The index of Farey sequence

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by

R. R. Hall and P. Shiu

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R. R. Hall
Department of Mathematics
York University
Heslington
York YO10 5DD
United Kingdom
Email: rrh1@york.ac.uk

P. Shiu
Department of Mathematical Sciences
Loughborough University
Loughborough
Leicestershire LE11 3TU
United Kingdom
Email: P.Shiu@lboro.ac.uk
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Abstract

We define an integer valued function of the Farey fractions which we call the index, and we prove two exact formulae involving this. We give asymptotic formulae for the second moment of the index and for the value distribution. Zagier communicated to us a remarkable formula relating the index to Dedekind sums and this yields further asymptotic formulae.

1. Introduction and statement of results

Let $\mathcal{F}_N = \{x_i, i = 1, 2, \ldots, R\}$ denote the Farey sequence of order $N$; here $\frac{1}{N} = x_1 < x_2 < \cdots < x_R = 1$ and

$$R = R_N = \sum_{1 \leq a \leq N} \phi(a) = \frac{3N^2}{\pi^2} + O(N \log N). \quad (1.1)$$

The sequence $(x_i)$ may be extended onto $\mathbb{Z}$ by defining $x_{i+R} = x_i + 1$ for all $i$. We suppose that $x_i = b/s$, and that the adjacent fractions are

$$x_{i-1} = \frac{a}{r} \quad \text{and} \quad x_{i+1} = \frac{c}{t},$$

we write $r = r(x_i)$, $s = s(x_i)$ and $t = t(x_i)$.

Definition. We define the index of the fraction $x_i$ as

$$\nu(x_i) := \frac{r + t}{s} = \frac{a + c}{b}. \quad (1.2)$$

Thus $\nu(x_i)$ is an integer because $br - as = cs - bt = 1$. In particular we have $\nu(x_1) = 1$ and $\nu(x_R) = 2N$. We are interested in some properties of the index, which is a periodic function on the extended Farey sequence $\{x_i : i \in \mathbb{Z}\}$.

There are two formulae for the index, expressing it as a function of $N, s$ and $r$, or of $N, s$ and $b$. For the first formula we recall from Hall and Tenenbaum [3] that

$$t = s \left[ \frac{N + r}{s} \right] - r. \quad (1.3)$$

We remark that recently Boca, Cobeli and Zaharescu [1] have made some very interesting applications of (1.3). It yields immediately

$$\nu(x_i) = \left[ \frac{N + r}{s} \right] \quad (1.4)$$

and since $r > N - s$ we see that

$$\left[ \frac{2N + 1}{s} \right] - 1 \leq \nu(x_i) \leq \left[ \frac{2N}{s} \right]. \quad (1.5)$$
It follows that if \( s | 2N + 1 \) then \( \nu(x_i) = \lfloor 2N/s \rfloor \), otherwise the index may take the two values \( \lfloor 2N/s \rfloor \) and \( \lfloor 2N/s \rfloor - 1 \). We refer to these as the upper and lower values of the index. As an example, we give a table for the indices of \( \mathcal{F}_9 \); here \( R = 28 \) and the index is symmetric, that is \( \nu(x_{R-i}) = \nu(x_i) \), so that we need only give the first 15 terms:

\[
\nu(x_i) = 1, 2, 2, 3, 1, 4, 1, 5, 5, 3, 2, 13, 9, 1.
\] (1.6)

The lower values have been underlined, and we remark that 1 is always a lower value, since \( \lfloor 2N/s \rfloor \geq 2 \), and that, as in the case \( s = 5 \) here, the index may be single-valued when \( s \) does not divide \( 2N + 1 \).

For the second formula for the index, we let \( \bar{b} \) and \( n \) be such that \( 1 \leq \bar{b} < s \), \( \bar{b} \equiv 1 \pmod{s} \) and \( 0 \leq n < s \), \( N \equiv n \pmod{s} \). Then \( r \equiv \bar{b} \pmod{s} \) giving \( r = ps + \bar{b} \) with

\[
p = \left\lfloor \frac{N}{s} \right\rfloor + \left\lfloor \frac{n - \bar{b}}{s} \right\rfloor.
\]

Similarly \( t = qs - \bar{b} \) with

\[
q = \left\lfloor \frac{N}{s} \right\rfloor + \left\lfloor \frac{n + \bar{b}}{s} \right\rfloor,
\]

so that

\[
\nu(x_i) = p + q = 2 \left\lfloor \frac{N}{s} \right\rfloor + \left\lfloor \frac{n - \bar{b}}{s} \right\rfloor + \left\lfloor \frac{n + \bar{b}}{s} \right\rfloor.
\] (1.7)

The second and third terms on the right of (1.7) can take the values \(-1, 0\) and \(0, 1\), respectively. Their sum can take the values \(0, \pm 1\), but not both the values \(\pm 1\).

Our investigation was initiated by one of us making a numerical observation while walking in a park. The observation led us to

**Theorem 1.** For all \( N \), we have

\[
\sum_{i=1}^{R} \nu(x_i) = 3R - 1.
\] (1.8)

We need to consider the frequency of the upper and the lower values of the index and this leads us to another exact formula.

**Definition.** The deficiency \( \delta(s) \) is the number of fractions \( x_i \in \mathcal{F}_N \) with denominator \( s \) such that \( \nu(x_i) \) takes its lower value.

**Theorem 2.** For all \( N \), we have

\[
\sum_{s=1}^{N} \delta(s) = N(2N + 1) - R_{2N} - 2R + 1.
\] (1.9)

Thus, for \( N = 9 \), the right-hand side of (1.9) has the value \( 171 - 102 - 56 + 1 = 14 \), which is in agreement with the table in (1.6). An immediate corollary of Theorem 2 is that the number of lower values is \( \sim (2\pi^2/3 - 6)R \).
The constant here is -57973..., so that the probability that $\nu(x_i)$ takes its lower value is rather more than $\frac{1}{2}$.

Of course there are quite a few indices taking the necessarily lower value 1.

A slightly more difficult result, which in the present treatment requires some analytic number theory, is

**Theorem 3.** For all $N$, we have

$$Z(N) := \sum_{i=1}^{R} \nu(x_i)^2 = \frac{24}{\pi^2}N^2 \left( \log 2N - \frac{\zeta'(2)}{\zeta(2)} - \frac{17}{8} + 2\gamma \right) + O(N \log^2 N). \quad (1.10)$$

We may enquire about the frequency with which $\nu(x_i)$ takes the value $k$. We define

$$F(N, k) := \sum_{\nu(x_i) = k} 1 = L(N, k) + U(N, k), \quad (1.11)$$

where $L(N, k)$ and $U(N, k)$ count, respectively, the number of occurrences of $k$ as a lower and upper value.

**Theorem 4.** For all $N$ we have, uniformly for $k \in \mathbb{N}$, that

$$L(N, k) = \ell_k R + O(k + \frac{N}{k} \log N), \quad U(N, k) = u_k R + O(k + \frac{N}{k} \log N), \quad (1.12)$$

in which

$$\ell_k = 4 \left( \frac{1}{(k+1)^2} - \frac{1}{k+1} + \frac{1}{k+2} \right), \quad k \geq 1, \quad \ell_1 = \frac{1}{3}, \quad \ell_k = 4 \left( \frac{1}{k} - \frac{2}{k+1} - \frac{1}{k+2} \right), \quad k \geq 2. \quad (1.13)$$

It follows at once that

$$F(N, k) = f_k R + O(k + \frac{N}{k} \log N), \quad (1.15)$$

where

$$f_1 = \frac{1}{3}, \quad f_k = 4 \left( \frac{1}{k} - \frac{2}{k+1} - \frac{1}{k+2} \right), \quad k \geq 2. \quad (1.16)$$

These results are useful only when $k^2 < N/\log N$, but we also have, in any case, that

$$\sum_{k \geq k} F(N, h) \leq \frac{4}{k^2} \left( 1 + O\left( \frac{\log N}{N} \right) \right) R \log R. \quad (1.17)$$

We next consider the partial sums of the index. One definition which seems appropriate is

$$D_j = D_j(N) := \sum_{i=0}^{j} \left( \nu(x_i) - 3 \right) + \frac{1}{2}, \quad (1.17)$$

where the star indicates that the end terms of the sum are each halved. For example, $D_1 = \frac{1}{2}(2N - 3) + \frac{1}{2}(1 - 3) + \frac{1}{2} = N - 2$. Notice that $D_j$ is odd, in the sense that $D_{R-j} = -D_j$. We were surprised to find in our numerical trials that $|D_j|$ seemed never to exceed $N - 2$ and apparently was much smaller than this on
average. The explanation lies in the following remarkable theorem which has been communicated to us by Don Zagier.

**Theorem 5 (D. Zagier).** We have

$$D_j = D(b, s) + \frac{t - r}{2s} + \frac{1}{2} - \frac{b}{s}, \quad (1.18)$$

where \(x_j = b/s\) and \(D(b, s)\) is 12 times Dedekind’s sum, that is

$$D(b, s) = 12 \sum_{\ell (\text{mod} \ s)} B_1\left(\frac{\ell}{s}\right) B_1\left(\frac{b\ell}{s}\right), \quad (1.19)$$

with

$$B_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

We give our proof, which is by induction on \(j\); it is merely a verification of the formula (1.18), and therefore does not explain how Zagier found the identity. The reader will find this secret, and much more information, in [8]. In an earlier version of our paper we had various conjectures which are now corollaries of Zagier’s theorem.

**Theorem 6.** We have \(|D_j| \leq N - 2\), with equality if and only if \(j = 1\) or \(j = R - 1\).

**Theorem 7.** We have

$$\sum_{j=1}^{R} D_j^2 = \frac{5\zeta(4)}{3\zeta(3)^2} N^3 + O(N^{3/2} \log^2 N). \quad (1.20)$$

**Theorem 8.** We have

$$\sum_{j=1}^{R} |D_j| \leq 2R \log^2 N + O(R). \quad (1.21)$$

In an earlier version of our paper, we also made following

**Conjecture.** There exists a function \(A : \mathbb{N} \to \mathbb{R}^+\) such that, for each fixed \(h \in \mathbb{N}\), we have

$$\nu(x_i)\nu(x_{i+h}) \sim A(h)R, \quad N \to \infty.$$
2. Proofs of Theorems 1, 2, 3 and 4

Theorems 1, 2 and 4 are entirely elementary, albeit Theorem 1 was not proved in the park. Theorem 3 is elementary except for our estimate of the sum appearing in (2.16) below, for which we require contour integration and the functional equation for the Riemann zeta-function. We may have overlooked something here and we should be interested to discover an elementary treatment of this sum.

Proof of Theorem 1. We write

\[ T(s) := \sum_{x_i = s} \nu(x_i) = \frac{1}{s} \sum (r + t) = \frac{2}{s} \sum r. \]  

(2.1)

The sum on the right is

\[ \sum_{N-s < r < N} \sum_{d|s} \mu(d) \sum_{d|n} n \]

\[ = \frac{1}{2} \sum_{d|s} \mu(d) \left( \left\lfloor \frac{N}{d} \right\rfloor^2 + \left\lfloor \frac{N}{d} \right\rfloor \left( \left\lfloor \frac{N}{d} \right\rfloor - \frac{s}{d} \right)^2 \left( \left\lfloor \frac{N}{d} \right\rfloor - \frac{s}{d} \right) \right) \]

\[ = \frac{1}{2} \sum_{d|s} d \mu(d) \left( \frac{2s}{d} \left\lfloor \frac{N}{d} \right\rfloor - \frac{s^2}{d^2} + \frac{s}{d} \right) \]

so that

\[ T(s) = 2 \sum_{d|s} \mu(d) \left\lfloor \frac{N}{d} \right\rfloor - \phi(s) + \epsilon(s), \]  

(2.2)

in which it is understood that \( \epsilon(1) = 1 \) and \( \epsilon(s) = 0 \) for \( s > 1 \). It follows that

\[ \sum_{i=1}^{R} \nu(x_i) = \sum_{s \leq N} T(s) = 2 \sum_{d \leq N} \mu(d) \left\lfloor \frac{N}{d} \right\rfloor^2 - R + 1, \]

where we have used the formula (1.1) for \( R \), and we remark that another formula is

\[ R = \frac{1}{2} \sum_{d \leq N} \mu(d) \left( \left\lfloor \frac{N}{d} \right\rfloor + 1 \right). \]

The required result (1.8) now follows from the fact that \( \sum_{d \leq N} \mu(d) \left[ \frac{N}{d} \right] = 1. \)

Proof of Theorem 2. Let \( s = s(x_i) \), and recall that \( \nu(x_i) \) takes at most two values \([2N/s]\) and \([2N/s] - 1\), the latter \( \delta(s) \) times. Hence the expression \( T(s) \) in (2.2) is given by

\[ T(s) = (\phi(s) - \delta(s)) \left\lfloor \frac{2N}{s} \right\rfloor + \delta(s) \left( \left\lfloor \frac{2N}{s} \right\rfloor - 1 \right) = \phi(s) \left\lfloor \frac{2N}{s} \right\rfloor - \delta(s). \]  

(2.3)

Applying Theorem 1, we find that

\[ \sum_{s \leq N} \delta(s) = \sum_{s \leq N} \phi(s) \left\lfloor \frac{2N}{s} \right\rfloor - 3R + 1, \]  

(2.4)

and since \( \left\lfloor \frac{2N}{s} \right\rfloor = 1 \) throughout the range \( N < s \leq 2N \), we may rewrite this as

\[ \sum_{s \leq N} \delta(s) = \sum_{s \leq 2N} \phi(s) \left\lfloor \frac{2N}{s} \right\rfloor - R_{2N} - 2R + 1. \]

The sum on the right-hand side is

\[ \sum_{s \leq 2N} \phi(s) \sum_{n \leq 2N \text{ (mod } s\text{)}} 1 = \sum_{n \leq 2N} \sum_{s|n} \phi(s) = \sum_{n \leq 2N} n = N(2N + 1), \]

so that the required result (1.9) follows from (2.3).
Proof of Theorem 3. We put

\[ V(s) := \sum_{x_i = s} \nu(x_i)^2, \]  

and we have

\[ V(s) = (\phi(s) - \delta(s)) \left( \left[ \frac{2N}{s} \right] \right)^2 + \delta(s) \left( \frac{2N}{s} - 1 \right)^2 = \phi(s) \left( \left[ \frac{2N}{s} \right] \right)^2 - \delta(s) \left( \frac{2N}{s} - 1 \right). \]

We write

\[ X_N := \sum_{s \leq N} \phi(s) \left( \frac{2N}{s} \right)^2, \]  

\[ Y_N := \sum_{s \leq N} \delta(s) \left( \frac{2N}{s} \right), \]

so that, by Theorem 2,

\[ \sum_{s \leq N} V(s) = X_N - 2Y_N + N(2N + 1) - R_{2N} - 2R + 1. \]  

(2.8)

Extending the range from \(1 \leq s \leq N\) to \(1 \leq s \leq 2N\) as in the proof of Theorem 2, we find that

\[ X_N = \sum_{s \leq 2N} \phi(s) \left( \frac{2N}{s} \right)^2 - R_{2N} + R \]

\[ = \sum_{s \leq 2N} \phi(s) \left( \left[ \frac{2N}{s} \right] \right)^2 \left( \frac{2N}{s} \right) + 1 - N(2N + 1) - R_{2N} + R. \]  

(2.9)

The sum in (2.9) is

\[ 2 \sum_{s \leq 2N} \frac{\phi(s)}{s} \sum_{n \leq 2N, n \equiv 0 \pmod{s}} n = 2 \sum_{n \leq 2N} n f(n), \]  

(2.10)

where

\[ f(n) := \sum_{s \mid n} \frac{\phi(s)}{s}. \]  

(2.11)

Assembling (2.8), (2.9) and (2.10), the sum \(Z(N)\) in the theorem becomes

\[ Z(N) = 2 \sum_{n \leq 2N} n f(n) - 2Y_N - 2R_{2N} - R + 1. \]  

(2.12)

We now turn our attention to the sum \(Y_N\) in (2.7), which we are unable to evaluate exactly. We recall from (2.3) and (2.2) that

\[ \delta(s) = \phi(s) \left( \left[ \frac{2N}{s} \right] \right) - T(s) = \phi(s) \left( \left[ \frac{2N}{s} \right] + 1 \right) - 2 \sum_{d \mid s} \mu(d) \left( \frac{N}{d} \right) - \epsilon(s) \]

so that

\[ \delta(s) = \phi(s) \left( \left[ \frac{2N}{s} \right] + 1 \right) - \frac{2N \phi(s)}{s} + O(\tau(s)), \]  

(2.13)
where $\tau$ is the divisor function. From (2.7) and (2.13), we now have

$$Y_N = \sum_{s \leq N} \phi(s) \left( \left\lfloor \frac{2N}{s} \right\rfloor \left( \left\lfloor \frac{2N}{s} \right\rfloor + 1 \right) - 2N \right) \phi(s) \left( \left\lfloor \frac{2N}{s} \right\rfloor \right) + O(N \log^2 N),$$

(2.14)
in which our largest error term arises. Extending the range of the sums here, we find that

$$Y_N = 2 \sum_{n \leq 2N} n f(n) - 2N \sum_{n \leq 2N} f(n) = \frac{6N^2}{\pi^2} + O(N \log^2 N).$$

(2.15)

Inserting this into (2.12) yields

$$Z(N) = 2 \sum_{n \leq 2N} (2N - n) f(n) - \frac{15N^2}{\pi^2} + O(N \log^2 N),$$

(2.16)

and it remains to consider the sum here.

In the following, it will be convenient to let the letters $s, \sigma, t$ and $T$ be the usual symbols used in the theory of the Riemann zeta-function. The Dirichlet series for $f(n)$ in (2.11) is given by

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(s+1)}.$$

(2.17)

Employing

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} ds = \max(x - 1, 0), \quad x > 0,$$

(2.18)

we find that

$$\sum_{n \leq 2N} (2N - n) f(n) = \sum_{n=1}^{\infty} \max \left( \frac{2N}{n} - 1, 0 \right) n f(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{(2N)^{s+1} \zeta^2(s)}{s(s+1)\zeta(s+1)} ds.$$

(2.19)

The integrand has a removable singularity at $s = 0$, and we move the line of integration to the contour $C$ comprising the five line segments $s = 2 + it$ ($|t| \geq T$), $s = \sigma \pm iT$ ($0 < \sigma \leq 2$), $s = it$ ($-T \leq t \leq T$). The residue of the integrand at the pole $s = 1$ is given by

$$\frac{12N^2}{\pi^2} \left( \log 2N - \frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} + 2\gamma \right),$$

(2.20)

and we proceed to estimate the integral along our contour $C$. On the segments on which $\sigma = 2$ the integrand is $\ll N^3/t^2$ and the integrals are

$$\ll \int_T^{\infty} N^3 \frac{dt}{t^2} \ll T^{-1} N^3.$$

(2.21)

On the line segments on which $s = \sigma \pm iT$ we have $\zeta(s) \ll T^{1+\epsilon}$, $|\zeta(s+1)| \gg 1/\log T$, so that the integrals are

$$\ll N^3 T^{-1+3\epsilon}.$$

(2.22)

We set $T = N^3$, so that the contributions from these integrals are $O(N^{3\epsilon})$. On the line $\sigma = 0$ we employ the functional equation. We have

$$|\Gamma(1-it) \sin \frac{it\pi}{2}| = \left( \frac{|t| \pi}{2} \tanh \frac{|t| \pi}{2} \right)^{\frac{1}{2}} \ll \min(|t|, \sqrt{|t|}),$$

(2.23)
and \(|\zeta(1 - it) = |\zeta(1 + it)|\) so that the integrand is
\[
\ll N \frac{\min(t^2, |t|)|\zeta(1 + it)|}{|t|(|t| + 1)}
\] (2.24)
and the integral is
\[
\ll N \left(1 + \int_1^T |\zeta(1 + it)| \frac{dt}{T}\right). 
\] (2.25)
We apply Cauchy’s inequality and the formula (Titchmarsh [7]: Theorem 7.2)
\[
\int_1^X |\zeta(1 + it)|^2 dt \sim \zeta(2) X
\] (2.26)
to see that the integral in (2.25) is \(\ll \log T \ll \log N\). Hence, by (2.19), (2.20), and the estimates (2.21) and (2.22),
\[
\sum_{n \leq 2N} (2N - n) f(n) = \frac{12N^2}{\pi^2} \left(\log 2N - \frac{\zeta'(2)}{\zeta(2)} - \frac{3}{2} + 2\gamma\right) + O(N \log N),
\] (2.27)
and the required result follows by inserting this into (2.16).

**Proof of Theorem 4.** It will be sufficient to consider \(L(N, k)\) as the other case is similar; we already saw that \(U(N, 1) = 0\). Let \(s(x_i) = s\) and \(\nu(x_i) = k\) take its lower value \([2N/s] - 1\). Thus \([2N/s] = k + 1\), so that
\[
\frac{2N}{k + 2} < s \leq \frac{2N}{k + 1};
\] (2.28)
moreover, from (1.4), we require that
\[
\left[\frac{N + r}{s}\right] = k,
\] (2.29)
that is
\[
\max(N - s + 1, sk - N) \leq r \leq \min(N, s(k + 1) - N).
\] (2.30)
From (2.28) this reduces to
\[
N - s + 1 \leq r \leq s(k + 1) - N
\] (2.31)
and since \(r\) is prime to \(s\) the number of choices for \(r\) in (2.31) is
\[
(s(k + 2) - 2N) \frac{\phi(s)}{s} + O(\tau(s)),
\] (2.32)
and we need to sum over the range in (2.28). We proceed by partial summation, writing
\[
\Phi(s) := \sum_{m \leq s} \frac{\phi(m)}{m} = \frac{6s}{\pi^2} + O(\log s),
\] (2.33)
and
\[
y = \left\lfloor \frac{2N}{k + 2} \right\rfloor, \quad z = \left\lfloor \frac{2N}{k + 1} \right\rfloor.
\] (2.34)
Assuming that $y < z$ to begin with, we find that
\[
\sum_{y<s\leq z} (s(k+2) - 2N) \frac{\phi(s)}{s} = -((y+1)(k+2) - 2N)\Phi(y) - (k+2) \sum_{y<s<z} \Phi(s) + (z(k+2) - 2N)\Phi(z). \tag{2.35}
\]
Notice that $0 < (y+1)(k+2) - 2N \leq z(k+2) - 2N \leq 2N/(k+1)$ so that the end terms contribute $\ll k^{-1}N \log N$ to the error. The middle terms contribute
\[
\ll (z - y)(k + 2) \log N \ll k^{-1}N \log N \tag{2.36}
\]
to the error, since
\[
z - y - 1 \leq \frac{2N}{(k+1)(k+2)}. \tag{2.37}
\]
The main term in (2.35) is
\[
\frac{6}{\pi^2} \sum_{y<s\leq z} (s(k+2) - 2N) = \frac{3(z - y)((z + y + 1)(k + 2) - 4N)}{\pi^2}, \tag{2.38}
\]
and we remark that the last factor in the numerator does not exceed $2N/(k+1)$, so that the error involved in (2.38) if we remove the square brackets in (2.34) is $\ll k + N/k$. Hence the sum in (2.35) equals
\[
\frac{12N^2}{\pi^2(k+1)^2(k+2)} + O(k + \frac{N \log N}{k}). \tag{2.39}
\]
It is easy to see that the error term arising from the divisor function in (2.32) is absorbed here; for example Dirichlet’s theorem gives
\[
\sum_{y<s\leq z} \tau(s) \ll (z - y) \log z + \sqrt{z} \ll \left(\frac{N}{k^2} + 1\right)N + \sqrt{N/k}. \tag{2.40}
\]
Therefore (2.39) provides a formula for $L(N,k)$ in the case $y < z$, and if $y = z$ then $L(N,k) = 0$ because the range for $s$ in (2.28) is empty and the formula remains valid. Finally we may replace $3N^2/\pi^2$ by $R$ in (2.39) without affecting the error term, and this gives the first asymptotic formula in (1.12) together with the formula for $\ell_k$. The remaining formulae can be established similarly, and (1.15) follows at once.

### 3. Proofs of Theorems 5, 6, 7 and 8

**Proof of Theorem 5.** We take as induction hypothesis that (1.18) holds at $j$, and we begin by checking it when $j = 1$. We already saw that $D_1 = N - 2$, and we have $s = N$, $b = 1$, $t = N - 1$, $r = 1$, $D(1,N) = N - 3 + 2/N$, so that (1.18) is correct.

Suppose now that (1.18) is true. We have
\[
D_{j+1} = D_j + \frac{1}{2}(\nu(x_j) - 3) + \frac{1}{2}(\nu(x_{j+1}) - 3) = D_j + \frac{r + t}{2s} + \frac{s + u}{2t} - 3,
\]

where \( u \) is the denominator of the fraction following \( c/t \) in \( \mathcal{F}_N \). From (1.18),

\[
D_{j+1} = D(b, s) + \frac{t}{s} + \frac{s + u}{2t} - \frac{5}{2} - \frac{b}{s} \tag{3.1}
\]

and we apply Lemma 2 of [5], which tells us that

\[
D(b, s) = D(c, t) - \frac{s^2 + t^2 + 1}{st} + 3 \tag{3.2}
\]

so that we have, from (3.1) and (3.2),

\[
D_{j+1} = D(c, t) + \frac{u - s}{2t} + 1 - \frac{1}{2} - \frac{b}{s} = D(c, t) + \frac{u - s}{2t} + 1 - \frac{c}{t},
\]

as required. This completes the induction and the proof.

**Proof of Theorem 6.** We first prove the following

**Lemma 1.** We have

\[
|t - r| \leq s - 2 + \text{pip}(s),
\]

where \( \text{pip}(s) = 1 \) if \( s|2N + 1 \), = 0 otherwise.

**Proof.** Since \( N - s + 1 \leq r, s \leq N \) it is evident that \( |t - r| \leq s - 1 \) with equality if and only if \( \max(r, t) = N \), \( \min(r, t) = N - s + 1 \), which implies \( r + t = 2N + 1 - s \). Since \( s|r + t \) this gives the result stated.

It will be sufficient to show that \( D_j \leq N - 2 \) with equality if and only if \( j = 1 \). We have

\[
D_j \leq D(b, s) + \frac{s - 2 + \text{pip}(s)}{2s} + \frac{1}{2} + \frac{1}{s} \tag{3.3}
\]

by Theorem 5 and the lemma; there is equality in (3.3) if and only if \( b = 1 \). We have

\[
D(b, s) \leq D(1, s) = s - 3 + \frac{2}{s}
\]

whence, from (3.3),

\[
D_j \leq s - 2 + \frac{\text{pip}(s)}{2s} \leq N - 2
\]

with equality if and only if \( s = N \). This is all we need.

**Proof of Theorem 7.** By Theorem 5 and Lemma 1, we have \( D_j = D(b, s) + O(1) \), and hence

\[
D_j^2 = D(b, s)^2 + O(|D_j|) + O(1).
\]

Hence the sums in (1.20) is

\[
\sum_{s=1}^{N} \sum_{(b, s)=1} D(b, s)^2 + O \left( \sum_{j=1}^{R} |D_j| \right) + O(R).
\]

We employ a recent theorem of Jia [6] to evaluate the inner sum, which is

\[
f_1(s)s^2 + O(s^2 \log^2 s)
\]

\[< 10>\]
where \( f_1(s) \) is defined as the coefficient in a Dirichlet series, viz
\[
\sum_{n=1}^{\infty} \frac{f_1(n)}{n^s} = 5 \zeta(z+3) \zeta(z+2) \zeta(z)
\]
from which it follows that our sum is
\[
\frac{5\zeta(4)}{3\zeta(3)^2} N^3 + O(N^{\frac{3}{2}} \log^2 N) + O\left(\sum_{j=1}^{R} |D_j|\right).
\]
(3.4)

As Theorem 8 shows, the second error term in (3.4) is of a smaller order than the first; in any case, for our purpose here, Cauchy’s inequality yields
\[
\sum_{j=1}^{R} |D_j| \ll N^{\frac{3}{2}},
\]
so that the theorem is proved.

Proof of Theorem 8. An alternative representation of the Dedekind sum, due to Eisenstein, is
\[
D(b, s) = 2 \sum_{\ell=1}^{s-1} \cot \left( \frac{\pi \ell}{s} \right) \cot \left( \frac{\pi b \ell}{s} \right)
\]
(3.5)
and it is a straightforward matter to deduce from (3.5) that
\[
\sum_{(b, s)=1} |D(b, s)| < \frac{12}{\pi^2} s \log^2 s.
\]
(3.6)
Hence
\[
\sum_{j=1}^{R} |D_j| \leq \frac{12}{\pi^2} \sum_{s=1}^{N} s \log^2 s + O(R)
\]
\[
\leq \frac{6}{\pi^2} N^2 \log^2 N + O(R)
\]
\[
\leq 2R \log^2 N + O(R),
\]
as required.

We end with a table of values for \( R, \sum_{j \leq R} |D_j| \) and \( \sum_{j \leq R} D_j^2 \). We remark that \( \frac{5\zeta(4)}{3\zeta(3)^2} \approx 1.24841 \), and that it appears from the table that (3.6) has a constant which is perhaps too large by a factor of about 4.

| \( N \) | \( R \) | \( \sum |D_j| \) | \( \sum D_j^2 \) | \( \frac{\sum |D_j|}{R \log^2 N} \) | \( \frac{\sum D_j^2}{N^3} \) |
|---|---|---|---|---|---|
| 10 | 32 | 80.5 | 384.25 | 0.47447 | 0.38425 |
| 50 | 774 | 5672.5 | 104831.1 | 0.47888 | 0.83865 |
| 100 | 3044 | 31093.5 | 971927 | 0.48165 | 0.97192 |
| 500 | 76116 | 1.41210 \times 10^6 | 1.44082 \times 10^8 | 0.48035 | 1.15266 |
| 1000 | 304192 | 6.97989 \times 10^6 | 1.18975 \times 10^9 | 0.48086 | 1.18975 |
| 5000 | 7600458 | 2.66173 \times 10^8 | 1.53870 \times 10^{11} | 0.48276 | 1.23096 |
| 10000 | 30397486 | 1.24780 \times 10^9 | 1.23822 \times 10^{12} | 0.48390 | 1.23822 |
References


R. R. Hall
Department of Mathematics
York University
Heslington
York YO10 5DD
United Kingdom
Email: rrh1@york.ac.uk

P. Shiue
Department of Mathematical Sciences
Loughborough University
Loughborough
Leicestershire LE11 3TU
United Kingdom
Email: P.Shiue@lboro.ac.uk