Behaviour of thin-walled structures under combined loads

This item was submitted to Loughborough University’s Institutional Repository by the/an author.

Additional Information:

- A Doctoral Thesis. Submitted in partial fulfillment of the requirements for the award of Doctor of Philosophy of Loughborough University.

Metadata Record: https://dspace.lboro.ac.uk/2134/7413

Publisher: © A.M.S. Al-Sheikh

Please cite the published version.
This item is held in Loughborough University’s Institutional Repository (https://dspace.lboro.ac.uk/) and was harvested from the British Library’s EThOS service (http://www.ethos.bl.uk/). It is made available under the following Creative Commons Licence conditions.

For the full text of this licence, please go to: http://creativecommons.org/licenses/by-nc-nd/2.5/
BEHAVIOUR OF THIN-WALLED STRUCTURES

UNDER COMBINED LOADS

by

ABDELRAOUF AL-SHEIKH, BSc, MTech

A Doctoral Thesis
Submitted in partial fulfilment of the requirements
for the award of the degree of
Doctor of Philosophy
of Loughborough University of Technology

May 1985

Supervisor: Dr P W Sharman
Department of Transport Technology

© by A.M.S. Al-Sheikh
TO MY MOTHER, AND IN MEMORY OF MY FATHER
SUMMARY

The thesis is concerned with the theory of thin-walled beams of open section. The aim is to formulate a general beam element for analysis of this type of structure. Thus a general stiffness matrix for the element, and a transformation matrix for loads and displacements with respect to centroid and shear centre were derived, by taking into consideration the value of bimoment due to an axial force offset from the shear centre. Internal forces including bimoments, and global displacements including warping were calculated, and the stress distributions on the cross-sections of a beam at each element node, were evaluated. The problem of buckling of thin-walled beams was treated using a finite strip program which was formulated to solve problems with the following combination of stresses:

a) Linearly distributed axial stresses
b) Uniform lateral stresses
c) Uniform shear stresses

The results for beams of cruciform, box and channel sections, under uniform axial stresses and linearly distributed axial stresses, also accounting for flexural stresses, were compared with other theoretical and some experimental results. The agreement was satisfactory.

A series of laboratory tests on beams of channel sections under compression were carried out. The recorded failure load and critical buckling load, computed by the Southwell plot method, were compared with the finite strip results and satisfactory agreement was observed.
ACKNOWLEDGEMENTS

I would like to express my thanks to the academic staff of the Department of Transport Technology, and to all technicians in the workshop and electronic laboratory for the help they provided me throughout this project.

My special thanks to my supervisor, Dr P W Sharman, for many useful discussions, help, encouragement and comradeship since we first met.

I wish to express my gratitude to my friend, Mr Joe Ogendo, for his advice on computing and programming and for many useful discussions and criticisms. I also wish to express my thanks to Mr D Crang for his assistance with the experimental work.

My gratitude to my wife Fatima and children, Etab, Sami and Mohammad who have patiently provided me with all the time I needed to complete this research and I hope that we will now see more of each other.

Finally, my sincere thanks to the people of the Syrian Arabic Republic to whom I am indebted for providing me with the financial support.
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page No</th>
</tr>
</thead>
<tbody>
<tr>
<td>Summary</td>
<td>i</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>ii</td>
</tr>
<tr>
<td>Contents</td>
<td>iii</td>
</tr>
<tr>
<td>Notations</td>
<td>vi</td>
</tr>
<tr>
<td>List of Figures</td>
<td>viii</td>
</tr>
<tr>
<td>List of Tables</td>
<td>xiii</td>
</tr>
</tbody>
</table>

1. INTRODUCTION

1.1 Literature Survey

1.1.1 Theory of Torsion and Flexure

1.1.2 Matrix Analysis of Thin-Walled Beams

1.1.3 Instability of Thin-Walled Beams

1.1.3.1 Local Instability

1.1.3.2 Interaction of Local and Member Buckling

1.2 Contribution of the Thesis

2. THEORY OF THIN-WALLED OPEN SECTION STRUCTURES

2.1 Introduction

2.2 Torsion Warping Theory

2.3 The Displacements and Strains on Thin-Walled Beams of Open Section

2.3.1 Out-of-plane and Lateral Displacements

2.3.2 The Kinematics of Displacements

2.4 The Law of Sectorial Area

2.4.1 The Determination of Sectorial Area Expressions

2.4.2 The Location of Zero Centroidal Pole $S_o$ and the Principal Sectorial Pole

2.5 Derivation of the Normal Stress Formulae

2.5.1 Assumptions of Elastic Beams and the Stress Formulae
<table>
<thead>
<tr>
<th>Section</th>
<th>Page No</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5.2 Bimoment</td>
<td>34</td>
</tr>
<tr>
<td>2.6 Stress Distribution on Thin-Walled Beams of Open Section</td>
<td>35</td>
</tr>
<tr>
<td>2.6.1 Assumptions</td>
<td>35</td>
</tr>
<tr>
<td>2.6.2 Determination of Tangential Stresses</td>
<td>36</td>
</tr>
<tr>
<td>2.7 The Differential Equations of Equilibrium for a Thin-Walled Beam</td>
<td>39</td>
</tr>
<tr>
<td>2.7.1 The State of Equilibrium of a Beam Under a Set of Internal Stresses and External Forces</td>
<td>39</td>
</tr>
<tr>
<td>2.8 Shear Centre</td>
<td>49</td>
</tr>
<tr>
<td>3. A THIN-WALLED BEAM FINITE ELEMENT</td>
<td>65</td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>65</td>
</tr>
<tr>
<td>3.2 Thin-Walled Beam Stiffness Matrix</td>
<td>66</td>
</tr>
<tr>
<td>3.3 Derivation of Torsion Warping Element Stiffness Matrix</td>
<td>67</td>
</tr>
<tr>
<td>3.4 Load Transformation from an Arbitrary Point to the Shear Centre of a Beam</td>
<td>71</td>
</tr>
<tr>
<td>3.4.1 The Effect of Applied Loads on Bimoment</td>
<td>71</td>
</tr>
<tr>
<td>3.4.1.1 Transformation of Transverse Forces</td>
<td>72</td>
</tr>
<tr>
<td>3.4.1.2 Transformation of Bending Moments</td>
<td>72</td>
</tr>
<tr>
<td>3.4.1.3 Transformation of Longitudinal Force</td>
<td>75</td>
</tr>
<tr>
<td>3.4.2 Transformation for Stiffness Matrix</td>
<td>81</td>
</tr>
<tr>
<td>3.5 Rotation for Principal Directions</td>
<td>84</td>
</tr>
<tr>
<td>3.6 Numerical Examples</td>
<td>86</td>
</tr>
<tr>
<td>3.6.1 Load Case 1</td>
<td>87</td>
</tr>
<tr>
<td>3.6.2 Load Case 2</td>
<td>88</td>
</tr>
<tr>
<td>3.6.3 Load Case 3</td>
<td>89</td>
</tr>
<tr>
<td>3.6.4 Load Case 4</td>
<td>89</td>
</tr>
<tr>
<td>4. FINITE STRIP METHOD</td>
<td>106</td>
</tr>
<tr>
<td>4.1 Introduction</td>
<td>106</td>
</tr>
<tr>
<td>4.2 Displacement Functions</td>
<td>108</td>
</tr>
<tr>
<td>4.3 Derivation of Strip Stiffness Matrix</td>
<td>111</td>
</tr>
<tr>
<td>4.3.1 Perturbation Forces and Displacements in a Strip</td>
<td>111</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Out-of-plane Stiffness Matrix $S_0$</td>
</tr>
<tr>
<td>4.3.3</td>
<td>In-plane Stiffness Matrix $S_I$</td>
</tr>
<tr>
<td>4.4</td>
<td>Rectangular Plate Stability</td>
</tr>
<tr>
<td>4.4.1</td>
<td>Buckling of Plate Under Pure Compression</td>
</tr>
<tr>
<td>4.4.2</td>
<td>Buckling of Plate Under Pure Bending</td>
</tr>
<tr>
<td>4.4.2.1</td>
<td>Both Sides are Simply Supported</td>
</tr>
<tr>
<td>4.4.3</td>
<td>Buckling of Plate Under Combined Compression and Bending</td>
</tr>
<tr>
<td>5.</td>
<td>BUCKLING OF THIN-WALLED BEAMS UNDER PURE COMPRESSION</td>
</tr>
<tr>
<td>5.1</td>
<td>Introduction</td>
</tr>
<tr>
<td>5.2</td>
<td>Buckling of Cruciform Type Structure</td>
</tr>
<tr>
<td>5.3</td>
<td>Buckling of Box Type Structure</td>
</tr>
<tr>
<td>5.4</td>
<td>Buckling of Channel Section Beams</td>
</tr>
<tr>
<td>5.5</td>
<td>Experimental Work</td>
</tr>
<tr>
<td>5.5.1</td>
<td>Buckling Load of Channel Beam</td>
</tr>
<tr>
<td>5.5.2</td>
<td>Buckling Mode</td>
</tr>
<tr>
<td>5.6</td>
<td>Calculations of Actual Node Displacements</td>
</tr>
<tr>
<td>6.</td>
<td>BUCKLING OF THIN-WALLED BEAMS UNDER COMBINED LOAD</td>
</tr>
<tr>
<td>6.1</td>
<td>Introduction</td>
</tr>
<tr>
<td>6.2</td>
<td>Buckling of Channel Under Combined Loading</td>
</tr>
<tr>
<td>6.3</td>
<td>Non-Linear Behaviour of Thin-Walled Beam Under Axial Compression</td>
</tr>
<tr>
<td>6.3.1</td>
<td>General</td>
</tr>
<tr>
<td>6.3.2</td>
<td>The Effect of Prebuckling Twist on Channel Local Instability</td>
</tr>
<tr>
<td>7.</td>
<td>CONCLUSIONS</td>
</tr>
<tr>
<td>7.1</td>
<td>Concluding Summary and Discussion</td>
</tr>
<tr>
<td>7.2</td>
<td>Suggestions for Further Research</td>
</tr>
<tr>
<td>7.2.1</td>
<td>Thin-Walled Beams Non-Linear Behaviour</td>
</tr>
<tr>
<td>7.2.2</td>
<td>Post-Buckling Behaviour of Thin-Walled Beams</td>
</tr>
<tr>
<td>7.3</td>
<td>Conclusions</td>
</tr>
<tr>
<td>REFERENCES</td>
<td></td>
</tr>
<tr>
<td>APPENDICES</td>
<td></td>
</tr>
</tbody>
</table>
## NOTATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Total cross-section area ([\text{mm}^2])</td>
</tr>
<tr>
<td>(a_y, b_y, d_y)</td>
<td>Linear coordinate in direction (Y) for points (A, B) and (D) respectively ([\text{mm}])</td>
</tr>
<tr>
<td>(a_z, b_z, d_z)</td>
<td>Linear coordinate in direction (Z) for points (A, B) and (D) respectively ([\text{mm}])</td>
</tr>
<tr>
<td>(B)</td>
<td>Bimoment ([\text{N.mm}^2])</td>
</tr>
<tr>
<td>(b)</td>
<td>Strip or plate breadth ([\text{mm}])</td>
</tr>
<tr>
<td>(B_f, B_w)</td>
<td>Flange and web breadth respectively ([\text{mm}])</td>
</tr>
<tr>
<td>(d_0, d_1)</td>
<td>Out-of-plane and in-plane strip deflection respectively ([\text{mm}])</td>
</tr>
<tr>
<td>(E)</td>
<td>Young's modulus ([\text{N/mm}^2], [\text{MN/m}^2])</td>
</tr>
<tr>
<td>(G)</td>
<td>Modulus of rigidity ([\text{N/mm}^2], [\text{MN/m}^2])</td>
</tr>
<tr>
<td>(I_y, I_z)</td>
<td>Second moment of area of the cross-section about (y) and (z) axis respectively ([\text{mm}^4])</td>
</tr>
<tr>
<td>(I_{yz})</td>
<td>Product moment of area of the cross-section ([\text{mm}^4])</td>
</tr>
<tr>
<td>(I_{pr})</td>
<td>Polar second moment of area ([\text{mm}^4])</td>
</tr>
<tr>
<td>(J)</td>
<td>Torsional moment of area ([\text{mm}^4])</td>
</tr>
<tr>
<td>(K)</td>
<td>Buckling coefficient</td>
</tr>
<tr>
<td>(K_b, K_c)</td>
<td>Pure bending and compression buckling coefficient respectively</td>
</tr>
<tr>
<td>(L, \lambda)</td>
<td>Half wavelength ([\text{mm}])</td>
</tr>
<tr>
<td>(L/B_w)</td>
<td>Aspect ratio</td>
</tr>
<tr>
<td>(M_y, M_z)</td>
<td>Direct applied bending moment about axes (y) and (z) respectively ([\text{N.mm}])</td>
</tr>
<tr>
<td>(M_y, N_z)</td>
<td>Total bending moments about axes (y) and (z) respectively ([\text{N.mm}])</td>
</tr>
<tr>
<td>(N)</td>
<td>Axial force (x)-direction ([\text{N}])</td>
</tr>
<tr>
<td>(P_o, P_i)</td>
<td>Out-of-plane and in-plane strip forces respectively</td>
</tr>
<tr>
<td>(P_x, P_y, P_z)</td>
<td>Forces in (x, y, z) directions for projected area on (x)-(s) plane ([\text{N/mm}^2])</td>
</tr>
<tr>
<td>(P_y, P_z)</td>
<td>Direct forces in (y) and (z) directions respectively ([\text{N}])</td>
</tr>
<tr>
<td>(q)</td>
<td>Shear flow ([\text{N/mm}])</td>
</tr>
<tr>
<td>(r)</td>
<td>Sectorial radius of a point on the contour ([\text{mm}])</td>
</tr>
<tr>
<td>(r_n)</td>
<td>Normal sectorial radius ([\text{mm}])</td>
</tr>
<tr>
<td>(S)</td>
<td>Profile direction</td>
</tr>
<tr>
<td>(S_o, S_i)</td>
<td>Out-of-plane and in-plane stiffness matrix respectively</td>
</tr>
</tbody>
</table>
\[ S_W \]  
First moment of sectorial area \([\text{mm}^4]\)

\[ S_{Wz}, S_{Wy} \]  
Product moment of sectorial area about \(y\) and \(x\) axis respectively \([\text{mm}^5]\)

\[ S_y, S_z \]  
First moment of area about \(y\) and \(z\) axis respectively \([\text{mm}^3]\)

\( t \)  
Wall thickness \([\text{mm}]\)

\( T_V \)  
St Venant twisting moment \([\text{N.mm}]\)

\( T_W \)  
Warping (flexural) twisting moment \([\text{N.mm}]\)

\( (T_W/T_f) \)  
Web-to-flange thickness ratio

\( u, v, w \)  
Displacements in \(x, y, z\) respectively \([\text{mm}]\)

\( u_0 \)  
Axial displacement of the origin of the principal coordinate (zero warping) \([\text{mm}]\)

\( x, y, z \)  
Linear global coordinate \([\text{mm}]\)

\( X, Y, Z \)  
Polynomial functions represent the displacement functions of \(u, v,\) and \(W\) in \(y\) direction

\( y_o, z_o \)  
Linear coordinates of the origin of the principal coordinate \([\text{mm}]\)

\( \alpha \)  
Angle of tangential direction of the contour to the \(x\)-axis

\( \Gamma \)  
Second moment of sectorial area (warping constant) \([\text{mm}^5]\)

\( \Delta \)  
Symbol indicates very small increment of a variable

\( \delta \)  
Partial differentiation

\( \varepsilon \)  
Strain \([\%]\)

\( \xi, \eta \)  
Dimensionless variables in the directions \(x\) and \(y\) respectively

\( \xi_n, \eta_n \)  
Tangential and perpendicular displacements of point \(S\) on the contour \([\text{mm}]\)

\( \nu \)  
Poisson's ratio

\( \sigma \)  
Axial stress \([\text{N/mm}^2]\)

\( \sigma_{cr} \)  
Critical axial stress \([\text{N/mm}^2]\)

\( \phi \)  
Angle of twist \([\text{rad, o}]\)

\( \phi' \)  
Rate of twist in \(x\) direction \([\text{rad/mm, o/mm}]\)

\( \omega \)  
Sectorial (area) coordinate \([\text{mm}^2]\)

Other symbols are also used occasionally, with the appropriate explanation.

\([K_{ij}]\)  
Warping stiffness matrix

\([K_{ij}]\)  
Element stiffness matrix
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure No</th>
<th>Title</th>
<th>Page No</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Warping of thin-walled open section under torsion (free ends)</td>
<td>50</td>
</tr>
<tr>
<td>2.2</td>
<td>Generalised forces on I-beam under eccentric axial load</td>
<td>51</td>
</tr>
<tr>
<td>2.3</td>
<td>Orthogonal axes for thin shell element</td>
<td>52</td>
</tr>
<tr>
<td>2.4</td>
<td>Displacements and rotation for an element of the cross-section of a thin-walled beam</td>
<td>53</td>
</tr>
<tr>
<td>2.5</td>
<td>Displacements and rotations of the contour of a thin-walled beam</td>
<td>54</td>
</tr>
<tr>
<td>2.6</td>
<td>Displacements and shear deformation of a thin-walled shell element</td>
<td>55</td>
</tr>
<tr>
<td>2.7</td>
<td>Sectorial coordinate $d\omega$ of point $S$</td>
<td>56</td>
</tr>
<tr>
<td>2.8a</td>
<td>Sectorial coordinate of point $S$ with respect to different sectorial poles</td>
<td>57</td>
</tr>
<tr>
<td>2.8b</td>
<td>Geometry of sectorial coordinate of point $S$</td>
<td>57</td>
</tr>
<tr>
<td>2.9</td>
<td>Cross-section of a channel beam</td>
<td>58</td>
</tr>
<tr>
<td>2.10</td>
<td>Relationship between sectorial coordinates with a common pole but different origins</td>
<td>59</td>
</tr>
<tr>
<td>2.11</td>
<td>Axial stress on a thin-walled beam cross-section</td>
<td>60</td>
</tr>
<tr>
<td>2.12</td>
<td>Axial and shear stress distribution</td>
<td>61</td>
</tr>
<tr>
<td>2.13</td>
<td>Shell element under general in-plane forces</td>
<td>62</td>
</tr>
<tr>
<td>2.14</td>
<td>Thin-walled open-section element under a general load system</td>
<td>63</td>
</tr>
<tr>
<td>2.15</td>
<td>External and internal forces on a shell element of thin-walled beam</td>
<td>64</td>
</tr>
<tr>
<td>CHAPTER 3</td>
<td>Page No</td>
<td></td>
</tr>
<tr>
<td>-----------</td>
<td>---------</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>Nodal torsional forces and displacements associated with a thin-walled beam element</td>
<td>94</td>
</tr>
<tr>
<td>3.2</td>
<td>Transformation of a transverse force</td>
<td>94</td>
</tr>
<tr>
<td>3.3</td>
<td>Transformation of a bending moment</td>
<td>94</td>
</tr>
<tr>
<td>3.4</td>
<td>Bending moment represented by a couple according to Vlasov (1961)</td>
<td>95</td>
</tr>
<tr>
<td>3.5</td>
<td>Bending moment represented by a couple</td>
<td>95</td>
</tr>
<tr>
<td>3.6</td>
<td>A load system applied at point D</td>
<td>96</td>
</tr>
<tr>
<td>3.7</td>
<td>Transformation of a load system to the centroid</td>
<td>96</td>
</tr>
<tr>
<td>3.8</td>
<td>Transformation of a load system applied at point D to the relevant centroid and shear centre</td>
<td>97</td>
</tr>
<tr>
<td>3.9</td>
<td>Local and global coordinates of an I-beam</td>
<td>98</td>
</tr>
<tr>
<td>3.10</td>
<td>Transformation of bending moment from global to local coordinates</td>
<td>98</td>
</tr>
<tr>
<td>3.11</td>
<td>The finite element model used in PAFEC</td>
<td>99</td>
</tr>
<tr>
<td>3.12</td>
<td>Cantilever under transverse force load case 1</td>
<td>100</td>
</tr>
<tr>
<td>3.13</td>
<td>Cantilever under axial force load case 2</td>
<td>100</td>
</tr>
<tr>
<td>3.14</td>
<td>Cantilever under transverse force load case 3</td>
<td>101</td>
</tr>
<tr>
<td>3.15</td>
<td>Cantilever under a bending force load case 4</td>
<td>101</td>
</tr>
<tr>
<td>3.16</td>
<td>Stress distribution on the cross-section load case 1</td>
<td>102</td>
</tr>
<tr>
<td>3.17</td>
<td>Stress distribution on the cross-section load case 2</td>
<td>103</td>
</tr>
<tr>
<td>3.18</td>
<td>Stress distribution on the cross-section load case 3</td>
<td>104</td>
</tr>
<tr>
<td>3.19</td>
<td>Stress distribution on the cross-section load case 4</td>
<td>105</td>
</tr>
<tr>
<td>CHAPTER 4</td>
<td>Page No</td>
<td></td>
</tr>
<tr>
<td>-----------</td>
<td>---------</td>
<td></td>
</tr>
<tr>
<td>4.1 Comparison between finite element method and finite strip method</td>
<td>125</td>
<td></td>
</tr>
<tr>
<td>4.2 Plate element under general in-plane and body forces</td>
<td>126</td>
<td></td>
</tr>
<tr>
<td>4.3a Out-of-plane displacements</td>
<td>127</td>
<td></td>
</tr>
<tr>
<td>4.3b In-plane displacements</td>
<td>127</td>
<td></td>
</tr>
<tr>
<td>4.4 Plate strip under general in-plane stresses</td>
<td>128</td>
<td></td>
</tr>
<tr>
<td>4.5 Edge forces and displacements of a strip</td>
<td>129</td>
<td></td>
</tr>
<tr>
<td>4.6 Plate under combined compression and bending ( \theta_2 = \theta_1 )</td>
<td>130</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 5</th>
<th>Page No</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1 Cruciform type structure</td>
<td>142</td>
</tr>
<tr>
<td>5.2 Torsional buckling mode of cruciform type structure</td>
<td>142</td>
</tr>
<tr>
<td>5.3 Box-type structure</td>
<td>143</td>
</tr>
<tr>
<td>5.4 Variation of ( K ) with ( B_f/B_w ) for box-type structure under pure compression</td>
<td>144</td>
</tr>
<tr>
<td>5.5a Variation of ( K ) with aspect ratio for box beam under pure compression</td>
<td>145</td>
</tr>
<tr>
<td>5.5b Variation of ( K ) with aspect ratio for box beam under pure compression</td>
<td>146</td>
</tr>
<tr>
<td>5.6 Channel section</td>
<td>147</td>
</tr>
<tr>
<td>5.7 Variation of ( K ) with ( (B_f/B_w) ) for channel under compression</td>
<td>148</td>
</tr>
<tr>
<td>5.8 Variation of ( K ) with aspect ratio for channel under compression ( T_w/T_f = 1 )</td>
<td>149</td>
</tr>
<tr>
<td>5.9 Variation of ( K ) with aspect ratio for channel under compression ( T_w/T_f = 1.2 )</td>
<td>150</td>
</tr>
<tr>
<td>5.10 Variation of ( K ) with aspect ratio for channel under compression ( T_w/T_f = 1.4 )</td>
<td>151</td>
</tr>
<tr>
<td>Section</td>
<td>Description</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>5.11</td>
<td>Variation of K with aspect ratio for channel under compression $\frac{T_w}{T_f} = 1.6$</td>
</tr>
<tr>
<td>5.12</td>
<td>Variation of K with aspect ratio for channel under compression $\frac{T_w}{T_f} = 1.6$</td>
</tr>
<tr>
<td>5.13</td>
<td>Variation of K with aspect ratio for channel under compression $\frac{T_w}{T_f} = 2.0$</td>
</tr>
<tr>
<td>5.14</td>
<td>Comparison of theoretical and experimental for variation of $P_{cr}$ with $\frac{B_f}{B_w}$</td>
</tr>
<tr>
<td>5.15</td>
<td>Description of End Condition of the Tested Channel Strut</td>
</tr>
<tr>
<td>5.16</td>
<td>Buckling mode for channel beam length, $L = 80, \text{mm}$</td>
</tr>
<tr>
<td>5.17</td>
<td>Buckling mode for channel beam length, $L = 280, \text{mm}$</td>
</tr>
<tr>
<td>5.18</td>
<td>Buckling mode for channel beam length, $L = 300, \text{mm}$</td>
</tr>
<tr>
<td>5.19</td>
<td>Buckling mode for channel beam length, $L = 1250, \text{mm}$</td>
</tr>
<tr>
<td>5.20</td>
<td>Channel strut under pure compression</td>
</tr>
<tr>
<td>5.21</td>
<td>Initiation of torsional buckling</td>
</tr>
<tr>
<td>5.22</td>
<td>Plate flange buckling</td>
</tr>
<tr>
<td>5.23</td>
<td>Buckling of second flange</td>
</tr>
<tr>
<td>5.24</td>
<td>Close-up picture of buckled channel</td>
</tr>
<tr>
<td>5.25</td>
<td>Plate Strip Geometry After Buckling</td>
</tr>
<tr>
<td>6.1</td>
<td>Combined bending load on channel section</td>
</tr>
<tr>
<td>6.2a,b</td>
<td>Variation of $K$ with $(B_f/B_w)$ for channel under different combined load</td>
</tr>
<tr>
<td>6.3</td>
<td>Variation of $K$ with $(B_f/B_w)$ for channel under combined load for $\alpha = 1.0$</td>
</tr>
<tr>
<td>6.4a,b</td>
<td>Variation of $K$ with aspect ratio for channel and combined load $\alpha = -1.0$</td>
</tr>
<tr>
<td>6.5a,b</td>
<td>Variation of $K$ with aspect ratio for channel and combined load $\alpha = -0.5$</td>
</tr>
<tr>
<td>6.6a,b</td>
<td>Variation of $K$ with aspect ratio for channel and combined load $\alpha = 0.5$</td>
</tr>
</tbody>
</table>
6.7a,b  Variation of $K$ with aspect ratio for channel under combined load $\alpha = 1.0$  
Page No 183

6.8a,b  Variation of $K$ with aspect ratio for channel under combined load $\alpha = 1.5$  
Page No 185

6.9  Variation of $P_{cr}$ with $\phi$  
Page No 187
# LIST OF TABLES

## CHAPTER 3

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page No</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Stiffness Matrix for Plane Beam</td>
<td>90</td>
</tr>
<tr>
<td>3.2</td>
<td>Stiffness Matrix for Beam-Bar</td>
<td>90</td>
</tr>
<tr>
<td>3.3</td>
<td>Axial Stress at the Beam Root for Load Cases 1 and 2</td>
<td>91</td>
</tr>
<tr>
<td>3.4</td>
<td>Axial Stress at the Beam Root for Cases 3 and 4</td>
<td>91</td>
</tr>
<tr>
<td>3.5</td>
<td>Displacements in X and Y Directions at the Tip of the Beam</td>
<td>92</td>
</tr>
<tr>
<td>3.6</td>
<td>Displacements in Z Direction and Rotation About X Direction at the Tip of the Beam</td>
<td>92</td>
</tr>
<tr>
<td>3.7</td>
<td>Thin-Walled Beam Element Stiffness Matrix</td>
<td>93</td>
</tr>
</tbody>
</table>

## CHAPTER 4

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page No</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Buckling Coefficient for Plate Under Pure Compression with Both Sides Simply Supported</td>
<td>120</td>
</tr>
<tr>
<td>4.2</td>
<td>Buckling Coefficient for Plate Under Pure Compression with One Side Simply Supported and Other is Free</td>
<td>120</td>
</tr>
<tr>
<td>4.3</td>
<td>Buckling Coefficient $K_c$ for Plate Under Pure Compression When One Side is Simply Supported and the Other is Clamped</td>
<td>120</td>
</tr>
<tr>
<td>4.4</td>
<td>Buckling Coefficient $K_c$ for Plate Under Pure Compression with One Side Clamped and the Other Free</td>
<td>121</td>
</tr>
<tr>
<td>4.5</td>
<td>Buckling Coefficient $K_b$ for Plate Under Pure Compression with Both Sides Clamped</td>
<td>121</td>
</tr>
<tr>
<td>4.6</td>
<td>Buckling Coefficient $K_b$ for Plate Under Pure Bending with Both Sides Simply Supported</td>
<td>122</td>
</tr>
<tr>
<td>4.7</td>
<td>Buckling Coefficient $K_b$ for Plate Under Pure Bending with One Side Simply Supported and the Other Clamped</td>
<td>122</td>
</tr>
<tr>
<td>Section</td>
<td>Description</td>
<td>Page No</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>4.8</td>
<td>Buckling Coefficient ( K_b ) for Plate Under Pure Bending with Both Sides Clamped</td>
<td>123</td>
</tr>
<tr>
<td>4.9</td>
<td>Buckling Coefficient ( K ) for Plate Under Combined Compression and Bending Load, with Both Sides Simply Supported</td>
<td>124</td>
</tr>
<tr>
<td>4.10</td>
<td>Buckling Coefficient ( K ) for Plate Under Combined Compression and Bending Load, with Both Sides Clamped</td>
<td>124</td>
</tr>
<tr>
<td><strong>CHAPTER 5</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.1</td>
<td>Buckling Coefficient of Box Type Structure of Different Geometric Ratios</td>
<td>139</td>
</tr>
<tr>
<td>5.2</td>
<td>Buckling Coefficient ( K_c ) for Channel Section Beams of Different Geometric Ratios</td>
<td>140</td>
</tr>
<tr>
<td>5.3</td>
<td>Buckling Coefficient ( K_c ) for Channel Section Beam with ( T_W/T_f = 1.0 ), and for Different ( B_f/B_w ) Ratios</td>
<td>141</td>
</tr>
<tr>
<td>5.4</td>
<td>Buckling Load of Channel Section Under Pure Compression</td>
<td>135</td>
</tr>
<tr>
<td><strong>CHAPTER 6</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.1</td>
<td>Buckling Coefficient for Channel Under Different Combined Load and for Different Flange-to-Web Width Ratios</td>
<td>170</td>
</tr>
<tr>
<td>6.2</td>
<td>Buckling Load Versus Angle of Twist ( \phi ) for Different Channel Lengths</td>
<td>172</td>
</tr>
</tbody>
</table>
1.0 INTRODUCTION

Thin-walled beams are popular with designers for many types of structures, since the manufacturing and economic possibilities are greater. However there is scant information on the performance of structures made from thin-walled beam elements, particularly those of open sections, and the behaviour is considerably complicated by the coupling of extensional, bending and torsional modes.

Thin-walled beams are characterised by the relative magnitudes of their dimensions. The wall thickness is small relative to the dimensions of the cross-section, and the dimensions of the cross-section are small compared with the length of the beam. Although the latter condition may not be very important. The first condition is important, and failing to satisfy this condition the theory of thin-walled beams may lead to erroneous numerical results.

It is useful to mention that for a thin-walled beam subjected to bending, torsion or combined loading, the value of direct stress at a point on the cross-section depends on the position of the point, the geometrical properties of the cross-section, and the applied loading. This may be true whether the cross-section of the thin-walled beam is closed or open. However, the study in this thesis will be mainly concerned with the theory of open sections.

1.1 Literature Survey

1.1.1 Theory of Torsion and Flexure

The theory of torsional warping and flexure is relatively new, previously both torsion and flexure were studied separately. The Euler-Bernoulli assumption of plane sections remaining plane while the structure is under bending, was the basic foundation of the theory of flexure. In this theory the axial force acting through the centroid produces only uniform tensile or compression stresses, and a transverse
force would provide only bending stresses. The shear stresses due
to the transverse force are calculated according to the approximate
theory of Jourawski (1856), which is discussed thoroughly by
Timoshenko (1953).

The theory of torsion was developed by St Venant in the 1850s, see
Timoshenko (1953). The new idea introduced by this theory, unlike the
Euler-Bernoulli assumption, is that the cross-section of a beam is
free to warp. The application of this theory was carried out for both
uniform and non-uniform torsion. The work of Bredt (1896), to esta-
lish the relationships between the shear flow and angle of twist with
the applied torque (which are usually referred to as Bredt's first
and second formulae) and the study by Prandtl in 1899, on narrow rect-
angular cross-sections, see Timoshenko (1953), made it possible to
apply St Venant's torsion theory to some types of thin-walled beams
used in engineering practice.

In 1905 further progress was due to Timoshenko who studied the effect
of restraining the warping of a symmetric I-beam at its ends when it
suffers a lateral buckling. This work resulted in the derivation of
the fundamental differential equation of torsion of symmetric I-beams
which enabled more investigation on lateral buckling of an I-beam under
transverse load, Timoshenko (1961). According to Timoshenko (1953),
Maillart explained the coupling of flexure and torsion in 1921 when he
introduced the concept of the shear centre and showed that the trans-
verse force and support reactions must act through the shear centre,
rather than the centroid of the cross-section in order to produce no
torsion. The use of open thin-walled beams in practice showed the
possibility of torsional or combined torsional and flexural failure.
In fact thin-walled beams tend to bend and twist simultaneously under
axial load.

Suspicion was directed at the Euler Buckling formula for failing to
predict the critical load for open sections correctly, and over-esti-
mating its value. The contributions made by Wagner in 1929 towards
the torsional buckling of thin-walled open sections to the then esta-
blished solutions for torsion and flexural buckling opened the way
for more investigation into the problem, Wagner (1936). However, in his theory Wagner introduced the concept of unit warping in the analyses and the theory was based on the assumption that the centre of rotation during buckling coincides with the shear centre, which subsequent investigators claimed to show that is not a general case, consequently the analysis based on this theory is not exact. According to Bleich (1952) Ostenfold was the first to present exact solutions for channel, angle and T-sections by considering the buckling due to torsion and flexure, but his analysis received no attention because of its complexity. Bleich and Bleich (1936), derived what they called the fundamental differential equations of bending, twisting and buckling of thin-walled structures of open section, using the theory of stationary potential energy. The study included unsymmetric sections, where the shear centre was taken as a reference instead of the centroid. Kappus (1938) pointed out that Bleich and Bleich differential equations are not applicable to beams of low torsional rigidity and he presented his refined theory using almost the same theory as used by Bleich and Bleich, in addition to Wagner's concept of unit warping. Lundquist and Fligg (1937) in a study of the primary failure of straight centrally loaded columns, introduced their theory of buckling by torsion and flexure, using the Wagner concept too, and added the assumption that the centre of rotation will be in such a position that the critical load is a minimum. Goodier (1941), extended the governing differential equations to include beam-columns under biaxial bending with identical loading conditions at each end. The assumption that the twisting and displacements of any cross-section of a beam are small compared with the loading eccentricities, made it possible to simplify the form of the differential equations.

A comprehensive theory on bending, torsion and buckling of thin-walled open section was published by Timoshenko (1945). In his study using Maxwell's reciprocal theorem in addition to the then accepted assumptions of combined torsional and flexural buckling, he established the identity of shear centre and centre of twist. Although a comprehensive
theory of combined torsion and flexure of thin-walled beam of open
section was published in the 1930's by Vlasov, it was not well-known
outside Russia for a long-time, Vlasov (1961). However, this theory
is approximate and was developed for engineering purposes, based on
certain assumptions to simplify the problem, as will be shown in
Chapter 2.

In Vlasov's theory the introduction of sectorial section properties,
which are based on geometrical considerations of a cross-section, was
essential to enable him to analyse a complete range of loading cases.
The main theme of this theory of thin-walled beams is based on three
assumptions:

i) The contour is undeformable in the plane of a cross-section.
   This assumption was considered by all authors, who investigated
   the non-uniform torsion of thin-walled beams. The wall thickness is
   a very important factor in justifying this assumption, since only
   primary warping is considered, and the theoretical investigations
   confirmed the validity of this assumption.

ii) The shear strain of the middle surface of the contour is neglected.
    This assumption was implemented by Vlasov (1961), and consequently
    a general set for stress and strain was derived, similar to those
    in simple beams.

iii) A thin-walled beam behaves as a thin shell, so that the longitudi-
    nal lines of the middle surface remain normal to the line of the
    middle surface after small rotation. This assumption was made
    by Vlasov (1961), and Goodier (1941) to calculate the longitudinal
    warping stress.
1.1.2 Matrix Analysis of Thin-Walled Beams

The analysis of thin-walled beams of open section, using transfer matrices was first approached by Vlasov (1961), the method was simply used by solving the general differential equations, and finding the arbitrary constants of the solution in terms of boundary forces, torque and bimoment, and boundary displacement twist, and rate of twist. Thus the general term of the torque, bimoment, twist and rate of twist throughout the length of the beam are functions of the distance \((x)\) and the initial parameters as they are called by Vlasov.

Renton (1960), presented a direct method in analysing the torsional-flexural buckling of axially-loaded thin-walled beams, using the solution of the governing differential equations developed by Vlasov. He also extended the application of this method for application to a general force system. Different methods of analysis and worked examples were presented, Renton (1967), such as Rayleigh-Ritz method, where the critical load parameter is given by the ratio of the least value of strain energy to the unit potential energy for any possible deflection, the answer is of course, dependent on the deflection mode chosen. Another energy method was used similar to the first called Rayleigh-Timoshenko method, for doubly symmetrical beams, where the shear centre and the centroid coincide, and the warping can be neglected. The strain energy in this method is expressed in terms of loads at any section. It is likely that the stiffness matrix of a thin-walled beam element was first presented by Krahula (1967). However, this matrix is only useful to analyse a beam under a load system applied at the shear centre. Krajcinovic (1969), applying Galerkin's method to the governing differential equation of a thin-walled beam under a general load system applied at an arbitrary point on the cross-section and by using the virtual work principle, he introduced a stiffness and geometric matrix, which may be used in thin-walled beam buckling problems. However it seems that Krajcinovic has used an inconsistent sign convention which led to different signs for off-diagonal stiffness matrix (see Chapter 3). Barsoum and Gallagher (1970) derived the stiffness and
geometric matrix using the principle of potential energy, where the strain energy expression used was made by Bleich (1952). They also used an assumed displacement function to derive their matrix. However the solution is approximate but it is useful for engineering purposes for certain load cases. Rajasekaran (1977), presented what he called 'element-wise approximations', when the beam was physically replaced by an assembly of discrete elements, where stiffness and geometric matrices were derived using an assumed shape of mode deflections, and by considering the second order terms in the strain expression, he was able to analyse the structure in the elastic and plastic regions. The most important feature of Rajasekaran's method is that he reiterated the assumptions of the thin-walled beam theory. Recently Gaafar and Tidbury (1981) in a series of tests on channel cantilevers observed that the thin-walled channel cantilever buckled at a significant distance from the fixed end, when loaded through the centroid of the free end, and a similar channel buckled at the fixed end when loaded through the shear centre. The study of stress distribution using strain gauges, showed that the stress due to bimoment could account for the maximum compressive stress being at a distance from the fixed end. A large deflection analysis was developed by them to agree with these experimental observations.

Baigent and Hancock (1982), presented a matrix method for the analysis of thin-walled beams, the non-uniform torsion effect was included in the matrix displacement analysis. The study also included the eccentricity of the load system from the shear centre, as well as joint types of thin-walled members. Pitched-roof portals constructed from channel sections bent about their major axis and supported by eccentric restraints were tested, and a good agreement between the theoretical and experimental results was observed. However the transformation matrix accounts for load eccentricities but does not include the effects for the axial force effect on the bimoment, when this force is applied at non-zero sectorial points, the good agreement mentioned was because the forces on the tested frame were applied at zero sectorial point.
1.1.3 Instability of Thin-Walled Beams

1.1.3.1 Local Instability

In the last century St Venant presented the general differential equation of equilibrium for a flat plate under compressive loads. See Timoshenko (1961). This equation has an exact solution for simply supported edges providing the plate is compressed along two opposite sides by a uniform stress distribution. However no exact analytical solution has yet been published for non-uniform load distribution. Using the hypothesis of neutral equilibrium Bryan (1891), for the first time, presented the energy method, which has proved to be a powerful tool for the problem of elastic stability, in cases where the analytical solution was most difficult. Thus the analysis of the rectangular plate was moved into a new era, when in the first decade of this century Timoshenko (1961) studied the problem of a rectangular plate under various load and support conditions. A Fourier series was used to represent the deflected shape, which made it possible to analyse the plate with clamped edges. In 1909 Reissner, independent of Timoshenko, presented the solution for the rectangular plate with two clamped edges, and for plate having one edge clamped and the other free, see Bleich (1952).

The first approximate method for the determination of buckling stresses of structures composed from plates, seems to have been presented by Bleich in 1924, see Bleich (1952), by means of the solution of St Venant's differential equation, assuming that the structure is a collection of single thin plate elements, each having a degree of elastic restraint of adjacent corners in the axial direction provided by the plate which has the smaller width to thickness ratio. The necessary conditions for such a solution are that the buckling wavelength of all plates forming the structure is the same. Bleich's solution was considered to be approximate because he assumed that the buckling wavelength is equal to the width of buckling plate, so that the flexibility of restraining plates is linearly proportional to the longitudinal compression stress. Bijlaard (1940) solved the problem taking into consideration the condition of equal buckling wavelength and accounting for the equilibrium and
compatibility between the adjacent plates. So the governing differential equation of equilibrium was found for each individual plate, which provides a number of simultaneous equations equal to the number of plates forming the structure.

The first extensive study of structures composed from thin plates was by Lundquist (1939), when he applied the moment distribution method to the problem of local stability, considering that the adjacent lines between the plates remain straight at the buckling load. The application of the method of moment distribution to the problem of stability of thin walled beams has been further developed by Lundquist and Stowell (1942). They introduced an approximate method for calculating the critical value of the plate coefficient by calculating the restraint coefficient of each plate and taking the average value. This proved to be in good agreement with the exact method in cases where the exact solution was available. One year later both these authors and Schette (1943), developed a procedure for calculating the critical value of plate buckling coefficient and derived expressions for the stiffness and carry-over factors, according to the method of moment distribution, James (1935). Extensive tables and charts were produced by Kroll, Fisher and Heimerl (1943), for the buckling coefficient of beams composed of box, channel, I-section and Z-section types, using moment distribution. However, a comparison between these more accurate results and Bleich's approximation shows that the latter is sufficiently accurate for many engineering purposes, see Winter (1949). Chilver (1951) presented a solution for a channel section under a uniform compression by solving a number of St Venant type simultaneous differential equations, equal to the number of plates forming the channel. He assumed that the longitudinal lines between the adjacent plates remain straight, so that the out-of-plane deflection at the corners is zero. Furthermore he assumed that the internal angle between these adjacent plates is unchanged during buckling and used the condition of equilibrium at the corner. Under the action of compression stress both flanges and the web buckle in m number of sinusoidal waves in the direction of compression, according to Timoshenko (1961). Thus by using all the boundary conditions,
the solution provides the critical stress value. Harvey (1953) followed similar procedures using Timoshenko's concept of a coefficient of edge fixity between adjacent plates, in the form of transcendental equation. A trial and error approximation was used to determine the buckling load coefficient, for a range of web to flange width ratios and beam length to web width ratios. This gave a set of charts for the buckling coefficient for plain, inwardly and outwardly lipped channel sections for different coefficients of edge fixity. Chilver (1953) presented an approach to the solution of problems of local instability of thin walled struts by considering different values of coefficient of edge fixity, where zero value corresponds to hinge support and infinite value corresponds to a fixed support, any value between these two limits represents the value of elastic support between the adjacent plates.

The instability of thin-walled structures under non-uniform compression seems to have been first investigated by Walker (1966). He assumed that the loaded edges are simply supported, while the unloaded edges are either completely restrained laterally and partly restrained in torsion, or are completely free both laterally and torsionally. A plain and lipped channel section was investigated, the web was uniformly loaded, while the flange was eccentrically loaded corresponding to a bending moment about an axis parallel to the web.

The coefficient of restraint (edge fixity) was considered equal in magnitude but different in sign between the web and the flange for the plain channel. In the lipped channel section, the common edge between the flange and the lip was considered to have a simply supported condition, which was justified by Pflüger (1959), for a lip-to-flange width ratio within a certain range. The solution of the equilibrium equation was considered to be a multiplication of two independent functions in both directions of the plate, which in the longitudinal direction was a sine series, while the other was a polynomial. The accuracy of this solution depends on the number of terms of the polynomial series. In addition to this theoretical study, Walker has
included experimental results which support his findings.

Przemieniecki (1968) introduced a method for stability analysis of thin-walled beams using a matrix displacement method. The non-linear total strain-displacement functions for an elastic continuum were considered in matrix form and the strain energy was expressed in terms of deflections and stress. The differentiation of the strain energy with respect to the deflection gives the elastic and geometric stiffness matrix. The instability criterion is that the sum of elastic and geometric stiffness is singular. Thus an eigen value problem is established, and the lowest value of the load factor is the critical load.

Although the finite element method is a powerful tool for analysing thin-walled structures, the fact is that a large size is needed for the stiffness matrix of the structure, which requires a large computer memory. Needless to say many complex structures cannot be solved using other methods, but in practice many simple structures do not need such an expensive technique. Many research workers in the engineering field have tried to explore a new method which is simpler, more economical, and as accurate to replace the finite element method for analysing these simple structures.

Cheung (1968) introduced a method for analysing thin-walled beams, called the finite strip method. This method is similar to the finite element method, but requires only that a beam is divided longitudinally into several strips. The deflections and forces within each strip are defined by the values at its two edges, see Chapter 4, or see Cheung (1976).

In the same year Wittrick (1968), introduced what he called a unified approach to the initial buckling of stiffened panels in compression. He assumed that the panels consisted of a series of long flat strips, rigidly connected together at their edges. Whatever the buckling mode, the individual strips are subjected to a set of forces and moments along their edges which vary sinusoidally in the lengthwise direction during
buckling. An out-of-plane and an in-plane stiffness matrix were computed, but no results were produced.

In a similar way Wittrick and Curzon (1968) presented another study for the local buckling of thin-walled beams under combined shear and compression loading. They assumed that when buckling occurs the lines between adjacent plates remain straight, and the plates are subjected on their long edges to sinusoidally varying edge moments, which in turn produce sinusoidally varying edge rotations. Relations between the edge moments and rotations were obtained in terms of two stability functions, and tables for these stability functions were included.

Przemieniecki (1971) seems to be the first author to introduce an explicit elastic and geometric stiffness matrix for the out-of-plane deflection, which can be used for local buckling of plates and structures. The in-plane deflection was neglected and the line between the adjacent plates considered to remain straight. The results on plate and channel sections were sufficiently accurate when compared with theoretical results. His work was later extended with no changes to the basic assumption, Przemieniecki (1973).

It seems that the most recognised work on the buckling problem by the finite strip method was by Plank and Wittrick (1974). They introduced explicit elastic and geometric stiffness matrices for both in-plane and out-of-plane deflections, for a strip element under combined compression, bending and shear forces. They also removed the restriction of straightness between the adjacent plates, and considered the compatibility of these lines. The results they produced when compared to the theoretical are in excellent agreement for a four strip plate (see Chapter 4).

The problem of local instability of thin-walled sections under combined compression and bending was thoroughly investigated by Rhodes and Harvey (1976). They utilized the restraint imposed by adjacent plates on each other. Allowing each plate to deform in any number of ways compatible to
the deformations of its adjacent plates. The principle of minimum potential energy is applied to establish the relative magnitude of the prescribed deflection functions to give the lowest possible buckling load for the section. This method is only suitable for the analysis of local buckling, of short struts, since the assumption of straight lines at the corner of adjacent plates was imposed. The experimental results on a channel section for different web-to-flange width ratios, showed good agreement with the theory for both compression and bending.

1.1.3.2 Interaction of Local and Member Buckling

Thin-walled beams may fail in any of the following modes:

a) Overall buckling: when the load reaches a certain magnitude, lateral deformation due to bending, torsion, or combined bending and torsion takes place suddenly. Thus the beam buckles, but its cross-section remains undeformed.

b) Local buckling: when the cross-section distorts under a certain applied load that point of the structure loses its loading capacity.

c) Interaction buckling: when both overall and local buckling occur, the beam may buckle laterally and its cross-section undergoes distortion at the same time.

The following is a review of interaction buckling.

Rajasekaran and Murray (1973) studied the coupling of load and member buckling in wide flange thin-walled beams using the virtual work equations in the finite element model in which the effects of local and member buckling are coupled. Results were obtained for the web restraining the flanges.
Rhodes and Harvey (1977) studied the buckling interaction problem of channel section under combined compression and bending. Change of cross-section during local buckling was taken into consideration in the post-buckling range in an approximate matter, which could have important effects on the behaviour of the beam.

A finite element method was used to study the lateral-torsional buckling of I-beams taking into consideration the effect of distortion of the beam cross-section by Johnson and Will (1974). A large number of plate elements were needed to achieve an accurate result. However the method can handle different loading and support conditions.

The finite strip method seems to have been first used for the problem of buckling interaction by Hancock (1977), in a study on I-beams under pure bending about the major axis. He discussed the phenomena of the number of half waves the beam forms before buckling occurs and suggested that local buckling occurs when a beam forms an integer number of half waves. However when a non-integer number of half waves is formed coupled local and torsional buckling occurs. This is true for intermediate length to web ratio, i.e. before the region of complete torsional or Euler buckling. The study also presented cases where a restraint is applied to prevent lateral deflection, or rotations at some edges of the beam. The same problem was treated using an approximate finite element method by Bradford and Trahair (1981), in their study on distortional buckling of doubly symmetric I-section beam columns. The elastic and geometric stiffness matrices were developed in terms of six nodal displacements of the cross-section. The method assumes that the flanges of the I-beam remain undistorted, so the local buckling is independently computed.
1.2 Contribution of the Thesis

The theoretical analysis of thin-walled beams of open sections is fairly well established, Vlasov (1961). However due to the wide application of this type of structure and the interest in buckling failure and collapse modes there was a perceived need for further research into the subject.

The survey of the published works has shown the following:

1. The assumptions used in deriving the underlying theory of thin-walled beams are not clearly stated or easily understood (see Chapter 2).

2. The transformation of a load system from an arbitrary point to the relevant centroid and shear centre is not complete. Thus incorrect stress distribution may result (see Chapter 3).

3. Several methods are found in the literature for analysing the local, torsional and member buckling of thin-walled beams under compression or bending stresses. But it seems no one has attempted to study the buckling behaviour under stress systems which are produced by the torsion theory of thin-walled beams, under arbitrary load systems - the case which is often found in practice.

This thesis attempts to cover the following areas:

a) Refinement to Vlasov's theory to clarify the confusions and ambiguities

b) Derive a general elastic stiffness matrix, and a general transformation matrix to account for a load system at an arbitrary point on the cross-section

c) Calculate the stress distribution on the cross-section of a thin-walled beam
d) Application of the finite strip method to investigate the buckling loads using an eigenvalue approach

e) Determine the eigen vector which represents the mode deformation and therefore establish the buckling type as local, torsional, Euler or interaction of more than one.
2. THEORY OF THIN-WALLED OPEN SECTION STRUCTURES

2.1 Introduction

Beams composed from thin plates are popular with designers, the advantages being that they are easy to produce and assemble, their performance under different force systems giving high efficiency in terms of their weight versus load.

The essential approach to thin-walled beams of open sections is by the theory of warping, developed by Wagner (1936), then Vlasov (1961), although they have mainly considered the problem of elastic deformation.

Before the study of the behaviour of thin-walled beams of open section can be carried out certain assumptions are made:

a) The cross-section is rigid (undeformable) so its middle line will move only as one body in its plane. This will cause normal and tangential stresses in the direction perpendicular to the wall to vanish.

b) Under twisting moments, points on an unrestrained cross-section will move at different rates in the axial direction, thus the plane of a cross-section does not remain plane, it is warped. This is in contrast with the Euler-Bernoulli assumption of plane sections remaining plane.

c) The application of a longitudinal force at an arbitrary position will cause warping, since this force may not be replaced by a statically equivalent longitudinal force. Thus the beam will be subjected to a self-balancing set of longitudinal forces. This is of course different from the elementary theory of beam bending, which does not produce any shear strains and stresses.

d) The shearing deformations in the cross-section defined by the plane between the longitudinal axis (x) and the periphery (s) will be infinitesimal.
2.2 Torsion Warping Theory

When a thin-walled beam of open-section is subjected to twisting moments at its ends, points on the cross-section of the beam move at different rates in the axial direction (Figure 2.1). Thus the plane of a cross-section does not remain plane, it is warped. If one or more cross-sections are restrained against warping, the state of stress in the beam will be totally different to that analysed by St. Venant's theory which states that a beam under twisting moment will suffer only torsional stresses. This theory, if applied to thin-walled beams of open section will result in serious errors. It is obvious that axial stresses due to warping restraint will develop in the beam. Thus the longitudinal rate of twist of the beam will not be constant in regions of constant torque.

There are also axial stresses produced in such a beam when subjected to a longitudinal force applied at an arbitrary point on the cross-section, and these stresses will vary according to the distance away from the point of application. To make the problem understood let us apply a longitudinal force at the tip of a cantilever positioned at the right end of the top flange of I-beam (see Figure 2.2a). It can be seen that the cantilever will suffer four different types of stresses in the axial direction:

a) normal stress due to the four direct forces applied at the far ends of the cross-section of \( P/4 \) (Figure 2.2b).

b) bending stresses due to bending moment about z-axis (Figure 2.2c)

c) bending stresses due to bending moment about y-axis (Figure 2.2d)

d) stresses due to skew-symmetry of four forces of value P/4, which form two bending moments equal in magnitude, but opposite in sign and acting in the plane of the two flanges of the I-beam. This will be known as warping stresses (Figure 2.2e).
### 2.3 The Displacements and Strains on Thin-Walled Beams of Open Section

#### 2.3.1 Out of plane and lateral displacements

The assumption of undeformable cross-section will lead to the fact that any out of plane displacements of the wall of a cross-section must vanish. Also the lateral displacements in the z and y directions as well as any rotations, the beam may undergo, are functions of the longitudinal direction x only. Also the axial displacement of the beam will be functions of x and the profile coordinate of the cross-section (S). It is also assumed that the x-axis and S stay orthogonal while the beam undergoes deformations i.e. $\gamma = 0$ (See Figure 2.3).

#### 2.3.2 The Kinematics of Displacements

If thin-walled beams of open section undergo deformation, and if point B is an arbitrary point on the profile rigidly connected to point A the instantaneous centre of twist (see Figure 2.4), then the kinematical relations for the displacements of point B and the centre of rotation A are as follows:

\[
\begin{align*}
    w_b &= w - (b_z - a_z) + [(b_z - a_z) \cos \phi - (b_y - a_y) \sin \phi] \\
    v_b &= v - (b_y - a_y) + [(b_z - a_z) \sin \phi + (b_y - a_y) \cos \phi]
\end{align*}
\]

If $\phi$ is small

\[
\cos \phi = 1, \quad \sin \phi = \phi
\]

hence

\[
\begin{align*}
    w_b &= w - (b_y - a_y) \phi \\
    v_b &= v + (b_z - a_z) \phi
\end{align*}
\]
The displacements of an arbitrary point $S$ in the middle surface of the cross-section when moved to point $S'$ (see Figure 2.5), are as follows:

$$v_S = v + (z-a_z)\phi$$  \hspace{1cm} (2.3)  

$$w_S = w - (y-a_y) \phi$$  \hspace{1cm} (2.4)  

where $A$ is the centre of twist of the cross-section

$a_y, a_z, y, z$ are the Cartesian coordinates of point $A$, and $S$ respectively.

$v, w, v_S, w_S$ are the displacements in the $y$ and $z$ directions of point $A$ and $S$ respectively.

$n, \xi$ are the displacements of point $S$ in the tangential and perpendicular directions of the contour.

From Figure 2.5 the following kinematic relations are observed.

$$r = (z-a_z) \sin \alpha - (y-a_y) \cos \alpha$$  \hspace{1cm} (2.5)  

$$r_n = (z-a_z) \cos \alpha + (y-a_y) \sin \alpha$$  \hspace{1cm} (2.6)  

$$\vec{n} = w_s \cos \alpha + v_s \sin \alpha$$  \hspace{1cm} (2.7)  

$$\vec{\xi} = w_s \sin \alpha - v_s \cos \alpha$$  \hspace{1cm} (2.8)  

Now substituting Equations (2.3) and (2.4) into Equations (2.7) and (2.8) we have:

$$\vec{n} = (w - (y-a_y)\phi) \cos \alpha + (v + (z-a_z) \phi) \sin \alpha$$  \hspace{1cm} (2.9)  

$$\vec{\xi} = (w - (y-a_y)\phi) \sin \alpha - (v + (z-a_z) \phi) \cos \alpha$$  \hspace{1cm} (2.10)  

Rewriting equations (2.9) and (2.10):
\[ \tilde{\eta} = v \sin \alpha + w \cos \alpha + [(z-a_z) \sin \alpha - (y-a_y) \cos \alpha] \phi \]  
(2.11)

\[ \tilde{\xi} = -v \cos \alpha + w \sin \alpha - [(z-a_z) \cos \alpha + (y-a_y) \sin \alpha] \phi \]  
(2.12)

Recalling equations (2.5) and (2.6) and substituting into equations (2.11) and (2.12):

\[ \tilde{\eta} = v \sin \alpha + w \cos \alpha + r \phi \]  
(2.13)

\[ \tilde{\xi} = -v \cos \alpha + w \sin \alpha - r \tilde{\eta} \phi \]  
(2.14)

To make clear what was said about the orthogonality between the longitudinal axis \(-x\) and the peripheral coordinate \(S\), we observe the displacements of an element cut-out from the beam, as in Figure 2.6, so the summation of angles \(\gamma_1\) and \(\gamma_2\) is the total change of angle between the two axes \(x\) and \(S\).

\[ \gamma_1 = \frac{\delta u}{\delta \gamma} \]  
(2.15)

\[ \gamma_2 = \frac{\delta \tilde{\eta}}{\delta x} \]

\[ \gamma = \gamma_1 + \gamma_2 = \frac{\delta u}{\delta \gamma} + \frac{\delta \tilde{\eta}}{\delta x} \]  
(2.16)

as \(\gamma = 0\) equation (2.16) can be written in the form:

\[ u = - \int \frac{\delta \tilde{\eta}}{\delta x} \, ds + u_0 \]  
(2.17)

where \(u_0\) is the integration constant.

Differentiating equation (2.13) with respect to the variable \(x\) and substituting in equation (2.17) noting that from Figure (2.7) the following relations,
dy = \sin \alpha \, ds

dz = \cos \alpha \, ds

d\omega = r \, ds

We have \( u = u_0 - v'y - w'z - \psi'\omega \)

where prime designates \( \left( \frac{3}{3x} \right) \)

where \( u \) in the last equation is the overall longitudinal displacement of an arbitrary point \( S \) of Cartesian coordinate \( (y, z) \) and sectorial coordinate \( \omega \).

\( u_0 \) is the integration constant, which represents the longitudinal displacement of the origin of the peripheral coordinate, \( S \), at \( x = \) const.

In other words \( u_0 \) is the longitudinal displacement of the beam due to pure extension, depending only on the variable \( x \).

\( v', w' \) are the rate of change of displacements of point \( S \) with respect to \( x \)-axis in the \( y \) and \( z \) directions respectively.

\( \phi' \) is the rate of change of twist of the beam with respect to the \( x \)-axis. The value of \( \phi'\omega \) represents the longitudinal displacement of point \( S \) due to the rate of change of twist. Note that the sectorial coordinate \( \omega \) is a function of the peripheral coordinate \( S \), thus the value of \( \phi'\omega \) at \( x = \) const, which we call the warping, varies linearly on the cross-section according to the law of sectorial area.
2.4 The Law of Sectorial Area

2.4.1 The Determination of Sectorial Area Expressions

Let $\omega_A$ and $\omega_D$ be sectorial areas of the profile shown in Figure 2.8a, their poles being A and D respectively. The sectorial area is twice the area swept by the sectorial radius $r$ from point $S_0 (y_0, z_0) = 0,$ to point $S (y_0 + \Delta y, z_0 + \Delta z),$ where $S_0$ is the point where the profile $S$ is measured, i.e. $\omega_A = \omega_D = 0,$ at $S = 0.$

Now from Figure 2.8b twice the shaded area is

$$\int d\omega_A = (z_0 - a_z)dy - (y_0 - a_y)dz = (2.20)$$

where $a_y, a_z$ are the coordinates of pole A referred to the system OYZ in Figure 2.8a.

Similarly for pole D

$$\int d\omega_D = (z_0 - d_z)dy - (y_0 - d_y)dz = (2.21)$$

Integrating equations 2.20 and 2.21:

$$\omega_A = (z_0 - a_z)y - (y_0 - a_y)z + C = (2.22)$$

Imposing the condition $\omega_A = 0,$ at $y=y_0,$ and $z=z_0$

$$0 = (z_0 - a_z)y_0 - (y_0 - a_y)z_0 + C$$

$$C = a_z y_0 - a_y z_0 = (2.23)$$

Substituting the value of $C$ into equation (2.22):

$$\omega_A = (z_0 - a_z)(y-y_0) - (y_0 - a_y)(z-z_0) = (2.24)$$

Similarly for $\omega_D$
\[ \omega_D = (z_0 - d_z)(y - y_0) - (y_0 - d_y)(z - z_0) \]  \hspace{1cm} (2.25)

Subtracting equation (2.25) from equation (2.24):

\[ \omega_A = \omega_D + (a_y - d_y)(z - z_0) - (a_z - d_z)(y - y_0) \]  \hspace{1cm} (2.26)

For the calculation of \( \omega_A \) and \( \omega_D \) we have \( A \) and \( D \) as two arbitrary poles. Now if we impose certain conditions on a pole point, which we will call the orthogonality conditions, which are in addition to the latter one, i.e. \( (\omega = 0, \ y_0 = y_0, \ \text{and} \ z = z_0) \). They are

\[
\begin{align*}
S_w &= \int_A \omega \ dA = 0 \\
S_{wy} &= \int_A \omega y \ dA = 0 \\
S_{wz} &= \int_A \omega z \ dA = 0 \\
r &= \int_A \omega^2 dA
\end{align*}
\]  \hspace{1cm} (2.27)

Let us call the pole \( A \) when satisfying equations (2.27), the principal sectorial pole. Pole \( D \) is any arbitrary pole, since from equation (2.26) we have

\[
\int_A \omega_A dA = \int_A \omega_D dA + (a_y - d_y) \int_A (z - z_0) dA - (a_z - d_z) \int_A (y - y_0) dA
\]  \hspace{1cm} (2.28)

If the origin of the coordinate system \( OYZ \) coincides with the centroid of the cross-section we have

\[
\int_A z \ dA = \int_A y \ dA = 0
\]

Equation 2.28 reads
Substituting equation (2.29) into (2.26) we have

\[
\omega_A = \omega_D + (a_y - d_y)z - (a_z - d_z)y - \frac{S_{\text{WD}}}{A} \tag{2.30}
\]

where \( S_{\text{WD}} = \int_A \omega_D \, dA \)

\[
A = \int_A dA
\]

Also by multiplying equation (2.26) by \( ydA \) and \( zdA \) respectively and integrating over the whole area of the cross-section \( A \), we have

\[
\int_A \omega_A \, ydA = \int_A \omega_D \, ydA + (a_y - d_y) \int_A (z - z_0) \, ydA - (a_z - d_z) \int_A (y - y_0) \, ydA
\]

\[
\int_A \omega_A \, zdA = \int_A \omega_D \, zdA + (a_y - d_y) \int_A (z - z_0) \, zdA - (a_z - d_z) \int_A (y - y_0) \, zdA
\]

Now imposing condition (2.27):

\[
S_{\text{WyD}} = -(a_y - d_y)I_{yz} + (a_z - d_z)I_z \tag{2.31}
\]

\[
S_{\text{WzD}} = -(a_y - d_y)I_y + (a_z - d_z)I_{yz} \tag{2.32}
\]

where

\[
S_{\text{WyD}} = \int_A \omega_D \, ydA, \quad S_{\text{WzD}} = \int_A \omega_D \, zdA
\]
\[ I_y = \int z^2 dA, \quad I_z = \int y^2 dA \]
\[ I_{yz} = \int yz dA \]

Solving equations (2.31) and (2.32)

\[ a_y - d_y = \frac{-(S_{yzD} I_z - S_{wyD} I_{yz})}{I_y I_z - I_{yz}^2} \quad (2.33) \]
\[ a_z - d_z = \frac{(S_{wyD} I_y - S_{wzD} I_{yz})}{I_y I_z - I_{yz}^2} \quad (2.34) \]

Another relation can be established from equation 2.26, by integrating the square of both sides of the equation with respect to the differential area dA along the whole area of the cross-section

\[ \Gamma_A = \Gamma_D + 2(a_y - d_y) S_{wzD} - 2(a_z - d_z) S_{wyD} + (a_y - d_y)^2 I_y \]
\[ - 2(a_y - d_y)(a_z - d_z) I_{yz} + (a_z - d_z)^2 I_z - \frac{S_{wD}^2}{A} \]

or by substituting the values of \( S_{wyD} \) and \( S_{wzD} \) from equations (2.31) and (2.32), we have

\[ \Gamma_A = \Gamma_D - (a_y - d_y)^2 I_y + 2(a_y - d_y)(a_z - d_z) I_{yz} - (a_z - d_z)^2 I_z - \frac{S_{wD}^2}{D} \quad (2.35) \]

or

\[ \Gamma_A = \Gamma_D + (a_y - d_y) S_{wzD} - (a_z - d_z) S_{wyD} - \frac{S_{wD}^2}{A} \quad (2.36) \]
It may be useful to mention that the point \( S_0(y_0, z_0) = 0 \), is in fact solely dependent upon the cross-section geometry, and its determination can be carried out by imposing the orthogonality conditions in addition to the first condition, i.e. \( \omega_A = \omega_D = 0 \) at \( S = 0 \), this point will be called the zero centroidal point.

2.4.2 The Location of Zero Centroidal Pole \( S_0 \) and the Principal Sectorial Pole

Each cross-section has one or more than one (up to infinity) zero centroidal points, depending on the geometry of the cross-section. To determine the location of zero centroidal pole it is advised firstly to determine the location of the principal sectorial pole, by imposing equations (2.32 and 2.34) where \( d_y \) and \( d_z \) are \( y \) and \( z \) coordinates of any point \( D \) on the cross-section (see Figure 2.9), where the values of \( S_{wyD} \) and \( S_{wzD} \) do not change when we carry the integration from any starting point (say \( S_1 \)) at the extreme of the cross-section.

If \( D \) is an arbitrary sectorial pole then,

\[
\omega_D(S_0, S) = \omega_D(S_1, S) - \omega_D(S_0, S_1) \tag{2.37}
\]

Multiplying both sides by \( y dA \) and \( z dA \) respectively and carrying the integration on the whole area \( A \), equation (2.37) reads:

\[
\int_A \omega_D(S_0, S) y dA = \int_A \omega_D(S_1, S) y dA - \omega_D(S_0, S_1) \int_A y dA \tag{2.38}
\]

\[
\int_A \omega_D(S_0, S) z dA = \int_A \omega_D(S_1, S) z dA - \omega_D(S_0, S_1) \int_A z dA \tag{2.39}
\]
Since the origin of the coordinate system coincides with the centroid

\[ \int_A y\, dA = \int_A z\, dA = 0 \]

Note that \( \omega_D(S_o, S_1) \) is a constant value, see Figure 2.10.
Hence equations (2.38 and 2.39) read as follows:

\[
S_{wyA}(S_o, S) = S_{wyA}(S_1, S) \quad (2.40)
\]

\[
S_{wzA}(S_o, S) = S_{wzA}(S_1, S) \quad (2.41)
\]

By determining the coordinate of principal pole \( a_y \) and \( a_z \), we use
equation (2.37) once more by multiplying both sides by \( dA \) and inte-
grating over the whole area of the cross-section \( A \), and changing
from pole \( D \) to the principal pole \( A \).

\[
\int_A \omega_A(S_o, S)\, dA = \int_A \omega_A(S_1, S)\, dA + \int_A \omega_A(S_o, S_1)\, dA \quad (2.42)
\]

\[
\omega_A(S_o, S_1) = -\frac{1}{A} \int_A \omega_A(S_1, S) \quad (2.43)
\]

where \( \int_A \omega_A(S_o, S)\, dA = 0 \)

From equation (2.43) we can determine the zero centroidal pole. coor-
dinates \( y_o \) and \( z_o \), which will help to find the sectorial properties
of such cross section. Other methods may be used to determine the
sectorial properties of such cross-section, such as Zbichowski-
Koscia (1967) and Timoshenko (1961).
2.5 Derivation of the Normal Stress Formulae

2.5.1 Assumptions of Elastic Beams and the Stress Formulae

It was assumed that the displacements of the thin-walled beams of open section are too small in comparison with the cross-sectional dimensions, thus the assumptions of elastic beams are valid. Hence

\[ \sigma = \varepsilon E \]  

(2.44)

where \( \sigma \) is the normal stress
\( \varepsilon \) is the strain
\( E \) is the modulus of elasticity

but \( \varepsilon = \frac{\partial u}{\partial x} = (u \text{ is the normal displacements in the } x \text{ direction}) \)

and from equation (2.9) we have

\[ \varepsilon = u_0' - v\gamma - w\zeta - \phi \omega \]  

(2.45)

Substituting the value of \( \varepsilon \) from equation (2.45) into equation (2.44):

\[ \sigma = -E (-u_0' + v\gamma + w\zeta + \phi \omega) \]  

(2.46)

The normal stress \( \sigma \) in equation (2.46) is a function of \( x \) and \( S \), and the representation can be seen in Figure (2.11), also we can introduce the system of generalized forces acting on the cross-section \( x = \text{const.} \)

\[ \bar{N} = \int A \sigma \, dA \]  

(2.47)

\[ \bar{M}_y = -\int A \sigma z \, dA \]  

(2.48)

\[ \bar{M}_z = \int A \sigma y \, dA \]  

(2.49)
where \( N \) is the axial force
\[
N = \sum \int t \, dS
\]
\( t \) is the wall thickness.

Substituting the value of normal stress \( \sigma \) from equation (2.46) into equations (2.47, 2.48 and 2.49), noting that \( u_0, v, w \) and \( \phi \) are functions of \( x \) only, we have the following equations:

\[
N = -E \left( - u_0 \int A \right. + v'' \int ydA + w'' \int zdA + \phi'' \int wdA \right)
\]

(2.50)

\[
M_y = E \left( - u_0 \int zdA + v'' \int yzdA + w'' \int z^2dA + \phi'' \int wzdA \right)
\]

(2.51)

\[
M_z = -E \left( - u_0 \int ydA + v'' \int y^2dA + w'' \int yzdA + \phi'' \int wydA \right)
\]

(2.52)

In these equations the sectorial coordinate \( \omega \) was chosen arbitrarily. Since we want to find the general solution for a structure rotating about any arbitrary point, which is different to the approach by Vlasov (1961), Oden (1981) and Gjelsvik (1981), as it will be shown later. Also \( \bar{M} \) was assumed to represent the total bending moment on the cross-section at \( x = \text{const} \), and this will include the bending moments due to the axial forces at positions remote from the centroid, in addition to the direct bending moment.

Let us assume the integrals in equations (2.50, 2.51 and 2.52) as follows:
\[ A \int dA = A \] total area of the cross section
\[ A \int ydA = S_z \] first moment of area of the cross-section about axis-z
\[ A \int zdA = S_y \] first moment of area of the cross-section about axis-y
\[ A \int y^2dA = I_z \] second moment of area of the cross-section about axis-z
\[ A \int z^2dA = I_y \] second moment of area of the cross-section about axis-y
\[ A \int yzdA = I_{yz} \] product moment of area of the cross-section

In addition to these expressions, which are well known from the strength of materials, there are new expressions to be considered, which are related to the law of sectorial area, these are:

\[ \int \omega dA = S_w \] first moment of sectorial area
\[ A \int \omega zdA = S_{wz} \] product moment of sectorial area about y-axis
\[ A \int \omega ydA = S_{wy} \] product moment of sectorial area about z-axis
\[ A \int \omega^2dA = I_r \] second moment of sectorial area

Substituting these expressions into equations (2.50, 2.51 and 2.52), we have

\[ N = -E (-u_0'A + v'S_z + w'S_y + \phi'S_w) \] (2.53)
\[ M_y = E (-u_0'S_y + v'I_{yz} + w'I_y + \phi'S_{wz}) \] (2.54)
\[ M_z = -E(-u_0'S_z + v'I_z + w'I_{yz} + \phi'S_{wy}) \] (2.55)

If the origin of the Cartesian coordinate system coincides with the centroid of the beam, as we can easily choose, then the expressions \( S_y \) and \( S_z \) will vanish, hence equations (2.53, 2.54 and 2.55) can be written as follows:
\[ N = -E (-uO' + \phi"S_w) \]  
\[ \bar{M}_y = E (v"I_{yz} + w"I_y + \phi"S_{wz}) \]  
\[ \bar{M}_z = -E (v"I_z + w"I_{yz} + \phi"S_{wy}) \]

or by solving these equations:

\[ -Eu' = -N/A - \phi"E S_w/A \]  
\[ Ew" = - [(\bar{M}_y I_z + \bar{M}_z I_{yz}) + E(S_{wy} I_y S_{wz} I_{yz})]\] \(/(I_y I_z - I_{yz}^2) \)

Substituting equations (2.59, 2.60, 2.61) into equation (2.46) and rewriting:

\[ \sigma = N/A + \frac{\bar{M}_z I_y + \bar{M}_y I_{yz}}{I_y I_z - I_{yz}^2} - \frac{\bar{M}_y I_z + \bar{M}_z I_{yz}}{I_y I_z - I_{yz}^2} \]

\[ - (\omega - \frac{S_{wz} I_z - S_{wy} I_{yz}}{I_y I_z - I_{yz}^2} - \frac{S_{wz} I_z - S_{wy} I_{yz}}{I_y I_z - I_{yz}^2})E \phi" \]  

Equation (2.62) is applicable for any sectorial pole, therefore let us now take an arbitrary sectorial pole D and write equation (2.62) accordingly:
\[
\sigma = \frac{N}{A} + \frac{M_I y + M_I yz}{I_{y^2} - I_{yz}^2} \quad y - \frac{M_{yz} + M_{yz}^2}{I_{y^2} - I_{yz}^2} \quad z
\]

- \left( \omega_D - \frac{S_{wyD} I_y - S_{wzD} I_{yz}}{I_{y^2} - I_{yz}^2} \right) \quad y - \frac{S_{wzD} I_z - S_{wyD} I_{yz}}{I_{y^2} - I_{yz}^2} \quad z - \frac{S_{wD}}{A} \quad E'' \]

(2.63)

Substitute equations (2.33) and (2.34) into equation (2.63):

\[
\sigma = \frac{N}{A} + \frac{M_I y + M_I yz}{I_{y^2} - I_{yz}^2} \quad y - \frac{M_{yz} + M_{yz}^2}{I_{y^2} - I_{yz}^2} \quad z
\]

- \left( \omega_D - (a_z - d_z) y + (a_y - d_y) z - \frac{S_{wD}}{A} \right) \quad E'' \]

(2.64)

It is obvious that the coefficient in the last term in the outer parentheses is equal to \( \omega_A \) as in equation (2.30) where \( A \) is the principal sectorial pole, hence equation (2.64) can be written in the form:

\[
\sigma = \frac{N}{A} + \frac{M_I y + M_I yz}{I_{y^2} - I_{yz}^2} \quad y - \frac{M_{yz} + M_{yz}^2}{I_{y^2} - I_{yz}^2} \quad z - \omega_A \quad E'' \]

(2.65)

Multiplying both sides of equation (2.65) by \( \omega_A dA \) and integrating on the whole area of the cross-section and applying the orthogonality condition from equations (2.27), while point \( A \) is the principal sectorial pole, we have:

\[
\int_A \sigma \omega_A dA = -E \Gamma_A \quad E'' \]

(2.66)
where \( \int \omega_A \, dA = \int \omega_A y \, dA = \int \omega_A z \, dA = 0 \)

and \( \Gamma_A = \int \omega_A^2 \, dA \)

Let us call the force \( \int \sigma \omega_A \, dA \) the bimoment and give it a symbol \( B \). Hence

\[ B = -E \Gamma_A \phi'' \]

or

\[ \phi'' = \frac{-B}{E\Gamma} \]  \hspace{1cm} (2.67)

Substituting \( \phi'' \) from equation (2.67) into equation (2.65) we have the final form of general normal stress expression in the thin-walled beams of open section:

\[ \sigma = \frac{N}{A} + \frac{M_z I_y + M_y I_{yz}}{I_y I_z - I_{yz}^2} z - \frac{M_y I_z + M_z I_{yz}}{I_y I_z - I_{yz}^2} y + B \frac{\omega}{\Gamma} \]  \hspace{1cm} (2.68)

where \( \omega = \omega_A \) the sectorial coordinate with respect to the principal sectorial pole.

It can be seen if the coordinate system OYZ is identical with the principal axes of a cross-section, then \( I_{yz} = 0 \) and equation (2.68) can be read in the following simpler form:

\[ \sigma = \frac{N}{A} + \frac{M_z}{I_z} y - \frac{M_y}{I_y} y + B \frac{\omega}{\Gamma} \]  \hspace{1cm} (2.69)

Equation (2.69) in fact was found by Vlasov (1961), but the difference in this approach is that he started the derivation by applying the orthogonality conditions from the beginning considering that the sectorial coordinate \( \omega \) is associated with the principal sectorial pole.
Our approach is to start with an arbitrary sectorial pole and consequently through mathematical verification come to the final result. Another point which can be mentioned here is that the definition of the bimoment automatically emerged from equations (2.65) and (2.66) where Vlasov had to define the bimoment one stage before.

It is also useful to mention that Oden (1981), did perhaps notice the lack of generality of Vlasov's approach, but his solution, in the author's opinion, was confusing for two reasons (see Oden (1981), pages 215-217):

1. The principal sectorial pole and the centre of rotation do coincide as equations (2.64 and 2.65) prove, since we have taken an arbitrary sectorial pole D. However a comparison between equations (2.64 and 2.65) shows that the cross-section rotates about the principal sectorial pole A.

2. The interpretation of the constant term $S_w/A$ in equation (2.59) was totally misjudged because the term $S_w = \int_A w dA$ was meant to be with respect to the arbitrary pole and not to the principal sectorial pole as was later interpreted. In any case if the calculation of the normal stress distribution is to start from a certain point, say $S_1$, the equations (2.68 and 2.69) can be used by substituting equation (2.37) into these equations.

The usefulness of the author's approach will also be seen later, as we establish the equation for the total bimoment when a force system is applied at a point different from the principal sectorial pole.

2.5.2 Bimoment

For thin-walled beams of open section, as was seen in equations (2.66) and (2.67) the bimoment is a representation of the axial stress distribution in terms of sectorial coordinate, along the profile of a cross-section. The physical representation of a bimoment can be by a
pair of equal and opposite moments, or can also be equivalent to four forces acting as shown in Figure 2.2e.

Thus bimoment is a vector quantity as it has a specific direction such as a bending moment, and therefore is dependent on the sign convention followed. However, the usual rules of vector resolution do not apply since a bimoment is a self-equilibrating force.

2.6 Stress Distribution on Thin Walled Beams of Open Section

2.6.1 Assumptions

The stress distributions in open-section are quite different from those in solid or closed sections, since a few assumptions were made due to the fact that the thickness of the wall is very small compared to the outer dimension, Vlasov (1961); Oden (1981). The first assumption is that the normal stresses $\sigma_x$ (Figure 2.12a), is uniform over the wall thickness; and the second assumption is that the stresses and strains normal to the wall surface are very small, hence they are neglected, i.e.

$$\sigma_n = \tau_{xn} = \tau_{sn} = \epsilon_n = \gamma_{xn} = \gamma_{sn} = 0$$

one only shearing stress is considered. $\tau_{xs}$, (Figure 2.12b) which is due to two different modes of deformations, Vlasov (1961). One mode is due to external torsional moments, transverse loads, with non-uniform axial deformation of a free to warp cross section, which result in a linear distribution of shearing stresses over the wall thickness, this will be given the symbol $(\tau_{xs})_T$ (Figure 2.12c). The other mode of deformations is due to the lateral shear forces in the direction of the tangent on the contour arc (Figure 2.12d), this will result in uniform tangential stresses over the wall thickness and will be given the symbol $(\tau_{xs})_V$, hence the total shear stresses over the cross-section will be:
\[ \tau_{xs} = (\tau_{xs})_T + (\tau_{xs})_V \]  

(2.70)

For the future we will denote \( \tau \) without the suffix xs.

### 2.6.2 Determination of Tangential Stresses

The condition of equilibrium of an infinitesimally small shell element of the beam will be used to determine the relationship between the normal stresses and tangential stresses as seen in Figure 2.13:

\[ \delta(\sigma t) \, dS + \delta(\tau t) \, dx + P_x \, dx \, dS = 0 \]  

(2.71)

where \( t = t(s) \) the wall thickness

\( P_x = P_x(x, s) \) the projection of external surface load on x-axis

Dividing equation (2.71) by \( dx \) and rewriting:

\[ \delta(\tau t) = - P_x \, dS - \frac{\partial \sigma}{\partial x} \, t \, dS \]  

(2.72)

Integrating equation (2.72) along the profile \( S \):

\[ \tau = \frac{1}{t} \left[ S_0(x) - \int P_x \, dS - \int \frac{\partial \sigma}{\partial x} \, t \, dS \right] \]  

(2.73)

where \( S_0(x) \) is an arbitrary function of \( x \) as a result of integration with respect to \( S \), which can be determined from the initial conditions at \( S = 0 \), \( \tau(x, 0) = \frac{1}{t(0)} \, S_0(x) \).
or \[ S_0(x) = t(o) \tau(x, o) \]

Recalling equation (2.46) and substituting into equation (2.73)

\[
\tau = \frac{1}{\xi} \left[ S_0(x) - \int_0^S P_x dS - \int_0^S E \left( u'' - v''y - w''z - \phi''\omega \right) tdS \right]
\]

or

\[
\tau = \frac{1}{\xi} \left[ S_0(x) - \int_0^S P_x dS - E(u''A(s) - v''S_z(s) - w''S_y(s) - \phi''S_w(s)) \right]
\]

where

\[
A(s) = \int_0^S dA, \quad S_z = \int_0^S ydA, \quad S_y = \int_0^S zdA, \quad S_w = \int_0^S \omega dA, \quad dA = tdS
\]

For more useful general equations we recall equation (2.68) and substitute it into equation (2.73):

\[
\tau = \frac{1}{\xi} \left[ S_0(x) - \int_0^S P_x dA - \int_0^S \left( N'_A + \frac{M'_I y + M'_I y z}{I_y I_z - I^2_{yz}} \right) y - \frac{M'_I y z + M'_I z y z}{I_y I_z - I^2_{yz}} z + B'_\omega \right) tdS \right]
\]

or

\[
\tau = \frac{1}{\xi} \left[ S_0(x) - \int_0^S P_x dS - N'_A A(s) + \frac{M'_I y + M'_I y z}{I_y I_z - I^2_{yz}} S_z(s) + \frac{M'_I y z + M'_I z y z}{I_y I_z - I^2_{yz}} S_y(s) - \frac{B'_\omega}{Î} S_w(s) \right]
\]

where \( \frac{dM}{dx} = N', \quad \frac{dN}{dx} = N', \quad \frac{dB}{dx} = B' \)}
For a more useful and less general expression take \( N' = 0 \), for a constant axial force along the x-axis, also if the axes \( y, z \) are the principal axes, \( I_{yz} = 0 \), hence equation (2.75) can be written as follows:

\[
\tau = \frac{1}{t} \left[ S_q(x) - \int_0^S P_x dS - \frac{M_z}{I_z} S_z(s) + \frac{M_y}{I_y} S_y(s) - \frac{B'}{r} S_w(s) \right] \tag{2.76}
\]

We note that in equations (2.75) and (2.76) the values \( S_z(s), S_y(s) \) and \( S_w(s) \) are the value of limited integration between \( S = 0 \), which represents the starting point on the profile, up to an arbitrary point \( S \) on the cross-section, at which the shearing stresses \( \tau \) to be determined. Also \( S_w \) is referred to as the principal sectorial pole (shear centre).

In Figure 2.14 the thin-walled beam of open section is subjected to a general system of transverse loads and a distributed external torque of intensity \( m \) per unit length. Since the bimoment \( B \) on the structure is self-equilibrating the equations of equilibrium for the rest of the forces will not be affected. Thus:

\[
\frac{dQ_y}{dx} = -p_y, \quad \frac{dQ_z}{dx} = -p_z
\]

\[
\frac{dM_z}{dx} = Q_y, \quad \frac{dM_y}{dx} = -Q_z
\]

Substituting equation (2.77) into equations (2.75 and 2.76) we have

\[
\tau = \frac{1}{t} \left[ S_q(x) - \int_0^S P_x dS - \frac{N'}{A} A(s) - \frac{Q_y I_y - Q_z I_{yz}}{I_y I_z - I_{yz}^2} S_z(s) - \frac{Q_z I_z - Q_y I_{yz}}{I_y I_z - I_{yz}^2} S_y(s) - \frac{T_w}{r} S_w(s) \right] \tag{2.78}
\]
and
\[ \tau = \frac{1}{t} \left[ S_0(x) - \int_0^s P_x \, ds - \frac{N'_s}{A(s)} - \frac{Q_y}{I_z} S_z(s) - \frac{Q_z}{I_y} S_y(s) - \frac{T_w}{T} S_w(s) \right] \quad (2.79) \]

where \( T_w \) can be defined by the equation
\[ T_w = \frac{dB}{dx} \quad (2.80) \]

Comparing equations (2.74 and 2.76) we have:
\[ \frac{dB}{dx} = -E \Gamma^\phi'' \quad (2.81) \]

or
\[ T_w = -E \Gamma^\phi'' \quad (2.82) \]

2.7 The Differential Equations of Equilibrium for a Thin-Walled Beam

2.7.1 The State of Equilibrium of a Beam Under a Set of Internal Stresses and External Forces

The element (Figure 2.15) of profile length \( S \) and axial length \( \Delta x \), and thickness \( t \) was examined for equilibrium under the prescribed forces and stresses, which include internal normal and shearing stresses, external torsional moment and body forces. These equilibrium conditions are:
\[ \sum F_x = 0, \int S \frac{\partial (\alpha t)}{\partial x} \, dx \, ds + (q_L - q_K + \int P_x \, ds) \, dx = 0 \quad (2.83) \]
\[ \sum F_y = 0, \int S \frac{\partial (\alpha t)}{\partial x} \, dx \, ds \sin \alpha + \int P_y \, ds \, dx = 0 \quad (2.83) \]
\[ \sum F_z = 0, \int_s \frac{\partial (\tau t)}{\partial x} \, dx \, dS \cos \alpha + \int_s P_z \, dS \, dx = 0 \]

\[ \sum M_D = 0, \int_s \frac{\partial (\tau t)}{\partial x} \, dx \, dS \left[ (z_0 - d_z) \sin \alpha - (y_0 - d_y) \cos \alpha \right] + T'_v \, dx + \bar{m} \, dx = 0 \]

In equations (2.83) the integrals are carried out with respect to the variables along the length of the profile \( L \). Dividing these equations by \( dx \) and recalling from equations (2.18 and 2.21):

\[ \sin \alpha \, dS = dy \]

\[ \cos \alpha \, dS = dz \]

\[ (z_0 - d_z) \, dy - (y_0 - d_y) \, dz = d\omega_D \]

we find

\[ \int_L \frac{\partial (\tau t)}{\partial x} \, dS + q_L - q_K + \int_L P_x \, dS = 0 \]

\[ \int_L \frac{\partial (\tau t)}{\partial x} \, dy + \int_L P_y \, dS = 0 \]

\[ \int_L \frac{\partial (\tau t)}{\partial x} \, dz + \int_L P_z \, dS = 0 \]

\[ \int_L \frac{\partial (\tau t)}{\partial x} \, d\omega_D + T'_v + \bar{m} = 0 \]
It is assumed that the thickness $t$ is a function of the variable $S$, hence $tdS$ in the first of equations (2.84) can be considered $dA$ and the limit of integration $L$ becomes $A$. In the other three equations integrate by parts, thus:

\[
\int_A \frac{\partial \sigma}{\partial x} dA + qL - q_K + \int_L P_x dS = 0
\]

\[
\left[ \frac{\partial (\tau t)}{\partial x} \right]_K^L - \int_L \frac{3}{\partial S} \left[ \frac{\partial (\tau t)}{\partial x} \right] dS + \int_L P_y dS = 0
\]

\[
\left[ \frac{\partial (\tau t)}{\partial x} \right]_K^L - \int_L \frac{3}{\partial S} \left[ \frac{\partial (\tau t)}{\partial x} \right] dS + \int_L P_z dS = 0
\]

\[
\left[ \frac{\partial (\tau t)}{\partial x} \omega_D \right]_K^L - \int_L \frac{3}{\partial S} \left[ \frac{\partial (\tau t)}{\partial x} \omega_D \right] dS + T_\nu + \bar{m} = 0
\]  

(2.85)

The value of the shear flow $\tau t=q$ per unit length in the longitudinal direction at the extreme points of the element in Figure 2.15 $S=S_K$ and $S=S_L$ must be equal to the shear stress value per unit length in the lateral direction, hence

\[ q_L \text{ (longitudinal)} = q_L \text{ (lateral)} \]  

\[ q_K \text{ (longitudinal)} = q_K \text{ (lateral)} \]  

(2.86)

hence from equations (2.86) we have

\[
\left[ \frac{\partial (\tau t)}{\partial x} \right]_K^L = q_L' Y_L - q_K' Y_K
\]

\[
\left[ \frac{\partial (\tau t)}{\partial x} \right]_K^L = q_L' Z_L - q_K' Z_K
\]  

(2.87)

\[
\left[ \frac{\partial (\tau t)}{\partial x} \omega_D \right]_K^L = q_L' \omega_D - q_K' \omega_D
\]
Recalling equations (2.46) and (2.74):

\[
\frac{\partial \sigma}{\partial x} = E_{U_0}'' - E_{V''}y - E_{W''}z - E_{\phi''} \omega_D
\]

\[
\frac{\partial}{\partial S} \left[ \frac{\partial (tt)}{\partial z} \right] dS = \frac{\partial}{\partial x} \left[ \frac{\partial (tt)}{\partial S} \right] dS =
\]

\[= - \frac{\partial P}{\partial x} dS - E_{U_0}'' dA + E_{V''} y dA + E_{W''} zdA + E_{\phi''} \omega_D dA
\]  

Substituting equations (2.87) and (2.88) into (2.85):

\[\quad - E_{U_0}'' \int A dA - E_{V''} \int A ydA - E_{W''} \int A zdA - E_{\phi''} \int A \omega_D dA
\]

\[+ q_L - q_K + \int_L P_x dS = 0 \]

(2.88)

\[\int_L \frac{\partial P}{\partial x} ydS + E_{U_0}'' \int A ydA - E_{V''} \int A y^2dA - E_{W''} \int A yzdA
\]

\[= - E_{\phi''} \int A \omega_D ydA + q_L' y_L - q_K' y_K + \int_L P_y dS = 0 \]

(2.89)

\[\int_L \frac{\partial P}{\partial x} zdS + E_{U_0}'' \int A zdA - E_{V''} \int A yzdA - E_{W''} \int A z^2dA
\]

\[= - E_{\phi''} \int A \omega_D zdA + q_L' z_L - q_K' z_K + \int_L P_z dS = 0 \]

\[\int_L \frac{\partial P}{\partial x} \omega_D dS + E_{U_0}'' \int A \omega_D dA - E_{V''} \int A \omega_D ydA - E_{W''} \int A \omega_D zdA
\]

\[= - E_{\phi''} \int A \omega_D^2 dA + q_L' \omega_D L - q_K' \omega_D K + \bar{T}_V + \bar{m} = 0 \]

or
\[
\begin{align*}
\text{Eu}_0''A - E\phi'''' & \quad S_{wd} + q_L - q_K + \int P_x dS = 0 \\
-E\phi''''I_z - E\phi''''I_{yz} - E\phi''''S_{wd} + q_L'y' - q_K'y_K + \\
+ \int P_y dS + \int \frac{3P_x}{\partial x} y dS = 0 \\
-E\phi''''I_{yz} - E\phi''''I_y - E\phi''''S_{wd} + q_L'z' - q_K'z_K + \\
+ \int P_z dS + \int \frac{3P_x}{\partial x} z dS = 0 \\
\text{Eu}_0''S_{wd} - E\phi''''S_{wd} - E\phi''''S_{wd} - E\phi''''\Gamma_D + q_L''\omega_{DL} \\
- q_K''\omega_{DK} + T_v' + \bar{m} + \int \frac{3P_x}{\partial x} \omega_{D} dS = 0
\end{align*}
\]

(2.90)

where

\[
\begin{align*}
A &= \int dA, \quad S_{wd} = \int \omega_d dA, \quad I_z = \int y^2 dA, \quad I_{yz} = \int yz dA, \quad S_{wd} = \int \omega_{D} dA, \\
S_{wd} &= \int \omega_{D}^2 dA, \quad I_y = \int z^2 dA, \quad \Gamma_D = \int \omega_{D}^2 dA
\end{align*}
\]

and also when the origin of the axes system coincides with the centroid of the cross-section

\[
\int y dA = \int z dA = 0
\]

Differentiating the first of equations (2.90) with respect to the variable \( x \), and substituting the values of \( u_0''', v''' \) and \( w''' \), from the first three into the fourth:
\[ S_{wD} \left( E \Phi'' S_{wD} - C_1 \right) + E S_{wyD} \left( \frac{S_{wyD} I_y - S_{wzD} I_{yz}}{I_y I_z - I_{yz}^2} \right) \phi'' + S_{wyD} \left( \frac{(C_3 I_{yz} - C_2 I_y)}{I_y I_z - I_{yz}^2} \right) \]

\[ + ES_{wzD} \left( \frac{S_{wzD} I_z - S_{wyD} I_{yz}}{I_y I_z - I_{yz}^2} \right) \phi'' + S_{wzD} \left( \frac{(C_3 I_{yz} - C_3 I_z)}{I_y I_z - I_{yz}^2} \right) - E \Gamma_D \phi'' + C_4 = 0 \]

(2.91)

where:

\[ C_1 = q'^L - q'^K + \int \frac{ap}{\partial x} ds \]

\[ C_2 = q'^{Iy}_L - q'^{Iy}_K + \int P_{ds} + \int \frac{ap}{\partial y} yds \]

\[ C_3 = q'^{Iz}_L - q'^{Iz}_K + \int P_{ds} + \int \frac{ap}{\partial z} zds \]

\[ C_4 = -q'^{JL}_{DK} - q'^{JL}_{DK} + T^I + \bar{m} + \int \frac{ap}{\partial x} w_{ds} \]

(2.92)

Rewriting equation (2.91):

\[ S_{wD}^2 A + \frac{S_{wyD} I_y - S_{wzD} I_{yz}}{I_y I_z - I_{yz}^2} S_{wyD} + \frac{S_{wzD} I_z - S_{wyD} I_{yz}}{I_y I_z - I_{yz}^2} S_{wzD} - \Gamma_D \phi'' = C_2 \frac{S_{wyD} I_y - S_{wzD} I_{yz}}{I_y I_z - I_{yz}^2} \]

\[ - C_3 \frac{S_{wzD} I_z - S_{wyD} I_{yz}}{I_y I_z - I_{yz}^2} - C_1 \frac{S_{wD}}{A} + C_4 = 0 \]

(2.93)

Substituting equations (2.33), (2.34), (2.36) and (2.92), equation (2.93) reads:
\[-E_A \Phi'' (a_z - d_z) \left( q_L^r y_L - q_K^r y_K \right) + \int L \frac{\partial P}{\partial x} y dS + \left( a_y - d_y \right) (q_L^r z_L - q_K^r z_K) + \int L P_z dS + \int L \frac{\partial P}{\partial x} z dS \]

\[-S_{WD}^r \left( q_L^r - q_K^r \right) + \int L \frac{\partial P}{\partial x} dS + q_L^r \omega_{DL} - q_K^r \omega_{DK} + T_V^r + \bar{m} + \int L \frac{\partial P}{\partial x} \omega_D dS = 0 \]

(2.94)

Or

\[-E_A \Phi''' + q_L^r \omega_{DL} - (a_z - d_z) y_L + (a_y - d_y) z_L - \frac{S_{WD}^r}{A} \]

\[+ q_K^r \omega_{DK} - (a_z - d_z) y_K + (a_y - d_y) z_K - \frac{S_{WD}^r}{A} \]

\[+ \bar{m} - (a_z - d_z) \int L P_y dS + (a_y - d_y) \int L P_z dS \]

\[+ \int L \frac{\partial P}{\partial x} \left( \omega_D - (a_z - d_z) y + (a_y - d_y) z - \frac{S_{WD}^r}{A} \right) dS \]

\[+ T_V^r = 0 \]

(2.95)

Recall equation (2.30) and substitute into (2.95):

\[-E_A \Phi'''' + \omega_{AL} \omega_{AK} + \bar{m} - (a_z - d_z) \int L P_y dS + (a_y - d_y) \int L P_z dS + \int L \frac{\partial P}{\partial x} \omega_A dS \]

\[+ T_V^r = 0 \]

(2.96)
In equation (2.96) we note that the fifth and sixth terms are the complementary contribution to the torsional moment \( \bar{m} \), by the external forces \( \mathcal{P}_y \) and \( \mathcal{P}_z \) when calculated about the principal sectorial pole A, rather than the arbitrary pole D, thus we have:

\[
m = \bar{m} - (a_z^2 - d_z) \int P_y \, dS + (a_y^2 - d_y) \int P_z \, dS
\]

(2.97)

The second and third terms are due to the shearing stresses at points L and K, and those two terms vanish if points L and K are the extreme points, since the edges of the structure are free of shear.

The seventh term \( \int \frac{\partial P_x}{\partial x} \omega \, dS \), is the contribution of the longitudinal surface force, to the flexural torsions, and would remind us with the effect of an axial force on the bimoment, which we will see its detail later. It is also important to note that, if the longitudinal force \( P_x \) is not a function of the profile variable \( S \), the whole term will vanish, as we can see from the following

\[
\int \frac{\partial P_x}{\partial x} \omega \, dS = \frac{\partial P_x}{\partial x} \int \omega \, dS
\]

but \( \int \omega \, dS = \frac{1}{t} \int \omega \, dA = 0 \)

where \( \omega \) is the sectorial coordinate about the principal sectorial pole A.

\( t \) thickness

That is to say a uniform axial stress on a cross-section, causes no flexural or torsional stresses.
Let us define now the term $T'_V$ from the conventional theory of pure torsion, which is related to St Venant hypothesis that, this moment will result only from the torsional shearing stresses, so the St Venant torque will be proportional to the rate of change of angle of twist as

$$T'_V = GJ\phi'$$  \hspace{1cm} (2.98)

where $G$ is the modulus of rigidity

$J$ is the torsional moment of area

$$\phi' = \frac{d\phi}{dx}$$

The torsional moment of area $J$ is calculated for a thin-walled beam according to the equation

$$J = \frac{1}{3} \sum_{i=1}^{n} t_i^3 b_i$$

where $n$ is the number of plates forming the cross-section

$t_i$ is the plate thickness

$b_i$ is the plate breadth

(for more detail see Timoshenko (1956), pages 240-246).

From equation (2.98) we have

$$T'_V = GJ\phi''$$  \hspace{1cm} (2.99)

Substitute equation (2.97) and (2.99) into equation (2.96) and let us call the term

$$\int \frac{\partial \omega}{\partial x} \omega dS$$

by the symbol $m_w$: \hspace{1cm} (for more detail see Timoshenko (1956), pages 240-246).
-\( \frac{d^3\phi}{d\xi^3} + GJ\phi'' = -m - m_w - (q_L \omega_{AL} - q_K \omega_{AK}) \)  \hspace{1cm} (2.100)

This is the final form of the differential equation, in general case. For the easier and more common case \( m_w \) will vanish also the shear stresses \( q_L \) and \( q_K \) will vanish if points \( L \) and \( K \) are the extreme points of the profile, which means that equation (2.100) may take the form:

\[-E\Gamma\phi'''' + GJ\phi'' = -m \]  \hspace{1cm} (2.101)

where \( \Gamma = \Gamma_A \) (since \( A \) is the principal sectorial pole).

The equation (2.69) and equation (2.101) were found by Vlasov (1961), but the difference in our approach is significant, in a way proving that the derivation of the differential equation which started by considering an arbitrary pole as a centre of rotation, and consequently led to the fact that the centre of rotation of such an open section is unique and that is the principal sectorial pole. Thus we can apply this on the rest of equations (2.90) so they may read as follows:

\[-EAu'' = q_x \]

\[EI_zv'''' + EI_{yz}w'''' = q_y \]

\[EI_yw'''' + EI_{yz}v'''' = q_z \]

\[-EI\phi'''' - GJ\phi'' = m \]

where \( q_x = \int P_x\,dS \)

\( q_y = \int P_y\,dS \)

\( q_z = \int P_z\,dS \)
are the total projections of the external surface forces in x, y and z directions respectively.

Also from equations (2.82 and 2.98) we find the total torsional moment on the beam, thus:

\[ T = T_v + T_w \]

\[ T = -E I \phi'' + G J \phi' \]  

(2.103)

2.8 Shear Centre

If a thin-walled beam of open section is subjected to a transverse force, passing through a certain point in order to subject the beam to a pure bending only, this point is called shear centre, where the shear centre in fact is the principal sectorial pole, which was defined and investigated in Section 2.5. The determination of the coordinates of the shear centre can be found in many texts, such as Vlasov (1961), Timoshenko (1961), and Oden (1980). In Section 2.5 equations (2.33), (2.34) and (2.43) may be enough to determine the shear centre coordinates, and the distribution of sectorial coordinate \( \omega \), considering that point A is the shear centre and D is any point on the cross-section of the beam, wherein equation (2.43) \( S_o \) is the zero centroid and \( S_1 \) is any arbitrary starting point.

The shear centre and the principal sectorial pole will be coincident only if we have a beam of constant cross-section lengthwise failing to have this condition will result in a complicated process which is beyond the scope of this thesis.
Fig. 2.1: WARping OF THIN-WALLED OPEN SECTION UNDER TORSION (FREE ENDS).
Fig. 2.2: GENERALISED FORCES ON I-BEAM UNDER ECCENTRIC AXIAL LOAD.
Fig. 2.3: ORTHOGONAL AXES FOR THIN SHELL ELEMENT.
Fig. 2.4: DISPLACEMENTS AND ROTATION FOR AN ELEMENT OF THE CROSS-SECTION OF A THIN-WALLED BEAM.
Fig. 2.5: DISPLACEMENTS AND ROTATIONS OF THE CONTOUR OF A THIN-WALLED BEAM.
Fig. 2.6: DISPLACEMENTS AND SHEAR DEFORMATION OF A THIN-WALLED SHELL ELEMENT.
Fig. 2.7: SECTORIAL CO-ORDINATE $dw$ OF POINT $S$
(a) SECTORIAL CO-ORDINATE OF POINT (S) WITH RESPECT TO DIFFERENT SECTORIAL POLES.

(b) GEOMETRY OF SECTORIAL CO-ORDINATE OF POINT S

Fig. 2. 8:
Fig. 2.9: CROSS SECTION OF A CHANNEL BEAM.
Fig. 2.10: RELATIONSHIP BETWEEN SECTORIAL CO-ORDINATES WITH A COMMON POLE BUT DIFFERENT ORIGINS.

Fig. 2.11: AXIAL STRESS ON A THIN-WALLED BEAM CROSS SECTION
Fig. 2.11: AXIAL STRESS ON A THIN-WALLED BEAM CROSS SECTION.
Fig. 2.12: AXIAL AND SHEAR STRESS DISTRIBUTION

Fig. 2.13: SHELL ELEMENT UNDER GENERAL IN-PLANE FORCES
Fig. 2.13: SHELL ELEMENT UNDER GENERAL IN-PLANE FORCES.
Fig 2.14: THIN-WALLED OPEN SECTION ELEMENT UNDER A GENERAL LOAD SYSTEM.
Fig. 2.15: EXTERNAL AND INTERNAL FORCES ON A SHELL ELEMENT OF THIN-WALLED BEAM.
3. A THIN-WALLED BEAM FINITE ELEMENT

3.1 Introduction

Assumptions of the Euler-Bernoulli theory of plane sections remaining plane may physically be interpreted as the shear deformations will be caused to vanish. This may be admissible on the basis that in most skeletal structures the members connecting the actual joints are relatively slender and the deformation is mainly due to bending. The alternative approach often called the Timoshenko theory, attempts to include the shear deformation in the solution which involves a complicated kinematic relationship of the boundary conditions, which results in a rather academic work, seldom available in practice, and thus is not attainable compared with the simpler method. It is often more convenient for the researcher or designer to adopt a detailed finite element analysis to account for these suspected shear deformations, since the boundary conditions can be simulated more closely.

In terms of the finite element method, distributed loads are usually replaced by nodal loads and sometimes moments, thus yielding cubic deformation modes for a uniform beam element and hence the simpler 4x4 stiffness matrix for beam structures appearing in many texts, presented in Table 3.1.

To be useful in plane structures, the axial deformation mode may be added, to give the general 6x6 stiffness matrix in Table 3.2. This step has the implicit assumption that the axial and shear forces pass through the same point i.e. the shear centre is coincident with the centroid. This is only true for a narrow range of sections with double symmetry and offset shear forces cause torsion.

To be applicable to three-dimensional structures, the bending modes about both principal axes must be considered, together with torsion, since the six degrees of freedom at each node necessarily include three rotations. The transformation matrix for a general orientation of the principal planes is presented in various texts, a suitable
coding for the element and its transformation is given by Sharman and Hawkins (1969). The torsional terms assume a linear twist mode consistent with the St Venant torsional theory, and are therefore restricted to solid or thick walled closed sections, or thin walled closed sections with closely spaced rigid diaphragms. Non-coincidence of the shear centre and centroid will cause axial and bending mode coupling, as well as transverse forces not passing through the shear centre cause torsion.

Thus the main purpose of this chapter is to remove these limitations and to present further illustrations of torsion warping theory.

3.2 Thin Walled Beam Stiffness Matrix

The stiffness matrix of the thin-walled beam of open section is the combination of bending and torsion matrices. The first will not be discussed, since it has been dealt with in many texts, such as Marten (1965), Przemieniecki (1968-), Livesley (1975) and many others. It appears that the latter matrix was first derived by Krahula (1967), using the coupling between twist mode (which is defined by \( \phi \)) and the warping mode (which is defined by \( \frac{d\phi}{dx} \)). The solution for the homogeneous differential equation (2.102) i.e. \( m=0 \), was used to define \( \phi \), and \( \frac{d\phi}{dx} \) as well as the corresponding torque and bimoment applied to the element. Kraicinovic (1969) derived an element stiffness matrix while investigating the problem of elastic stability and dynamic response of a structure assembled from thin walled members, using Galerkin's method to find the work done by external and internal forces through a set of assumed virtual displacements. Some of Kraicinovic's matrix element signs are not the same as Krahula's when \( i \neq j \) in the matrix (i.e. off-diagonal terms). The current author reproduced Kraicinovic's work and found inconsistency in sign convention, since a positive twisting mode at \( x=0 \), was associated with a negative twisting moment while a positive twisting mode at \( x=l \), was associated with a positive twisting moment. In the author's opinion, this inconsistency
of sign convention caused the reported different stiffness matrix. Renton (1974), in a study of transmission of non-uniform torsion through joints, derived the torsion warping element stiffness matrix. Although Renton's matrix is the same as Krahula's, the latter was not mentioned in the list of references. It is useful to mention that a complete derivation of the stiffness matrix was not produced in the literature, apart from Krajcinovic's work, which is now proven to be incorrect. In the following using a simple method, and a consistent sign convention, the author derives the element stiffness matrix, according to the known theory of torsion warping.

3.3 Derivation of Torsion Warping Element Stiffness Matrix

The theory of torsion warping couples the twist mode defined by $\phi$, with the warping mode defined by $\frac{db}{dx}$, thus the stiffness matrix for a number of nodes $i$ and $j$, see Figure (3.1), may be defined by

$$\begin{bmatrix} T_i, B_i, T_j, B_j \end{bmatrix}^T = [K][\phi_i, \phi'_i, \phi_j, \phi'_j]^T$$

Recalling the fourth of equations (2.10) we have

$$\phi'' - K^2\phi'' = m/Er \quad (3.2)$$

where $K^2 = GJ/Er$

The solution of the homogeneous differential equations (3.2) i.e. for $m=0$, is

$$\phi = A_1\sinh (Kx) + A_2 \cosh (Kx) + A_3x + A_4 \quad (3.3)$$
where $A_1$, $A_2$, $A_3$ and $A_4$ are arbitrary constants, dependent on the boundary conditions.

Differentiating $\phi$ with respect to $x$ three times, we have

\[
\phi' = A_1 K \cosh (Kx) + A_2 K \sinh (Kx) + A_3 \tag{3.4}
\]

\[
\phi'' = A_1 K^2 \sinh (Kx) + A_2 K^2 \cosh (Kx) \tag{3.5}
\]

\[
\phi''' = A_1 K^3 \cosh (Kx) + A_2 K^3 \sinh (Kx) \tag{3.6}
\]

Recalling now equations (2.67) and (2.96), for the bimoment and torsional moment:

\[
B = -E I \phi'' \tag{3.7}
\]

\[
T = G J \phi' - E I \phi''' \tag{3.8}
\]

Or, in view of equations (3.3) to (3.6) we have

\[
T = A_3 E I K^2 \tag{3.9}
\]

\[
B = -E I K^2 \phi + A_3 E I K^2 x + A_4 E I K^2 \tag{3.10}
\]
Using Figure (3.1), and equations (3.3) and (3.4) we define the constants $A_1$, $A_2$, $A_3$ and $A_4$ in terms of nodal displacements $\phi_i$, $\phi_i'$, $\phi_j$ and $\phi_j'$.

$$A_1 = \frac{1}{K D_o} \left[-K \sinh\alpha \phi_i + (\cosh\alpha - \alpha \sinh\alpha)\phi_j' + K \sinh\alpha \phi_j' - (\cosh\alpha - 1)\phi_j'\right]$$  
(3.11)

$$A_2 = \frac{1}{K D_o} \left[K (\cosh\alpha - 1)\phi_i + n \phi_i' - K (\cosh\alpha - 1)\phi_j + (\sinh\alpha - \alpha)\phi_j'\right]$$  
(3.12)

$$A_3 = \frac{1}{D_o} \left[K \sinh\alpha \phi_i + (\cosh\alpha - 1)\phi_i' - K \sinh\alpha \phi_j + (\cosh\alpha - 1)\phi_j'\right]$$  
(3.13)

$$A_4 = \frac{1}{K D_o} \left[K (\cosh\alpha - 1 - \alpha \sinh\alpha)\phi_i + n \phi_i' + K (\cosh\alpha - 1)\phi_j - (\sinh\alpha - \alpha)\phi_j'\right]$$  
(3.14)

where $D_o = 2\cosh\alpha - 2 - \alpha \sinh\alpha$

$n = \alpha \cosh\alpha - \sinh\alpha$

$\alpha = K \ell$

Substituting equations (3.11) to (3.14) into equations (3.9) and (3.10):

$$T = \frac{E I}{L_o} K^2 \left[K \sinh\alpha \phi_i + (\cosh\alpha - 1)\phi_i' - K \sinh\alpha \phi_j + (\cosh\alpha - 1)\phi_j'\right]$$  
(3.15)
\[ B = -\frac{E}{\kappa}K^2\phi + \frac{E}{\kappa}K^2 \left[ (\cosh \alpha - 1 - \frac{x}{L} \alpha \sinh \alpha) \phi_i \right. \]
\[ + (x(\cosh \alpha - 1) - \frac{\alpha \cosh \alpha - \sinh \alpha}{K}) \phi_i \]
\[ + ((\cosh \alpha - 1) - x \sinh \alpha) \phi_j \]
\[ + (x(\cosh \alpha - 1) - \frac{\sinh \alpha - \alpha}{K}) \phi_j \]

Equations (3.15) and (3.16) represent general equations of the torsional moments and bimoments along the beam element from \( x=0 \) to \( x=L \), the sign convention will now be used in order that a positive twisting angle \( \phi_i \) will be associated with a positive twisting moment \( T_i \), where the other three degrees of freedom \( \phi, \phi', \phi'' \) have a value of zero, and positive twisting angle \( \phi_j \), will be associated with a positive twisting moment \( T_j \), where \( \phi, \phi', \phi'' \) are zeros. The same rule will be used for the warping mode and the bimoment, i.e. a positive warping mode \( \phi'_i \), will be associated with a positive bimoment \( B_i \), where \( \phi, \phi', \phi'' \) are zeros, and the same for \( \phi'_j \), and \( B_j \). Applying this rule to equations (3.15) and (3.16), bearing in mind that the other signs in the equations follow suit, we have

\[ T_i = \frac{E}{\kappa}K^2 \left[ -K \sinh \alpha \phi_i - (\cosh \alpha - 1) \phi'_i + K \sinh \alpha \phi_j - (\cosh \alpha - 1) \phi'_j \right] \]

\[ (3.17) \]

\[ T_j = \frac{E}{\kappa}K^2 \left[ K \sinh \alpha \phi_i + (\cosh \alpha - 1) \phi'_i - K \sinh \alpha \phi_j + (\cosh \alpha - 1) \phi'_j \right] \]

\[ (3.18) \]

\[ B_i = \frac{E}{\kappa}K \left[ -K(\cosh \alpha - 1) \phi_i - n \phi''_i + K(\cosh \alpha - 1) \phi'_j - (\sinh \alpha \alpha) \phi''_j \right] \]

\[ (3.19) \]
\[ B_j = \frac{ErK}{D_o} [-K(\cosh-1)\phi_i - (\sinh-\alpha)\phi_i + K(\cosh-1)\phi_j - n\phi_j] \]  

Writing equations 3.17 to 3.20 in matrix form:

\[
\begin{align*}
T_i & = \begin{bmatrix} -K^2\sinh & -K(\cosh-1) & K^2\sinh & -K(\cosh-1) \\ -K(\cosh-1) & -n & K(\cosh-1) & -(\sinh-\alpha) \\ K^2\sinh & K(\cosh-1) & -K^2\sinh & K(\cosh-1) \\ -K(\cosh-1) & -(\sinh-\alpha) & K(\cosh-1) & -n \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_i^t \\ \phi_j \\ \phi_j^t \end{bmatrix} \\
B_j & = \begin{bmatrix} -K^2\sinh & -K(\cosh-1) & K^2\sinh & -K(\cosh-1) \\ -K(\cosh-1) & -n & K(\cosh-1) & -(\sinh-\alpha) \\ K^2\sinh & K(\cosh-1) & -K^2\sinh & K(\cosh-1) \\ -K(\cosh-1) & -(\sinh-\alpha) & K(\cosh-1) & -n \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_i^t \\ \phi_j \\ \phi_j^t \end{bmatrix}
\end{align*}
\]  

(3.21)

N.B. The numerical magnitude of the constant \(D_o\) is always negative as defined above.

One must note that this element stiffness matrix is the same as was produced by Krahula and Renton.

3.4 Load Transformation from an Arbitrary Point to the Shear Centre of a Beam

3.4.1 The Effect of Applied Loads on Bimoment

It was seen in Chapter 2 that the displacements and forces generated by the theory of torsion warping, are related to the principal sectorial pole, which we will call from now on, the shear centre, but from the known theory of simple beams, axial force applied away from the centroid, will in fact produce bending moments on the structure. In other words, in a complete stiffness matrix, which includes bending and twisting modes (14x14 matrix), the terms from the bending modes are relative to forces through the shear centre, and those due to axial modes are relative to forces through the centroid. The load
transformation from an arbitrary point on the structure, must therefore consider the two modes separately.

### 3.4.1.1 Transformation of transverse forces

According to the hypothesis which considers the beam section as rigid, the stresses at a section will not change when an external transverse load is replaced by another set of forces statically equivalent to the first. Thus, in general, a transverse force will cause a torsional moment about the shear centre. The torsional moment may also cause flexural twist if the boundary conditions are appropriate (see Figure 3.2).

### 3.4.1.2 Transformation of bending moments

The bimoment due to an applied bending moment is given by $M'e$, where $M$ is the bending moment applied at an arbitrary point $D$, and $e$ is the distance between the plane of the moment and the shear centre (see Figure 3.3).

The bimoment produced by a bending moment offset from the shear centre is a self-balancing longitudinal load, whether the bending moment consists of a transverse or longitudinal couple with a distance ($\Delta s > 0$). According to Vlasov only the first case is admissible (see Vlasov (1961) pages 112-113). The following theoretical work proves that a bending moment applied at any point on the structure may be replaced by a pair of longitudinal forces at an infinitesimal distance apart ($\Delta s \to 0$). The distance between any couple has to be very small in order to consider the action of this couple as moment, whether the couple is longitudinally or transversally directed.

In his comparison between Figures 3.4a and 3.4b, Vlasov has omitted the important fact that the distance $B_1$ must tend to zero, in order not to change the state of stresses when a bending moment, acting at
the midpoint distance between points B and S₁ is replaced by a longitudinal couple. The example of Figures 3.4c and 3.4d, gives the same result, since the superposition of the transverse couple on the structure will give a result identical to the one for a bending moment, only when the distance between the couple tends to zero. The solution for a moment in a transverse plane consisting of a pair of forces an infinitesimal distance apart thoroughly explained by Vlasov, we now try to solve the same problem using a longitudinal couple, assuming a beam under a bending moment at point C as in Figure 3.5a and replace it by a pair of longitudinal forces, one at point C and the other at point D, where CD can be considered as a rigid bracket of infinitesimal width and thickness in order not to change the torsion warping constant \( r \), the forces at C and D are \( P_D = -P_C = P \), as in Figure 3.5b.

The bimoment due to the couple is \( B = B_C + B_D \) \( \text{(a)} \)

\[
B_C = -P \omega_C \quad \text{(b)}
\]

\[
B_D = P \omega_D \quad \text{(c)}
\]

where \( \omega_C \) and \( \omega_D \) are the sectorial coordinates of points C and D respectively with respect to the shear centre.

But \( \omega_D = \omega_C + \Delta \omega_{CD} \) \( \text{(d)} \)

and \( \Delta \omega_{CD} = e \Delta S \) \( \text{(e)} \)

where \( e \) is the distance between the shear centre and the plane of the bending moment.

Substituting the values of \( B_C \), \( B_D \) and \( \Delta \omega_{CD} \) into equation (a):

\[
B = P e \Delta S \quad \text{(f)}
\]
when \( \Delta S = 0, M = P \Delta S \) \hspace{1cm} (g)

where \( M \) is bending moment

or

\[
B = M e
\] \hspace{1cm} (h)

The distribution of the bimoment along the \( x \) axis is always similar to that of longitudinal force, this also possible if a bending moment is not applied on the cross-section but at any arbitrary point connected to the section by a rigid bracket.

We will solve here the example given by Vlasov (page 107) for a beam hinged at both ends subjected to a concentrated bending moment \( M \) in the plane \( x = t \) at an eccentricity \( e \) (from the shear centre). We use the solution from Vlasov (page 123) and after substituting the initial values of kinematic and static parameters at \( x = 0, \phi_0, \phi'_0, B_0, T_0 \) \((\theta_0, \theta'_0, B_0, H_0 \) according to Vlasov's terminology) and \( B = -B = -Me \), and also using the hyperbolic relations we find:

For the part \( 0 < x < t \)

\[
\phi(x) = \frac{Me}{GJ} \left[ -\frac{x}{t} + \frac{\cosh \frac{\alpha}{t} (x-t)}{\sinh \alpha} \sinh \frac{\alpha}{t} x \right]
\]

\[
\phi'(x) = \frac{Me}{GJ} \left[ -\frac{1}{t} + \frac{\alpha \cosh \frac{\alpha}{t} (x-t)}{\sinh \alpha} \cosh \frac{\alpha}{t} x \right]
\]

\[
B(x) = \frac{cosh \frac{\alpha}{t} (x-t)}{\sinh \alpha} \sinh \frac{\alpha}{t} x
\]

\[
T(x) = -\frac{Me}{\alpha}
\]  \hspace{1cm} (3.22)
For the part $t \leq x \leq \lambda$

$$\phi(x) = \frac{M_e}{GJ} \left[ \frac{\xi - x}{\xi} - \frac{\cosh \frac{\alpha}{\xi} t}{\sinh \alpha} \sinh \frac{\alpha}{\xi} (\xi - x) \right]$$

$$\phi'(x) = \frac{M_e}{GJ} \left[ -\frac{1}{\xi} + \frac{\alpha \cosh \frac{\alpha}{\xi} t}{\xi \sinh \alpha} \cosh \frac{\alpha}{\xi} (\xi - x) \right]$$

$$B(x) = \frac{\cosh \frac{\alpha}{\xi} t}{\sinh \alpha} \sinh \frac{\alpha}{\xi} (\xi - x)$$

$$T(x) = -\frac{M_e}{\xi}$$

These two sets of equations are the same as those derived by Vlasov (page 107). By solving further examples, it can be seen that we can always replace a bending moment by a pair of longitudinal forces a small distance apart, and in the same way by a pair of transverse couples. Thus when a bending moment is applied to a structure at the point $x=t$, the structure will suffer due to the flexural twist effect the same stress distribution as if that bending moment acts at any point of the intersection line of its plane and yz plane.

### 3.4.1.3 Transformation of longitudinal force

When a longitudinal force applied at an arbitrary point is transferred to the shear centre, it will produce an additional bimoment proportional to the value of the sectorial coordinate at this point. According to Vlasov "any replacement of a longitudinal force by another force statically equivalent to it amounts to subjecting the beam to an additional self-balancing force system". This additional force system is the bimoment which is defined by:
where $P$ is a longitudinal force applied at point $D$

$\omega_D$ is the sectorial coordinate of point $D$ with respect to the shear centre (principal sectorial pole).

Rajasekaran (1977), in a study of plastic beam-columns using the finite element method has introduced a transformation matrix to the beam element in order to account for the effects of forces applied at an arbitrary point and for a coordinate transformation. Baigent and Hancock (1982) introduced the same transformation matrix which studied the problem of assemblages of thin-walled beams. Both Rajasekaran and Baigent and Hancock did not include in their transformation matrix, the term which accounts for the effect of a longitudinal force acting at an arbitrary point on the bimoment. The theoretical study following shows that this term must be included in the transformation matrix.

If a thin-walled beam of open section is subjected to a longitudinal force $N$, bending moments $M_y$ and $M_z$ and a bimoment $B_D$ acting at an arbitrary point $D$ on the structure (Figure 3.6). The axial stress at an arbitrary point on the profiles will be according to equation (2.46) as follows:

$$\sigma = E \left( u''_y - v''y - w''z - \phi''\omega_D \right)$$

(3.25)

where $\omega_D$ is the sectorial coordinate with respect to the sectorial pole $D$. The generalized forces on the structure at point $D$ will be:

$$N = \int_A \sigma \, dA$$

(3.26)

$$M_y = - \int_A \sigma \, zdA$$

(3.27)

$$M_z = \int_A \sigma \, ydA$$

(3.28)
and the bimoment which was found by equations (2.66) and (2.67)

\[ B_D = \int_A \sigma_D \, dA \]  

(3.29)

Substituting equation (3.25) into equations (3.26 to 3.29) and re-writing we have

\[ \text{EA}_U = N + E S_{\text{wD}} \phi'' \]  

(3.30)

\[ E(I_y I_z - I_{yz}^2)w'' = \bar{M}_y I_z + \bar{M}_z I_y - E(S_{\text{wzD}} I_z - S_{\text{wyD}} I_{yz}) \phi'' \]  

(3.31)

\[ E(I_y I_z - I_{yz}^2)v'' = -\bar{M}_z I_y - \bar{M}_y I_{yz} - E(S_{\text{wyD}} I_y - S_{\text{wzD}} I_{yz}) \phi'' \]  

(3.32)

\[ B_D = E S_{\text{wD}} u''_o - E S_{\text{wyD}} v'' - E S_{\text{wzD}} w'' - E \Gamma_D \phi'' \]  

(3.33)

where \( A = \int_A dA, \ S_{\text{wD}} = \int_A \omega_D dA, \ S_{\text{wyD}} = \int_A \omega_D y dA, \ S_{\text{wzD}} = \int_A \omega_D z dA \)

\[ I_D = \int_A \omega_D^2 dA, \ \int y dA = 0, \ \int z dA = 0, \ I_y = \int y^2 dA, \ I_z = \int z^2 dA \]

\[ I_{yz} = \int_A y z dA \]

Substituting equations (3.30), (3.31) and (3.32) into equation (3.33) we have

\[ B_D = S_{\text{wD}} (N/A + E S_{\text{wD}} \phi''/A) + S_{\text{wyD}} [(\bar{M}_z I_y + \bar{M}_y I_{yz})/(I_y I_z - I_{yz}^2)] \]

\[ + E(S_{\text{wyD}} I_y - S_{\text{wzD}} I_{yz}) \phi''/(I_y I_z - I_{yz}^2) \]
\[ -S_{\text{wzD}}\left(\overline{M}_{y'z} + \overline{M}_{z'yz}\right)/(I_{y'z} - I_{yz}^2) - E(S_{\text{wzD}} I_{z} - S_{\text{wyD}} I_{yz})\phi''/(I_{y'z} - I_{yz}^2) \]

- \[ E\Gamma_D \phi'' \]  

(3.34)

Rewrite equation 3.34:

\[ B_D = \frac{N}{A} S_{\text{wD}} + \frac{(\overline{M}_{y'z} + \overline{M}_{y'yz})}{I_{y'z} - I_{yz}^2} S_{\text{wyD}} - \frac{(\overline{M}_{y'z} + \overline{M}_{z'yz})}{I_{y'z} - I_{yz}^2} S_{\text{wzD}} \]

\[ - [\Gamma_D - \frac{(S_{\text{wyD}} I_{y'z} - S_{\text{wzD}} I_{yz})}{I_{y'z} - I_{yz}^2} S_{\text{wyD}} - \frac{(S_{\text{wzD}} I_{z} - S_{\text{wyD}} I_{yz})}{I_{y'z} - I_{yz}^2} S_{\text{wzD}} - \frac{S_{\text{wD}}^2}{A}] E\phi'' \]

(3.35)

Recalling from Chapter 2 equations (2.31,2.32,2.33,2.34 and 2.36)

\[ S_{\text{wyD}} = -(a_y - d_y) I_{yz} + (a_z - d_z) I_z \]  

(3.36)

\[ S_{\text{wzD}} = -(a_y - d_y) I_y + (a_z - d_z) I_{yz} \]  

(3.37)

\[ a_y - d_y = \frac{-(S_{\text{wzD}} I_z - S_{\text{wyD}} I_{yz})}{I_{y'z} - I_{yz}^2} \]  

(3.38)

\[ a_z - d_z = \frac{S_{\text{wyD}} I_y - S_{\text{wzD}} I_{yz}}{I_{y'z} - I_{yz}^2} \]  

(3.39)
Substituting equations (3.36) to (3.40) into (3.35) we have

\[ B_D = \frac{N}{A} S_{wD} + (a_y - d_y) \bar{M}_y + (a_z - d_z) \bar{M}_z - E \Gamma_A \phi'' \] (3.41)

From equation (2.67)

\[ B = -E \Gamma_A \phi'' \]

where \( A \) is the shear centre.

And equation (2.41) can be written as follows:

\[ B = - \frac{N}{A} S_{wD} - (a_y - d_y) \bar{M}_y - (a_z - d_z) \bar{M}_z + B_D \] (3.42)

In equation (3.42) the bending moments \( \bar{M}_y \) and \( \bar{M}_z \) represent the total generalized bending moment on the structure at point \( D \), and these can be represented by the applied bending moments \( M_y \) and \( M_z \) and the bending moment due to the generalized longitudinal force \( N \), positioned at point \( D \), as Figure (3.7).

\[ \bar{M}_y = M_y - d_z N \] (3.43)

\[ \bar{M}_z = M_z + d_y N \] (3.44)

Substituting equations (3.43) and (3.44) into (3.42)
\[ B = - \frac{N}{A} S_{wD} - (a_y - d_y)(M_y - d_z N) - (a_z - d_z)(M_z + d_y N) + B_D \]

or

\[ B = N \left( a_y d_z - a_z d_y - \frac{S_{wD}}{A} \right) - (a_y - d_y)M_y - (a_z - d_z)M_z + B_D \quad (3.45) \]

We can clearly see in equation (3.45) that the first term is a multiplication of the longitudinal force \( N \) by the sectorial coordinate of the point \( D \) with respect to the shear centre \( A \), as we can see from the following. Recall equations (2.24) and (2.29):

\[ \omega_A = (z_0 - a_z)(y - y_0) - (y_0 - a_y)(z - z_0) \quad (3.46) \]

\[ \frac{S_{wD}}{A} = (a_y - d_y)z_0 - (a_z - d_z)y_0 \quad (3.47) \]

The sectorial coordinate of point \( D \) with respect to the shear centre \( A \) can be found from equation (3.46) by substituting the variables \( y \) and \( z \) by \( d_y \) and \( d_z \) respectively, thus

\[ \omega_A(D) = (z_0 - a_z)(d_y - y_0) - (y_0 - a_y)(d_z - z_0) \]

or

\[ \omega_A(D) = a_y d_z - a_z d_y - [(a_y - d_y)z_0 - (a_z - d_z)y_0] \quad (3.48) \]

Substituting equation (3.47) into (3.48)

\[ \omega_A(D) = a_y d_z - a_z d_y - \frac{S_{wD}}{A} \quad (3.49) \]
Substitute now equation (3.49) into (3.45) we have the final equation of force transformation

\[ B = N_\omega(D) - (a_\text{y} - d_\text{y})M_\text{y} - (a_\text{z} - d_\text{z})M_\text{z} + B_D \]  

Equation (3.50) does not only give the effect of a longitudinal load applied at an arbitrary point D, to the bimoment on the structure, but also gives the effect of the bending moment on the bimoment in terms of the coordinate of point D and the shear centre multiplied by the value of the bending moment. This is as proved in the earlier work by Al-Sheikh and Sharman (1983) in their report on thin-walled beam finite elements.

### 3.4.2 Transformation for Stiffness Matrix

The transformation for the stiffness matrix of a thin-walled beam of open section, comprises the coordinate transformation of local systems of each node to the global system, and the transformation of node actions acting at an arbitrary point on the cross-section to the centroid and shear centre. The first can be found in many texts, see Beaufait et al (1970), the latter will be defined as in Figure (3.8) and by means of equation (3.50) as follows:

\[ P_{\text{xC}} = P_{xD} \]
\[ P_{\text{yA}} = P_{yD} \]
\[ P_{\text{zA}} = P_{zD} \]
\[ M_{\text{xA}} = P_{yD} \cdot (d_\text{z} - a_\text{z}) - P_{zD} (d_\text{y} - a_\text{y}) + M_{xD} \]
\[ M_{\text{yC}} = P_{xD} d_\text{z} + M_{yD} \]
Writing equations (3.51) in matrix form we have:

\[
\begin{bmatrix}
\bar{M}_y & M_x & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{M}_y & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{M}_z & 0 & 0 & 0 & 0 \\
0 & -(a_y-d_y) & (a_y-d_y) & 1 & 0 & 0 & 0 \\
-d_z & 0 & 0 & 0 & 1 & 0 & 0 \\
d_y & 0 & 0 & 0 & 0 & 1 & 0 \\
\omega(D) & 0 & 0 & -a_y-d_y & -a_z-d_z & 0 & 0 \\
\end{bmatrix}
\]

In equations (3.52) we see that the second and third terms of fourth row is the contributions of the transverse force applied at point D to the twisting moment. The first term, in the fifth row, is the contribution of the longitudinal force to the bending moment about the y-axis. The first term of the sixth row is the contribution of the longitudinal force to the bending moment about the z-axis. In row seven, the first term is the contribution of the axial force at point D to the total bimoment while the fifth and sixth terms are the contributions of the two bending moments \( M_{yD} \) and \( M_{zD} \) at point D, to the total bimoment.

Equations (3.52) may be written in the form:
\[ [P_i] = [T_i][P_{Di}] \]

where \([P_i] = [P_{xC} \ P_{yA} \ P_{zA} \ M_{xA} \ N_{yC} \ N_{zC} \ B_{A}]^T_i \] \hfill (3.53)

\[
[T_i] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -(a_z-d_z) & (a_y-d_y) & 1 & 0 & 0 & 0 \\
-d_z & 0 & 0 & 0 & 1 & 0 & 0 \\
d_y & 0 & 0 & 0 & 0 & 1 & 0 \\
\omega(D) & 0 & 0 & 0 & -(a_y-d_y) & -(a_z-d_z) & 1 \\
\end{bmatrix}
\]

\[ [P_{Di}] = [P_{xD} \ P_{yD} \ P_{zD} \ M_{xD} \ M_{yD} \ M_{zD} \ B_{D}]^T_i \] \hfill (3.54)

The transformation for stiffness matrix is

\[ [K_i] = [T_{i,j}]^T[K_{i,j}][T_{i,j}] \] \hfill (3.55)

where \(K_{i,j}\) is the element stiffness matrix which is the combination of bending modes as given in many texts, see Zienkiewicz (1971) or Weaver and Gere (1980), and the torsional mode given in equation (3.21). The element stiffness matrix will be as shown in Table 3.7.
and 

\[
[T_{ij}] = \begin{bmatrix} [T_i] & 0 \\ 0 & [T_j] \end{bmatrix}
\]  \hspace{1cm} (3.56)

where \([T_j] = [T_i] \)  \hspace{1cm} (3.57)

and 

\[
[T_{ij}^T] = \begin{bmatrix} [T_i^T] & 0 \\ 0 & [T_j^T] \end{bmatrix}
\]  \hspace{1cm} (3.58)

3.5 Rotation for Principal Directions

Equations (3.52) may be applied only for translations of node actions i.e. when the principal axis coincides with the global axis. When failing to secure this condition, a matrix for rotation of principal direction may be needed.

The rotation matrix may be derived from Figure 3.9 as follows:

\[
r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}
\]  \hspace{1cm} (3.59)

This rotation matrix is to actions associated with translation displacements \(u, v\) and \(w\). It is also applicable to the actions associated with rotations \(\phi_x, \phi_y\) and \(\phi_z\). The actions associated with rate of twist, i.e. bimoment, will have the same value after rotation, this can be seen easily in Figure 3.10, as follows. If a symmetric I-beam of web width
d is subjected to a bimoment $B$, this bimoment may be represented by two equal and opposite bending moments in $y$ direction as shown

$$B = M_y b$$  \hspace{1cm} (3.60)$$
$$B = M_y d \cos \beta$$  \hspace{1cm} (3.61)$$

where:  \hspace{1cm} b = d \cos \beta  \hspace{1cm} (3.62)$$

The projections of $M_y$ on the principal axes $y_m$ and $z_m$ are,

$$M_{y_m} = M_y \cos \beta$$  \hspace{1cm} (3.63)$$

$$M_{z_m} = M_y \sin \beta$$  \hspace{1cm} (3.64)$$

It is also seen from the figure that the bimoment due to $M_z$ vanishes because its plane $(x_m, y_m)$ passes through the shear centre, while the bimoment due to $M_{y_m}$ is

$$B_m = M_{y_m} d \text{ (note that } M_{y_m} \text{ acts in the plane of the flange)}$$

but

$$M_{y_m} = M_y \cos \beta$$

Thus

$$B_m = M_y d \cos \beta$$  \hspace{1cm} (3.65)$$

Comparing equations (3.61) and (3.65) we find them identical, which means that the term in the rotation matrix, associated with the bimoment will be:

$$r_{44} = 1$$
Thus a rotation matrix for actions at node $i$ will be,

$$
[R_i] = \begin{bmatrix}
[r] \\
1 
\end{bmatrix}
$$

(3.66)

A rotation for beam element, will be,

$$
[R_{ij}] = \begin{bmatrix}
[R_i] \\
[R_j] 
\end{bmatrix}
$$

(3.67)

where

$$
[R_j] = [R_i]
$$

(3.68)

A comprehensive transformation for element stiffness matrix will be,

$$
[K_{ij}]_R = [R_{ij}^T][T_{ij}^T][K_{ij}][T_{ij}][R_{ij}]
$$

(3.69)

3.6 Numerical Examples

Whilst it is desirable to show the correlation of theoretical methods with experimental results, confidence may also be obtained by comparing one theoretical method with another. For thin shell structures, the finite element method is generally accepted as convergent to the true solution provided reasonably fine meshes are used and the element formulation is known to be accurate for the particular application. Therefore, some exercises were completed on open section beams using the flat shell elements contained in the finite element program PAFEC. The basis of the element formulation is the isoparametric technique, which is known to be accurate when the elements are rectangular or almost rectangular, and of reasonable aspect ratio.
The model used was a beam 1000 mm long, of channel section with web depth of 80 mm and flanges of 40 mm width and of uniform thickness 1.0 mm. The boundary condition was as a cantilever subjected to various tip forces.

The division into elements is shown in Figure 3.11 with sufficient elements to account for the gradients of stress to be expected. The elements were eight noded quadrilaterals and additional elements were introduced at the tip to prevent cross-sectional distortion under the applied forces. This diaphragm had sufficient nodes to connect all the existing modes at the tip and was of the same thickness as the beam member.

The stress output of PAFEC is unfortunately in terms of the principal stresses together with the angle of the principal planes from the reference axis. It was noted that in the region of the root, this angle was within approximately 5° of the longitudinal axis and therefore the maximum principal stress closely corresponds to the axial stress. The stresses at the element centroid were taken, and were extrapolated to the root using a linear relationship. The eight noded flat shell element (which is called Type 44210) was used, which has 48 degrees of freedom in an arbitrary axis set. Four load cases were considered and the results of the tip deflections and the root stresses from PAFEC were compared with the torsional warping theoretical results. In all different force cases applied, a satisfactory agreement between both methods is observed, details of which follow.

3.6.1 Load Case 1

A transverse force of 1000N acts at the tip, at the extreme end of the flange and parallel to the web. The description of this case is shown in Figure 3.12. The stress distribution results on the cross-section at the root of the beam, are shown in Table 3.3, together with the error resulting between the finite element method and the warping theory. The error computed is within 0.9%. This slight discrepancy
may be caused by the fact that PAFEC stress distribution results were extrapolated to the root. Figure 3.16 shows the stress distribution diagramatically. The displacements in three directions and the twist at the tip of the beam were calculated and stated in Tables 3.5 and 3.6, together with those calculated by PAFEC, and the error between the two methods is within 2.8%.

3.6.2 Load Case 2

A longitudinal load of 1000N was applied at the same point as Case 1, as shown in Figure 3.13. The results of both stress distributions at the root and the deflections at the tip, are in Tables 3.3, 3.5 and 3.6. The error is within 2.9% to the PAFEC results. This case is a significant one, because it proves the author's claim of the lack of generality of the transformation matrices of Rajasekaran (1977) and Baigent and Hancock (1982). This is because the point at which the longitudinal load is applied has a sectorial coordinate value of \( \omega = -1000 \, \text{mm}^2 \), which gives rise to a bimoment at the tip of the beam of value \( B = p \omega = -10^6 \, \text{N. mm}^2 \), and at the root of the beam by

\[
B_{\text{root}} = \frac{-p \omega}{\cosh k \lambda}
\]

where \( k = \sqrt{\frac{GJ}{EF}} \)

Thus:

\[
B_{\text{root}} = 725000 \, \text{N.mm}^2
\]

This bimoment will in turn give rise to the maximum stress according to equation (2.69)

\[
\sigma = \frac{B}{I} \omega
\]
\[ \sigma = \frac{725000}{29866667} \times 1000 = 24.3 \text{ N/mm}^2 \]

this value represents about one-half of the total stress if we use the previously mentioned matrices of Rajasekaran or Baigent and Hancock. Thus the computation using the finite element method of PAFEC agrees with the author's anticipation, since the error between the author's result and the PAFEC result is within 1.5%.

The stress distribution at the root of the beam is shown in Figure 3.17.

3.6.3 Load Case 3
A transverse force of 1000N was applied at the tip of the beam and in the plane of the web, as in Figure 3.14. This is similar to Case 1, and the deflection errors are within 2% (Tables 3.5 and 3.6) and the stresses errors of the root are within 0.8% (Table 3.4). The stress distribution is shown in Figure 3.18.

3.6.4 Load Case 4
A bending moment of 10000 N.mm with vector in the y-direction is applied at point 2, as shown in Figure 3.15. The deflections computed are in Tables 3.5 and 3.6, the errors to those computed by PAFEC are within 2%. The stress distributions at the root are in Table 3.4 and in Figure 3.19. The error is within 1.5%.
TABLE 3.1: Stiffness Matrix for Plane Beam

\[
\begin{bmatrix}
S_1 & M_1 \\
S_2 & M_2
\end{bmatrix} =
\begin{bmatrix}
12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\
-12EI/L^3 & -6EI/L^2 & 2EI/L \\
4EI/L & -6EI/L^2 & 2EI/L \\
12EI/L^3 & 6EI/L^2 & 4EI/L
\end{bmatrix}
\]

\[v_1 \quad \theta_1 \quad v_2 \quad \theta_2\]

TABLE 3.2: Stiffness Matrix for Beam-Bar

\[
\begin{bmatrix}
P_1 & S_1 & M_1 \\
P_2 & S_2 & M_2
\end{bmatrix} =
\begin{bmatrix}
EA/L & 12EI/L^3 & 6EI/L^2 \\
0 & 0 & -12EI/L^3 \\
0 & -6EI/L^2 & 2EI/L \\
12EI/L^3 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\]

\[u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad \theta_2\]
<table>
<thead>
<tr>
<th>Node No</th>
<th>CASE 1</th>
<th>CASE 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>W.T.*</td>
<td>F.E.M.</td>
</tr>
<tr>
<td>2</td>
<td>1270</td>
<td>1260</td>
</tr>
<tr>
<td>1</td>
<td>-1130</td>
<td>-1120</td>
</tr>
<tr>
<td>110</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>1130</td>
<td>1120</td>
</tr>
<tr>
<td>4</td>
<td>-1270</td>
<td>-1260</td>
</tr>
</tbody>
</table>

**TABLE 3.3:** AXIAL STRESS AT THE BEAM ROOT FOR CASE 1 AND CASE 2

* Warping theory method
** Finite element method

<table>
<thead>
<tr>
<th>Node No</th>
<th>CASE 3</th>
<th>CASE 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>W.T.</td>
<td>F.E.M.</td>
</tr>
<tr>
<td>2</td>
<td>175</td>
<td>174</td>
</tr>
<tr>
<td>1</td>
<td>-480</td>
<td>-476</td>
</tr>
<tr>
<td>110</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>480</td>
<td>476</td>
</tr>
<tr>
<td>4</td>
<td>-175</td>
<td>-174</td>
</tr>
</tbody>
</table>

**TABLE 3.4:** AXIAL STRESS AT THE BEAM ROOT FOR CASE 3 AND CASE 4
<table>
<thead>
<tr>
<th>Load Case</th>
<th>Node No</th>
<th>$U_x$ (mm)</th>
<th>$U_y$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>W.T.</td>
<td>F.E.M.</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>2.84</td>
<td>2.762</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>0.368</td>
<td>0.3713</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>-1.12</td>
<td>-1.1061</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>-0.0495</td>
<td>-0.0503</td>
</tr>
</tbody>
</table>

**TABLE 3.5: DISPLACEMENTS IN X AND Y DIRECTIONS AT THE TIP OF THE BEAM**

<table>
<thead>
<tr>
<th>Load Case</th>
<th>Node No</th>
<th>$U_z$ (mm)</th>
<th>$\phi_x$ (rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>W.T.</td>
<td>F.E.M.</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>-91.9</td>
<td>-90.317</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>-5.18</td>
<td>-5.1062</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>-25.1</td>
<td>-24.595</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>-1.89</td>
<td>-1.8792</td>
</tr>
</tbody>
</table>

**TABLE 3.6: DISPLACEMENT IN Z DIRECTION AND ROTATION ABOUT X DIRECTION AT THE TIP OF THE BEAM**
TABLE 3.7: Thin-Walled Beam Element Stiffness Matrix
Fig. 3.1: NODAL TORSIONAL FORCES AND DISPLACEMENTS ASSOCIATED WITH A THIN-WALLED BEAM ELEMENT.

Fig. 3.2: TRANSFORMATION OF A TRANSVERSE FORCE.

Fig. 3.3: TRANSFORMATION OF A BENDING MOMENT.
Fig. 3.4: BENDING MOMENT REPRESENTED BY A COUPLE ACCORDING TO VLASOV (1961).

Fig. 3.5: BENDING MOMENT REPRESENTED BY A COUPLE.
Fig. 3.6: A LOAD SYSTEM APPLIED AT POINT D.

Fig. 3.7: TRANSFORMATION OF A LOAD SYSTEM TO THE CENTROID.
Fig. 3.8: TRANSFORMATION OF A LOAD SYSTEM APPLIED AT POINT D TO THE RELEVANT CENTROID AND SHEAR CENTRE.
Fig. 3.9: LOCAL AND GLOBAL CO-ORDINATES OF AN I-BEAM.

Fig. 3.10: TRANSFORMATION OF BENDING MOMENT FROM GLOBAL TO LOCAL CO-ORDINATES.
Fig. 3.11: THE FINITE ELEMENT MODEL USED IN P A F E C
Fig. 3.12: CANTILEVER UNDER TRANSVERSE FORCE.
LOAD CASE 1.

Fig. 3.13: CANTILEVER UNDER AXIAL FORCE.
LOAD CASE 2.
Fig. 3.14: CANTILEVER UNDER TRANSVERSE FORCE. LOAD CASE 3.

Fig. 3.15: CANTILEVER UNDER A BENDING FORCE. LOAD CASE 4.
Fig. 3.16: STRESS DISTRIBUTION ON THE CROSS SECTION. LOAD CASE 1.
Fig. 3.17: STRESS DISTRIBUTION ON THE CROSS SECTION, LOAD CASE 2.
Fig. 3.18: STRESS DISTRIBUTION ON THE CROSS SECTION. LOAD CASE 3.
Fig. 3.19: STRESS DISTRIBUTION ON THE CROSS SECTION, LOAD CASE 4.
4. FINITE STRIP METHOD

4.1 Introduction

The finite strip method in structural analysis was developed mainly by Cheung between 1968 and 1976, also other authors may be mentioned as main contributors, such as Przemieniecki, Wittrick, Cusens, Loo. The main purpose of finite strip method development was to replace the finite element method which is the most general tool of solution in structural analysis. However for structures having simple boundary conditions and regular geometric plans, the finite element method is very often expensive and would require a large number of finite elements, particularly if a fine mesh for greater accuracy is required. This also implies very large matrices for the calculation of the lowest eigenvalue representing the critical buckling stress. The finite strip method eliminates the necessity of a large number of elements in the lengthwise direction of the structure (see Figure 4.1) by introducing a lengthwise function for displacements, in order to form a special elastic and geometric stiffness matrices, which depend upon the buckling wave length and boundary stress distribution of the strip. The finite strip method differs from the finite element method in the following features:

i) the displacement functions are simple polynomials in some directions and continuously differentiable functions in the other directions which satisfy the boundary conditions. This compares with the finite element method which uses the polynomial displacement functions in all directions;
ii) the general form of the displacement function is given as a product of the simple polynomials and the trigonometric series which reduces the two-dimensional problem to a one-dimensional one;

iii) the lengthwise fictitious line between two strips is called the nodal line, so the ends of the line always have a part of the boundary conditions;

iv) the degrees of freedom at a nodal line are called nodal displacement parameters, and connected with the displacements and their first partial derivatives in the transverse direction, also non-displacement terms such as strains may be included;

v) the number of degrees of freedom at each nodal line in the finite strip method are usually less than those of the element node in the finite element method, because of the use of continuous displacement functions in the lengthwise direction. For example in plate bending, the out-of-plane displacements $w$ and the rotation about the axial axis $\phi_x$ exist at each nodal line of the strips while $w, \phi_x, \phi_y$ exist at each element node;

vi) a displacement function is to represent the displacement, strain and stress within the strip, which makes it possible too for an elastic and geometric stiffness matrix to include all concentrated and distributed loads on the strip, using either virtual work
or minimum total potential energy principles. The overall stiffness matrix is simply the assembly of all stiffness matrices for the strips which can be solved in a similar way to the stiffness analysis of a structure.

4.2 Displacement Functions

It is very important to choose a suitable function to represent the displacement throughout the field of the strip, in addition to satisfying a priori the boundary conditions of the strip, and to satisfy the compatibility conditions between the adjacent strips. One more feature of the displacement function is very important, that is to lead to a standard eigenvalue problem, in which the coefficients of the overall stiffness matrix are linear functions of the load factor, so the buckling load may easily be extracted, Cheung (1976), Plank and Wittrick (1974).

The out-of-plane deflection of a plate (see Figure 4.2) is governed by St Venant's differential equation (4.1):

\[
\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{1}{D} \left( q + N_x \frac{\partial^2 W}{\partial x^2} + N_y \frac{\partial^2 W}{\partial y^2} + 2 N_{xy} \frac{\partial^2 W}{\partial x \partial y} \right) \tag{4.1}
\]

where \( w \) is out-of-plane displacements

\( q \) is the lateral load
$N_x, N_y$ the direct forces per unit length in the $x$ and $y$ directions respectively

$N_{xy}$ is the shear force in the $x$-$y$ plane per unit length

$$D = \frac{Et^3}{12 (1-v^2)}$$

where $E$ is Young's modulus, $t$ plate thickness and $v$ is Poisson's ratio.

If the lateral load, $q = 0$, equation (4.1) can be written in the form

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{t}{E t^3} \left( \sigma_x \frac{\partial^2 w}{\partial x^2} + \sigma_y \frac{\partial^2 w}{\partial y^2} + 2\tau \frac{\partial^2 w}{\partial x \partial y} \right)$$ (4.2)

where $t$ = the plate thickness

$N_x = t \sigma_x$

$N_y = t \sigma_y$

$N_{xy} = t \tau$

The solution which satisfies the differential equation, and the condition of simply supported ends is:

$$w(x, y) = Re (Z(n)e^{i\xi})$$ (4.3)

where $\xi = \frac{\pi x}{\lambda}$

$n = \frac{2y}{b}$

$\lambda = \text{half wavelength}$

$b = \text{plate width}$
If \( \tau \neq 0 \), the function \( Z(n) \) is complex.

The polynomial function \( Z(n) \), can be determined independently in order to describe the displacement field in the cross-section of a strip, when a nodal displacement function is given a unit value and a value of zero to the other displacements at both nodes of a strip, this condition is necessary to satisfy the compatibility conditions between two adjacent strips.

Many polynomial functions are available, Cheung (1976), but we are using the most common Przemieniecki (1968), Cheung and Cheung (1971) and Plank and Wittrick (1974), which is a function of the nodal displacement, (see Figure 4.3).

\[
Z(n) = \frac{b}{8} (1-n-n^2+n^3) \phi_i + \frac{1}{4} (2-3n+n^3) w_i - \frac{b}{8} (1+n-n^2-n^3) \phi_j + \frac{1}{4} (2+3n-n^3) w_j
\]

or

\[
Z = \left[ \frac{b}{8} (1-n-n^2+n^3) , \frac{1}{4} (2-3n+n^3) , - \frac{b}{8} (1+n-n^2-n^3) , \frac{1}{4} (2+3n-n^3) \right] \tag{4.4}
\]

where \( \phi_i, \phi_j \) are the rotations about the x-axis of nodal lines i and j respectively

\( w_i, w_j \) are the out-of-plane displacements of nodal lines i and j respectively.
Similarly if we choose the polynomial in-plane displacements functions we may have:

\[
X = \left[ 0, \frac{1}{2} (1-n), 0, \frac{1}{2} (1+n) \right] \quad (4.5)
\]

\[
Y = \left[ \frac{1}{2} (1-n), 0, \frac{1}{2} (1+n), 0 \right] \quad (4.6)
\]

where \( u_i, u_j \) are the displacements in the x-direction of nodal lines \( i, j \) respectively

\( v_i, v_j \) are the displacements in the y-direction of nodal lines \( i, j \) respectively.

4.3 Derivation of Strip Stiffness Matrix

4.3.1 Perturbation Forces and Displacements in a Strip

Conventional finite element procedures are used in deriving a stiffness matrix for out-of-plane and in-plane displacements of a strip, using the principle of virtual work. Each strip in the structure is treated as a single element, and its components of displacements vary sinusoidally lengthwise, thus the governing differential equations are ordinary and can be solved. The corresponding stiffness and geometric matrices for both out-of-plane and in-plane are derived for each strip to relate to the forces which vary sinusoidally to the displacements at the longitudinal edges of the strip.
In Figure (4.4), the state of stress on a strip is shown; it consists of a linearly varying longitudinal stress \((\sigma_1 + n\sigma_2)\), a uniform lateral direct stress \(\sigma_y\) on both edges, and a uniform shear stress \(\tau\). The nodal lines (i and j) y-coordinate is \(-\frac{b}{2}\), and \(\frac{b}{2}\) respectively, and the thickness of the plate (strip) is \(t\).

In Figure (4.5), the edge displacements and the corresponding forces are shown, as each edge (nodal line) has four degrees of freedom, three translations \(u_x, v_y, w_z\) and one rotation \(\phi_x\). The direct forces \(P_x, P_y, P_z\) and bending moment \(m_x\), together with the displacements are multiplied by \(e^{i\xi}\), to represent the actual value of displacements or forces at an arbitrary point \(x\).

Plank and Wittrick (1974) derived the elastic and stability matrices, for the problem above, as follows. The perturbation forces per unit length of edge are defined by:

\[
\text{Re} \{(m_i P_{zi} P_{yi} P_{xi} m_j P_{zj} P_{yj} P_{xj})e^{i\xi}\}
\]

and the corresponding displacements of the edges are similarly defined by

\[
\text{Re} \{(\phi_i W_i v_i u_i \phi_j W_j v_j u_j)e^{i\xi}\}
\]

where \(\text{Re}\{\quad\}\) denotes the real part of the quantity inside the brackets.
The presence of shear stress \( \tau \) may cause the quantities \( m_i, \phi_i \) etc to be complex.

For small buckling displacements we may consider uncoupled in-plane and out-of-plane effects, and they may be studied separately. Thus the perturbation forces and displacement vectors can be defined by:

\[
P_0 = \{m_i \ P_zi \ m_j \ P_{zj}\} \quad d_0 = \{\phi_i \ w_i \ \phi_j \ w_j\}
\]

\[
P_I = \{P_yi \ ipzi \ P_yj \ ip_{xj}\}, d_I = \{v_i \ iu_i \ v_j \ iu_j\}
\]

where 0 and I denote out-of-plane and in-plane respectively.

Perturbation stiffness matrices \( S_0 \) and \( S_I \) may be defined by the equations

\[
P_0 = S_0d_0
\]

\[
P_I = S_Id_I
\]

where \( S_0 \) and \( S_I \) are 4x4 stiffness matrices.

The introduction of \( i = \sqrt{-1} \) in the in-plane force and displacement vectors implicitly incorporates a right-angle phase difference between the \( u \) and \( v \) displacements and between \( P_x \) and \( P_y \) forces. As a result the stiffness matrix \( S_I \) is real and symmetrical.
4.3.2 Out-of-Plane Stiffness Matrix $S_0$

The principle of virtual work was used to derive the out-of-plane stiffness matrix $S_0$, of the individual stiffness matrix, as follows

$$\delta W_{eo} + \delta W_{mo} = \delta W_{io} \quad (4.11)$$

where $\delta W_{eo}$ is the virtual work due to the edge forces over a wavelength of $2\lambda$, where the edges of the strip undergo virtual displacements corresponding to $\delta d_0$ in the vector $d_0$.

$$\delta W_{eo} = \frac{\lambda}{\pi} \int_0^{2\pi} \text{Re} \{\delta d_0^T e^{i\xi}\} \text{Re} \{P_0 e^{i\xi}\} d\xi \quad (4.12)$$

where $\text{Re} \{\delta d_0^T e^{i\xi}\}$ are the actual edge displacements

$\text{Re} \{P_0 e^{i\xi}\}$ are the actual edge forces

and $T$ denotes transpose.

$\delta W_{mo}$ is the virtual work done by the in-plane stresses acting on the four edges of a strip of width $b$ and length $2\lambda$

$$\delta W_{mo} = \frac{t}{2} \int \int \left[ \sigma_x \left( \frac{\partial w}{\partial x} \right)^2 + \sigma_y \left( \frac{\partial w}{\partial y} \right)^2 + 2\tau \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy \quad (4.13)$$

where $\sigma_x$ is the local value of longitudinal stress defined by
\[ \sigma_x = \sigma_1 + \eta \sigma_2 \quad (4.14) \]

and \( \delta W_{i0} \) is the internal virtual work due to the virtual displacements \( \delta d_0 \).

\[ \delta W_{i0} = \frac{\lambda b d}{2\pi} \int_{-1}^{2\pi} +1 \delta K^T \delta K \, d\xi \, d\eta \quad (4.15) \]

where

\[ K_o = \left\{ \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, \frac{\partial^2 w}{\partial x \partial y} \right\} \]

\[ F = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \quad (4.16) \]

Equations (4.11), (4.12), (4.13) and (4.15) lead to

\[ S_0 = \left( \frac{D}{b^3} \right) \left[ H_0 - \nu \Omega^2 H_0 + \Omega^2 (\Omega^2 - K_1) H_1 + (2\Omega^2 - K_y) H_y - \Omega^2 K_2 H_2 - \Omega \lambda K_3 H_3 \right] \quad (4.17) \]

where

\[ [K_1 \ K_2 \ K_y \ K_3] = \frac{b^2 t}{D} \left[ \sigma_1, \sigma_2, \sigma_y, \tau \right] \]

\[ \Omega = \pi b / \lambda \]
and the six 4x4 stiffness matrices are given by:

\[
H_0 = 8 \int_{-1}^{+1} Z^n T Z^n \, d\eta, \quad H_y = 2 \{(Z^T Z)'^1\}_{-1}^{+1}
\]

\[
H_1 = \frac{1}{2} \int_{-1}^{+1} Z^T Z \, d\eta, \quad H_y = 2 \int_{-1}^{+1} Z'^T Z' \, d\eta
\]

\[
H_2 = \frac{1}{2} \int_{-1}^{+1} nZ^T Z \, d\eta, \quad H_s = \int_{-1}^{+1} (Z'^T Z - Z T Z') \, d\eta
\]

where \( Z \) is defined by equation (4.4) and prime and double prime denote differentiation with respect to \( \eta \). Carrying out the integration in equations (4.18) gives the explicit expressions in Appendix I.

### 4.3.3 In-plane Stiffness Matrix \( S_I \)

A parallel argument was used in deriving the stiffness matrix for in-plane action, to that of out-of-plane, the application of virtual work was put as

\[
\delta W_{eI} + \delta W_{mI} = \delta W_{iI}
\]

where

\[
\delta W_{eI} = (\lambda/\pi) \int_0^{2\pi} Re \{d^T \xi^J J_p \xi \} Re \{J J_p \xi \} \, d\xi
\]

and \( J_p \) is a 4x4 diagonal matrix defined by

\[
J_p = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}
\]
and

\[ \delta W_{II} = \frac{\lambda b t}{2\pi} \int_0^1 \int_0^{-1} \delta \mathbf{e}^T \sigma d\xi d\eta \]  \hfill (4.21)

\[ \mathbf{e} = \{ \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \} \]  \hfill (4.22)

and \( \sigma \) is the corresponding stress vector, so that

\[ \sigma = \mathbf{E}' \mathbf{F} \mathbf{e} \]  \hfill (4.23)

where \( \mathbf{E}' = \frac{\mathbf{E}}{1 - \nu^2} \)

and \( \mathbf{F} \) defined by equation (4.16).

Substituting equation (4.23) into equation (4.21):

\[ \delta W_{II} = \frac{\lambda \mathbf{E}' b t}{2\pi} \int_0^1 \int_0^{-1} \delta \mathbf{e}^T \mathbf{F} \mathbf{e} d\xi d\eta \]  \hfill (4.24)

From equations (4.5) (4.6) and (4.22):

\[ \mathbf{e} = \text{Re} \{ \mathbf{F}_I e^{i\xi} \} \]  \hfill (4.25)

where \( \mathbf{F}_I = \begin{pmatrix} \frac{1}{b} & 2Y' \\ 2X' + i\eta Y \end{pmatrix} \)  \hfill (4.26)
\[ \delta W_{mi} = \frac{1}{2} \sigma_1 t \delta \int \left\{ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right\} dx \, dy \]  
\hfill (4.27)

or
\[ \delta W_{mi} = \lambda \Re \{ \delta d_1^T B_1 d_1 \} \]  
\hfill (4.28)

where the bar denotes the complex conjugate, and
\[ B_1 = (\eta^2 \sigma_1 t/2b) \int_{-1}^{+1} \overline{\delta d_1^T (X^T X + Y^T Y)} \, \delta d_1 \]  
\hfill (4.29)

From equations (4.19) to (4.29):
\[ S_I = \left( E'/t \right) \int_{-1}^{+1} \overline{\delta d_1} \left( \eta^2 (1-\varepsilon) X^T X + \frac{1}{2} \eta^2 (1-\nu-2\varepsilon) Y^T Y + 2(1-\nu)X^T X' \right. \]  
\[ + 4Y^T Y' + 2i\nu \eta (Y^T X - X^T Y') + i(1-\nu)\eta (X^T Y - Y^T X') \} \, \delta d_1 \]  
\hfill (4.30)

where \( \varepsilon = \sigma_1 / E' \).

Explicit expressions for the elements of \( S_I \), can be found by substituting equations (4.5) and (4.6), and integrating with respect to \( \eta \), which can be found in Appendix I.
4.4 Rectangular Plate Stability

A rectangular plate with length \(a\), width \(b\) and thickness \(t\), was studied under different load conditions and different side support conditions, the accuracy obtained for buckling coefficient \(K\) was dependent upon the number of strips the plate is divided into, and on the load and support conditions. In the case of pure compression, a four strip plate was enough to give an error of about 0.02% for simply supported sides, and 0.3% for clamped sides, compared with an approximate theoretical result Timoshenko (1961). In the case of pure bending a four strip plate gives 0.3% for simply supported sides and 2.0% for clamped sides.

\[
\sigma_{cr} = \frac{n^2}{12(1-\nu^2)} KE(t) t^2
\]

4.4.1 Buckling of Plate Under Pure Compression

In Tables 4.1 to 4.5 a plate under pure compression was analysed for buckling with different side plate conditions and the close agreement with theoretical results was clearly observed, with an error less than 1.0% in all cases for four strips plate. It is seen that dividing the plate into more strips results in increased accuracy.

4.4.2 Buckling of Plate Under Pure Bending

4.4.2.1 Both sides are simply supported

In bending the number of strips has to be at least four before good results are obtained. Four strips results in a maximum error of 0.6%, while eight strips gives a maximum error of 0.26%. Bearing in mind that the theoretical results are rounded to the first decimal the finite strip method results would be sufficiently accurate for most purposes compared with the exact results. It is clear that the theoretical results in Table 4.7, deviated about 2% from the finite strip method results.
<table>
<thead>
<tr>
<th>$\ell/b$</th>
<th>No. of Strips</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>Max Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>4.5354</td>
<td>4.2083</td>
<td>4.0517</td>
<td>4.0086</td>
<td>4.1466</td>
<td>4.4864</td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4.5311</td>
<td>4.2029</td>
<td>4.0450</td>
<td>4.0005</td>
<td>4.1352</td>
<td>4.4712</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4.5309</td>
<td>4.2026</td>
<td>4.0447</td>
<td>4.0001</td>
<td>4.1346</td>
<td>4.4704</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>4.5308</td>
<td>4.2025</td>
<td>4.0446</td>
<td>4.0000</td>
<td>4.1345</td>
<td>4.4703</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Exact Result</td>
<td>4.5308</td>
<td>4.2025</td>
<td>4.0446</td>
<td>4.0000</td>
<td>4.1344</td>
<td>4.4702</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 4.1:** Buckling coefficient for plate under pure compression with both sides simply supported

<table>
<thead>
<tr>
<th>$\ell/b$</th>
<th>No. of Strips</th>
<th>0.5</th>
<th>1.0</th>
<th>1.4</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>Max Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>4.4126</td>
<td>1.4357</td>
<td>0.9531</td>
<td>0.6983</td>
<td>0.5632</td>
<td>0.5162</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4.4049</td>
<td>1.4343</td>
<td>0.9525</td>
<td>0.6980</td>
<td>0.5630</td>
<td>0.5161</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>4.4039</td>
<td>1.4342</td>
<td>0.9524</td>
<td>0.6980</td>
<td>0.5630</td>
<td>0.5161</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>4.4037</td>
<td>1.4342</td>
<td>0.9524</td>
<td>0.6980</td>
<td>0.5630</td>
<td>0.5161</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>Theoretical Results</td>
<td>4.40</td>
<td>1.434</td>
<td>0.952</td>
<td>0.698</td>
<td>0.564</td>
<td>0.516</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 4.2:** Buckling coefficients for plate under pure compression with one side simply supported and the other is free

<table>
<thead>
<tr>
<th>$\ell/b$</th>
<th>No. of Strips</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>5.991</td>
<td>5.482</td>
<td>5.820</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>5.925</td>
<td>5.415</td>
<td>5.746</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>5.919</td>
<td>5.411</td>
<td>5.741</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>5.918</td>
<td>5.410</td>
<td>5.740</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Theoretical Results</td>
<td>5.92</td>
<td>5.41</td>
<td>5.74</td>
<td>5.92</td>
<td>5.51</td>
<td>5.41</td>
</tr>
</tbody>
</table>

**TABLE 4.3:** Buckling coefficient $K_c$ for a plate under pure compression when one side is simply supported and the other is clamped
### TABLE 4.4: BUCKLING COEFFICIENT $K_{b}$ FOR PLATE UNDER PURE COMPRESSION WITH ONE SIDE CLAMPED AND THE OTHER FREE

<table>
<thead>
<tr>
<th>$z/b$</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.7</th>
<th>1.8</th>
<th>2.0</th>
<th>Max Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Strips</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.7065</td>
<td>1.4735</td>
<td>1.3681</td>
<td>1.3345</td>
<td>1.3358</td>
<td>1.3465</td>
<td>1.3903</td>
<td>0.7</td>
</tr>
<tr>
<td>4</td>
<td>1.6989</td>
<td>1.4674</td>
<td>1.3629</td>
<td>1.3299</td>
<td>1.3315</td>
<td>1.3423</td>
<td>1.3865</td>
<td>0.5</td>
</tr>
<tr>
<td>6</td>
<td>1.6984</td>
<td>1.4671</td>
<td>1.3626</td>
<td>1.3298</td>
<td>1.3314</td>
<td>1.3422</td>
<td>1.3864</td>
<td>0.5</td>
</tr>
<tr>
<td>8</td>
<td>1.6983</td>
<td>1.4681</td>
<td>1.3633</td>
<td>1.3302</td>
<td>1.3318</td>
<td>1.3426</td>
<td>1.3867</td>
<td>0.5</td>
</tr>
<tr>
<td>Theoretical results</td>
<td>1.7</td>
<td>1.47</td>
<td>1.36</td>
<td>1.33</td>
<td>1.34</td>
<td>1.38</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 4.5: BUCKLING COEFFICIENTS $K_b$ FOR PLATE UNDER PURE COMPRESSION WITH BOTH SIDES CLAMPED

<table>
<thead>
<tr>
<th>$z/b$</th>
<th>0.5</th>
<th>0.6</th>
<th>0.661</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>Max Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Strips</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7.9454</td>
<td>7.3069</td>
<td>7.2261</td>
<td>7.2598</td>
<td>7.5775</td>
<td>8.1517</td>
<td>4.1</td>
</tr>
<tr>
<td>4</td>
<td>7.7164</td>
<td>7.0763</td>
<td>6.9908</td>
<td>7.0202</td>
<td>7.3226</td>
<td>7.8765</td>
<td>0.6</td>
</tr>
<tr>
<td>6</td>
<td>7.6973</td>
<td>7.0600</td>
<td>6.9753</td>
<td>7.0050</td>
<td>7.3077</td>
<td>7.8609</td>
<td>0.4</td>
</tr>
<tr>
<td>8</td>
<td>7.6935</td>
<td>7.0568</td>
<td>6.9724</td>
<td>7.0022</td>
<td>7.3050</td>
<td>7.8582</td>
<td>0.36</td>
</tr>
<tr>
<td>Theoretical Results</td>
<td>7.69</td>
<td>7.05</td>
<td>6.97*</td>
<td>7.00</td>
<td>7.29</td>
<td>7.83</td>
<td></td>
</tr>
</tbody>
</table>

* This result is an exact result, which was taken from Wittrick and Curzon (1968).
### TABLE 4.6: BUCKLING COEFFICIENT K₆ FOR PLATE UNDER PURE BENDING WITH BOTH SIDES SIMPLY SUPPORTED

<table>
<thead>
<tr>
<th>( \frac{a}{b} )</th>
<th>No. of Strips</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.667</th>
<th>0.8</th>
<th>0.9</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>29.222</td>
<td>25.624</td>
<td>24.207</td>
<td>23.965</td>
<td>24.556</td>
<td>25.669</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>29.118</td>
<td>25.544</td>
<td>24.137</td>
<td>23.897</td>
<td>24.486</td>
<td>25.594</td>
<td>-0.2</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>29.105</td>
<td>25.533</td>
<td>24.126</td>
<td>23.886</td>
<td>24.475</td>
<td>25.582</td>
<td>-0.26</td>
</tr>
<tr>
<td>Theoretical Results</td>
<td></td>
<td>29.1</td>
<td>25.6</td>
<td>24.1</td>
<td>23.9</td>
<td>24.4</td>
<td>25.6</td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 4.7: BUCKLING COEFFICIENT K₆ FOR PLATE UNDER PURE BENDING WITH ONE SIDE SIMPLY SUPPORTED AND THE OTHER CLAMPED

<table>
<thead>
<tr>
<th>( \frac{a}{b} )</th>
<th>No. of Strips</th>
<th>0.5</th>
<th>0.6</th>
<th>0.667</th>
<th>0.7</th>
<th>0.8</th>
<th>1.0</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>25.538</td>
<td>24.154</td>
<td>23.947</td>
<td>24.001</td>
<td>24.661</td>
<td>27.674</td>
<td>-2.5</td>
</tr>
<tr>
<td>Theoretical Results</td>
<td></td>
<td>26.0</td>
<td>24.65</td>
<td>24.48</td>
<td>24.6</td>
<td>25.3</td>
<td>28.3</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 4.6: BUCKLING COEFFICIENT K₆ FOR PLATE UNDER PURE BENDING WITH BOTH SIDES SIMPLY SUPPORTED

TABLE 4.7: BUCKLING COEFFICIENT K₆ FOR PLATE UNDER PURE BENDING WITH ONE SIDE SIMPLY SUPPORTED AND THE OTHER CLAMPED
### TABLE 4.8: BUCKLING COEFFICIENT $K_b$ FOR PLATE UNDER PURE BENDING WITH BOTH SIDES CLAMPED

<table>
<thead>
<tr>
<th>$z/b$</th>
<th>0.4</th>
<th>0.45</th>
<th>0.47</th>
<th>0.48</th>
<th>0.5</th>
<th>0.6</th>
<th>Max Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>40.970</td>
<td>39.996</td>
<td>39.891</td>
<td>39.890</td>
<td>39.983</td>
<td>42.021</td>
<td>0.75</td>
</tr>
<tr>
<td>6</td>
<td>40.768</td>
<td>39.821</td>
<td>39.724</td>
<td>39.726</td>
<td>39.825</td>
<td>41.860</td>
<td>0.32</td>
</tr>
<tr>
<td>8</td>
<td>40.634</td>
<td>39.706</td>
<td>39.614</td>
<td>39.619</td>
<td>39.721</td>
<td>41.767</td>
<td>-0.16</td>
</tr>
<tr>
<td>Theoretical Results</td>
<td>40.7</td>
<td>39.7</td>
<td>39.6</td>
<td>39.6</td>
<td>39.7</td>
<td>41.8</td>
<td></td>
</tr>
</tbody>
</table>

#### 4.4.3 Buckling of Plate Under Combined Compression and Bending

In the combined stress condition only one load case was considered consisting of triangular loading when at the edges we have zero and maximum respectively as in Figure 4.6. Two support conditions were considered, as in Tables 4.9 and 4.10, which show an excellent agreement between the theoretical results and the finite strip method results. A maximum error of 0.47% for four strips was observed.
### TABLE 4.9: BUCKLING COEFFICIENT K FOR PLATE UNDER COMBINED COMPRESSION AND BENDING LOAD, WITH BOTH SIDES SIMPLY SUPPORTED

<table>
<thead>
<tr>
<th>$l/b$</th>
<th>0.4</th>
<th>0.6</th>
<th>0.7</th>
<th>0.75</th>
<th>0.8</th>
<th>1.0</th>
<th>Max Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>15.156</td>
<td>9.746</td>
<td>8.699</td>
<td>8.370</td>
<td>8.134</td>
<td>7.814</td>
<td>0.47</td>
</tr>
<tr>
<td>6</td>
<td>15.152</td>
<td>9.744</td>
<td>8.698</td>
<td>8.368</td>
<td>8.132</td>
<td>7.812</td>
<td>0.45</td>
</tr>
<tr>
<td>8</td>
<td>15.151</td>
<td>9.743</td>
<td>8.697</td>
<td>8.368</td>
<td>8.132</td>
<td>7.812</td>
<td>0.44</td>
</tr>
<tr>
<td>Theoretical Results</td>
<td>15.1</td>
<td>9.7</td>
<td>N.A.</td>
<td>8.4</td>
<td>8.1</td>
<td>7.8</td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 4.10: BUCKLING COEFFICIENT K FOR PLATE UNDER COMBINED COMPRESSION AND BENDING LOAD, WITH BOTH SIDES CLAMPED

<table>
<thead>
<tr>
<th>$l/b$</th>
<th>0.6</th>
<th>0.64</th>
<th>0.65</th>
<th>0.66</th>
<th>0.7</th>
<th>0.8</th>
<th>Max Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical Results</td>
<td>13.7</td>
<td>13.57</td>
<td>13.56</td>
<td>13.57</td>
<td>13.65</td>
<td>14.3</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 4.1a:
Finite Element Method
elements per flange = 21

Fig. 4.1b:
Finite Strip Method
strips per flange = 3

Fig. 4.1
Fig. 4.2: PLATE ELEMENT UNDER GENERAL IN-PLANE AND BODY FORCES.
Fig. 4.3a: Out-of-Plane Displacement.

Fig. 4.3b: In-Plane Displacement.
Fig. 4.4: PLATE STRIP UNDER GENERAL IN-PLANE STRESSES.
Fig. 4.5: EDGE FORCES AND DISPLACEMENTS OF A STRIP.
Fig. 4.6: PLATE UNDER COMBINED COMPRESSION AND BENDING $\sigma_2 = \sigma_1$. 
5. BUCKLING OF THIN-WALLED BEAMS UNDER PURE COMPRESSION

5.1 Introduction

In Chapter 4 the finite strip method has proved its accuracy for the calculation of buckling coefficient of thin flat plates under uniform and linear stress distribution, and for different types of edge support. In this chapter the method is tested for cruciform, box and channel sections under uniform stress. The strips in these structures do not lie in the same plane which means that a transformation matrix for the displacements, forces and stiffness matrix of a strip is needed, see Cheung (1976). The results obtained are compared with published results, and excellent agreement was observed. Each flat section was divided into four strips.

5.2 Buckling of Cruciform Type Structure

A cruciform type beam of 100 mm width and 1 mm wall thickness as shown in Figure 5.1, with simply supported ends, and subjected to a uniform stress, was studied for torsional buckling. The theoretical solution is as follows, see Megson (1979):

\[ \sigma_{CR} = \frac{GJ}{I_{PR}} \]  

where \( I_{PR} \) is the polar second moment of area

\( J \) is the torsional constant

\( G \) is the modulus of rigidity

\[ I_{PR} = I_z + I_y = 2 I_z \]  

where \( I_z = \frac{b^3 t}{12} \)

\( b = 100 \text{ mm and } t = 1 \text{ mm} \)
\[
I_{PR} = 166.7 \times 10^3 \text{ mm}^4
\]

and
\[
J = 2 \frac{bt^3}{3}
\]

\[
J = 66.67 \text{ mm}^4
\]

\[
G = \frac{E}{2(1+v)}
\]

Taking \( E = 208000 \text{ MN/m}^2 \)

and \( v = 0.25 \)
gives

\[
G = 83200 \text{ MN/m}^2
\]

Thus
\[
\sigma_{CR} = 33.28 \text{ N/mm}^2
\]

The value of \( \sigma_{CR} \) computed by the finite strip method was:

\[
\sigma_{CR(FSM)} = 33.32 \text{ N/mm}^2
\]

which gives an error of approximately 0.1%. The buckling mode computed is shown in Figure 5.2 which is clearly a torsional mode.
5.3 Buckling of Box Type Structure

A box type structure of different flange-to-web width ratio and different flange-to-web thickness ratio, as shown in Figure 5.3, was tested by the finite strip method. The buckling coefficients computed for local buckling were compared with the results of Kroll et al (1943), as in Table 5.1, which shows maximum error of 1%. Figure 5.4 represents a design chart for local buckling coefficient of a box for flange-to-width ratios $B_f/B_w$ between 0.1 and 1.0 and for different thickness ratios. It is clear that the local buckling coefficient decreases very slowly for flange-to-web thickness ratio $T_f/T_w > 1.5$ when $B_f/B_w$ increases, while rapidly decreasing when this value increases for $T_f/T_w < 1.5$. Figures 5.5a, 5.5b show the coefficient of buckling versus the aspect ratio $L/B_w$, for $T_f/T_w = 2.0$, and for different values of $B_f/B_w$. When the value of $B_f/B_w > 0.6$ the curve has two troughs, because local and flange buckling are different, while for $B_f/B_w < 0.6$ the curve has one trough only because torsional-flexural buckling occurs at a low value of aspect ratio $L/B_w$ thus eliminating flange buckling.

5.4 Buckling of Channel Section Beams

The same computer program used in previous computation of buckling coefficient or buckling stresses of plate, cruciform and box type structures is used for channel sections under pure compression. Only one half wave over the length of the beam is considered to form during buckling. The channel beam geometry is shown in Figure 5.6. Table 5.2 shows local buckling coefficients for a wide variety of flange-to-web width and thickness ratios, compared with Kroll's results, Kroll et al (1943). The maximum error recorded was 6.4%. This is not a satisfactory achievement considering that Kroll's results are more accurate. For one web-to-flange thickness ratio $T_w/T_f = 1.0$ the buckling coefficients were compared with three other results, Harvey (1953), Chilver (1951) and Gobara and Tsu (1969), in addition to Kroll's results. This comparison is presented in Table 5.3, which includes also the error between the results of finite strip methods and each of those author's results. It is clear that Harvey's results are the closest ones to the
author's results, with a maximum error of 1.4%. A design chart for local buckling coefficients for channel section versus flange-to-web width ratio $B_f/B_w$ between 0.2 to 0.7 and for different web-to-flange ratios $T_w/T_f$ is given in Figure 5.7.

A set of graphs were plotted for buckling coefficients $K_c$ for channel sections of different flange-to-web ratio $B_f/B_w$, thickness ratio $T_w/T_f$ and for different aspect ratios $L/B_w$. These graphs are in Figures 5.8 to 5.13. The local buckling occurs at low aspect ratio between 0.5 to 2.0, where the minimum buckling coefficient represents a purely local buckling. The region of interaction of local and torsional buckling is between the first minimum value and the peak of the curve where pure torsional buckling occurs. The third region represents the interaction between torsional and Euler buckling. Thus as the ratio $B_f/B_w$ increases the value of local buckling coefficient decreases. However the value of torsional buckling coefficient increases until a certain limit ($B_f/B_w = 0.5$). At greater values than this ratio, the value of torsional buckling coefficient decreases, because of the dominancy of Euler buckling. This particular value is significant since it shows clearly the change from Euler mode dominancy to torsional mode dominancy, and it also indicates an envelope of the Euler buckling coefficient changing with the aspect ratio. However in Figures 5.11, 5.12 and 5.13 we note a strange trend in the local buckling region for the curve of ratio $B_f/B_w = 0.3$, as it seems that there are two local buckling coefficients at two different aspect ratios. This is clear in Figure 5.13, where $T_w/T_f = 2.0$. The results of the finite strip method were compared with both theoretical and experimental results published by Rhodes and Harvey (1976). As shown in Figure 5.14, where a channel of uniform thickness $t$ equals to $1/50$ of the web width, and for different flange-to-web width ratios, was tested for minimum local buckling. Dimensionless critical load versus $B_f/B_w$ was plotted, and excellent agreement observed.
5.5 Experimental Work

5.5.1 Buckling Load of Channel Beam

The initial work was achieved using channel section beams of uniform thickness of 1.62 mm mild steel plate, web width of 52.4 mm and flange width of 35.2 mm. Beams were tested in pure compression, which was achieved by applying the load through three zero warping points on the cross-section connected to a rigid plate, this in turn is applied upon through a ball aligned with the centroid. This arrangement will satisfy simply supported end conditions where the cross-section is free to twist and warp, see Figure 5.15. Three different beam lengths were tested \( L = 750, 1000 \) and 1250 mm and load-axial deflection, and load-angle of twist were plotted continuously by an XY plotter, connected to LVDT's. The Southwell plot was used to compute the buckling load, as the gradient of the angle of twist per unit load plotted against the total angle of twist. This was compared with the buckling corresponding to the finite strip method results, which are summarised in Table 5.4 showing a maximum error of 14% between the finite strip method results and the experimental buckling load for the relatively short strut (750 mm). The error computed for the longest beam (1250 mm) is within 1%, which may indicate that the improvement of the results between the theory and the experiment is by providing suitable instruments to record twist (which was much larger in this case).

<table>
<thead>
<tr>
<th>Length (mm)</th>
<th>Failure Load (KN)</th>
<th>Southwell Plot Method (KN)</th>
<th>Finite Strip Method (KN)</th>
<th>Error to 1st Value (%)</th>
<th>Error to 2nd Value (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>750</td>
<td>41.3</td>
<td>44.5</td>
<td>47.1</td>
<td>14%</td>
<td>6%</td>
</tr>
<tr>
<td>1000</td>
<td>33.2</td>
<td>34.5</td>
<td>31.1</td>
<td>-6%</td>
<td>-10%</td>
</tr>
<tr>
<td>1250</td>
<td>23.2</td>
<td>23.4</td>
<td>23.45</td>
<td>1%</td>
<td>0.2%</td>
</tr>
</tbody>
</table>

TABLE 5.4: Buckling Load of Channel Section Under Pure Compression
5.5.2 Buckling Mode

The theoretical buckling modes were examined for different lengths considering the strut forms one half wave during the buckling, taking these lengths as 80, 280, 300 and 1250 mm. The results are plotted in Figures 5.16 to 5.19. In Figure 5.16 for L = 80 mm, the buckling mode is clearly local, and a cross-sectional distortion occurs. However the distortion is symmetric about the axis of symmetry of the cross-section and the value of the corner displacements are very close to zero, while in Figure 5.17 for L = 280, the buckling is not pure local since the whole cross-section has moved slightly in the lateral direction. In Figure 5.18, for L = 300 mm, the buckling mode is combined of torsional and web local buckling. It is astonishing to compare Figures 5.17 and 5.18, which clearly reveal different buckling modes, although the lengths of the two beams are only slightly different showing the importance of the aspect ratio. In Figure 5.19, for L = 1250 mm, the buckling mode is almost pure torsional, since the difference between the angle of twist of the elements is less than 0.01%. For a beam of length greater than 1900 mm, pure Euler buckling occurs, where the cross-section laterally translates as an undeformed cross-section. In the experiment the buckling of a beam of L = 1250 mm was closely observed, and photographs were obtained during different loading stages as shown in Figures 5.20 to 5.24. In Figure 5.20 the beam is fairly straight at the early stage of load application. Figure 5.21 shows clearly the rotation and lateral displacement of the beam notably at its centre. In Figure 5.22, flange buckling is observed accompanied with straightening of the beam and the rotation is less obvious than in Figure 5.21, because when local buckling occurred to the flange it reduced its axial stiffness.

Thus the section is more in the form of an angle, which has its centre of rotation at the adjacent corner. This makes the beam change the centre and magnitude of twist, until the second flange buckles locally, as shown in Figures 5.23 and 5.24 which show the locally buckled flanges and torsionally buckled web. Note that there is only one complete
plastic hinge line in each flange at the crest of the buckle while the rest is a combination of elastic and plastic deformation, because of the interaction of the torsional and local buckling. The author believes that the occurrence of the local buckling at this particular length may be due to a partial rotational restraint imposed on the beam by the large (25 mm diameter) loading ball. This may be minimised by lubrication and reducing the contact surface between the ball and the supporting rigid plate.

5.6 Calculations of Actual Node Displacements

The eigenvector part of the solution provides the relationship between the deflection and rotation of the cross-section nodes. This relationship is used to determine the actual shape of the cross-section when the buckling occurs. The method involves certain assumptions:

a) The actual deflections are small relative to the size of the cross-section.

b) The angle of twist $\phi_x$ is less than $\pi/2$, and the average of the angle of twist of two nodes is the approximate value of the strip's angle of twist.

c) The plane of each strip remains flat, so the strip width $b$ is maintained.

Let us consider now a strip of width $b$ which suffers an out-of-plane deflection and rotation $f_m w_1, f_m \phi_1, f_m w_2, f_m \phi_2$ at nodes 1 and 2 respectively, where $f_m$ is a factor relating the actual displacements and rotations to the modes given by the eigenvector solution. From Figure 5.20 we can see the geometric relationship:

$$ f_m \frac{(w_1 - w_2)}{b} = \sin (f_m \phi) $$

(5.3)
where \( \phi = \frac{1 + \phi}{2} \)

The only unknown term in this equation is the factor \( f_m \), which has a single value for \( 0 < \phi < \pi/2 \), and the simplest method to find \( f_m \) is by trial and error.

For the particular problem described in the last paragraph, the factor \( f_m \) was calculated for two cases, the first was for \( L = 80 \), where the approximate out-of-plane deflection at the far end of the flange was 8.23 mm and the second case was for \( L = 1250 \), where a torsional buckling occurs for the same point, the deflection is 24.62 mm.
<table>
<thead>
<tr>
<th>$T_f/T_w$</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>Max Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_f/B_w$</td>
<td>FSM*</td>
<td>Kroll</td>
<td>FSM</td>
<td>Kroll</td>
<td>FSM</td>
<td>Kroll</td>
<td>FSM</td>
</tr>
<tr>
<td>0.1</td>
<td>6.08</td>
<td></td>
<td>6.37</td>
<td></td>
<td>6.57</td>
<td></td>
<td>6.70</td>
</tr>
<tr>
<td>0.2</td>
<td>5.69</td>
<td>5.66</td>
<td>6.05</td>
<td>6.02</td>
<td>6.30</td>
<td>6.31</td>
<td>6.49</td>
</tr>
<tr>
<td>0.3</td>
<td>5.44</td>
<td></td>
<td>5.87</td>
<td></td>
<td>6.15</td>
<td></td>
<td>6.36</td>
</tr>
<tr>
<td>0.4</td>
<td>5.29</td>
<td>5.3</td>
<td>5.72</td>
<td>5.72</td>
<td>6.06</td>
<td>6.05</td>
<td>6.29</td>
</tr>
<tr>
<td>0.5</td>
<td>5.16</td>
<td></td>
<td>5.61</td>
<td></td>
<td>6.00</td>
<td></td>
<td>6.25</td>
</tr>
<tr>
<td>0.6</td>
<td>5.04</td>
<td>5.03</td>
<td>5.51</td>
<td>5.52</td>
<td>5.92</td>
<td>5.92</td>
<td>6.22</td>
</tr>
<tr>
<td>0.7</td>
<td>4.87</td>
<td></td>
<td>5.41</td>
<td></td>
<td>5.86</td>
<td></td>
<td>6.20</td>
</tr>
<tr>
<td>0.8</td>
<td>4.65</td>
<td>4.64</td>
<td>5.29</td>
<td>5.28</td>
<td>5.79</td>
<td>5.79</td>
<td>6.17</td>
</tr>
<tr>
<td>0.9</td>
<td>4.38</td>
<td></td>
<td>5.09</td>
<td></td>
<td>5.71</td>
<td></td>
<td>6.13</td>
</tr>
<tr>
<td>1.0</td>
<td>4.00</td>
<td>4.00</td>
<td>4.82</td>
<td>4.81</td>
<td>5.57</td>
<td>5.56</td>
<td>6.09</td>
</tr>
</tbody>
</table>

TABLE 5.1: Buckling coefficient of box type structure of different geometric ratios

*The finite strip method.
<table>
<thead>
<tr>
<th>$T_{w}/T_{f}$</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>Max Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{f}/B_{w}$</td>
<td>FSM</td>
<td>Kroll</td>
<td>FSM</td>
<td>Kroll</td>
<td>FSM</td>
<td>Kroll</td>
<td>FSM</td>
</tr>
<tr>
<td>0.2</td>
<td>4.51</td>
<td>5.49*</td>
<td>4.28</td>
<td>-</td>
<td>4.09</td>
<td>-</td>
<td>3.97</td>
</tr>
<tr>
<td>0.3</td>
<td>4.41</td>
<td>4.41</td>
<td>4.02</td>
<td>3.98</td>
<td>3.78</td>
<td>3.76</td>
<td>3.62</td>
</tr>
<tr>
<td>0.4</td>
<td>3.83</td>
<td>3.74</td>
<td>3.33</td>
<td>3.25</td>
<td>2.97</td>
<td>2.89</td>
<td>2.63</td>
</tr>
<tr>
<td>0.5</td>
<td>3.00</td>
<td>2.90</td>
<td>2.48</td>
<td>2.40</td>
<td>2.08</td>
<td>2.02</td>
<td>1.75</td>
</tr>
<tr>
<td>0.6</td>
<td>2.28</td>
<td>7.2</td>
<td>1.83</td>
<td>1.76</td>
<td>1.49</td>
<td>1.46</td>
<td>1.23</td>
</tr>
<tr>
<td>0.7</td>
<td>1.76</td>
<td>1.7</td>
<td>1.39</td>
<td>1.32</td>
<td>1.16</td>
<td>1.09</td>
<td>0.91</td>
</tr>
</tbody>
</table>

* These values were produced by the energy method.

TABLE 5.2: Buckling Coefficient $K_c$ for channel section beams of different geometric ratios
<table>
<thead>
<tr>
<th>$T_w/T_f$</th>
<th>$T_w/T_f = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_f/B_w$</td>
<td>Author</td>
</tr>
<tr>
<td>0.2</td>
<td>4.51</td>
</tr>
<tr>
<td>0.3</td>
<td>4.41</td>
</tr>
<tr>
<td>0.4</td>
<td>3.83</td>
</tr>
<tr>
<td>0.5</td>
<td>3.00</td>
</tr>
<tr>
<td>0.6</td>
<td>2.28</td>
</tr>
<tr>
<td>0.7</td>
<td>1.76</td>
</tr>
</tbody>
</table>

**TABLE 5.3:** Buckling coefficient $K_c$ for channel section beam with $T_w/T_f = 1.0$, and for different $B_f/B_w$ ratios
Fig. 5.1. CRUCIFORM TYPE STRUCTURE.

Fig. 5.2. TORSIONAL BUCKLING MODE OF CRUCIFORM TYPE STRUCTURE.
Fig. 5.3: BOX TYPE STRUCTURE.
Fig. 5.4: Buckling Coeff. k versus B/B_w ratio for Box Type Structure under Pure Compression
FIG.5.5a BUCKLING COEFF. $K_c$ VERSUS ASPECT RATIO FOR BOX BEAM UNDER COMPRESSION FOR $TF/Tw = 2.0$

**SYMBOL KEY**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>BF/Bw</th>
<th>$\Delta$</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\nabla$</td>
<td>=0.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+</td>
<td>=0.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>x</td>
<td>=0.4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>□</td>
<td>=0.5</td>
<td></td>
</tr>
</tbody>
</table>

Buckling stress $\sigma_{cr} = \frac{\pi^2}{12(1-\nu^2)} \frac{t_w}{b_w} (\frac{L}{Bw})^2$
FIG. 5.5b BUCKLING COEFF. Kc VERSUS ASPECT RATIO FOR BOX BEAM UNDER COMPRESSION FOR TF/Tw = 2.0

<table>
<thead>
<tr>
<th>SYMBOL KEY</th>
<th>BF/Bw</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
</tr>
</tbody>
</table>

Kc x 10^-2 vs. L/Bw

1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0

10^-3 10^0 10^1 10^2

L/Bw
Fig. 5.6. CHANNEL SECTION.
FIG. 5.7 BUCKLING COEFF. Kc VERSUS FLANGE TO WEB WIDTH RATIO FOR CHANNEL UNDER COMPRESSION

SYMBOL KEY

\[
\begin{array}{c}
\text{T/f} = 1.0 \\
\text{T/f} = 1.2 \\
\text{T/f} = 1.4 \\
\text{T/f} = 1.8 \\
\text{T/f} = 2.0
\end{array}
\]

\[
\begin{array}{c}
\triangle \\
\ast \\
+x \\
\square \\
\circ
\end{array}
\]
FIG. 5.8 BUCKLING COEFF. Kc VERSUS ASPECT RATIO FOR CHANNEL UNDER COMPRESSION FOR Tw/TF = 1.0

SYMBOL KEY

Δ + ○ × ▽

BF/Bw = 0.2
= 0.3
= 0.4
= 0.5
= 0.6
= 0.7
FIG. 5.10 BUCKLING COEFF. $K_c$ VERSUS ASPECT RATIO FOR CHANNEL UNDER COMPRESSION FOR $T_w/T_F = 1.4$

<table>
<thead>
<tr>
<th>SYMBOL KEY</th>
<th>BF/Bw</th>
</tr>
</thead>
<tbody>
<tr>
<td>△</td>
<td>0.2</td>
</tr>
<tr>
<td>▽</td>
<td>0.3</td>
</tr>
<tr>
<td>△</td>
<td>0.4</td>
</tr>
<tr>
<td>×</td>
<td>0.5</td>
</tr>
<tr>
<td>□</td>
<td>0.6</td>
</tr>
<tr>
<td>◆</td>
<td>0.7</td>
</tr>
</tbody>
</table>
FIG. 5.11 BUCKLING COEFF. Kc VERSUS ASPECT RATIO FOR CHANNEL UNDER COMPRESSION FOR Tw/TF = 1.6

<table>
<thead>
<tr>
<th>SYMBOL KEY</th>
<th>BF/Bw</th>
</tr>
</thead>
<tbody>
<tr>
<td>△</td>
<td>0.2</td>
</tr>
<tr>
<td>▽</td>
<td>0.3</td>
</tr>
<tr>
<td>⊕</td>
<td>0.4</td>
</tr>
<tr>
<td>✗</td>
<td>0.5</td>
</tr>
<tr>
<td>□</td>
<td>0.6</td>
</tr>
<tr>
<td>●</td>
<td>0.7</td>
</tr>
</tbody>
</table>
FIG. 5.12 BUCKLING COEFF. VERSUS ASPECT RATIO FOR CHANNEL UNDER COMPRESSION FOR Tw/TI = 1.8

SYMBOL KEY

<table>
<thead>
<tr>
<th>BF/Bw</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>▲</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>▲</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>▲</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>▲</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>▲</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1.2
1.0
0.8
0.6
0.4
0.2
0.0

KC × 10^1

10^1
10^2

L/Bw

μf pull

μf pull
FIG. 5.14 BUCKLING LOAD Vs BF/Bw FOR CHANNEL UNDER PURE COMPRESSION

SYMBOL KEY

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δ</td>
<td>Rhodes Theor. Res.</td>
</tr>
<tr>
<td>▽</td>
<td>F.S.M.</td>
</tr>
<tr>
<td>+</td>
<td>Rhode Exp. Res.</td>
</tr>
</tbody>
</table>

\[ D_1 = \pi^2 D \]
FIGURE 5.15: Description of End Condition of the Tested Channel Strut
Fig. 5.16. BUCKLING MODE FOR BEAM LENGTH, 
L = 80 mm.
Fig. 5.17. BUCKLING MODE FOR BEAM LENGTH, $L = 280$ mm.
Fig. 5.18. BUCKLING MODE FOR BEAM LENGTH, L = 300 mm.
Fig. 5.19. BUCKLING MODE FOR BEAM LENGTH, 
L = 1250 mm.
FIGURE 5.20: Channel Strut Under Pure Compression
FIGURE 5.21: Initiation of Torsional Buckling
FIGURE 5.22: Plate Flange Buckling
FIGURE 5.23: Buckling of Second Flange
FIGURE 5.24 Close-up Picture of Buckled Channel
Figure 5.25: PLATE STRIP GEOMETRY AFTER BUCKLING
6.0 BUCKLING OF THIN-WALLED BEAMS UNDER COMBINED LOADING

6.1 Introduction

Many studies on the instability of thin-walled beams under pure compression can be found in the literature, but only a few of these study the structures under combined loading. Walker (1966) has presented results and a design chart for both plain and lipped channels under combined compression and bending, for different flange-to-web ratios. Buckling coefficients were computed and compared with experimental results, and good agreement was observed. Rhodes and Harvey presented, in a series of papers, an extensive study for channel beams under combined compression and bending loads, the ratio of compression to bending load was defined by $\bar{a}$, as explained in Figure 6.1. Results for different values of $\bar{a}$ were presented and some were experimentally tested, where good agreement was observed, Rhodes and Harvey (1976).

6.2 Buckling of Channel Under Combined Loading

The finite strip method was used to find the buckling coefficient for plain channel sections of uniform thickness and subjected to combined compression and bending. For a range of different values of $\bar{a}$, and for different flange-to-web width ratios the buckling coefficient was found and compared with Rhodes and Harvey's results, as shown in Table 6.1 and in Figures 6.2a and 6.2b. It is clear that some of these results have an error of approximately 9.0%, but some results are very close to each other with errors between 0.0 to 1.0%, bearing in mind that Rhodes' results were measured from a graph, in their paper which makes the possibility of error greater. For the particular case $\bar{a} = 1$, where the flange edge has zero loading and the web has a uniform compression, the author's results were compared with Rhodes and Harvey's experimental and theoretical results, and a good agreement is observed as can be seen in Figure 6.3.
Five different combined load cases on a channel section of uniform thickness were tested by the finite strip method, and design charts were presented in Figures 6.4a to 6.8b. The load cases were chosen in order to have a body about the minor axis of the channel, and zero bending about the major axis. The arrangement of the loading is that the web is under pure compression, while the flange is under linearly varying compression or tension depending on the value of $\alpha$. In Figures 6.4a to 6.8b the following features may be seen:

a) For lower values of $\alpha$ the structure is susceptible to flange local buckling at moderately high aspect ratios of up to 3 in Figure 6.4b. This will also reduce the value of torsional buckling load which occurs at higher aspect ratios.

b) As the value of $\alpha$ increases the value of buckling coefficient increases, and the value of the critical aspect ratio decreases.

c) In all load cases for $B_f/B_w = 0.1$, there is no local buckling for the flange, and Euler buckling is predominate.

d) As the flange-to-web width ratio $B_f/B_w$ increases the torsional buckling load increases (peak of the curve); this is always within a limit and dependent on the load case itself. In cases of $\alpha = -1.0$ and 0.5 the ratio $B_f/B_w = 0.4$ is the peak. In this case we see that the local and torsional buckling are in balance; if the ratio $B_f/B_w$ increases the local buckling will be more dominant than the torsional one, thus Euler buckling may occur suddenly. It is also noted that the curve of this particular ratio indicates an envelope for the Euler buckling limit.

e) As the value of $\alpha$ increases the value of flange-to-web width ratio increases for the curve which joins the peaks, as may be seen in Figure 6.6a and 6.6b. This curve is for $B_f/B_w = 0.5$, even for the higher value of $\alpha$ as in Figure 6.7b $B_f/B_w = 0.7$ for $\alpha = 1.0$, and in Figure 6.8b $B_f/B_w = 0.8$ for $\alpha = 1.5$. 

f) As the ratio $B_f/B_w$ increases the local buckling coefficient value decreases, for low values of $\bar{a}$, while for $\bar{a} > 1.0$ the value of buckling coefficient increases with increasing $B_f/B_w$. Figure 6.8a. In Figure 6.8b the appearance of another trough on the curve as $B_f/B_w$ increases, is because the edge of the flanges are under tension while the web and part of the flanges are under compression, so the possibility exists that the web or the flanges may buckle locally at two different aspect ratios.
<table>
<thead>
<tr>
<th>$\bar{\alpha}$</th>
<th>$B_f/B_w$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>Author</td>
<td>-</td>
<td>4.7</td>
<td>4.93</td>
<td>5.01</td>
<td>5.01</td>
<td>4.96</td>
<td>4.83</td>
<td>4.45</td>
<td>3.92</td>
<td>3.40</td>
</tr>
<tr>
<td></td>
<td>Rhodes</td>
<td>4.53</td>
<td>4.79</td>
<td>4.96</td>
<td>5.04</td>
<td>5.00</td>
<td>4.92</td>
<td>4.7</td>
<td>4.31</td>
<td>3.8</td>
<td>3.28</td>
</tr>
<tr>
<td>0.5</td>
<td>Author</td>
<td>-</td>
<td>4.61</td>
<td>4.69</td>
<td>4.52</td>
<td>4.04</td>
<td>3.34</td>
<td>2.68</td>
<td>2.16</td>
<td>1.77</td>
<td>1.47</td>
</tr>
<tr>
<td></td>
<td>Rhodes</td>
<td>4.52</td>
<td>4.71</td>
<td>4.68</td>
<td>4.43</td>
<td>3.94</td>
<td>3.27</td>
<td>2.61</td>
<td>2.05</td>
<td>1.62</td>
<td>1.37</td>
</tr>
<tr>
<td>-0.5</td>
<td>Author</td>
<td>-</td>
<td>4.42</td>
<td>4.08</td>
<td>2.21</td>
<td>2.33</td>
<td>1.71</td>
<td>1.3</td>
<td>1.02</td>
<td>0.82</td>
<td>0.67</td>
</tr>
<tr>
<td></td>
<td>Rhodes</td>
<td>4.39</td>
<td>4.44</td>
<td>4.00</td>
<td>3.14</td>
<td>2.28</td>
<td>1.63</td>
<td>1.26</td>
<td>0.97</td>
<td>0.77</td>
<td>0.67</td>
</tr>
<tr>
<td>-1</td>
<td>Author</td>
<td>-</td>
<td>4.32</td>
<td>3.75</td>
<td>2.72</td>
<td>1.90</td>
<td>1.57</td>
<td>1.07</td>
<td>0.81</td>
<td>0.65</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>Rhodes</td>
<td>4.37</td>
<td>4.35</td>
<td>3.71</td>
<td>2.61</td>
<td>1.82</td>
<td>1.32</td>
<td>1.00</td>
<td>0.77</td>
<td>0.61</td>
<td>0.53</td>
</tr>
<tr>
<td>1.5</td>
<td>Author</td>
<td>-</td>
<td>4.78</td>
<td>5.1</td>
<td>5.25</td>
<td>5.31</td>
<td>5.32</td>
<td>5.31</td>
<td>5.29</td>
<td>5.25</td>
<td>5.21</td>
</tr>
<tr>
<td></td>
<td>Rhodes</td>
<td>4.53</td>
<td>4.91</td>
<td>5.18</td>
<td>5.3</td>
<td>5.37</td>
<td>5.37</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0</td>
<td>Author</td>
<td>-</td>
<td>4.51</td>
<td>4.41</td>
<td>3.83</td>
<td>3.00</td>
<td>2.28</td>
<td>1.75</td>
<td>1.36</td>
<td>1.09</td>
<td>0.87</td>
</tr>
<tr>
<td></td>
<td>Rhodes</td>
<td>4.44</td>
<td>4.58</td>
<td>4.36</td>
<td>3.77</td>
<td>2.91</td>
<td>2.25</td>
<td>1.72</td>
<td>1.34</td>
<td>1.08</td>
<td>0.85</td>
</tr>
</tbody>
</table>

**TABLE 6.1:** Buckling Coefficients for Channel Under Different Combined Load and for Different Flange-to-Web Width Ratios
6.3 Non-Linear Behaviour of Thin-Walled Beam Under Axial Compression

6.3.1 General

It is usual to assume that a thin-walled beam under such a load behaves linearly up to the point of bifurcation, thus eliminating all pre-buckling behaviour which may occur due to either initial imperfection, and non-linear deformation during the early stages of load application. However, this effect may be important in thin-walled beams which are liable to torsional buckling because the twist which may occur before the torsional buckling due to initial deformations on the larger scale twist occurring after torsional buckling results in a relatively different position from load axis (which passes through the centroid of the ends) and the cross-section axes at various positions along the length. This will result in a totally different stress distribution on the cross-section which will vary along the length. Thus the local buckling may be different to that predicted for uniform stress.

6.3.2 The Effect of Pre-Buckling Twist on Channel Local Instability

The behaviour of a channel was studied theoretically by considering the change of both the load position relative to the sectional properties along the length. The channel is the same which was tested in Chapter 5, paragraph 5.5.1. When the load is applied along the line of the original (untwisted) centroids, the cross-section away from the ends rotates about the shear centre. Thus the load axis will not pass through the centroid any more. The coordinate and the sectorial properties of the proposed new position of the point load was calculated for various angles of twist and was fed back into the computer program for the calculation of the new stress distribution on the cross-section. This stress distribution was used to calculate the buckling load of the channel using the finite strip program. Four different load positions were assumed in addition to the first one which represented a pure compression.
The change of load position taken in terms of twisting angle $\phi$ used 0, 10, 20, 30, 40 degrees. This limit is suggested by the experiments in which the angle of twist did not exceed this value. For those cases and for three different beam lengths 750, 1000 and 1250 mm, the results are presented in Table 6.2 and Figure 6.9 which show a significant decrease in the value of buckling load for increasing angle of twist.

<table>
<thead>
<tr>
<th>$L$ (mm)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>750</td>
<td>47.1</td>
<td>45.25</td>
<td>41.55</td>
<td>38.00</td>
<td>35.34</td>
</tr>
<tr>
<td>1000</td>
<td>31.1</td>
<td>29.50</td>
<td>26.6</td>
<td>24.1</td>
<td>22.2</td>
</tr>
<tr>
<td>1250</td>
<td>23.45</td>
<td>21.82</td>
<td>19.25</td>
<td>17.28</td>
<td>15.8</td>
</tr>
</tbody>
</table>

TABLE 6.2: Buckling Load Versus Angle of Twist $\phi$ for Different Channel Lengths.
Figure 6.1: COMBINED BENDING LOAD ON CHANNEL SECTION.
FIG. 6.2 a, BUCKLING COEFF. K VS FLANGE TO WEB RATIO FOR CHANNEL UNDER DIFFERENT COMBINED COMPRESSION AND BENDING LOAD

<table>
<thead>
<tr>
<th>SYMBOL KEY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δ</td>
</tr>
<tr>
<td>▽</td>
</tr>
<tr>
<td>+</td>
</tr>
<tr>
<td>×</td>
</tr>
<tr>
<td>□</td>
</tr>
<tr>
<td>◆</td>
</tr>
</tbody>
</table>
FIG. 6.2 b, BUCKLING COEFF. Vs FLANGE TO WEB WIDTH RATIO FOR CHANNEL UNDER DIFFERENT COMBINED COMPRESSION AND BENDING LOAD

<table>
<thead>
<tr>
<th>SYMBOL KEY</th>
</tr>
</thead>
<tbody>
<tr>
<td>△ Rhodes</td>
</tr>
<tr>
<td>▽ Author</td>
</tr>
<tr>
<td>+ Rhodes</td>
</tr>
<tr>
<td>× Author</td>
</tr>
<tr>
<td>□ Rhodes</td>
</tr>
<tr>
<td>□ Author</td>
</tr>
</tbody>
</table>
FIG. 6.3, BUCKLING LOAD VS BF/BW FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING $\alpha=1.0$

### SYMBOL KEY

- $\Delta$ Rhodes Theor. Res.
- $\triangledown$ F.S.M.
- $+$ Rhode Exp. Res.

$$d_i = \pi^2 D$$

$$D = \frac{E t^3}{12(1-\mu^2)}$$
FIG. 6.4a, BUCKLING COEFF. K VS ASPECT RATIO FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING LOAD $\alpha = 1.0$

<table>
<thead>
<tr>
<th>SYMBOL KEY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
</tr>
<tr>
<td>$\triangleright$</td>
</tr>
<tr>
<td>$+$</td>
</tr>
<tr>
<td>$\times$</td>
</tr>
<tr>
<td>$\Box$</td>
</tr>
</tbody>
</table>
FIG. 6.4b, BUCKLING COEFF. K Vs ASPECT RATIO FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING LOAD $\lambda = -1.0$

<table>
<thead>
<tr>
<th>SYMBOL KEY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\diamond$ BF/Bw = 0.6</td>
</tr>
<tr>
<td>$\circ$    = 0.7</td>
</tr>
<tr>
<td>$\ast$     = 0.8</td>
</tr>
<tr>
<td>$\triangle$ = 0.9</td>
</tr>
<tr>
<td>$\triangledown$ = 1.0</td>
</tr>
</tbody>
</table>
FIG. 6.5a, BUCKLING COEFFICIENT $K$ VS ASPECT RATIO FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING LOAD $\alpha = -0.5$

**SYMBOL KEY**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\triangle$</td>
<td>$BF/Bw=0.1$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\triangledown$</td>
<td>$=0.2$</td>
<td></td>
</tr>
<tr>
<td>$+$</td>
<td>$=0.3$</td>
<td></td>
</tr>
<tr>
<td>$\times$</td>
<td>$=0.4$</td>
<td></td>
</tr>
<tr>
<td>$\square$</td>
<td>$=0.5$</td>
<td></td>
</tr>
</tbody>
</table>
FIG. 6.6 a BUCKLING COEFF. K VS ASPECT RATIO FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING LOAD $\tilde{\nu}=0.5$

**SYMBOL KEY**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>BF/Bw</th>
<th>v</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>$+$</td>
<td></td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td>$\times$</td>
<td></td>
<td></td>
<td>0.5</td>
</tr>
</tbody>
</table>

$K \times 10^7$

$L/B_w$

Tufty.Plot
FIG. 6.6b, BUCKLING COEFF. K vs ASPECT RATIO FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING LOAD $\bar{\alpha}=0.5$

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>BF/Bw</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Diamond$</td>
<td>0.6</td>
</tr>
<tr>
<td>$\bigcirc$</td>
<td>0.7</td>
</tr>
<tr>
<td>$\times$</td>
<td>0.8</td>
</tr>
<tr>
<td>$\bigtriangleup$</td>
<td>0.9</td>
</tr>
<tr>
<td>$\triangledown$</td>
<td>1.0</td>
</tr>
</tbody>
</table>
FIG. 6.7a, BUCKLING COEFF. K vs ASPECT RATIO FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING LOAD $\alpha=1.0$

<table>
<thead>
<tr>
<th>SYMBOL KEY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
</tr>
<tr>
<td>$\nabla$</td>
</tr>
<tr>
<td>$+$</td>
</tr>
<tr>
<td>$\times$</td>
</tr>
<tr>
<td>$\Box$</td>
</tr>
</tbody>
</table>

$K \times 10^{-1}$

$L/Bw$
FIG. 6.7 b, BUCKLING COEFF. K Vs ASPECT RATIO FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING LOAD $\xi=1.0$

**SYMBOL KEY**

- $\Diamond$ BF/Bw = 0.6
- $\bigcirc$ = 0.7
- $\times$ = 0.8
- $\triangle$ = 0.9
- $\blacktriangle$ = 1.0
FIG. 6.8a, BUCKLING COEFF. $K$ vs ASPECT RATIO FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING LOAD $\bar{\alpha}=1.5$

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>KEY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta$</td>
<td>$BF/Bw=0.1$</td>
</tr>
<tr>
<td>$\triangledown$</td>
<td>$=0.2$</td>
</tr>
<tr>
<td>$+$</td>
<td>$=0.3$</td>
</tr>
<tr>
<td>$\times$</td>
<td>$=0.4$</td>
</tr>
<tr>
<td>$\Box$</td>
<td>$=0.5$</td>
</tr>
</tbody>
</table>

![Graph of $K \times 10^{-1}$ vs $L/Bw$ for different aspect ratios](image)
FIG. 6.8b, BUCKLING COEFF. K VS ASPECT RATIO FOR CHANNEL UNDER COMBINED COMPRESSION AND BENDING LOAD \( \alpha = 1.5 \)

**SYMBOL KEY**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>BF/Bw</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \diamond )</td>
<td>BF/Bw = 0.6</td>
<td>( 0.7 )</td>
</tr>
<tr>
<td>( \circ )</td>
<td>BF/Bw = 0.7</td>
<td>( 0.7 )</td>
</tr>
<tr>
<td>( \times )</td>
<td>BF/Bw = 0.8</td>
<td>( 0.8 )</td>
</tr>
<tr>
<td>( \triangle )</td>
<td>BF/Bw = 0.9</td>
<td>( 0.9 )</td>
</tr>
<tr>
<td>( \triangledown )</td>
<td>BF/Bw = 1.0</td>
<td>( 1.0 )</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>KE'</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-----</td>
<td></td>
</tr>
<tr>
<td>$\Delta L = 750$ mm</td>
<td>$\Delta L = 1000$ mm</td>
<td></td>
</tr>
<tr>
<td>$\nabla L = 1250$ mm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 6.8:** Buckling Load $P_{cr}$ vs. Cross-Section Angle of Twist for Channel Under a Load at Fixed Position Originally at the Centroid

The governing differential equations of the buckled channel cross section were derived, and the governing differential equations of the channel cross-section were also derived. The differential equations were then given.

An overall transformation matrix for load applied at a arbitrary point of the cross-section, and the shear stress transformation matrix is established for a complete and general system of forces. A finite-element method is used by a finite-element program (FEME) and by the theory stated in this thesis, for different tip load cases. The comparison, between both cases, for both stress distribution and deflections, gave excellent agreement, which indicated the validity of the analysis.
7. **CONCLUSIONS**

7.1 **Concluding Summary and Discussion**

The work undertaken throughout this research involved extensive theoretical work to develop adequate theory for the analysis of thin-walled beams of open sections. The work also included developing a computer program and also experimental tests to study the buckling of this type of structure. In particular, in Chapters 2 and 3, the following theoretical studies were conducted:

a) Thin-walled beam theory introduced by Vlasov was refined and rederived for an arbitrary sectorial pole.

b) Axial and tangential stresses were derived accordingly.

c) The governing differential equations of thin-walled beams of open section were also derived and each term of flexural and torsional loading was identified.

d) A thin-walled beam finite element stiffness matrix was introduced similar to Krahula (1967), and detailed discussion was then given.

e) An overall transformation matrix for loads applied at an arbitrary point on the cross-section, and for the first time this matrix is established for a complete and general system of forces, and verified by mathematical analysis.

f) A channel section cantilever was solved by a finite element program PAFEC and by the theory adopted in this thesis, for different tip load cases. The comparison, between both results for both stress distribution and deflections, gives excellent agreement, which indicated the validity of the analysis.
In Chapter 4 a comprehensive finite strip method program according to Plank and Wittrick (1974) was written and a rederivation of the stiffness matrices was done, which resulted in a change of sign in one of the stiffness matrices. This particular stiffness matrix does not affect the magnitude of buckling coefficients of plates, and box-type structures at short wavelengths. This is in fact not the case when this is applied to the channel sections, since an error of 7% to 15% was noted when compared to the theoretical results presented by Kroll et al (1943).

In Chapter 5 the method was tested for cruciform, box, and channel types of structure, which proved accurate for all possible buckling modes. A set of design charts for channel section was presented for different flange-to-web width and thickness ratios respectively. Experimental work has been carried out for a channel section under a load acting through the centroid which results in pure compression load on the channel. The Southwell plot method was used to calculate the critical buckling load which was compared with the recorded failure load and the buckling load computed by the finite strip method and a good agreement for longer struts was observed. However an error of up to 14% was observed for shorter struts.

The finite strip method was used again in Chapter 6 for the calculation of buckling coefficient, for a channel of uniform wall thickness and different flange-to-web width, and under a different type of combined load. The results were compared with Rhodes and Harvey (1976), and good agreement with both theoretical and experimental results was observed. An attempt was made to take into account specifically the true, non-linear nature of the behaviour, when gradually applied through the lines of the original centroid. The change of coordinates and sectorial properties of the load point due to the beam twist was taken into consideration, and that resulted in a linear stress distribution for each of the flanges and the web. This stress distribution was used to calculate the buckling load which was less than the case of pure compression.
7.2 Suggestions for Further Research

7.2.1 Thin-Walled Beams Non-Linear Behaviour

Further studies are required of thin-walled beams by means of experimental results, and by a finite element method, to include the non-linearity behaviour of thin-walled beams under continuous load up to the failure point. This may be possible by obtaining a theory for large deflections as was attempted by Gaafar and Tidbury (1981). Although the attempt made by the author in Chapter 6 indicates a trend for the non-linear behaviour of thin-walled beams, it does not give a comprehensive solution to the problem, and further theoretical and experimental work is needed.

7.2.2 Post-Buckling Behaviour of Thin-Walled Beams

The conducted research has essentially taken into consideration the elastic stage of buckling, which may prove to be an important factor in the failure of many structural systems. The attempt made to predict the actual deflections during the buckling needs to be proved correct or at least to be corrected, since the prediction of such deflection is important to the post-buckling stages and to the amount of energy absorbed by the structure. The work by Murray and Koo (1981) on the plastic mechanisms in the load buckling, can be proved useful in the post-buckling stages. The observation of many experiments on thin-walled beams subjected to either pure compression or combined compression and bending loads, convinced the author that such a theory to combine the pre- and post-buckling of thin-walled beams may be possible. This is because most (if not all) the strain energy which is produced during the pre-buckling stage caused the flanges of a channel to form a number of sine waves, their edges straighten after buckling and the whole energy dissipates into the plastic hinge lines. If this phenomenon can be analysed it becomes easy to predict the actual deflection of the cross-section during the buckling.
7.3 Conclusions

Theory of thin-walled beams of open sections was refined, and adequate definitions were made, an element stiffness matrix and transformation matrix were introduced. The comparison with other well known methods proved to give good results. Thus the stress distribution may be used in computing the buckling load of a general structure by means of the finite strip method, which proved to be very accurate and suitable for all types of buckling modes.
REFERENCES

A Thin-Walled Beam Finite Element.
Internal Report TT 8302, April 1983, Department of Transport Technology,
Loughborough University of Technology

BAIGENT, A.H. and HANCOCK, G.J. (1982).
Structural Analysis of Assemblages of Thin-Walled Members.

Finite Element Analysis of Torsional and Torsional-Flexural Stability
Problems.

Computer Methods of Structural Analysis.
Civil Engineering and Engineering MechanicsSeries, N.M. Newmark and

Warping Torson in Commercial Vehicle Frames, Taking into Consideration
Flexible Joints.

BIJLAARD, P.P. (1940).
Theory of the Plastic Stability of Thin Plates.

BLEICH, F. (1952).
McGraw-Hill Book Co. Inc.

Bending Torsion and Buckling of Bars Composed of Thin Walls.
Distortional Buckling of I-Beams.

BRYAN, G.H. (1891).

Finite Strip Method Analysis of Elastic Slabs.

Finite Strip Method in Structural Analysis.

Natural Vibrations of Thin, Flat-Walled Structures with Different Boundary Conditions.

CHILVER, A.H. (1951).
The Behaviour of Thin-Walled Structural Members in Compression.
Engineering, CLXXII, pp 281-282.

CHILVER, A.H. (1953).
The Stability and Strength of Thin-Walled Steel Struts.
The Engineering, August 7, 1953.

CHILVER, A.H. (1953).
Maximum Strength of the Thin-Walled Channel.
Large Deflections Analysis of a Thin-Walled Channel Section Cantilever Beam.

Overall and Local Buckling of Channel Columns.
6513, April, pp 447-462.

GJELSVIK, Atle.
The Theory of Thin-Walled Beams.
John Wiley and Sons, 1981.

GOODIER, J. N. (1941).
The Buckling of Compressed Bars by Torsion and Flexure.

Local, Distortional and Lateral Buckling of I-beams.
Research Report R312, Univ. of Sydney, School of Civil Eng., Sydney, Australia, December.

HARVEY, J. M. (1953).
Structural Strength of Thin-Walled Channel Sections.
Engineering, CLXXV, pp 291-293.

JAMES, B. W. (1935).
Principal Effects of Axial Load on Moment Distribution Analysis of Rigid Structures.
NACA, TN, No. 534.

Beam Buckling by Finite Element Procedure.
March, pp 669-685.

JOURAWSKI, D. J. (1856).
Sur la resistance d'un corps prismatique.
Annales des Ponts et Chaussees, Memoires et Documents, 3rd Series, Vol. 12
Part 2, 1856, pp 328-351.
KAPPUS, R. (1938).
Drillknicken zentrisch gedrückter Stäbemit offenem profil im elastischen
Bereich, Luftfahrt-Forschung, 1937.
Translated by NACA as TM No. 851, 1938.

KRAHULA, J.L. (1967).
Analysis of Bent and Twisted Bars Using the Finite Element Method.

A Consistent Discrete Elements Technique for Thin-Walled Assemblages.

Charts for the Calculation of the Critical Stress for Local Instability
of Columns with I-, Z-, Channel, and Rectangular Tube Section.
ARR 3KD4, WR.L-429, Nov. 1943, NACA.

Matrix Methods of Structural Analysis.

Local Stability of Channel and Z-Sections.
NACA, TN, No. 722.

Local Stability of Symmetrical Rectangular Tubes.
NACA, TN. No. 743.

A Theory for Primary Failure of Straight Centrally Loaded Columns.
NACA, Tech. Rep. 582.
Critical Compressive Stress for Flat Rectangular Plates Elastically
Restrained.
NACA, R. No. 733, and 734.

Principle of Moment Distribution Applied to Stability of Structures
Composed of Bars and Plates.
NACA, ARR, No. 3K06.

Aircraft Structures for Engineering Students.
Edward Arnold.

Introduction to Matrix Methods of Structural Analysis.

Some Basic Plastic Mechanisms in the Local Buckling of Thin-Walled
Steel Structures.


Thin-Walled Compression Members, Parts I, II and III.
Publications of Technische Hochschule, Hannover, January 1959 and
March 1961.

Buckling under Combined Loading of Thin, Flat-Walled Structures by a
Complex Finite Strip Method.
The Aeronautical Journal of the Royal Aeronautical Society, Vol. 72,  
pp 1077-1086.

Theory of Matrix Structural Analysis.  

Matrix Analysis of Local Instability in Plates, Stiffened Panels and  
Columns.

Finite Element Structural Analysis of Local Instability.  

Finite Element Method for Plastic Beam-Columns.  
Theory of Beam Columns, Vol. 2, Space Behaviour and Design. Edited by  
Wai-Fah Chem and Toshie Atsuta.  

Coupled Local Buckling in Wide-Flange Beam-Columns.  

The Local Instability of Thin-Walled Sections Under Combined Compression  
and Bending.  
Third Specialty Conference, 1976, pp 52-70.

Plain Channel Section Struts in Compression and Bending Beyond the Local  
Buckling Load.  
Interaction Behaviour of Plain Channel Columns Under Concentric or
Eccentric Loading.
2nd Int. Colloquium on the Stability of Steel Structures, Liege, 1977,
pp 439-444.

RENTON, J.D. (1960).
A Direct Solution of the Torsional-Flexural Buckling of Axially-
Loaded Thin-Walled Bars.
Structural Engineer, September.

RENTON, J.D. (1967).
Buckling of Frames Composed of Thin-Walled Members.
Thin Walled Structures edited by A.H. Chilver, Chatto and Windus,
London.

On the Transmission of Non-Uniform Torsion Through Joints.
Internal Report No. 1086/74. Oxford University Press.

Derivation of Stiffness Matrices for Beam Type Elements.
Internal Report, Department of Transport Technology, Loughborough Uni-
versity of Technology.

TIMOSHENKO, S. (1945).
Theory of Bending, Torsion and Buckling of Thin-Walled Members of Open
Cross-Section.
Journal Franklin Inst., pp 559-609.

History of Strength of Materials.

McGraw-Hill Book Co. Inc.

Thin-Walled Elastic Beams.
English Translation, National Science Foundation, Washington, DC.
London, Oldbourne Press.

Torsion and Buckling of Open Sections.

Local Instability in Plates and Channel Struts.
Journal of the Structural Division, ASCE, ST3, June, pp 4339-55.

D. Van Nostrand Co.

Performance of Compression Plates as Parts of Structural Members.

A Unified Approach to the Initial Buckling of Stiffened Panels in Compression.
The Aeronautical Quarterly, August, pp 265-283.
Stability Functions for the Local Buckling of Thin Flat-Walled Structures with the Wall in Combined Shear and Compress.
Aeronautical Quarterly, 19, 327-351.

ZBIROHOSKI-KO'SCIA, K. (1967).
Thin-Walled Beams from Theory to Practice.

The Finite Element Method in Engineering Science.

PAFEC 75
PAFEC Ltd, Strelley Hall, Strelley, Nottingham NG8 6PE.
**APPENDIX I**

I-1 Expressions for the out-of-plane stiffness matrices in equations 4.18:

- \( H_0 = \begin{pmatrix} 4b^2 & 6b & 2b^2 & -6b \\ 12 & 6b & -12 & \\ 4b^2 & -6b & & \\ SYM & 12 & & \end{pmatrix} \)

- \( H_v = \begin{pmatrix} 0 & -b & 0 & 0 \\ 0 & 0 & 0 & \\ 0 & b & & \\ SYM & 0 & & \end{pmatrix} \)

\[ H_1 = \begin{pmatrix} 4b^2 & 22b & -3b^2 & 13b \\ 156 & -13b & 54 & \\ 4b^2 & -22b & & \\ SYM & 156 & & \end{pmatrix} \]

\[ H_y = \begin{pmatrix} 4b^2 & 3b & -b^2 & -3b \\ 36 & 3b & -36 & \\ 4b^2 & -3b & & \\ SYM & 36 & & \end{pmatrix} \]

\[ H_2 = \begin{pmatrix} -b^2 & -8b & 0 & b \\ -84 & b & 0 & \\ b^2 & -8b & & \\ SYM & 84 & & \end{pmatrix} \]

\[ H_s = \begin{pmatrix} 0 & -6b & -6^2 & 6b \\ 0 & -6b & 30 & \\ 0 & -6b & & \\ SKEW & SYM & 0 & \end{pmatrix} \]

I-2 Expressions for the in-plane stiffness matrices in equation 4.30:

\[ S_1(1,1) = S_1(3,3) = (E't/b)[1 + (1 - \nu - 2\varepsilon)a^2/6] \]

\[ S_1(2,2) = S_1(4,4) = (E't/b)[(1-\nu)/2 + (1-\varepsilon)a^2/3] \]
\[ S_{I}(1,2) = -S_{I}(3,4) = \left( E't/b \right)(1-3\nu)\alpha/4 \]

\[ S_{I}(1,3) = \left( E't/b \right)[-1 + (1 - \nu - 2\epsilon)\alpha^2/12] \]

\[ S_{I}(1,4) = -S_{I}(2,3) = -\left( E't/b \right)(1+\nu)\alpha/4 \]

\[ A_{I}(2,4) = \left( E't/b \right)[-(1-\nu)/2 + (1-\epsilon)\alpha^2/6] \]

where \[ \epsilon = \frac{(1-\nu^2)\sigma_1}{E} \]