Stochastic Infinity-Laplacian equation and One-Laplacian equation in image processing and mean curvature flows: finite and large time behaviours

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Stochastic Infinity-Laplacian Equation and One-Laplacian Equation in Image Processing and Mean Curvature Flows: Finite and Large Time Behaviours

by

Fajin WEI

A doctoral thesis
submitted in partial fulfillment of the requirements
for the award of the degree of
Doctor of Philosophy
of Loughborough University,
22nd July 2010

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Abstract

In Part I, the existence and uniqueness of solutions for the initial-boundary value and Cauchy problems of the ‘stochastic parabolic infinity-Laplacian equation’

\[ du = \frac{\langle D^2 u Du, Du \rangle}{|Du|^2} dt + g dW_t \]

are investigated. The existence of pathwise stationary solutions of this stochastic partial differential equation (SPDE, for abbreviation) is demonstrated.

In Part II, a connection between certain kind of state constrained controlled Forward-Backward Stochastic Differential Equations (FBSDEs) and Hamilton-Jacobi-Bellman equations (HJB equations) are demonstrated. The special case provides a probabilistic representation of some geometric flows, including the mean curvature flows.

Part II includes also a probabilistic proof of the finite time existence of the mean curvature flows.

Acknowledgment

I would like to express my deep and sincere gratitude to my two supervisors, Professors Huaizhong Zhao and Shige Peng.

Professor Huaizhong Zhao led me to this field and guided me with fatigueless help and warm-hearted encouragement throughout these years at Loughborough University UK and eventually to the final fulfillment of this doctoral thesis.

Professor Shige Peng is my mentor whom I followed into the fascinating world of stochastic analysis and mathematics in general since I began my study at Shandong University China. His academic capability and personality is truly an exemplar for me.

Also I would like to thank so many professors, doctors, staffs, classmates, and friends at Loughborough and Shandong Universities and from elsewhere for their generous help and friendliness during these years.

I would like to give my special gratitude to my beloved wife, Ms. Shiyan Guo, and my parents for their extraordinary support and patience during this period.
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Part I

Stochastic Parabolic

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Chapter 1

Introduction

Pathwise stationary solutions of stochastic partial differential equations

The notion of pathwise stationary solutions of random dynamical systems \cite{Arnold1998} is a natural extension of equilibria or fixed points in deterministic systems.

A simple but nontrivial example is the Ornstein-Uhlenbeck process defined by the stochastic differential equation (SDE)

\[ dz(t) = -z(t) dt + dW_t. \]

It defines a random dynamical system

\[ z(z_0, t) = z_0 e^{-t} + \int_0^t e^{s-t} dW_s. \]

And its pathwise stationary solution is given by

\[ z(\omega) = \int_{-\infty}^0 e^s dW_s, \]

that is, a.s.

\[ z(z(\omega), t) = z(\theta_t \omega). \]

Here \( \theta_t \) is the shift operator of the path of Wiener process, i.e.,

\[ (\theta_t W)_s := W_{t+s} - W_t, \]
for all $s \in \mathbb{R}$. Moreover, for any $z_0$, $z(z_0, t, \theta_{t-\omega}) \to z(\omega)$ when $t \to \infty$. As a ‘one-force, one-solution’ setting, a pathwise stationary solution describes the pathwise invariance of the stationary solution over time along the measurable and $\mathbb{P}$-preserving transformation $\theta_t : \Omega \to \Omega$, and the pathwise limit of the solutions of random dynamical systems. It is among the fundamental questions of basic importance and has been studied by (Arnold, 1998; Caraballo, Kloeden, & Schmalfuß, 2004; E, Khanin, Mazel, & Sinai, 2000; Liu & Zhao, 2009; Mattingly, 1999; Mohammed, Zhang, & Zhao, 2008; Sinai, 1991, 1996; Zhang & Zhao, 2007, 2010; Zhao & Zheng, 2009), among others. For random dynamical systems generated by SPDEs such random fixed points consist of infinitely many random moving invariant surfaces on the configuration space caused by the random external force pumped to the system constantly. They demonstrate some complicated phenomena such as turbulence and their existence and stability are of great interests in both mathematics and physics. However the existence of pathwise stationary solutions of random dynamical systems is a difficult and subtle problem. There have been extensive works on random dynamical systems in which researchers usually first assume there is an invariant set then prove invariant manifolds and stability results at a point of the invariant set; see, for instance, (Arnold, 1998; Duan, Lü, Schmalfuß, 2003; Mohammed, Zhang, & Zhao, 2008; Ruelle, 1982), and references therein. But the invariant manifolds theory does not provide the existence results of the invariant set and the pathwise stationary solution, or a method to find them. For the existence of pathwise stationary solutions for SPDEs, only very few results are known. In (Sinai, 1991, 1996) Hopf-Cole transformation was used to establish the stationary strong solution of the stochastic Burgers equation with periodic or random forcing ($C^3$ in the spatial variable). In (Mohammed, Zhang, & Zhao, 2008) the pathwise stationary solution of the stochastic evolution equations was identified as a solution of the corresponding integral equation up to time $+\infty$ and the existence was obtained for certain SPDEs. In (Zhang & Zhao, 2007) a novel approach, i.e., the method of infinite horizon backward doubly stochastic differential equations (BDSDEs) was proposed to study the existence of pathwise stationary solutions for a large class of SPDEs.
Infinity-Laplacian: theoretical background

Let \( T > 0 \) and \( U \subset \mathbb{R}^n \) be a bounded domain. Denote
\[
U_T := U \times (0, T]
\]
and
\[
\Gamma_T := \overline{U_T} - U_T = (U \times \{0\}) \cup (\partial U \times [0, T])
\]
the parabolic boundary. Suppose \( 1 < p < \infty \) and recall the problem in the calculus of variations in \( L^p \) to minimise the functional
\[
I_p[v] := \frac{1}{p} \int_U |Dv|^p dx
\]
over all the functions \( v : \overline{U} \to \mathbb{R} \) subject to some given boundary conditions. The associated Euler-Lagrange equation is the elliptic \('p\)-Laplacian' equation
\[
\Delta_p u := \text{div}(|Du|^{p-2} Du) = 0.
\]

One extreme case is the \('1\)-Laplacian' when \( p = 1 \). It appears in the level set equation of the mean curvature flow \cite{Chen, Giga, Goto, 1991, Evans, Spruck, 1991},
\[
\partial_t u = \Delta_1 u := \Delta u - \frac{\langle D^2 u Du, Du \rangle}{|Du|^2}.
\]

Another extreme case is the elliptic \('[\infty]\)-Laplacian' equation when \( p \to \infty \), which arises in the calculus of variations in \( L^\infty \),
\[
\Delta_\infty u := \frac{\langle D^2 u Du, Du \rangle}{|Du|^2} = 0 \text{ in } U, \quad u = \psi \text{ on } \partial U,
\]
where \( \psi : \overline{U} \to \mathbb{R} \) is bounded and Lipschitz. Its solution \( u \) is called the \('\infty\)-harmonic' function. To see that \('[\infty]\)-Laplacian' is, formally, the limit case of \('[p]\)-Laplacian' when \( p \to \infty \) just divide
\[
\text{div}(|Du|^{p-2} Du) = |Du|^{p-2} \Delta u + (p - 2)|Du|^{p-4} \langle D^2 u Du, Du \rangle
\]
by \((p - 2)|Du|^{p-2}\) and take the limit as \(p \to \infty\). The equation (1.1) was investigated in the ‘absolute minimal Lipschitz extension’ problem: suppose \(\psi\) is Lipschitz and we look for a Lipschitz function \(u : \bar{U} \to \mathbb{R}\) such that \(u|_{\partial U} = \psi|_{\partial U}\) and on all open set \(V \subset U\),

\[
\text{ess sup}_{V} |Du| \leq \text{ess sup}_{V} |Dv|
\]

for all \(v : \bar{V} \to \mathbb{R}\) with \(v|_{\partial V} = u|_{\partial V}\). This equation has been extensively studied by many authors since (Aronsson, 1967; Jensen, 1993). More information can be found in the survey articles (Aronsson, Crandall, & Juutinen, 2004; Evans, 2007), and references therein.

Consider the related parabolic ‘\(\infty\)-Laplacian’ equation of \(u : \bar{U}_T \to \mathbb{R}\)

\[
\partial_t u = \frac{(D^2 u Du, Du)}{|Du|^2} \quad \text{in} \quad U_T, \quad u = \psi \quad \text{on} \quad \Gamma_T. \tag{1.2}
\]

It is worth mentioning that the one-dimensional case of the above equation is just the classic heat equation. The existence and uniqueness of viscosity solutions of equations in a more general form are investigated in (Alvarez, Lions, & Morel, 1992). This equation is studied in details in (Juutinen & Kawohl, 2006).

**Infinity-Laplacian: applications in the image processing**

The equations (1.1) and (1.2) have applications in image restoration problem in image processing (Caselles, Morel, & Sbert, 1998). Assume that an image is mainly made of areas of constant or smoothly varying intensity which are separated by discontinuities represented by sharp edges. The geometric structure of the discontinuities is the information to be decoded. Thus the decoder is required to reconstruct the smooth areas in between by making use of the data of edges. This can be formulated as a scattered scalar data interpolation problem from an initial set of points and curves in the plane under some constraints about the smoothness. Assume that \(u \in C^2(\mathbb{R}^2, \mathbb{R})\) is known at all pixels but \(x_0 \in \mathbb{R}^2\). Then, among many possibilities, \(u(x_0)\) can be obtained by
propagation from neighbouring pixels. More precisely, we choose the propagation to be along the direction of gradient, that is,

$$u(x_0) = \frac{1}{2} \left[ u(x_0 + h \frac{Du}{|Du|}) + u(x_0 - h \frac{Du}{|Du|}) \right] + o(h^2).$$

Letting $h$ tend to zero, by the Taylor expansion we learn immediately that $u$ satisfies the equation (1.1).

It is worth remarking that the equation (1.2) has noticeable asymptotic behaviour. In fact, the solution of the parabolic equation (1.2) will converge to the solution of the elliptic equation (1.1) when the time $t$ approaches infinity. This property is utilised to compute the solution of the equation (1.1) numerically by iteratively solving a sequence of equations relating to the equation (1.2). In other words, starting from the initial image $u(x, 0)$ we solve the parabolic equation step by step as time increases and the solution approaches asymptotically to $u(x, \infty)$, which satisfies the elliptic equation and is the desired image after processing (Caselles, Morel, & Sbert, 1998).

In this aspect, noise abounds both intrinsically and from the external environment during the process of image restoration by the parabolic equation (1.2) as described above. Second, the lack of information of the areas in between edges makes it reasonable in the image processing to add noise representing the uncertainty or various possibilities of the original image. So we put forward the following random model

$$u(x, t+h) = \frac{1}{2} \left[ u \left( x + \sqrt{h} \frac{Du}{|Du|}, t \right) + u \left( x - \sqrt{h} \frac{Du}{|Du|}, t \right) \right] + g(x)(W_{t+h} - W_t) + o(h^2).$$

Here we propose use the Gaussian random variable to approximate the variation of the original image at time $t + h$ from the linear interpolation of the image at time $t$ along the direction of gradient. Rewriting the above equality by Taylor expansion and taking limit as $h \to 0$ we thus obtain that $u$ satisfies the ‘stochastic parabolic infinity-Laplacian equation’ (1.3).

**Stochastic parabolic infinity-Laplacian equation**

In this part we investigate the equation (1.2) in an environment with noise, namely, the ‘stochastic parabolic infinity-Laplacian equation’, or the ‘stochastic infinity heat
equation’ of the form \([1.3]\).

\[
du = \frac{\langle D^2 u D u, Du \rangle}{|D u|^2} dt + g dW_t \text{ in } U_T, \tag{1.3}
\]

\[
u = \psi \text{ on } \Gamma_T.
\]

This stochastic PDE is of importance both theoretically and in applications as afore-
mentioned. The one-dimensional equation is just the stochastic heat equation which has
been studied as a typical SPDE ever since. As to the multidimensional equation inter-
esting properties arise from the degeneracy and nonlinearity of the infinity-Laplacian.

We first investigate the Cauchy problem of the SPDE \([1.3]\) when \(g := g(t)\) and
\(U := \mathbb{R}^n\). The existence and uniqueness of its solutions, in the viscosity sense, are
discussed. Then the existence of pathwise stationary solutions \([Arnold, 1998]\) of this
SPDE is proved. Second, we study the initial-boundary value problem of this SPDE
when \(g := g(x)\) and \(U\) is a bounded domain. The existence and uniqueness of its
solutions, and also the existence of pathwise stationary solutions are demonstrated.

Difficulties mainly arise from the nonlinearity and degeneracy of \(\Delta_\infty\) in this partic-
ular equation. Above all, via the Ornstein-Uhlenbeck transform we change this SPDE
into a random PDE which is more convenient to tackle. Then viscosity solutions of
this random PDE are defined in a similar way as in the deterministic case, regard-
ing the variable \(\omega\) as a parameter. The idea to prove the uniqueness of solutions is
rather standard in the theory of viscosity solutions \([Crandall, Ishii, & Lions, 1992]\). Here we essentially adopt the deduction in \([Alvarez, Lions, & Morel, 1992]\) but in
a more complicated fashion, along with which the boundedness of solutions is also
proved. The existence of solutions is demonstrated by approximations of smooth so-
lutions for a family of parameterised equations. But then the Cauchy problem and
initial-boundary value problem are treated differently. For the Cauchy problem we fol-
low the classical Bernstein method as in \([Alvarez, Lions, & Morel, 1992]\) to obtain
estimates of the gradients of the smooth solutions, which leads to the Lipschitz es-
timate of these solutions. Thus we take limit when the parameters tend to zero by
virtue of the Arzelà-Ascoli lemma. While in order to achieve estimates of the smooth
solutions for the initial-boundary value problem, barrier functions are constructed in a

\(\S 1.0\)
similar way as in (Juutinen & Kawohl 2006), but followed by rather more complicated
deductions to obtain Hölder estimates. Then again Arzelà-Ascoli lemma is utilised to
deduce the existence of the limit function. In light of the general consistence-stability
properties of viscosity solutions (Crandall, Ishii, & Lions 1992) the limit function is
indeed the viscosity solution of our random PDE. The existence of pathwise stationary
solutions is showed by applying the techniques to prove the asymptotic property of
solutions for the equation (1.2) in (Caselles, Morel, & Sbert 1998), combined with a
time reversion procedure in order to take the limit when the time approaches infinity.
Chapter 2

Cauchy Problem on the Whole Space

2.1 Preliminaries

Let \((W_t, t \in \mathbb{R})\) be a standard one-dimensional Wiener process on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the filtration \((\mathcal{F}_t, t \in \mathbb{R})\) generated by the Wiener process, i.e., \(\mathcal{F}_t := \sigma(W_s: 0 \leq s \leq t)\) for all \(t \geq 0\) and \(\mathcal{F}_t := \sigma(W_s: t \leq s \leq 0)\) for all \(t \leq 0\). Let us define for \(t \in \mathbb{R}\) the time shift operator for the path of Wiener process as \((\theta_t W)_s := W_{t+s} - W_t\), where \(s \in \mathbb{R}\). Now we introduce the notion of pathwise stationary solution of random dynamical systems, which can be regarded as the counterpart of fixed point of deterministic systems.

**Definition 2.1** Let \(u: U \times I \times \Omega \to U\) be a measurable random dynamical system on a measurable space \((U, \mathcal{B}(U))\) over a metric dynamical system \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in I})\), then a pathwise stationary solution of \(u\) is a \(\mathcal{F}\)-measurable random variable \(Y: \Omega \to U\) such that

\[ u(Y(\omega), t, \omega) = Y(\theta_t \omega) \]

for all \(t \in I\) and \(\mathbb{P}\)-a.s..
For a stochastic differential equation in general form on some abstract space
\[ dX(t, \omega) = F(X(t, \omega), t, \omega)dt + G(X(t, \omega), t, \omega)dW_t, \]
a solution \( X(t, \omega) \) is called a pathwise stationary solution if a.s.
\[ X(X_0(\omega), t, \omega) := X(t, \omega) = X(0, \theta(t) \omega) =: X_0(\theta(t) \omega). \]

Recall the cocycle property for random dynamical systems.

**Definition 2.2** We say that a random dynamical system \( u : U \times I \times \Omega \to U \) satisfies the **cocycle property** if a.s.

(i). \( u(0, \omega) = id : U \to U \); and

(ii). \( u(t+s, \omega) = u(t, \theta_s \omega) \circ u(s, \omega) \), for all \( t, s \in I \).

Recall that the solution of the stochastic differential equation (SDE)
\[ dz_t = -z_t dt + dW_t \]
is called the Ornstein-Uhlenbeck process. We know that
\[ z(\omega) := \int_{-\infty}^{0} e^s dW_s \quad (2.1) \]
is its pathwise stationary solution, which means that the Ornstein-Uhlenbeck process starting from \( z_0 = z(\omega) \) or \( z_{-\infty} := \lim_{t \to -\infty} = 0 \) has the solution
\[ z_t = z(\theta_t \omega) = \int_{-\infty}^{0} e^s dW_{t+s} = \int_{-\infty}^{t} e^{s-t} dW_s. \]
This can be easily verified, since
\[ dz_t = -e^{-t} \left( \int_{-\infty}^{t} e^s dW_s \right) dt + dW_t = -z_t dt + dW_t. \]

In light of the Birkhoff-Khinchin ergodic theorem (see [Sinai 1994] for instance) it holds readily that
\[ \bar{z} := \bar{z}(\omega) := \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} |z(\theta_s \omega)| ds \]
exists and \( \bar{z} = E\bar{\bar{z}} = E|z| \).

Here we cite a technical theorem in the theory of viscosity solutions which will be used below in the proofs of the uniqueness for solutions of the equations. Let \( \mathcal{O} \subset \mathbb{R}^N \) be locally compact, \( T > 0 \). Denote

\[
P_{\mathcal{O}}^{2,+} u(t,x) := \{ (\partial_t \phi(t,x), D\phi(t,x), D^2\phi(t,x)) : \phi \text{ is } C^{1,2} \text{ and } u - \phi \text{ has a local maximum at } (t,x) \}.
\]

\( P_{\mathcal{O}}^{2,-} u := -P_{\mathcal{O}}^{2,+} (-u) \) is defined similarly and \( \bar{P}_{\mathcal{O}}^{2,+}, \bar{P}_{\mathcal{O}}^{2,-} \) are their closures respectively.

The technical proposition is as follows.

**Proposition 2.3** *(Crandall, Ishii, & Lions, 1992)* Let \( u_i \in USC((0,T) \times \mathcal{O}_i) \) (upper semicontinuous functions) for \( i = 1, \ldots, k \) where \( \mathcal{O}_i \subset \mathbb{R}^{N_i} \) are locally compact. Let \( \phi \) be defined on an open neighborhood of \( (0,T) \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_k \) and such that \( (t,x_1,\cdots,x_k) \mapsto \phi(t,x_1,\cdots,x_k) \) is once continuously differentiable in \( t \) and twice continuously differentiable in \( (x_1,\cdots,x_k) \). Suppose that \( \hat{t} \in (0,T) \), \( \hat{x}_i \in \mathcal{O}_i \) and

\[
w(t,x_1,\cdots,x_k) := u_1(t,x_1) + \cdots + u_k(t,x_k) - \phi(t,x_1,\cdots,x_k) \leq w(\hat{t},\hat{x}_1,\cdots,\hat{x}_k)
\]

for \( 0 < t < T, x_i \in \mathcal{O}_i \). Assume, moreover, that there is an \( r > 0 \) such that for every \( M > 0 \) there is a \( C \) such that for \( i = 1, \ldots, k \)

\[
b_i \leq C \text{ whenever } (b_i,q_i,X_i) \in P_{\mathcal{O}_i}^{2,+} u_i(t,x_i),
\]

\[
|t - \hat{t}| \leq r \text{ and } |u_i(t,x_i)| + |q_i| + |X_i| \leq M.
\]

Then for each \( \epsilon > 0 \) there are \( b_i, X_i \in \mathcal{S}^{N_i}, \mathcal{S}^{N_i} \) being the set of symmetric \( N_i \times N_i \) matrices, such that

(i). \((b_i,D_x \phi(\hat{t},\hat{x}_1,\cdots,\hat{x}_k),X_i) \in P_{\mathcal{O}_i}^{2,+} u_i(\hat{t},\hat{x}_i) \) for \( i = 1, \ldots, k \),

(ii).

\[
-\left( \frac{1}{\epsilon} + |A| \right) I \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \leq A + \epsilon A^2,
\]

\[\S 2.1 \cdot 11\]
(iii). $b_1 + \cdots + b_k = \partial_t \phi(\hat{t}, \hat{x}_1, \cdots, \hat{x}_k),$

where $A := (D^2 \phi)(\hat{t}, \hat{x}_1, \cdots, \hat{x}_k)$.

Observe that the condition (2.2) is guaranteed by having each $u_i$ be a subsolution of a parabolic equation.

### 2.2 Uniqueness

Let’s consider the Cauchy problem of the one-dimensional stochastic PDE for $\bar{u}$ on $\mathbb{R}^n \times [0, \infty)$ in which $g := g(t, \omega)$ for $t \in (-\infty, \infty)$ is a given stochastic process with Lebesgue integrable paths a.s.,

$$
\begin{align*}
    d\bar{u} &= \frac{\langle D^2 \bar{u} D\bar{u}, Du \rangle}{|Du|^2} \, dt + g(t) \, dW_t \text{ in } \mathbb{R}^n \times (0, \infty), \\
    \bar{u} &= \bar{u}_0 \text{ on } \mathbb{R}^n \times \{0\}.
\end{align*}
$$

Via the Ornstein-Uhlenbeck transform

$$
\bar{u}(x, t) =: u(x, t) + g(t) (\theta t \omega) - \int_{-\infty}^{t} g(s) \, d\theta s \omega, \quad (2.4)
$$

the SPDE (2.3) becomes the following random PDE (2.5)

$$
\begin{align*}
    \partial_t u &= \frac{\langle D^2 u Du, Du \rangle}{|Du|^2} + g(t) z(\theta t \omega) \text{ in } \mathbb{R}^n \times (0, \infty), \\
    u &= u_0 := \bar{u}_0 - \int_{-\infty}^{0} g(s) \, d\theta s \omega \text{ on } \mathbb{R}^n \times \{0\}.
\end{align*}
$$

In fact, it is easy to see if $\bar{u}$ satisfies (2.3),

$$
\begin{align*}
    d\bar{u} &= du + z(\theta t \omega) \, dg(t) + g(t)(-z(\theta t \omega) dt + dW_t) - z(\theta t \omega) dg(t) \\
    &= \frac{\langle D^2 u Du, Du \rangle}{|Du|^2} \, dt + g(t) z(\theta t \omega) dt + g(t)(-z(\theta t \omega) dt + dW_t) \\
    &= \frac{\langle D^2 \bar{u} D\bar{u}, D\bar{u} \rangle}{|D\bar{u}|^2} \, dt + g(t) dW_t,
\end{align*}
$$

§2.2 · 12 ·
and 
\[ \bar{u}(x, 0) = u(x, 0) + \int_{-\infty}^{0} g(s) \, dz(\theta_s \omega) = \bar{u}_0(x). \]

Note that the Ornstein-Uhlenbeck transform in a more general form is applied in (Duan, Lü, Schmalfuß, 2003).

Regarding \( \omega \) as a parameter in the random PDE (2.5) we define the viscosity solutions of the random PDE in a similar way as for the deterministic PDEs. We denote \( USC(Domain, Range) \) and \( LSC(Domain, Range) \) as the classes of upper semi-continuous functions mapping from the Domain to the Range and lower semi-continuous functions mapping from the Domain to the Range respectively. Note that \( USC(Domain, Range) \cap LSC(Domain, Range) = C(Domain, Range) \).

**Definition 2.4** A function 
\[ u(\omega, \cdot, \cdot) \in USC(\mathbb{R}^n \times [0, \infty), \mathbb{R}) \]

\( (u(\omega, \cdot, \cdot) \in LSC(\mathbb{R}^n \times [0, \infty), \mathbb{R}), \text{ resp.}, \) 

a.a. \( \omega \in \Omega \), is called a **viscosity subsolution (supersolution, resp.)** of the equation (2.5) if a.s. 
\[ u(\omega, x, 0) \leq u_0 \ (u(\omega, x, 0) \geq u_0, \text{ resp.}), \]

and for all \( \phi(\omega, \cdot, \cdot) \in C^{2,1}(\mathbb{R}^n \times [0, \infty), \mathbb{R}) \), a.a. \( \omega \in \Omega \), such that \( u - \phi \) attains its local maximum (minimum, resp.) at \((x_0, t_0)\), then at \((x_0, t_0)\), a.s. 
\[ \partial_t \phi \leq \frac{\langle D^2 \phi D\phi, D\phi \rangle}{|D\phi|^2} + gz(\theta_t \omega), \]

\[ \left( \partial_t \phi \geq \frac{\langle D^2 \phi D\phi, D\phi \rangle}{|D\phi|^2} + gz(\theta_t \omega), \text{ resp.} \right) \]

if \( D\phi(x_0, t_0) \neq 0 \); and 
\[ \partial_t \phi \leq \sup_{|\alpha| \leq 1, \alpha \in \mathbb{R}^n} \langle D^2 \phi \alpha, \alpha \rangle + gz(\theta_t \omega), \]

\[ \left( \partial_t \phi \geq \sup_{|\alpha| \leq 1, \alpha \in \mathbb{R}^n} \langle D^2 \phi \alpha, \alpha \rangle + gz(\theta_t \omega), \text{ resp.} \right) \]

if \( D\phi(x_0, t_0) = 0 \). A viscosity solution of the equation (2.5) is a viscosity subsolution and supersolution of the equation (2.5) simultaneously.
Definition 2.5 A function \( \bar{u} : \Omega \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) is called a \textit{viscosity solution} of the equation (2.3) if \( u : \Omega \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \), related to \( \bar{u} \) by the Ornstein-Uhlenbeck transform (2.4), is a viscosity solution of the equation (2.5). In other words, \( \bar{u} \) satisfies the statement in Definition 2.4 with \( u - \phi \) replaced by \( \bar{u} - \bar{g} - \phi \) when the extrema are considered.

We now show the uniqueness of solutions for the equation (2.5), in the spirit of the proof for a certain class of deterministic PDEs arising from image processing problems (Alvarez, Lions, & Morel, 1992).

**Proposition 2.6 (Uniqueness)** Assume that functions \( u_0, v_0 : \Omega \times \mathbb{R}^n \to \mathbb{R} \) are bounded and Lipschitz continuous. Suppose that functions \( u, v : \Omega \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) are the viscosity solutions of the equation (2.5) with initial functions \( u_0, v_0 \) respectively. And assume that \( u, v \) are bounded and Lipschitz continuous. Then \( \forall T > 0 \) there exists a constant \( C > 0 \), which depends on the Lipschitz constants of \( u_0 \) and \( u \), and the bounds of \( u_0 \) and \( v_0 \), such that

\[
\sup_{\mathbb{R}^n \times [0, T]} |u - v| \leq C \sup_{\mathbb{R}^n} |u_0 - v_0|.
\]

This implies that the equation (2.5) has at most one viscosity solution and that the solution is continuously dependent on the initial value function.

**Proof.** Let \( T > 0 \) be fixed. Suppose \( u, v \) are solutions to the equation (2.5) with initial value functions \( u_0, v_0 \) respectively.

Define

\[
\Phi(x, y, t) := u(x, t) - v(y, t) - \phi(x, y, t)
\]

\[
:= u(x, t) - v(y, t) - \left( \frac{|x - y|^4}{4\epsilon} + \lambda t + \rho \left( \frac{|x|^2 + |y|^2}{2} + \frac{1}{T - t} \right) \right),
\]

for \( x, y \in \mathbb{R}^n, t \in [0, T] \) where \( \epsilon, \lambda, \rho > 0 \) are constants to be determined. Simple calculations imply that

\[
\partial_t \phi = \lambda + \frac{\rho}{(T - t)^2},
\]

\[\S 2.2 \quad \cdot 14 \cdot\]
\[ D\phi = \left( \begin{array}{c} D_x \phi \\ D_y \phi \end{array} \right) = \left( \begin{array}{c} \frac{1}{\epsilon} |x - y|^2 (x - y) + \rho x \\ -\frac{1}{\epsilon} |x - y|^2 (x - y) + \rho y \end{array} \right), \]

and

\[ D^2 \phi = \left( \begin{array}{cc} \frac{2}{\epsilon} (x - y)(x - y)^* + \frac{1}{\epsilon} |x - y|^2 I_n + \rho I_n & \frac{2}{\epsilon} (x - y)(x - y)^* - \frac{1}{\epsilon} |x - y|^2 I_n \\ -\frac{2}{\epsilon} (x - y)(x - y)^* - \frac{1}{\epsilon} |x - y|^2 I_n & \frac{2}{\epsilon} (x - y)(x - y)^* + \frac{1}{\epsilon} |x - y|^2 I_n + \rho I_n \end{array} \right) \]

:= \left( \begin{array}{cc} A + \rho I_n \\ -A \end{array} \right) \left( \begin{array}{c} \frac{2}{\epsilon} \rho I_n \\ -A \\ A + \rho I_n \end{array} \right).

Now from the assumption it is readily deduced that a maximum point, denoted by \((x_0, y_0, t_0)\), of \(\Phi(x, y, t)\) exists. By the definition of \(\Phi\) it follows that

\[ \Phi(x_0, y_0, t_0) = u(x_0, t_0) - v(y_0, t_0) - \frac{|x_0 - y_0|^4}{4\epsilon} - \lambda t_0 - \rho \left( \frac{|x_0|^2 + |y_0|^2}{2} + \frac{1}{T - t_0} \right) \]

\[ \leq u(x_0, t_0) - v(y_0, t_0) \]

\[ \leq |u|_\infty + |v|_\infty. \]

We require that \(\rho \leq T\), then obviously we have that

\[ \Phi(x_0, y_0, t_0) = u(x_0, t_0) - v(y_0, t_0) - \frac{|x_0 - y_0|^4}{4\epsilon} - \lambda t_0 - \rho \left( \frac{|x_0|^2 + |y_0|^2}{2} + \frac{1}{T - t_0} \right) \]

\[ \geq u(0, 0) - v(0, 0) - \frac{\rho}{T} \]

\[ \geq -|u_0|_\infty - |v_0|_\infty - 1. \]

Thus

\[ \rho \left( \frac{|x_0|^2 + |y_0|^2}{2} + \frac{1}{T - t_0} \right) = u(x_0, t_0) - v(y_0, t_0) - \frac{|x_0 - y_0|^4}{4\epsilon} - \lambda t_0 - \Phi(x_0, y_0, t_0) \]

\[ \leq u(x_0, t_0) - v(y_0, t_0) - \Phi(x_0, y_0, t_0) \]

\[ \leq 2|u|_\infty + 2|v|_\infty + 1 \]
where $C_1 := C_1(|u|_\infty, |v|_\infty)$. Consequently,
\[
|x_0|^2 + |y_0|^2 \leq \frac{2C_1}{\rho}.
\]
And by the assumption we have $\Phi(x_0, y_0, t_0) \geq \Phi(y_0, y_0, t_0)$, so that
\[
u(x_0, t_0) - v(y_0, t_0) - \lambda t_0 - \rho \left( \frac{|x_0|^2 + |y_0|^2}{2} + \frac{1}{T - t_0} \right) \geq \nu(y_0, t_0) - v(y_0, t_0) - \lambda t_0 - \rho \left( \frac{|y_0|^2}{2} + \frac{1}{T - t_0} \right).
\]
Thus we have that
\[
\frac{|x_0 - y_0|^4}{4\epsilon} \leq \nu(x_0, t_0) - u(y_0, t_0) + \rho \left( |x_0|^2 - |y_0|^2 \right) \leq L|x_0 - y_0| + \rho \left( \frac{|y_0|^2}{2} - |y_0|^2 \right)
\]
which leads to
\[
|x_0 - y_0|^3 \leq 4L\epsilon + 2\epsilon \rho |x_0 + y_0|
\]
That is,
\[
|x_0 - y_0|^3 \leq 4\epsilon(L + \sqrt{C_1 \rho}), \tag{2.6}
\]
where $L$ is the Lipschitz constant of the solution $u$ w.r.t. $x$.

We claim that $t_0 = 0$. To the contrary we assume temporarily that $t_0 > 0$. Now by virtue of Proposition 2.3 for all $\mu > 0$, we find $a, b \in \mathbb{R}$ and $X, Y \in \mathcal{S}^n, (n \times n)$ symmetric matrices, such that
\[
a - b = \lambda + \frac{\rho}{(T - t)^2}
\]
and
\[
\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2 \phi(x_0, y_0, t_0) + \mu(D^2 \phi(x_0, y_0, t_0))^2,
\]
and it holds that a.s.
\[
a \leq \frac{\langle XD_x \phi(x_0, y_0, t_0), D_x \phi(x_0, y_0, t_0) \rangle}{|D_x \phi(x_0, y_0, t_0)|^2} + g(t_0)z(\theta_t \omega), \tag{2.7}
\]
when \(D_x \phi(x_0, y_0, t_0) = \epsilon^{-1}|x_0 - y_0|^2(x_0 - y_0) + \rho x_0 \neq 0\), or
\[
a \leq \sup_{\alpha \in \mathbb{R}^n, |\alpha| = 1} \langle X \alpha, \alpha \rangle + g(t_0)z(\theta_t \omega), \tag{2.8}
\]
when \(D_x \phi(x_0, y_0, t_0) = 0\).

and
\[
b \geq \frac{\langle YD_y \phi(x_0, y_0, t_0), D_y \phi(x_0, y_0, t_0) \rangle}{|D_y \phi(x_0, y_0, t_0)|^2} + g(t_0)z(\theta_t \omega), \tag{2.9}
\]
when \(D_y \phi(x_0, y_0, t_0) = -\epsilon^{-1}|x_0 - y_0|^2(x_0 - y_0) + \rho y_0 \neq 0\), or
\[
b \geq \inf_{\alpha \in \mathbb{R}^n, |\alpha| = 1} \langle Y \alpha, \alpha \rangle + g(t_0)z(\theta_t \omega), \tag{2.10}
\]
when \(D_x \phi(x_0, y_0, t_0) = 0\).

Now we have to deduce a contradiction from every combination of (2.7)-(2.9), (2.7)-(2.10), (2.8)-(2.9), and (2.8)-(2.10). For simplicity we denote
\[
\beta_1 := \frac{D_x \phi(x_0, y_0, t_0)}{|D_x \phi(x_0, y_0, t_0)|} = \frac{\frac{1}{\epsilon}|x_0 - y_0|^2(x_0 - y_0) + \rho x_0}{\frac{1}{\epsilon}|x_0 - y_0|^2(x_0 - y_0) + \rho x_0},
\]
\[
\beta_2 := \frac{D_y \phi(x_0, y_0, t_0)}{|D_y \phi(x_0, y_0, t_0)|} = \frac{-\frac{1}{\epsilon}|x_0 - y_0|^2(x_0 - y_0) + \rho y_0}{-\frac{1}{\epsilon}|x_0 - y_0|^2(x_0 - y_0) + \rho y_0}.
\]

For instance we now consider the case (2.7)-(2.9) with \(x_0 \neq y_0\). Put the two inequalities together and we have, for all \(\mu > 0\),
\[
\lambda + \frac{\rho}{(T - t_0)^2}
\]
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\[
\begin{align*}
&= a - b \\
&\leq \langle X\beta_1, \beta_1 \rangle - \langle Y\beta_2, \beta_2 \rangle \\
&= \text{trace} \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \begin{pmatrix} \beta_1 \beta_1^* & \beta_1 \beta_2^* \\ \beta_2 \beta_1^* & \beta_2 \beta_2^* \end{pmatrix} \\
&= \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \\
&\leq \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \left( D^2\phi(x_0, y_0, t_0) + \mu(D^2\phi(x_0, y_0, t_0))^2 \right) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \\
&= \text{trace} \left\{ \begin{bmatrix} A + 2\mu A^2 + 2\mu \rho A & -A - 2\mu A^2 - 2\mu \rho A \\ -A - 2\mu A^2 - 2\mu \rho A & A + 2\mu A^2 + 2\mu \rho A \end{bmatrix} + \rho(1 + \mu \rho)I_n \right\} \begin{pmatrix} \beta_1 \beta_1^* & \beta_1 \beta_2^* \\ \beta_2 \beta_1^* & \beta_2 \beta_2^* \end{pmatrix} \\
&= \text{trace} \left[ (A + 2\mu A^2 + 2\mu \rho A)(\beta_1 - \beta_2)(\beta_1 - \beta_2)^* \right] + 2\rho(1 + \mu \rho).
\end{align*}
\]

Note that \(a, b, X, Y\) do depend on \(\mu\), whereas the constants and variables, except \(\mu\) itself, in the two sides of the above inequality do not. Because \(\mu > 0\) is arbitrary letting it tend to 0 implies that

\[
\lambda + \frac{\rho}{(T - t_0)^2} \leq \text{trace} (A(\beta_1 - \beta_2)(\beta_1 - \beta_2)^*) + 2\rho.
\]

Now we compute that

\[
\begin{align*}
\text{trace}(A(\beta_1 - \beta_2)(\beta_1 - \beta_2)^*) &\leq \sqrt{\text{trace}(AA^*)}|\beta_1 - \beta_2|^2 \\
&= |\beta_1 - \beta_2|^2 \sqrt{\frac{8}{\epsilon^2}|x_0 - y_0|^4 + \frac{1}{\epsilon^2}|x_0 - y_0|^4} \\
&= \frac{3}{\epsilon} |\beta_1 - \beta_2|^2 |x_0 - y_0|^2 \\
&\leq \frac{12}{\epsilon} |x_0 - y_0|^2 \\
&\leq \frac{12}{\epsilon} (4\epsilon)^{2/3}(L + \sqrt{C_1\rho})^{2/3} \\
&= 24\sqrt{2} \epsilon^{-1/3}(L + \sqrt{C_1\rho})^{2/3}.
\end{align*}
\]
Eventually we obtain explicitly that

\[
\lambda + \frac{\rho}{(T - t_0)^2} \leq 24 \sqrt[3]{\frac{2}{\epsilon}} \sqrt[3]{\left( L + \sqrt{C_1 \rho} \right)^2} + 2\rho.
\]

Assume \( \sup_{\mathbb{R}^n \times [0,T]} |u - v| > 0; \) for if \( \sup_{\mathbb{R}^n \times [0,T]} |u - v| = 0 \) our conclusion follows. And notice that \( \sup_{\mathbb{R}^n \times [0,T]} |u - v| \leq |u|_{\infty} + |v|_{\infty}. \) Let

\[
\epsilon^{1/3} := \delta \sup_{\mathbb{R}^n \times [0,T]} |u - v|
\]

for some \( \delta > 0 \) to be determined, and let

\[
24 \sqrt[3]{\frac{2}{\epsilon}} \left( L + \sqrt{C_1 \rho} \right)^{2/3} =: C_2 \left( \sup_{\mathbb{R}^n \times [0,T]} |u - v| \right)^2,
\]

where \( C_2 := C_2(C_1, L, \rho, \sup_{\mathbb{R}^n \times [0,T]} |u - v|) = C_2(|u_0|_{\infty}, |v_0|_{\infty}, L, \rho, \sup_{\mathbb{R}^n \times [0,T]} |u - v|). \) Consequently the above inequality becomes

\[
\lambda + \frac{\rho}{(T - t_0)^2} \leq \delta^{-1} C_2 \sup_{\mathbb{R}^n \times [0,T]} |u - v| + 2\rho.
\]

Now let

\[
\lambda := \delta^{-1} C_2 \sup_{\mathbb{R}^n \times [0,T]} |u - v| + 3\rho
\]

and we get immediately

\[
\rho + \frac{\rho}{(T - t_0)^2} \leq 0.
\]

Hence we encounter a contradiction and learn that actually \( t_0 = 0. \)

The combination (2.7)-(2.9) with \( x_0 = y_0 \) is much simpler. The other combinations (2.7)-(2.10), (2.8)-(2.9), and (2.8)-(2.10) are similarly discussed; we omit them here.

Therefore, for all \( \rho > 0 \) we get

\[
\begin{align*}
&u(x, t) - v(y, t) - \frac{|x - y|^4}{4\epsilon} - \lambda t - \rho \left( \frac{|x|^2 + |y|^2}{2} + \frac{1}{T - t} \right) \\
&\leq \sup_{x, y \in \mathbb{R}^n} \left( u_0(x) - v_0(y) - \frac{|x - y|^4}{4\epsilon} - \rho \left( \frac{|x|^2 + |y|^2}{2} + \frac{1}{T} \right) \right).
\end{align*}
\]

Taking limits as \( \rho \downarrow 0 \) we deduce that

\[
u(x, t) - v(y, t) - \frac{|x - y|^4}{4\epsilon} - \lambda t \leq \sup_{x, y \in \mathbb{R}^n} \left( u_0(x) - v_0(y) - \frac{|x - y|^4}{4\epsilon} \right), \]
with

$$\lambda = \delta^{-1} C_2 \sup_{\mathbb{R}^n \times [0,T]} |u - v|.$$ 

Thus

$$u(x, t) - v(x, t) \leq \sup_{x, y \in \mathbb{R}^n} \left( u_0(x) - u_0(y) + u_0(y) - v_0(y) - \frac{|x - y|^4}{4\epsilon} \right) + \lambda t.$$ 

Hence

$$\sup_{\mathbb{R}^n \times [0,T]} (u - v) \leq \sup_{\mathbb{R}^n} |u_0 - v_0| + \lambda T + \sup_{x, y \in \mathbb{R}^n} \left( u_0(x) - u_0(y) - \frac{|x - y|^4}{4\epsilon} \right) \leq \sup_{\mathbb{R}^n} |u_0 - v_0| + \lambda T + \sup_{r \geq 0} \left( Lr - \frac{r^4}{4\epsilon} \right) ,$$

$$= \sup_{\mathbb{R}^n} |u_0 - v_0| + \lambda T + L^{4/3} \epsilon^{1/3} - (L\epsilon)^{4/3} / 4\epsilon , \text{ (letting } L = \frac{r^3}{\epsilon} \text{ or } r = (L\epsilon)^{1/3}, \text{ )}$$

$$= \sup_{\mathbb{R}^n} |u_0 - v_0| + \lambda T + 3L^{4/3} \epsilon^{1/3}$$

$$= \sup_{\mathbb{R}^n} |u_0 - v_0| + \delta^{-1} TC_2 \sup_{\mathbb{R}^n \times [0,T]} |u - v| + \frac{3}{4} L^{4/3} \delta \sup_{\mathbb{R}^n \times [0,T]} |u - v|.$$

Let

$$\delta := \frac{1}{3} L^{-4/3}$$

and we obtain that

$$\sup_{\mathbb{R}^n \times [0,T]} (u - v) \leq \sup_{\mathbb{R}^n} |u_0 - v_0| + 3TL^{4/3}C_2 \sup_{\mathbb{R}^n \times [0,T]} |u - v| + \frac{1}{4} \sup_{\mathbb{R}^n \times [0,T]} |u - v|.$$

Then let

$$T := \frac{L^{4/3}}{12C_2}$$

and we get that

$$\sup_{\mathbb{R}^n \times [0,T]} (u - v) \leq \sup_{\mathbb{R}^n} |u_0 - v_0| + \frac{1}{2} \sup_{\mathbb{R}^n \times [0,T]} |u - v|.$$

Changing the role of $u$ and $v$ gives that

$$\sup_{\mathbb{R}^n \times [0,T]} |u - v| \leq \sup_{\mathbb{R}^n} |u_0 - v_0| + \frac{1}{2} \sup_{\mathbb{R}^n \times [0,T]} |u - v|.$$

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that is,
\[ \sup_{\mathbb{R}^n \times [0,T]} |u - v| \leq 2 \sup_{\mathbb{R}^n} |u_0 - v_0|. \]

Finally, for all \( T' \in [0, \infty) \) there exists \( N \geq 1 \) such that \( NT \geq T' \) for the \( T \) above. Then by iteration it holds that
\[ \sup_{\mathbb{R}^n \times [0,T']} |u - v| \leq 2^N \sup_{\mathbb{R}^n} |u_0 - v_0|. \]

This completes the proof. \( \square \)

2.3 Boundedness

The uniform boundedness of the solutions of the random PDE \((2.5)\) is showed in the following proposition.

**Proposition 2.7** Suppose that the equation \((2.5)\) with initial value function \( u_0 : \Omega \times \mathbb{R}^n \to \mathbb{R} \) has a viscosity solution \( u : \Omega \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \), then it is uniformly bounded in \( x \). More precisely, we have that
\[ \inf_{\mathbb{R}^n} u_0 - \int_0^t g(s) z(\theta s \omega) ds \leq u(x, t) \leq \sup_{\mathbb{R}^n} u_0 + \int_0^t g(s) z(\theta s \omega) ds. \quad (2.11) \]

In particular, it follows that
\[ |u(x, t)| \leq |u_0|_\infty + \left| \int_0^t g(s) z(\theta s \omega) ds \right|; \]
and for \( T > 0 \) it holds that for \( 0 \leq t \leq T \),
\[ |u(x, t)| \leq |u_0|_\infty + \int_0^T |g(s) z(\theta s \omega)| ds. \]

**Proof.** Let
\[ \phi(x, t) := \sup_{\mathbb{R}^n} u_0(x) + \delta t + \int_0^t g(s) z(\theta s \omega) ds, \]
for arbitrary \( \delta > 0 \). Suppose that \( u - \phi \) attains its local maximum at \((x_0, t_0)\). We claim that \( t_0 = 0 \). If not, \( t_0 > 0 \). Then at \((x_0, t_0)\)
\[ \partial_t \phi(x_0, t_0) \leq \sup_{|\alpha| \leq 1} \langle D^2 \phi(x_0, t_0) \alpha, \alpha \rangle + g(t_0) z(\theta t_0 \omega), \]
that is,

\[ \delta + g(t_0)z(\theta_0\omega) \leq 0 + g(t_0)z(\theta_0\omega), \]

or \( \delta \leq 0 \), which contradicts our assumption. Thus \( t_0 = 0 \); the local maximum is actually taken at \((x_0, 0)\). Now,

\[
\sup_{\mathbb{R}^n \times [0, \infty)} (u(x, t) - \phi(x, t)) = \sup_{\mathbb{R}^n} (u_0(x) - \phi(x, 0)) = \sup_{\mathbb{R}^n} (u_0(x) - \sup_{\mathbb{R}^n} u_0(x)) = 0.
\]

It follows that \( u(x, t) - \phi(x, t) \leq 0 \) for all \( \delta > 0 \), namely, for all \( \delta > 0 \)

\[ u(x, t) \leq \sup_{\mathbb{R}^n} u_0(x) + \delta t + \int_0^t g(s)z(\theta_s\omega)ds. \]

As \( \delta \) is arbitrary we get immediately

\[ u(x, t) \leq \sup_{\mathbb{R}^n} u_0(x) + \int_0^t g(s)z(\theta_s\omega)ds. \]

The lower bound can be obtained analogously. \( \Box \)

2.4 Existence

In this section the existence of solutions of the random PDE (2.5) is deduced through the classical Bernstein method for the estimation of the gradients. This is inspired by the deterministic case studied in (Alvarez, Lions, & Morel, 1992).

First we cite here a classical result of the maximum principle which will be used later. See (Ladyzhenskaya, Solonnikov, & Uraltseva, 1968, Chapter I, §2, Theorem 2.9).

**Proposition 2.8** Consider the following quasilinear parabolic second order partial differential equations of general form for an unknown function \( u: \overline{U_T} \to \mathbb{R} \)

\[ \mathcal{L}u \equiv u_t - a_{ij}(x, t, u, u_x)u_{x_i}u_{x_j} + a(x, t, u, u_x) = 0, \]
Let $u(x, t)$ be a classical solution of the above equation in $U_T$. Suppose that the functions $a_{ij}(x, t, u, p)$ and $a(x, t, u, p)$ take finite values for any finite $u, p$, and $(x, t) \in U_T$, and that for $(x, t) \in U_T$ and arbitrary $u$,

$$a_{ij}(x, t, u, 0)\xi_i\xi_j \geq 0,$$

and

$$ua(x, t, u, 0) \geq -b_1u^2 - b_2,$$  \hspace{1cm} (2.12)

where $b_1$ and $b_2$ are nonnegative constants. Then

$$\max_{U_T} |u(x, t)| \leq \inf_{\lambda > b_1} e^{\lambda T} \left[ \max_{\Gamma_T} |u|, \sqrt{\frac{b_2}{\lambda - b_1}} \right].$$

If in place of (2.12) the condition

$$ua(x, t, u, 0) \geq -\Phi(|u|)u - b_2$$

is fulfilled, where $b_2 \geq 0$ and $\Phi(r)$ is a nondecreasing positive function of $r \geq 0$ satisfying the condition

$$\int_0^{\infty} \frac{d\tau}{\Phi(\tau)} = \infty,$$

then the estimate

$$\max_{U_T} |u(x, t)| \leq \inf_{\lambda \geq 1} \phi(\xi),$$

$$\xi = e^{\lambda T} \max \left\{ 1; \phi^{-1} \left( \frac{b_2}{(\lambda - 1)\Phi(0)} \right); \phi^{-1} \left( \max_{\Gamma_T} |u| \right) \right\}$$

is valid, in which $\phi^{-1}(\xi)$ is the inverse of the function $\phi(\xi)$ defined by the equation

$$\int_0^{\phi(\xi)} \frac{d\tau}{\Phi(\tau)} = \ln \xi.$$
Proof. We prove by approximation. Consider the following random PDE.

\[
\frac{\partial u^\epsilon}{\partial t} = \frac{\langle D^2 u^\epsilon D u^\epsilon, Du^\epsilon \rangle}{|Du^\epsilon|^2 + \epsilon} + \epsilon \Delta u^\epsilon + g(t)\sigma(\theta_t \omega), \quad (2.13)
\]

where \( \epsilon > 0 \), \( u^\epsilon_0 \in C^\infty(\mathbb{R}^n, \mathbb{R}) \), \( u^\epsilon_0(x) \to u_0(x) \) uniformly, \( |Du^\epsilon_0|_{L^\infty} \leq |Du_0|_{L^\infty} \), and \( |u^\epsilon_0|_{L^\infty} \leq |u_0|_{L^\infty} \). Here for \( f : \mathbb{R}^n \to \mathbb{R}^n \) we denote \( |f|_{L^\infty} := |f|_{L^\infty(\mathbb{R}^n)} := \text{ess sup}_{x \in \mathbb{R}^n} |f(x)| \) where \( | \cdot | \) is the usual Euclidean norm for \( \mathbb{R}^n \). From the theory of general quasilinear uniformly parabolic PDE, the equation (2.13) has a unique classical solution \( u^\epsilon \in C^\infty(1)(\mathbb{R}^n \times [0, \infty)) \) (see Ladyzhenskaya, Solonnikov, & Uraltseva [1968], for instance).

Next we deduce an a priori estimate of the norm of the gradient \( |Du^\epsilon|_{L^\infty} \). For simplicity we write \( w := u^\epsilon \) and denote \( w_k \) the partial derivative of \( w \) w.r.t. \( x_k \), and so on. Differentiating both sides of the equation (2.13) w.r.t. \( x_k \) gives

\[
\frac{\partial}{\partial t} w_k = (|Dw|^2 + \epsilon)^{-1} \left( \langle D^2 w_k Dw, Dw \rangle + 2 \langle D^2 w Dw, Dw_k \rangle \right) - (|Dw|^2 + \epsilon)^{-2} 2 \langle D^2 w Dw, Dw \rangle \langle Dw, Dw_k \rangle + \epsilon \sum_i w_{kii}.
\]

Multiply both sides of the above equation by \( 2w_k \) and take the summation over \( k \), then we get that

\[
\frac{\partial}{\partial t} |Dw|^2 = (|Dw|^2 + \epsilon)^{-1} \left[ \sum_k 2w_k D^2 w_k Dw, Dw \right] + 2 \langle D^2 w Dw, \sum_k 2w_k Dw_k \rangle
\]

\[
-(|Dw|^2 + \epsilon)^{-2} 2 \langle D^2 w Dw, Dw \rangle \sum_k 2w_k Dw_k + 2 \epsilon \sum_{k,i} w_{kii} w_k
\]

\[
= (|Dw|^2 + \epsilon)^{-1} \left[ \sum_k 2w_k D^2 w_k Dw, Dw \right] + 4 \langle D^2 w Dw, D^2 w Dw \rangle
\]

\[
-(|Dw|^2 + \epsilon)^{-2} 4 \langle D^2 w Dw, Dw \rangle \langle Dw, D^2 w Dw \rangle + 2 \epsilon \sum_{k,i} w_{kii} w_k
\]

\[
= (|Dw|^2 + \epsilon)^{-1} 2 \sum_{i,j,k} w_{ijk} w_j w_k + (|Dw|^2 + \epsilon)^{-1} 4 |D^2 w Dw|^2
\]

\[
-(|Dw|^2 + \epsilon)^{-2} 4 |(Dw)^* D^2 w Dw|^2 + 2 \epsilon \sum_{k,i} w_{kii} w_k.
\]
Rewrite the right hand side of the above equation in terms of $|Dw|^2$ and we have that

$$\partial_t |Dw|^2$$

$$= (|Dw|^2 + \epsilon)^{-1} (\langle D^2(|Dw|^2) Dw, Dw \rangle - 2|D^2w Dw|^2) + 4(|Dw|^2 + \epsilon)^{-1}|D^2w Dw|^2$$

$$- 4(|Dw|^2 + \epsilon)^{-2}|(Dw)^* D^2w Dw|^2 + \epsilon \left(\Delta(|Dw|^2) - 2|D^2w|^2\right)$$

$$= \frac{\langle D^2(|Dw|^2) Dw, Dw \rangle}{|Dw|^2 + \epsilon} + \epsilon \Delta(|Dw|^2) + \frac{2|D^2w Dw|^2}{|Dw|^2 + \epsilon} - \frac{4|\langle Dw \rangle^* D^2w Dw|^2}{(|Dw|^2 + \epsilon)^2} - 2\epsilon |D^2w|^2$$

This implies that in $\mathbb{R}^n \times (0, \infty)$

$$\partial_t |Dw|^2 \leq \frac{\langle D^2(|Dw|^2) Dw, Dw \rangle}{|Dw|^2 + \epsilon} + \epsilon \Delta(|Dw|^2) + \frac{|\langle Dw \rangle^* D^2w Dw|^2}{2(|Dw|^2 + \epsilon)}.$$

Now we consider $(x, t)$ in two cases. If $|Dw(x, t)| < 1$ the desired bound is just achieved.

For those $\{(x, t) : |Dw(x, t)| \geq 1\}$ the above inequality is immediately followed by

$$\partial_t |Dw|^2 \leq \frac{\langle D^2(|Dw|^2) Dw, Dw \rangle}{|Dw|^2 + \epsilon} + \epsilon \Delta(|Dw|^2) + \frac{1}{2}|D^2w Dw|^2.$$

Hence, by the maximum principle Proposition 2.8 for the Cauchy problem of the quasilinear parabolic PDE (see also (Brézis 1987)) we obtain that there exists a constant $C > 0$ such that

$$|Dw(x, t)| \leq C|Dw(x, 0)| = C|Du_0(x)|.$$

Hence,

$$|Du^\epsilon(\cdot, t)|_{L^\infty(\mathbb{R}^n)} \leq C|Du_0^\epsilon|_{L^\infty(\mathbb{R}^n)}.$$  \hfill (2.14)

Because of the general consistence-stability properties of viscosity solutions we finally only need to show there exists a subsequence of $u^\epsilon$ converging uniformly on $\mathbb{R}^n \times [0, T]$ to some function $u$ in $C(\mathbb{R}^n \times [0, T], \mathbb{R}) \cap L^\infty(0, T; W^{1, \infty}(\mathbb{R}^n, \mathbb{R}))$ for any $T > 0$. This follows from the Ascoli-Arzelà theorem. Actually, (2.14) implies that there exists a constant $C_T$, independent of $\epsilon, x, y, t$, such that $\forall t \in [0, T], x, y \in \mathbb{R}^n$

$$|u^\epsilon(x, t) - u^\epsilon(y, t)| \leq C_T|x - y|.$$
Lastly we show the continuity in $t$. Fix $s \in [0,T]$. For arbitrary $\delta > 0$ there exists $u_\delta^\epsilon \in W^{2, \infty}(\mathbb{R}^n, \mathbb{R})$ such that $|u^\epsilon(\cdot, s) - u_\delta^\epsilon|_{L^\infty} \leq C_T \delta$ and $|D^2 u_\delta^\epsilon|_{L^\infty(\mathbb{R}^n)} \leq C_T / \delta$. Let us denote

$$\tilde{u}(x, t) := u^\epsilon(x, t) - u_\delta^\epsilon(x).$$

Regarding the known term $Du^\epsilon(x, t)$ as a coefficient we have that

$$\partial_t \tilde{u} = \partial_t u^\epsilon = \langle D^2 (\tilde{u} + u_\delta^\epsilon) Du^\epsilon, Du^\epsilon \rangle / |Du^\epsilon|^2 + \epsilon \Delta (\tilde{u} + u_\delta^\epsilon) + g(t) z(\theta_t \omega)$$

$$\leq \langle D^2 \tilde{u} Du^\epsilon, Du^\epsilon \rangle / |Du^\epsilon|^2 + \epsilon \Delta \tilde{u} + (1 + \sqrt{n\epsilon}) |D^2 u_\delta^\epsilon|_{L^\infty(\mathbb{R}^n)} + g(t) z(\theta_t \omega).$$

Then from the maximum principle it is deduced that for $s \leq t \leq T$,

$$|u^\epsilon(\cdot, t) - u_\delta^\epsilon|_{L^\infty(\mathbb{R}^n)} \leq |u^\epsilon(\cdot, s) - u_\delta^\epsilon|_{L^\infty(\mathbb{R}^n)} + \frac{(1 + \sqrt{n\epsilon}) C_T}{\delta} (t - s) + \int_s^t |g(r) z(\theta_r \omega)| dr$$

$$\leq C_T \delta + \frac{(1 + \sqrt{n\epsilon}) C_T}{\delta} (t - s) + \sup_{s \leq r \leq t} |g(r) z(\theta_r \omega)| (t - s).$$

Choosing $\delta := (t - s)^{1/2}$ we finally obtain that

$$|u^\epsilon(x, t) - u^\epsilon(x, s)| \leq (2 + \sqrt{n\epsilon}) C_T |t - s|^{1/2} + \sup_{s \leq r \leq t} |g(r) z(\theta_r \omega)||t - s|.$$ 

\[\square\]

**Remark 2.10** For later convenience we choose the constants $C_T$ to be increasing in $T$.

### 2.5 Pathwise stationary solutions

Now we study the asymptotic property of the solution of the equation (2.5) as time tends to infinity, and prove that the limit function is a pathwise stationary solution of the random PDE (2.5). By virtue of the Ornstein-Uhlenbeck transform this limit function gives the corresponding pathwise stationary solution for the stochastic PDE.
To do this we follow essentially the same techniques in (Caselles, Morel, & Sbert, 1998) where the asymptotic property of the deterministic PDE is demonstrated. A time reversion technique is applied to deal with the limiting procedure when the time approaches infinity.

**Proposition 2.11** The equation (2.5) has at least one pathwise stationary solution.

**Proof.** Let $T > 1$ and set $K$ to be a constant such that

$$K > 2 \left( |u_0|_{\infty} + \int_0^T |g(s)z(\theta_s \omega)| ds \right) + \frac{1}{T}.$$ 

Let $\rho : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\lim_{T \to \infty} \rho(T) = \infty$. Define a smooth function $f(t, T)$ of $t \in [0, \infty)$ as $f(t, T) = -K$ on $[0, T - 1/\rho(T)] \cup [T + 1/\rho(T), \infty)$, $f(T, T) = 0$ and increasing on $[T - 1/\rho(T), T]$, decreasing on $[T, T + 1/\rho(T)]$. Now let

$$v_T^+(x, \omega) := \sup_{t \geq 0} \left[ u(x, t, \theta_{-T} \omega) - \frac{t}{T^2} + f(t, T) \right],$$
$$v_T^-(x, \omega) := \inf_{t \geq 0} \left[ u(x, t, \theta_{-T} \omega) + \frac{t}{T^2} - f(t, T) \right],$$

where $u(x, t, \omega)$ is the solution of equation (2.5) with initial value function $u_0$. Observing that

$$|v_T^+(x, \omega) - v_T^+(y, \omega)| \leq \sup_{t \geq 0} |u(x, t, \theta_{-T} \omega) - u(y, t, \theta_{-T} \omega)| \leq L|x - y|,$$

and that

$$|v_T^-(x, \omega) - v_T^-(y, \omega)| \leq \sup_{t \geq 0} |u(x, t, \theta_{-T} \omega) - u(y, t, \theta_{-T} \omega)| \leq L|x - y|,$$

we learn that $v_T^+$ and $v_T^-$ are also Lipschitz in $x$ with the same Lipschitz constants as $u$. From previous results we know that there exists a sequence $T_k \uparrow \infty$ such that $v_{T_k}^+, v_{T_k}^-$ converge to some Lipschitz functions $\mu, \nu$ respectively.
Suppose the supremum in the definition of $v^+_T$ is obtained at $t^+_T$. We claim that $t^+_T \in [T - 1/\rho(T), T + 1/\rho(T)]$. If not, the definitions of $v^+_T$ imply that

$$u(x, t^+_T, \theta - T\omega) - \frac{t^+_T}{T^2} + f(t^+_T, T) \leq |u_0|_\infty + \int_0^T |g(s) z(\theta_s \omega)|ds - K,$$

$$u(x, T, \theta - T\omega) - \frac{T}{T^2} + f(T, T) \geq -|u_0|_\infty - \int_0^T |g(s) z(\theta_s \omega)|ds - \frac{1}{T}.$$

Thus

$$K \leq 2 \left( |u_0|_\infty + \int_0^T |g(s) z(\theta_s \omega)|ds \right) + \frac{1}{T},$$

which contradicts the assumption. Similar claim is true for $t^-_T$ at which the infimum in the definition of $v^-_T$ is attained, i.e., $t^-_T \in [T - 1/\rho(T), T + 1/\rho(T)]$.

And we know that

$$v^k_T(x, \omega) \geq u(x, T_k, \theta - T_k\omega) - \frac{1}{T_k},$$

$$v^k_T(x, \omega) \leq u(x, T_k, \theta - T_k\omega) + \frac{1}{T_k},$$

so that

$$\mu(x, \omega) \geq \limsup_{k \to \infty} u(x, T_k, \theta - T_k\omega) \geq \liminf_{k \to \infty} u(x, T_k, \theta - T_k\omega) \geq \nu(x, \omega).$$

On the other side, noting that $f$ by definition is non-positive, we have

$$v^+_T(x, \omega) - v^-_T(x, \omega) = \sup_{t \geq 0} \left[ u(x, t, \theta - T\omega) - \frac{t}{T^2} + f(t, T) \right] - \inf_{t \geq 0} \left[ u(x, t, \theta - T\omega) + \frac{t}{T^2} - f(t, T) \right] = u(x, t^+_T, \theta - T\omega) - u(x, t^+_T, \theta - T\omega) - \frac{t^+_T + t^-_T}{T^2} + f(t^+_T, T) + f(t^-_T, T) \leq u(x, t^+_T, \theta - T\omega) - u(x, t^-_T, \theta - T\omega) - \frac{t^+_T + t^-_T}{T^2}.$$

Since $[T - 1/\rho(T), T + 1/\rho(T)] \ni t^T_\pm$, it follows that $t^T_\pm \to \infty$ and

$$\frac{2T - 2/\rho(T)}{T^2} \leq \frac{t^+_T + t^-_T}{T^2} \leq \frac{2T + 2/\rho(T)}{T^2},$$

which implies that

$$\frac{t^+_T + t^-_T}{T^2} \to 0.$$
as $T \to \infty$. And from the continuity of $u$ in $t$, obtained in Proposition 2.9, we have that

$$|u(x, t^+_T, \theta_{-T}\omega) - u(x, t^-_T, \theta_{-T}\omega)| \leq C_{T+1/\rho(T)} |t^+_T - t^-_T|^{1/2}$$

for all $T \in (1, \infty)$. Now we further require $\rho$ is such that $\rho(T) \geq 1$ for $T > 1$ and

$$\frac{C_{T+1}}{\sqrt{\rho(T)}} \to 0$$

as $T \to \infty$. Then it follows from Remark 2.10 that

$$|u(x, t^+_T, \theta_{-T}\omega) - u(x, t^-_T, \theta_{-T}\omega)| \leq C_{T+1} |t^+_T - t^-_T|^{1/2} \leq C_{T+1} \frac{2}{\rho(T)} \to 0$$

when $T \to \infty$.

Hence

$$\lim_{k \to \infty} (v^T_{t_k}(x, \omega) - v^{-T}_{t_k}(x, \omega)) \leq 0,$$

which means $\mu(x, \omega) \leq \nu(x, \omega)$. Therefore

$$\mu(x, \omega) = \nu(x, \omega) = \lim_{k \to \infty} v^T_{t_k}(x, \omega) = \lim_{k \to \infty} u(x, T_k, \theta_{-T_k}\omega).$$

For simplicity we denote $\tau$ the sequence $T_k$. Thus

$$\lim_{\tau \to \infty} u(x, \tau, \theta_{-\tau}\omega) =: \hat{u}(x, 0, \omega) =: \hat{u}_0(x, \omega) \quad (2.15)$$

exists.

Define for all $t \geq 0$

$$\hat{u}(x, t, \omega) := \hat{u}_0(x, \theta_t\omega) = \lim_{\tau \to \infty} u(x, \tau, \theta_{t-\tau}\omega) = \lim_{\tau \to \infty} u(x, t + \tau, \theta_{-\tau}\omega). \quad (2.16)$$

The next task is to prove that the function $\hat{u}$ solves the equation (2.5) with initial value function $\hat{u}_0(x, \omega)$. It is readily seen that as a consequence of the uniqueness the solution $u$ of our equation (2.5), as a random dynamical system $u : C(\mathbb{R}^n, \mathbb{R}) \times [0, \infty) \times \Omega \to C(\mathbb{R}^n, \mathbb{R})$, satisfies the cocycle property. Therefore it implies that for $t, r \geq 0$,

$$u(t + r, \theta_{-r}\omega) = u(t, \omega) \circ u(r, \theta_{-r}\omega).$$
By virtue of Lemma 2.6 we learn that $u$ is continuous as a functional of the initial value function. So in particular letting $r := \tau \to \infty$ we get

$$u(\hat{u}_0(\omega), t, \omega) := u(t, \omega) \circ \hat{u}_0(\omega) = \hat{u}(t, \omega) = \hat{u}_0(\theta_t \omega).$$

This means that $\hat{u}$ solves the equation (2.5) with initial value function $\hat{u}_0$. □

Recall the Ornstein-Uhlenbeck transform for the solution $\bar{u}$ of the SPDE (2.3)

$$\bar{u}(x, t, \omega) = u(x, t, \omega) + \int_{-\infty}^{t} g(s, \omega)dz(\theta_s \omega).$$

For $t, r \geq 0$ it follows that

$$\bar{u}(x, t + r, \theta_{-r} \omega) = u(x, t + r, \theta_{-r} \omega) + \int_{-\infty}^{t+r} g(s, \theta_{-r} \omega)dz(\theta_{s-r} \omega),$$

and in particular when $t = 0$

$$\bar{u}(x, r, \theta_{-r} \omega) = u(x, r, \theta_{-r} \omega) + \int_{-\infty}^{r} g(s, \theta_{-r} \omega)dz(\theta_{s-r} \omega) = u(x, r, \theta_{-r} \omega) + \int_{-\infty}^{0} g(s + r, \theta_{-r} \omega)dz(\theta_{s} \omega) = u(x, r, \theta_{-r} \omega) + \int_{-\infty}^{0} g(s + r, \theta_{-r} \omega)[-z(\theta_s \omega)ds + dW_s].$$

Now we further impose an assumption on $g$ that a.s.

$$\lim_{\tau \to \infty} g(t + \tau, \theta_{-\tau} \omega) = g(t, \omega), \quad (2.17)$$

for all $t \geq 0$. Consequently, taking limit as $\tau \to \infty$ we have

$$\bar{u}_0(x, \omega) := \bar{u}(x, 0, \omega) := \lim_{\tau \to \infty} \bar{u}(x, \tau, \theta_{-\tau} \omega) = \bar{u}_0(x, \omega) + \int_{-\infty}^{0} g(s, \omega)dz(\theta_s \omega).$$

Similarly it holds that

$$\bar{u}(x, t + r, \theta_{-r} \omega) = u(x, t + r, \theta_{-r} \omega) + \int_{-\infty}^{t+r} g(s, \theta_{-r} \omega)dz(\theta_{s-r} \omega) = u(x, t + r, \theta_{-r} \omega) + \int_{-\infty}^{t} g(s + r, \theta_{-r} \omega)dz(\theta_s \omega).$$
Therefore,

\[ \tilde{u}(x, t, \omega) := \tilde{u}_0(\theta_t \omega) \]

\[ = \lim_{\tau \to \infty} \tilde{u}(x, \tau, \theta_{t-r} \omega) \]

\[ = \lim_{\tau \to \infty} \tilde{u}(x + \tau, \theta_{-\tau} \omega) \]

\[ = \hat{u}(x, t, \omega) + \int_{-\infty}^{t} g(s, \omega)dz(\theta_s \omega). \]

And it is readily seen that the relation of \( \tilde{u} \) and \( \hat{u} \) is actually the Ornstein-Uhlenbeck transform, which means that \( \tilde{u} \) solves the SPDE (2.3) with initial value function \( \tilde{u}_0 \).

**Corollary 2.12** Under the condition (2.17) the SPDE (2.3) has at least one pathwise stationary solution.
Chapter 3

Initial-boundary Value Problems on Bounded Domains

3.1 Uniqueness

Let us recall the following Ornstein-Uhlenbeck process discussed before.

$$dz_t = -z_t dt + dW_t,$$

and $z_{-\infty} = 0$. Its solution is

$$z_t = \int_{-\infty}^{t} e^{s-t} dW_s = z(\theta_t \omega),$$

where $z(\omega)$ is as in the expression (2.1).

Let $T > 0$ and $U \subset \mathbb{R}^n$ be a bounded domain. Denote $U_T := U \times (0, T]$ and

$\Gamma_T := \overline{U_T} - U_T = (U \times \{0\}) \cup (\partial U \times [0, T])$ the parabolic boundary. Now we consider the stochastic PDE (3.1) for $\bar{u} : \overline{U_T} \to \mathbb{R}$ in which $g := g(x)$ involving the spatial variable but not the time, i.e.,

$$d\bar{u} = \frac{\langle D^2 \bar{u} \bar{D}u, D\bar{u} \rangle}{|D\bar{u}|^2} dt + g(x) dW_t \text{ in } U_T,$$

$$\bar{u} = \bar{\psi} \text{ on } \Gamma_T,$$

where $g \in C^2(\bar{U}, \mathbb{R})$. 

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Similarly as before we define the Ornstein-Uhlenbeck transform
\[ \bar{u}(x,t) =: u(x,t) + \int_{-\infty}^{t} g(x) dz_s \]
\[ = u(x,t) + \int_{-\infty}^{t} g(x) dz(\theta_s \omega) \]
\[ = u(x,t) + g(x)z_t \]
\[ = u(x,t) + g(x)z(\theta_t \omega) \]
\[ =: u(x,t) + \bar{g}(x,t). \quad (3.2) \]

Then,
\[ du = d\bar{u} - g(x)(-z_t)dt + dW_t \]
\[ = \frac{\langle D^2 \bar{u}, D \bar{u} \rangle}{|D\bar{u}|^2} dt + g(x)dz_t + g(x)z_tdt - g(x)dW_t \]
\[ = \frac{\langle D^2(u + \bar{g}), D(u + \bar{g}) \rangle}{|D(u + \bar{g})|^2} dt + \bar{g}(x,t)dt, \]
and
\[ u(x,0) = \bar{u}(x,0) - \bar{g}(x,0) = \bar{\psi}(x) - g(x)z_0 = \bar{\psi}(x) - g(x)z(\omega). \]

Thus the SPDE (3.1) becomes the following random PDE (3.3)
\[ \partial_t u = \frac{\langle D^2(u + \bar{g}), D(u + \bar{g}) \rangle}{|D(u + \bar{g})|^2} + g(x)z_t \]
\[ = \Delta_\infty(u + \bar{g})(x,t) + \bar{g}(x,t) \text{ in } U_T, \]
\[ u = \psi := \bar{\psi} - g z(\omega) \text{ on } \Gamma_T. \]

We define the viscosity solutions for the random PDE (3.3) as before.

**Definition 3.1** A function
\[ u(\omega, \cdot, \cdot) \in USC(U \times [0, \infty), \mathbb{R}) \]
\[ (u(\omega, \cdot, \cdot) \in LSC(U \times [0, \infty), \mathbb{R}), \text{ resp.}), \]
a.a. \( \omega \in \Omega \), is called a *viscosity subsolution* (*supersolution*, resp.) of the equation (3.3) if a.s.
\[ u|_{\Gamma_T} \leq \psi \ (u|_{\Gamma_T} \geq \psi, \text{ resp.}), \]
and for all $\phi(\omega, \cdot, \cdot) \in C^{2,1}(U \times [0, \infty), \mathbb{R})$, a.a. $\omega \in \Omega$, such that $u - \phi$ attains its local maximum (minimum, resp.) at $(x_0, t_0)$, then at $(x_0, t_0)$, a.s.

$$\partial_t \phi \leq \frac{\langle D^2(\phi + \bar{g})D(\phi + \bar{g}), D(\phi + \bar{g}) \rangle}{|D(\phi + \bar{g})|^2} \bar{g},$$

if $D(\phi + \bar{g})(x_0, t_0) \neq 0$; and

$$\partial_t \phi \geq \sup_{|\alpha| \leq 1, \alpha \in \mathbb{R}^n} \langle D^2(\phi + \bar{g})\alpha, \alpha \rangle \bar{g},$$

if $D(\phi + \bar{g})(x_0, t_0) = 0$. A viscosity solution of the equation (3.3) is a viscosity subsolution and supersolution of the equation (3.3) simultaneously.

**Definition 3.2** A function $\bar{u} : \Omega \times U \times [0, \infty) \to \mathbb{R}$ is called a viscosity solution of the equation (3.1) if $u : \Omega \times U \times [0, \infty) \to \mathbb{R}$, related to $\bar{u}$ by the Ornstein-Uhlenbeck transform (3.2), is a viscosity solution of the equation (3.3). In other words, $\bar{u}$ satisfies the statement in Definition 3.1 with $u - \phi$ replaced by $\bar{u} - \bar{g} - \phi$ when the extrema are considered.

Now we set off to demonstrate the uniqueness of the solutions for the random PDE (3.3). Similarly to before the proof is in the spirit of its deterministic counterpart, however slightly more complexity arises here when the Hölder continuity is dealt with, and with different exponents for the initial-boundary value functions and the corresponding solutions.

**Proposition 3.3 (Uniqueness)** Assume that functions $u_0, v_0 : \Omega \times U \to \mathbb{R}$ are bounded and Hölder-$\alpha_0$ continuous with $0 < \alpha_0 \leq 1$. Suppose that functions $u, v : \Omega \times U \times [0, \infty) \to \mathbb{R}$ are the solutions of the equation (3.3) with initial value functions $u_0, v_0$ respectively. And assume that $u, v$ are bounded and Hölder-$\alpha$ continuous in $x$. 

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with $0 < \alpha \leq 1$. Then $\forall T > 0$ there exists a constant $C > 0$, which depends on the Hölder constants of $u_0$ and $u$, and the bounds of $u_0$ and $v_0$, such that

$$
sup_{U \times [0,T]} |u - v| \leq C \sup_{U} |u_0 - v_0|.
$$

This implies that equation (3.3) has at most one solution, and that the solution is continuously dependent on the initial-boundary value function.

**Proof.** Let $T > 0$. Suppose $u, v$ are solutions to the equation (3.3) with initial-boundary value functions $u_0, v_0$ respectively.

Define

$$
\Phi(x, y, t) := u(x, t) - v(y, t) - \phi(x, y, t)
$$

for $x, y \in U, t \in [0, T]$ where $\epsilon, \lambda, \rho > 0$ are constants to be determined. Simple calculations imply that

$$
\partial_t \phi = \lambda + \frac{\rho}{(T - t)^2},
$$

$$
D \phi = \begin{pmatrix} D_x \phi \\ D_y \phi \end{pmatrix} = \begin{pmatrix} \frac{1}{\epsilon} |x - y|^2 (x - y) \\ -\frac{1}{\epsilon} |x - y|^2 (x - y) \end{pmatrix},
$$

and

$$
D^2 \phi = \begin{pmatrix} \frac{2}{\epsilon} (x - y)(x - y)^* + \frac{1}{\epsilon} |x - y|^2 I & -\frac{2}{\epsilon} (x - y)(x - y)^* - \frac{1}{\epsilon} |x - y|^2 I \\ -\frac{2}{\epsilon} (x - y)(x - y)^* - \frac{1}{\epsilon} |x - y|^2 I & \frac{2}{\epsilon} (x - y)(x - y)^* + \frac{1}{\epsilon} |x - y|^2 I \end{pmatrix} =: \begin{pmatrix} A & -A \\ -A & A \end{pmatrix}.
$$

Now from the assumption it is readily deduced that a maximum point, denoted by $(x_0, y_0, t_0)$, of $\Phi(x, y, t)$ exists. By the definition of $\Phi$ it follows that

$$
\Phi(x_0, y_0, t_0)
$$
\[ u(x_0, t_0) - v(y_0, t_0) - \frac{|x_0 - y_0|^4}{4\epsilon} - \lambda t_0 - \frac{\rho}{T - t_0} \leq u(x_0, t_0) - v(y_0, t_0) \leq |u|_\infty + |v|_\infty. \]

If we require that \( \rho \leq T \) and, without loss of generality, assume that \( 0 \in U \), then we obviously have that

\[ \Phi(x_0, y_0, t_0) \]
\[ = u(x_0, t_0) - v(y_0, t_0) - \frac{|x_0 - y_0|^4}{4\epsilon} - \lambda t_0 - \frac{\rho}{T - t_0} \]
\[ \geq u(0, 0) - v(0, 0) - \frac{\rho}{T} \]
\[ \geq -|u_0|_\infty - |v_0|_\infty - 1. \]

Thus

\[ \frac{\rho}{T - t_0} \]
\[ = u(x_0, t_0) - v(y_0, t_0) - \frac{|x_0 - y_0|^4}{4\epsilon} - \lambda t_0 - \Phi(x_0, y_0, t_0) \]
\[ \leq u(x_0, t_0) - v(y_0, t_0) - \Phi(x_0, y_0, t_0) \]
\[ \leq 2|u|_\infty + 2|v|_\infty + 1 \]
\[ =: C_1, \]

where \( C_1 := C_1(|u|_\infty, |v|_\infty) \). And by the assumption we have that \( \Phi(x_0, y_0, t_0) \geq \Phi(y_0, y_0, t_0) \), therefore,

\[ u(x_0, t_0) - v(y_0, t_0) - \frac{|x_0 - y_0|^4}{4\epsilon} - \lambda t_0 - \frac{\rho}{T - t_0} \]
\[ \geq u(y_0, t_0) - v(y_0, t_0) - \lambda t_0 - \frac{\rho}{T - t_0}, \]

which implies that

\[ \frac{|x_0 - y_0|^4}{4\epsilon} \]
\[ \leq u(x_0, t_0) - u(y_0, t_0) \]
\[ \leq L|x_0 - y_0|^{\alpha}, \]
where \( L \) is the Hölder constant of the solution \( u \) w.r.t. \( x \). Immediately it follows that

\[
|x_0 - y_0|^{4-\alpha} \leq 4L\epsilon,
\]

that is,

\[
|x_0 - y_0| \leq (4L\epsilon)^{\frac{1}{4-\alpha}}. \tag{3.4}
\]

We claim that \( t_0 = 0 \). To the contrary we assume temporarily that \( t_0 > 0 \). Note that the term \( \frac{\rho}{T-t_0} \) in the expression of \( \Phi \) guarantees that \( t_0 < T \). Now in light of Proposition 2.3 as before, for all \( \mu > 0 \), we find \( a,b \in \mathbb{R} \) and \( X,Y \in S^n, (n \times n) \) symmetric matrices, such that

\[
a - b = \partial_t \phi(x_0, y_0, t_0) = \lambda + \frac{\rho}{(T - t_0)^2}
\]

and

\[
\begin{pmatrix}
X & 0 \\
0 & -Y
\end{pmatrix}
\leq D^2 \phi(x_0, y_0, t_0) + \mu[D^2 \phi(x_0, y_0, t_0)]^2
\]

\[
= \begin{pmatrix}
A + 2\mu A^2 & -A - 2\mu A^2 \\
-A - 2\mu A^2 & A + 2\mu A^2
\end{pmatrix},
\]

and it holds that a.s.

\[
a \leq \frac{\langle (X + D^2 \bar{g}(x_0, t_0)) \left( D_x \phi(x_0, y_0, t_0) + D \bar{g}(x_0, t_0) \right), D_x \phi(x_0, y_0, t_0) + D \bar{g}(x_0, t_0) \rangle}{|D_x \phi(x_0, y_0, t_0) + D \bar{g}(x_0, t_0)|^2}
\]

+ \bar{g}(x_0, t_0), \tag{3.5}

when \( D_x \phi(x_0, y_0, t_0) + D \bar{g}(x_0, t_0) \neq 0 \), or

\[
a \leq \sup_{\alpha \in \mathbb{R}^n, ||\alpha|| = 1} \langle (X + D^2 \bar{g}(x_0, t_0))\alpha, \alpha \rangle + \bar{g}(x_0, t_0), \tag{3.6}
\]

when \( D_x \phi(x_0, y_0, t_0) + D \bar{g}(x_0, t_0) = 0 \),

and

\[
b \geq \frac{\langle (Y + D^2 \bar{g}(y_0, t_0)) \left( D_y \phi(x_0, y_0, t_0) + D \bar{g}(y_0, t_0) \right), D_y \phi(x_0, y_0, t_0) + D \bar{g}(y_0, t_0) \rangle}{|D_y \phi(x_0, y_0, t_0) + D \bar{g}(y_0, t_0)|^2}
\]

+ \bar{g}(y_0, t_0), \tag{3.7}

\$3.1 \cdot 37 \cdot$
when \( D_y \phi(x_0, y_0, t_0) + D\bar{g}(y_0, t_0) \neq 0 \), or

\[
\begin{align*}
\beta_1 &:= \frac{D_x \phi(x_0, y_0, t_0) + D\bar{g}(x_0, t_0)}{|D_x \phi(x_0, y_0, t_0) + D\bar{g}(x_0, t_0)|} \\
&= \frac{1}{\epsilon} |x_0 - y_0|^2 (x_0 - y_0) + D\bar{g}(x_0, t_0) \\
&= \frac{1}{\epsilon} |x_0 - y_0|^2 (x_0 - y_0) + D\bar{g}(x_0, t_0) \\
\beta_2 &:= \frac{D_y \phi(x_0, y_0, t_0) + D\bar{g}(y_0, t_0)}{|D_y \phi(x_0, y_0, t_0) + D\bar{g}(y_0, t_0)|} \\
&= -\frac{1}{\epsilon} |x_0 - y_0|^2 (x_0 - y_0) + D\bar{g}(y_0, t_0) \\
&= -\frac{1}{\epsilon} |x_0 - y_0|^2 (x_0 - y_0) + D\bar{g}(y_0, t_0).
\end{align*}
\]

For instance we now consider the case \((3.5)-(3.7)\) with \( x_0 \neq y_0 \). Put the two inequalities together and we have, for all \( \mu > 0 \), that

\[
\lambda + \frac{\rho}{(T - t_0)^2} = a - b \leq \langle (X + D^2 \bar{g}(x_0, t_0)) \beta_1, \beta_1 \rangle - \langle (Y + D^2 \bar{g}(y_0, t_0)) \beta_2, \beta_2 \rangle + \bar{g}(x_0, t_0) - \bar{g}(y_0, t_0)
\]

\[
= \text{trace} \left[ \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \begin{pmatrix} \beta_1 \beta_1^* & \beta_1 \beta_2^* \\ \beta_2 \beta_1^* & \beta_2 \beta_2^* \end{pmatrix} \right] + \langle D^2 \bar{g}(x_0, t_0) \beta_1, \beta_1 \rangle - \langle D^2 \bar{g}(y_0, t_0) \beta_2, \beta_2 \rangle \\
&\quad + \bar{g}(x_0, t_0) - \bar{g}(y_0, t_0)
\]

\[
\leq \text{trace} \left[ \begin{pmatrix} A + 2\mu A^2 & -A - 2\mu A^2 \\ -A - 2\mu A^2 & A + 2\mu A^2 \end{pmatrix} \begin{pmatrix} \beta_1 \beta_1^* & \beta_1 \beta_2^* \\ \beta_2 \beta_1^* & \beta_2 \beta_2^* \end{pmatrix} \right] + \langle D^2 \bar{g}(x_0, t_0) \beta_1, \beta_1 \rangle - \langle D^2 \bar{g}(y_0, t_0) \beta_2, \beta_2 \rangle \\
&\quad + \bar{g}(x_0, t_0) - \bar{g}(y_0, t_0).
\]

Note that \( a, b, X, Y \) do depend on \( \mu \), whereas the constants and variables, except \( \mu \) itself, in the two sides of the above inequality do not. Because \( \mu > 0 \) is arbitrary letting...
it tend to 0 implies that
\[
\lambda + \frac{\rho}{(T-t_0)^2} \leq \text{trace } (A(\beta_1 - \beta_2)(\beta_1 - \beta_2)^*) + \langle D^2 g(x_0, t_0)\beta_1, \beta_1 \rangle - \langle D^2 g(y_0, t_0)\beta_2, \beta_2 \rangle + \bar{g}(x_0, t_0) - \bar{g}(y_0, t_0).
\]

Now noting that $|\beta_1| = |\beta_2| = 1$ and (3.4), we compute that
\[
\text{trace}(A(\beta_1 - \beta_2)(\beta_1 - \beta_2)^*) 
\leq \sqrt{\text{trace}(AA^*)}|\beta_1 - \beta_2|^2
= |\beta_1 - \beta_2|^2 \sqrt{\frac{8}{c^2}|x_0 - y_0|^4 + \frac{1}{c^2}|x_0 - y_0|^4}
= \frac{3}{c}|\beta_1 - \beta_2|^2|x_0 - y_0|^2
\leq \frac{12}{c}|x_0 - y_0|^2
\leq \frac{12}{c}(4L)e^\frac{2}{4-\alpha} L^{\frac{2}{4-\alpha}}.
\]

Eventually we obtain explicitly that
\[
\lambda + \frac{\rho}{(T-t_0)^2} \leq 12^{4-\sqrt{16}e^\frac{2}{4-\alpha} L^{\frac{2}{4-\alpha}}} + [(D^2 g(x_0)\beta_1, \beta_1) - \langle D^2 g(y_0)\beta_2, \beta_2 \rangle + g(x_0) - g(y_0)] z_{t_0}
\leq 12^{4-\sqrt{16}e^\frac{2}{4-\alpha} L^{\frac{2}{4-\alpha}}} + [\|D^2 g(x_0)\| + |D^2 g(y_0)| + |g(x_0) - g(y_0)|] |z_{t_0}|
\leq 12^{4-\sqrt{16}e^\frac{2}{4-\alpha} L^{\frac{2}{4-\alpha}}} + 2\|[D^2 g]_\infty + |g|\|_\infty |z_{t_0}|.
\]

Assume sup$_{U \times [0, T]} |u - v| > 0$; for if sup$_{U \times [0, T]} |u - v| = 0$ the conclusion follows.

And notice that sup$_{U \times [0, T]} |u - v| \leq |u|_\infty + |v|_\infty$. Let
\[
\frac{2-\alpha}{\epsilon^{4-\alpha}} =: \delta \left( \sup_{U \times [0, T]} |u - v| \right)^{\frac{4-a_0(2-a)}{a_0(4-a)}},
\]
for some $\delta > 0$ to be determined, and let
\[
12^{4-\sqrt{16}L^{\frac{2}{4-\alpha}}} =: C_2 \left( \sup_{U \times [0, T]} |u - v| \right)^{1+\frac{4-a_0(2-a)}{a_0(4-a)}},
\]
where $C_2 := C_2(\alpha_0, \alpha, L, \sup_{U \times [0,T]} |u-v|)$. Consequently the above inequality becomes

$$\lambda + \frac{\rho}{(T-t_0)^2} \leq \delta^{-1} C_2 \sup_{U \times [0,T]} |u-v| + 2(|D^2 g|_\infty + |g|_\infty) z_{t_0}.$$ 

Now let

$$\lambda := \delta^{-1} C_2 \sup_{U \times [0,T]} |u-v|$$

and we get immediately

$$\frac{\rho}{(T-t_0)^2} \leq 2(|D^2 g|_\infty + |g|_\infty) z_{t_0}.$$

Note that the previous inequalities hold almost surely, namely, for almost all $\omega \in \Omega$. But the r.v. $z_{t_0}$ is Gaussian; therefore there exists $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') > 0$ such that for all $\omega \in \Omega'$ we have

$$\frac{\rho}{(T-t_0)^2} > 2(|D^2 g|_\infty + |g|_\infty) z_{t_0}(\omega).$$

Hence we encounter a contradiction, which actually leads to the fact that $t_0 = 0$.

The combination (3.5)-(3.7) with $x_0 = y_0$ is much simpler. The other combinations (3.5)-(3.8), (3.6)-(3.7), and (3.6)-(3.8) are similarly discussed; we omit them here.

Therefore, for all $\rho > 0$ we get that

$$u(x,t) - v(y,t) - \frac{|x-y|^4}{4\epsilon} - \lambda t - \frac{\rho}{T-t} \leq \sup_{x,y \in U} \left( u_0(x) - v_0(y) - \frac{|x-y|^4}{4\epsilon} - \frac{\rho}{T} \right).$$

Taking limits as $\rho \downarrow 0$ we deduce that

$$u(x,t) - v(y,t) - \frac{|x-y|^4}{4\epsilon} - \lambda t \leq \sup_{x,y \in U} \left( u_0(x) - v_0(y) - \frac{|x-y|^4}{4\epsilon} \right),$$

with

$$\epsilon^{\frac{\alpha_0}{4-\alpha_0}} = \delta^{-\frac{\alpha_0(4-\alpha)}{2(2-\alpha)}} \sup_{U \times [0,T]} |u-v| \text{ and } \lambda = \delta^{-1} C_2 \sup_{U \times [0,T]} |u-v|.$$ 

Thus, letting $y = x$ in the left hand side of the above inequality entails that

$$u(x,t) - v(x,t) \leq \sup_{x,y \in U} \left( u_0(x) - u_0(y) + u_0(y) - v_0(y) - \frac{|x-y|^4}{4\epsilon} \right) + \lambda t.$$
Without loss of generality we assume that the Hölder constants of $u_0$ and $u$ are the same $L$. Hence,

$$
\sup_{U \times [0, T]} (u - v) 
\leq \sup_{\mathbb{R}^n} |u_0 - v_0| + \lambda T + \sup_{x, y \in U} \left(u_0(x) - u_0(y) - \frac{|x - y|^4}{4\epsilon}\right)
\leq \sup_{U} |u_0 - v_0| + \lambda T + \sup_{r \geq 0} \left(L r^{\alpha_0 - \frac{4}{4 \epsilon}} \right)
= \sup_{U} |u_0 - v_0| + \lambda T + \frac{3}{4} L^{\frac{4}{4 - \alpha_0}} \epsilon^{\frac{\alpha_0}{4 - \alpha_0}}
= \sup_{U} |u_0 - v_0| + \lambda T + \frac{3}{4} L^{\frac{4}{4 - \alpha_0}} \epsilon^{\frac{\alpha_0}{4 - \alpha_0}} \sup_{U \times [0, T]} |u - v|,
$$

where the supremum in the third line is attained when $L = \frac{4 - \alpha_0}{\epsilon}$ or $r = (L \epsilon)^{\frac{1}{4 - \alpha_0}}$. Let

$$
\delta^{(4 - \alpha_0)(2 - \alpha)} := \frac{1}{3} L^{\frac{4}{4 - \alpha_0}} \epsilon^{\frac{1}{4 - \alpha_0}}, \quad \text{i.e.,} \quad \delta^{-1} = 3 \frac{(4 - \alpha_0)(2 - \alpha)}{\alpha_0(4 - \alpha)} L^{\frac{4(2 - \alpha)}{4 - \alpha}},
$$

and we obtain that

$$
\sup_{U \times [0, T]} (u - v) \leq \sup_{U} |u_0 - v_0| + \frac{3}{4} L^{\frac{4(2 - \alpha)}{\alpha_0(4 - \alpha)}} \epsilon^{\frac{1}{4 - \alpha_0}} C_2 \sup_{U \times [0, T]} |u - v| + \frac{1}{4} \sup_{U \times [0, T]} |u - v|.
$$

Then let

$$
T := \frac{L^{\frac{4(2 - \alpha)}{\alpha_0(4 - \alpha)}}}{4(3) \frac{(4 - \alpha_0)(2 - \alpha)}{\alpha_0(4 - \alpha)}} C_2
$$

and we achieve that

$$
\sup_{U \times [0, T]} (u - v) \leq \sup_{U} |u_0 - v_0| + \frac{1}{2} \sup_{U \times [0, T]} |u - v|.
$$

Changing the role of $u$ and $v$ gives that

$$
\sup_{U \times [0, T]} |u - v| \leq \sup_{U} |u_0 - v_0| + \frac{1}{2} \sup_{U \times [0, T]} |u - v|,
$$

that is,

$$
\sup_{U \times [0, T]} |u - v| \leq 2 \sup_{U} |u_0 - v_0|.
$$
for the $T$ previously specified.

Finally, for all $T' \in [0, \infty)$ there exists $N \geq 1$ such that $NT \geq T'$ for the $T$ above. Then by iteration it holds that

$$\sup_{U \times [0, T']} |u - v| \leq 2^N \sup_U |u_0 - v_0|.$$ 

This completes the proof. \hfill \Box

## 3.2 Boundedness

Now we show the uniform boundedness of the solutions of the random PDE \((3.3)\).

**Proposition 3.4** Suppose that the equation \((3.3)\) with initial value function $u_0 : \Omega \times U \to \mathbb{R}$ has a solution $u : \Omega \times U \times [0, \infty) \to \mathbb{R}$, then $u$ is uniformly bounded in $x$. More precisely, we have

$$\inf u_0(x) - (|D^2g|_\infty + |g|_\infty) \int_0^t |z_s| ds \leq u(x, t) \leq \sup u_0(x) + (|D^2g|_\infty + |g|_\infty) \int_0^t |z_s| ds.$$ 

(3.9)

In particular, for all $T > 0$ it follows that for $0 \leq t \leq T$,

$$|u(x, t)| \leq |u_0|_\infty + (|D^2g|_\infty + |g|_\infty) \int_0^T |z_s| ds.$$ 

**Proof.** Let

$$\phi(x, t) := \sup u_0(x) + (|D^2g|_\infty + |g|_\infty + \delta) \int_0^t |z_s| ds,$$

for arbitrary $\delta > 0$. Suppose $u - \phi$ attains local maximum at $(x_0, t_0)$ and $t_0 > 0$. Then at $(x_0, t_0)$ we have a.s.

$$\partial_t \phi \leq \sup_{|\alpha| \leq 1} \langle D^2(\phi + \bar{g})\alpha, \alpha \rangle + \bar{g} = \sup_{|\alpha| \leq 1} \langle D^2g(x_0)\alpha, \alpha \rangle z_{t_0} + g(x_0)z_{t_0},$$

that is, a.s.

$$(|D^2g|_\infty + |g|_\infty + \delta)|z_{t_0}| \leq (|D^2g(x_0)| + g(x_0))z_{t_0}.$$
But
\[(|D^2g(x_0) + g(x_0)|)z_{t_0} \leq (|D^2g|_\infty + |g|_\infty)z_{t_0} \cdot\]
Therefore,
\[\delta |z_{t_0}| \leq 0.\]
Since \(z_{t_0}\) is a Gaussian r.v., the above inequality does not hold a.s.; it contradicts our assumption on \(\delta\). Thus \(t_0 = 0\); the local maximum is actually attained at \((x_0, 0)\). Now,
\[
\sup_{U \times [0, \infty)} (u(x, t) - \phi(x, t)) = \sup_U (u_0(x) - \phi(x, 0)) = \sup_U (u_0(x) - \sup_U u_0(x)) = 0.
\]
It follows that \(u(x, t) - \phi(x, t) \leq 0\) for all \(\delta > 0\), namely,
\[u(x, t) \leq \sup_U u_0(x) + (|D^2g|_\infty + |g|_\infty + \delta) \int_0^t |z_s| ds.\]
As \(\delta > 0\) is arbitrary we get immediately
\[u(x, t) \leq \sup_U u_0(x) + (|D^2g|_\infty + |g|_\infty) \int_0^t |z_s| ds.\]
The lower bound can be obtained analogously. \(\square\)

### 3.3 Existence

We start to deal with the existence for solutions of the initial-boundary value problem of the random PDE (3.3). Approximation by smooth solutions of a family of parameterised PDEs is utilised. Similarly as in \cite{JuutinenKawohl2006} the main idea is to construct proper barrier functions to deduce estimates for the spatial and temporal regularities of the solutions of the approximating PDEs. Here Hölder continuity is considered. Then, as before, the Arzelà-Ascoli lemma is called in to ensure the existence of the limit when the parameters of approximation approach zero. Finally as usual the general consistence-stability properties of viscosity solutions guarantee that the limit
function is actually the viscosity solution of our random PDE (3.3). The main goal of this section is to prove the following proposition.

**Proposition 3.5 (Existence)** Assume that \( \psi \) is bounded and continuous. Then the equation (3.3) has at least one solution.

We prove by approximation. Consider the following random PDE.

\[
\partial_t u^{\epsilon,\delta} = L^{\epsilon,\delta} u^{\epsilon,\delta} \quad \text{(3.10)}
\]

\[
:= \frac{(D^2(u^{\epsilon,\delta} + \bar{g})D(u^{\epsilon,\delta} + \bar{g}))}{|D(u^{\epsilon,\delta} + \bar{g})|^2 + \delta} + \epsilon \Delta u^{\epsilon,\delta} + g(x) z_t
\]

\[
=: \frac{(D^2 w Dw, Dw)}{|Dw|^2 + \delta} + \epsilon \Delta u^{\epsilon,\delta} + \bar{g} z_t
\]

\[
=: \frac{(D^2 u^{\epsilon,\delta} Dw, Dw)}{|Dw|^2 + \delta} + \epsilon \Delta u^{\epsilon,\delta} + \left( \frac{(D^2 g Dw, Dw)}{|Dw|^2 + \delta} + g \right) z_t
\]

\[
=: \sum_{i,j=1}^n a_{ij}(x,t) u^{\epsilon,\delta}_{ij}(x,t) + \left[ (D^2 g(x) \beta(x,t), \beta(x,t)) + g(x) \right] z_t
\]

\[
= \text{trace}(a D^2 u^{\epsilon,\delta}) + \text{trace}(\beta^* D^2 g) + g z_t
\]

\[
= \epsilon \Delta u^{\epsilon,\delta} + \text{trace}(\beta^* D^2 u^{\epsilon,\delta}) + \text{trace}(\beta^* D^2 g) + g z_t \text{ in } U_T,
\]

where \( \epsilon, \delta > 0 \) are constants and \( C^{2,1}(U_T, \mathbb{R}) \ni \psi^{\epsilon,\delta}(x,t) \to \psi(x,t) \) uniformly, \(|D\psi^{\epsilon,\delta}|_{L^\infty} \leq |D\psi|_{L^\infty}, |\psi^{\epsilon,\delta}|_{L^\infty} \leq |\psi|_{L^\infty} \). And here

\[
a := \epsilon I + \beta^* := \epsilon I + \frac{Dw(Dw)^*}{|Dw|^2 + \delta},
\]

\[
\beta := \frac{Dw}{\sqrt{|Dw|^2 + \delta}},
\]

and \( w := u + \bar{g} \). From the theory of general quasilinear uniformly parabolic PDE, the equation (3.10) has a unique smooth classical solution \( u^{\epsilon,\delta} \) (please refer to (Ladyzhenskaya, Solonnikov, & Uraltseva, 1968)).

**Spatial boundary regularity at \( \partial U \times [0, T] \)**

We begin with the Hölder estimation for the boundary regularity of the solutions. First we consider smooth initial-boundary value functions.

§3.3
Lemma 3.6 Suppose that \( u^\epsilon,\delta(x,t) \) is a smooth solution of the equation (3.10) where the initial-boundary value function \( \psi \in C^{2,1}(U \times \mathbb{R}, \mathbb{R}) \). Then for each \( 0 < \alpha < 1 \) there exists a constant \( C > 0 \), depending on the diameter of the domain \( U \), denoted by \( |U| \), \( \alpha \), \( |D\psi|_\infty \), \( |\psi|_\infty \), \( |D^2g|_\infty \), \( |Dg|_\infty \), \( |g|_\infty \), and \( \sup_{t \leq T} |z_t| \), but independent of \( \epsilon \) and \( \delta \), such that

\[
|u^\epsilon,\delta(x,t_0) - u^\epsilon,\delta(x_0,t_0)| = |u^\epsilon,\delta(x,t_0) - \psi(x_0,t_0)| \leq C|x - x_0|^\alpha,
\]

for all \((x_0,t_0) \in \partial U \times (0,T)\) and \( x \in U \) and \( \epsilon, \delta > 0 \) sufficiently small.

Proof. Let

\[
v(x,t) := \psi(x_0,t_0) + C|x - x_0|^\alpha + (|D^2g|_\infty + |g|_\infty) \int_{t_0}^t |z_s|ds - K(t-t_0), \tag{3.11}
\]

where \((x_0,t_0) \in \partial U \times (0,T), x \in U, |t-t_0| \leq 1, \) and \( 0 < \alpha < 1 \) is given, while \( C, K > 0 \) are constants to be determined later. Hereafter we denote \( w, a, \) and \( \beta \) the same as in the equation (3.10) but with \( u^\epsilon,\delta \) replaced by \( v \). It clearly holds that

\[
\partial_t v = (|D^2g|_\infty + |g|_\infty)|z_t| - K.
\]

And by simple calculations it follows that

\[
\begin{align*}
Dv &= C\alpha|x - x_0|^{\alpha-2}(x - x_0), \\
D^2v &= C\alpha|x - x_0|^{\alpha-2}I + C\alpha(\alpha - 2)|x - x_0|^\alpha - 4(x - x_0)^*(x - x_0), \\
\Delta v &= C\alpha(n + \alpha - 2)|x - x_0|^{\alpha-2}, \\
\langle D^2v, \beta \rangle &= \langle (C\alpha|x - x_0|^{\alpha-2}I + C\alpha(\alpha - 2)|x - x_0|^\alpha - 4(x - x_0)^*)(x - x_0)^*, \beta \rangle \\
&= C\alpha|x - x_0|^{\alpha-2} \left( I + (\alpha - 2) \frac{(x - x_0)(x - x_0)^*}{|x - x_0|^2} \right) \beta, \\
\text{trace}(aD^2v) &= \text{trace}([\epsilon I + \beta^*\beta^{\epsilon}])D^2v = \epsilon \Delta v + \langle D^2v, \beta \rangle \\
&= \epsilon C\alpha(n + \alpha - 2)|x - x_0|^{\alpha-2} \\
&\quad + C\alpha|x - x_0|^{\alpha-2} \left( I + (\alpha - 2) \frac{(x - x_0)(x - x_0)^*}{|x - x_0|^2} \right) \beta, \\
&= C\alpha|x - x_0|^{\alpha-2} \left[ \epsilon(n + \alpha - 2) + |\beta|^2 + (\alpha - 2) \frac{(x - x_0)(x - x_0)^*}{|x - x_0|^2} \right]^2.
\end{align*}
\]
We want to find appropriate constants $C, K$ such that
\[
\partial_t v \geq \text{trace}(a D^2 v) + |\text{trace}(\beta^* D^2 g) + g| z_t.
\]
Noticing that $|\beta| \leq 1$ we have readily that
\[
|\text{trace}(\beta^* D^2 g) + g| z_t \leq (|D^2 g|_\infty + |g|_\infty)|z_t|.
\]
Thus obviously it suffices to show that
\[
\text{trace}(a D^2 v) = \epsilon \Delta v + \text{trace}(\beta^* D^2 v) \leq -K,
\]
that is,
\[
C\alpha |x - x_0|^{n-2} \left[ \epsilon (n + \alpha - 2) + |\beta|^2 + (\alpha - 2) \left\langle \frac{x - x_0}{|x - x_0|}, \beta \right\rangle^2 \right] \leq -K.
\]
Necessarily, when $n \geq 2$, it is required that
\[
|\beta|^2 + (\alpha - 2) \left\langle \frac{x - x_0}{|x - x_0|}, \beta \right\rangle^2 < 0,
\]
i.e.,
\[
\left| \left\langle \frac{x - x_0}{|x - x_0|}, \beta \right\rangle \right| > \frac{1}{\sqrt{2} - \alpha}.
\]
For simplicity we also require the above inequality when $n = 1$. Because of $0 < \alpha < 1$ we have that
\[
\frac{\sqrt{2}}{2} < \frac{1}{\sqrt{2} - \alpha} < 1.
\]
Now choose $\gamma \in \left(\frac{1}{\sqrt{2} - \alpha}, 1\right)$ to be a constant and our task is to find proper conditions such that
\[
\left| \left\langle \frac{x - x_0}{|x - x_0|}, \beta \right\rangle \right| \geq \gamma.
\]
Further we compute that
\[
\begin{align*}
\left\langle \frac{x - x_0}{|x - x_0|}, \beta \right\rangle & = \left\langle \frac{x - x_0}{|x - x_0|}, \frac{D w}{|D w|} \right\rangle \\
& = \left\langle \frac{x - x_0}{|x - x_0|}, \frac{C\alpha |x - x_0|^{\alpha-2}(x - x_0) + D g(x) z_t}{|C\alpha |x - x_0|^{\alpha-2}(x - x_0) + D g(x) z_t|} \right\rangle.
\end{align*}
\]
\[ \begin{align*}
&= C_{\alpha}|x - x_0|^{\alpha - 1} \\
&\quad \left| C_{\alpha}|x - x_0|^{\alpha - 2}(x - x_0) + Dg(x)_t \right| \\
&\quad + \frac{1}{|x - x_0|} \left| C_{\alpha}|x - x_0|^{\alpha - 2}(x - x_0) + Dg(x)_t \right| \\
&= \frac{1}{|x - x_0|} + \frac{Dg(x)_t}{C_{\alpha}|x - x_0|^{\alpha - 1}} + \left\langle \frac{x - x_0}{|x - x_0|}, \frac{Dg(x)_t}{C_{\alpha}|x - x_0|^{\alpha - 1}} \right\rangle.
\end{align*} \]

Since
\[ 1/ \left[ 1 + \frac{|Dg(x)_t|}{C_{\alpha}|x - x_0|^{\alpha - 1}} \right] \leq 1/ \left\langle \frac{x - x_0}{|x - x_0|}, \frac{Dg(x)_t}{C_{\alpha}|x - x_0|^{\alpha - 1}} \right\rangle, \]
and
\[ \left\langle \frac{x - x_0}{|x - x_0|}, \frac{Dg(x)_t}{C_{\alpha}|x - x_0|^{\alpha - 1}} \right\rangle \leq \frac{|Dg(x)_t|}{|C_{\alpha}|x - x_0|^{\alpha - 1} - |Dg(x)_t|}, \]

hence we have that
\[ \left\langle \frac{x - x_0}{|x - x_0|}, \frac{\beta}{|\beta|} \right\rangle \geq 1/ \left[ 1 + \frac{|Dg(x)_t|}{C_{\alpha}|x - x_0|^{\alpha - 1}} \right] - \frac{|Dg(x)_t|}{|C_{\alpha}|x - x_0|^{\alpha - 1} - |Dg(x)_t|}. \]

Since \(-1 < \alpha - 1 < 0\) it holds that
\[ C_{\alpha}|x - x_0|^{\alpha - 1} \geq C_{\alpha}|U|^{\alpha - 1}. \]

Let
\[ C_{\alpha}|U|^{\alpha - 1} \geq C_0|Dg|_\infty \sup_{t \leq T} |z_t| + 1 \]

for some constant \(C_0 \geq 1\) to be determined. Thus we deduce that
\[ \left\langle \frac{x - x_0}{|x - x_0|}, \frac{\beta}{|\beta|} \right\rangle \geq \frac{1}{1 + \frac{1}{C_0} - \frac{1}{C_0 - 1}} = \frac{C_0}{C_0 + 1} - \frac{1}{C_0 - 1}. \]

Now in order to satisfy
\[ \frac{C_0}{C_0 + 1} - \frac{1}{C_0 - 1} \geq \gamma, \]

\[ \S \]
we require that
\[ \frac{C_0}{C_0 + 1} \geq \frac{1 + \gamma}{2} \quad \text{and} \quad \frac{1}{C_0 - 1} \leq \frac{1 - \gamma}{2}, \]
viz.,
\[ C_0 \geq \frac{3 - \gamma}{1 - \gamma}. \]
The above implies that
\[ C \geq \frac{|U|^{1-\alpha}}{\alpha} (C_0 |Dg|_\infty \sup_{t \leq T} |z_t| + 1) \]
\[ = \frac{|U|^{1-\alpha}}{\alpha} \left( \frac{3 - \gamma}{1 - \gamma} |Dg|_\infty \sup_{t \leq T} |z_t| + 1 \right), \]
where \( \gamma \in \left( \frac{1}{\sqrt{2-\alpha}}, 1 \right). \)
Therefore, it follows that
\[ \left( \frac{x - x_0}{|x - x_0|}, |\beta| \right)^2 \geq \gamma^2 > \frac{1}{2 - \alpha}, \]
which in turn gives
\[ |\beta|^2 + (\alpha - 2) \left( \frac{x - x_0}{|x - x_0|}, |\beta| \right)^2 \leq (\alpha - 2) \gamma^2 + 1 |\beta|^2. \]
Then
\[
\text{trace}(aD^2v) = C\alpha |x - x_0|^{\alpha-2} \left[ \epsilon(n + \alpha - 2) + |\beta|^2 + (\alpha - 2) \left( \frac{x - x_0}{|x - x_0|}, |\beta| \right)^2 \right] \\
\leq C\alpha |x - x_0|^{\alpha-2} \left[ \epsilon(n + \alpha - 2) + (\alpha - 2) \gamma^2 + 1 |\beta|^2 \right] \\
= C\alpha |x - x_0|^{\alpha-2} \left[ \epsilon(n + \alpha - 2) - (2 - \alpha) \gamma^2 - 1 |\beta|^2 \right].
\]
We have that
\[
|\beta|^2 = \frac{|Dw|^2}{|Dw|^2 + \delta} \\
= \frac{1}{1 + \frac{\delta}{|Dw|^2}} \\
= \frac{1}{1 + \frac{|C\alpha |x - x_0|^{\alpha-2}(x - x_0) + Dg(x)z_t|^2}{|Dw|^2}}.
\]
Since
\[
|C\alpha |x - x_0|^{\alpha-2}(x - x_0) + Dg(x)z_t|
\]
\[§3.3 \quad \cdot \, 48\]
it follows immediately that

$$\beta^2 \geq 1 / \left[ 1 + \frac{\delta}{\left( \frac{2}{3-\gamma} \right)^2} \right].$$

Continuing with the previous deduction we obtain that

$$\text{trace}(aD^2v) \leq C\alpha |x - x_0|^{\alpha - 2} \left\{ \epsilon(n + \alpha - 2) - [(2 - \alpha)\gamma^2 - 1] / \left[ 1 + \frac{\delta}{\left( \frac{2}{3-\gamma} \right)^2} \right] \right\}. $$

Now in order to achieve our aim that $\text{trace}(aD^2v) \leq -K$ we only need to consider the sufficient condition that

$$C\alpha |x - x_0|^{\alpha - 2} \left\{ \epsilon(n + \alpha - 2) - [(2 - \alpha)\gamma^2 - 1] / \left[ 1 + \frac{\delta}{\left( \frac{2}{3-\gamma} \right)^2} \right] \right\} \leq -K,$$

or equivalently,

$$\epsilon(n + \alpha - 2) \leq [(2 - \alpha)\gamma^2 - 1] / \left[ 1 + \frac{\delta}{\left( \frac{2}{3-\gamma} \right)^2} \right] - \frac{K}{C\alpha |x - x_0|^{\alpha - 2}}. $$

Again it suffices to show that

$$\epsilon(n + \alpha - 2) \leq [(2 - \alpha)\gamma^2 - 1] / \left[ 1 + \frac{\delta}{\left( \frac{2}{3-\gamma} \right)^2} \right] - \frac{K}{C\alpha |U|^{\alpha - 2}}. $$

Let

$$0 \leq \delta \leq \left( \frac{2}{3-\gamma} \right)^2$$
and let
\[ \frac{K}{Ca|U|^\alpha} \leq \frac{(2 - \alpha)\gamma^2 - 1}{4}, \]
i.e.,
\[ C \geq \frac{4|U|^{2-\alpha}K}{\alpha[(2 - \alpha)\gamma^2 - 1]}. \]
Then for the right hand side of the previous inequality we have that
\[
\frac{[(2 - \alpha)\gamma^2 - 1]}{\left[1 + \delta \left(\frac{2}{3 - \gamma}\right)^2\right]} - \frac{K}{Ca|U|^\alpha} \geq \frac{1}{2}[(2 - \alpha)\gamma^2 - 1] - \frac{1}{4}[(2 - \alpha)\gamma^2 - 1] \]
\[= \frac{1}{4}[(2 - \alpha)\gamma^2 - 1]. \]
Further our sufficient condition turns out to be
\[ \epsilon(n + \alpha - 2) \leq \frac{1}{4}[(2 - \alpha)\gamma^2 - 1]. \]
There are two cases. If \( n = 1 \), the above inequality is true for all \( \epsilon > 0 \) because \(-1 < \alpha - 1 < 0 \) and the right hand side of the above inequality is positive. If \( n \geq 2 \), the above inequality changes to be
\[ \epsilon \leq \frac{(2 - \alpha)\gamma^2 - 1}{4(n + \alpha - 2)}. \]
In summary, we have shown the following. For \( K > 0 \) and
\[ C \geq \max \left\{ \frac{|U|^{1-\alpha}}{\alpha} \left(\frac{3 - \gamma}{1 - \gamma}\right)|Dg|_{\infty} \sup_{t \in T}|z_t| + 1, \frac{4|U|^{2-\alpha}K}{\alpha[(2 - \alpha)\gamma^2 - 1]} \right\}, \]
where
\[ \gamma \in \left(\frac{1}{\sqrt{2 - \alpha}}, 1\right), \]
and for small \( \epsilon, \delta > 0 \), viz.,
\[ 0 < \epsilon \leq \frac{(2 - \alpha)\gamma^2 - 1}{4(n + \alpha - 2)}, \]
and
\[ 0 < \delta \leq \left(\frac{2}{3 - \gamma}\right)^2, \]
it holds that

\[
\partial_t v \geq L^\epsilon,\delta v
\]

\[
= \text{trace}(a D^2 v) + |\text{trace}(\beta \beta^* D^2 g) + g|z_t
\]

\[
= \frac{(D^2 v Dw, Dw)}{|Dw|^2 + \delta} + \epsilon \Delta v + \left( \frac{(D^2 g Dw, Dw)}{|Dw|^2 + \delta} + g \right) z_t,
\]

where \((x_0, t_0) \in \partial U \times (0, T]\) and \(x \in U\).

Next we investigate \(v\) on the parabolic boundary. The first case is when \(t_0 > 1\) and we consider the domain \(U \times (t_0 - 1, t_0)\). There are two subcases.

(i). \((x, t) \in \partial U \times (t_0 - 1, t_0]\). Then

\[
u(x, t) \leq \psi(x_0, t_0) + |D\psi|_\infty |x - x_0| + |\psi_t|_\infty |t - t_0|
\]

\[
\leq \psi(x_0, t_0) + |D\psi|_\infty |U|^{1 - \alpha}|x - x_0|^{\alpha} + |\psi_t|_\infty (t_0 - t)
\]

\[
\leq \psi(x_0, t_0) + C|x - x_0|^\alpha + (|D^2 g|_\infty + |g|_\infty) \int_{t_0}^{t} |z_s| ds - K(t - t_0)
\]

\[
= v(x, t),
\]

if \(C \geq |U|^{1 - \alpha}|D\psi|_\infty\) and \(K \geq |\psi_t|_\infty\).

(ii). \((x, t) \in U \times \{t_0 - 1\}\). By the maximum principle we know that

\[
u(x, t) \leq |\psi|_\infty + (|D^2 g|_\infty + |g|_\infty) \int_{0}^{t} |z_s| ds.
\]

Thus,

\[
u(x, t_0 - 1) \leq |\psi|_\infty + (|D^2 g|_\infty + |g|_\infty) \int_{0}^{t_0 - 1} |z_s| ds
\]

\[
\leq \psi(x_0, t_0) + C|x - x_0|^\alpha + (|D^2 g|_\infty + |g|_\infty) \int_{t_0}^{t_0 - 1} |z_s| ds + K
\]

\[
= v(x, t_0 - 1),
\]

if \(K \geq 2|\psi|_\infty + (|D^2 g|_\infty + |g|_\infty) \int_{0}^{T} |z_s| ds\).

The second case is when \(t_0 \leq 1\) and we consider the domain \(U \times (0, t_0)\). Two subcases are followed.

(i). \((x, t) \in \partial U \times (0, t_0]\). The same as in the first case \(t_0 > 1\).
(ii). \((x, t) \in U \times \{0\}\). Then,

\[
\begin{align*}
\quad \quad u(x, 0) &= \psi(x, 0) \\
&\leq \psi(x_0, t_0) + |D\psi|_\infty |x - x_0| + |\psi_t|_\infty t_0 \\
&\leq \psi(x_0, t_0) + |D\psi|_\infty |U|^{1-\alpha} |x - x_0|^\alpha + |\psi_t|_\infty t_0 \\
&\leq \psi(x_0, t_0) + C|x - x_0|^\alpha + (|D^2g|_\infty + |g|_\infty) \int_{t_0}^{0} |z_s| ds + Kt_0 \\
&= v(x, 0),
\end{align*}
\]

if \(C \geq |U|^{1-\alpha}|D\psi|_\infty\) and \(K \geq |\psi_t|_\infty + (|D^2g|_\infty + |g|_\infty) \int_{0}^{1} |z_s| ds\).

Now eventually we have obtain the following. Recall that \(0 < \alpha < 1\) in \((3.11)\), the definition of \(v(x, t)\). If we assume that

\[
C \geq \max \left\{ \frac{|U|^{1-\alpha}}{\alpha} \left( \frac{3 - \gamma}{1 - \gamma} |Dg|_\infty \sup_{t \leq T} |z_t| + 1 \right), \frac{4|U|^{2-\alpha} K}{\alpha(2 - \alpha) \gamma^2 - 1} |U|^{1-\alpha}|D\psi|_\infty \right\},
\]

where

\[
\gamma \in \left( \frac{1}{\sqrt{2 - \alpha}}, 1 \right),
\]

and that

\[K \geq \max \{2|\psi|_\infty, |\psi_t|_\infty \} + (|D^2g|_\infty + |g|_\infty) \int_{0}^{T} |z_s| ds,
\]

then for

\[0 < \epsilon \leq \frac{(2 - \alpha) \gamma^2 - 1}{4(n + \alpha - 2)},\]

and

\[0 < \delta \leq \left( \frac{2}{3 - \gamma} \right)^2,
\]

we have

\[
\partial_t v(x, t) \geq L^{\epsilon \delta} v(x, t),
\]

in the domain \(U \times (t_0 - 1, t_0)\) or \(U \times (0, t_0)\) where \(x_0 \in \partial U\), and \(v(x, t) \geq u(x, t)\) on the parabolic boundary. Hence by the comparison theorem we learn that \(v(x, t) \geq u(x, t)\) in the above domain. In particular when \(t = t_0\) it follows immediately that

\[
u(x, t_0) \leq \psi(x_0, t_0) + C|x - x_0|^\alpha.
\]
Analogously, by setting
\[ v(x,t) := \psi(x_0,t_0) - C|x - x_0|^\alpha - (|D^2g|_\infty + |g|_\infty) \int_{t_0}^t |z_s| ds + K(t-t_0) \]
with the same assumptions about the constants \( C, K, \) and \( \epsilon, \delta \) as specified previously, it emerges that
\[ u(x,t_0) \geq \psi(x_0,t_0) - C|x - x_0|^\alpha. \]
Finally we conclude that
\[ |u(x,t_0) - \psi(x_0,t_0)| \leq C|x - x_0|^\alpha, \]
which completes the proof.

Remark 3.7 Note that in the above proof the choice of the constant \( \gamma \) plays an important rôle.

Remark 3.8 We can not obtain the Lipschitz regularity by letting \( \alpha = 1 \) in Lemma 3.6 for \( v(x,t) \) with \( \alpha = 1 \) is not a viscosity solution of the equation (3.3). To circumvent this difficulty one way worthy of trying is to consider the approximating equation
\[ \partial_t^\epsilon \psi^0,\delta = L^0,\delta \psi^0,\delta \] instead of \( \partial_t^\epsilon \psi^\epsilon,\delta = L^\epsilon,\delta \psi^\epsilon,\delta \) with \( \epsilon > 0 \). But we have verified that at least for the function constructed in the form
\[ v(x,t) := \psi(x_0,t_0) + C_1|x - x_0| - C_2|x - x_0|^\alpha - K(t-t_0) + (|D^2g|_\infty + |g|_\infty) \int_{t_0}^t |z_s| ds \]
the desired result can not be achieved, whatever \( \alpha \in \mathbb{R} \) is chosen.

Next, boundary regularity is obtain when the initial-boundary value function is only continuous, based on the previous Lemma 3.6 that is just proved.

Corollary 3.9 Suppose \( \psi(x,t) \) is only continuous and other conditions are the same as in Lemma 3.6. Then the modulus of continuity of \( u^\epsilon,\delta \) on \( \partial U \times (0,T) \) can be estimated in terms of \( |U|, |\psi|_\infty, \) the modulus of continuity of \( \psi, |D^2g|_\infty, |Dg|_\infty, |g|_\infty, \) and \( \sup_{t \leq T} |z_t| \).
Proof. Fix \((x_0, t_0) \in \partial U \times (0, T)\). Given \(\rho > 0\) we choose \(0 < r < t_0\) such that 
\(|\psi(x, t) - \psi(x_0, t_0)| < \rho\) as long as \(|x - x_0| \vee |t - t_0| < r\). Define 
\[
\psi^\pm(x, t) := \psi(x_0, t_0) \pm \rho \pm \frac{2|\psi|_{\infty}}{r^2}|x - x_0|^2 \pm \frac{2|\psi|_{\infty}}{r}|t - t_0|. 
\]
Then on the parabolic boundary \(\Gamma_t\),
\[
\psi^-(x, t) \leq \psi(x_0, t_0) - \rho \leq \psi(x, t) \leq \psi(x_0, t_0) + \rho \leq \psi^+(x, t)
\]
if \(|x - x_0| \vee |t - t_0| < r\), and
\[
\psi^-(x, t) \leq -|\psi|_{\infty} \leq \psi(x, t) \leq |\psi|_{\infty} \leq \psi^+(x, t)
\]
if \(|x - x_0| \vee |t - t_0| \geq r\). So \(\psi^- \leq \psi \leq \psi^+\) on \(\Gamma_t\).

Suppose that \(u^\epsilon_\pm\) are the unique solutions of the equation (3.10) with initial-boundary value functions \(\psi^\pm\) respectively. Then by the comparison theorem we obtain that
\(u^\epsilon_- \leq u^\epsilon \leq u^\epsilon_+\) on \(U_T\).

By the estimate in Lemma 3.6 there exists a constant \(C > 0\) such that
\[
|u^\epsilon_\pm(x_0, t_0) - \psi(x_0, t_0)| \leq \frac{C}{r^2}|x - x_0|^\alpha.
\]
Finally we deduce that
\[
|u^\epsilon(x_0, t_0) - \psi(x_0, t_0)| \leq \rho + \frac{C}{r^2}|x - x_0|^\alpha,
\]
which completes the proof. \(\Box\)

Now a little further effort leads to the following global Hölder estimate. In other words, the boundary regularity that we just obtained can be extended to the interior of the domain \(U\).

Corollary 3.10 Given assumptions as in Lemma 3.6 we assume \(\psi \in C^{2,1}(U \times \mathbb{R}, \mathbb{R})\). Then there exists a constant \(C > 0\) depending on \(|U|, \alpha, \ |D\psi|_{\infty}, \ |\psi_t|_{\infty}, \ |\psi|_{\infty}, \ |D^2 g|_{\infty}, \ |D g|_{\infty}, \ |g|_{\infty}, \ \text{and sup}_{t \leq T} |z_t|\), but independent of \(0 < \epsilon < 1\) and \(0 < \delta < 1\) such that
\[
|u^\epsilon(x, t) - u^\epsilon(y, t)| \leq C|x - y|^\alpha
\]
for all \( x, y \in U \) and \( t \in (0, T) \). Moreover if \( \psi \) is only continuous, then the modulus of continuity of \( u^{\epsilon, \delta} \) in \( x \) on \( U_T \) can be estimated in terms of \( |\psi|_\infty \), the modulus of continuity of \( \psi \), \( |D^2 g|_\infty \), \( |D \psi|_\infty \), \( |g|_\infty \), and \( \sup_{t \leq T} |z_t| \).

**Proof.** Assume \( \psi \in C^{2,1}(U \times \mathbb{R}, \mathbb{R}) \) and denote

\[
v^\xi(x, t) := u^{\epsilon, \delta}(x + \xi, t) \quad \text{and} \quad U^\xi := \{ x \in \mathbb{R}^n : x + \xi \in U \}
\]

for \( t \in (0, T) \) and \( \xi \in \mathbb{R}^n \). From Lemma 3.6 we learn that

\[
|v^\xi(x, t) - u^{\epsilon, \delta}(x, t)| \leq C|\xi|^\alpha,
\]

or equivalently,

\[
v^\xi(x, t) - C|\xi|^\alpha \leq u^{\epsilon, \delta}(x, t) \leq v^\xi(x, t) + C|\xi|^\alpha,
\]

for \( t \in (0, T) \) and \( x \in \partial U \cap U^\xi \), or equivalently, \( x \in \partial [U \cap U^\xi] \). Thus by the comparison theorem we deduce that

\[
v^\xi(x, t) - C|\xi|^\alpha \leq u^{\epsilon, \delta}(x, t) \leq v^\xi(x, t) + C|\xi|^\alpha,
\]

or equivalently,

\[
|v^\xi(x, t) - u^{\epsilon, \delta}(x, t)| \leq C|\xi|^\alpha,
\]

for \( t \in (0, T) \) and \( x \in U \cap U^\xi \). Since \( \xi \in \mathbb{R}^n \) is arbitrary the desired assertion is verified.

\( \Box \)

**Temporal boundary regularity at \( U \times \{0\} \)**

We consider the regularity at the initial time. Here a Lipschitz estimate is achieved.

**Lemma 3.11** Suppose \( u^{\epsilon, \delta} \) is a smooth solution of the equation \( (3.10) \) where the initial-boundary value function \( \psi \in C^{2,1}(U \times \mathbb{R}, \mathbb{R}) \). Then there exists a constant \( C > 0 \) depending on \( |D^2 \psi|_\infty \), \( |\psi_t|_\infty \), \( |D^2 g|_\infty \), \( |g|_\infty \), \( \sup_{t \leq T} |z_t| \), but independent of \( 0 < \epsilon < 1 \) and \( 0 < \delta < 1 \) such that

\[
|u^{\epsilon, \delta}(x, t) - u^{\epsilon, \delta}(x, 0)| = |u^{\epsilon, \delta}(x, t) - \psi(x, 0)| \leq Ct,
\]
for all \((x,t) \in U \times (0,T)\). Moreover if \(\psi\) is only continuous in \(x\), then the modulus of continuity of \(u^{\epsilon,\delta}\) on \(U \times \{0\}\) can be estimated in terms of \(|\psi|_\infty, |D^2g|_\infty, |g|_\infty, \sup_{t \leq T} |z_t|\), and the modulus of continuity of \(\psi\) in \(x\).

**Proof.** First suppose that \(\psi \in C^{2,1}(U \times \mathbb{R}, \mathbb{R})\). Let 
\[ v(x,t) := \psi(x,0) + Ct, \]
where \(C > 0\) is a constant to be determined. Direct calculations give that
\[ L^{\epsilon,\delta}v = \epsilon \Delta v + \text{trace}(\beta \beta^* D^2v) + \text{trace}(\beta \beta^* D^2g) + g|z_t| \]
and
\[ L^{\epsilon,\delta}v = \epsilon \Delta \psi(x,0) + \text{trace}(\beta \beta^* D^2\psi) + \text{trace}(\beta \beta^* D^2g) + g|z_t| \]
\[ \leq \epsilon \sqrt{\sum_{i=1}^{n} \psi_{ii}(x,0)^2} + |D^2\psi|_\infty + (|D^2g|_\infty + |g|_\infty) \sup_{t \leq T} |z_t| \]
\[ \leq \epsilon \sqrt{n \sum_{i=1}^{n} \psi_{ii}(x,0)^2} + |D^2\psi|_\infty + (|D^2g|_\infty + |g|_\infty) \sup_{t \leq T} |z_t| \]
\[ \leq (\epsilon \sqrt{n} + 1)|D^2\psi|_\infty + (|D^2g|_\infty + |g|_\infty) \sup_{t \leq T} |z_t|. \]
Hence if we choose
\[ C \geq (\sqrt{n} + 1)|D^2\psi|_\infty + (|D^2g|_\infty + |g|_\infty) \sup_{t \leq T} |z_t| \]
then it follows that
\[ v_t - L^{\epsilon,\delta}v \geq 0. \]

Now we compare \(v\) and \(u^{\epsilon,\delta}\) on the parabolic boundary. For \((x,t) \in U \times \{0\}\) we have
\[ u^{\epsilon,\delta}(x,0) = \psi(x,0) = v(x,0). \]
For \((x,t) \in \partial U \times (0,T)\) we have
\[ u^{\epsilon,\delta}(x,t) = \psi(x,t) \]
\[ \psi(x, 0) + \psi_t(0) + |\psi_t(\infty) t \leq \psi(x, 0) + C t \]

Thus by the comparison principle it follows that

\[ u^{\epsilon, \delta}(x, t) \leq v(x, t) = \psi(x, 0) + C t, \]

for all \((x, t) \in U \times (0, T)\) where we select the constant

\[ C \geq \max \left\{ \left( \sqrt{n} + 1 \right) \left| D^2 \psi \right|_{\infty} + \left( |D^2 g|_{\infty} + |g|_{\infty} \right) \sup_{t \leq T} |z_t|, |\psi_t|_{\infty} \right\}. \]

Analogously by letting

\[ v(x, t) := \psi(x, 0) - C t \]

we deduce that

\[ u^{\epsilon, \delta}(x, t) \geq v(x, t) = \psi(x, 0) - C t \]

with the same constant \(C\). Hence the Lipschitz estimate is obtained, i.e.,

\[ |u^{\epsilon, \delta}(x, t) - \psi(x, 0)| \leq C t, \]

with the constant \(C\) defined above.

Second we suppose that \(\psi\) is only continuous. Let \(x_0 \in U\) be fixed. By the continuity, for \(\rho > 0\) given we choose \(0 < r < \text{dist}(x_0, \partial U)\) such that \(|\psi(x, 0) - \psi(x_0, 0)| < \rho\) as long as \(|x - x_0| < r\). Define the smooth functions

\[ \psi_{\pm}(x, t) := \psi(x_0, 0) + \rho \pm \frac{2|\psi|_{\infty}}{r^2} |x - x_0|^2. \]

Immediately we have \(\psi_- \leq \psi \leq \psi_+\) on the parabolic boundary. To see this just notice that

\[ \psi(x, 0) \leq \psi(x_0, 0) + \rho \leq \psi_+(x, 0) \]

for \(|x - x_0| < r\) while

\[ \psi_+(x, 0) \geq \psi(x_0, 0) + 2|\psi|_{\infty} \geq |\psi|_{\infty} \geq \psi(x, 0) \]
for $|x - x_0| \geq r$, and that

$$\psi_+(x, t) \geq \psi(x_0, 0) + 2|\psi|_\infty \geq |\psi|_\infty \geq \psi(x, t)$$

for $(x, t) \in \partial U \times [0, T]$.

Suppose that $u_{\epsilon, \delta}^\pm$ are the unique solutions with initial-boundary value functions $\psi_\pm$ respectively. Then by the comparison theorem we get that $u_{\epsilon, \delta}^- \leq u^\epsilon \leq u_{\epsilon, \delta}^+$ in $U_T$. Thus apply the Lipschitz estimate we have obtained to $u_{\epsilon, \delta}^\pm$ and we have that

$$|u_{\epsilon, \delta}^\pm(x, 0) - \psi_\pm(x_0, 0)| \leq \max \left\{ \left(\sqrt{n} + 1\right) D^2 \psi_\pm + (|D^2 g|_\infty + |g|_\infty) \sup_{t \leq T} |z_t|, (|\psi_\pm|)|t|_\infty \right\} t$$

Finally we deduce that

$$|u^\epsilon(x, 0) - \psi(x_0, 0)| \leq \max \left\{ \left(\sqrt{n} + 1\right) D^2 \psi_\pm + (|D^2 g|_\infty + |g|_\infty) \sup_{t \leq T} |z_t|, \rho \right\} t,$$

which proves the assertion as desired. \hfill \Box

Next the global Lipschitz regularity w.r.t. the time is achieved.

**Corollary 3.12** Given the same assumptions as in Lemma 3.11 we assume $\psi \in C^{2,1}(U \times \mathbb{R}, \mathbb{R})$. Then there exists a constant $C > 0$ depending on $|D^2 \psi|_\infty$, $|\psi|_\infty$, $|D^2 g|_\infty$, $|g|_\infty$, $\sup_{t \leq T} |z_t|$, but independent of $0 < \epsilon < 1$ and $0 < \delta < 1$ such that

$$|u^\epsilon(x, t) - u^\epsilon(x, s)| \leq C|t - s|,$$

for all $x \in U$ and $t, s \in (0, T)$. Moreover if $\psi$ is only continuous, then the modulus of continuity of $u^\epsilon$ in $t$ on $U \times (0, T)$ can be estimated in terms of $|\psi|_\infty$, $|D^2 g|_\infty$, $|g|_\infty$, $\sup_{t \leq T} |z_t|$, and the modulus of continuity of $\psi$ in $x$ and $t$. 

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Proof. Let

\[ v(x, t) := u^{\epsilon, \delta}(x, t + r) \]

for \( r > 0 \). Then it is easy to see that both \( u^{\epsilon, \delta} \) and \( v \) are solutions of the equation (3.10) in \( U_{T-r} := U \times (0, T-r) \). Thus for \( \psi \in C^{2,1}(U \times \mathbb{R}, \mathbb{R}) \) we learn that

\[
\sup_{U_{T-r}} |u^{\epsilon, \delta} - v| = \max \left\{ \left| \psi(\cdot, 0) - u^{\epsilon, \delta}(\cdot, r) \right|_{L^\infty(U)}, \sup_{x \in \partial U} \left| u^{\epsilon, \delta}(x, \cdot) - u^{\epsilon, \delta}(x, \cdot + r) \right|_{L^\infty((0,T))} \right\}
\]

\[
\leq \max \left\{ CR, |\psi_t|_{\infty, r} \right\}
\]

by the comparison theorem and Lemma 3.11. This implies that

\[
|u^{\epsilon, \delta}(x, t + r) - u^{\epsilon, \delta}(x, t)| \leq Cr,
\]

which is the desired Lipschitz estimate.

The case when \( \psi \) is only continuous can be proved analogously, as in the proof of Lemma 3.11.

□

Existence

Now eventually we show the proof of our main claim in Proposition 3.5.

Proof of Proposition 3.5. Given a family of functions \( (f_\eta)_{\eta \geq 0} \) which are uniformly bounded and Hölder-\( \alpha \) equicontinuous, that is,

\[
|f_\eta(x) - f_\eta(y)| \leq C|x - y|^\alpha
\]

for some constant \( C > 0 \) independent of \( \eta \) and \( x, y \). Then by the Arzelà-Ascoli lemma we deduce that there exists a uniformly convergent subsequence, still denoted by \( f_\eta \), with \( \lim_{\eta \to 0} f_\eta =: f \). Moreover, the limit function \( f \) is also Hölder-\( \alpha \) continuous; since

\[
|f(x) - f(y)| \leq |f(x) - f_\eta(x)| + |f_\eta(x) - f_\eta(y)| + |f_\eta(y) - f(y)|
\]

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\[ C|x - y|^\alpha + o(1), \]
as \( \eta \to 0. \)

Corollaries \[ 3.10 \] \[ 3.12 \] and the comparison theorem guarantee that the solutions of the equation \[ (3.10) \], \( u^{\epsilon, \delta} \), are Hölder-\( \alpha \) equicontinuous as a sequence indexed by \( \epsilon, \delta \). So the above argument applies and the limit function \( u \) as \( \epsilon, \delta \to 0 \) is a viscosity solution of the equation \[ (3.3) \] by virtue of the stability properties of viscosity solutions (Crandall, Ishii, & Lions \[ 1992 \]).

The existence of solutions of the equation \[ (3.3) \] with \( \psi \) only continuous follows by approximation by smooth functions using the Corollaries \[ 3.10 \] and \[ 3.12 \]. □

### 3.4 Pathwise stationary solutions

Now, as before, we study the asymptotic property of the solution of the equation \[ (3.3) \] as time tends to infinity and prove that the limit function is the pathwise stationary solution of the random PDE \[ (3.3) \]. In light of the Ornstein-Uhlenbeck transform this limit function gives the corresponding pathwise stationary solution for the stochastic PDE \[ (3.1) \]. Again techniques in (Caselles, Morel, & Sbert \[ 1998 \]) are applied, along with the random time pull-back.

**Proposition 3.13** The equation \[ (3.3) \] has at least one pathwise stationary solution.

**Proof.** Let \( T > 1 \) and

\[
K > 2 \left( |\psi|_\infty + (|D^2 g|_\infty + |g|_\infty) \int_0^T |z_s|ds \right) + \frac{1}{T}.
\]

Let \( \rho : [0, \infty) \to [0, \infty) \) be a strictly increasing function with \( \lim_{T \to \infty} \rho(T) = \infty \). Define a smooth function \( f(t, T) \) of \( t \in [0, \infty) \) as \( f(t, T) = -K \) on \( [0, T - 1/\rho(T)] \cup [T + 1/\rho(T), \infty) \), \( f(T, T) = 0 \) and increasing on \( [T - 1/\rho(T), T] \), decreasing on \( [T, T + 1/\rho(T)] \). Now let

\[
\psi^T_t(x, \omega) := \sup_{t \geq 0} \left[ u(x, t, \theta_{-T} \omega) - \frac{t}{T^2} + f(t, T) \right],
\]
\( v_T^-(x, \omega) := \inf_{t > 0} \left[ u(x, t, \theta - T \omega) + \frac{t}{T^2} - f(t, T) \right], \)

where \( u(x, t, \omega) \) is the solution of the equation (3.3) with initial boundary value function \( \psi \). Observing that

\[
\begin{align*}
|v_T^+(x, \omega) - v_T^+(y, \omega)| &\leq \sup_{t \geq 0} |u(x, t, \theta - T \omega) - u(y, t, \theta - T \omega)| \\
&\leq L|x - y|^\alpha,
\end{align*}
\]

and that

\[
\begin{align*}
|v_T^-(x, \omega) - v_T^-(y, \omega)| &\leq \sup_{t \geq 0} |u(x, t, \theta - T \omega) - u(y, t, \theta - T \omega)| \\
&\leq L|x - y|^\alpha,
\end{align*}
\]

we learn that \( v_T^+ \) and \( v_T^- \) are also Hölder continuous in \( x \) with the same constant and exponent as \( u \). From previous results we know that there exists a sequence \( T_k \uparrow \infty \) such that \( v_{T_k}^+, v_{T_k}^- \) converge to some Hölder-\( \alpha \) continuous functions \( \mu, \nu \) respectively.

Suppose the supremum in the definition of \( v_T^+ \) is obtained at \( t_T^+ \), and the infimum of \( v_T^- \) is obtained at \( t_T^- \). We claim that \( t_T^+ \in [T - 1/\rho(T), T + 1/\rho(T)] \). If not, the definitions of \( v_T^\pm \) imply that

\[
\begin{align*}
&u(x, t_T, \theta - T \omega) - \frac{t_T}{T^2} + f(t_T, T) \quad \leq \quad |\psi|_\infty + (|D^2 g|_\infty + |g|_\infty) \int_0^T |z_s| ds - K, \\
&u(x, T, \theta - T \omega) - \frac{T}{T^2} + f(T, T) \quad \geq \quad -|\psi|_\infty - (|D^2 g|_\infty + |g|_\infty) \int_0^T |z_s| ds - \frac{1}{T},
\end{align*}
\]

Thus

\[
K \leq 2 \left( |\psi|_\infty + (|D^2 g|_\infty + |g|_\infty) \int_0^T |z_s| ds \right) + \frac{1}{T},
\]

which contradicts the assumption. Similar claim is true for \( t_T^- \) of \( v_T^- \), that is, \( t_T^- \in [T - 1/\rho(T), T + 1/\rho(T)] \).

And we know that

\[
\begin{align*}
v_{T_k}^+ (x, \omega) &\geq u(x, T_k, \theta - T_k \omega) - \frac{1}{T_k},
\end{align*}
\]

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\[ v_{T_k}^-(x, \omega) \leq u(x, T_k, \theta_{-T_k} \omega) + \frac{1}{T_k}, \]

so that

\[ \mu(x, \omega) \geq \limsup_k u(x, T_k, \theta_{-T_k} \omega) \geq \liminf_k u(x, T_k, \theta_{-T_k} \omega) \geq \nu(x, \omega). \]

On the other side, noting that \( f \) by definition is non-positive, we have

\[ v_{T_k}^-(x, \omega) - v_{T_k}^- (x, \omega) = \sup_{t \geq 0} \left[ u(x, t, \theta_{-T} \omega) - \frac{t}{T^2} + f(t, T) \right] - \inf_{s \geq 0} \left[ u(x, s, \theta_{-T} \omega) + \frac{s}{T^2} - f(s, T) \right] = u(x, t_{T_k}^+, \theta_{-T} \omega) - u(x, t_{T_k}^-, \theta_{-T} \omega) - \frac{t_{T_k}^+ + t_{T_k}^-}{T^2} + f(t_{T_k}^+, T) + f(t_{T_k}^-, T) \leq u(x, t_{T_k}^+, \theta_{-T} \omega) - u(x, t_{T_k}^-, \theta_{-T} \omega) - \frac{t_{T_k}^+ + t_{T_k}^-}{T^2}. \]

Since \( [T - 1/\rho(T), T + 1/\rho(T)] \ni t_{T_k}^\pm \) it follows that \( t_{T_k}^\pm \to \infty \) and

\[ \frac{2T - 2/\rho(T)}{T^2} \leq \frac{t_{T_k}^+ + t_{T_k}^-}{T^2} \leq \frac{2T + 2/\rho(T)}{T^2}, \]

which implies that

\[ \frac{t_{T_k}^+ + t_{T_k}^-}{T^2} \to 0 \]

as \( T \to \infty \). From Corollary 3.12 we know that \( u \) is Lipschitz continuous in \( t \) and the constant \( C = C_T \) depends on \( T \) via the supremum \( \sup_{t \leq T} |z_t| \). Now we require \( \rho \) is such that \( \rho(T) \geq 1 \) when \( T \geq 1 \), and

\[ \frac{C_{T+1}}{\rho(T)} \to 0 \]

as \( T \to \infty \). Thus it follows that

\[ |u(x, t_{T_k}^+, \theta_{-T} \omega) - u(x, t_{T_k}^-, \theta_{-T} \omega)| \leq C_{T+1} |t_{T_k}^+ - t_{T_k}^-| \leq C_{T+1} \frac{2}{\rho(T)} \to 0 \]

when \( T \to \infty \). Hence

\[ \lim_{k \to \infty} (v_{T_k}^+(x, \omega) - v_{T_k}^-(x, \omega)) \leq 0, \]

which means \( \mu(x, \omega) \leq \nu(x, \omega) \). Therefore

\[ \mu(x, \omega) = \nu(x, \omega) = \lim_{k \to \infty} v_{T_k}^+(x, \omega) = \lim_{k \to \infty} v_{T_k}^-(x, \omega) = \lim_{k \to \infty} u(x, T_k, \theta_{-T_k} \omega). \]
For simplicity we denote $\tau$ the sequence $T_k$. Thus
\[
\lim_{\tau \to \infty} u(x, \tau, \theta_{-r} \omega) =: \hat{u}(x, 0, \omega) =: \hat{u}_0(x, \omega)
\] (3.12)
exists.

Define for all $t \geq 0$
\[
\hat{u}(x, t, \omega) := \hat{u}_0(x, \theta_t \omega) = \lim_{\tau \to \infty} u(x, \tau, \theta_{t-r} \omega) = \lim_{\tau \to \infty} u(x, t + \tau, \theta_{-r} \omega).
\] (3.13)

The next task is to prove that the function $\hat{u}$ solves the equation (3.3) with initial value function $\hat{u}_0(x, \omega)$. It is readily known that as a consequence of the uniqueness the solution $u$ of our equation (3.3), as a random system $u : C(U, \mathbb{R}) \times [0, \infty) \times \Omega \to C(U, \mathbb{R})$, satisfies the cocycle property. Therefore it implies that for $t, r \geq 0$,
\[
u(t + r, \theta_{-r} \omega) = u(t, \omega) \circ u(r, \theta_{-r} \omega).
\]
By virtue of Lemma 3.3 we learn that $u$ is continuous as a functional of the initial value function. So in particular letting $r := \tau \to \infty$ we get
\[
u(\hat{u}_0(\omega), t, \omega) = u(t, \omega) \circ \hat{u}_0(\omega) = \hat{u}(t, \omega) = \hat{u}_0(\theta_t \omega).
\]
This means that $\hat{u}$ solves the equation (3.3) with initial value function $\hat{u}_0$. \hfill \Box

Recall the Ornstein-Uhlenbeck transform for the solution $\ddot{u}$ of the SPDE (3.1)
\[
\ddot{u}(x, t, \omega) = u(x, t, \omega) + g(x, t, \omega) = u(x, t, \omega) + g(x) z_t(\omega).
\]
For $t, r \geq 0$ it follows that
\[
\ddot{u}(x, t + r, \theta_{-r} \omega) = u(x, t + r, \theta_{-r} \omega) + g(x) z_{t+r}(\theta_{-r} \omega),
\]
and in particular when $t = 0$
\[
\ddot{u}(x, r, \theta_{-r} \omega) = u(x, r, \theta_{-r} \omega) + g(x) z_r(\theta_{-r} \omega).
\]
Because of $z_t(\omega) = z(\theta_t \omega)$ we know that
\[
z_r(\theta_{-r} \omega) = z(\theta_r \theta_{-r} \omega) = z(\omega),
\]
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so that
\[
\lim_{r \to \infty} z_r(\theta_r \omega) = z(\omega),
\]
where the r.v. $z(\omega)$ is the same as in the expression (2.1). Consequently, we have the existence of the limit
\[
\tilde{u}_0(x, \omega) := \tilde{u}(x, 0, \omega) := \lim_{\tau \to \infty} \tilde{u}(x, \tau, \theta_{-\tau} \omega) = \tilde{u}_0(x, \omega) + g(x)z(\omega).
\]
Similarly, since
\[
z_{t+r}(\theta_{-r} \omega) = z(\theta_{t+r} \theta_{-r} \omega) = z(\theta_t \omega),
\]
we get that
\[
\lim_{r \to \infty} z_{t+r}(\theta_{-r} \omega) = z(\theta_t \omega).
\]
Therefore the limit exists and
\[
\tilde{u}(x, t, \omega) := \tilde{u}_0(\theta_t \omega) = \lim_{\tau \to \infty} \tilde{u}(x, \tau, \theta_{t-\tau} \omega) = \lim_{\tau \to \infty} \tilde{u}(x, t+\tau, \theta_{-\tau} \omega) = \tilde{u}(x, t, \omega) + g(x)z(\theta_t \omega).
\]
And it is readily seen that the relation of $\tilde{u}$ and $\hat{u}$ is actually the Ornstein-Uhlenbeck transform, which means that $\tilde{u}$ solves the SPDE (3.1) with initial value function $\tilde{u}_0$.

**Corollary 3.14** The SPDE (3.1) has at least one pathwise stationary solution.
Part II

Backward SDEs and Mean Curvature Type Flows
Chapter 4

Introduction

Backward Stochastic Differential Equations (BSDEs) in general form were introduced by E. Pardoux and S. Peng in (Pardoux & Peng, 1990) and they proved the existence and uniqueness of the solution. Then in 1999 S. Peng proved in (Peng, 1999) that the limit of a monotonic increasing sequence of RCLL supersolutions of BSDEs is also the solution of the same equation, from which he obtained the construction of the smallest supersolution on condition that there exists at least one special solution. For general review of BSDEs, please see (Yan, Peng, Wu, & Fang, 2000, Part II).

Mean curvature flows were studied early by M. Gage, R. S. Hamilton, M. Grayson, G. Huisken, and T. Ilmanen and others through classic methods of differential geometry and partial differential equations; see (Gage, 1983, 1984; Gage & Hamilton, 1986; Grayson, 1987, 1989; Hamilton, 1993; Huisken, 1984; Huisken & Ilmanen, 1997, 2001; Ilmanen, 1994), and (Huisken, 1998; Struwe, 1996). However, the three-dimensional mean curvature flow may develop singularity in finite time even if the initial condition is smooth. To overcome this difficulty, one approach is the geometric measure theory, started by K. A. Brakke (Brakke, 1978); see B. White (White, 2000, 2002, 2003) for updated progresses. Another way is the level set method, or the theory of viscosity solutions initiated by L. C. Evans, J. Spruck and Y. G. Chen, Y. Giga, S. Goto at the same time. In their papers (Chen, Giga, & Goto, 1991; Evans & Spruck, 1991) they proved that the level set equation of mean curvature flow has unique solution in
the weak viscosity sense (the solution may not be continuous). Later, R. Buckdahn, P. Cardaliaguet, M. Quincampoix and H. M. Soner, N. Touzi found simultaneously a connection between mean curvature flows and some controlled stochastic differential equations thus obtained a probabilistic representation of the mean curvature flow; see their papers (Buckdahn, Cardaliaguet, & Quincampoix, 2002; Soner & Touzi, 2002b). One can refer to the expository article (Crandall, Ishii, & Lions, 1992) for general knowledge of viscosity solutions.

In this part we prove that under some assumptions the supersolution of the state constrained controlled Forward Backward Stochastic Differential Equation (FBSDE) is also the viscosity supersolution of a certain Hamilton-Jacobi-Bellman (HJB) equation. Thus by S. Peng (Peng, 1999) this is also the case for the smallest supersolution. Especially this relation implies a representation of some geometric flows (e.g. mean curvature flows). The main result is Proposition 5.9.

This part also includes a verification of the finite time existence of mean curvature flows, by utilising its probabilistic representation.
Chapter 5

Backward SDEs and Hamilton-Jacobi-Bellman Equations

5.1 Backward SDEs

Controlled Forward-Backward SDEs

Let $T \geq 0$ be fixed. Suppose there is a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ given, and a $d$-dimensional standard Wiener process $\{W := W_t : 0 \leq t \leq T\}$ generating the natural filtration $\mathcal{F}_t := \sigma\{W_s : 0 \leq s \leq t\}, 0 \leq t \leq T$.

Let $0 \leq t \leq T$ and $s \in [t, T]$; we consider the following controlled Forward-Backward SDE

$$
X_{s}^{t,x,v} = x + \int_{t}^{s} \mu(r, X_{r}^{t,x,v}, v_{r})dr + \int_{t}^{s} \sigma(r, X_{r}^{t,x,v}, v_{r})dW_{r},
$$

$$
Y_{s}^{t,x,v} = g(X_{T}^{t,x,v}) + \int_{s}^{T} f(r, X_{r}^{t,x,v}, Y_{r}^{t,x,v}, Z_{r}^{t,x,v}, v_{r})dr - \int_{s}^{T} Z_{r}^{t,x,v}dW_{r}. \tag{5.1}
$$

Here we state the hypotheses.

(H1) The admissible control set is denoted by $\mathcal{U}$, in which all $v(\cdot) \in \mathcal{U}$ satisfies: $v_t \in$
$U \subset \mathbb{R}^k$ for all $t$; $U$ is a compact set; $v(\cdot)$ is progressively measurable; and

$$\mathbb{E} \int_0^\tau \|v_t\|^2 dt \leq \infty,$$

for all stopping time $\tau \in [0, T]$.

\[ \text{(H2)} \] The functions

$$\mu : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,$$

$$\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{n \times d}$$

are measurable and of linear growth w.r.t. $(x, v)$, Lipschitz continuous w.r.t. $x$, Hölder-$\gamma$ continuous w.r.t. $v$ where $\gamma \in (0, 1]$ given. That is to say, for all $t \in [0, T]$ and $x, x_1, x_2 \in \mathbb{R}^n$ there is a constant $C$ such that

$$\|\mu(t, x, v)\| + \|\sigma(t, x, v)\| \leq C(1 + \|x\| + \|v\|),$$

and

$$\|\mu(t, x_1, v_1) - \mu(t, x_2, v_2)\| + \|\sigma(t, x_1, v_1) - \sigma(t, x_2, v_2)\| \leq C(\|x_1 - x_2\| + \|v_1 - v_2\|^{\gamma}).$$

\[ \text{(H3)} \] Let

$$f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R},$$

$$g : \mathbb{R}^n \to \mathbb{R},$$

and for all $(x, y, z, v), f(\cdot, x, y, z, v)$ is $\mathcal{F}_t$-adapted, bounded; $g(\cdot)$ is $\mathcal{F}_T$-measurable, bounded. $f$ is Lipschitz continuous w.r.t. $(y, z)$ and is continuous w.r.t. $v$; $f, g$ are Hölder-$\gamma$ continuous w.r.t. $x$; namely, there is a constant $C$ such that for all $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, x_1, y_1, z_1, v) - f(t, x_2, y_2, z_2, v)| + |g(x_1) - g(x_2)| \leq C(\|x_1 - x_2\|^\gamma + |y_1 - y_2| + \|z_1 - z_2\|).$$
And $f$, $g$ are of linear growth w.r.t. $x$, i.e. there is a constant $C$ such that for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^k$,

$$|f(0,x,0,0,v)| + |g(x)| \leq C(1 + \|x\|).$$

(H4) $\mu$, $\sigma$, $f$, $g$ are deterministic functions of $(t,x,y,z,v)$.

Now we define the value function as the essential supremum of the cost function

$$u(t,x) := \text{ess sup}_{v(\cdot) \in \mathcal{U}} Y^t_{t,x,v}.$$  \hspace{1cm} (5.2)

And consider the Hamilton-Jacobi-Bellman equation

$$-\partial_t u(t,x) = F(t,x,u(t,x),Du(t,x),D^2u(t,x)), \ t \in [0,T)$$  \hspace{1cm} (5.3)

$$u(T,x) = g(x),$$

where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$, $S^n$ denoting the set of all $n$-dimensional symmetric matrices, and

$$F(t,x,r,p,S) := \sup_{v \in \mathcal{U}} \left\{ \frac{1}{2} \text{trace}(\sigma \sigma^T(t,x,v)S) + \langle \mu(t,x,v),p \rangle + f(t,x,r,\sigma^T(t,x,v)p,v) \right\}.$$

**Remark 5.1** The HJB equation can also be written as

$$-\sup_{v \in \mathcal{U}} \{ \mathcal{L}^v u(t,x) + f(t,x,u(t,x),\sigma(t,x,v)Du(t,x),v) \} = 0,$$

where

$$\mathcal{L}^v u(t,x) := \partial_t u(t,x) + \frac{1}{2} \text{trace}[\sigma \sigma^T(t,x,v)D^2u(t,x)] + \langle \mu(t,x,v),Du(t,x) \rangle$$

is the usual second order elliptic differential operator with control $v$ involved.

Then, from S. Peng (Yan, Peng, Wu, & Fang, 2000, Theorem 7.3) we have the following lemma.
Lemma 5.2 Under the hypotheses (H1)-(H4), \( u(t,x) \) defined as (5.2) is the unique viscosity solution of the HJB equation (5.3).

Similarly we can define another value function after the FBSDE (5.1) as
\[
u(t,x) := \text{ess inf}_{v(.) \in \mathcal{U}} Y_{t,x,v}(t),
\]
and consider the corresponding Hamilton-Jacobi-Bellman equation
\[-\partial_t u(t,x) = F(t,x,u(t,x),Du(t,x),D^2u(t,x)), t \in [0,T) \] (5.5)
\[u(T,x) = g(x),\]
where
\[
F(t,x,r,p,S) := \inf_{v(.) \in \mathcal{U}} \left\{ \frac{1}{2} \text{trace}(\sigma\sigma^T(t,x,v)S) + \langle \mu(t,x,v),p \rangle + f(t,x,r,\sigma^T(t,x,v)p,v) \right\}.
\]

By check through the proof of Lemma 5.2 we have the following result analogously.

Lemma 5.3 Under the hypotheses (H1)-(H4), \( u(t,x) \) defined as (5.4) is the unique viscosity solution of the HJB equation (5.5).

Supersolutions of Forward-Backward SDEs

In this section we turn to the properties of supersolutions of Forward-Backward SDEs.

Consider the state constrained controlled FBSDE
\[
X_{t,x,v}^s = x + \int_t^s \mu(r,X_{t,x,v}^r,v_r)dr + \int_t^s \sigma(r,X_{t,x,v}^r,v_r)dW_r,
\]
\[
Y_{t,x,v}^s = g(X_{T,x,v}^T) + \int_s^T f(r,X_{t,x,v},Y_{t,x,v}^r,Z_{t,x,v}^r,v_r)dr + A_{t,x,v}^T - A_{t,x,v}^s - \int_s^T Z_{t,x,v}^r dW_r,
\]
\[s.t. \quad \Phi(s,X_{t,x,v}^s,Y_{t,x,v}^s,Z_{t,x,v}^s) = 0, \quad \forall s \geq t, \text{ a.s.,} \]
where \( A_{t,x,v}^T \) is an increasing process, \( E\|A_{T,x,v}^T\|^2 < \infty \) and \( \Phi \geq 0. \)
Definition 5.4 If there exists \((A^{t,x,v}, Y^{t,x,v}, Z^{t,x,v})\) which solves the FBSDE (5.6) then we call that \((A^{t,x,v}, Y^{t,x,v}, Z^{t,x,v})\) is a supersolution of the FBSDE (5.6). A supersolution \((\overline{A}^{t,x,v}, \overline{Y}^{t,x,v}, \overline{Z}^{t,x,v})\) is called the smallest supersolution of the equation (5.6) if \(\overline{Y}^{t,x,v} \leq Y^{t,x,v}, s \geq t, \) a.s., for any supersolution \((A^{t,x,v}, Y^{t,x,v}, Z^{t,x,v})\) of the equation (5.6).

Based on the result of S. Peng (Peng, 1999) we readily have the following lemma.

Lemma 5.5 In addition to the conditions above we assume

(i). that \((H3)-(H4)\) hold with \(f\) replaced by \(f + \alpha \Phi\) for all \(\alpha \in \mathbb{R}_+;\)

(ii). that for all \((t,x,r,p)\), there exists \(v \in U\) such that \(\Phi(t,x,r,\sigma(t,x,v)p) = 0, \) a.s.;

(iii). and that there exists at least one supersolution for the FBSDE (5.6).

Then there exists the smallest supersolution \((\overline{Y}^{t,x,v}, \overline{Z}^{t,x,v})\), which is obtained as the limit of an increasing sequence of solutions for a family of FBSDEs. That is to say, there exists \((Y^{\alpha,t,x,v}, Z^{\alpha,t,x,v})_{\alpha > 0}\) such that \(Y^{\alpha,t,x,v} \uparrow Y^{t,x,v}\) when \(\alpha \uparrow \infty,\) where, for all \(\alpha > 0,\) \((Y^{\alpha,t,x,v}, Z^{\alpha,t,x,v})\) is the solution of the following FBSDE.

\[
X^{t,x,v}_s = x + \int_t^s \mu(r, X^{t,x,v}_r, v_r) dr + \int_t^s \sigma(r, X^{t,x,v}_r, v_r) dW_r,
\]

\[
Y^{\alpha,t,x,v}_s = g(X^{t,x,v}_T) + \int_s^T f(r, X^{t,x,v}_r, Y^{\alpha,t,x,v}_r, Z^{\alpha,t,x,v}_r, v_r) dr
+ \int_s^T \alpha \Phi(r, X^{t,x,v}_r, Y^{\alpha,t,x,v}_r, Z^{\alpha,t,x,v}_r) dr - \int_s^T Z^{\alpha,t,x,v}_r dW_r, \quad s \in [t,T].
\]

Remark 5.6 It is worth mentioning that the approach to construct the FBSDEs (5.7), whose solutions approximate the solution of the equation (5.6), is called the penalisation method.

5.2 Generalised Hamilton-Jacobi-Bellman equations

M. Crandall, H. Ishii and P.-L. Lions (Crandall, Ishii, & Lions, 1992) states the stability properties of limits of viscosity solutions. We include it here as a lemma.
Definition 5.7 For a function $\phi : \text{Domain} \to \mathbb{R}$ we define its upper semicontinuous envelope as

$$
\phi^*(z) := \lim_{r \downarrow 0} \sup \left\{ \phi(x) : x \in \text{Domain}, \ |x - z| < r \right\}.
$$

Its lower semicontinuous envelope is similarly defined as

$$
\phi_*(z) := \lim_{r \downarrow 0} \inf \left\{ \phi(x) : x \in \text{Domain}, \ |x - z| < r \right\}.
$$

Lemma 5.8 (i). Let $u_n \in USC(O)$ the upper semicontinuous functions on the open set $O$ for $n \in \mathbb{N}$; and define for all $z \in O$,

$$
\bar{u}(z) := \limsup_{n \to \infty} u_n(z) := \lim_{j \to \infty} \sup \left\{ u_n(x) : n \geq j, \ x \in O, \ |x - z| \leq \frac{1}{j} \right\};
$$

assume $\bar{u}(z) < \infty$ for all $z \in O$; suppose $u_n$ is the viscosity solution of

$$
F(x, u, Du, D^2u) \leq 0.
$$

Then $\bar{u}$ is the viscosity solution of

$$
F(x, u, Du, D^2u) \leq 0
$$
on $O$.

(ii). Let $u_n \in LSC(O)$ the lower semicontinuous functions on the open set $O$ for $n \in \mathbb{N}$; and define for all $z \in O$,

$$
u(z) := \liminf_{n \to \infty} u_n(z) := \lim_{j \to \infty} \inf \left\{ u_n(x) : n \geq j, \ x \in O, \ |x - z| \leq \frac{1}{j} \right\};
$$

assume $\nu(z) > -\infty$ for all $z \in O$; suppose $u_n$ is the viscosity solution of

$$
F(x, u, Du, D^2u) \geq 0.
$$

Then $\nu$ is the viscosity solution of

$$
F(x, u, Du, D^2u) \geq 0
$$
on $O$.
(iii). Let $u_n$ be the viscosity solution of $F_n(x, u, Du, D^2u) \leq 0$ for all $n \in \mathbb{N}$; define

$$F(x, r, p, S) := \liminf_{n \to \infty} F_n(x, r, p, S);$$

suppose $\bar{u}(z) = \limsup_{n \to \infty} u_n(z) < \infty$ for all $z \in \mathcal{O}$. Then $\bar{u}$ is the viscosity solution of $F(x, u, Du, D^2u) \leq 0$. Note that $F_n$ may not be continuous; if $F_n = F$ independently of $n$ and $F$ is discontinuous, then $\bar{F} = F^*.$

(iv). Let $u_n$ be the viscosity solution of $F_n(x, u, Du, D^2u) \geq 0$ for all $n \in \mathbb{N}$; define

$$\bar{F}(x, r, p, S) := \limsup_{n \to \infty} F_n(x, r, p, S);$$

suppose $\underline{u}(z) = \liminf_{n \to \infty} u_n(z) > -\infty$ for all $z \in \mathcal{O}$. Then $\underline{u}$ is the viscosity solution of $\bar{F}(x, u, Du, D^2u) \geq 0$. Similarly $F_n$ may not be continuous; if $F_n = F$ independent of $n$, discontinuous, then $\bar{F} = F^*.$

Forward-Backward SDEs and generalised HJB equations

Define after the FBSDE (5.6)

$$u(t, x) := \text{ess inf}_{v \in \mathcal{U}} Y^{t, x, v}_s, \quad \text{(5.8)}$$

where $Y^{t, x, v}_s, s \in [t, T]$ is the smallest supersolution of (5.6). And consider the HJB equation

$$- \inf_{v \in U_0(t, x)} \{ \mathcal{L}^v u(t, x) + f(t, x, u, \sigma(t, x, v)Du(t, x), v) \} = 0, \quad t \in [0, T) \quad \text{(5.9)}$$

$$u(T, x) = g(x),$$

where

$$U_0(t, x) := \{ v \in \mathcal{U} : \Phi(t, x, u(t, x), \sigma(t, x, v)Du(t, x)) = 0, \ a.s. \},$$

and $\Phi \geq 0$

is the control set dependent on $(t, x)$, and as before $\mathcal{L}^v$ is the second order linear elliptic differential operator involving control $v$. We denote the left hand side of the equation (5.9) as $F(t, x, u(t, x), \partial_t u(t, x), Du(t, x), D^2u(t, x))$ for simplicity. And for all $\alpha \in \mathbb{R}_+$ we denote

$$F_0(t, x, r, q, p, S)$$

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\[ := - \inf_{v \in U} \left\{ q + \frac{1}{2} \text{trace}(\sigma^T(t, x, v) S) + \langle \mu(t, x, v), p \rangle 
 + f(t, x, r, \sigma(t, x, v)p, v) + \alpha \Phi(t, x, r, \sigma(t, x, v)p) \right\}. \]

Thus,
\[
F^\alpha(t, x, u(t, x), \partial_t u(t, x), Du(t, x), D^2 u(t, x)) = - \inf_{v \in U} \{ L^v u(t, x) + f(t, x, u(t, x), \sigma(t, x, v)Du(t, x), v) 
+ \alpha \Phi(t, x, u(t, x), \sigma(t, x, v)Du(t, x)) \}.\]

**Proposition 5.9** Notations as above, we have that the function \( u \) in (5.8) defined after the FBSDE (5.6) as above is the viscosity supersolution and weak viscosity subsolution of the HJB equation (5.9) under the following conditions.

(i). (H3)-(H4) hold with \( f \) replaced by \( f + \alpha \Phi \) for all \( \alpha \in \mathbb{R}_+ \).

(ii). There exists at least one supersolution of FBSDE (5.6).

(iii). For all \((t, x, r, p)\), there exists \( v \in U \) such that \( \Phi(t, x, r, \sigma(t, x, v)p) = 0 \), a.s.. This guarantees that \( U_0 \) is always nonempty, i.e., \( U_0(t, x) \neq \emptyset \) for all \((t, x)\).

**Proof.** Let us define
\[ u^\alpha(t, x) := \text{ess inf} \ Y_t^{\alpha, t, x, v}, \]
where \( Y^{\alpha, t, x, v} \) is the solution of the FBSDE (5.7). Then by Lemma 5.3 it follows that \( u^\alpha \) is the viscosity solution of following HJB equation
\[
F^\alpha(t, x, u(t, x), \partial_t u(t, x), Du(t, x), D^2 u(t, x)) = 0, \quad t \in [0, T) \quad (5.10)
\]
\[ u(T, x) = g(x). \]

We claim that for all \((t, x)\) it holds that, when \( \alpha \to \infty \),
\[ u^\alpha(t, x) = \text{ess inf} \ Y_t^{\alpha, t, x, v} \uparrow \text{ess inf} \ Y_t^{t, x, v} = u(t, x). \]

In fact, by Lemma 5.5 we have that
\[ Y_t^{\alpha, t, x, v} \uparrow Y_t^{t, x, v}. \]
when $\alpha \to \infty$ for all $(t, x)$ fixed. And it is obvious that, for all $\alpha \in \mathbb{R}_+$,

$$\text{ess inf}_{v(\cdot) \in \mathcal{U}} Y^\alpha_{t,t,x,v} \leq \text{ess inf}_{v(\cdot) \in \mathcal{U}} Y_{t,t,x,v},$$

thus,

$$\lim_{\alpha \uparrow \infty} u^\alpha(t, x) \leq u(t, x).$$

On the other hand we have

$$\text{ess inf}_{v(\cdot) \in \mathcal{U}} Y_{t,t,x,v} - \text{ess inf}_{v(\cdot) \in \mathcal{U}} Y^\alpha_{t,t,x,v} \leq \text{ess sup}_{v(\cdot) \in \mathcal{U}} (Y_{t,t,x,v} - Y^\alpha_{t,t,x,v}).$$

Since $\mathcal{U}$ is a compact set we deduce that, for all $\alpha \in \mathbb{R}_+$, there exists a control process $v^\alpha(\cdot) \in \mathcal{U}$ such that

$$\text{ess sup}_{v(\cdot) \in \mathcal{U}} (Y_{t,t,x,v} - Y^\alpha_{t,t,x,v}) = Y_{t,t,x,v} - Y^\alpha_{t,t,x,v}.$$

Thus we obtain that

$$\lim_{\alpha \uparrow \infty} (Y_{t,t,x,v} - Y^\alpha_{t,t,x,v}) = \lim_{\alpha \uparrow \infty} \lim_{\alpha' \uparrow \infty} (Y_{t,t,x,v} - Y'^\alpha_{t,t,x,v}) = 0.$$

This implies that

$$u(t, x) - \lim_{\alpha \uparrow \infty} u^\alpha(t, x) = \text{ess inf}_{v(\cdot) \in \mathcal{U}} Y_{t,t,x,v} - \lim_{\alpha \uparrow \infty} \text{ess inf}_{v(\cdot) \in \mathcal{U}} Y^\alpha_{t,t,x,v} \leq 0.$$

Hence we learn that

$$\lim_{\alpha \uparrow \infty} u^\alpha(t, x) = u(t, x)$$

as desired.
Next we claim that

$$F^\alpha(t, x, r, q, p, S) \downarrow F(t, x, r, q, p, S)$$

when $\alpha \to \infty$. Above all, since $U_0(t, x) \subset U$ and $\Phi$ is nonnegative we have clearly that

$$F^\alpha(t, x, r, q, p, S) = -\inf_{v \in U} \left\{ q + \frac{1}{2} \text{trace} (\sigma \sigma^T(t, x, v) S) + \langle \mu(t, x, v), p \rangle + f(t, x, r, \sigma(t, x, v) p, v) + \alpha \Phi(t, x, r, \sigma(t, x, v) p) \right\} \geq -\inf_{v \in U_0(t, x)} \left\{ q + \frac{1}{2} \text{trace} (\sigma \sigma^T(t, x, v) S) + \langle \mu(t, x, v), p \rangle + f(t, x, r, \sigma(t, x, v) p, v) \right\} = F(t, x, r, q, p, S),$$

and that $F^\alpha(t, x, r, q, p, S)$ is decreasing as $\alpha \to \infty$. So it follows that the limit

$$\lim_{\alpha \uparrow \infty} F^\alpha(t, x, r, q, p, S) \geq F(t, x, r, q, p, S)$$

exists. On the other side, because of the compactness of the control set $U$ we learn that for all $\alpha \in \mathbb{R}_+$ there exists a $v^\alpha \in U$ such that

$$\inf_{v \in U} \left\{ q + \frac{1}{2} \text{trace} (\sigma \sigma^T(t, x, v) S) + \langle \mu(t, x, v), p \rangle + f(t, x, r, \sigma(t, x, v) p, v) + \alpha \Phi(t, x, r, \sigma(t, x, v) p) \right\} = q + \frac{1}{2} \text{trace} (\sigma \sigma^T(t, x, v^\alpha) S) + \langle \mu(t, x, v^\alpha), p \rangle + f(t, x, r, \sigma(t, x, v^\alpha) p, v^\alpha) + \alpha \Phi(t, x, r, \sigma(t, x, v^\alpha) p).$$

Then by virtue of the compactness of $U$, the continuity of $\mu, \sigma, f$, and $\Phi$, and the nonnegativeness of $\Phi$ we know that for fixed $(t, x, r, q, p, S)$ the above quantities are bounded, that is, there exist positive constants $C_1$ and $C_2$, dependent on $(t, x, r, q, p, S)$, such that

$$q + \frac{1}{2} \text{trace} (\sigma \sigma^T(t, x, v^\alpha) S) + \langle \mu(t, x, v^\alpha), p \rangle.$$
\[ + f(t, x, r, \sigma(t, x, v^\alpha)p, v^\alpha) \right| \]
\[ \leq C_1(t, x, r, q, p, S), \]

and that
\[ -C_2(t, x, r, q, p, S) \]
\[ \leq q + \frac{1}{2} \text{trace}(\sigma \sigma^T(t, x, v^\alpha)S) + \langle \mu(t, x, v^\alpha), p \rangle \]
\[ + f(t, x, r, \sigma(t, x, v^\alpha)p, v^\alpha) + \alpha \Phi(t, x, r, \sigma(t, x, v^\alpha)p) \]
\[ \leq \inf_{v \in U_0(t, x)} \left\{ q + \frac{1}{2} \text{trace}(\sigma \sigma^T(t, x, v)S) + \langle \mu(t, x, v), p \rangle \right. \]
\[ \left. + f(t, x, r, \sigma(t, x, v)p, v) \right\}. \]

Thus there exists a constant \( C_3(t, x, r, q, p, S) > 0 \) such that
\[ |\alpha \Phi(t, x, r, \sigma(t, x, v^\alpha)p)| \leq C_3(t, x, r, q, p, S). \]

Now letting \( \alpha \to \infty \) we obtain that
\[ \Phi(t, x, r, \sigma(t, x, v^\alpha)p) \to 0. \]

And this immediately implies that there exists an \( \alpha_0 > 0 \) such that for all \( \alpha \geq \alpha_0 \) we have that
\[ v^\alpha \in U_0(t, x), \]

in light of the compactness of the control set \( U_0(t, x) \). Hence it follows that
\[ \lim_{\alpha \to \infty} \inf_{v \in U} \left\{ q + \frac{1}{2} \text{trace}(\sigma \sigma^T(t, x, v)S) + \langle \mu(t, x, v), p \rangle \right. \]
\[ \left. + f(t, x, r, \sigma(t, x, v)p, v) + \alpha \Phi(t, x, r, \sigma(t, x, v)p) \right\} \]
\[ = \lim_{\alpha \to \infty} \left\{ q + \frac{1}{2} \text{trace}(\sigma \sigma^T(t, x, v^\alpha)S) + \langle \mu(t, x, v^\alpha), p \rangle \right. \]
\[ \left. + f(t, x, r, \sigma(t, x, v^\alpha)p, v^\alpha) + \alpha \Phi(t, x, r, \sigma(t, x, v^\alpha)p) \right\} \]
\[ = \inf_{v \in U_0(t, x)} \left\{ q + \frac{1}{2} \text{trace}(\sigma \sigma^T(t, x, v)S) + \langle \mu(t, x, v), p \right\)
which verifies the assertion that $F^\alpha(t, x, r, q, p, S) \downarrow F(t, x, r, q, p, S)$ when $\alpha \to \infty$.

Now we claim that: if the function $u^\alpha$ is the viscosity solution of the equation $F^\alpha = 0$ then (i) the function $u$ is a viscosity supersolution of the equation $F = 0$, and that (ii) $\bar{u}$ is a viscosity subsolution of the equation $\bar{F} = 0$. For these, on account of Lemma 5.8 we have to prove that (i) $\bar{u} = u$, $\bar{u} > -\infty$, $\bar{F} = F$; and that (ii) $\bar{u} = u^*$, $\bar{u} < \infty$, $\bar{F} = F_\eta$.

For the assertion (i), note that $u^\alpha$ converges increasingly to $u$ when $\alpha \to \infty$. Thus we have that

$$
\inf \left\{ u^\alpha(s, y) : \alpha \geq j, |s - t| + \|y - x\| \leq \frac{1}{j} \right\}
= \inf \left\{ u^j(s, y) : |s - t| + \|y - x\| \leq \frac{1}{j} \right\}
\leq u^j(t, x)
\leq u(t, x),
$$

and after taking the limit as $j$ tends to infinity we get that $\bar{u}(t, x) \leq u(t, x)$. But on the other hand, by virtue of the continuity of $u^\alpha$ we know that for every $\epsilon > 0$ there exists $j' \geq j$ such that

$$
u^j(t, x) - \epsilon
< \inf \left\{ u^j(s, y) : |s - t| + \|y - x\| \leq \frac{1}{j} \right\}
\leq \inf \left\{ u^{j'}(s, y) : |s - t| + \|y - x\| \leq \frac{1}{j'} \right\}.
$$

So by letting $j$, thus $j'$, goes to infinity it follows that

$$
u(t, x) - \epsilon < \bar{u}(t, x),
$$

and since $\epsilon$ is arbitrary we actually obtain that

$$
u(t, x) \leq \bar{u}(t, x).
$$

Hence it holds that $u(t, x) = \bar{u}(t, x)$. 

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Clearly one has for $(t,x) \in [0,T] \times \mathbb{R}^n$ that

\[
\begin{align*}
y(t,x) &= \lim_{j \to \infty} \inf \left\{ u^\alpha(s,y) : \alpha \geq j, \ |s-t| + \|y-x\| \leq \frac{1}{j} \right\} \\
&\geq \inf \{ u^1(s,y) : |s-t| + \|y-x\| \leq 1 \} \\
&> -\infty,
\end{align*}
\]

since $u^1$ is continuous on the compact set $\{(s,y) : |s-t| + \|y-x\| \leq 1\}$ thus is bounded on it.

For the assertion (ii), we have that

\[
\begin{align*}
\sup \left\{ u^\alpha(s,y) : \alpha \geq j, \ |s-t| + \|y-x\| \leq \frac{1}{j} \right\} &= \sup \left\{ u(s,y) : |s-t| + \|y-x\| \leq \frac{1}{j} \right\}.
\end{align*}
\]

Thus it follows that $\bar{u}(t,x) = u^*(t,x)$ when $j \to \infty$.

For all $(t,x) \in [0,T] \times \mathbb{R}^n$ since $u^*$ is upper semicontinuous we have that

\[
u^*(t,x) \leq u(t,x) < \infty.
\]

In light of the lower semicontinuity of $F^\alpha$ and the fact that $F^\alpha \downarrow F$ when $\alpha \to \infty$, $\bar{F} = F$ and $\bar{F} = F^\alpha$ follow similarly as $\bar{u} = u$ and $\bar{u} = u^*$.

Hence all the conditions required for Lemma 5.8 are satisfied thus our claims are proven. \(\square\)

### 5.3 Mean curvature flows

As an example we apply Proposition 5.9 to the level set equation of the mean curvature flow. To do that, let us consider a simple case of our result, namely, when

\[
f(t,x,r,z,v)|_{z=0} = 0, \ a.e., \quad (5.11)
\]

\[
\Phi(t,x,r,z) = |z|, \ a.s., \ a.e.
\]

for the HJB equation (5.9), or

\[
f(s,X^t,x,v_s,Y^t,x,v_s,Z^t,x,v_s,v)_{|Z^t,x,v_s=0} = 0, \ a.s. \ a.e.,
\]

for the HJB equation (5.9), or

\[
f(s,X^t,x,v_s,Y^t,x,v_s,Z^t,x,v_s,v)_{|Z^t,x,v_s=0} = 0, \ a.s. \ a.e.,
\]
\[
\Phi(s, X_{s}^{t,x,v}, Y_{s}^{t,x,v}, Z_{s}^{t,x,v}) = |Z_{s}^{t,x,v}|, \ a.s., \ a.e.
\]

where \( s \in [t, T] \) for the FBSDE (5.6). We have the following lemma.

**Lemma 5.10** Suppose that \( 0 \leq t \leq T \), and the r.v. \( \xi \in L^2(\mathcal{F}_T) \), then the following BSDE

\[
-dY_t = dA_t, \\
Y_T = \xi,
\]

has the smallest supersolution \((A, Y, Z)\). Here

\[
Y_t = \text{ess sup}_\omega \xi, \ 0 \leq t < T, \\
Y_T = \xi, \ t = T.
\]

The increasing process \( A \) is

\[
A_t = 1_{(t-T)}(t)(\text{ess sup}_\omega \xi - \xi).
\]

And \( Z_t = 0 \) for \( t \in [0, T] \).

**Proof.** Obviously the above \((Y, A)\) satisfy the equation (5.12). Now suppose in the contrary that \( Y \) is not the smallest supersolution, that is to say, there exist \((\bar{Y}, \bar{A})\) satisfying (5.12), and exist \( t_0 < T \) and \( \delta > 0 \), such that

\[
\bar{Y}_{t_0} < Y_{t_0} - \delta, \ a.s.. \\
\]

Then,

\[
\xi + \bar{A}_T - \bar{A}_{t_0} < \xi + A_T - A_{t_0} - \delta, \ a.s.,
\]

or

\[
\bar{A}_T - \bar{A}_{t_0} < A_T - A_{t_0} - \delta, \ a.s..
\]

And since \( \bar{A} \) is an increasing process, we have \( 0 \leq \bar{A}_T - \bar{A}_{t_0}, \ a.s., \) thus

\[
0 < \text{ess sup}_\omega \xi - \xi - \delta, \ a.s.,
\]

\( \S 5.3 \cdot 81 \cdot \)
or
\[ \xi < \text{ess sup}_\omega \xi - \delta, \text{a.s..} \]
Now apply the essential supremum w.r.t. \( \omega \in \Omega \) to both sides of the above inequality and then we obtain that
\[ \text{ess sup}_\omega \xi < \text{ess sup}_\omega \xi - \delta, \]
i.e.,
\[ 0 < -\delta, \]
which is a contradiction. Hence \((Y, A)\) is the smallest supersolution of the BSDE (5.12).
\( \square \)

In the classical setup of the evolution of a hypersurface by its mean curvature, we are given initially a smooth connected hypersurface \( \Gamma_0 \) which is the boundary of a bounded open subset \( S_0 \) in \( \mathbb{R}^n \). As time progresses we allow the hypersurface to evolve, by moving each point in the opposite direction of the mean curvature vector, at a velocity equal to \((n-1)\) times the absolute value of the mean curvature at that point. Assuming this evolution exists, we thereby define for each \( t > 0 \) a new hypersurface \( \Gamma_t \).

Then the primary task is to study the geometric properties of \( \{\Gamma_t\}_{t \geq 0} \). We know that there is a bounded uniformly continuous function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) such that
\[ \Gamma_0 = \{x \in \mathbb{R}^n : g(x) = 0\}, \]
and
\[ S_0 = \{x \in \mathbb{R}^n : g(x) < 0\}, \mathbb{R}^n \setminus \overline{S_0} = \{x \in \mathbb{R}^n : g(x) > 0\}. \]

Suppose temporarily \( u : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a smooth function whose spatial gradient \( Du \) does not vanish in some open region of \( \mathbb{R}^n \times (0, \infty) \). We regard the level sets
\[ \Gamma_t := \{x \in \mathbb{R}^n : u(t, x) = 0\} \]
as the evolution of \( \Gamma_0 \) at time \( t \geq 0 \). Denote
\[ S_t := \{x \in \mathbb{R}^n : u(t, x) < 0\}. \]
Let \( \nu = \nu(t, x) \) be a smooth unit normal vector field to \( \Gamma_t \) then

\[
- \frac{1}{n-1} \text{div}(\nu)\nu
\]

is the mean curvature vector field. Thus for fixed \((t, x)\) the point \(x\) evolves according to the ordinary differential equation (ODE)

\[
\dot{x}(s) = -[\text{div}(\nu)(s, x(s)), \ s > t, \quad \ (5.15) \\
x(t) = x.
\]

As \(x(s) \in \Gamma_s\), we have \(u(s, x(s)) = 0\). Thus

\[
0 = \frac{d}{ds}u(s, x(s)) = -[(Du, \nu)\text{div}(\nu)](s, x(s)) + \partial_t u(s, x(s)).
\]

Setting \(s := t\) we discover that at \((t, x)\)

\[
\partial_t u = (Du, \nu)\text{div}(\nu).
\]

Choosing

\[
\nu := \frac{Du}{|Du|}
\]

it follows that

\[
\partial_t u = |Du|\text{div}\left( \frac{Du}{|Du|} \right),
\]

which is the level set equation of the mean curvature flow \((5.16)\) as follows.

\[
\partial_t u(t, x) = \Delta u(t, x) - \frac{(D^2 u(t, x)Du(t, x), Du(t, x))}{|Du(t, x)|^2}
\]

\[
= \sum_{i,j=1}^{n}(\delta_{ij} - u_{x_i}u_{x_j}/|Du|^2)u_{x_i,x_j} \text{ in } \mathbb{R}^n \times (0, \infty),
\]

\[
u = g \text{ on } \mathbb{R}^n \times \{t = 0\}.
\]

Conversely, assume \(u\) is a smooth of \((5.16)\) in some region with \(Du\) nonvanishing. Fix \((t, x)\) and then solve the ODE \((5.15)\). Since \(u\) solves \((5.16)\) we deduce as above that, for \(s > t\),

\[
u(s, x(s)) = 0.
\]

Consequently the level sets of \(u\) evolve according their mean curvature.
If the initial condition is replaced by a terminal condition the equation (5.16) turns out to be

\[- \partial_t u(t, x) = \sum_{i,j=1}^{n} (\delta_{ij} - u_{x_i}u_{x_j}) |Du|^2 u_{x_i x_j} \text{ in } \mathbb{R}^n \times [0, T), \quad (5.17)\]

\[u = g \text{ on } \mathbb{R}^n \times \{t = T\}.\]

Observe that (5.17) can also be written in the form of the following generalised HJB equation.

\[- \partial_t u(t, x) = \inf_{v \in U_0(t, x)} \frac{1}{2} \trace \left\{ \left[ \sqrt{2}(I_n - vv^*) \right]^2 D^2 u \right\} \text{ in } \mathbb{R}^n \times [0, T), \quad (5.18)\]

\[u = g \text{ on } \mathbb{R}^n \times \{T\},\]

where

\[U_0(t, x) := \{v \in \mathbb{R}^n : |v| = 1, \sigma(v)Du(t, x) := \sqrt{2}(I_n - vv^*)Du(t, x) = 0\}. \quad (5.19)\]

In fact, from

\[\sigma Du = \sqrt{2}(I_n - vv^*)Du = 0\]

we learn that

\[Du = \langle Du, v \rangle v,\]

that is to say \(v\) is parallel to \(Du\). Since \(|v| = 1\) we must have

\[v = \pm \frac{Du}{|Du|}.\]

Thus we see that the equation (5.18) is actually the equation (5.17).

**Corollary 5.11** Let \(0 \leq t \leq T\) and \(s \in [t, T]\). Suppose that \(Y^{t,x,v}\) is the smallest supersolution of the following FBSDE.

\[dX^{t,x,v}_s = \sqrt{2}(I_n - v_s v_s^*)dW_s, \quad (5.20)\]

\[X^{t,x,v}_t = x,\]

\[-dY^{t,x,v}_s = dA_s, \quad (5.21)\]

\[Y^{t,x,v}_T = g(X^{t,x,v}_T).\]
where \( v(\cdot) \in \mathcal{U} \), and
\[
\mathcal{U} := \{ v(\cdot) : |v_s| = 1, \; \forall s \geq 0 \}. \tag{5.22}
\]

Then
\[
u(t, x) := \inf_{v(\cdot) \in \mathcal{U}} Y_{t,x}^{t,v} \tag{5.23}
\]
is the unique weak viscosity solution of the level set equation of the mean curvature flow (5.17).

**Proof.** By virtue of Lemma 5.10 when \( \xi = g(X_T^{t,x,v}) \) the smallest supersolution of the FBSDE (5.20), (5.21) is constructed as follows.
\[
Y_{t,x,v}^s = \esssup_{\omega} g(X_T^{t,x,v}), \; t \leq s < T, \tag{5.24}
\]
\[
Y_{t,x,v}^T = g(X_T^{t,x,v}), \; s = T.
\]

By our main result Proposition 5.9 we have immediately that (5.23) is a viscosity solution of the level set equation of the mean curvature flow (5.17).

The uniqueness of the viscosity solution was first achieved in (Chen, Giga, & Goto, 1991; Evans & Spruck, 1991).

**Remark 5.12** The function (5.23) is actually
\[
u(t, x) = \inf_{v \in \mathcal{U}} \esssup_{\omega} g(X_T^{t,x,v}).
\]

This probabilistic representation of the mean curvature flow (5.17) was first provided in (Buckdahn, Cardaliaguet, & Quincampoix, 2002; Soner & Touzi, 2002b).
Chapter 6

Finite Time Existence of Mean Curvature Flows

Here we give a proof that the existence of the mean curvature flow is of finite time.

In the classical setup of the evolution of a hypersurface by its mean curvature, we are given initially a smooth connected hypersurface $\Gamma_0$ which is the boundary of a bounded open subset $S_0$ in $\mathbb{R}^n$. As time progresses we allow the hypersurface to evolve, by moving each point in the opposite direction of the mean curvature vector, at a velocity equal to $(n-1)$ times the absolute value of the mean curvature at that point. Assuming this evolution exists, we thereby define for each $t > 0$ a new hypersurface $\Gamma_t$. Then the primary task is to study the geometric properties of $\{\Gamma_t\}_{t \geq 0}$.

We know that there is a bounded uniformly continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Gamma_0 = \{ x \in \mathbb{R}^n : g(x) = 0 \},$$

and

$$S_0 = \{ x \in \mathbb{R}^n : g(x) < 0 \}, \quad \mathbb{R}^n \setminus \overline{S_0} = \{ x \in \mathbb{R}^n : g(x) > 0 \}.$$

Define the evolution of $\Gamma_0$ at time $t$ as

$$\Gamma_t := \{ x \in \mathbb{R}^n : u(t, x) = 0 \}.$$
for some unknown function \( u : [0, \infty) \times \mathbb{R}^n \to \mathbb{R} \), and also denote

\[
S_t := \{ x \in \mathbb{R}^n : u(t, x) < 0 \}.
\]

Then we recall the level set equation of the mean curvature flow \((5.16)\) as follows.

\[
\partial_t u = \Delta u - \frac{(D^2 u D u, D u)}{|D u|^2}
= \sum_{i,j=1}^n (\delta_{ij} - u_{x_i} u_{x_j} / |D u|^2) u_{x_i x_j}
\text{ in } (0, \infty) \times \mathbb{R}^n,
\]

\[
u = g \text{ on } \{0\} \times \mathbb{R}^n.
\]

Given a complete stochastic basis \((\Omega, \mathcal{F}, \mathbb{P})\) with a standard Brownian motion \(\{B_t\}_{t \geq 0}\) adapted to the natural filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), let \(s \geq 0\) and consider the following controlled SDE.

\[
dX_{s}^{0, x, v} = \sqrt{2} (I_n - v_s v_s^T) dB_s,
\]

\[
X_{0}^{0, x, v} = x \in \mathbb{R}^n,
\]

where

\[
v(\cdot) \in \mathcal{U} := \{ v(\cdot) : \forall s \geq 0, v_s \in \mathbb{R}^n, |v_s| = 1 \},
\]

the control process taking values in the \((n - 1)\)-dimensional sphere \(S^{n-1}\). In light of Corollary \([5.11]\) we immediately have the following lemma by a simple time change between \(t\) and \(T - t\).

**Lemma 6.1** Define from \((6.2)\)

\[
u(t, x) := \inf_{v(\cdot) \in \mathcal{U}} \text{esssup}_{\omega \in \Omega} g(X_{t}^{0, x, v}).
\]

Then \(\nu\) is the unique viscosity solution of the level set equation of the mean curvature flow \((6.1)\).

Now we state the main result.

**Proposition 6.2** The mean curvature evolution of a smooth connected hypersurface \(\Gamma_0\), which is the boundary of a bounded open subset \(D_0 \subset \mathbb{R}^n\), exists only for finite time. That is, there exists some \(T > 0\) such that \(\Gamma_t = \emptyset\), the empty set, for all \(t \geq T\).
Proof. Without confusion we will suppress the superscripts of $X_{t}^{0,x,v}$. First it is easy to check that for every $x \in \mathbb{R}^{n}$,

$$\lim_{t \to \infty} \mathbb{E}|X_{t}|^{2} = \infty.$$ 

In fact, applying the Itô’s Lemma we obtain that

$$d|X_{t}|^{2} = 2X_{t}\sqrt{2}(I_{n} - v_{s}v_{s}^{T})dB_{s} + \frac{1}{2}\text{trace}\left\{ \left[ \sqrt{2}(I_{n} - v_{s}v_{s}^{T}) \right]^{2} \right\} ds$$

$$= 2X_{t}\sqrt{2}(I_{n} - v_{s}v_{s}^{T})dB_{s} + (n - 1)ds.$$ 

Thus

$$\mathbb{E}|X_{t}|^{2} = |x|^{2} + (n - 1)t \to \infty, \text{ when } t \to \infty.$$ 

But we know that for any r.v. $\xi : \Omega \to \mathbb{R}$, $|\xi|_{L^{p}} \uparrow |\xi|_{L^{\infty}}$ when $p \to \infty$. So,

$$(\mathbb{E}|X_{t}|^{2})^{1/2} \leq \text{ess sup}_{\omega \in \Omega} X_{t}$$

and then

$$\lim_{t \to \infty} \text{ess sup}_{\omega \in \Omega} X_{t} = \infty.$$ 

It implies that there exists $\hat{\Omega} \subset \Omega$ with $\mathbb{P}(\hat{\Omega}) > 0$ such that

$$\lim_{t \to \infty} |X_{t}| = \infty \text{ on } \hat{\Omega}.$$ 

For $\omega \in \hat{\Omega}$, since $D_{0}$ is bounded $X_{t}(\omega)$ will surely travel into $\mathbb{R}^{n} \setminus D_{0}$ for $t$ sufficiently large. Thus by the beforehand selection of $g$ we have

$$\lim_{t \to \infty} g(X_{t}) > 0.$$ 

And as $g$ is a bounded function we utilise the dominated convergence theorem to deduce that

$$\lim_{t \to \infty} \inf_{v(\cdot) \in U} \text{ess sup}_{\omega \in \Omega} g(X_{t})$$

$$= \lim_{t \to \infty} \inf_{v(\cdot) \in U} (\mathbb{E}|g(X_{t})|^{p})^{1/p}$$

$$= \inf_{v(\cdot) \in U} (\mathbb{E}(\lim_{t \to \infty} |g(X_{t})|^{p})^{1/p}$$
\[ \inf_{v(\cdot) \in \mathcal{U}} \, \text{ess sup}_{\omega \in \Omega} \lim_{t \to \infty} g(X_t) \geq 0. \]

That means that there exist a \( T > 0 \) such that for all \( t \geq T \),

\[ \inf_{v(\cdot) \in \mathcal{U}} \, \text{ess sup}_{\omega \in \Omega} g(X_t) > 0. \]

Hence \( \Gamma_t = \emptyset \) for all \( t \geq T \). \( \square \)

**Remark 6.3** The classical PDE approach to prove the finite time existence of the mean curvature flow is to deduce a comparison theorem between the hypersurface \( \Gamma_t \) and a sphere whose interior contains it \([\text{Evans \\ & Spruck, 1991}]\).
Bibliography


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