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Pathwise Stationary Solutions of Stochastic Partial Differential Equations and Backward Doubly Stochastic Differential Equations on Infinite Horizon

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Summary. The main purpose of this paper is to study the existence of stationary solution for stochastic partial differential equations. We establish a new connection between backward doubly stochastic differential equations on infinite time horizon and the stationary solution of the SPDEs. For this we study the existence of the solution of the associated BDSDEs on infinite time horizon and prove it is a stationary viscosity solution of the corresponding SPDEs.

Keywords: backward doubly stochastic differential equations, stochastic partial differential equations, pathwise stationary solution, random dynamical systems.

1 Introduction

As for the deterministic dynamical systems, the pathwise stationary solution of a stochastic dynamical system is one of the fundamental questions of basic importance and a generic phenomenon ([1], [9], [12], [13], [16], [18]-[20], [28], [29]). However, in contrast to the deterministic dynamical systems, the existence of stationary solutions of stochastic dynamical systems generated e.g. by stochastic differential equations (SDEs) or stochastic partial differential equations (SPDEs), is a difficult problem and results are only known in some special cases. In recent years, substantial results on the existence and uniqueness of invariant measures for SPDEs and weak convergence of the law of the solutions as time tends to $\infty$ have been proved ([4], [5], [11], [14], [15] to name, but a few). The invariant measure describes the invariance of a certain solution in law when time changes. As for the pathwise stationary solutions, some significant progresses have been made for certain SPDEs ([9], [12], [13], [16], [18]-[20], [28], [29]), however their existence still remained unsolved for many SPDEs. Their existence and stabilities are of great interests both in mathematics and physics. Let $u : [0,\infty) \times U \times \Omega \rightarrow U$ be a
measurable random dynamical system on a measurable space \((U, \mathcal{B})\) over a metric dynamical system \((\Omega, \mathcal{F}, P, (\theta_t)_{t \geq 0})\), then a stationary solution is a \(\mathcal{F}\)-measurable random variable \(Y : \Omega \to X\) such that (Arnold [1])

\[
u(t, Y(\omega), \omega) = Y(\theta_t \omega) \quad t \geq 0, \text{ a.s.}
\] (1.1)

This "one-force, one solution" setting is a natural extension of equilibrium in deterministic systems to stochastic counterparts. Different from the invariant measure, it describes the pathwise invariance of the stationary solution over time along the measurable and \(P\)-preserving transformation \(\theta_t : \Omega \to \Omega\) and the pathwise limit of the solutions of SPDEs. The simplest nontrivial example is the Ornstein-Uhlenbeck process defined by the stochastic differential equation

\[
du(t) = -u(t)dt + dB_t.
\]

It defines a random dynamical system \(u(t, u_0) = u_0 e^{-t} + \int_0^t e^{-(t-s)} dB_s\) and its stationary point is given by \(Y(\omega) = \int_{-\infty}^0 e^s dB_s\). But unlike deterministic cases, in general, the stationary solutions of the stochastic systems cannot be given explicitly. It was even unthinkable to represent them as solutions of certain differential or functional equations. The existence of the stationary solutions for SPDEs is a subtle problem. There is no method which can be applied to SPDEs with great generalities. In [28], [29], the stationary strong solution of the stochastic Burgers’ equations with periodic or random forcing \((C^3\text{ in the space variable})\) was established by Sinai using the Hopf-Cole transformation. In [18], the existence of the \(H^1[0,1]\)-valued solution of the stochastic Burgers’ integral equation, as the stationary solution of the stochastic Burgers’ equation on \([0, 1]\) with \(L^2[0,1]\)-valued noise, was obtained by Liu and Zhao. In [12], the existence of stationary solution for certain SPDEs with linear one-dimensional noise was obtained by Duan, Lu and Schmalfuss. In [20], the stationary solution of the stochastic evolution equations was identified as a solution of a corresponding integral equation up to time \(+\infty\) and the existence was established for certain SPDEs by Mohammed, Zhang and Zhao. But the existence of solution of such a stochastic integral equation in general is far from clear. The local existence of stable/unstable manifold near a stationary solution for a stochastic dynamical system generated by SPDEs was studied by Mohammed, Zhang and Zhao in [19], [20], and by Duan, Lu and Schmalfuss in [12], [13].

In this paper, we use a new approach to study the stationary solutions of SPDEs via the solutions of backward doubly stochastic differential equations (BDSDEs) on infinite horizon. Backward stochastic differential equations (BSDEs) have been studied extensively in the last 15 years since the pioneering work of Pardoux and Peng [23]. The connection of BSDEs and quasilinear parabolic partial differential equations (PDEs) was discovered by Pardoux and Peng in [24] and Peng in [26]. The BDSDEs and their connections with the strong solutions of the SPDEs were studied by Pardoux and Peng in [25], also recently by Backdahn and Ma for the stochastic viscosity solutions in [6], [7], and by Bally and Matoussi for the weak solutions in [3]. On the other hand, the infinite horizon BSDE was first studied by Peng in
It was shown that the corresponding PDE is a Poisson equation (elliptic equation). This was studied systematically by Pardoux in [22]. Coupled forward-backward stochastic differential equations (FBSDEs) on infinite horizon and their connections with PDEs were recently studied by Shi and Zhao in [27]. Notice that the solutions of the Poisson equations can be regarded as the stationary solutions of the parabolic PDEs. Deepening this idea, it would not be unreasonable to conjecture that the solutions of infinite time horizon BDSDEs (if exists) be the stationary solutions of the corresponding SPDEs. Of course, we cannot write it as Poisson equation or stochastic Poisson equation like in the deterministic cases. However, it is very natural to describe the stationary solutions of SPDEs by infinite horizon BDSDEs. In this sense, BDSDEs (or BSDEs) can be regarded as more general SPDEs (or PDEs).

We first study the existence of the solution to the BDSDE on the infinite time horizon in the space \( S^{p,-K}([0, \infty); \mathbb{R}^1) \cap M^{2,-K}([0, \infty); \mathbb{R}^1) \times M^{2,-K}([0, \infty); \mathbb{R}^d) \) (see Section 2 for definitions). The equation can be written in the following integral form: for any \( s \geq t \),

\[
e^{-\frac{K'}{2}t} Y^t_{s,x} = \int_s^\infty e^{-\frac{K'}{2}\tau} f(X^t_{\tau,x}, Y^t_{\tau,x}, Z^t_{\tau,x}) d\tau + \int_s^\infty K' \frac{1}{2} e^{-\frac{K'}{2}\tau} Y^t_{\tau,x} d\tau
\]

\[
+ \int_s^\infty e^{-\frac{K'}{2}\tau} g(X^t_{\tau,x}, Y^t_{\tau,x}, d1B_\tau) - \int_s^\infty e^{-\frac{K'}{2}\tau} \langle Z^t_{\tau,x}, dW_\tau \rangle. \tag{1.2}
\]

Here \( f, g \) and \( K' \) satisfy conditions (H.1)', (H.2)', (H.3)', (H.4)' in Section 3, \( \{B_t\} \) is a \( \mathbb{R}^d \)-Brownian motion on a probability space \( (\Omega^1, F^1, P^1) \) and \( \{W_t\} \) is a \( \mathbb{R}^d \)-Brownian motion on a probability space \( (\Omega^2, F^2, P^2) \); the stochastic integral

\[
\int_s^\infty e^{-\frac{K'}{2}\tau} \langle g(X^t_{\tau,x}, Y^t_{\tau,x}), d1B_\tau \rangle = \lim_{n \to \infty} \int_s^n e^{-\frac{K'}{2}\tau} \langle g(X^t_{\tau,x}, Y^t_{\tau,x}), d1B_\tau \rangle
\]

in \( L^2(dP^1 \times dP^2) \), where \( \int_s^n e^{-\frac{K'}{2}\tau} \langle g(X^t_{\tau,x}, Y^t_{\tau,x}), d1B_\tau \rangle d1B_\tau \) is a backward stochastic integral;

\[
\int_s^\infty e^{-\frac{K'}{2}\tau} \langle Z^t_{\tau,x}, dW_\tau \rangle = \lim_{n \to \infty} \int_s^n e^{-\frac{K'}{2}\tau} \langle Z^t_{\tau,x}, dW_\tau \rangle
\]

in \( L^2(dP^1 \times dP^2) \), where \( \int_s^n e^{-\frac{K'}{2}\tau} \langle Z^t_{\tau,x}, dW_\tau \rangle d1B_\tau \) is a forward stochastic integral; \( X^t_{s,x} \) is generated by the following stochastic differential equation

\[
\left\{ \begin{array}{l}
dX^t_{s,x} = b(X^t_{s,x}) ds + \sigma(X^t_{s,x}) dW_s, \quad s > t \\
X^t_{s,x} = x, \quad 0 \leq s \leq t.
\end{array} \right. \tag{1.3}
\]

Here \( b(\cdot), \sigma(\cdot) \) satisfy the condition (H.5) in Section 3. Consider the following SPDEs

\[
v(t, x) = v(0, x) + \int_0^t [L v(s, x) + f(x, v(s, x), \sigma^*(x) Dv(s, x))] ds
\]

\[
- \int_0^t \langle g(x, v(s, x)), d\hat{B}_s \rangle. \tag{1.4}
\]
Here \( \hat{B}_s \) is a two-sided Brownian motion; \( \int_0^t \langle g(x, v(s, x)), dB_s \rangle \) is a forward stochastic integral; \( \mathcal{L} \) is infinitesimal generator of the process \( X^{t,x}_s \) given by

\[
\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}
\]

with \( (a_{ij}(x)) = \sigma \sigma^*(x) \). Take \( B_s \) in Eq.(1.2) to be \( \hat{B}_T - s - \hat{B}_T \) for all \( 0 \leq s < \infty \). We then prove that, for arbitrary \( T > 0 \) and \( 0 \leq t < T \), \( v(t, x, \omega) = Y^{T-t,x}_t(\omega) \) is a stationary viscosity solution of Eq.(1.4). For this, we show in Section 4 that the shift operator \( \theta_1 \) of \( B_s \), defined from \( \hat{B} \) through time reversal at \( T \), satisfies that \( \theta_1 B_{t-t} B \) is independent of \( T \). This remarkable fact is used to show \( Y^{T-t,x}_t(\omega) \) is independent of \( T \). It is proved that the stationary solution is a stochastic viscosity solution of Eq.(1.4). Stochastic viscosity solutions of such SPDEs were introduced by Buckdahn and Ma in [6]-[8].

As far as we know, the connection of the pathwise stationary solutions of the SPDEs and infinite horizon BDSDEs we study in this paper is new. We believe this new method can be used to many SPDEs such as those with quadratic or polynomial growth nonlinear terms, more types of noises, weak solutions etc. We don’t intend to include all these results in the current paper, but only study Lipschitz continuous nonlinear term and finite dimensional noise for simplicity in order to initiate this intrinsic method to the study of this basic problem in dynamics of SPDEs. We would like to point out that our BDSDE method depend on neither the continuity of the random dynamical system (continuity means \( \Phi(t, \cdot, \omega) : U \rightarrow U \) is a.s. continuous) nor on the method of the random attractors. The continuity problem for the SPDE (1.4) with the nonlinear noise considered in this paper still remains open mainly due to the failure of Kolmogorov’s continuity theorem in infinite dimensional setting as pointed out by many researchers (e.g. [12], [19]).

2 BDSDE on infinite time interval

Let \( (\Omega^1 \times \Omega^2, \mathcal{F}^1 \times \mathcal{F}^2, P^1 \times P^2) \) be a product probability space, \( (B_t)_{t \geq 0} \) and \( (W_t)_{t \geq 0} \) be two mutually independent standard Brownian motion processes with values on \( \mathbb{R}^l \) and \( \mathbb{R}^d \), defined on \( (\Omega^1, \mathcal{F}^1, P^1) \) and \( (\Omega^2, \mathcal{F}^2, P^2) \) respectively. Let \( \mathcal{N}^1 \times \mathcal{N}^2 \) denote the class of \( P^1 \times P^2 \)-null sets of \( \mathcal{F}^1 \times \mathcal{F}^2 \). For each \( t \geq 0 \), we define

\[
\mathcal{F}_t \triangleq \mathcal{F}_{t\infty} \times \mathcal{F}_{t} \bigvee \mathcal{N}^1 \times \mathcal{N}^2.
\]

Here for any process \( \{ \eta_t \}, \mathcal{F}_{s\leq r} = \sigma \{ \eta_r - \eta_s; 0 \leq s \leq r \}, \mathcal{F}_t^\eta = \mathcal{F}_0^\eta, \mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t, \mathcal{F}_\infty = \bigvee_{T \geq 0} \mathcal{F}_T. \) Take \( K > 0, q \geq 2 \), then we denote \( M^{q-K}(0,\infty);\mathbb{R}^d \) the set of jointly measurable random processes \( \{ \varphi_t, t \geq 0 \} \) with values on \( \mathbb{R}^d \) satisfying
(i) \( \varphi_t \) is \( \mathcal{F}_t \)-measurable for \( t \geq 0 \);

(ii) \( E[\int_0^\infty e^{-Kt} e^{-|\varphi|} dt] < \infty \).

Also we denote \( S_{q,-K}^0 ([0, \infty) ; \mathbb{R}^1) \) the set of jointly measurable random processes \( \{ \psi_t, t \geq 0 \} \) with values on \( \mathbb{R}^1 \) satisfying

(i) \( \psi_t \) is \( \mathcal{F}_t \)-measurable for \( t \geq 0 \) and continuous for \( \omega \)-a.s. \( \omega \);

(ii) \( E[\sup_{t \geq 0} e^{-Kt} e^{-|\psi_t|}] < \infty \).

Similarly, \( M^{2,0}_p ([0, T] ; \mathbb{R}^d) \) is the set of jointly measurable random processes \( \{ \tilde{\psi}_t, 0 \leq t \leq T \} \) with values on \( \mathbb{R}^d \) satisfying

(i) \( \tilde{\psi}_t \) is \( \mathcal{F}_t \)-measurable for \( 0 \leq t \leq T \) and continuous for \( \omega \)-a.s. \( \omega \);

(ii) \( E[\sup_{0 \leq t \leq T} |\tilde{\psi}_t|] < \infty \).

We assume that

(H.1). \( f : \Omega^1 \times \Omega^2 \times [0, \infty) \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1 \), \( g : \Omega^1 \times \Omega^2 \times [0, \infty) \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1 \) are jointly measurable and \( f(0, 0) \in M^{p,-K}([0, \infty); \mathbb{R}^1) \), \( g(0, 0) \in M^{p,-K}([0, \infty); \mathbb{R}^d) \), where \( p > d + 2 \).

(H.2). There exist three constants \( C_0, C > 0 \), \( 0 < \alpha < \frac{1}{4} \), s.t. for any \( (\omega^1, \omega^2, t) \in \Omega^1 \times \Omega^2 \times [0, \infty) \), \( (y_1, z_1), (y_2, z_2) \in \mathbb{R}^1 \times \mathbb{R}^d \),

\[
|f_t(y_1, z_1) - f_t(y_2, z_2)| \leq C_0|y_1 - y_2| + C|z_1 - z_2|, \\
|g_t(y_1, z_1) - g_t(y_2, z_2)| \leq C|y_1 - y_2| + \alpha|z_1 - z_2|.
\]

(H.3). There exists a constant \( \mu > 0 \), s.t. for \( K < K' < 2K, 2\mu - K' - p(2p - 1)C^2 > 0 \) and for any \( (\omega^1, \omega^2, t) \in \Omega^1 \times \Omega^2 \times [0, \infty) \), \( y_1, y_2 \in \mathbb{R}^1 \), \( z \in \mathbb{R}^d \),

\[
(y_1 - y_2)(f_t(y_1, z) - f_t(y_2, z)) \leq -\mu|y_1 - y_2|^2.
\]

The main purpose of this section is to prove the existence and uniqueness of the solutions of BDSDEs on infinite horizon. First, we briefly include the results obtained by Pardoux and Peng ([25]) and Buckdahn-Ma ([6]–[8]) on finite time BDSDE. The following BDSDE was first introduced by Pardoux and Peng in [25]: for \( 0 \leq t \leq T \),
\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T (g(s, Y_s, Z_s), d^1 B_s) - \int_t^T (Z_s, dW_s). \] (2.1)

Here the integral w.r.t. \{B_t\} is a "backward Itô’s integral" and the integral w.r.t. \{W_t\} is a standard "forward Itô’s integral". The backward integral is particular case of Itô-Skorohod integral ([21]). An important fact is that, for \(0 \leq T \leq T'\) and arbitrary \(a, b\) satisfying \(0 \leq a \leq b \leq T'\), if a process \(h\) with values on \(\mathbb{R}^1\) satisfies \(\int_0^b |h_s|ds < \infty\) a.s., then

\[ \int_t^T \langle h_s, d^1 B_s \rangle = - \int_{T'-T}^{T'-t} \langle h_{T'-s}, d\hat{B}_s \rangle \quad \text{a.s.}. \] (2.2)

Here \(\hat{B}_s = B_{T'-s} - B_{T'}\).

The following existence and uniqueness theorem for \(\mathcal{F}_t\)-measurable solution \((Y_t, Z_t)_{0 \leq t \leq T}\) of Eq.(2.1) was proved in [25]. It will be used in the proof of Theorem 2.3.

**Theorem 2.1** ([25]) Under conditions (H.1), (H.2), for any given \(\mathcal{F}_t\)-measurable \(\xi \in L^2(d\mathbb{P}^1 \times d\mathbb{P}^2)\), Eq.(2.1) has a unique \(\mathcal{F}_t\)-measurable solution \[(Y_t, Z_t) \in S^{2,0}([0,T];\mathbb{R}^1) \times M^{2,0}([0,T];\mathbb{R}^d).\]

Let \((X^t_s, x)_{s \leq T}\) be defined by Eq.(1.3), and consider Eq.(2.1) in the following form for \(t \leq s \leq T\)

\[ Y^t_s = h(X^t_s) + \int_s^T f(X^t_r, Y^t_r, Z^t_r)dr + \int_s^T \langle g(X^t_r, Y^t_r, Z^t_r), d^1 B_r \rangle - \int_s^T \langle Z^t_r, dW_r \rangle. \] (2.3)

Pardoux and Peng also proved that under some strong smoothness conditions of \(h, b, \sigma, f\) and \(g\) (for details see [25]), \(u(t, x) = Y^t_s, (t, x) \in [0, T] \times \mathbb{R}^d\), is independent of \(\omega^s \in \mathcal{G}^2\) and is the unique classical solution of the following backward SPDE

\[ u(t, x) = h(x) + \int_t^T [\mathcal{L}u(s, x) + f(x, u(s, x), \sigma(x)Du(s, x))]ds + \int_t^T \langle g(x, u(s, x), \sigma(x)Du(s, x)), d^1 B_s \rangle, \quad 0 \leq t \leq T. \] (2.4)

It is easy to see from (2.2) this equation is the time reversal version of a forward SPDE. We are particularly interested in the case that \(g(x, y, z) = g(x, y)\) where the stochastic viscosity solution can be defined. In such a case, \(v(t, x) = u(T - t, x)\) is a solution of Eq.(1.4) with \(\hat{B}_s = B_{T-s} - B_T\). The stochastic viscosity solution of SPDE (1.4) was studied by Buckdahn and Ma in [6]-[8] and it was proved that the solution \(Y^t_s\) of Eq.(2.3), \((t, x) \in [0, T] \times \mathbb{R}^d\), is
a viscosity solution of Eq.(2.4) under conditions (H.1)', (H.2)', (H.4)', (H.5) stated in Sections 3 and 4. Therefore it gives the stochastic viscosity solution of Eq.(1.4) through the time reversal argument. To benefit readers, we include briefly Buckdahn-Ma’s main idea of the stochastic viscosity solutions for SPDEs in [6]-[8]. Define \( \lambda(t, x, y) \) as the solution of the following SDE

\[
\lambda(t, x, y) = y - \int_0^t \langle g(x, \lambda(s, x, y)), dB_s \rangle.
\]

Under the condition (H.4)', the equation can define a stochastic flow \( \lambda(t, x, y) \), i.e., for fixed \( x \), the random field \( \lambda(\cdot, x, \cdot) \) is continuously differentiable in the variable \( y \), and the mapping \( y \rightarrow \lambda(t, x, y) \) defines a diffeomorphism for all \((t, x) \), a.s. \( \omega^1 \). Denote its inverse by \( \zeta(t, x, y) = \lambda(t, x, \cdot) \). Let \( v(t, x) = \zeta(t, x, v(t, x)) \), the so-called Doss-Sussmann transformation, then \( \hat{v}(t, x) \) satisfies the following random PDE

\[
\hat{v}(t, x) = \hat{v}(0, x) + \int_0^t \left[ \mathcal{L}\hat{v}(s, x) + \hat{f}(s, x, \hat{v}(s, x), \sigma^*(x)D\hat{v}(s, x)) \right] ds. \tag{2.5}
\]

Here

\[
\hat{f}(t, x, y, z) = \frac{1}{D_y\lambda(t, x, y)} \left( f(x, \lambda(t, x, y), \sigma^*(x)D_x\lambda(t, x, y)) + D_y\lambda(t, x, y)z 
+ \mathcal{L}_x\lambda(t, x, y) + (\sigma^*(x)D_{xy}\lambda(t, x, y), z) + \frac{1}{2}D_{yy}\lambda(t, x, y)|z|^2 \right).
\]

Then the stochastic viscosity solution \( v(t, x) \) of Eq.(1.4) was defined in [6]-[8] through the "deterministic" viscosity solution \( \hat{v}(t, x) \) of Eq.(2.5) for a.s. \( \omega^1 \in \Omega^1 \) via the relation \( v(t, x) = \lambda(t, \hat{v}(t, x)) \). The notion of viscosity solutions of partial differential equations was first introduced by Crandall and Lions in [10]. Buckdahn and Ma established the connection of the solution of BDSDE (2.3) and the stochastic viscosity solution of the SPDE (1.4). In particular, the following result of Buckdahn and Ma will be used in Section 4.

**Theorem 2.2** ([6]) *Assume Conditions (H.1), (H.2) and that the function \( h : \mathbb{R}^d \rightarrow \mathbb{R} \) is continuous and for some constants \( C, \delta > 0 \), \( |h(x)| \leq C(1 + |x|) \). Then \( v(t, x) = u(T - t, x) = Y_{T-t,x}^T \) is a stochastic viscosity solution of Eq. (1.4).*

Now we consider the following BDSDE with infinite time horizon, for \( t \geq 0 \),

\[
e^{-\frac{K'}{2}t}Y_t = \int_t^\infty e^{-\frac{K'}{2}s}f_s(Y_s, Z_s)ds + \int_t^\infty \frac{K'}{2}e^{-\frac{K'}{2}s}Y_s ds
+ \int_t^\infty e^{-\frac{K'}{2}s}\langle g_s(Y_s, Z_s), dB_s \rangle - \int_t^\infty e^{-\frac{K'}{2}s}\langle Z_s, dW_s \rangle. \tag{2.6}
\]
Note that the integral w.r.t. \( \{B_t\} \) and the integral w.r.t. \( \{W_t\} \) are the same as in Eq.(2.1).

The main objective of this Section is to prove

**Theorem 2.3** Under the above conditions (H.1), (H.2), (H.3), Eq.(2.6) has a unique solution

\[(Y, Z) \in S^{p,-K}([0, \infty); \mathbb{R}^1) \cap M^{2,-K}([0, \infty); \mathbb{R}^1) \times M^{2,-K}([0, \infty); \mathbb{R}^d).\]

**Proof.** Rewrite Eq.(2.6) in differential form

\[
\begin{cases}
    dY_t = -f_t(Y_t, Z_t)dt - \langle g_t(Y_t, Z_t), dB_t \rangle + \langle Z_t, dW_t \rangle, \\
    \lim_{T \to \infty} e^{-Kt} \langle Y, Z \rangle_T = 0 \quad \text{a.s.}
\end{cases}
\]

So for arbitrary \( T > 0 \) and \( 0 \leq t \leq T \), Eq.(2.6) is equivalent to

\[
\begin{cases}
    Y_t = Y_T + \int_0^T f_s(Y_s, Z_s)ds + \int_0^T \langle g_s(Y_s, Z_s), dB_s \rangle - \int_0^T \langle Z_s, dW_s \rangle, \\
    \lim_{T \to \infty} e^{-Kt} \langle Y, Z \rangle_T = 0 \quad \text{a.s.}
\end{cases}
\]

We are able to investigate Eq.(2.7) instead of Eq.(2.6).

**Uniqueness.** Let \((Y^1_t, Z^1_t)\) and \((Y^2_t, Z^2_t)\) be two solutions of Eq.(2.7). Define

\[
\begin{align*}
    \bar{Y}_t &= Y^1_t - Y^2_t, \\
    \bar{Z}_t &= Z^1_t - Z^2_t, \\
    t &\geq 0
\end{align*}
\]

Then for \( 0 \leq t \leq T \),

\[
\begin{cases}
    \bar{Y}_t = \bar{Y}_T + \int_0^T \left( f_s(Y^1_s, Z^1_s) - f_s(Y^2_s, Z^2_s) \right)ds \\
    + \int_0^T \left( g_s(Y^1_s, Z^1_s) - g_s(Y^2_s, Z^2_s), dB_s \right) - \int_0^T \langle Z_s, dW_s \rangle, \\
    \lim_{T \to \infty} e^{-Kt} \langle \bar{Y}, \bar{Z} \rangle_T = 0 \quad \text{a.s.}
\end{cases}
\]

Rewrite it by differential form and note that \( \frac{K^\prime}{2} \leq K \),

\[
\begin{cases}
    d\bar{Y}_s = -\left( f_s(Y^1_s, Z^1_s) - f_s(Y^2_s, Z^2_s) \right)ds \\
    -\langle g_s(Y^1_s, Z^1_s) - g_s(Y^2_s, Z^2_s), dB_s \rangle + \langle Z_s, dW_s \rangle, \\
    \lim_{T \to \infty} e^{-Kt} \langle \bar{Y}, \bar{Z} \rangle_T = 0 \quad \text{a.s.}
\end{cases}
\]

Apply Itô’s formula to \( e^{-Ks} |\bar{Y}_s|^2 \), then

\[
\begin{align*}
    d(e^{-Ks} |\bar{Y}_s|^2) &= -K e^{-Ks} |\bar{Y}_s|^2 ds + 2e^{-Ks} \bar{Y}_s d\bar{Y}_s \\
    &\quad -e^{-Ks} g_s(Y^1_s, Z^1_s) - g_s(Y^2_s, Z^2_s))^2 ds + e^{-Ks} |\bar{Z}_s|^2 ds.
\end{align*}
\]
Therefore,
\[
E[e^{-Kt}|\tilde{Y}_{t}|^2] + E\int_t^T e^{-Ks}|\tilde{Z}_s|^2 ds - E\int_t^T Ke^{-Ks}|\tilde{Y}_s|^2 ds
\]
\[
= E[e^{-KT}|\tilde{Y}_{T}|^2] + E\int_t^T 2e^{-Ks}\tilde{Y}_s(f_s(Y^1_s, Z^1_s) - f_s(Y^2_s, Z^1_s))ds
\]
\[
+ E\int_t^T 2e^{-Ks}\tilde{Y}_s(f_s(Y^2_s, Z^2_s) - f_s(Y^2_s, Z^1_s))ds
\]
\[
+ E\int_t^T e^{-Ks}[g_s(Y^1_s, Z^1_s) - g_s(Y^2_s, Z^2_s)]^2 ds
\]
\[
\leq E[e^{-KT}|\tilde{Y}_{T}|^2] - E\int_t^T 2\mu e^{-Ks}|\tilde{Y}_s|^2 ds + E\int_t^T 2e^{-Ks}C|\tilde{Y}_s| |\tilde{Z}_s| ds
\]
\[
+ E\int_t^T e^{-Ks}(C|\tilde{Y}_s| + \alpha|\tilde{Z}_s|)^2 ds.
\]
That is,
\[
E[e^{-Kt}|\tilde{Y}_{t}|^2] + E\int_t^T (1 - \frac{1}{2} - 2\alpha^2)e^{-Ks}|\tilde{Z}_s|^2 ds
\]
\[
+ E\int_t^T (2\mu - K - 4C^2)e^{-Ks}|\tilde{Y}_s|^2 ds
\]
\[
\leq E[e^{-KT}|\tilde{Y}_{T}|^2].
\]  
(2.8)
So
\[
E[e^{-Kt}|\tilde{Y}_{t}|^2] \leq E[e^{-KT}|\tilde{Y}_{T}|^2].
\]

With $K$ replaced by $K'$ it is also valid, so
\[
E[e^{-K't}|\tilde{Y}_{t}|^2] \leq e^{(K'-K)T}E[e^{-KT}|\tilde{Y}_{T}|^2].
\]

Since $Y^1_t, Y^2_t \in S^{p,K}([0,\infty);\mathbb{R}^1)$ and $S^{p,K}([0,\infty);\mathbb{R}^1) \subset S^{2,-K}([0,\infty);\mathbb{R}^1)$, $E[e^{-KT}|\tilde{Y}_T|^2] \leq C_p$ for all $T \geq 0$. Here and in the following, $C_p$ is a generic constant. Taking the limit of $T$, we have
\[
E[e^{-K't}|\tilde{Y}_{t}|^2] = 0.
\]
That is
\[
\tilde{Y}_t = 0 \quad \text{a.s.}
\]

From (2.8), for a.e. $t$,
\[ Z_t = 0 \quad \text{a.s.} \]

**Existence.** For each \( n \in \mathbb{N} \), we define a sequence of BDSDEs as follows,

\[
Y^n_t = \int_t^\infty f_s(Y^n_s, Z^n_s)ds + \int_t^\infty \langle g_s(Y^n_s, Z^n_s), dB_s \rangle - \int_t^\infty \langle Z^n_s, dW_s \rangle. \tag{2.9}
\]

Also let \((Y^n_t, Z^n_t)_{t \geq n} = (0, 0)\), and according to \([25]\) (or Theorem 2.1), Eq.(2.9) has a unique solution \((Y^n_t, Z^n_t) \in S^{2-K}([0, \infty); \mathbb{R}^n) \cap M^{2-K}([0, \infty); \mathbb{R}^n \times M^{2-K}([0, \infty); \mathbb{R}^n \times d)\). Here we want to prove under conditions of Theorem 2.3 that \(Y^n \in S^{p-K}([0, \infty); \mathbb{R}^n)\). We first prove Lemma 2.4 below.

**Lemma 2.4** Let \((Y^n_t)_{t \geq 0}\) be the solution of Eq.(2.9), then under the conditions of Theorem 2.3, \(Y^n_t \in S^{p-K}([0, \infty); \mathbb{R}^n)\).

**Proof.** Apply Itô’s formula to \(e^{-K^r}[Y^n_r]^p\), we have

\[
d(e^{-K^r}[Y^n_r]^p) = -Ke^{-K^r}[Y^n_r]^p + \frac{p}{2}e^{-K^r}[Y^n_r]^{p-2}(-2Y^n_r f_r(Y^n_r, Z^n_r))dr
\]

\[
-|g_r(Y^n_r, Z^n_r)|^2 dr + |Z^n_r|^2 dr - 2Y^n_r (g_r(Y^n_r, Z^n_r), dB_r)
\]

\[
+2Y^n_r (Z^n_r, dW_r) + \frac{p(p-2)}{8}e^{-K} |Y^n_r|^{p-4}(-4|Y^n_r|^2 |g_r(Y^n_r, Z^n_r)|^2 dr
\]

\[
+4|Y^n_r|^2 |Z^n_r|^2 dr).
\]

Then

\[
e^{-K^s}[Y^n_s]^p + \frac{p(p-1)}{2} \int_s^\infty e^{-K^r}[Y^n_r]^{p-2} |Z^n_r|^2 dr - K \int_s^\infty e^{-K^r}[Y^n_r]^{p}dr
\]

\[
= p \int_s^\infty e^{-K^r}[Y^n_r]^{p-2} Y^n_r f_r(Y^n_r, Z^n_r)dr
\]

\[
+ p \int_s^\infty e^{-K^r}[Y^n_r]^{p-2} g_r(Y^n_r, Z^n_r), dB_r
\]

\[
+ \frac{p(p-1)}{2} \int_s^\infty e^{-K^r}|Y^n_r|^{p-2} |g_r(Y^n_r, Z^n_r)|^2 dr
\]

\[
- p \int_s^\infty e^{-K^r}[Y^n_r]^{p-2} Y^n_r (Z^n_r, dW_r)
\]

\[
\leq p \int_s^\infty e^{-K^r}|Y^n_r|^{p-2} Y^n_r (f_r(Y^n_r, Z^n_r) - f_r(0, Z^n_r))dr
\]

\[
+ p \int_s^\infty e^{-K^r}|Y^n_r|^{p-2} Y^n_r (f_r(0, Z^n_r) - f_r(0, 0))dr
\]

\[
+ p \int_s^\infty e^{-K^r}|Y^n_r|^{p-2} Y^n_r f_r(0, 0)dr
\]
+ \mu \int_s^n e^{-K_r} |Y_r|^p \, dr + \frac{\mu}{2} \int_s^n e^{-K_r} |Y_r|^{p-2} \langle f_r(0,0), |Y_r|^2 \rangle \, dr \\
+ \nu \int_s^n e^{-K_r} |Y_r|^p \, dr + p \int_s^n e^{-K_r} |Y_r|^{p-2} \langle B_r, |Y_r|^2 \rangle \, dr \\
+ 2\nu (p-1) \int_s^n e^{-K_r} |Y_r|^p \, dr - \nu \int_s^n e^{-K_r} |Y_r|^{p-2} \langle B_r, |Y_r|^2 \rangle \, dr \\
\leq -\mu \int_s^n e^{-K_r} |Y_r|^p \, dr + pC^2 \int_s^n e^{-K_r} |Y_r|^{p-2} \langle B_r, |Y_r|^2 \rangle \, dr \\
+ \frac{3\mu p}{2} \int_s^n e^{-K_r} |Y_r|^{p-2} \langle f_r(0,0), |Y_r|^2 \rangle \, dr \\
+ \nu \int_s^n e^{-K_r} |Y_r|^p \, dr + p \int_s^n e^{-K_r} |Y_r|^{p-2} \langle B_r, |Y_r|^2 \rangle \, dr \\
+ 2\nu (p-1)C^2 \int_s^n e^{-K_r} |Y_r|^p \, dr + 2p(p-1)\alpha^2 \int_s^n e^{-K_r} |Y_r|^{p-2} \langle B_r, |Y_r|^2 \rangle \, dr \\
+ p\nu \int_s^n e^{-K_r} |Y_r|^p \, dr - \nu \int_s^n e^{-K_r} |Y_r|^{p-2} \langle B_r, |Y_r|^2 \rangle \, dr.

Here $\varepsilon$ is an arbitrary positive real number. Just as above, using the Hölder inequality and Young inequality, we have
Due to the arbitrariness of $\varepsilon$ and (H.3), we can fix $\varepsilon > 0$ s.t. $p\mu - K - p(2p - 1)C^2 - \varepsilon > 0$. Note here $C_p$ depends on $\varepsilon$, but for fixed $\varepsilon$, $C_p$ is a fixed number. Taking expectation on (2.10), we have

\[
E[\int_0^\infty e^{-Kt} |Y_r^n|^p |Z_r^n|^2 dr] + E[\int_0^\infty e^{-Kt} |Y_r^n|^p dr] \leq C_p E[\int_0^\infty e^{-Kt} |f_r(0,0)|^p dr] + C_p E[\int_0^\infty e^{-Kt} |g_r(0,0)|^p dr] < \infty.
\]

By B-D-G inequality, we get from (2.10) that

\[
E[\sup_{t \geq 0} e^{-Kt} |Y_t^n|^p] \\
\leq C_p E[\int_0^\infty e^{-Kt} (|f_r(0,0)|^p + |g_r(0,0)|^p) dr] \\
+ pE[\int_0^\infty e^{-2Kt} |Y_r^n|^2 |Z_r^n|^2 |Z_r^n|^2 |Z_r^n|^2 dr] \\
+ \frac{1}{3} E[\sup_{t \geq 0} e^{-Kt} |Y_t^n|^p] + C_p E[\int_0^\infty e^{-Kt} |Y_r^n|^p |Z_r^n|^2 |Z_r^n|^2 dr] \\
+ \frac{1}{3} E[\sup_{t \geq 0} e^{-Kt} |Y_t^n|^p] + C_p E[\int_0^\infty e^{-Kt} |Y_r^n|^p |Z_r^n|^2 dr].
\] (2.12)

That is

\[
\frac{1}{3} E[\sup_{t \geq 0} e^{-Kt} |Y_t^n|^p]
\]
The proof of Lemma 2.4 also works with Remark 2.5
By (H.1) and (2.11),

\[ Y_n \in S^{p,-K}([0, \infty); \mathbb{R}^1). \]

Lemma 2.4 is proved.

\[ \diamond \]

**Remark 2.5** The proof of Lemma 2.4 also works with \( p \) replaced by 2. Note that if \( f,(0,0) \in M^{p,-K}([0, \infty); \mathbb{R}^1) \), then by Hölder inequality

\[
E[\int_0^\infty e^{-Kr}|f_s(0,0)|^2 ds] \\
\leq E[\left( \int_0^\infty e^{\frac{p-2}{2}Kr} ds \right)^\frac{1}{2} \left( \int_0^\infty e^{-Kr}|f_s(0,0)|^2 ds \right)^\frac{1}{2}] < \infty,
\]

therefore \( f,(0,0) \in M^{2,-K}([0, \infty); \mathbb{R}^1) \). By the same method we have \( g,(0,0) \in M^{2,-K}([0, \infty); \mathbb{R}^1) \) as well. So it can be easily seen in (2.11) with \( p \) replaced by 2 or follows from [25] that

\[ (Y^n, Z^n) \in M^{2,-K}([0, \infty); \mathbb{R}^1) \times M^{2,-K}([0, \infty); \mathbb{R}^d). \]

Then back to the proof of Theorem 2.3. We want to show that \((Y^n, Z^n)\) is a Cauchy sequence in the space of \( S^{p,-K}([0, \infty); \mathbb{R}^1) \cap M^{2,-K}([0, \infty); \mathbb{R}^1) \times M^{2,-K}([0, \infty); \mathbb{R}^d)\) with the norm

\[
\left( \left( E[\sup_{t \geq 0} e^{-Kt}|\cdot|^p] \right)^\frac{1}{p} + E[\int_0^\infty e^{-Kr}|\cdot|^2 dr] + E[\int_0^\infty e^{-Kr}|\cdot|^2 dr] \right)^\frac{1}{2},
\]

as in Pardoux [22]. Firstly we show that, for \( m, n \in \mathbb{N} \) and \( m \geq n \),

\[
\lim_{n,m \to \infty} E[\sup_{t \geq 0} e^{-Kt}|Y^m_t - Y^n_t|^p] = 0.
\]

Define \( \tilde{Y}^m,n = Y^m_t - Y^n_t, \tilde{Z}^m,n = Z^m_t - Z^n_t \).

(i) When \( n \leq t \leq m \),

\[
\tilde{Y}^m,n_t = Y^m_t - Y^n_t = \int_t^m f_s(Y^m_s, Z^m_s) ds + \int_t^m \langle g_s(Y^m_s, Z^m_s), d^s B_s \rangle - \int_t^m \langle Z^m_s, dW_s \rangle.
\]

By (2.10), we have

\[
E[\int_n^m e^{-Kr}|Y^m_r|^2 dr] + E[\int_n^m e^{-Kr}|Y^m_r|^2 dr] \\
\leq C_p E[\int_n^m e^{-Kr}(|f_r(0,0)|^2 + |g_r(0,0)|^2) dr].
\]

(2.14)
As $n, m \to \infty$, the right hand side converges to 0. Similar method as in (2.12) and (2.13), by (2.10) and (2.14), we have

$$
E[ \sup_{n \leq t \leq m} e^{-Kt}|Y^m_t|^p] \leq C_p E\left[ \int_{n}^{m} e^{-Kr}(|f_r(0,0)|^p + |g_r(0,0)|^p)dr \right] \to 0, \text{ as } n, m \to \infty.
$$

(ii) When $0 \leq t \leq n$,

$$
\tilde{Y}^m_n = Y^m_n + \int_{t}^{n} \tilde{f}_r dr + \int_{t}^{n} \langle \tilde{g}_r, d^1 B_r \rangle - \int_{t}^{n} \langle \tilde{Z}^m_n, dW_r \rangle.
$$

Here

$$
\tilde{f}_r = f_r(Y^m_r, Z^m_r) - f_r(Y^m_r, Z^m_r),
$$
$$
\tilde{g}_r = g_r(Y^m_r, Z^m_r) - g_r(Y^m_r, Z^m_r).
$$

Applying Itô’s formula to $e^{-Kr}|\tilde{Y}^m_r|^p$, Hölder inequality and Young inequality, we have for $s \leq n$,

$$
e^{-Ks}|\tilde{Y}^m_s|^p + \frac{p(p-1)}{2} \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p - 2|\tilde{Y}^m_r|^2 \tilde{f}_r dr
$$
$$
- K \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p dr
$$
$$
= e^{-Kn}|Y^m_n|^p + p \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p - 2|\tilde{Y}^m_r|^2 \tilde{f}_r dr
$$
$$
+ p \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p - 2|\tilde{Y}^m_r|^2 \tilde{g}_r dr
$$
$$
+ \frac{p(p-1)}{2} \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p dr
$$
$$
- p \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p - 2|\tilde{Y}^m_r|^2 \tilde{Z}^m_n, dW_r
$$
$$
\leq e^{-Kn}|Y^m_n|^p + p \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p - 2|\tilde{Y}^m_r|^2 (f_r(Y^m_r, Z^m_r) - f_r(Y^m_r, Z^m_r)) dr
$$
$$
+ p \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p - 2|\tilde{Y}^m_r|^2 (f_r(Y^m_r, Z^m_r) - f_r(Y^m_r, Z^m_r)) dr
$$
$$
+ p \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p - 2|\tilde{Y}^m_r|^2 \langle \tilde{g}_r, d^1 B_r \rangle
$$
$$
+ \frac{p(p-1)}{2} \int_{s}^{n} e^{-Kr}|\tilde{Y}^m_r|^p - 2|\tilde{Y}^m_r|^2 (C|\tilde{Y}^m_r|^p + \alpha|\tilde{Z}^m_n|^2) dr
$$
From (i), the right hand side converges to 0, as $n, m \to \infty$.

By (2.16) and (2.17), similar method as in (2.12) and (2.13), we have

\[
-p \int_s^n e^{-Kr} |Y_{r,m}^n|^p |p - 2Y_{r,m}^n| \{Z_r, dW_r\} \\
\leq e^{-Kn} |Y_n^m|^p - p \mu \int_s^n e^{-Kr} |Y_{r,m}^n|^p dr \\
+ p \int_s^n e^{-Kr} |Y_{r,m}^n|^p - 2C |Y_{r,m}^n| \{Z_r, dW_r\} \\
+ p \int_s^n e^{-Kr} |Y_{r,m}^n|^p - 2Y_{r,m}^n \{\bar{g}_r, d^1 B_r\} \\
+ p(p - 1) \int_s^n e^{-Kr} |Y_{r,m}^n|^p - 2(2C^2 |Y_{r,m}^n|^2 + 1/2 |Z_r|^2) dr \\
- p \int_s^n e^{-Kr} |Y_{r,m}^n|^p - 2Y_{r,m}^n \{Z_r, dW_r\}.
\]

That is

\[
e^{-Kr} |Y_{s,m}^n|^p + \frac{p}{36} (14p - 23) \int_s^n e^{-Kr} |Y_{r,m}^n|^p - 2Y_{r,m}^n|^2 dr \\
+(p \mu - K - p^2 C^2) \int_s^n e^{-Kr} |Y_{r,m}^n|^p dr \\
\leq e^{-Kn} |Y_n^m|^p + p \int_s^n e^{-Kr} |Y_{r,m}^n|^p - 2Y_{r,m}^n \{\bar{g}_r, d^1 B_r\} \\
- p \int_s^n e^{-Kr} |Y_{r,m}^n|^p - 2Y_{r,m}^n \{Z_r, dW_r\}. \tag{2.16}
\]

So

\[
E \int_0^n e^{-Kr} |Y_{r,m}^n|^p |p - 2Y_{r,m}^n|^2 dr + E \int_0^n e^{-Kr} |Y_{r,m}^n|^p dr \\
\leq C_p E[e^{-Kn} |Y_n^m|^p]. \tag{2.17}
\]
\[
\frac{1}{\delta} E[ \sup_{0 \leq t \leq n} e^{-Kt} |\hat{Y}_t^{m,n}|^p] \\
\leq E[e^{-Kn} |Y^n_t|^p] + C_p E[\int_0^n e^{-Kr} |\hat{Y}_r^{m,n}|^{p-2} |\hat{Z}_r^{m,n}|^2 dr] \\
+ C_p E[\int_0^n e^{-Kr} |\hat{Y}_r^{m,n}|^{p}] \\
\leq C_p E[e^{-Kn} |Y^n_t|^p] \to 0, \quad \text{as } n, m \to \infty.
\]

From (i) (ii), we have for \(m, n \in \mathbb{N}\),
\[
\lim_{n,m \to \infty} E[\sup_{t \geq 0} e^{-Kt} |Y^n_t - Y^n_t|^p] = 0.
\]

Furthermore the above arguments also hold for \(p = 2\) in (2.15) and (2.17), by Remark 2.5, as \(n, m \to \infty\) we have
\[
E[\int_0^\infty e^{-Kr} |\hat{Y}_r^{m,n}|^2 dr] + E[\int_0^\infty e^{-Kr} |\hat{Z}_r^{m,n}|^2 dr] \to 0.
\]

Therefore, \((Y^n, Z^n)\) is a Cauchy sequence in the Banach space
\[
S^{p,-K}([0, \infty); \mathbb{R}^n) \cap M^{2,-K}([0, \infty); \mathbb{R}^n) \times M^{2,-K}([0, \infty); \mathbb{R}^{n \times d}).
\]

We take \((Y_t, Z_t)_{t \geq 0}\) as the limit of \((Y^n_t, Z^n_t)_{t \geq 0}\) in the space mentioned above. We will show that \((Y_t, Z_t)_{t \geq 0}\) is the solution of Eq.(2.6). Since for \(t \leq n\),
\[
Y^n_t = \int_t^n f_s(Y^n_s, Z^n_s) ds + \int_t^n \langle g_s(Y^n_s, Z^n_s), d^\dagger B_s \rangle - \int_t^n \langle Z^n_s, dW_s \rangle,
\]

it turns out that for \(t \leq n\),
\[
e^{-K't}Y^n_t = \int_t^n e^{-K's} f_s(Y^n_s, Z^n_s) ds + \int_t^n \frac{K'}{2} e^{-K's} Y^n_s ds \\
+ \int_t^n e^{-K's} \langle g_s(Y^n_s, Z^n_s), d^\dagger B_s \rangle \\
- \int_t^n e^{-K's} \langle Z^n_s, dW_s \rangle.
\]

We will show that Eq.(2.18) converges to Eq.(2.6) as \(m \to \infty\). We verify this term by term. For the first term,
\[
E[|e^{-K't}Y^n_t - e^{-K't}Y_t|^2] \leq E[\sup_{t \geq 0} e^{-Kt} |Y^n_t - Y_t|^2] \to 0.
\]
For the second term, by Hölder inequality,
\begin{align*}
E & \left[ \int_t^n e^{-\frac{K'}{2}s} f_s(Y^n_s, Z^n_s) ds - \int_t^\infty e^{-\frac{K'}{2}s} f_s(Y_s, Z_s) ds \right]^2 \\
& \leq 2E \left[ \left| \int_t^n e^{-\frac{K'}{2}s} (f_s(Y^n_s, Z^n_s) - f_s(Y_s, Z_s)) ds \right|^2 \\
& + 2E \left[ \int_t^\infty e^{-\frac{K'}{2}s} f_s(Y_s, Z_s) ds \right]^2 \\
& \leq 2E \left[ \int_t^n e^{-(K' - K)s} ds \int_t^n e^{-Ks} |f_s(Y^n_s, Z^n_s) - f_s(Y_s, Z_s)|^2 ds \\
& + 2E \int_t^\infty e^{-(K' - K)s} ds \int_t^n e^{-Ks} |f_s(Y_s, Z_s)|^2 ds \\
& \leq C_p E \left[ \int_0^n e^{-Ks} |Y^n_s - Y_s|^2 ds \right] + C_p E \left[ \int_0^n e^{-Ks} |Z^n_s - Z_s|^2 ds \right] \\
& + C_p E \left[ \int_t^n e^{-Ks} |f_s(Y_s, 0)|^2 ds \right] \to 0.
\end{align*}

The third term can be seen similarly
\begin{align*}
E & \left[ \int_t^n K' \frac{K'}{2} e^{-\frac{K'}{2}s} Y^n_s ds - \int_t^\infty K' \frac{K'}{2} e^{-\frac{K'}{2}s} Y_s ds \right]^2 \\
& \leq 2E \left[ \left| \int_t^n K' \frac{K'}{2} e^{-\frac{K'}{2}s} (Y^n_s - Y_s) ds \right|^2 \\
& + 2E \left[ \int_t^\infty K' \frac{K'}{2} e^{-\frac{K'}{2}s} Y_s ds \right]^2 \\
& \leq 2E \left[ \int_t^n K' \frac{K'}{4} e^{-(K' - K)s} ds \int_t^n e^{-Ks} |Y^n_s - Y_s|^2 ds \\
& + 2E \int_t^\infty K' \frac{K'}{4} e^{-(K' - K)s} ds \int_t^n e^{-Ks} |Y_s|^2 ds \\
& \leq C_p E \left[ \int_0^n e^{-Ks} |Y^n_s - Y_s|^2 ds \right] + C_p E \left[ \int_t^n e^{-Ks} |Y_s|^2 ds \right] \to 0.
\end{align*}

For the fourth term, noting $K' > K$ and by Itô’s isometry, we have
\begin{align*}
E & \left[ \int_t^n e^{-\frac{K'}{2}s} \langle g_s(Y^n_s, Z^n_s), d^1 B_s \rangle - \int_t^\infty e^{-\frac{K'}{2}s} \langle g_s(Y_s, Z_s), d^1 B_s \rangle \right]^2 \\
& \leq 2E \left[ \left| \int_t^n e^{-\frac{K'}{2}s} \langle g_s(Y^n_s, Z^n_s), d^1 B_s \rangle - g_s(Y_s, Z_s), d^1 B_s \rangle \right|^2 \\
& + 2E \left[ \int_t^\infty e^{-\frac{K'}{2}s} \langle g_s(Y_s, Z_s), d^1 B_s \rangle \right|^2 \\
& = 2E \int_t^n e^{-K's} |g_s(Y^n_s, Z^n_s) - g_s(Y_s, Z_s)|^2 ds.
\end{align*}
The last term is
\begin{align*}
E[ & \int_{t}^{n} e^{-\frac{K'}{2}s}(Z^n_s, dW_s) - \int_{t}^{n} e^{-\frac{K'}{2}s}(Z_s, dW_s)^2] \\
& \leq 2E[ \int_{t}^{n} e^{-\frac{K'}{2}s}(Z^n_s - Z_s, dW_s)^2] + 2E[ \int_{n}^{\infty} e^{-\frac{K'}{2}s}(Z_s, dW_s)^2] \\
& = 2E[ \int_{t}^{n} e^{-K's}|Z^n_s - Z_s|^2 ds] + 2E[ \int_{n}^{\infty} e^{-K's}|Z_s|^2 ds] \\
& \leq 2E[ \int_{t}^{n} e^{-K's}|Z^n_s - Z_s|^2 ds] + 2E[ \int_{n}^{\infty} e^{-K's}|Z_s|^2 ds] \longrightarrow 0.
\end{align*}

That is to say $(Y_t, Z_t)_{t \geq 0}$ is the solution of Eq.(2.6). The proof of Theorem 2.3 is completed.

### 3 The stationary solution of BDSDE

In Section 2, we proved the existence and uniqueness of the solution of Eq.(1.1). In this section we will show that the solution is stationary under some conditions.

Let $\theta_t = (\theta^1_t, \theta^2_t): \Omega^1 \times \Omega^2 \longrightarrow \Omega^1 \times \Omega^2$, $t \geq 0$, be defined by $\theta_t(\omega^1, \omega^2) = (\theta^1_t(\omega^1), \theta^2_t(\omega^2))$. Here $\theta^1_t$ and $\theta^2_t$, $t \geq 0$, are shifts on $(\Omega^1, \mathcal{F}^B_t, P^1)$ and $(\Omega^2, \mathcal{F}^W_t, P^2)$ respectively, s.t. for arbitrary $s, t, u \geq 0$,

(i) $(P^1 \times P^2) \cdot (\theta^1_t, \theta^2_t)^{-1} = P^1 \times P^2$;

(ii) $\theta^1_0 = I^1$, $\theta^2_0 = I^2$, where $I^1$ and $I^2$ are the identity transformation on $\Omega^1$ and $\Omega^2$ respectively;

(iii) $\theta^1_s \circ \theta^1_t = \theta^1_{s+t}$, $\theta^2_s \circ \theta^2_t = \theta^2_{s+t}$;

(iv) $\theta^1_s \circ B_t = B_{s+t} - B_t$, $\theta^2_s \circ W_s = W_{s+t} - W_t$;
(v) \((\theta^1)^{-1}(\mathcal{F}_{s,u}^B) = \mathcal{F}_{s+t,u+t}^B\), \((\theta^2)^{-1}(\mathcal{F}_{s,u}^W) = \mathcal{F}_{s+t,u+t}^W\).

Also for arbitrary measurable \(\phi : (\Omega^1 \times \Omega^2, \mathcal{F}^1 \times \mathcal{F}^2, P^1 \times P^2) \rightarrow \mathbb{R}\), we define

\[
\theta \circ \phi(\omega^1, \omega^2) = \phi(\theta(\omega^1, \omega^2)).
\]

Then we give the added condition which makes \(f_1(y, z)\) and \(g_1(y, z)\) stationary w.r.t. \(\theta_r\).

(H.4). \(\theta_r \circ f_s(y, z) = f_{s+r}(y, z), \theta_r \circ g_s(y, z) = g_{s+r}(y, z)\) for arbitrary \(r, s \geq 0\) and fixed \(y, z\).

We will prove

**Proposition 3.1** Under conditions (H.1), (H.2), (H.3), (H.4), Eq. (2.6) has a stationary solution \((Y_t, Z_t)_{t \geq 0}\) s.t. for fixed \(r \geq 0\),

\[
\theta_r \circ Y_t = Y_{t+r}, \quad \theta_r \circ Z_t = Z_{t+r} \quad \text{a.s.}
\]

**Proof.** Let \(\hat{B}_s = B_{T'-s} - B_T\) for arbitrary \(T' > 0\) and \(-\infty < s \leq T'\). Then \(\hat{B}_s\) is a Brownian motion with \(\hat{B}_0 = 0\). For fixed \(r \geq 0\), acting \(\theta^1_r\) on \(\hat{B}_s\), we have

\[
\theta^1_r \circ \hat{B}_s = \theta^1_r \circ (B_{T'-s} - B_T) = B_{T'-s+r} - B_{T+r}
\]

\[
= (B_{T'-s+r} - B_T') - (B_{T+r} - B_T) = \hat{B}_{s-r} - \hat{B}_{-r}.
\]

So for \(0 \leq t \leq T \leq T'\) and a process \(h\), which satisfies the condition in (2.2),

\[
\theta_r \circ \int_t^T \langle h_s, d^1 B_s \rangle = -\theta_r \circ \int_{T'-T}^{T'-t} \langle h_{T'-s}, d\hat{B}_s \rangle
\]

\[
= -\int_{T'-T}^{T'-t} (\theta_r \circ h_{T'-s}, d\hat{B}_{s-r})
\]

\[
= -\int_{T'-T}^{T'-t-r} (\theta_r \circ h_{T'-s-r}, d\hat{B}_s)
\]

\[
= \int_{T+r}^{T+r} \langle \theta_r h_{s-r}, d^1 B_s \rangle. \tag{3.1}
\]

As \(T'\) can be chosen arbitrarily, so we can get for arbitrary \(T \geq 0, 0 \leq t \leq T, r \geq 0\),

\[
\theta_r \circ \int_t^T \langle h_s, d^1 B_s \rangle = \int_{T+r}^{T+r} \theta_r \langle h_{s-r}, d^1 B_s \rangle.
\]

By (H.1) and (H.2), it is easy to see that \(g(Y, Z)\) satisfies the condition in (2.2), hence by (H.4) and (3.1)
\[
\theta_r \circ \int_t^{T_r} (g_s(Y_s, Z_s), d^1 B_s) = \int_t^{T_r} (g_s(\theta_r Y_{s-r}, \theta_r Z_{s-r}), d^1 B_s) \quad \text{a.s.} \quad (3.2)
\]

We consider Eq.(2.7) which is equivalent to Eq.(2.6). Pass \( \theta_r \) on both sides of Eq.(2.7) and by (3.2), \( \theta_r \circ Y_t \) satisfies the following equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
\theta_r \circ Y_t = \theta_r \circ Y_T + \int_t^{T_r} f_s(\theta_r \circ Y_{s-r}, \theta_r \circ Z_{s-r}) ds \\
+ \int_t^{T_r} (g_s(\theta_r \circ Y_{s-r}, \theta_r \circ Z_{s-r}), d^1 B_s) \\
- \int_t^{T_r} (\theta_r \circ Z_{s-r}, dW_s), \\
\lim_{T \to \infty} e^{-\frac{K}{2}(T+r)}(\theta_r \circ Y_T) = 0 \\
\end{array} \right. \\
\end{aligned}
\]

(3.3)

On the other hand, from Eq.(2.7), it follows that

\[
\begin{aligned}
\left\{ \begin{array}{l}
Y_{t+r} = Y_{T+r} + \int_t^{T_r} f_s(Y_s, Z_s) ds + \int_t^{T_r} (g_s(Y_s, Z_s), d^1 B_s) \\
- \int_t^{T_r} (Z_s, dW_s), \\
\lim_{T \to \infty} e^{-\frac{K}{2}(T+r)}Y_{T+r} = 0 \\
\end{array} \right. \\
\end{aligned}
\]

(3.4)

Let \( \hat{Y} = \theta_r \circ Y_{r-r}, \hat{Z} = \theta_r \circ Z_{r-r} \). Compare (3.3) with (3.4), by the uniqueness of solution of Eq.(2.7),

\[
\hat{Y} = Y, \quad \hat{Z} = Z \quad \text{a.s.}
\]

We proved the desired result.

\( \diamond \)

Before we give a type of FBDSDE, we first consider the SDE on \((\Omega^2, \mathcal{F}_\infty^W, \mathbb{P}^2)\) below. For \(0 \leq t \leq s\),

\[
X_{t,x}^s = x + \int_t^s b(X_u^{t,x}) du + \int_t^s \sigma(X_u^{t,x}) dW_u. \quad (3.5)
\]

The coefficients in Eq.(3.5) satisfy

(H.5) : \( b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d, \sigma(\cdot) : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) are globally Lipschitz continuous with Lipschitz constant \( L \) which satisfies \( K - pL - \frac{p(p-1)}{2} L^2 > 0 \) for \( p, K \) in conditions of considered BDSDE.

We regulate that \( X_{t,x}^s \) is defined on \([0, \infty)\) by defining \( (X_{t,x}^s)_{s < t} = x \). For fixed \( r \geq 0 \), act \( \theta_r^2 \) on both sides of Eq.(3.5), and it follows that

\[
\theta_r^2 \circ X_{t,x}^s = x + \int_t^s b(\theta_r^2 \circ X_u^{t,x}) du + \int_t^s \sigma(\theta_r^2 \circ X_u^{t,x}) dW_u. \\
\]

Equivalently,
\[
\theta^2_r \circ X_{t,x}^s = x + \int_{t+r}^{s+r} b(\theta^2_r \circ X_{t,x}^{s-r})du + \int_{t+r}^{s+r} \sigma(\theta^2_r \circ X_{t,x}^{s-r})dW_u.
\]

By the same method as in the proof of Proposition 3.1, we have for fixed \( r \geq 0 \),

\[
\theta^2_r \circ X_{t,x}^s = X_{t,x}^{s+r} \text{ a.s.} \tag{3.6}
\]

We assume that

\[
(H.1)' : f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^1, \ g : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \rightarrow \mathbb{R}^l \text{ are jointly measurable.}
\]

\[
(H.2)' : \text{There exist four constants } C_0, C_1, C > 0, 0 < \alpha < \frac{1}{3}, \text{ s.t. for any (} x_1, y_1, z_1, (x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d, \]

\[
\left| f(x_1, y_1, z_1) - f(x_2, y_2, z_2) \right| \leq C_1 |x_1 - x_2| + C_0 |y_1 - y_2| + C |z_1 - z_2|, \]

\[
\left| g(x_1, y_1, z_1) - g(x_2, y_2, z_2) \right| \leq C_1 |x_1 - x_2| + C |y_1 - y_2| + \alpha |z_1 - z_2|.
\]

\[
(H.3)' : \text{There exists a constant } \mu > 0, \text{ s.t. for } p > d + 2, K < K' < 2K, 2 \mu - \frac{1}{p} K' - p(2p - 1)C^2 > 0 \text{ and for any } (\omega^1, \omega^2, t) \in \Omega^1 \times \Omega^2 \times [0, \infty), \]

\[
y_1, y_2 \in \mathbb{R}^1, z \in \mathbb{R}^d, \]

\[
(y_1 - y_2)(f_t(y_1, z) - f_t(y_2, z)) \leq -\mu |y_1 - y_2|^2.
\]

Now we give FBDSDE on infinite time interval below. For \( s \geq 0 \),

\[
e^{-\frac{K'}{2} r} Y_{t,x}^s = \int_s^\infty e^{-\frac{K'}{2} r} f(X_{t,x}^r, Y_{t,x}^r, Z_{t,x}^r)dr + \int_s^\infty \frac{K'}{2} e^{-\frac{K'}{2} r} Y_{t,x}^r dr
\]

\[
+ \int_s^\infty e^{-\frac{K'}{2} r} g(X_{t,x}^r, Y_{t,x}^r, Z_{t,x}^r) d^1 B_r - \int_s^\infty e^{-\frac{K'}{2} r} (Z_{t,x}^r, dW_r),
\]

which is equivalent to for \( 0 \leq s \leq T \),

\[
\begin{cases}
Y_{t,x}^s = Y_{T,x}^s + \int_s^T f(X_{t,x}^u, Y_{t,x}^u, Z_{t,x}^u)du \\
+ \int_s^T g(X_{t,x}^u, Y_{t,x}^u, Z_{t,x}^u) d^1 B_u - \int_s^T (Z_{t,x}^u, dW_u),
\end{cases}
\tag{3.7}
\]

We have the following proposition.

**Proposition 3.2** Under conditions \((H.1)' , (H.2)' , (H.3)' , (H.5)\), Eq.(3.7) has a unique solution

\((Y_{t,x}^s, Z_{t,x}^s) \in S^{p,-K}([0, \infty); \mathbb{R}^1) \cap M^{2,-K}([0, \infty); \mathbb{R}^1) \times M^{2,-K}([0, \infty); \mathbb{R}^d)\).

**Proof.** Let
\[ \hat{f}_s(y, z) = f(X_{s}^{t,x}, y, z), \quad \hat{g}_s(y, z) = g(X_{s}^{t,x}, y, z). \]

We just need to verify that \( \hat{f}, \hat{g} \) satisfy conditions (H.1), (H.2), (H.3) in Theorem 1.1. It is obvious that \( \hat{f}, \hat{g} \) satisfy (H.2), (H.3), and we show that \( \hat{f}, \hat{g} \) satisfy (H.1) as well, i.e.

\[
E[\int_0^\infty e^{-Ks}|\hat{f}_s(0, 0)|^p ds] < \infty, \quad E[\int_0^\infty e^{-Ks}|\hat{g}_s(0, 0)|^p ds] < \infty.
\]

Notice

\[
E[\int_0^\infty e^{-Ks}|\hat{f}_s(0, 0)|^p ds] = E[\int_0^\infty e^{-Ks}|f(X_{s}^{t,x}, 0, 0)|^p ds]
\leq C_p E[\int_0^\infty e^{-Ks}|f(X_{s}^{t,x}, 0, 0) - f(0, 0, 0)|^p ds] + C_p E[\int_0^\infty e^{-Ks}|f(0, 0, 0)|^p ds]
\leq C_p E[\int_0^\infty e^{-Ks}C_1^p|X_{s}^{t,x}|^p ds] + C_p E[\int_0^\infty e^{-Ks}|f(0, 0, 0)|^p ds].
\]

Now applying Itô’s formula to \( e^{-K_t}|X_{t}^{t,x}|^p \), we have

\[
d(e^{-K_t}|X_{t}^{t,x}|^p)
= -Ke^{-K_t}|X_{t}^{t,x}|^p dr + \frac{p}{2}e^{-K_t}|X_{t}^{t,x}|^{p-2} \langle 2(X_{t}^{t,x}, b(X_{s}^{t,x}))_utils\rangle dr
+ \|\sigma(X_{t}^{t,x})\|^2 dr + 2\langle X_{t}^{t,x}, \sigma(X_{t}^{t,x}) dW_t \rangle
+ \frac{p(p-2)}{2}e^{-K_t}|X_{t}^{t,x}|^{p-4} \langle \sigma(X_{t}^{t,x}) \sigma^*(X_{t}^{t,x}) X_{t}^{t,x}, X_{t}^{t,x} \rangle_{dr}.
\]

Then by Lipschitz condition and Young inequality, for \( 0 \leq t \leq s \),

\[
e^{-K_s}|X_{s}^{t,x}|^p + \int_t^s Ke^{-K_r}|X_{r}^{t,x}|^p dr
\leq e^{-K_t}|x|^p + \int_t^s pe^{-K_r}|X_{r}^{t,x}|^{p-2} \langle X_{r}^{t,x}, b(X_{s}^{t,x}) \rangle_{dr}
+ \int_t^s \frac{p(p-1)}{2}e^{-K_r}|X_{r}^{t,x}|^{p-2}\|\sigma(X_{t}^{t,x})\|^2_{dr}
+ \int_t^s pe^{-K_r}|X_{r}^{t,x}|^{p-2} \langle X_{r}^{t,x}, \sigma(X_{t}^{t,x}) dW_t \rangle
\leq e^{-K_t}|x|^p + \int_t^s pe^{-K_r}|X_{r}^{t,x}|^{p-2}|X_{r}^{t,x}||b(X_{s}^{t,x})||_{dr}
Due to the arbitrariness of $\varepsilon$, we have

$$E\left[\int_t^s e^{-Kr} |X_{t,r}|^p dr \right] \leq e^{-Kt} |x|^p + C_p E\left[\int_t^s e^{-Kr} |b(0)|^p + \|\sigma(0)\|^p dr \right] < \infty.$$
\[ E\left[ \int_0^\infty e^{-Kr}|X^t_x|^{p}dr \right] < \infty. \tag{3.8} \]

So

\[ E\left[ \int_0^\infty e^{-Ks}|\hat{f}(0,0)|^{p}ds \right] < \infty. \]

By the same method,

\[ E\left[ \int_0^\infty e^{-Ks}|\hat{g}(0,0)|^{p}ds \right] < \infty. \]

Furthermore by (H.1)' and (H.2)', it is easy to see that \( g(X, Y, Z) \) satisfies the condition in (2.2). Hence by (3.1) and (3.6), we have for \( 0 \leq s \leq T, r \geq 0, \), \( \theta_r \circ Y^{t,x} \) satisfies the following equation

\[
\begin{cases}
\theta_r \circ Y^{t,x}_s = \theta_r \circ Y^{t,x}_T + \int_{s}^{T+r} f(X^{t+r,x}_{u}, \theta_r \circ Y^{t,x}_u, \theta_r \circ Z^{t,x}_u)du \\
+ \int_{s}^{T+r} g(X^{t+r,x}_{u}, \theta_r \circ Y^{t,x}_u, \theta_r \circ Z^{t,x}_u), d^\dagger B_u \\
- \int_s^{T+r} \theta_r \circ Z^{t,x}_{u-r}, dW_u, \\
\lim_{T \to \infty} e^{-K'(T+r)}(\theta_r \circ Y^{t,x}_T) = 0 \quad \text{a.s.}
\end{cases}
\]

By the same method in Proposition 3.1, we have for fixed \( r \geq 0, \)

\[ \theta_r \circ Y^{t,x}_s = Y^{t+r,x}_{s+r}, \quad \theta_r \circ Z^{t,x}_s = Z^{t+r,x}_{s+r} \quad \text{a.s.} \]

In particular,

\[ \theta^1_r \circ Y^{t,x}_t(\omega^1) = Y^{t+r,x}_{t+r}(\omega^1) \quad \text{a.s.} \tag{3.9} \]

In next section, we will prove perfection result i.e. (3.9) holds for all \( r \geq 0 \) a.s. This will be used to discuss the stationary solution of the corresponding SPDE.

4 The stationary solution of the corresponding SPDE

Consider a simpler form of Eq.(7.7). For \( s \geq 0, \)

\[
e^{-K' r} Y^{t,x}_s = \int_s^\infty e^{-K' r} f(X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr + \int_s^\infty \frac{K'}{2} e^{-K' r} Y^{t,x}_r dr \\
+ \int_s^\infty e^{-K' r} g(X^{t,x}_r, Y^{t,x}_r), d^\dagger B_r - \int_s^\infty e^{-K' r} (Z^{t,x}_r, dW_r),
\]
equivalently, for \( 0 \leq t \leq s \leq T, \)
Here \( C_b^{2,3}(\mathbb{R}^d, \mathbb{R}^1) \) is the space of bounded functions defined on \( \mathbb{R}^d \times \mathbb{R}^1 \) which are twice continuously differentiable with respect to \( x \in \mathbb{R}^d \) and three times continuously differentiable with respect to \( y \in \mathbb{R}^1 \).

For arbitrary \( T > 0 \), we will show that under conditions (H.1)', (H.2)', (H.3)', (H.4)', (H.5) and for arbitrary \( t \in [0, T] \), \( x \) belonging to an arbitrary bounded set in \( \mathbb{R}^d \), the solution \( Y^{T-t,x}_s \) of Eq.(4.1) is the viscosity solution ([6]) of the following SPDE

\[
v(t, x) = v(t, v_0)(x) = v_0(x) + \int_0^t [\mathcal{L}v(s, x) + f(x, v(s, x), \sigma^*(x)Du(s, x))]ds - \int_0^t \langle g(x, v(s, x)), dB_s \rangle, \quad t \geq 0.
\]

Here \( \{B_s\} \) is a two-sided Brownian motion. Denote by \( U \) the Banach space of \( C^{0, \delta}(0 < \delta < 1 - \frac{2\alpha}{p}) \) functions \( f: \mathbb{R}^d \rightarrow \mathbb{R}^1 \) for which the norm

\[
|f| = \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{1 + |x|} + \sup_{x \neq y, x, y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|^\delta}
\]

is finite and consider its Borel \( \sigma \)-field \( B \). If we assume that Eq. (4.2) has a unique stochastic viscosity solution, then it is not difficult to see that (4.2) defines a measurable random dynamical system \( v: [0, T] \times U \times \Omega \rightarrow U \) for any \( T > 0 \). This can be seen from (2.5), the Doss-Sussmann transformation, the cocycle property of the stochastic flow \( \lambda \) and some standard arguments. The uniqueness of the stochastic viscosity solution was studied in Buckdahn and Ma [7]. Note here we don’t need the continuity of random dynamical system. In fact the continuity of the map \( v(t, \cdot, \omega^1) : U \rightarrow U \) a.s. remains open for Eq. (4.2).

For arbitrary \( T > 0 \), define \( u(t, x) = v(T-t, x) \) and \( B_s = \hat{B}_{T-s} - \hat{B}_T \) for \( 0 \leq s < \infty \). Then \( \{B_s\} \) is a standard Brownian motion and \( g(x, u(\cdot, x)) \) satisfies condition in (2.2). Hence by (2.2) and integral transformation in Eq.(4.2), we can see that \( u \) satisfies the backward SPDE of the following form

\[
\begin{align*}
u(t, x) = & u(T, x) + \int_t^T [\mathcal{L}u(s, x) + f(x, u(s, x), \sigma^*(x)Du(s, x))]ds \\
& + \int_t^T \langle g(x, u(s, x)), dB_s \rangle, \quad 0 \leq t \leq T.
\end{align*}
\]
Lemma 4.1 Under condition (H.5), let $(X_{s}^{t,x})_{s \geq 0}$ be the solution of Eq.(3.5), then for arbitrary $T > 0$, $t, t' \in [0,T]$ and $x, x'$ belonging to an arbitrary bounded set in $\mathbb{R}^{d}$, 

$$E[\int_{0}^{\infty} e^{-Kr}|X_{t'}^{t,x'} - X_{t}^{t,x}|^p dr] \leq C_{p}(|x' - x|^p + |t' - t|^{\frac{p}{2}}) \quad \text{a.s.}$$

Proof. Without loss of generality, we assume $t' \geq t \geq 0$. For $r \geq 0$, let

$$\ddot{X}_{r} = X_{r}^{t',x'} - X_{r}^{t,x},$$

$$\ddot{b}_{r} = b_{r}(X_{r}^{t',x'}) - b_{r}(X_{r}^{t,x}),$$

$$\ddot{\sigma}_{r} = \sigma_{r}(X_{r}^{t',x'}) - \sigma_{r}(X_{r}^{t,x}).$$

Then for $r \geq t'$,

$$\begin{cases} d\ddot{X}_{r} = \ddot{b}_{r} dr + \ddot{\sigma}_{r} dW_{r}, \\ \dddot{X}_{r} = x' - X_{t'}^{t,x}. \end{cases}$$

Apply Itô’s formula to $e^{-Kr}|\dddot{X}_{r}|^p$, we have

$$e^{-Kr}|\dddot{X}_{s}|^p + \int_{t'}^{s} Ke^{-Kr}|\dddot{X}_{r}|^p dr$$

$$= e^{-Kt'}|x' - X_{s}^{t,x}|^p + \int_{t'}^{s} pe^{-Kr}|\dddot{X}_{r}|^{p-2}\langle \dddot{X}_{r}, \dddot{b}_{r} \rangle dr$$

$$+ \int_{t'}^{s} \frac{p(p-1)}{2} e^{-Kr}|\dddot{X}_{r}|^{p-4}\langle \dddot{\sigma}_{r}^{*}, \dddot{X}_{r} \rangle dr + \int_{t'}^{s} pe^{-Kr}|\dddot{X}_{r}|^{p-2}\langle \dddot{X}_{r}, \dddot{\sigma}_{r} dW_{r} \rangle$$

$$\leq e^{-Kt'}|x' - X_{s}^{t,x}|^p + \int_{t'}^{s} pe^{-Kr}|\dddot{X}_{r}|^{p-2}\langle \dddot{X}_{r}, \dddot{b}_{r} \rangle dr$$

$$+ \int_{t'}^{s} \frac{p(p-1)}{2} e^{-Kr}|\dddot{X}_{r}|^{p-2}\|\dddot{\sigma}_{r}\|^2 dr + \int_{t'}^{s} pe^{-Kr}|\dddot{X}_{r}|^{p-2}\langle \dddot{X}_{r}, \dddot{\sigma}_{r} dW_{r} \rangle$$

$$\leq e^{-Kt'}|x' - X_{s}^{t,x}|^p + \int_{t'}^{s} pLe^{-Kr}|\dddot{X}_{r}|^p dr + \int_{t'}^{s} \frac{p(p-1)}{2} L^2 e^{-Kr}|\dddot{X}_{r}|^p dr$$

$$+ \int_{t'}^{s} pe^{-Kr}|\dddot{X}_{r}|^{p-2}\langle \dddot{X}_{r}, \dddot{\sigma}_{r} dW_{r} \rangle$$

$$\leq |x' - X_{s}^{t,x}|^p + (pL + \frac{p(p-1)}{2} L^2) \int_{t'}^{s} e^{-Kr}|\dddot{X}_{r}|^p dr$$

$$+ p \int_{t'}^{s} e^{-Kr}|\dddot{X}_{r}|^{p-2}\langle \dddot{X}_{r}, \dddot{\sigma}_{r} dW_{r} \rangle.$$
By (3.8), then
\[
E[e^{-Ks}|\bar{X}_s|^p] + (K - pL - \frac{p(p-1)}{2}L^2)E[\int_t^s e^{-Kr}|\bar{X}_r|^p dr] \\
\leq E[|x' - X_{t,x}^r|^p].
\]

Now for arbitrary \(T > 0\), we consider \(t, t' \in [0, T]\) and \(x, x'\) belonging to an arbitrary bounded set in \(\mathbb{R}^d\). Referring to P184 in [17], we have
\[
E[|x' - X_{t,x}^r|^p] \leq C_p(|x' - x|^p + |t' - t|^\frac{p}{2}).
\]

Noting that \(K - pL - \frac{p(p-1)}{2}L^2 > 0\), we have
\[
E[\int_t^\infty e^{-Kr}|\bar{X}_r|^p dr] \leq C_p(|x' - x|^p + |t' - t|^\frac{p}{2}). \tag{4.4}
\]

Also referring to the same formula in [17], we have
\[
E[|x' - X_{t,x}^r|^p] \leq C_p(|x' - x|^p + |t' - t|^\frac{p}{2} + |t' - r|\frac{p}{2}).
\]

Therefore
\[
E[\int_t^{t'} e^{-Kr}|\bar{X}_r|^p dr] \\
= E[\int_t^t e^{-Kr}|x' - x|^p dr] + E[\int_t^{t'} e^{-Kr}|x' - X_{t,x}^r|^p dr] \\
\leq C_p|x' - x|^p + \int_t^{t'} e^{-Kr}E[|x' - X_{t,x}^r|^p] dr \\
\leq C_p|x' - x|^p + \int_t^{t'} e^{-Kr}C_p(|x' - x|^p + |t' - t|^\frac{p}{2} + |t' - r|\frac{p}{2}) dr \\
\leq C_p|x' - x|^p + C_p|x' - x|^p + C_p|t' - t|^\frac{p}{2} + C_p|t' - t|^\frac{p}{2 + 1}. \tag{4.5}
\]

By (4.4) and (4.5), together with bounded \(|t' - t|\), it follows that
\[
E[\int_0^\infty e^{-Kr}|\bar{X}_r|^p dr] \leq C_p(|x' - x|^p + |t' - t|^\frac{p}{2}) \quad a.s.,
\]
for \(t, t' \in [0, T]\) and \(x, x'\) belonging to an arbitrary bounded set in \(\mathbb{R}^d\). Lemma 4.1 is proved. \(\diamondsuit\)

**Proposition 4.2** Under conditions (H.1)', (H.2)', (H.3)', (H.5), let \((Y_{s,x}^{t,x})_{s \geq 0}\) be the solution of Eq. (4.1), then for arbitrary \(T > 0\), \(t \in [0, T]\) and \(x\) belonging to an arbitrary bounded set in \(\mathbb{R}^d\), \((t, x) \mapsto Y_{t,x}^{r,s}\) is a.s. continuous. In particular, \(Y_{t,x}^{r,s}\) is a.s. \(\delta\)-Hölder continuous w.r.t. \(x\) for any \(0 < \delta < 1 - \frac{d+2}{p}\) and of linear growth, so \(Y_{t,x}^{r,s} \in U\).
Proof. For $t, t' \geq 0$, let $\tilde{Y}_r = Y^{t',x'}_r - Y^{t,x}_r$, $\bar{Z}_r = Z^{t',x'}_r - Z^{t,x}_r$,

$$\begin{align*}
\tilde{f}_r &= f(X^{t',x'}_r, Y^{t',x'}_r, Z^{t',x'}_r) - f(X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r), \\
\tilde{g}_r &= g(X^{t',x'}_r, Y^{t',x'}_r) - g(X^{t,x}_r, Y^{t,x}_r).
\end{align*}$$

Then

$$\begin{align*}
\left\{ d\tilde{Y}_r &= -\tilde{f}_r dr - \langle \tilde{g}_r, d\bar{B}_r \rangle + \langle \bar{Z}_r, dW_r \rangle, \\
\lim_{t \to \infty} e^{-\frac{pK'}{2} T}\tilde{Y}_T &= 0 \quad \text{a.s..}
\end{align*}$$

Apply Itô's formula to $e^{-\frac{pK'}{2} t}|\tilde{Y}_r|^p$, for $0 \leq s \leq T$, we have

$$\begin{align*}
e^{-\frac{pK'}{2} s}|\tilde{Y}_s|^p + \frac{p(p-1)}{2} \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}|\tilde{f}_r|^2 dr - \frac{pK'}{2} \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^p dr \\
= e^{-\frac{pK'}{2} T}|\tilde{Y}_T|^p + p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\tilde{f}_r dr \\
+ p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\tilde{Y}_r \langle \tilde{g}_r, d\bar{B}_r \rangle + \frac{p(p-1)}{2} \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}|\tilde{g}_r|^2 dr \\
- p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\tilde{Y}_r \langle \bar{Z}_r, dW_r \rangle \\
\leq e^{-\frac{pK'}{2} T}|\tilde{Y}_T|^p \\
+ p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\langle f(X^{t',x'}_r, Y^{t',x'}_r, Z^{t',x'}_r) - f(X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r), d\tilde{Y}_r \rangle \\
+ p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\langle f(X^{t',x'}_r, Y^{t',x}_r, Z^{t',x}_r) - f(X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r), d\tilde{Y}_r \rangle \\
+ p \frac{p(p-1)}{2} \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}(C_1|\bar{X}_r| + C|\tilde{Y}_r|)^2 dr \\
+ p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\langle \tilde{g}_r, d\bar{B}_r \rangle - p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\tilde{Y}_r \langle \bar{Z}_r, dW_r \rangle \\
\leq e^{-\frac{pK'}{2} T}|\tilde{Y}_T|^p - \mu \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^p dr \\
+ p \frac{2}{2} \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}(2C^2|\tilde{Y}_r|^2 + C_1^2|\bar{X}_r|^2 + |\tilde{Z}_r|^2)^2 dr \\
+ p \frac{p(p-1)}{2} \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}(C_2^2|\bar{X}_r|^2 + C^2|\tilde{Y}_r|^2)^2 dr \\
+ p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\tilde{Y}_r \langle \tilde{g}_r, d\bar{B}_r \rangle - p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\tilde{Y}_r \langle \bar{Z}_r, dW_r \rangle.
\end{align*}$$

Therefore, for $0 \leq s \leq T$, 

$$e^{-\frac{pK'}{2} T}|\tilde{Y}_T|^p - \mu \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^p dr \\
+ p \frac{2}{2} \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}(2C^2|\tilde{Y}_r|^2 + C_1^2|\bar{X}_r|^2 + |\tilde{Z}_r|^2)^2 dr \\
+ p \frac{p(p-1)}{2} \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}(C_2^2|\bar{X}_r|^2 + C^2|\tilde{Y}_r|^2)^2 dr \\
+ p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\tilde{Y}_r \langle \tilde{g}_r, d\bar{B}_r \rangle - p \int_s^T e^{-\frac{pK'}{2} r}|\tilde{Y}_r|^{p-2}\tilde{Y}_r \langle \bar{Z}_r, dW_r \rangle.$$
\begin{align*}
e^{-\frac{pK^\prime}{2} s} |\bar{Y}_s|^p + \frac{p(p-2)}{2} \int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^{p-2} |\bar{Z}_r|^2 dr \\
+ (p \mu - \frac{pK^\prime}{2} - p^2 C^2) \int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^p dr \\
\leq e^{-\frac{pK^\prime}{2} T} |\bar{Y}_T|^p + \frac{pC^2}{2C^2} + p(p-1)C^2 \int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^{p-2} |\bar{X}_r|^2 dr \\
+p \int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^{p-2} \bar{Y}_r^2 \bar{g}_r, d\beta_r) - p \int_s^T p e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^{p-2} \bar{X}_r, dW_r) \\
\leq e^{-\frac{pK^\prime}{2} T} |\bar{Y}_T|^p + \varepsilon \int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^p dr + C_p \int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{X}_r|^p dr \\
+p \int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^{p-2} \bar{Y}_r(\bar{g}_r, d\beta_r) - p \int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^{p-2} \bar{Y}_r(\bar{Z}_r, dW_r). \\
\text{(4.6)}
\end{align*}

Then, for \(0 \leq s \leq T\),

\begin{align*}
E[e^{-\frac{pK^\prime}{2} s} |\bar{Y}_s|^p] + \frac{p(p-2)}{2} E[\int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^{p-2} |\bar{Z}_r|^2 dr] \\
+ (p \mu - \frac{pK^\prime}{2} - p^2 C^2 - \varepsilon) E[\int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{Y}_r|^p dr] \\
\leq E[e^{-\frac{pK^\prime}{2} T} |\bar{Y}_T|^p] + C_p E[\int_s^T e^{-\frac{pK^\prime}{2} r} |\bar{X}_r|^p dr]. \\
\text{(4.7)}
\end{align*}

Since \((Y^{t, \omega}) \in S^{p,-K}([0, \infty); \mathbb{R}^1)\), for arbitrary \(T \geq 0\),

\(E[e^{-\frac{pK^\prime}{2} T} |\bar{Y}_T|^p] \leq E[\sup_{s \geq 0} e^{-Ks} |\bar{Y}_s|^p] < \infty\).

Therefore we can apply Lebesgue's dominated convergence theorem by taking the limit of \(T\) to have

\[
\lim_{T \to \infty} E[e^{\frac{pK^\prime}{2} T} |\bar{Y}_T|^p] = E[\lim_{T \to \infty} e^{\frac{pK^\prime}{2} T} |\bar{Y}_T|^p] = 0. \\
\text{(4.8)}
\]
So we take the limit of $T$ in (4.7) and by Lemma 4.1 and monotone convergence theorem to have

$$E[e^{-\frac{pK'}{2}s} | \bar{Y}_s|^p] + \frac{p(p-2)}{2} E[\int_0^\infty e^{-\frac{pK'}{2}r} |\bar{Y}_r|^p|\bar{Z}_r|^2 dr]$$

$$+ (p\mu - \frac{pK'}{2} - pC^2 - \varepsilon) E[\int_0^\infty e^{-\frac{pK'}{2}r} |\bar{Y}_r|^p dr]$$

$$\leq C_p E[\int_0^\infty e^{-Kr} |\bar{X}_r|^p dr]$$

$$\leq C_p (|x' - x|^p + |t' - t|^\frac{p}{2}).$$

By the arbitrariness of $\varepsilon$, it follows that

$$E[\int_0^\infty e^{-\frac{pK'}{2}r} |\bar{Y}_r|^p-2 |\bar{Z}_r|^2 dr] + E[\int_0^\infty e^{-\frac{pK'}{2}r} |\bar{Y}_r|^p dr]$$

$$\leq C_p (|x' - x|^p + |t' - t|^\frac{p}{2}).$$

(4.9)

From (4.6),

$$E[\sup_{0 \leq s \leq T} e^{-\frac{pK'}{2}s} |\bar{Y}_s|^p]$$

$$\leq E[e^{-\frac{pK'}{2}T} |\bar{Y}_T|^p] + C_p E[\int_0^\infty e^{-\frac{pK'}{2}r} |\bar{X}_r|^p dr]$$

$$+ pE[\int_0^\infty e^{-pK'r} |\bar{Y}_r|^2p-2 |\bar{g}_r|^2 dr]$$

$$+ pE[\int_0^\infty e^{-pK'r} |\bar{Y}_r|^2p-2 |\bar{Z}_r|^2 dr].$$

(4.10)

Noting (4.8), Taking the limit of $T$ on both sides of (4.10) and using monotone convergence theorem we have

$$E[\sup_{s \geq 0} e^{-\frac{pK'}{2}s} |\bar{Y}_s|^p]$$

$$\leq C_p E[\int_0^\infty e^{-\frac{pK'}{2}r} |\bar{X}_r|^p dr] + \frac{1}{3} E[\sup_{s \geq 0} e^{-\frac{pK'}{2}s} |\bar{Y}_s|^p]$$

$$+ C_p E[\int_0^\infty e^{-\frac{pK'}{2}r} |\bar{Y}_r|^p dr] + \frac{1}{3} E[\sup_{s \geq 0} e^{-\frac{pK'}{2}s} |\bar{Y}_s|^p]$$

$$+ C_p E[\int_0^\infty e^{-\frac{pK'}{2}r} |\bar{Y}_r|^p-2 |\bar{Z}_r|^2 dr].$$

From this inequality, by Lemma 4.1 and (4.9), for arbitrary $T > 0, t, t' \in [0, T]$ and $x, x'$ belonging to an arbitrary bounded set in $\mathbb{R}^d$, we have
\[
E[\sup_{s \geq 0} e^{-pKs}|Y_s|^p] \leq C_p(|x'| - x|p + |t' - t|^\frac{p}{2}).
\] (4.11)

Noting \( p > d + 2 \) in (4.11), by Kolmogorov Lemma (see [17]), we have \( Y^{t, \cdot}_s \) has a continuous modification for \( t \in [0, T] \) and \( x \) belonging to an arbitrary bounded set in \( \mathbb{R}^d \) under the norm
\[
\sup_{s \geq 0} e^{-Ks}|Y^{t, \cdot}_s|.
\]
In particular,
\[
\lim_{t' \rightarrow t, x' \rightarrow x} e^{-Kt}|Y^{t', x'}_t - Y^{t, x}_t| = 0.
\]

Then we have a.s.
\[
\lim_{t' \rightarrow t, x' \rightarrow x} \left| e^{-Kt'}Y^{t', x'}_t - e^{-Kt}Y^{t', x}_t \right| \\
\leq \lim_{t' \rightarrow t, x' \rightarrow x} \left| (e^{-Kt'}Y^{t', x'}_t - e^{-Kt}Y^{t', x}_t) + (e^{-Kt}Y^{t', x'}_t - e^{-Kt}Y^{t, x}_t) \right| = 0.
\]

The convergence of the first term follows from the continuity of \( Y^{t, \cdot}_t \) in \( s \). That is to say, \( e^{-Kt}Y^{t, x}_t \) is a.s. continuous, therefore \( Y^{t, x}_t \) is continuous in \( t \in [0, T] \) and \( x \) belonging to an arbitrary bounded set in \( \mathbb{R}^d \). Furthermore, from (4.11) by Kolmogorov Lemma, \( Y^{t, x}_t \) is a.s. \( \delta \)-Hölder continuous w.r.t. \( x \) for any \( 0 < \delta < 1 - \frac{d + 2}{p} \) and of linear growth for \( x \) belonging to an arbitrary bounded set in \( \mathbb{R}^d \). Proposition 4.2 is proved.

With the above results, we claim that

**Theorem 4.3** Assume conditions (H.1'), (H.2'), (H.3'), (H.4'), (H.5) and Eq. (4.3) has a unique stochastic viscosity solution. Define \( u(t, x, \omega^1) = Y^{t, x}_t(\omega^1) \) for \( t \in [0, T] \) and \( x \in \mathbb{R}^d \), where \( Y^{t, x}_t(\omega^1) \) is the solution of Eq. (4.1). Then \( u(t, x, \omega^1) \) is a stationary stochastic viscosity solution of Eq. (4.3) w.r.t. \( \theta^1 \) defined in Section 3, i.e. there exists \( \Delta^{1, *}_t \subset \Omega^1_t \), s.t. \( P^1(\Delta^{1, *}_t) = 1 \) and for all \( \omega^1 \in \Omega^{1, *}_t \), \( t, r \geq 0, x \in \mathbb{R}^d \), \( u(t + r, x, \omega^1) = u(t, x, \theta^1_t\omega^1) \).

**Proof.** By Proposition 4.2, \( u(\cdot, \cdot, \omega^1) \) is a.s. continuous for \( t \in [0, T] \) and \( x \) belonging to an arbitrary bounded set in \( \mathbb{R}^d \) and of linear growth in \( x \). In particular, \( u(T, x) \) is continuous in \( x \) and of linear growth in \( x \). So, referring to [6] (or Theorem 2.2) for the time reveral version, we see that \( u(t, x, \omega^1) \) is a stochastic viscosity solution of Eq. (4.3). To prove the perfection version of the stationary solution, define \( \bar{B}(0, N) \) the closed ball in \( \mathbb{R}^d \) of radius \( N \) centred at 0. It is obvious that \( \bigcup_{N=1}^{\infty} \bar{B}(0, N) = \mathbb{R}^d \). First consider when \( x \) is in a ball \( \bar{B}(0, N) \). Since \( Y^{t, \cdot}_t \in C(\bar{B}(0, N)) \) and \( C(\bar{B}(0, N)) \) is a separable space,
by [2] and (3.9), we know there exist a modification of \( Y^t \), still denoted by \( Y^t \), and a full measure set \( \Omega^{1,N} \subset \Omega^1 \) such that for all \( \omega^1 \in \Omega^{1,N} \), \( t \geq 0 \), \( x \in \mathbb{B}(0, N) \), \( Y^t_{i+r,x}(\omega^1) = Y^t_{i,x}(\theta_i^1 \omega^1) \). Take \( \Omega^{1,*} = \bigcap_{N=1}^{\infty} \Omega^{1,N} \), then \( P^1(\Omega^{1,*}) = 1 \). Now for any \( x \in \mathbb{R}^d \), there exists an \( N \) s.t. \( x \in \mathbb{B}(0, N) \), but for all \( \omega^1 \in \Omega^{1,*} \), it is obvious that \( \omega^1 \in \Omega^{1,N} \), so for any \( t, r \geq 0 \), \( x \in \mathbb{R}^d \), \( u(t + r, x, \omega^1) = u(t, x, \theta_i^1 \omega^1) \).

Note that \( v(t, x) = u(T - t, x) \), hence \( v(t, x)(\omega^1) = Y^{T-t,x}_T(\omega^1) \) is the solution of Eq.(4.2) for arbitrary \( T > 0 \). In fact \( Y^{T-t,x}_T(\omega^1) \) does not depend on \( T \). For example, if we take \( T' \geq T \), then we can show that \( Y^{T-t,x}_T(\omega^1) = Y^{T'-t,x}_{T'}(\omega^1) \) when \( 0 \leq t \leq T \), where \( \omega^1(s) = \hat{B}_{T-s} - \hat{B}_T \), \( 0 \leq s < \infty \), and \( \hat{\omega}^1(s) = \hat{B}_{T-s} - \hat{B}_T \), \( 0 \leq s < \infty \). Let \( \theta_\cdot \) and \( \hat{\theta}_\cdot \) are the shifts of \( \omega^1(\cdot) \) and \( \hat{\omega}^1(\cdot) \) respectively. Since by (3.9) we have

\[
\begin{align*}
Y^{T-t,x}_{T-t}(\omega^1) &= \theta_{T-t}^1 Y^0_{T}(\omega^1) = Y^0_{0}(\theta_{T-t}^1 \omega^1) \\
Y^{T'-t,x}_{T'-t}(\hat{\omega}^1) &= \hat{\theta}_{T'-t}^1 Y^0_{T'}(\hat{\omega}^1) = Y^0_{0}(\hat{\theta}_{T'-t}^1 \hat{\omega}^1),
\end{align*}
\]

we just need to show that \( \theta_{T-t}^1 \omega^1 = \hat{\theta}_{T-t}^1 \hat{\omega}^1 \). In fact we have

\[
(\theta_{T-t}^1 \omega^1)(s) = \omega^1(T - t + s) - \omega^1(T - t) = (\hat{B}_{T-(T-t+s)} - \hat{B}_T) - (\hat{B}_{T-(T-t)} - \hat{B}_T) = \hat{B}_{t+s} - \hat{B}_t.
\]

The right side of the above formula does not depend on \( T \), therefore \( \theta_{T-t}^1 \omega^1(s) = \hat{\theta}_{T-t}^1 \hat{\omega}^1(s) = \hat{B}_{t+s} - \hat{B}_t \). That is to say \( Y^{T-t,x}_{T-t}(\omega^1) \) does not depend on the choice of \( T \).

On probability space \( (\Omega^1, \mathcal{F}\hat{\mathbb{B}}^\infty, P^1) \), we define \( \hat{\theta}_t^1 = (\theta^1_t)^{-1} \), \( t \geq 0 \). Note that \( \{B_s\} \) is a two-sided Brownian motion due to the two-sidedness of Brownian motion \( \{B_s\} \), so \( (\theta^1_t)^{-1} = \hat{\theta}_{-t} \) is well defined (see [1]). It is easy to see that \( \hat{\theta}_t^1 \) is a shift w.r.t. \( \{\hat{B}_t\} \) satisfying

(i) \( P^1 \cdot (\hat{\theta}_t^1)^{-1} = P^1 \);
(ii) \( \hat{\theta}_0^1 = I^1 \);
(iii) \( \hat{\theta}_t^1 \circ \hat{\theta}_s^1 = \hat{\theta}_{s+t}^1 \);
(iv) \( \hat{\theta}_t^1 \circ \hat{B}_s = \hat{B}_{s+t} - \hat{B}_t \);
(v) \( (\hat{\theta}_t^1)^{-1}(\mathcal{F}\hat{\mathbb{B}}_{s,u}) = \mathcal{F}\hat{\mathbb{B}}_{s+r,u+r} \).

Since \( v(t, x)(\omega^1) = u(T - t, x)(\omega^1) = Y^{T-t,x}_{T-t}(\omega^1) \) a.s.,
\[
\tilde{\theta}_t^1 v(t, x)(\omega^1) = \theta_1 u(T-t, x)(\omega^1) = u(T-t-r, x)(\omega^1) = v(t+r, x)(\omega^1),
\]

for \(r \geq 0\) and \(T > t + r\) a.s. In particular, let \(Y(\omega^1) = v_0(\omega^1) = Y_{T^\cdot}^T(\omega^1)\) which can be regarded as a point in \(U\). Then above formula implies that

\[
\tilde{\theta}_t^1 Y(\omega^1) = Y(\tilde{\theta}_t^1 \omega^1) = v(t, \omega^1) = v(t, v_0(\omega^1), \omega^1) = v(t, Y(\omega^1), \omega^1), \quad t \geq 0, \text{ a.s.}
\]

Here as we have seen before, \(v\) is a measurable random dynamical system. That is to say \(v(t, \omega^1)(x) = Y(\tilde{\theta}_t^1 \omega^1)(x) = Y_{T-t}^{T-t, x}(\omega^1)\) is a stationary solution of the SPDE (4.2). Therefore we have the following theorem

**Theorem 4.4** Assume conditions (H.1)', (H.2)', (H.3)', (H.4)', (H.5) and Eq. (4.2) has a unique stochastic viscosity solution. Define \(v(t, x, \omega^1) = Y_{T-t}^{T-t, x}(\omega^1)\) for \(t \in [0, T]\) a.s. and \(x\) belonging to an arbitrary bounded set in \(\mathbb{R}^d\), where \(Y_{T-t}^{T-t, x}(\omega^1)\) is the solution of Eq.(4.1). Then \(v(t, x, \omega^1)\) is a stationary stochastic viscosity solution of Eq.(4.2) w.r.t. \(\tilde{\theta}^1\).

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