Unit roots and double smooth transitions

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Unit Roots and Double Smooth Transitions

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Abstract

Techniques for testing the null hypothesis of difference stationarity against stationarity around some deterministic function have received much attention. In particular, unit root tests where the alternative is stationarity around a smooth transition in linear trend have recently been proposed to permit the possibility of non-instantaneous structural change. In this paper we extend such an approach to admit more than one structural change, allowing the model under the alternative hypothesis to be stationary about two smooth transitions in linear trend. Tests involving this added generality are developed and their properties investigated; application of the tests to two interesting time series highlights the potential benefits of this double transition extension.

Keywords. Unit root tests; non-linear trend; structural change.
1. INTRODUCTION

The issue of characterising a time series process as (trend) stationary or difference stationary has received much attention in the literature, and consequently, following the seminal work of Dickey and Fuller (1979), many unit root tests have been developed. In a recent paper, Leybourne, Newbold and Vougas (1998) proposed a set of unit root tests where the process under the alternative hypothesis is stationary around a smooth transition in linear trend. This generalisation of the standard Dickey-Fuller procedure comprises an important extension to unit root testing on two grounds. First, the possibility of structural change in the trend function under the alternative is a likely phenomenon, and is an issue that has been extensively studied by, for example, Perron (1989, 1990) and Zivot and Andrews (1992). Secondly, it is intuitively appealing to permit such potential structural shifts to occur gradually over time rather than the instantaneous breaks assumed in the aforementioned papers. The Leybourne-Newbold-Vougas tests make use of the logistic smooth transition function, following work by Bacon and Watts (1971), Maddala (1977), Granger and Teräsvirta (1993) and Lin and Teräsvirta (1994), allowing the speed and midpoint of the transition to be determined endogenously.

In this paper, we consider a further extension to these unit root tests in terms of the specification of the alternative hypothesis. It is highly plausible that more than one structural change may have occurred during the observation period of the time series being investigated. We therefore consider tests where the unit root null is tested against an alternative of stationarity around two smooth transitions in linear trend. Section 2 presents such unit root tests, their simulated critical values, and power comparisons with their single transition counterparts. In Section 3 we conduct empirical applications using average global temperature data and the Nelson and Plosser (1982) US consumer price series which illustrate the value of the double transition extension. The paper is concluded in Section 4.

2. UNIT ROOT TESTS

Following the precedent of Leybourne, Newbold and Vougas (1998), we consider three models for the alternative hypothesis against which the unit root null could be tested. Each model represents a stationary process around two smooth transitions in linear trend; the differences are in the order of the deterministics: Model A contains no trend and involves transitions in mean only, Model B has transitions in intercept only but permits a fixed trend...
component, Model C allows most generality with transitions in both intercept and trend:

Model A: \[ y_t = \alpha_1 + \alpha_2 S_{lt}(\gamma_1, \tau_1) + \alpha_3 S_{lt}(\gamma_2, \tau_2) + \nu_t \]

Model B: \[ y_t = \alpha_1 + \beta_1 t + \alpha_2 S_{lt}(\gamma_1, \tau_1) + \alpha_3 S_{lt}(\gamma_2, \tau_2) + \nu_t \]

Model C: \[ y_t = \alpha_1 + \beta_1 t + \alpha_2 S_{lt}(\gamma_1, \tau_1) + \beta_2 t S_{lt}(\gamma_1, \tau_1) + \alpha_3 S_{2lt}(\gamma_2, \tau_2) + \beta_3 t S_{2lt}(\gamma_2, \tau_2) + \nu_t . \]

The disturbance term \( \nu_t \) in each model is a stationary process with zero mean, and the transition functions \( S_{lt}(\gamma_i, \tau_i) \) are logistic smooth transition functions defined by:

\[ S_{lt}(\gamma_i, \tau_i) = \left[ 1 + \exp\left( -\frac{\gamma_i - (t - \tau_i T)}{T} \right) \right]^{-1} \quad i = 1, 2 \]

for a sample size \( T \). The midpoints of the two transitions are given by \( \tau_1 T \) and \( \tau_2 T \) respectively; the transitions speeds are allowed to differ, and are respectively determined by \( \gamma_1 \) and \( \gamma_2 \).

Tests of a unit root null hypothesis against one of the above models as the alternative can be conducted using the two-step procedure employed by Leybourne, Newbold and Vougas (1998). The first step involves non-linear estimation of model A, B or C, minimizing the sum of squared residuals (analytically over \( \alpha_i, \beta_i \), numerically over \( \gamma_i, \tau_i \)). The resulting residuals \( \hat{\nu}_t \) are then used to estimate the augmented Dickey-Fuller regression:

\[ \Delta\hat{\nu}_t = \rho\hat{\nu}_{t-1} + \sum_{i=1}^{k} \delta_i \Delta\hat{\nu}_{t-i} + \eta_t \]

where the number of lagged difference terms, \( k \), is determined by some method of order selection. The test statistic is then the \( t \)-ratio associated with the ordinary least squares estimate of \( \rho \). Modifying the Leybourne-Newbold-Vougas notation, we denote the test statistics associated with use of models A, B and C as \( s_{2\alpha} \), \( s_{2\alpha(\beta)} \) and \( s_{2\alpha}\beta \) respectively.

Table I presents critical values for these three tests at the 10%-, 5%- and 1%-level, obtained by Monte Carlo simulation using 10,000 replications. The null hypothesis was generated as a random walk (without drift) with errors drawn from the standard normal distribution. When optimizing numerically in the first step of the test procedure, the Broyden, Fletcher, Goldfarb and Shanno algorithm in the OPTMUM subroutine for GAUSS was employed, and a grid of starting values for the midpoint fractions \( \tau_1, \tau_2 \) were considered each time. In the subsequent augmented Dickey-Fuller regressions, the value of \( k \) was set equal to its true value of zero. Critical values for a number of sample sizes are reported, including one large sample to approximate the tests’ asymptotic critical values. The critical values are larger in absolute value than those for the single transition tests, as would be expected: for example,
with $T = 200$, the 5%-level critical values are $-5.07$, $-5.53$ and $-6.01$ for $s_{2\alpha}$, $s_{2\alpha(\beta)}$ and $s_{2\alpha\beta}$, respectively, compared to $-4.16$, $-4.63$ and $-4.87$ for their single transition counterparts.

In an extension to the Leybourne-Newbold-Vougas unit root tests, Sollis, Leybourne and Newbold (1999) considered the further possibility that the transition function under the alternative may often be asymmetric, with the adjustments into and out of the transition phase occurring at different rates. Using this notion, these authors proposed three tests of the unit root null corresponding to those of Leybourne, Newbold and Vougas (1998), with the transition determined by the generalised logistic function (Nelder, 1961):

$$G_t(\gamma, \tau, \theta) = [1 + \exp\{-\gamma(t - \tau)/\theta\}]^{-\theta} \quad 0 < \theta \leq 1.$$  

The additional parameter $\theta$ controls the degree of asymmetry, with $\theta = 1$ corresponding to the symmetric logistic function. Application of this generalisation to our framework of double smooth transitions is equally desirable, allowing both transition functions to be potentially asymmetric, and to different degrees. In order to estimate a model with two asymmetric smooth transitions in linear trend, we must use a grid of possible values for the asymmetry parameters, along the lines of Sollis, Leybourne and Newbold (1999). However, the number of possible pairings of these parameters for an appropriately fine grid is very large. Whilst this is not problematic theoretically, the limits of computing power constrain the number of numerical optimizations that can be performed in a realistic time frame. Unfortunately, these limits currently prevent simulation of critical values for such tests.

In order to investigate the power of the double transition unit root tests, it is useful to conduct comparisons with the Leybourne-Newbold-Vougas single transition tests. While the greater generality of the double transition approach may result in a loss of power for series that have at most one transition, there is potential for power gains to be present when the true data generating process is stationary around two smooth transitions in linear trend. It is instructive therefore to simulate empirical powers for such cases. Focusing on the $s_\alpha$ and $s_{2\alpha}$ tests for purposes of tractability, we generated series from the following model:

$$y_t = 1 + \sqrt{T}S_{1t}(\gamma_1, \tau_1) + \sqrt{T}S_{2t}(\gamma_2, \tau_2) + \mu_t, \quad \mu_t = \phi\mu_{t-1} + \epsilon_t,$$

where $\epsilon_t \sim IN(0, \sigma^2_\epsilon)$, i.e. $y_t$ is stationary around two transitions in mean with a break size of $\sqrt{T}$. For a given sample size (and proportional break sizes), the relative powers depend primarily on the speed and timing of the transitions; test power is also determined by the parameters of the underlying stationary process (the autoregressive parameter $\phi$ and the error
standard deviation $\sigma_\varepsilon$). In order to obtain a range of interesting power comparisons, we set $\phi = 0.8$ and $\sigma_\varepsilon = 0.2$, fixed the timing of the transitions at $\tau_1 = 0.3$ and $\tau_2 = 0.7$, and conducted experiments for different transition speeds, letting $\gamma_1 = \gamma_2$ for simplicity. Table II provides test rejection frequencies for experiments based on 5,000 replications and sample sizes $T = 100$ and $T = 200$. As with the simulation of the critical values, the value of $k$ in the Dickey-Fuller regressions was set equal to its correct value of zero in the computation of the test statistics.

Considering the results for $T = 200$, $s_{2t}$ has substantially greater power than $s_t$ for all but the slowest transitions. The difference in the powers is dramatic, the power of $s_t$ rapidly decreasing to zero as the transition speeds increase, while that of $s_{2t}$ remains much higher. For the very slowest transitions ($\gamma_1 = \gamma_2 = 0.01$), the generated process is very close to stationarity about a linear trend, so it is not surprising that $s_{2t}$ has less power than $s_t$ in this extreme case. The power of $s_{2t}$ decreases steadily as the transitions become more rapid and approximate instantaneous structural breaks, but moderate power is still achieved for the fastest transitions considered, in marked contrast to $s_t$. For $T = 100$, the power gains of $s_{2t}$ over $s_t$ are less striking. The single transition test now has superior power for the two slowest transition speeds, before again dropping to trivial levels. For $s_{2t}$, a similar pattern is observed to the $T = 200$ case, although now the powers are lower overall and the decrease for near-instantaneous transitions is more severe. In general, the double smooth transition test performs best for larger samples and intermediate speeds for the two transitions.

A range of other power simulations were also carried out. Altering the midpoints of the smooth transitions $\tau_1$ and $\tau_2$ generally resulted in very similar powers to those reported, although the relative power of $s_t$ improved slightly when the transition midpoints were closer together. Allowing the transition speeds $\gamma_1$ and $\gamma_2$ to differ produced powers roughly the same as when these parameters were both set at an inbetween value, for example results for $\gamma_1 = 0.01$ and $\gamma_2 = 5$ with $T = 200$ were close to the empirical powers reported for $\gamma_1 = \gamma_2 = 0.5$.

3. EMPIRICAL APPLICATIONS

In this section we consider applications of the unit root tests to two interesting time series. The first is annual average global temperature data for the period 1856-1998 (143 observations), obtained from the Climate Research Unit at the University of East Anglia.
The series is a combination of land air and sea surface temperatures, expressed as deviations from the average over 1961-90; further details regarding its construction can be found in Jones (1994), Parker, Jones, Bevan and Folland (1994) and Parker, Folland and Jackson (1995). The second time series considered is annual data on the US consumer price index for 1860-1970 (111 observations), as studied by Nelson and Plosser (1982) in their influential work on characterising economic time series.

In previous work, Galbraith and Green (1992) and Seater (1993) examined global average temperature from the perspective of testing for unit roots using data up to the late 1980s. In this regard their findings are limited by the fact that the time span did not cover the continued substantial temperature rises in the last decade. Galbraith and Green studied a monthly series for the period 1880-1988, and found sufficient evidence, using an augmented Dickey-Fuller $\tau_\tau$ test, to reject the unit root null in favour of stationarity about a linear trend. However, these authors speculated about the possibility that a longer time series may result in a nonlinear trend being a better model for the data. Seater, on the other hand, using annual data for 1854-1989, did not find evidence against the null using the $\tau_\tau$ test, but argued in favour of trend stationarity on other grounds. Nelson and Plosser (1982) applied the $\tau_\tau$ test to the US consumer price series, while Leybourne, Newbold and Vougas (1998) also applied their most general test $s_{a\beta}$; the conclusion of both these analyses was that the series was best characterised as a unit root process, with the tests failing to reject the null.

Table III presents results from our application of the augmented Dickey-Fuller unit root test ($\tau_\tau$), the Leybourne-Newbold-Vougas tests ($s_{a\alpha}, s_{a\alpha(\beta)}$, $s_{a\beta}$) and the double transition variants proposed in this paper ($s_{2a\alpha}, s_{2a\alpha(\beta)}, s_{2a\beta}$), to the two time series. In conducting each test, the lag order used in the augmented Dickey-Fuller regressions was determined by sequential downward testing at the 5%-level, starting with $k = 8$. For the tests involving smooth transitions, a grid of starting values for the transition midpoint fractions were considered, as in the Monte Carlo experiments of the previous section.

The unit root null hypothesis is not rejected for either series at the 5%-level when the augmented Dickey-Fuller and Leybourne-Newbold-Vougas tests are employed, although rejection at the 10%-level occurs for the US consumer price series for the $s_{a\beta}$ test. In contrast, rejections are obtained at the 1%-level for all the double transition tests for average global temperature and for the most general $s_{2a\beta}$ test for consumer prices. Thus when an appropriately general trend function is permitted under the alternative, sufficient evidence exists to reject the unit root null for these data. The series therefore appear to be best
characterised as stationary around two smooth transitions in linear trend, with obvious implications for climatic and economic modelling and forecasting.

In addition to conducting unit root tests, it is interesting to estimate the implied models for the two series. For the average global temperature series it is first necessary to decide which of the three models A, B or C is most appropriate, since rejections were obtained in favour of each alternative hypothesis. We therefore estimated each model with autoregressive errors of a common order, determined by the maximum lag order required in the augmented Dickey-Fuller regressions, i.e. AR(2) errors. Likelihood ratio tests were then performed to compare the models. Testing the restrictions of model A relative to models B and C resulted in probability values of 0.008 and 0.005 respectively, clearly indicating the importance of the trend component. The probability value associated with testing the model B restrictions relative to model C was 0.055; although the restrictions are not quite rejected at the 5%-level, the decision is marginal and our preference is for the more general model with transitions in both intercept and trend. For the US consumer price series the only rejection in the unit root tests was in favour of model C, thus the most general model was also adopted for this series, also with AR(2) errors following the results of the augmented Dickey-Fuller regressions.

Estimation of these models resulted in the fitted double smooth transitions plotted in Figures 1 and 2. The estimated transition midpoint fractions \((\tau_1, \tau_2)\) were \((0.335, 0.756)\) for average global temperature, corresponding to the years 1903 and 1963, and \((0.547, 0.788)\) for US consumer prices, corresponding to the years 1920 and 1946. The associated transition speeds \((\gamma_1, \gamma_2)\) were \((1.450, 0.109)\) and \((0.416, 0.126)\) respectively. The fitted trend lines track the data well in general, clearly picking up the structural changes visible in the time series plots, the only exception being the relatively high prices at the beginning of the US consumer price series. Of particular interest is the clear evidence of increases in trend average global temperature from the early 1900s and again from 1970.

4. CONCLUSION

Testing for a unit root against an alternative of stationarity around some deterministic function has an important role in time series analysis. In this paper we have broadened the class of trend functions against which the unit root null hypothesis can be tested, allowing for double smooth transitions in linear trend. Our tests are not as powerful as others with simpler
alternatives when the process under consideration has at most one transition, and as a result our tests should not be treated as encompassing Dickey-Fuller and Leybourne-Newbold-Vougas type tests, even though the simpler trend functions involved are special cases of the double transitions we consider. Rather, we recommend use of the new tests in addition to those mentioned, as a further alternative hypothesis to be considered, especially if one suspects two smooth transitions to be present in the series and the unit root null is not rejected by other tests. If the true generating process is indeed stationary around two smooth transitions in linear trend, our tests can strongly reject the unit root null hypothesis. Further generalisations to permit asymmetry in the smooth transition functions using similar techniques to Sollis, Leybourne and Newbold (1999) may also be worthwhile in the future.
REFERENCES


TABLE I

**NULL CRITICAL VALUES FOR UNIT ROOT TESTS AGAINST STATIONARITY AROUND DOUBLE SMOOTH TRANSITIONS IN LINEAR TREND**

<table>
<thead>
<tr>
<th></th>
<th>$s_{2n}$</th>
<th></th>
<th>$s_{2n}\beta$</th>
<th></th>
<th>$s_{2n\beta}$</th>
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<td>0.10</td>
<td>0.05</td>
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<tr>
<td>50</td>
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<td>-6.48</td>
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<td>-5.25</td>
<td>-5.53</td>
<td>-6.05</td>
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TABLE II

EMPIRICAL POWERS OF UNIT ROOT TESTS AGAINST STATIONARITY AROUND SINGLE AND DOUBLE SMOOTH TRANSITIONS IN LINEAR TREND ($\tau_1 = 0.3, \tau_2 = 0.7$)

<table>
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<tr>
<th>$\gamma_1 = \gamma_2$</th>
<th>$s_{\alpha} \ 0.10$</th>
<th>$s_{\alpha} \ 0.05$</th>
<th>$s_{2\alpha} \ 0.10$</th>
<th>$s_{2\alpha} \ 0.05$</th>
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<td>0.367</td>
<td>0.413</td>
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</tr>
<tr>
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<td>0.619</td>
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<tr>
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<td>0.058</td>
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<td>0.000</td>
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<tr>
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<td>0.081</td>
</tr>
<tr>
<td>5.00</td>
<td>0.000</td>
<td>0.000</td>
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<table>
<thead>
<tr>
<th>$s_{\alpha} \ 0.10$</th>
<th>$s_{\alpha} \ 0.05$</th>
<th>$s_{2\alpha} \ 0.10$</th>
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<td>0.996</td>
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<td>0.931</td>
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<td>0.239</td>
<td>0.106</td>
<td>0.913</td>
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<td>0.000</td>
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<td>0.000</td>
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<td>0.521</td>
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### TABLE III

**EMPIRICAL APPLICATIONS OF UNIT ROOT TESTS**

<table>
<thead>
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<th></th>
<th>Average global temperature</th>
<th>US consumer prices</th>
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<td>$T = 111$</td>
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<tr>
<td>$k$</td>
<td>Test statistic</td>
<td>$k$</td>
</tr>
<tr>
<td>$\tau_t$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$s_{at}$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$s_{at(\beta)}$</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>$s_{a\beta}$</td>
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<td>1</td>
</tr>
<tr>
<td>$s_{2at}$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$s_{2at(\beta)}$</td>
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<td>5</td>
</tr>
<tr>
<td>$s_{2a\beta}$</td>
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<td>1</td>
</tr>
</tbody>
</table>

*Note:* * and *** denote significance at the 10%- and 1%-levels respectively.
Figure 1. Average global temperature and fitted double smooth transition function.
Figure 2. US consumer prices and fitted double smooth transition function.