*CNC*DCNC(I)-3*188.113*(CNC**2)
$*DCNC(I)

DDWMX(I)=-4.974*DCNC(l)

DPCMX(I)=-89.0484*DCNC(I)+2*109.72*CNC*DCNC(I)-3*36.57*(CNC**2)*DCNC(l)

WAC1=(P21/P2)/DSQRT(T2lfM)

WAC2=WCNLA, X+DWCMAX*(1.0-EXP(-0.36*(PCN4AX-(P3/P21))))

DWAC1(1)=(-(P21/P2)*0.5*((T2lfM)**(-0.5))*(DT21(I)fM)
+$+DSQRT('r2lfM)*(DP21(I)/P2))/(T21fr2)

DWAC2(I)=DWCMX(I)+DWCMAX*(-EXP(-0.36*(PCMAX-(P3/P21)))*

$*(-0.36*(DPCMX(I)+P3*(P21**(-2.0))*DP21(I))))+

$*(1.0-EXP(-0.36*(PCMAX-(P3/P21))))*DDWMX(I)

IF (I. EQ. 3) DWAC2(3)=DWCMAX*(-EXP(-0.36*(PCMAX-(P3/P21))))*+

$(-0.3*(DPCMX(3)-(P21*DP3(3)-P3*DP21(3))/(P21**2.0)))+

$*(1.0-EXP(-0.36*(PCMAX-(P3/P21))))*DDWMX(3) +DWCMX(3)

DWAC(I)=WAC1*DWAC2(1)+WAC2*DWAC1(1)

DWA3(1)=(1.0-PCBLC)*DWAC(l)

DWG50(3)=279.218/DSQRT(T4)

DWG50(4)=3.96992

DWG50(5)=(-0.5*279.21*P4/DSQRT(T4**3))*DT4(5)

DWG4(I)=DWG50(I)-PCBLHP*PCBLC*DWAC(I)

DT55(1)=-0.10458*(((CNC*(0.5)*DT21(I)*T50)/DSQRT(121*T50))+

$+DSQRT(T21*T50)*DCNC(I)

IF (I. EQ. 5) DT55(5)=0.8615*DT50(5)-0.5*0.10458*CNC*((T21*DT50(5)

+$+T50*DT21(5))/DSQRT(T21*T50))+(1.0-EXP(-0.36*(PCMAX-(P3/P21))))*+

$*(1.0-EXP(-0.36*(PCMAX-(P3/P21))))*DDWMX(5)+

$*(1.0-EXP(-0.36*(PCMAX-(P3/P21))))*DDWMX(5)

DDT(1)=T55(1)/SP77

DDT(4)=(T55(4)+414.582)/SP77

SK1=P7*A8

DSK1(I)=P7*DA8(4)

IF (I. EQ. 4) DSK1(4)=A8+P7*DA8(4)

DWG7(I)=-0.5*1121.784*A8*P7*DT7(I)/DSQRTM**3)+(1121.78*P7/

$DSQRTM))*DA8(I)

IF (I. EQ. 4) DWG7(4)=-0.5* 1 121.784*A8*P7*DT7(4)/DSQRTM**3)+

$*(1121.78*P7/DSQRT(T'7))*DA8(4)+A8*(1121.78/DSQRTM))

PP1=EXP(-2.31368*(PFMAX-(P21/P2)))

DPP1(I)=2.31368*(DP21(I)/P2)*PP1

IF (I. EQ. 2) DPP1(2)=2.31368*((DPFMX(2))-((DP21(2)/P22)))*PP1

PP2=EXP(-0.36*(PCMAX-(P3/P21)))

DPP2(I)=0.36*(DPCMX(I)+(P3*DP21(I)/P21**2))*PP2

IF (I. EQ. 3) DPP2(3)=-0.36*(DPCMX(3)-(P21*DP3(3)-P3*DP21(3))

$*(P21**2.0)))*PP2

FIP=-0.5*(P21/P2)*((T21/T2)**(-0.5))

DFIP(I)=-0.5*(T21/T2)**(-0.5))*((P21/I/P2)+0.5*(P21/P2)

$*(-0.5)*((T21/T2)**(-1.5))*((P21/I/T2)

$*(-P21)/(T21/T2)**(-0.5))*((T21/T2)**(-1.5))*((P21/I/T2)

$*(P21/(T21/T2)**(-2.0))*((P21/I/T2)

IF (I. EQ. 3) PK(3)=-0.36*(DPCMX(3)-(P21*DP3(3)-P3*DP21(3))

$(P21**2.0)

SSNC=91.189*AJ/(PMIHP*NC)

SSSC=-91.189*AJ/(PMILP*NF)

SSP4=(RA/VCOMB)*GAMSTR*T4

SSP7=(RA/GAMSTR/VAFBN)

10 CONTINUE
\[ SP7 = SS7P \times (WG4 - WA3 + WAF - WG7) \]
\[ SSU4 = (CBV \times RA \times T4) / (VCOMB \times P4) \]
\[ DNCDT(1) = SSNC \times (CPC \times WAC \times (DT21(1) - DT3(1)) + CPC \times (T21 - T3) \times DWAC(1)) \]
\[ + (CPC \times WAC \times (T21 - T3) + CPHT \times WG50 \times (T4 - T50)) \times (SSNC/NC) \]
\[ DNCDT(2) = SSNC \times (CPC \times WAC \times (DT21(2) - DT3(2)) + CPC \times (T21 - T3) \times DWAC(2)) \]
\[ DNCDT(3) = SSNC \times (CPC \times (T21 - T3) \times DWAC(3) + CPHT \times (T4 - T50) \times DWG50(3) + \]
\[ CPC \times WAC \times (DT21(3) - DT3(3))) \]
\[ DNCDT(4) = SSNC \times (CPHT \times (T4 - T50) \times DWG50(4) + CPC \times WAC \times (DT21(4) - DT3(4)) \]
\[ - CPC \times (T21 - T3) \times DWAC(4)) \]
\[ DNCDT(5) = SSNC \times (CPHT \times (T4 - T50) \times DWG50(5) + CPHT \times WG50 \]
\[ \times (DT4(5) - DT50(5)) + CPC \times WAC \times (DT21(5) - DT3(5)) + CPC \times (T21 - T3) \times DWAC(5)) \]

\[ DO \ 40 \ I = 1, 5 \]
\[ DNFDT(I) = SSNF \times (-CPF \times WAF \times DT21(I) + CPF \times (T2 - T21) \times DWAF(I) - CPR \times WAF \times DT21(I)) + \]
\[ CPF \times WAF \times (T2 - T21) \times DWA3(I) - CPR \times WAF \times DT21(I) + \]
\[ CPA \times WAF \times DT21(I) + CPR \times WAF \times DT21(I) \]
\[ DP4DT(1) = SSP4 \times (DWA3(I) + DWFB(I) - DWG4(I)) \]
\[ DP7DT(1) = SSP7 \times (DWG4(I) - DWA3(I) + DWAF(I) - DWG7(I)) \]
\[ DU4DT(1) = SSU4 \times (T4 \times (DWG4(I) - DWFB(I) - DWA3(I)) + \]
\[ GAMSTR \times (T3 \times DWA3(I) + WA3 \times DT3(I) - T4 \times DWG4(I) + \]
\[ (1.0 + EFFIC) \times DWFB(I) \times T4)) \]

\[ DO 40 I = 1, 5 \]
\[ DNFDT(2) = SSNF \times (-CPF \times WAF \times DT21(2) + CPF \times (T2 - T21) \times DWAF(2) - CPR \times WAF \times DT21(2)) + \]
\[ CPF \times WAF \times (T2 - T21) \times DWA3(2) - CPR \times WAF \times DT21(2) + \]
\[ CPA \times WAF \times DT21(2) + CPR \times WAF \times DT21(2) \]
\[ DP4DT(2) = SSP4 \times (DWA3(2) + DWFB(2) - DWG4(2)) + (WA3 + WFB - WG4) \]
\[ DP7DT(2) = SSP7 \times (DWG4(2) - DWA3(2) + DWAF(2) - DWG7(2)) + \]
\[ DU4DT(2) = SSU4 \times (T4 \times (DWG4(2) - DWFB(2) - DWA3(2)) + \]
\[ GAMSTR \times (T3 \times DWA3(2) + WA3 \times DT3(2) - T4 \times DWG4(2) - \]
\[ WG4 \times DT4(2) + (1.0 + EFFIC) \times DWFB(2) \times T4)) \]

\[ DO 99 I = 1, 5 \]
\[ AMAT(1, 1) = DNCDT(I) \]
\[ AMAT(2, 1) = DNFDT(I) \]
\[ AMAT(3, 1) = DP4DT(I) \]
\[ AMAT(4, 1) = DP7DT(I) \]
\[ AMAT(5, 1) = DU4DT(I) \]

\[ 99 \ CONTINUE \]

Where AMAT represents A matrix
Appendix D

Analytical Partial Derivatives with Respect to the Control Variables

This appendix show a computer listing of the analytical work leading to the partial derivatives of the variables NC, NF, P4, P7 and U4 with respect to the control variables WFB and A8.

CNF=NF/XNLPDS
T21=(T2+TX*(CNF**2)-48.0*(A8-2.9482558))
CNF=NC/(XNHPDS*DSQRT(T21/T2))
T3=(T21+TX3*(CNF**2)-68.0*(A8-2.9482558))
T4=(U4/CVB)
T50=ST4*T4
P3=(1.05944*P4)
P21=((XX+0.0129774*T21)-0.0185376*P3)
WFN1AX=(261.01*CNF-63.916)
PFMAX=(26.1.01*CNF-63.916)
PFMAX=(3.516739*CNF-0.23561)
WAF=(WFMAX+28.502*(1.0-EXP(-313268*(PFMAX-(P21/P2)))))
WCMAX=(137.54-457.987*CNC+564.325*(CNC**2)-188.113*(CNC**3))
DWCMAX=(6.492-4.974*CNC)
PCMAX=(YY-89.0484*CNC+109.72*(CNC**2)-36.57*(CNC**3))
WAC=((P21/P2)/DSQRT(T21/T2))*(WCMAX+DWCMAX*(1.0-EXP(-0.36*
$(PCMAX-(P3/P21)))))
RAFAN=P21/P2
RACOMP=P3/P2
EFFIC=20.71175
WA3=(1.0-PCBLC)*WAC
WG50=SP4*P4/DSQRT(T4)+3.96992*P7
WG4=WG50-PCBLHP*PCBLC*WAC
WG5=WG50+PCBLLP*PCBLC*WAC
T50=106.002+ST5*T50-0.10458*CNC*DSQRT(T21*T50)
T7=(T55+ST7*P7)/SP77
WG7=(1121.784*A8*P7)/DSQRTM)
PCBLDU=1.0-PCBLHP-PCBLLP
PS8=0.539782*P7
FGP=CAPSF*(PS8-P2)*A8
V8=DSQRT(ST8*T7+68558.365)
FGM=CVMNOZ*V8*WG7/G
FG=FGP+FGM
ZC=((P3/P21)-1.0)/(PCMAX-1.0)
ZF=((P21/P2)-1.0)/(PFMAX-1.0)

DO 10 I=1,2

DT21(I)= -48.00
DCNC(I)=-(NC*DT21(I))/(2*T2*XNHPDS*(T21/T2)**1.5)
DT3(I)=DT21(I)+743.2722*2.0*CNC*DCNC(I)-68.0
DP21(I)=0.0129774*DT21(I)
DT50(I)=0.727*DT4(I)
DWFMX(I)=261.01*DCNF(I)
DPFMX(I)=3.516739*DCNF(I)
DWAF(I)=(-28.502*2.313268*DP21(I)/P2)*EXP(-2.31368*(PFMAX
*$-(P21/P2)))
DWCMX(I)=457.987*DCNC(I)+2.0*564.325*CNC*DCNC(I)-3*188.113
*$*(CNC**2)*DCNC(I)
DDWMX(I)=4.974*DCNC(I)
DPCMX(I)=89.0484*DCNC(I)+2*109.72*CNC*DCNC(I)-3*36.57*(CNC**2)
*$*DCNC(I)
WAC1=(P21/P2)/DSQRT(T21/T2)
WAC2=WCMAX+DWCMAX*(1.0-EXP(-0.36*(PCMAX-(P3/P21))))

297
DWAC1(I) = (-P21/P2)*0.5*((T21fr2)**(-0.5))*(DT21(I)*fM)
$+DSQRT(T21/r2)*(DP21(l)/P2))/Ml/T2)

DWAC2(I) = DWCMX(I) + DWCMAX*(-EXP(-0.36*(PCMAX-(P3/P21)))*
$(-0.36*(PCMX(I)+P3*(P21**(-2.0)))*DP21(I)))+
$(1.0-EXP(-0.36*(PCMAX-(P3/P21))))*DDWMX(I)

DWAC(I) = WAC1*DWAC2(I) + WAC2*DWAC1(I)

DWA3(I) = (1.0-PCBLCC)*DWAC(I)

DWG4(I) = DWG50(I)-PCBLLP*PCBLC*DWAC(I)

DWG55(I) = DWG50(I)+PCBLLP*PCBLC*DWAC(I)

DT55(I) = -0.10458*((((CNC*(0.5))DT21(I)*T50)/DSQRT(T21*T50))+
$DSQRT(T21*T50)*DCNC(I))

DT7(I) = DT55(I)/SP77

DWG7(I) = (1121.784*P7)/SQRTM)

SSN4 = 91.1890*AJ/(PMIHP*NC)

SSNF = (91.1890*AJ/(PMIILP*NF))

SSP4 = (RA/VCOMB)*GAMSTR*T4

SSP7 = (RA*GAMSTR/VAFBN)

SP7 = SSP7*(WG4-WA3+WAF-WG7)

SSU4 = (CVB*RA*T4)/(VCOMB*P4)

SSU4 = (CVB*RA*T4)/(VCOMB*P4)

DNCDT(I) = SSNC*(CPC*WAC*(DT21(I)-DT3(I))+CPC*(T21-T3)*DWAC(I))

DNFTD(I) = SSNF*(CPF*(T2-T21)*DWAF(I)-CPF*WAF*DT21(I)+CPLT
$*(T50-T55)*DWG55(I)-CPLT*WG55*DT55(I))

DPP4DT(I) = SSP4*(DWA3(I)+1-DWG4(I))

DP4DT(I) = SSP4*(DWA3(I)+1-DWG4(I))

DSP7(I) = SSP7*(DWG4(I)-DA3(I)+DAF(I)-DWG7(I))

DP7DT(I) = SSP7*T7*(DWG4(I)-DA3(I)+DAF(I)-DWG7(I)+SP7*DT7(I)

DU4DT(I) = SSU4*(T4*(DWG4(I)-DA3(I)+GAMSTR*(T3*DA3(I)+WA3*
$DT3(I) - T4*DWG4(I))

DU4DT(I) = SSU4*(T4*(DWG4(I)-1 - DA3(I))

+$ GAMSTR*(T3*DA3(I) - T4*DWG4(I) + T4*(1.0 + EFFIC)))

10 CONTINUE

DO 88 I = 1,2

BMAT(1,1) = DNCDT(I)

BMAT(2,1) = DNFTD(I)

BMAT(3,1) = DP4DT(I)

BMAT(4,1) = DP7DT(I)

BMAT(5,1) = DU4DT(I)

88 CONTINUE

Where BMAT (I,J) represent DXU(J,I) of section (7.2).
Appendix E

Jacobian Element Evaluation

The following equations describe the formation of the elements of the Jacobian matrix from equations (7.99) to (7.102) for values of j=1 to j=5. The remarks in section (7.2) refer here.

\[ J_R(m+5,j) = -(Q(m,j)+AFF(m,j)) \]

990 CONTINUE

DO 500 m=1,5
DO 500 j=1,5

\[ AFF(m,n) = \psi_1 \cdot ANCDT(m,j) + \psi_2 \cdot ANFDT(m,j) + \psi_3 \cdot AP4DT(m,j) + \psi_4 \cdot AP7DT(m,j) + \psi_5 \cdot AU4DT(m,j) \]

500 CONTINUE

How ANCDT(m,j), ANFDT(m,j), AP4DT(m,j), AP7DT(m,j) and AU4DT(m,j) are constructed is shown below:

\[ AT21(2,2) = 214.273 \times 2 \times (DCNF(2))^{**2} \]
\[ AP21(2,2) = 0.0129774 \times AT21(2,2) \]
\[ APPI(2,2) = -2.31368 \times ((DPFMX(2)-((DP21(2)/P2))) \times DPP1(2)-PP1 \times AP21(2,2)) \]

DO 100 I=1,5
DO 100 J=1,5

\[ ACNC(I,J) = ((2 \times T2 \times XNHPSD \times (T21/T2)^{**1.5}) \times (-NC \times AT21(I,J)) \times (-NC \times DT21(I)) \times 2 \times T2 \times \]
\[ XNHPDS \times 1.5 \times (T21/T2)^{0.5} \\
\times (DT21(I)/T2)/((2 \times T2 \times XNHPDS \times \\
(T21/T2)^{1.5})^2) \]

\[ ACNC(I,1) = ((2 \times T2 \times XNHPDS \times (T21/T2)^{1.5})^2) \]

\[ ACNC(1,J) = (XNHPDS \times (T21/T2)^{0.5} \times ((T21/T2)^{-0.5}) \times \\
(DT21(I)/T2)-0.5 \times NC \times ((DT21(I)/T2)^{-0.5}) \times \\
((T21/T2)^{0.5} \times (DT21(J)/T2)+((T21/T2)^{0.5} \times \\
(721/T2)^{-0.5} \times AT21(1,J)/2)) \]

\[ ADWMX(I,J) = -4.974 \times ACNC(I,J) \]
APCMX(I, J) = -89.0484*ACNC(I, J)+2*109.72*(CNC*ACNC(I, J)+DCNC(I)*DCNC(J))-3*36.57*((CNC**2)*ACNC(I, J)+2.0*CNC*DCNC(J)*DCNC(I))

APIP(I, J) = -(DT21(I)*DFIP(J)/T2)-FIP*(AT21(I, J)/T2)+((P21(I)/P2)*0.5*(DT21(J)/T2)**(-0.5)+SQRT(T21/J)*(AP21(I, J)/T2)

AWAC1(I, J) = ((T21/T2)*APIP(I, J)-PI*(DT21(J)/T2))/((T21/T2)**2)

APK(I, J) = -0.36*(APCMX(I, J)+(P21**(-2.0))*(P3*AP21(I, J)+DP21(I)*DP3(J))+(DP21(I)*P3*(-2.0*(P21**(-3.0))*DP21(J)))

APK(3, J) = -0.36*((APCMX(3, J)-(DP3(3)*DP21(J)-P3*AP21(3, J)-DP21(3)*DP3(J)))*(P21**2)-(DP21(J)/(P21**4))

AWAC2(I, J) = AWCMX(I, J)-DWCMX*(PP2*APK(I, J)+PK(I)*DPP2(J))+PP2*(PK(I)*DDWMX(J)+(1-PP2)*ADWMX(I, J)-DDWMX(I)*DPP2(J)

AWAC(I, J) = WAC1*AWAC2(I, J)+DWAC2(I)*DWAC1(J)+WAC2*AWAC1(I, J)+DWAC1(I)*DWAC2(J)

AWA3(I, J) = (1.0-PCBLHC)*AWAC(I, J)

AWG50(3, 5) = -279.218*0.5*((T4**(-1.5)))*DT4(5)

AWG50(5, 3) = (-0.5*279.21/SQRT(T4**3))*DT4(5)

AWG50(5, 5) = (-0.5*279.21*DT4(5)*P4*(-1.5)*(T4**(-2.5)))*DT4(5)

AWG4(I, J) = AWG50(I, J)-PCBLH*PCBLC*AWAC(I, J)
\[ \text{AWG55}(I,J) = \text{AWG50}(I,J) + \text{PCBLLP} \cdot \text{PCBL}\cdot \text{AWAC}(I,J) \]

\[ \text{AT55}(I,J) = -0.10458 \times \left( ((\sqrt{T21 \cdot T50} \cdot 0.5 \cdot T50 \cdot (CNC \cdot \text{AT21}(1, J) + DT71(I) \cdot DCNC(J)) - CNC \cdot 0.5 \cdot DT21(I) \cdot T50 \cdot 0.5 \cdot T50 \cdot DT21(J) \cdot (T50 \cdot T21) ** (-0.5)) / (T21 \cdot T50) \right) + \sqrt{n} \cdot T50 \cdot ACNC(I, J) + DCNC(I) \cdot 0.5 \cdot T50 \cdot DT21(J) \cdot (T21 \cdot T50)** (-0.5) \]

\[ \text{AT55}(I,5) = -0.10458 \times \left( ((\sqrt{T21 \cdot T50} \cdot 0.5 \cdot DT21(1) \cdot (T50 \cdot DCNC(5) + CNC \cdot DT50(5)) + CNC \cdot T50 \cdot AT21(5,5)) - CNC \cdot 0.5 \cdot DT21(I) \cdot T50 \cdot 0.5 \cdot (T50 \cdot DI21(5) + T21 \cdot DT50(5)) \cdot (T50 \cdot T21)** (-0.5)) / (T21 \cdot T50) \right) + \sqrt{T21 \cdot T50} \cdot ACNC(5, J) + DCNC(I) \cdot 0.5 \cdot (T50 \cdot DT21(5) + T21 \cdot DT50(5)) \cdot (T21 \cdot T50)** (-0.5) \]

\[ \text{AT55}(5,J) = -0.10458 \times \left( ((\sqrt{T21 \cdot T50}) \cdot 0.5 \cdot DT21(5) \cdot (T50 \cdot DCNC(5) + CNC \cdot DT50(5)) + CNC \cdot T50 \cdot AT21(5,J)) - CNC \cdot 0.5 \cdot DT21(I) \cdot T50 \cdot 0.5 \cdot (T50 \cdot DI21(5) + T21 \cdot DT50(5)) \cdot (T50 \cdot T21)** (-0.5)) / (T21 \cdot T50) \right) + \sqrt{T21 \cdot T50} \cdot ACNC(5, J) + DCNC(J) \cdot 0.5 \cdot (T50 \cdot DT21(5) + T21 \cdot DT50(5)) \cdot (T21 \cdot T50)** (-0.5) \]

\[ \text{AT55}(5,5) = -0.5 \cdot 0.10458 \cdot CNC \cdot ((\sqrt{T21 \cdot T50}) \cdot DT50(5) \cdot DT21(5) \cdot DT50(5) + T50 \cdot AT21(5,5) - (T21 \cdot DT50(5) + T50 \cdot DT21(5)) \cdot 0.5 \cdot (DT21(5) \cdot T50 + T21 \cdot DT50(5)) \cdot (T21 \cdot T50)** (-0.5)) / (T21 \cdot T50) - 0.5 \cdot 0.10458 \cdot ((T21 \cdot DT50(5) + T50 \cdot DT21(5)) / \sqrt{T21 \cdot T50}) \cdot DCNC(5) - 0.10458 \cdot DCNC(5) \cdot 0.5 \cdot (T50 \cdot DT21(5) + T21 \cdot DT50(5)) \cdot (T21 \cdot T50)** (-0.5) - 0.10458 \cdot \sqrt{T21 \cdot T50} \cdot ACNC(5, 5) \]

\[ \text{AT7}(I,J) = \text{AT55}(I,J) / \text{SP77} \]

302
AWG7(I, J) = -0.5*1121.784*(SK1*((AT7(I, J)*SQRT(T7**3)-DT7(I)\*1.5)*DT7(J)*(T7**(0.5)))/T7**3))+(DT7(I)/SQRT(T7**3)))*DSK1(J))-(0.5*1121.784*P7*DA8(I))*T7**(0.5)*DT7(J)+(1121.784*P7/SQRT(T7))*AA8(I, J)

AWG7(I,4) = -0.5*1121.784*(SK1*((AT7(I,4)*SQRT(T7**3)-DT7(4)*T7**3)/(T7**(0.5)))/T7**3))+(DT7(4)/SQRT(T7**3)))*DSK1(I))-(0.5*1121.784*P7*DA8(4))*(T7**(-1.5)*DT7(I)+1121.784*((SQRT(T7)*DA8(I)-A8*0.5*DT7(I)*(M**(-0.5))))(M)+(1121.784*P7/SQRT(T7))*AA8(1,4)

AWG7(4, J) = -0.5*1121.784*(SK1*((AT7(4, J)*SQRT(T7**3)-DT7(J)*T7**3)/(T7**(0.5)))/T7**3))+(DT7(J)/SQRT(T7**3)))*DSK1(4))-(0.5*1121.784*P7*DA8(4))*(T7**(-1.5)*DT7(4)+(1121.784*DA8(J)/SQRT(T7)))+(1121.784*P7/SQRT(T7))*AA8(4, J)

AWG7(4,4) = -0.5*1121.784*(SK1*((AT7(4,4)*SQRT(T7**3)-DT7(4)*T7**3)/(T7**(0.5)))/T7**3))+(DT7(4)/SQRT(T7**3)))*DSK1(4))-(0.5*1121.784*P7*DA8(4))*(T7**(-1.5)*DT7(4)+(1121.784*DA8(4)/SQRT(T7)))+(1121.784*P7/SQRT(T7))*AA8(4,4)

DPP10(I) = CPC*(WAC*(AT21(I, I)-AT3(I, I))+((DT21(I)-DT3(I))*DWAC(I)))+CPC*((T21-T3)*AWAC(1, I)+((DT21(I)-DT3(I))*DWAC(I))

ANCDT(I, J) = SSNC*CPC*(WAC*(AT21(I, J)-AT3(I, J))
\begin{align*}
\text{ANCDT}(1, J) &= SSNC*DP10(J)-(DNCDT(J)/NC) \\
\text{ANCDT}(1, 1) &= SSNC*DP10(1)+PP10*DSSNC(1) \\
&\quad - (NC*DNCDT(1)-NCDOT)/(NC**2) \\
\text{ANCDT}(2, J) &= SSNC*CPC*(WAC*(AT21(2, J)-AT3(2, J)) \\
&\quad + (DT21(2)-DT3(2))*DWAC(J)+(T21-T3) \\
&\quad *AWAC(2, J)+DWAC(2)*(DT21(J)-DT3(J))) \\
\text{ANCDT}(2, 1) &= SSNC*CPC*(WAC*(AT21(2,1)-AT3(2,1)) \\
&\quad + (DT21(2)-DT3(2))*DWAC(1)+(T21-T3) \\
&\quad *AWAC(2,1)+DWAC(2)*(DT21(1)-DT3(1))) \\
&\quad + (DNCDT(2)/SSNC)*DSSNC(1) \\
\text{ANCDT}(3, J) &= (DNCDT(3)/SSNC)*DSSNC(J)+SSNC \\
&\quad *(CPC*((T21-T3)*AWAC(3, J)+DWAC(3) \\
&\quad *(DT21(3)-DT3(3)))+CPC*((DT21(3)-DT3(3)) \\
&\quad *DWAC(J)+WAC*(AT21(3, J)-AT3(3, J)))) \\
&\quad +SSNC*CPHT*((T4-T50)*AWG50(3, J) \\
&\quad +DWG50(3)*(DT4(J)-DT50(J))) \\
\text{ANCDT}(4, J) &= (DNCDT(4)/SSNC)*DSSNC(J) \\
&\quad +SSNC*(CPC*((DT21(4)-DT3(4)) \\
&\quad *DWAC(J)+WAC*(AT21(4, J)-AT3(4, J))) \\
&\quad +CPC*((T21-T3)*AWAC(4, J)+DWAC(4)*(DT21(J) \\
&\quad -DT3(J)))+SSNC*CPHT*((T4-T50)*AWG50(4, J) \\
&\quad +DWG50(4)*(DT4(J)-DT50(J))) \\
\text{ANCDT}(5, J) &= SSNC*(CPC*(WAC*(AT21(5, J) \\
&\quad -AT3(5,J))+(DT21(5)-DT3(5)) \\
&\quad *DWAC(J)+CPC*((T21-T3)*AWAC(5, J) \\
&\quad +DWAC(5)*(DT21(J)-DT3(J)))) \\
&\quad +SSNC*CPHT*((T4-T50)*AWG50(5, J) \\
&\quad +DWG50(5)*(DT4(J)-DT50(J))) \\
&\quad +(DT4(5)-DT3(5))*DWG50(5) \\
&\quad +(DNCDT(5)/SSNC)*DSSNC(J) \\
\text{ANFDT}(I, J) &= SSNF*(CPF*(-DT21(J)*DWAFG)
\end{align*}
ANFDT(5,J) = SSNF*(CPF*((T2-T21)*AWAF(5,J)
-DWAF(5)*DT21(J)
-WAF*AT21(5,J)-DT21(5)*DWAF(5))
+CPLT*((T50-T55)*AWG55(5,J)
+DWG55(5)*(DT50(J)-DT55(J))
-DT55(5))*DWG55(J)-WG55*AT55(J)))
+(DNFDT(5)/SSNF)*DSSNF(J)

AP4DT(I,J) = SSP4*(AWA3(I,J)+AWFB(I,J)-AWG4(I,J))
+(RA/VCOMB)*GAMSTR*DT4(J)
*(DWA3(I)+DWFB(I)-DWG4(I))

AP7DT(I,J) = SSP7*(T7*(AWG4(I,J)-AWA3(I,J)+AWAF(I,J)
-AWG7(I,J))+DT7(J)*(DSP7(I)/SSP7))
+SP7*AT7(I,J)+DT7(I)*DSP7(J)

AU4DT(I,J) = SSU4*(T4*(AWG4(I,J)-AWFB(I,J)-AWA3(I,J))+(DWG4(I)-DWFB(I)-DWA3(I))*DT4(J)+
GAMSTR*(T3*AWA3(I,J)+DWA3(I)*DT3(J)
+WA3*AT3(I,J)+DT3(I)*DWA3(J)-T4*AWG4(I,J)
-DW4(I)*DT4(J)-DWG4(I)*DT4(J)+(1.0+EFFIC)*
(AWFB(I,J)*T4+DT4(J)*DWFB(I))))

AU4DT(I,3) = SSU4*(T4*(AWG4(I,3)-AWFB(I,3)-AWA3(I,3))+(DWG4(I)-DWFB(I)-DWA3(I))*DT4(3)+
GAMSTR*(T3*AWA3(I,3)+DWA3(I)*DT3(3)
+WA3*AT3(I,3)+DT3(I)*DWA3(3)
-T4*AWG4(I,3)-DW4(I)*DT4(3)-DWG4(I)
*DT4(3)+(1.0+EFFIC)*(AWFB(I,3)*T4
+DT4(3)*DWFB(I))))+(DU4DT(I)/SSU4)*
*DSSU4(3)

AU4DT(I,5) = SSU4*(T4*(AWG4(I,5)-AWFB(I,5)-AWA3(I,5))+(DWG4(I)-DWFB(I)-DWA3(I))*DT4(5)+
GAMSTR*(T3*AWA3(I,5)+DWA3(I)*DT3(5)+WA3
*AT3(I,5)+DT3(I)*DWA3(5)-T4*AWG4(I,5)

305
\[-DWG4(I)*DT4(5)-DWG4(I)*DT4(5)+(1.0+EFFIC)*
(AWFB(I,5)*T4+DT4(5)*DWFB(I)))+\[(DU4DT(I)
/SSU4)*DSSU4(5)\]

\[AU4DT(3,J) = SSU4*(T4*(AWG4(3,J)-AWFB(3,J)-AWA3(3,J))+
(DWG4(3)-DWFB(3)-DWA3(3))*DT4(J)+
GAMSTR*(T3*AWA3(3,J)+DWA3(3)*DT3(J)
+WA3*AT3(3,J)+DT3(J)*DWA3(J)
-T4*AWG4(3,J)-DWG4(3)*DT4(J)-DWG4(3)
*DT4(J)+(1.0+EFFIC)*(AWFB(3,J)*T4
+DT4(J)*DWFB(3)))+(DU4DT(3)+
(U4DOT/P4))/SSU4*DSSU4(J)-(DU4DT(J)/P4)\]

\[AU4DT(3,3) = SSU4*(T4*(AWG4(3,3)-AWFB(3,3)-AWA3(3,3))+
(DWG4(3)-DWFB(3)-DWA3(3))*DT4(J)+
GAMSTR*(T3*AWA3(3,3)+DWA3(3)*DT3(3)
+WA3*AT3(3,3)+DT3(3)*DWA3(3)
-T4*AWG4(3,3)-DWG4(3)*DT4(3)-DWG4(3)
*DT4(3)+(1.0+EFFIC)*(AWFB(3,3)*T4
+DT4(3)*DWFB(3)))+(DU4DT(3)+
(U4DOT/P4)/SSU4)*DSSU4(3)-(P4*DU4DT(3)
-U4DOT)/P4**2)\]

\[AU4DT(5,J) = SSU4*(T4*(AWG4(5,J)-AWFB(5,J)-AWA3(5,J))+
(DWG4(5)-DWFB(5)-DWA3(5))*DT4(J)
+(DWG4(J)-DWFB(J)-DWA3(J))*DT4(5)
+GAMSTR*(T3*AWA3(5,J)+DWA3(5)*DT3(J)
+WA3*AT3(5,J)+DT3(5)*DWA3(J)
-T4*AWG4(5,J)-DWG4(5)*DT4(J)-DWG4(5)
*DT4(J)+(1.0+EFFIC)*(AWFB(5,J)*T4
+DT4(J)*DWFB(5)+DT4(5)*DWFB(J)))+\[(DU4DT(5)-(U4DOT*DT4(5)/T4))/SSU4
*DSSU4(J)+DT4(5)*((T4*DU4DT(J)-U4DOT
*DT4(J))/T4**2)\]

\[DSSNF(2) = -SSNF/NF\]
\[
ANFDT(2,J) = \text{SSNF}*(-\text{CPF}*\text{DT21}(2)*\text{DWAF}(J) +\text{CPF}*(\text{T2-T21})*\text{AWAF}(2,J) +\text{CPF}*(\text{DT21}(2))*((-\text{DT21}(J)))-\text{CPF}*(\text{WAF})
\]
\[
*\text{AT21}(2,J)-\text{CPLT}*(\text{WG55})*\text{AT55}(2,J) +\text{DT55}(2)*\text{DWG55}(J)+\text{CPLT}*((\text{T50-}T\text{55}))
\]
\[
*\text{AWG55}(2,J)+\text{DWG55}(2)*((\text{DT50}(J)-\text{DT55}(J)))) -(\text{DNFDT}(J)/\text{NF})
\]
\[
ANFDT(1,2) = \text{SSNF}*(\text{CPF}*(-\text{DT21}(2)*\text{DWAF}(I) +\text{AWAF}(1,2)*(\text{T2-T21})-\text{DT21}(I)
\]
\[
*\text{WAF})*\text{AT21}(I,2)) +\text{CPLT}*((\text{T50-}T\text{55})*\text{AWG55}(I,2)
\]
\[
+\text{DWG55}(I)*((\text{T50-}T\text{55})*\text{AWG55}(I,2)
\]
\[
+\text{DWG55}(2)*(\text{DT50}(2)-\text{DT55}(2)) -\text{AT55}(I,2)*\text{AWG55}(2,2)
\]
\[
+\text{DWG55}(2)*((\text{DNFDT}(I)+(\text{NFDOT}/\text{NF})))/\text{SSNF}) +D\text{SSNF}(2)-(\text{NF})*\text{DNFDT}(2)(\text{NF})/\text{NF}**2
\]
\[
ANFDT(2,2) = \text{SSNF}*(-\text{CPF})*\text{DT21}(2)*\text{DWAF}(2)
\]
\[
+\text{CPF}*(\text{T2-T21})*\text{AWAF}(2,2)
\]
\[
+\text{CPF}*(\text{DT21}(2))-\text{CPF}*(\text{WAF})*\text{AT21}(2,2)
\]
\[
-C\text{PLT}*(\text{WG55})*\text{AT55}(2,2)+\text{DT55}(2)*\text{DWG55}(2)) +\text{CPLT}*((\text{T50-}T\text{55})*\text{AWG55}(2,2)
\]
\[
+\text{DWG55}(2)*((\text{DT50}(2)-\text{DT55}(2)))+((\text{DNFDT}(2)+(\text{NFDOT}/\text{NF})))/\text{SSNF})
\]
\[
*D\text{SSNF}(2)-(\text{NF})*\text{DNFDT}(2)-\text{NF}**2\]
\[
AP4DT(5,J) = \text{SSP4}*(\text{AWA3}(5,J)+\text{AWFB}(5,J)
\]
\[
-\text{AWG4}(5,J))/((\text{DWA3}(5)
\]
\[
+\text{DWFB}(5)-\text{DWG4}(5))*(\text{RA}/\text{VCOMB})
\]
\[
*\text{GAMSTR}*(\text{DT4}(J)/((\text{RA})/\text{VCOMB})
\]
\[
*\text{GAMSTR}*(\text{DT4}(5))*(\text{DWA3}(J)
\]
\[
+\text{DWFB}(J)-\text{DWG4}(J))
\]

100 CONTINUE

The other elements for the values of j=6 to j=10 were obtained as follows:

DO 200 j=6,10

\[
J_R(6,j) = -\text{DNCDT}(j-5)
\]
\[
J_R(7,j) = -\text{DNFDT}(j-5)
\]
\[ J_R(8,j) = -DP4\text{DT}(j-5) \]
\[ J_R(9,j) = -DP7\text{DT}(j-5) \]
\[ J_R(10,j) = -DU4\text{DT}(j-5) \]

200 CONTINUE

There is, however, an extra observation to be added to those already made in section (7.2) viz:

\[ \frac{\partial D\text{VAR}(i)}{\partial X_j} = A\text{VAR}(i,j) \]

where the character \( A \) denotes \( \partial/\partial X_j \). If \( j \) has a fixed value, then the term will mean the partial derivative of \( \text{VAR}(i) \) with respect to the independent variable which has the same index value. For example

\[ AT21(2,2) = \frac{\partial (\partial T21)}{\partial X_2} = \frac{\partial D\text{T21}(2)}{\partial X_2} = \frac{\partial D\text{T21}(2)}{\partial NF} \]
\[ ACNC(3,1) = \frac{\partial (\partial \text{CNC})}{\partial X_3} = \frac{\partial D\text{CNC}(3)}{\partial X_1} = \frac{\partial D\text{CNC}(3)}{\partial NC} \]
\[ ACNC(4,3) = \frac{\partial (\partial \text{CNC})}{\partial X_4} = \frac{\partial D\text{CNC}(4)}{\partial X_3} = \frac{\partial D\text{CNC}(4)}{\partial P4} \]
\[ AT55(1,5) = \frac{\partial (\partial T55)}{\partial X_5} = \frac{\partial D\text{T55}(1)}{\partial X_5} = \frac{\partial D\text{T55}(1)}{\partial U4} \]
\[ ANCDT(5,4) = \frac{\partial (\partial \text{CDT})}{\partial X_5} = \frac{\partial D\text{CDT}(4)}{\partial X_4} = \frac{\partial D\text{CDT}(4)}{\partial P7} \]

and so on.